



Boundedness of non-local operators with spatially dependent coefficients and L_p -estimates for non-local equations

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Abstract

We prove the boundedness of the non-local operator

$$\mathcal{L}^a u(x) = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \chi_\alpha(y)(\nabla u(x), y)) a(x, y) \frac{dy}{|y|^{d+\alpha}}$$

from $H_{p,w}^\alpha(\mathbb{R}^d)$ to $L_{p,w}(\mathbb{R}^d)$ for the whole range of $p \in (1, \infty)$, where w is a Muckenhoupt weight. The coefficient $a(x, y)$ is bounded, merely measurable in y , and Hölder continuous in x with an arbitrarily small exponent. We extend the previous results by removing the largeness assumption on p as well as considering weighted spaces with Muckenhoupt weights. Using the boundedness result, we prove the unique solvability in L_p spaces of the corresponding parabolic and elliptic non-local equations.

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1 Introduction

We consider non-local operators of the form

$$\mathcal{L}^a u(x) = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \chi_\alpha(y)(\nabla u(x), y)) a(x, y) \frac{dy}{|y|^{d+\alpha}}, \quad (1.1)$$

where $\alpha \in (0, 2)$ and $\chi_\alpha = 1_{\alpha>1} + 1_{\alpha=1} 1_{B_1}$. Note that the coefficient $a(x, y)$ in the operator $\mathcal{L}^a$ is a function of not only $y \in \mathbb{R}^d$ but also $x \in \mathbb{R}^d$. Non-local operators as above appear in the equations describing various phenomena in physics, mathematical finance, biology, and fluid dynamics (see, for instance, [3, 6]).

The main focus of this paper is the boundedness (i.e., continuity) of the operator $\mathcal{L}^a$ from $H_{p,w}^\alpha(\mathbb{R}^d)$ to $L_{p,w}(\mathbb{R}^d)$, $1 < p < \infty$, under the assumption that the coefficient $a(x, y)$ is measurable in y and continuous in x with an integrability condition on the modulus of continuity (cf. Assumption 2.1 and Remark 2.4), where $H_{p,w}^\alpha(\mathbb{R}^d)$ denotes the Bessel potential space with a Muckenhoupt weight $w \in A_p(\mathbb{R}^d)$. See Theorem 2.5. This result then implies the boundedness of the parabolic operator $\partial_t - \mathcal{L}^a$ with a time-dependent coefficient $a(t, x, y)$ from $\mathcal{H}_{p,q,w}^{1,\alpha}(T)$ to $L_{p,q,w}(T)$, where $\mathcal{H}_{p,q,w}^{1,\alpha}(T)$ and $L_{p,q,w}(T)$ are mixed norm spaces with weights for parabolic equations. See Corollary 6.3. We remark that even in the unweighted case, i.e., $w = 1$, our result is new.

It is well known that, in the study of partial differential equations, a prerequisite is the boundedness of the corresponding differential operators in the considered function spaces. For instance, in the L_p -theory, one usually obtains a priori estimates of solutions and uses the method of continuity along with the unique solvability of a simple equation. However, all of these steps require the boundedness of the involved operator in an L_p type function space. In the local case, for instance, if the operator $\mathcal{L}$ is of the form

$$\mathcal{L}u = a^{ij} D_{ij}u,$$

the boundedness of the operator $\mathcal{L}$ from $W_p^2(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ is straightforward, requiring no regularity assumptions on the coefficients a^{ij} , provided that they are bounded measurable functions. This is also the case when the L_p space is replaced with a weighted L_p space. Similarly, for a given $f \in L_p(\mathbb{R}^d)$, to find a solution $u \in H_p^\alpha(\mathbb{R}^d)$ of the following non-local elliptic equation

$$\mathcal{L}^a u = f,$$

it is essential to have

$$\|\mathcal{L}^a u\|_{L_p(\mathbb{R}^d)} \leq N \|u\|_{H_p^\alpha(\mathbb{R}^d)},$$

where $H_p^\alpha(\mathbb{R}^d)$ is a Bessel potential space without weights. Contrary to the local operator case, the boundedness of non-local operators in L_p spaces is far from being obvious even in the simple case when $a = a(y)$. Indeed, for non-local operators as in (1.1), as far as the authors are aware, there is no result in the literature about the boundedness of the operators if the coefficient $a(x, y)$ is merely bounded measurable without any regularity assumptions as a function of $x \in \mathbb{R}^d$.

There have been considerable studies about the boundedness of non-local operators as in (1.1) in L_p spaces along with L_p estimates of solutions to non-local operators. If the coefficient $a(x, y)$ is constant, the operator $\mathcal{L}^a$ in (1.1) is, by definition, the fractional Laplacian of order α , whose boundedness is guaranteed for $u \in H_p^\alpha(\mathbb{R}^d)$. However, even in the aforementioned simple case, i.e., $a = a(y)$ in the elliptic case and $a = a(t, y)$ in the parabolic case, in

early work about the boundedness of $\mathcal{L}^a$, some conditions were imposed on $a(y)$ or $a(t, y)$ as a function of $y \in \mathbb{R}^d$. See [1, 16], the results of which can be applied to show boundedness of $\mathcal{L}^a$ only when $a = a(y)$ or $a = a(t, y)$ is either symmetric, i.e., $a(-y) = a(y)$, or homogeneous of order zero and sufficiently smooth as a function of $y \in \mathbb{R}^d$. In fact, the papers [1, 16] deal with more general forms of non-local operators. These restrictions on $a(y)$ were removed in [7] so that $a(y)$ can be bounded measurable, but still needs to be a function of only $y \in \mathbb{R}^d$ (one can also consider separable coefficients $a(x, y) = a_1(x)a_2(y)$ by writing $\mathcal{L}^a = a_1(x)\mathcal{L}^{a_2}$).

The boundedness of non-local operators with x -dependent kernels was considered in [17] for L_p spaces with sufficiently large p . In terms of the operator $\mathcal{L}^a$ in (1.1), the coefficient a can be a function of $x \in \mathbb{R}^d$ as well as of $y \in \mathbb{R}^d$ provided that $a(x, y)$ is measurable in y and Hölder continuous in x . Using this boundedness result, the authors also proved the existence and uniqueness in Sobolev spaces of solutions to the Cauchy problem for parabolic integro-differential equations with variable coefficients of the order $\alpha \in (0, 2)$ for sufficiently large p . The main results in [17] have been applied, for instance, in [18, Theorem 2.5] to non-local operators with so-called variable densities, where the investigation was also limited to sufficiently large p .

In this paper, we prove the boundedness of the operator $\mathcal{L}^a$ in (1.1) when $a(x, y)$ is measurable in y and Hölder continuous in x (see Assumption 2.1 and Remark 2.4) for all $p \in (1, \infty)$. That is, we remove the restriction in [17] that p has to be sufficiently large. Moreover, we show the boundedness of the operator in weighted L_p spaces with weight $w \in A_p(\mathbb{R}^d)$. Nevertheless, whether the boundedness of non-local operators still holds for general $a(x, y)$ (for instance, discontinuous or uniformly continuous $a(x, y)$ in x) remains to be an open problem. It is worth mentioning that if Hölder spaces are considered, the boundedness of the operator $\mathcal{L}^a$ under optimal conditions is obtained relatively easily by using perturbation arguments. See, for instance, [8, 15].

As an application of the operator boundedness result, we obtain L_p estimates as in [17] for all $p \in (1, \infty)$. See Theorem 6.5. Since our boundedness result is proved in weighted L_p spaces, one can consider weighted L_p estimates for non-local operators. Regarding results in the weighted setting, see a recent paper [10].

To prove the boundedness of non-local operators, we first make an observation that $\mathcal{L}^a$ with $a = a(y)$ (i.e., a is independent of x) is a bounded operator in *weighted* spaces. This is done by refining the argument in [17] based on singular integral operator theory. Next, we prove the boundedness of the operator with $a = a(x, y)$ in weighted L_p spaces with $A_p(\mathbb{R}^d)$ weights for a sufficiently large p . Then, we use the Rubio de Francia extrapolation theorem to obtain the boundedness for all $p \in (1, \infty)$. While dealing with the operator with $a = a(x, y)$, we first consider $\varphi\mathcal{L}^a$ instead of $\mathcal{L}^a$, where φ is a suitable cut-off function. Similarly as in [17], we employ a cut-off function φ because we write $\mathcal{L}^a u$ as an integral operator with a kernel having an insufficient decay at infinity. See the proof of Lemma 5.1. After that, we apply a partition of unity type argument to obtain the boundedness of $\mathcal{L}^a$. See the proof of Theorem 2.5.

The remaining part of the paper is organized as follows. In the next section, we introduce some notation and state the main result of the paper. In Sect. 3, we present some auxiliary results for the proof of the main result. In Sect. 4, we obtain the boundedness of the operator $\mathcal{L}^a$ in weighted L_p spaces when the coefficient a is a function of y only. We present the proof of the main result in Sect. 5. Finally, using the main result, we obtain the L_p estimates for non-local elliptic and parabolic equations.

2 Notation and main result

For $x \in \mathbb{R}^d$ and $R \in (0, \infty)$, we denote

$$B_R(x) = \{y \in \mathbb{R}^d : |x - y| < R\}, \quad B_R = B_R(0),$$

where d is a positive integer. As usual, $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz function space in $\mathbb{R}^d$ and $C_0^\infty(\mathbb{R}^d)$ is the set of all infinitely differentiable functions with compact support in $\mathbb{R}^d$. By $(-\Delta)^{\alpha/2}$ we mean the fractional Laplacian of order α . That is,

$$\begin{aligned} -(-\Delta)^{\alpha/2}u(x) &= c \text{ P. V. } \int_{\mathbb{R}^d} (u(x+y) - u(x)) \frac{dy}{|y|^{d+\alpha}} \\ &= \frac{c}{2} \int_{\mathbb{R}^d} (u(x+y) + u(x-y) - 2u(x)) \frac{dy}{|y|^{d+\alpha}} \end{aligned}$$

for sufficiently regular u defined on $\mathbb{R}^d$, where

$$c = c(d, \alpha) = \frac{\alpha(2-\alpha)\Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{d/2}2^{2-\alpha}\Gamma\left(2-\frac{\alpha}{2}\right)}.$$

For $1 < p < \infty$, we set $A_p(\mathbb{R}^k)$ to be the set of all nonnegative functions w on $\mathbb{R}^k$ such that

$$\begin{aligned} [w]_{A_p(\mathbb{R}^k)} &:= \sup_{x_0 \in \mathbb{R}^k, R > 0} \left(\frac{1}{|B_R|} \int_{B_R(x_0)} w \, dx \right) \left(\frac{1}{|B_R|} \int_{B_R(x_0)} w^{-1/(p-1)} \, dx \right)^{p-1} < \infty, \end{aligned}$$

where k is either 1 or d in this paper. For $w \in A_p(\mathbb{R}^d)$, by $L_{p,w}(\mathbb{R}^d)$ we mean the space of all measurable functions in $\mathbb{R}^d$ with the norm

$$\|f\|_{L_{p,w}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f|^p w \, dx \right)^{\frac{1}{p}}.$$

For $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^d)$, and $\alpha \in \mathbb{R}$, recall the definition of the weighted Bessel potential space

$$H_{p,w}^\alpha(\mathbb{R}^d) = \{u \in L_{p,w}(\mathbb{R}^d) : (1-\Delta)^{\alpha/2}u \in L_{p,w}(\mathbb{R}^d)\}$$

with

$$\|u\|_{H_{p,w}^\alpha(\mathbb{R}^d)} = \|(1-\Delta)^{\alpha/2}u\|_{L_{p,w}(\mathbb{R}^d)}.$$

As one may expect, we have

$$\|u\|_{H_{p,w}^\alpha(\mathbb{R}^d)} \approx \|u\|_{L_{p,w}(\mathbb{R}^d)} + \|(-\Delta)^{\alpha/2}u\|_{L_{p,w}(\mathbb{R}^d)}. \quad (2.1)$$

In Lemma 3.4 we show the equivalence of the two norms with constants independent of $\alpha \in (0, 2)$. As usual, when $w = 1$, in the above notation we simply remove w so that, for instance, $L_p = L_{p,w}$ and $H_p^\alpha = H_{p,w}^\alpha$.

Throughout the paper, we denote

$$\nabla_y^\alpha u(x) = u(x+y) - u(x) - (\nabla u(x), y)(1_{B_1}(y)1_{\alpha=1} + 1_{\alpha>1})$$

for $x, y \in \mathbb{R}^d$.

We now state our assumption on the coefficient $a(x, y)$ in (1.1).

Assumption 2.1 (i) There is a positive constant K such that for all $x, y \in \mathbb{R}^d$,

$$|a(x, y)| \leq (2 - \alpha)K.$$

(ii) There exist $\beta \in (0, 1)$ and an increasing function $\omega(\tau)$, $\tau > 0$, such that

$$|a(x_1, y) - a(x_2, y)| \leq (2 - \alpha)\omega(|x_1 - x_2|)$$

for $x_1, x_2, y \in \mathbb{R}^d$ and

$$\int_{|y| \leq 1} \omega(|y|) \frac{dy}{|y|^{d+\beta}} < \infty. \quad (2.2)$$

(iii) If $\alpha = 1$, then for any $x \in \mathbb{R}^d$ and $0 < r < R$,

$$\int_{r \leq |y| \leq R} ya(x, y) \frac{dy}{|y|^{d+\alpha}} = 0.$$

Remark 2.2 One noticeable difference between the assumptions on a in [17] and Assumption 2.1 is the term $(2 - \alpha)$. We add $(2 - \alpha)$ to guarantee that the estimates do not blow up when α goes to 2.

Remark 2.3 The condition (iii) in Assumption 2.1 is satisfied, for instance, if $a(x, y)$ is symmetric in y , i.e., $a(x, y) = a(x, -y)$.

Remark 2.4 The condition (ii) in Assumption 2.1 is equivalent to a Hölder continuity condition on $a(x, y)$ as a function of x . Since $\omega(\tau)$ is increasing, we see that

$$\int_{r \leq |y| \leq 2r} \omega(|y|) \frac{dy}{|y|^{d+\beta}} \geq \omega(r) \int_{r \leq |y| \leq 2r} \frac{dy}{|y|^{d+\beta}} \geq Nr^{-\beta} \omega(r),$$

which together with (2.2) indicates that $a(x, y)$ is Hölder continuous in x .

Our main result is about the boundedness of non-local operators. Let us consider the non-local operator $\mathcal{L}^a$ in (1.1).

Theorem 2.5 Let $\beta \in (0, 1)$, $\alpha \in (0, 2)$, $1 < p < \infty$, and $w \in A_p(\mathbb{R}^d)$. Suppose that the coefficient $a = a(x, y)$ satisfies Assumption 2.1. Then $\mathcal{L}^a$ is a bounded operator from $H_{p,w}^\alpha(\mathbb{R}^d)$ to $L_{p,w}(\mathbb{R}^d)$. More precisely, there exists a constant $N = N(d, p, \alpha, \beta, K, \omega, [w]_{A_p})$ such that, for any $u \in H_{p,w}^\alpha(\mathbb{R}^d)$,

$$\|\mathcal{L}^a u\|_{L_{p,w}(\mathbb{R}^d)} \leq N \|u\|_{H_{p,w}^\alpha(\mathbb{R}^d)}.$$

Moreover, the constant N can be chosen so that $N = N(d, p, \alpha_0, \alpha_1, \beta, K, \omega, [w]_{A_p})$ if $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$ and $N = N(d, p, \alpha_0, \beta, K, \omega, [w]_{A_p})$ if $1 < \alpha_0 \leq \alpha < 2$, respectively.

Remark 2.6 In particular, if $1 < \alpha_0 \leq \alpha < 2$, the constant N is independent of α so that the constant N does not blow up as $\alpha \nearrow 2$. Regarding this observation, see also, for instance, [2, 7].

Throughout the paper, we may omit $\mathbb{R}^d$ in $C_0^\infty(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, or $L_p(\mathbb{R}^d)$ whenever the omission is clear from the context. Sometimes we use ‘sup’ to represent the essential supremum. We write $N(d, \delta, \dots)$ in the estimates to express that the constant N is determined only by the parameters $d, \delta, \dots$. The constant N can differ from line to line.

3 Auxiliary results

In this section, we present some auxiliary results.

Lemma 3.1 For any $\alpha \in (0, 1)$ and $f \in \mathcal{S}(\mathbb{R}^d)$,

$$f(x) = N_0 \int_{\mathbb{R}^d} |z|^{-d+\alpha} (-\Delta)^{\alpha/2} f(x-z) dz$$

and

$$f(x+y) - f(x) = N_0 \int_{\mathbb{R}^d} k^\alpha(z, y) (-\Delta)^{\alpha/2} f(x-z) dz,$$

where

$$k^\alpha(z, y) = |z+y|^{-d+\alpha} - |z|^{-d+\alpha}$$

is an integrable function of z and

$$N_0 = N_0(\alpha, d) = \frac{\Gamma((d-\alpha)/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)}.$$

Proof The first assertion follows from

$$\mathcal{F}^{-1}[[\xi]^{-\alpha}](x) = N_0 |x|^{-d+\alpha}, \quad (3.1)$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. See Lemma 2.1 in [11] for the constant N_0 and the integrability of k^α . $\square$

Lemma 3.2 Let $\gamma \in (0, 2)$, $R \in (0, \infty)$, and $u \in L_{1,\text{loc}}(\mathbb{R}^d)$. Then

$$\int_{B_R^c} |u(x+y)| \frac{dy}{|y|^{d+\gamma}} \leq N R^{-\gamma} \gamma^{-1} \mathcal{M}u(x) \quad (3.2)$$

and

$$\int_{B_R} |u(x+y)| \frac{dy}{|y|^{d-\gamma}} \leq N R^\gamma \gamma^{-1} \mathcal{M}u(x),$$

where N depends only on d .

Proof Denote $B_j = B_{2^j R}$ for $j = 0, 1, \dots$, and observe that

$$\begin{aligned} \int_{B_R^c} |u(x+y)| \frac{dy}{|y|^{d+\gamma}} &= \sum_{j=0}^{\infty} \int_{B_{j+1} \setminus B_j} |u(x+y)| \frac{dy}{|y|^{d+\gamma}} \\ &\leq \sum_{j=0}^{\infty} R^{-(d+\gamma)} 2^{-j(d+\gamma)} \int_{B_{j+1} \setminus B_j} |u(x+y)| dy \\ &\leq N R^{-\gamma} \sum_{j=0}^{\infty} 2^{-\gamma j} \int_{B_{j+1}} |u(x+y)| dy \\ &\leq N R^{-\gamma} (1 - 2^{-\gamma})^{-1} \mathcal{M}u(x) \leq N R^{-\gamma} \gamma^{-1} \mathcal{M}u(x), \end{aligned}$$

which shows the first inequality (3.2). For the second inequality, we set $B_j = B_{2^{-j} R}$ for $j = 0, 1, \dots$, and proceed similarly as above with $B_j \setminus B_{j+1}$ in place of $B_{j+1} \setminus B_j$. $\square$

The following lemma is a weighted- L_p norm inequality for multiplier operators in [13]. From the proofs there, one can see that the constant N depends only on the parameters described below. As usual, we denote a multi-index by $\gamma = (\gamma_1, \dots, \gamma_d)$ with $|\gamma| = \gamma_1 + \dots + \gamma_d$, where γ_i is a nonnegative integer.

Lemma 3.3 *Let m be a bounded multiplier of an operator T defined on $\mathcal{S}(\mathbb{R}^d)$. If there is a constant C such that for any $|\gamma| \leq d$,*

$$\sup_{R>0} \left(R^{2|\gamma|-d} \int_{R<|\xi|<2R} |D^\gamma m(\xi)|^2 d\xi \right)^{1/2} \leq C, \quad (3.3)$$

then for any $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$, there exists a constant $N = N(d, p, C, [w]_{A_p})$ such that

$$\|Tf\|_{L_{p,w}} \leq N\|f\|_{L_{p,w}}$$

for any $f \in \mathcal{S}(\mathbb{R}^d)$.

The next lemma is about the equivalence of the $H_{p,w}^\alpha$ norms stated in (2.1).

Lemma 3.4 *Let $\alpha \in [0, \alpha_0]$ and $1 < p < \infty$. Then for any $w \in A_p(\mathbb{R}^d)$, there exists a constant $N = N(d, p, \alpha_0, [w]_{A_p})$ such that*

$$N^{-1}(\|u\|_{L_{p,w}} + \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}) \leq \|u\|_{H_{p,w}^\alpha} \leq N(\|u\|_{L_{p,w}} + \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}) \quad (3.4)$$

for any $u \in H_{p,w}^\alpha(\mathbb{R}^d)$.

Proof To prove (3.4), it suffices to show that, for $u \in \mathcal{S}(\mathbb{R}^d)$,

$$\|(1 - \Delta)^{-\alpha/2} u\|_{L_{p,w}} \leq N\|u\|_{L_{p,w}}, \quad \|(-\Delta)^{\alpha/2} (1 - \Delta)^{-\alpha/2} u\|_{L_{p,w}} \leq N\|u\|_{L_{p,w}} \quad (3.5)$$

for the first inequality, and

$$\|(1 - \Delta)^{\alpha/2} (1 + (-\Delta)^{\alpha/2})^{-1} u\|_{L_{p,w}} \leq N\|u\|_{L_{p,w}}$$

for the second inequality. We set

$$m_1(\xi) = (1 + |\xi|^2)^{\alpha/2}, \quad m_2(\xi) = 1 + |\xi|^\alpha.$$

Then, by differentiating, one can check that the following multipliers

$$\frac{1}{m_1(\xi)}, \quad \frac{|\xi|^\alpha}{m_1(\xi)}, \quad \frac{m_1(\xi)}{m_2(\xi)}$$

satisfy the assumptions of Lemma 3.3, especially, the inequality (3.3). For instance, we use

$$|D^\gamma m_1| \leq N(1 + |\xi|^2)^{(\alpha-|\gamma|)/2}, \quad |D^\gamma m_2| \leq N|\xi|^{\alpha-|\gamma|} + 1_{|\gamma|=0},$$

$$\left| D^\gamma \left(\frac{1}{m_2(\xi)} \right) \right| \leq N|\xi|^{-|\gamma|} 1_{|\xi| \leq 2} + N|\xi|^{-\alpha-|\gamma|} 1_{|\xi| \geq 1}$$

to obtain that

$$\left| D^\gamma \left(\frac{m_1(\xi)}{m_2(\xi)} \right) \right| \leq N|\xi|^{-|\gamma|},$$

where $N = N(d, \alpha_0, \gamma)$. In particular, N can be chosen depending on α_0 instead of α . The lemma is proved. $\square$

The following lemma is a classical result for the Riesz transforms and the Hilbert transform (see, for instance, [19]).

Lemma 3.5 *Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$. There exists $N = N(d, p, [w]_{A_p})$ such that, for any $u \in L_{p,w}(\mathbb{R}^d)$,*

$$\|(-\Delta)^{-1/2} \nabla u\|_{L_{p,w}} \leq N \|u\|_{L_{p,w}}.$$

Lemma 3.6 *Let $\alpha \in (0, 2)$, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^d)$. There exists a constant $N = N(d, p, \alpha, [w]_{A_p})$ such that, for any $u \in H_{p,w}^\alpha(\mathbb{R}^d)$, $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |u(x+y) - u(x)| |\varphi(x+y-z) - \varphi(x-z)| \frac{2-\alpha}{|y|^{d+\alpha}} dy \right)^p w(x) dx dz \\ & \leq N \|u\|_{H_{p,w}^\alpha}^p \|\varphi\|_{H_1^1}^p. \end{aligned} \quad (3.6)$$

Moreover, the constant N can be chosen so that $N = N(d, p, \alpha_0, \alpha_1, [w]_{A_p})$ if $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$ and $N = N(d, p, [w]_{A_p})$ if $1 \leq \alpha < 2$, respectively.

Proof We write

$$\begin{aligned} & \int_{\mathbb{R}^d} |u(x+y) - u(x)| |\varphi(x+y-z) - \varphi(x-z)| \frac{dy}{|y|^{d+\alpha}} \\ & = \left(\int_{B_1} + \int_{B_1^c} \right) \left(\cdots \frac{dy}{|y|^{d+\alpha}} \right). \end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned} & \int_{B_1^c} |u(x+y) - u(x)| |y|^{-(d+\alpha)} dy \\ & \leq \int_{B_1^c} |u(x+y)| |y|^{-(d+\alpha)} dy + N \alpha^{-1} |u(x)| \leq N \alpha^{-1} \mathcal{M}u(x). \end{aligned} \quad (3.7)$$

Using the Minkowski inequality in z , (3.7), and the weighted Hardy-Littlewood maximal function theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{B_1^c} |u(x+y) - u(x)| |\varphi(x+y-z) - \varphi(x-z)| \frac{dy}{|y|^{d+\alpha}} \right)^p dz w(x) dx \\ & \leq N \|\varphi\|_{L_p}^p \int_{\mathbb{R}^d} \left(\int_{B_1^c} |u(x+y) - u(x)| \frac{dy}{|y|^{d+\alpha}} \right)^p w(x) dx \\ & \leq N \alpha^{-p} \|\varphi\|_{L_p}^p \int_{\mathbb{R}^d} (\mathcal{M}u(x))^p w(x) dx \leq N \alpha^{-p} \|\varphi\|_{L_p}^p \|u\|_{L_{p,w}}^p, \end{aligned} \quad (3.8)$$

where $N = N(d, p, [w]_{A_p})$.

Note that for $\alpha < 1$, by Lemma 3.2,

$$\begin{aligned} & \int_{B_1} |u(x+y) - u(x)| |y|^{-(d+\alpha-1)} dy \\ & \leq \int_{B_1} |u(x+y)| |y|^{-(d+\alpha-1)} dy + N(1-\alpha)^{-1} |u(x)| \\ & \leq N(1-\alpha)^{-1} \mathcal{M}u(x), \end{aligned} \quad (3.9)$$

where $N = N(d)$. By using (3.9) and the Minkowski inequality, for $\alpha < 1$, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{B_1} |u(x+y) - u(x)| |\varphi(x+y-z) - \varphi(x-z)| \frac{dy}{|y|^{d+\alpha}} \right)^p dz w(x) dx \\ & \leq \|D\varphi\|_{L_p}^p \int_{\mathbb{R}^d} \left(\int_{B_1} |u(x+y) - u(x)| \frac{dy}{|y|^{d+\alpha-1}} \right)^p w(x) dx \\ & \leq N(1-\alpha)^{-p} \|D\varphi\|_{L_p}^p \int_{\mathbb{R}^d} (\mathcal{M}u(x))^p w(x) dx \\ & \leq N(1-\alpha)^{-p} \|D\varphi\|_{L_p}^p \|u\|_{L_{p,w}}^p, \end{aligned} \quad (3.10)$$

where $N = N(d, p, [w]_{A_p})$.

For $1 \leq \alpha < 2$, we observe that by Lemma 3.2 with $R = s$,

$$\begin{aligned} & \int_{B_1} |u(x+y) - u(x)| \frac{dy}{|y|^{d+\alpha-1}} \leq \int_0^1 \int_{B_1} |Du(x+sy)| \frac{dy}{|y|^{d+\alpha-2}} ds \\ & \leq \int_0^1 s^{\alpha-2} \int_{B_s} |Du(x+y)| \frac{dy}{|y|^{d+\alpha-2}} ds \leq N(2-\alpha)^{-1} \mathcal{M}(Du)(x), \end{aligned} \quad (3.11)$$

where the constant N depends only on d . By utilizing (3.11) and the Minkowski inequality, we see that

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{B_1} |u(x+y) - u(x)| |\varphi(x+y-z) - \varphi(x-z)| \frac{dy}{|y|^{d+\alpha}} \right)^p dz w(x) dx \\ & \leq \|D\varphi\|_{L_p}^p \int_{\mathbb{R}^d} \left(\int_{B_1} |u(x+y) - u(x)| \frac{dy}{|y|^{d+\alpha-1}} \right)^p w(x) dx \\ & \leq N(2-\alpha)^{-p} \|D\varphi\|_{L_p}^p \int_{\mathbb{R}^d} (\mathcal{M}Du(x))^p w(x) dx \\ & \leq N(2-\alpha)^{-p} \|D\varphi\|_{L_p}^p \|Du\|_{L_{p,w}}^p, \end{aligned} \quad (3.12)$$

where $N = N(d, p, [w]_{A_p})$. Combining (3.8), (3.10), (3.12), and Lemmas 3.4 and 3.5, we obtain (3.6). In particular, when $\alpha \in [1, 2)$, by Lemma 3.4 and the first inequality in (3.5),

$$\|Du\|_{L_{p,w}} \leq N\|u\|_{H_{p,w}^1} \leq N\|u\|_{H_{p,w}^\alpha}. \quad (3.13)$$

Finally, due to the presence of the term $2-\alpha$ in (3.6), from the above proof we see that N can be chosen as described in the lemma if $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$ or $1 \leq \alpha < 2$. $\square$

Lemma 3.7 *Let $\alpha \in (0, 2)$, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^d)$. There exists a constant $N = N(d, p, \alpha, [w]_{A_p})$ such that for any $u \in H_{p,w}^\alpha(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \left(\int_{|y| \geq 1} \left| \nabla_y^\alpha u(x) \right| \frac{dy}{|y|^{d+\alpha}} \right)^p w(x) dx \leq N \left(\|u\|_{L_{p,w}}^p 1_{\alpha \leq 1} + \|Du\|_{L_{p,w}}^p 1_{\alpha > 1} \right).$$

Moreover, the constant N can be chosen so that $N = N(d, p, \alpha_0, [w]_{A_p})$ if $0 < \alpha_0 \leq \alpha \leq 1$ or $1 < \alpha_0 \leq \alpha < 2$. (An upper bound α_1 for $\alpha \in (0, 1)$ does not appear in this case.)

Proof Observe that for $|y| \geq 1$,

$$\begin{aligned} |\nabla_y^\alpha u(x)| & \leq \left(\int_0^1 |\nabla u(x+sy)| ds + |\nabla u(x)| \right) |y| 1_{\alpha > 1} \\ & \quad + (|u(x+y)| + |u(x)|) 1_{\alpha \leq 1}, \end{aligned}$$

which implies that

$$\int_{|y| \geq 1} \left| \nabla_y^\alpha u(x) \right| \frac{dy}{|y|^{d+\alpha}} \leq N\alpha^{-1} \mathcal{M}u(x) 1_{\alpha \leq 1} + N(\alpha - 1)^{-1} \mathcal{M}Du(x) 1_{\alpha > 1}$$

by Lemma 3.2. Then our assertion follows directly from the weighted Hardy-Littlewood maximal function theorem. $\square$

Lemma 3.8 *Let $0 < \alpha < 2$, $p \in (1, \infty)$, and $w \in A_p(\mathbb{R}^d)$. Then, there is a constant $N = N(d, p, \alpha, [w]_{A_p})$ such that for any $u \in H_{p,w}^2(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left| \nabla_y^\alpha u(x) \right| (2 - \alpha) \frac{dy}{|y|^{d+\alpha}} \right)^p w(x) dx \leq N \|u\|_{H_{p,w}^2}^p.$$

Moreover, the constant N can be chosen so that $N = N(d, p, \alpha_0, \alpha_1, [w]_{A_p})$ if $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$ and $N = N(d, p, \alpha_0, [w]_{A_p})$ if $1 < \alpha_0 \leq \alpha < 2$, respectively.

Proof For $\alpha \geq 1$, by using a similar argument as in (3.11) and Lemma 3.2, we see that there is a constant $N = N(d)$ such that

$$\begin{aligned} & \int_{B_1} |u(x+y) - u(x) - (\nabla u(x), y)| \frac{dy}{|y|^{d+\alpha}} \\ & \leq \int_0^1 \int_0^1 \int_{B_1} s |D^2 u(x+sty)| \frac{dy}{|y|^{d-(2-\alpha)}} dt ds \\ & \leq N(2-\alpha)^{-1} \mathcal{M}(D^2 u)(x). \end{aligned} \quad (3.14)$$

By Lemma 3.7, (3.14), and the weighted Hardy-Littlewood maximal function theorem, we reach our assertion. For $\alpha < 1$, we proceed similarly upon obtaining

$$\int_{B_1} |u(x+y) - u(x)| \frac{dy}{|y|^{d+\alpha}} \leq N(1-\alpha)^{-1} \mathcal{M}(Du)(x).$$

$\square$

4 Boundedness of non-local operator with $a = a(y)$ in A_p weighted spaces

If the coefficient a of the non-local operator $\mathcal{L}^a$ in (1.1) is a function of y only, $\mathcal{L}^a$ is a bounded operator from H_p^α to L_p . See [7, 17]. We extend this result in the setting of A_p weighted spaces, which is an essential ingredient to the proof of the main theorem.

Proposition 4.1 *Let $p \in (1, \infty)$ and $\alpha \in (0, 2)$. Suppose that the x -independent coefficient $a = a(y)$ of a non-local operator $\mathcal{L}^a$ satisfies Assumption 2.1. Then, for each $w \in A_p(\mathbb{R}^d)$, there is a constant $N = N(d, p, \alpha, [w]_{A_p})$ such that for any $u \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|\mathcal{L}^a u\|_{L_{p,w}} \leq NK \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}, \quad (4.1)$$

where K is the constant in Assumption 2.1. Moreover, the constant N can be chosen so that $N = N(d, p, \alpha_0, \alpha_1, [w]_{A_p})$ if $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$ and $N = N(d, p, \alpha_0, [w]_{A_p})$ if $1 < \alpha_0 \leq \alpha < 2$, respectively. Furthermore, if the condition (iii) in Assumption 2.1 is satisfied for $\alpha \in (0, 2)$, the constant N depends only on $d, p, [w]_{A_p}$, and α_0 , where $0 < \alpha_0 \leq \alpha < 2$.

Before proving Proposition 4.1, we introduce a well known result from harmonic analysis. We refer the reader to [4, 5, 14].

Lemma 4.2 *Let T be a singular integral operator of the type*

$$Tf(x) = \int_{\mathbb{R}^d} K(x-z)f(z) dz.$$

Assume that there are positive constants γ and C_1 such that

$$|K(x+z) - K(x)| \leq C_1 \frac{|z|^\gamma}{|x|^{d+\gamma}}, \quad |x| \geq 4|z|.$$

Also assume that there is a constant C_2 such that

$$\|Tf\|_{L_2} \leq C_2 \|f\|_{L_2}$$

for any $f \in L_2(\mathbb{R}^d)$. Then, for any $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^d)$, and $f \in L_{p,w}(\mathbb{R}^d)$, there is a constant $N = N(d, p, \gamma, C_1, C_2, [w]_{A_p})$ such that

$$\|Tf\|_{L_{p,w}} \leq N \|f\|_{L_{p,w}}.$$

Now, we prove Proposition 4.1.

Proof of proposition 4.1

To show Proposition 4.1, we exploit an idea in [17]. We first consider the case when $\alpha \in (0, 1)$. For $\varepsilon \in (0, 1)$, set $a_\varepsilon(y) = a(y)1_{\varepsilon \leq |y| \leq \varepsilon^{-1}}$. Using Fubini's Theorem and Lemma 3.1,

$$\begin{aligned} \mathcal{L}^a u(x) &= \int_{\mathbb{R}^d} (u(x+y) - u(x))a(y) \frac{dy}{|y|^{d+\alpha}} \\ &= \lim_{\varepsilon \rightarrow 0} N_0 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k^\alpha(z, y)(-\Delta)^{\alpha/2} u(x-z) dz \right) a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \\ &= \lim_{\varepsilon \rightarrow 0} N_0 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k^\alpha(z, y)a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \right) (-\Delta)^{\alpha/2} u(x-z) dz. \end{aligned}$$

For $\varepsilon \in (0, 1)$, set

$$T^\varepsilon f(x) = \int_{\mathbb{R}^d} k_\varepsilon(x-z)f(z) dz,$$

where

$$k_\varepsilon(x) = \int_{\mathbb{R}^d} k^\alpha(x, y)a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}},$$

so that in the pointwise sense,

$$\mathcal{L}^a u(x) = N_0 \lim_{\varepsilon \rightarrow 0} T^\varepsilon (-\Delta)^{\alpha/2} u(x).$$

From now on, we check that T^ε satisfies the conditions in Lemma 4.2.

Lemma 4.3 *Let $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$. Assume that $|a(y)| \leq 1$ for all $y \in \mathbb{R}^d$. Then, there is a constant $N = N(d, \alpha_0, \alpha_1)$ such that for any $f \in L_2(\mathbb{R}^d)$,*

$$\|T^\varepsilon f\|_{L_2} \leq N \|f\|_{L_2}.$$

Proof Since

$$\mathcal{F}[T^\varepsilon f] = \mathcal{F}[k_\varepsilon]\mathcal{F}[f],$$

we only show that $\mathcal{F}[k_\varepsilon]$ is bounded. Bearing (3.1) in mind, we see that, by denoting $\hat{\xi} = \xi/|\xi|$,

$$\begin{aligned} |\mathcal{F}[k_\varepsilon](\xi)| &= \left| \int_{\mathbb{R}^d} \mathcal{F}[k^\alpha(\cdot, y)](\xi) a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \right| \\ &\leq N |\xi|^{-\alpha} \int_{\mathbb{R}^d} |e^{i(\hat{\xi}, y)} - 1| \frac{dy}{|y|^{d+\alpha}} = N \int_{\mathbb{R}^d} |e^{i(\hat{\xi}, y)} - 1| \frac{dy}{|y|^{d+\alpha}} \\ &\leq N \int_{B_1} \frac{dy}{|y|^{d+\alpha-1}} + N \int_{B_1^c} \frac{dy}{|y|^{d+\alpha}} \leq N(d, \alpha_0, \alpha_1). \end{aligned}$$

The lemma is proved. $\square$

Lemma 4.4 Let $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$. Suppose that for all $y \in \mathbb{R}^d$, $|a(y)| \leq 1$. Then, for all $|x| > 4|z|$,

$$|k_\varepsilon(x+z) - k_\varepsilon(x)| \leq N \frac{|z|^\gamma}{|x|^{d+\gamma}}, \quad (4.2)$$

where $N = N(d, \alpha_0, \alpha_1)$ and $\gamma = \gamma(\alpha_0, \alpha_1)$.

Proof In fact, the lemma follows from the proof of [17, Lemma 14], where the authors prove, instead of (4.1), the boundedness of $\mathcal{L}^a$ in L_p spaces *without weights* by obtaining

$$\int_{|x|>4|z|} |k_\varepsilon(x+z) - k_\varepsilon(x)| dx \leq C, \quad z \in \mathbb{R}^d$$

instead of (4.2). However, the proof there also shows the inequality (4.2). For the reader's convenience, here we present a slightly simplified version of the proof of [17, Lemma 14] dedicated to verify the inequality (4.2).

Fix $\varepsilon \in (0, 1)$. For notational simplicity, we replace $a_\varepsilon, k_\varepsilon$ with a, k , respectively. For any $r > 0$,

$$\begin{aligned} k(rx) &= \int_{\mathbb{R}^d} k^\alpha(rx, y) a(y) \frac{dy}{|y|^{d+\alpha}} \\ &= \int_{\mathbb{R}^d} (|rx+y|^{-d+\alpha} - |rx|^{-d+\alpha}) a(y) \frac{dy}{|y|^{d+\alpha}} \\ &= r^{-d+\alpha} \int_{\mathbb{R}^d} (|x+\frac{y}{r}|^{-d+\alpha} - |x|^{-d+\alpha}) a(y) \frac{dy}{|y|^{d+\alpha}} \\ &= r^{-d} \int_{\mathbb{R}^d} (|x+y|^{-d+\alpha} - |x|^{-d+\alpha}) a(r y) \frac{dy}{|y|^{d+\alpha}} \\ &=: r^{-d} k(x, r). \end{aligned}$$

Hence, for $|x| > 4|z|$, by writing

$$k(x+z) - k(x) = k\left(|x| \frac{x+z}{|x|}\right) - k\left(|x| \frac{x}{|x|}\right)$$

and using the above $k(\cdot, \cdot)$, we have

$$|k(x+z) - k(x)| = |x|^{-d} \left(k\left(\frac{x+z}{|x|}, |x|\right) - k\left(\frac{x}{|x|}, |x|\right) \right)$$

$$\leq |x|^{-d} \int \left| \left| \hat{x} + \frac{\hat{z}}{r} + y \right|^{-d+\alpha} - \left| \hat{x} + \frac{\hat{z}}{r} \right|^{-d+\alpha} - |\hat{x} + y|^{-d+\alpha} + |\hat{x}|^{-d+\alpha} \right| \frac{dy}{|y|^{d+\alpha}},$$

where $r = |x|/|z| (> 4)$, $\hat{x} = x/|x|$ and $\hat{z} = z/|z|$. By using an orthogonal transformation, we may assume that $\hat{x} = e = (1, 0, \dots, 0) \in \mathbb{R}^d$. We split

$$\begin{aligned} I &= \int \left| \left| e + \frac{\hat{z}}{r} + y \right|^{-d+\alpha} - \left| e + \frac{\hat{z}}{r} \right|^{-d+\alpha} - |e + y|^{-d+\alpha} + |e|^{-d+\alpha} \right| \frac{dy}{|y|^{d+\alpha}} \\ &= \int_{|y| \leq \frac{1}{r}} \dots + \int_{\frac{1}{r} \leq |y|, \frac{1}{2} \leq |y+e|} \dots + \int_{|y+e| \leq \frac{2}{r}} \dots + \int_{\frac{2}{r} \leq |y+e| \leq \frac{1}{2}} \dots \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For any $a, b \in \mathbb{R}^d$ satisfying $|a + tb| > 0$ for any $t \in [0, 1]$, it follows that

$$\begin{aligned} \left| |a + b|^{-d+\alpha} - |a|^{-d+\alpha} \right| &\leq N|b| \int_0^1 |a + tb|^{-d+\alpha-1} dt \\ &\leq N|b| \left(\inf_{0 \leq t \leq 1} |a + tb| \right)^{-d+\alpha-1}. \end{aligned} \quad (4.3)$$

Estimate of I_1 . For any $0 \leq t \leq 1$ and $|y| \leq 1/r$,

$$|e + ty|, \quad |e + \frac{\hat{z}}{r} + ty| \geq \frac{1}{2}.$$

Thus, bearing (4.3) in mind,

$$\begin{aligned} I_1 &\leq \int_{|y| \leq \frac{1}{r}} \left| \left| e + \frac{\hat{z}}{r} + y \right|^{-d+\alpha} - \left| e + \frac{\hat{z}}{r} \right|^{-d+\alpha} \right| + \left| |e + y|^{-d+\alpha} - |e|^{-d+\alpha} \right| \frac{dy}{|y|^{d+\alpha}} \\ &\leq N \int_{|y| \leq \frac{1}{r}} \frac{dy}{|y|^{d+\alpha-1}} \leq Nr^{\alpha-1}. \end{aligned}$$

Estimate of I_2 . For any $0 \leq t \leq 1$ and $|y + e| \geq \frac{1}{2}$,

$$|e + t\frac{\hat{z}}{r} + y|, \quad |e + t\frac{\hat{z}}{r}| \geq \frac{1}{4}.$$

By (4.3), I_2 is bounded by

$$\begin{aligned} &\int_{\frac{1}{r} \leq |y|, \frac{1}{2} \leq |y+e|} \left| \left| e + \frac{\hat{z}}{r} + y \right|^{-d+\alpha} - |e + y|^{-d+\alpha} \right| + \left| \left| e + \frac{\hat{z}}{r} \right|^{-d+\alpha} - |e|^{-d+\alpha} \right| \frac{dy}{|y|^{d+\alpha}} \\ &\leq \frac{N}{r} \int_{\frac{1}{r} \leq |y|} \frac{dy}{|y|^{d+\alpha}} \leq Nr^{\alpha-1}. \end{aligned}$$

Estimate of I_3 . For $|y + e| \leq \frac{2}{r}$, we see that

$$|y| \geq \frac{1}{2}, \quad |e + \frac{\hat{z}}{r} + y| \leq \frac{3}{r}.$$

Therefore, using $|e + \hat{z}/r| \geq 1/2$,

$$I_3 \leq N \int_{|e + \frac{\hat{z}}{r} + y| \leq \frac{3}{r}} \left(|e + \frac{\hat{z}}{r} + y|^{-d+\alpha} + 1 \right) dy + N \int_{|e + y| \leq \frac{2}{r}} \left(|e + y|^{-d+\alpha} + 1 \right) dy$$

$$\leq N \int_{|y| \leq \frac{3}{r}} |y|^{-d+\alpha} dy \leq Nr^{-\alpha}.$$

Estimate of I_4 . For $\frac{2}{r} \leq |y+e| \leq \frac{1}{2}$, $t \in [0, 1]$, we have

$$\frac{1}{r} \leq |e + t\frac{\hat{z}}{r} + y| \leq 1, \quad \frac{1}{2} \leq |y|, \quad |e + t\frac{\hat{z}}{r}| \geq \frac{1}{2}.$$

By (4.3), we arrive at

$$\begin{aligned} I_4 &\leq \int_{\frac{2}{r} \leq |e+y| \leq \frac{1}{2}} \left| |e + \frac{\hat{z}}{r} + y|^{-d+\alpha} - |e+y|^{-d+\alpha} \right| + \left| |e + \frac{\hat{z}}{r}|^{-d+\alpha} - |e|^{-d+\alpha} \right| \frac{dy}{|y|^{d+\alpha}} \\ &\leq Nr^{-1} \int_0^1 \int_{\frac{1}{r} \leq |e+t\frac{\hat{z}}{r}+y| \leq 1} |e + t\frac{\hat{z}}{r} + y|^{-d+\alpha-1} dy dt \\ &\quad + Nr^{-1} \int_0^1 \int_{\frac{2}{r} \leq |e+y| \leq 1/2} |e + t\frac{\hat{z}}{r}|^{-d+\alpha-1} dy dt \\ &\leq Nr^{-1} \int_{\frac{1}{r} \leq |y| \leq 1} |y|^{-d+\alpha-1} dy + Nr^{-1} \leq Nr^{-\alpha}. \end{aligned}$$

Finally, by taking $\gamma = \min\{1 - \alpha_1, \alpha_0\}$, we have

$$I \leq Nr^{-\gamma}.$$

The lemma is proved. $\square$

Let us continue the proof of Proposition 4.1. Clearly, we can assume that $K = 1$. By combining Lemmas 4.2, 4.3 and 4.4, we see that the assertion in Proposition 4.1 holds when $\alpha \in (0, 1)$. Note that the constant N in (4.1) can be chosen depending only on α_0 and α_1 , but independent of α , provided that $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$.

To complete the proof of Proposition 4.1, we proceed as follows. First, we prove again the inequality (4.1) for $\alpha \in [3/4, 1)$ to present the exact dependency of the constant N on α as $\alpha \nearrow 1$. Then, using this result, we prove the inequality (4.1) for $\alpha \in (1, 2)$. In particular, we show that the constant N can be chosen independent of α if $1 < \alpha_0 \leq \alpha < 2$. Finally, we prove the case $\alpha = 1$. In the steps below, we use the idea in the proof of [17, Lemma 15].

As above, we set $a_\varepsilon(y) = a(y)I_{\varepsilon < |y| < \varepsilon^{-1}}$ and denote

$$\mathcal{L}_\varepsilon^\alpha u(x) = \int_{\mathbb{R}^d} \nabla_y^\alpha u(x) a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}}.$$

Then we have

$$\mathcal{L}^\alpha u(x) = \lim_{\varepsilon \rightarrow 0} \mathcal{L}_\varepsilon^\alpha u(x).$$

Thus, to prove (4.1) it suffices to show that

$$\|\mathcal{L}_\varepsilon^\alpha u\|_{L_{p,w}} \leq N \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}, \quad (4.4)$$

where $N = N(d, p, \alpha, [w]_{A_p})$, but independent of ε . Let $\alpha \in [3/4, 1)$. By Lemma 3.1, we see that

$$\begin{aligned} \mathcal{L}_\varepsilon^\alpha u(x) &= N_0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k^{1/2}(z, y) (-\Delta)^{1/4} u(x-z) dz a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \\ &= N_0 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k^{1/2}(z, y) ((-\Delta)^{1/4} u(x-z) - (-\Delta)^{1/4} u(x)) dz a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \end{aligned}$$

$$= N_0 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k^{1/2}(z, y) a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} \right) ((-\Delta)^{1/4} u(x-z) - (-\Delta)^{1/4} u(x)) dz, \quad (4.5)$$

where $N_0 = N_0(d)$ and the second equality is due to

$$\int_{\mathbb{R}^d} k^{1/2}(z, y) dz = 0.$$

Note that, if we set

$$m_\varepsilon(z) := \int_{\mathbb{R}^d} \left(\left| \frac{z}{|z|} + y \right|^{-d+1/2} - 1 \right) a_\varepsilon(|z|y) \frac{dy}{|y|^{d+\alpha}},$$

then, for $z \neq 0$,

$$\int_{\mathbb{R}^d} k^{1/2}(z, y) a_\varepsilon(y) \frac{dy}{|y|^{d+\alpha}} = \frac{1}{|z|^{d-1/2+\alpha}} m_\varepsilon(z).$$

From (4.5), we get that

$$\mathcal{L}_\varepsilon^\alpha u(x) = N_0 \int_{\mathbb{R}^d} ((-\Delta)^{1/4} u(x-z) - (-\Delta)^{1/4} u(x)) m_\varepsilon(z) \frac{dz}{|z|^{d-1/2+\alpha}}. \quad (4.6)$$

If we have

$$|m_\varepsilon(z)| \leq (1-\alpha)^{-1} N \quad (4.7)$$

for any $z \neq 0$, where N depends only on d , but independent of ε , then the representation (4.6) along with Proposition 4.1 for order $\alpha - 1/2$ proves (4.4). Thus, it remains to show (4.7). Upon denoting $\hat{z} = z/|z|$, we write

$$\begin{aligned} m_\varepsilon(z) &= \int_{|y| \leq 1/2} \left(\frac{1}{|\hat{z} + y|^{d-1/2}} - 1 \right) a_\varepsilon(|z|y) \frac{dy}{|y|^{d+\alpha}} \\ &\quad + \int_{|y| > 1/2} \left(\frac{1}{|\hat{z} + y|^{d-1/2}} - 1 \right) a_\varepsilon(|z|y) \frac{dy}{|y|^{d+\alpha}} =: I_\varepsilon + J_\varepsilon. \end{aligned}$$

Set

$$f(t) = \frac{1}{|\hat{z} + ty|^{d-1/2}}, \quad t \in [0, 1].$$

Then, since $|\hat{z} + ty| \geq 1/2$ for $|y| \leq 1/2$,

$$\left| \frac{1}{|\hat{z} + y|^{d-1/2}} - 1 \right| = |f(1) - f(0)| \leq \sup_{t \in [0, 1]} |f'(t)| \leq N|y|.$$

Therefore,

$$|I_\varepsilon| \leq N \int_{|y| \leq 1/2} |a_\varepsilon(|z|y)| \frac{dy}{|y|^{d+\alpha-1}} \leq (1-\alpha)^{-1} N,$$

where $N = N(d)$. To estimate J_ε , we split

$$\begin{aligned} J_\varepsilon &= \int_{|y| > 1/2, |\hat{z} + y| \geq 1/4} \left(\frac{1}{|\hat{z} + y|^{d-1/2}} - 1 \right) a_\varepsilon(|z|y) \frac{dy}{|y|^{d+\alpha}} \\ &\quad + \int_{|\hat{z} + y| \leq 1/4} \left(\frac{1}{|\hat{z} + y|^{d-1/2}} - 1 \right) a_\varepsilon(|z|y) \frac{dy}{|y|^{d+\alpha}}, \end{aligned}$$

where both integrals are bounded by a constant depending only on d because $\alpha \geq 3/4$,

$$\left| (|\hat{z} + y|^{-d+1/2} - 1) a_\varepsilon(|z|y) \right| \leq N$$

in the first integral, and $|y| \geq 1/2$ in the second integral. Hence, we have obtained (4.7). Since $1/4 \leq \alpha - 1/2 < 1/2$, it follows that (4.4) holds with

$$N = (1 - \alpha)^{-1} N(d, p, \alpha - 1/2, [w]_{A_p}) = (1 - \alpha)^{-1} N(d, p, [w]_{A_p}). \quad (4.8)$$

For $\alpha \in (1, 2)$, $\mathcal{L}^\alpha u$ can be written in the form

$$\sum_{i=1}^d \int_{\mathbb{R}^d} (D_i u(x + y) - D_i u(x)) a_i(y) \frac{dy}{|y|^{d+\alpha-1}} = \sum_{i=1}^d \mathcal{L}^{a_i} (D_i u)(x),$$

where

$$a_i(y) = \frac{y^i}{|y|} \int_0^1 a(y/s) s^{-1+\alpha} ds.$$

Note that the order of $\mathcal{L}^{a_i}$ is $\alpha - 1$. By Proposition 4.1 for $\alpha \in (0, 1)$ just proved above and Lemma 3.5,

$$\|\mathcal{L}^\alpha u\|_{L_{p,w}} \leq \sum_{i=1}^d \|\mathcal{L}^{a_i} D_i u\|_{L_{p,w}} \leq N \|(-\Delta)^{(\alpha-1)/2} \nabla u\|_{L_{p,w}} \leq N \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}.$$

From $|a| \leq (2 - \alpha)$ and (4.8), we see that $N = N(d, p, \alpha_0, [w]_{A_p})$ if $1 < \alpha_0 \leq \alpha < 2$ with no blow-up as $\alpha \nearrow 2$.

Finally, we prove (4.4) for $\alpha \in (3/4, 5/4)$ with $N = N(d, p, [w]_{A_p})$ under the additional assumption that $a(y)$ satisfies the condition (iii) in Assumption 2.1.

Using the condition (iii) in Assumption 2.1, we have

$$\mathcal{L}_\varepsilon^\alpha u(x) = N_0 \int_{\mathbb{R}^d} ((-\Delta)^{1/4} u(x - z) - (-\Delta)^{1/4} u(x)) m_\varepsilon(z) \frac{dz}{|z|^{d+\alpha-1/2}},$$

where

$$\begin{aligned} m_\varepsilon(z) &= \int_{|y| \leq 1/2} \left(\frac{1}{|\hat{z} + y|^{d-1/2}} - 1 - (-d + 1/2)(\hat{z}, y) \right) a_\varepsilon(|z|y) \frac{dy}{|y|^{d+\alpha}} \\ &\quad + \int_{|y| > 1/2} \left(\frac{1}{|\hat{z} + y|^{d-1/2}} - 1 \right) a_\varepsilon(|z|y) \frac{dy}{|y|^{d+\alpha}} =: I_\varepsilon + J_\varepsilon. \end{aligned}$$

Then, since $|\hat{z} + ty| \geq 1/2$ for $|y| \leq 1/2$ (recall $f(t) = |\hat{z} + ty|^{-d+1/2}$),

$$\begin{aligned} \left| \frac{1}{|\hat{z} + y|^{d-1/2}} - 1 - (-d + 1/2)(\hat{z}, y) \right| &= |f(1) - f(0) - f'(0)| \\ &\leq \frac{1}{2} \sup_{t \in [0,1]} |f''(t)| \leq N|y|^2, \end{aligned}$$

we have

$$|I_\varepsilon| \leq N \int_{|y| \leq 1/2} |a_\varepsilon(|z|y)| \frac{dy}{|y|^{d+\alpha-2}} \leq N,$$

where $N = N(d)$. Note that the estimate of J_ε still depends only on d . By using the proposition for $1/4 < \alpha - 1/2 < 3/4$ proved above, we have (4.4) with

$$N = N(d, p, \alpha - 1/2, [w]_{A_p}) = N(d, p, [w]_{A_p}).$$

Note that the constant N in (4.4) for the above three cases ($\alpha \in (0, 1)$, $\alpha \in (3/4, 5/4)$, $\alpha \in (1, 2)$) can be chosen depending only on a_0 if $0 < a_0 \leq \alpha < 2$, which is the last assertion of the proposition. In particular, the constant N does not blow up as $\alpha \rightarrow 1$ if the condition (iii) in Assumption 2.1 is satisfied. This finishes the proof of Proposition 4.1.

5 Proof of theorem 2.5

We start with a key lemma.

Lemma 5.1 *Let $\beta \in (0, 1)$, $\alpha \in (0, 2)$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^d)$, $\varepsilon \in (0, 1)$, and $\varphi \in C_0^\infty(B_\varepsilon)$. Also let $a(x, y)$ be the coefficient of a non-local operator $\mathcal{L}^a$ satisfying conditions (ii) and (iii) of Assumption 2.1. Set*

$$\gamma(r) = \int_{|y| \leq r} \omega(|y|) \frac{dy}{|y|^{d+\beta}}, \quad 0 < r < \infty.$$

Suppose that there is a positive constant K such that

$$\sup_{x \in B_{2\varepsilon}, y \in \mathbb{R}^d} |a(x, y)| \leq (2 - \alpha)K.$$

Then there is a constant $N = N(d, p, \alpha, \beta, [w]_{A_p})$ such that for $u \in \mathcal{S}(\mathbb{R}^d)$,

$$\|\varphi \mathcal{L}^a u\|_{L_{p,w}}^p \leq NC(\varepsilon, \varphi, p, \beta, \gamma, K) \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}^p, \quad (5.1)$$

where C is a constant determined as follows.

$$C(\varepsilon, \varphi, p, \beta, \gamma, K) = K^p \|\varphi\|_{L_\infty}^p + \varepsilon^p K^p \|D\varphi\|_{L_\infty}^p + \varepsilon^{\beta p} \|\varphi\|_{L_\infty}^p \gamma^p(\varepsilon).$$

The constant N can be chosen so that $N = N(d, p, \alpha_0, \alpha_1, \beta, [w]_{A_p})$ if $0 < \alpha_0 \leq \alpha \leq \alpha_1 < 1$ and $N = N(d, p, \alpha_0, \beta, [w]_{A_p})$ if $1 < \alpha_0 \leq \alpha < 2$, respectively.

Proof We first consider the case when $p \in (d/\beta, \infty)$. To clarify the dependency of N with respect to α , we use N_1 when the constants in the inequalities below do not depend on α . For a measurable function f on $\mathbb{R}^d \times \mathbb{R}^d$ and $r > 0$, denote

$$\mathcal{K}(r, f) = (2 - \alpha)^{-1} \sup_{z \in B_r, y \in \mathbb{R}^d} |f(z, y)|.$$

Assume that $a(x, y)$ are infinitely differentiable in x . By Lemma 3.1, we see that, for any $x \in \mathbb{R}^d$,

$$\varphi(x)a(x, y) = N_1 \int_{\mathbb{R}^d} |x - z|^{-d+\beta} (-\Delta)^{\beta/2} (\varphi(\cdot) a(\cdot, y))(z) dz.$$

Using this, we observe that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}
\varphi(x)\mathcal{L}^a u(x) &= \int_{\mathbb{R}^d} \nabla_y^\alpha u(x) \varphi(x) a(x, y) \frac{dy}{|y|^{d+\alpha}} \\
&= N_1 \int_{\mathbb{R}^d} \nabla_y^\alpha u(x) \left(\int_{\mathbb{R}^d} |x-z|^{-d+\beta} (-\Delta)^{\beta/2} (\varphi(\cdot) a(\cdot, y))(z) dz \right) \frac{dy}{|y|^{d+\alpha}} \\
&= N_1 \int_{\mathbb{R}^d} |x-z|^{-d+\beta} \left(\int_{\mathbb{R}^d} \nabla_y^\alpha u(x) (-\Delta)^{\beta/2} (\varphi(\cdot) a(\cdot, y))(z) \frac{dy}{|y|^{d+\alpha}} \right) dz \\
&=: N_1 \int_{\mathbb{R}^d} |x-z|^{-d+\beta} \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot) a(\cdot, y))(z)} u(x) dz.
\end{aligned} \tag{5.2}$$

For $|x| \leq \varepsilon$,

$$|x-z|^{-d+\beta} \leq N_1 |z|^{-d+\beta}, \quad |z| \geq 2\varepsilon. \tag{5.3}$$

Hence, from (5.2) and (5.3), we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |\varphi(x)\mathcal{L}^a u(x)|^p w(x) dx &= \int_{B_\varepsilon} |\varphi(x)\mathcal{L}^a u(x)|^p w(x) dx \\
&\leq N_1 \int_{B_\varepsilon} \left(\int_{\mathbb{R}^d} |x-z|^{-d+\beta} \left| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot) a(\cdot, y))(z)} u(x) \right| dz \right)^p w(x) dx \\
&\leq N_1 \int_{B_\varepsilon} \left(\int_{B_{2\varepsilon}} |x-z|^{-d+\beta} \left| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot) a(\cdot, y))(z)} u(x) \right| dz \right)^p w(x) dx \\
&\quad + N_1 \int_{B_\varepsilon} \left(\int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \left| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot) a(\cdot, y))(z)} u(x) \right| dz \right)^p w(x) dx \\
&=: J_1 + J_2.
\end{aligned}$$

Estimate of J_1 . Using Hölder's inequality, we have

$$J_1 \leq N_1 \int_{B_\varepsilon} \left(\int_{B_{2\varepsilon}} |x-z|^{(-d+\beta)\frac{p}{p-1}} dz \right)^{p-1} \left(\int_{B_{2\varepsilon}} \left| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot) a(\cdot, y))(z)} u(x) \right|^p dz \right) w(x) dx.$$

Note that, for $|x| \leq \varepsilon$,

$$\left(\int_{B_{2\varepsilon}} |x-z|^{(-d+\beta)\frac{p}{p-1}} dz \right)^{p-1} \leq \left(\int_{B_{3\varepsilon}} |z|^{(-d+\beta)\frac{p}{p-1}} dz \right)^{p-1} \leq N_1 \varepsilon^{\beta p-d}.$$

Hence, we have that

$$\begin{aligned}
J_1 &\leq N_1 \varepsilon^{\beta p-d} \int_{B_\varepsilon} \int_{B_{2\varepsilon}} \left| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot) a(\cdot, y))(z)} u(x) \right|^p w(x) dz dx \\
&\leq N_1 \varepsilon^{\beta p-d} \int_{B_{2\varepsilon}} \int_{\mathbb{R}^d} \left| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot) a(\cdot, y))(z)} u(x) \right|^p w(x) dx dz.
\end{aligned}$$

By Proposition 4.1, we see that

$$\begin{aligned}
J_1 &\leq N_1 N^p \varepsilon^{\beta p-d} (2-\alpha)^{-p} \int_{B_{2\varepsilon}} \left(\sup_{y \in \mathbb{R}^d} |(-\Delta)^{\beta/2} (\varphi(\cdot) a(\cdot, y))(z)|^p \right) \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2} u|^p w(x) dx dz \\
&\leq N_1 N^p \varepsilon^{\beta p} (2-\alpha)^{-p} \left(\sup_{y \in \mathbb{R}^d, z \in B_{2\varepsilon}} |(-\Delta)^{\beta/2} (\varphi(\cdot) a(\cdot, y))(z)|^p \right) \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2} u|^p w(x) dx,
\end{aligned} \tag{5.4}$$

where N is the constant in Proposition 4.1. To estimate the supremum of

$$|(-\Delta)^{\beta/2}(\varphi(\cdot)a(\cdot, y))(z)|$$

in (5.4), we observe that, for any $y, z \in \mathbb{R}^d$,

$$\begin{aligned} & |(-\Delta)^{\beta/2}(\varphi(\cdot)a(\cdot, y))(z)| \\ & \leq N_1 \int_{|h|>\varepsilon} |\varphi(z+h)a(z+h, y) - \varphi(z)a(z, y)| \frac{dh}{|h|^{d+\beta}} \\ & \quad + N_1 \int_{|h|<\varepsilon} |\varphi(z+h)a(z+h, y) - \varphi(z)a(z, y)| \frac{dh}{|h|^{d+\beta}} \\ & \leq N_1 \int_{|h|>\varepsilon} |\varphi(z+h)a(z+h, y) - \varphi(z)a(z, y)| \frac{dh}{|h|^{d+\beta}} \\ & \quad + N_1 |a(z, y)| \int_{|h|\leq\varepsilon} |\varphi(z+h) - \varphi(z)| \frac{dh}{|h|^{d+\beta}} \\ & \quad + N_1 |\varphi(z)| \int_{|h|\leq\varepsilon} |a(z+h, y) - a(z, y)| \frac{dh}{|h|^{d+\beta}} \\ & \quad + N_1 \int_{|h|\leq\varepsilon} |a(z+h, y) - a(z, y)| |\varphi(z+h) - \varphi(z)| \frac{dh}{|h|^{d+\beta}} \\ & =: I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (5.5)$$

For any $y \in \mathbb{R}^d$ and $z \in B_{2\varepsilon}$,

$$\begin{aligned} I_1 & \leq N_1 \sup_{z, y} |\varphi(z)a(z, y)| \varepsilon^{-\beta} \\ & \leq N_1 \sup_{z \in B_{2\varepsilon}, y \in \mathbb{R}^d} |a(z, y)| \|\varphi\|_{L_\infty} \varepsilon^{-\beta} \leq N_1 (2 - \alpha) \mathcal{K}(2\varepsilon, a) \|\varphi\|_{L_\infty} \varepsilon^{-\beta}, \\ I_2 & \leq N_1 \sup_{z \in B_{2\varepsilon}, y \in \mathbb{R}^d} |a(z, y)| \|D\varphi\|_{L_\infty} \varepsilon^{1-\beta} \leq N_1 (2 - \alpha) \mathcal{K}(2\varepsilon, a) \|D\varphi\|_{L_\infty} \varepsilon^{1-\beta}, \\ I_3, I_4 & \leq N_1 (2 - \alpha) \|\varphi\|_{L_\infty} \gamma(\varepsilon). \end{aligned}$$

From the above estimates of I_i , $i = 1, 2, 3, 4$, together with (5.4), we obtain that

$$J_1 \leq NC(\varepsilon, \varphi, p, \beta, \gamma, \mathcal{K}(2\varepsilon, a)) \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}^p, \quad (5.6)$$

where N can be chosen as described in the lemma.

Estimate of J_2 . By the Minkowski inequality

$$\begin{aligned} J_2^{1/p} & = N_1 \left\| \int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \left| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot)a(\cdot, y))(z)} u(\cdot) \right| dz \right\|_{L_{p,w}(B_\varepsilon)} \\ & \leq N_1 \int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \left\| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot)a(\cdot, y))(z)} u(\cdot) \right\|_{L_{p,w}(B_\varepsilon)} dz \\ & \leq N_1 \int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \left\| \mathcal{L}^{(-\Delta)^{\beta/2}(\varphi(\cdot)a(\cdot, y))(z)} u(\cdot) \right\|_{L_{p,w}(\mathbb{R}^d)} dz. \end{aligned}$$

Then by Proposition 4.1, it follows that

$$J_2^{1/p} \leq N_1 N \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}} \int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \sup_{y \in \mathbb{R}^d} |(-\Delta)^{\beta/2}(\varphi(\cdot)a(\cdot, y))(z)| dz,$$

where N is the constant in Proposition 4.1. For $z \in B_{2\varepsilon}^c$, we see from (5.5) that

$$I_2 = I_3 = I_4 = 0$$

and

$$I_1 = N_1 \int_{|h|>\varepsilon} |\varphi(z+h)a(z+h, y)| \frac{dh}{|h|^{d+\beta}},$$

which imply that $J_2^{1/p}$ is bounded by

$$\begin{aligned} & N_1 N (2-\alpha)^{-1} \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}} \int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \int_{|h|>\varepsilon} \sup_{y \in \mathbb{R}^d} |\varphi(z+h)a(z+h, y)| \frac{dh}{|h|^{d+\beta}} dz \\ &= N_1 N (2-\alpha)^{-1} \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}} \int_{|h|>\varepsilon} \int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \sup_{y \in \mathbb{R}^d} |\varphi(z+h)a(z+h, y)| dz \frac{dh}{|h|^{d+\beta}}, \end{aligned}$$

where

$$|z|^{-d+\beta} \sup_{y \in \mathbb{R}^d} |\varphi(z+h)a(z+h, y)| = 0$$

for $z+h \in B_\varepsilon^c$ and

$$|z|^{-d+\beta} \sup_{y \in \mathbb{R}^d} |\varphi(z+h)a(z+h, y)| \leq |z|^{-d+\beta} \sup_{z \in B_\varepsilon, y \in \mathbb{R}^d} |a(z, y)| \|\varphi\|_{L_\infty}$$

for $z+h \in B_\varepsilon$. Thus,

$$\begin{aligned} & \int_{B_{2\varepsilon}^c} |z|^{-d+\beta} \sup_{y \in \mathbb{R}^d} |\varphi(z+h)a(z+h, y)| dz \\ & \leq \int_{B_{2\varepsilon}^c \cap \{z+h \in B_\varepsilon\}} \varepsilon^{-d+\beta} \sup_{z \in B_\varepsilon, y \in \mathbb{R}^d} |a(z, y)| \|\varphi\|_{L_\infty} dz \\ & \leq N_1 \varepsilon^\beta \sup_{z \in B_\varepsilon, y \in \mathbb{R}^d} |a(z, y)| \|\varphi\|_{L_\infty}. \end{aligned}$$

This shows that

$$\begin{aligned} J_2^{1/p} & \leq N_1 N (2-\alpha)^{-1} \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}} \int_{|h|>\varepsilon} \varepsilon^\beta \sup_{z \in B_\varepsilon, y \in \mathbb{R}^d} |a(z, y)| \|\varphi\|_{L_\infty} \frac{dh}{|h|^{d+\beta}} \\ & = N_1 N (2-\alpha)^{-1} \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}} \sup_{z \in B_\varepsilon, y \in \mathbb{R}^d} |a(z, y)| \|\varphi\|_{L_\infty} \\ & \leq N_1 N \mathcal{K}(2\varepsilon, a) \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}} \|\varphi\|_{L_\infty}. \end{aligned}$$

By combining the above estimate for J_2 and (5.6) for J_1 , we see that the inequality (5.1) holds for $a(x, y)$ which is infinitely differentiable in x with $C(\varepsilon, \varphi, p, \beta, \gamma, \mathcal{K}(2\varepsilon, a))$.

For general a , set

$$a_n(x, y) = a(x, y) * \eta_n(x), \quad \eta(x) = n^d \eta(nx)$$

for $n = 1, 2, \dots$, where η is a standard mollifier on $\mathbb{R}^d$. Note that

$$|a_n(z_1, y) - a_n(z_2, y)| \leq (2-\alpha)\omega(|z_1 - z_2|), \quad z_1, z_2, y \in \mathbb{R}^d.$$

By the inequality (5.1) proved above for a_n , we have

$$\|\varphi \mathcal{L}^{a_n} u\|_{L_{p,w}}^p \leq NC(\varepsilon, \varphi, p, \beta, \gamma, \mathcal{K}(2\varepsilon, a_n)) \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}^p.$$

Since for any $\rho > 0$,

$$\mathcal{K}(2\varepsilon, a_n) \leq \mathcal{K}(2\varepsilon + \rho, a)$$

for large n , by Fatou's Lemma, we see that for any $\rho > 0$,

$$\begin{aligned} \|\mathcal{L}^a u\|_{L_{p,w}}^p &\leq \liminf_{n \rightarrow \infty} \|\mathcal{L}^{a_n} u\|_{L_{p,w}}^p \\ &\leq \liminf_{n \rightarrow \infty} NC(\varepsilon, \varphi, p, \beta, \gamma, \mathcal{K}(2\varepsilon, a_n)) \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}^p \\ &\leq NC(\varepsilon, \varphi, p, \beta, \gamma, \mathcal{K}(2\varepsilon + \rho, a)) \|(-\Delta)^{\alpha/2} u\|_{L_{p,w}}^p. \end{aligned}$$

Upon noting that

$$\lim_{\rho \rightarrow 0} C(\varepsilon, \varphi, p, \beta, \gamma, \mathcal{K}(2\varepsilon + \rho, a)) \leq C(\varepsilon, \varphi, p, \beta, \gamma, K),$$

we finish the proof when $p > d/\beta$.

For general $p \in (1, \infty)$, we apply the Rubio de Francia extrapolation theorem. See, for instance, [9, Theorem 2.5]. The lemma is proved. $\square$

Remark 5.2 From our arguments, it is not clear if the constant N in Lemma 5.1 can be independent of β . For instance, if we take $p = 2d/\beta$ and let β go to zero, then even the constant in Proposition 4.1 might blow up. On the other hand, if one can the dependency on β in Lemma 5.1, the boundedness of the operators $\mathcal{L}^a$ with $a(x, y)$ having assumptions can be achieved by mollifying the coefficients.

Proof of theorem 2.5

Here, we use a version of the partition of unity argument in the proof of Theorem 1.6.4 in [12].

We may assume that $u \in \mathcal{S}$ as $\mathcal{S}$ is dense in $H_{p,w}^\alpha$. Let

$$\mathcal{L}_1^a u(x) = \int_{|y| \leq 1} (u(x+y) - u(x) - \chi_\alpha(y)(\nabla u(x), y)) a(x, y) \frac{dy}{|y|^{d+\alpha}}.$$

According to Lemmas 3.7, 3.4, and 3.5, we obtain

$$\|(\mathcal{L}^a - \mathcal{L}_1^a)u\|_{L_{p,w}} \leq N \|u\|_{H_{p,w}^\alpha}.$$

Therefore, it is sufficient to show that

$$\|\mathcal{L}_1^a u\|_{L_{p,w}} \leq N \|u\|_{H_{p,w}^\alpha}.$$

Let $\varphi \in C_0^\infty(B_1)$, $0 \leq \varphi \leq 1$, and $\varphi(x) = 1$ if $|x| \leq 1/2$. Then for any $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\mathcal{L}_1^a u(x)|^p w(x) dx = N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\varphi(x-z) \mathcal{L}_1^a u(x)|^p w(x) dz dx.$$

Observe that

$$\begin{aligned} \varphi(x-z) \mathcal{L}_1^a u(x) &= \mathcal{L}_1^a (u(\cdot) \varphi(\cdot - z))(x) - u(x) \mathcal{L}_1^a (\varphi(\cdot - z))(x) \\ &\quad - \int_{|y| \leq 1} (u(x+y) - u(x)) (\varphi(x+y-z) - \varphi(x-z)) a(x, y) \frac{dy}{|y|^{d+\alpha}} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We first estimate I_2 . Using the Minkowski inequality in z and Lemma 3.8 to φ without weights, we see that

$$\int \int I_2^p w(x) dz dx \leq N \int |u(x)|^p \|\varphi\|_{H_p^2}^p w(x) dx \leq N \|u\|_{L_{p,w}}^p.$$

For I_3 , using Lemma 3.6 and the fact that $|a(x, y)| \leq (2 - \alpha)K$, we have

$$\int \int I_3^p w(x) dz dx \leq N \|u\|_{H_{p,w}^\alpha}^p.$$

To estimate I_1 , we observe that

$$\mathcal{L}_1^a(u(\cdot)\varphi(\cdot - z))(x) = \varphi\left(\frac{x - z}{4}\right) \mathcal{L}_1^a(u(\cdot)\varphi(\cdot - z))(x).$$

Thus by Lemma 5.1, we see that

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{L}_1^a(u(\cdot)\varphi(\cdot - z))(x)|^p w(x) dx dz \\ & \leq N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2}(u(\cdot)\varphi(\cdot - z))(x)|^p w(x) dx dz \\ & \leq N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2}u(x)\varphi(x - z)|^p w(x) dx dz \\ & \quad + N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2}(\varphi(\cdot - z))(x)u(x)|^p w(x) dx dz \\ & \quad + N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |u(x + y) - u(x)| |\varphi(x + y - z) - \varphi(x - z)| \frac{dy}{|y|^{d+\alpha}} \right) w(x) dz dx \\ & =: J_1 + J_2 + J_3. \end{aligned}$$

By the same calculations as in the estimates of I_2 and I_3 , we derive that

$$J_2 + J_3 \leq N \|u\|_{H_{p,w}^\alpha}^p.$$

Note that $J_1 = N \|(-\Delta)^{\alpha/2}u\|_{L_{p,w}}^p$. Consequently, by Lemma 3.4 we reach the result.

6 Application: L_p estimates for non-local equations

In this section we study the following non-local parabolic equations of the form

$$u_t - \mathcal{L}^a u = f \quad \text{in } \mathbb{R}_T^d := (0, T) \times \mathbb{R}^d \quad (6.1)$$

with the non-local operator $\mathcal{L}^a$ given by

$$\mathcal{L}^a u(t, x) = \int_{\mathbb{R}^d} (u(t, x + y) - u(t, x) - \chi_\alpha(y)(\nabla u(t, x), y)) a(t, x, y) \frac{dy}{|y|^{d+\alpha}}.$$

We introduce some function spaces related to parabolic equations. Set $w(t, x) = w_1(t)w_2(x)$, where $w_1 \in A_q(\mathbb{R})$ and $w_2(x) \in A_p(\mathbb{R}^d)$, $p, q \in (1, \infty)$. Then, we define $L_{p,q,w}(\mathbb{R}_T^d)$, also denoted by $L_{p,q,w}(T)$, to be the space of integrable functions with the mixed norm

$$\|u\|_{L_{p,q,w}(T)} = \left(\int_0^T \left(\int_{\mathbb{R}^d} |u(t, x)|^p w_2(x) dx \right)^{q/p} w_1(t) dt \right)^{\frac{1}{q}}.$$

We write $H_{p,q,w}^\alpha(T)$ to indicate the class of $H_{p,w_2}^\alpha(\mathbb{R}^d)$ -valued functions u on $(0, T)$ equipped with the norm

$$\|u\|_{H_{p,q,w}^\alpha(T)} = \left(\int_0^T \|u(t, \cdot)\|_{H_{p,w_2}^\alpha(\mathbb{R}^d)}^q w_1(t) dt \right)^{\frac{1}{q}}.$$

As solution spaces for parabolic equations, $\mathcal{H}_{p,q,w}^{1,\alpha}(T)$ denotes the collection of functions $u \in H_{p,q,w}^\alpha(T)$ such that $u_t \in L_{p,q,w}(T)$. It is a Banach space with respect to the norm

$$\|u\|_{\mathcal{H}_{p,q,w}^{1,\alpha}(T)} = \|u\|_{H_{p,q,w}^\alpha(T)} + \|u_t\|_{L_{p,q,w}(T)}.$$

We write only p instead of p, q when $p = q$ so that, for instance, $\mathcal{H}_p^{1,\alpha}(T)$ means $\mathcal{H}_{p,p,w}^{1,\alpha}(T)$ with $w = 1$.

We now state our assumption on the coefficient $a(t, x, y)$ in (6.1). Throughout the section, we set K and δ as positive constants and assume that the function a_0 satisfies the following conditions.

Assumption 6.1 (i) The function $a_0 = a_0(t, y) \geq 0$ is measurable, homogeneous in y with index zero, differentiable in y up to the order $[\frac{d}{2}] + 1$, and

$$|D_y^\gamma a_0(t, y)| \leq K$$

for all $t \in (0, T)$, $y \in \mathbb{R}^d$ and multi-indices, $\gamma \in \mathbb{N}^d$ with $|\gamma| \leq [\frac{d}{2}] + 1$.

(ii) When $\alpha = 1$, for all $t \in (0, T)$

$$\int_{S^{d-1}} y a_0(t, y) \mu_{d-1}(dy) = 0,$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and μ_{d-1} is the standard Lebesgue measure on it.

(iii) For all $t \in (0, T)$,

$$\inf_{\xi \in S^{d-1}} \int_{S^{d-1}} |(y, \xi)|^\alpha a_0(t, y) \mu_{d-1}(dy) \geq \delta > 0.$$

Assumption 6.2 (i) For all $x, y \in \mathbb{R}^d$ and $t \in (0, T)$,

$$(2 - \alpha)a_0(t, y) \leq a(t, x, y) \leq (2 - \alpha)K.$$

(ii) There exist $\beta \in (0, 1)$ and a increasing function $\omega(\tau)$, $\tau > 0$, such that

$$|a(t, x_1, y) - a(t, x_2, y)| \leq (2 - \alpha)\omega(|x_1 - x_2|)$$

for $x_1, x_2, y \in \mathbb{R}^d$ and $t \in (0, T)$, and

$$\int_{|y| \leq 1} \omega(|y|) \frac{dy}{|y|^{d+\beta}} < \infty.$$

(iii) If $\alpha = 1$, then for any $(t, x) \in \mathbb{R}_T^d$ and $0 < r < R$,

$$\int_{r \leq |y| \leq R} y a(t, x, y) \frac{dy}{|y|^{d+\alpha}} = 0.$$

By using Theorem 2.5 for each $t \in (0, T)$ and then integrating with respect to t , we obtain the following boundedness of parabolic non-local operators.

Corollary 6.3 Let $\beta \in (0, 1)$, $\alpha \in (0, 2)$, $1 < p, q < \infty$, and $w(t, x) = w_1(t)w_2(x)$, $w_1 \in A_q(\mathbb{R})$ and $w_2 \in A_p(\mathbb{R}^d)$. Suppose that the coefficient $a(t, x, y)$ of $\mathcal{L}^a$ satisfies the conditions (ii) and (iii) in Assumption 6.2 and is bounded by $(2 - \alpha)K$. Then the operator $\partial_t - \mathcal{L}^a$ is continuous from $\mathcal{H}_{p,q,w}^{1,\alpha}(T)$ to $L_{p,q,w}(T)$.

Remark 6.4 As is easy to see, in Corollary 6.3 one can have any nonnegative function $w_1(t)$ instead of $w_1 \in A_q(\mathbb{R})$.

By utilizing the above results, especially, when $w = 1$ and $p = q$, we prove the following L_p -estimates for parabolic and elliptic equations with spatial non-local operators with Hölder continuous coefficients. As noted in the introduction, the same type of results are proved in [17] but for sufficiently large p . That is, we have removed the largeness assumption on p in [17]. On the other hand, our boundedness results for operators in the weighted space setting also make it possible to derive weighted versions of the results below. Indeed, such results are proved recently in [10] using Theorem 2.5 and Corollary 6.3 in this paper after detailed preparations for weighted L_p -estimates.

We first state the parabolic case. Note that in the results below we do not pursue whether the constant N stays bounded, for instance, as $\alpha \nearrow 2$ (see Remark 2.6). However, it is possible to investigate the dependency of N on α as described in Theorem 2.5. See [10] for a priori estimates with more informative dependency of N on α .

Theorem 6.5 Let $\beta \in (0, 1)$, $\alpha \in (0, 2)$, $1 < p < \infty$, and the coefficient a of the operator $\mathcal{L}^a$ satisfy Assumption 6.2. Suppose that $u \in \mathcal{H}_p^{1,\alpha}(T)$ satisfies

$$\begin{cases} u_t - \mathcal{L}^a u + \lambda u = f & \text{in } \mathbb{R}_T^d, \\ u(0, x) = 0, & x \in \mathbb{R}^d, \end{cases} \quad (6.2)$$

where $f \in L_p(T)$. Then, there exists $\lambda_0 = \lambda_0(d, p, \alpha, \beta, \delta, K, \omega)$ such that for any $\lambda \geq \lambda_0$,

$$\|u_t\|_{L_p(T)} + \|(-\Delta)^{\alpha/2} u\|_{L_p(T)} \leq N \|f\|_{L_p(T)}, \quad (6.3)$$

$$\|u\|_{L_p(T)} \leq N(T \wedge \lambda^{-1}) \|f\|_{L_p(T)}, \quad (6.4)$$

where $N = N(d, p, \alpha, \beta, \delta, K, \omega)$. Moreover, for any $f \in L_p(T)$ there exists a unique $u \in \mathcal{H}_p^{1,\alpha}(T)$ satisfying (6.2), (6.3), and (6.4).

Before presenting the proof of Theorem 6.5, we first give some corollaries derived from Theorem 6.5 for parabolic equations with lower-order coefficients and an elliptic counterpart of Theorem 6.5.

Corollary 6.6 Let $\beta \in (0, 1)$, $\alpha \in (0, 2)$, $1 < p < \infty$ and $a = a(t, x, y)$ satisfy Assumption 6.2. Also let $b^i(t, x)$ and $c(t, x)$ be bounded by K . Suppose that $u \in \mathcal{H}_p^{1,\alpha}(T)$ satisfies

$$\begin{cases} u_t - \mathcal{L}^a u + b^i D_i u I_{\alpha>1} + cu = f & \text{in } \mathbb{R}_T^d, \\ u(0, x) = 0, & x \in \mathbb{R}^d, \end{cases} \quad (6.5)$$

where $f \in L_p(T)$. Then

$$\|u\|_{\mathcal{H}_p^{1,\alpha}(T)} \leq N \|f\|_{L_p(T)}, \quad (6.6)$$

where $N = N(d, p, \alpha, \beta, \delta, K, \omega, T)$. Moreover, for any $f \in L_p(T)$, there exists a unique $u \in \mathcal{H}_p^{1,\alpha}(T)$ satisfying (6.5) and (6.6).

Proof By the method of continuity, it is sufficient to show (6.6). We only consider the case when $\alpha > 1$ as the other case is simpler. Fix $\lambda \geq \lambda_0$, where λ_0 is the constant in the statement of Theorem 6.5. Let $\tau \in (0, T)$. Due to Theorem 6.5, we have

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{1,\alpha}(\tau)} &\leq N\|u_t - \mathcal{L}^\alpha u + \lambda u\|_{L_p(\tau)} \\ &\leq N\|u_t - \mathcal{L}^\alpha u + b^i D_i u + cu\|_{L_p(\tau)} + N\|u\|_{L_p(\tau)} + N\|Du\|_{L_p(\tau)}, \end{aligned}$$

where the constant N is independent of τ . By (3.13) and Lemma 3.4,

$$\|Du\|_{L_p(\mathbb{R}^d)} \leq N\|(-\Delta)^{\alpha/2}u\|_{L_p(\mathbb{R}^d)} + N\|u\|_{L_p(\mathbb{R}^d)}.$$

Then, by dilation in x (see, for instance, the proof of [12, Theorem 1.5.1]) and integration with respect to the t variable,

$$\|Du\|_{L_p(\tau)} \leq \varepsilon\|(-\Delta)^{\alpha/2}u\|_{L_p(\tau)} + N(\varepsilon)\|u\|_{L_p(\tau)}.$$

By taking sufficiently small ε , we see that

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{1,\alpha}(\tau)} &\leq N\|u_t - \mathcal{L}^\alpha u + b^i D_i u + cu\|_{L_p(\tau)} + N\|u\|_{L_p(\tau)} \\ &= N\|f\|_{L_p(\tau)} + N\|u\|_{L_p(\tau)}. \end{aligned}$$

Observe that

$$\begin{aligned} \|u(t, \cdot)\|_{L_p(\mathbb{R}^d)}^p &\leq \left\| \int_0^t |u_t|(s, \cdot) ds \right\|_{L_p(\mathbb{R}^d)}^p \\ &\leq t^{p-1} \left(\int_0^t \|u_t(s, \cdot)\|_{L_p(\mathbb{R}^d)}^p ds \right) = t^{p-1} \|u_t\|_{L_p(t)}^p \end{aligned}$$

for any $t \in (0, T)$. Hence, by taking integration over $(0, \tau)$, we have

$$\|u\|_{L_p(\tau)}^p \leq \left(\int_0^\tau t^{p-1} \|u_t\|_{L_p(t)}^p dt \right) \leq \left(\int_0^\tau t^{p-1} \|u\|_{\mathcal{H}_p^\alpha(t)}^p dt \right).$$

Thus,

$$\|u\|_{\mathcal{H}_p^{1,\alpha}(\tau)}^p \leq N\|f\|_{L_p(\tau)}^p + N \left(\int_0^\tau t^{p-1} \|u\|_{\mathcal{H}_p^\alpha(t)}^p dt \right).$$

Set

$$\mathcal{A}(s) = \|u\|_{\mathcal{H}_p^{1,\alpha}(s)}^p, \quad \mathcal{B}(s) = N\|f\|_{L_p(s)}^p, \quad \mathcal{C}(s) = Ns^{p-1}$$

for $s \in (0, T)$. Then, we see that

$$\mathcal{A}(\tau) \leq \mathcal{B}(\tau) + \int_0^\tau \mathcal{C}(t)\mathcal{A}(t) dt$$

for any $\tau \in (0, T)$. By Grönwall's inequality,

$$\mathcal{A}(T) \leq \mathcal{B}(T)e^{\int_0^T \mathcal{C}(t) dt}.$$

This proves the corollary. $\square$

Corollary 6.7 Let $\beta \in (0, 1)$, $\alpha \in (0, 2)$, $1 < p < \infty$, and the operator $\mathcal{L}^a$ be independent of the time variable with $a(x, y)$ satisfying Assumption 6.2. Then, there are $\lambda_0 = \lambda_0(d, p, \alpha, \beta, \delta, K, \omega)$ and $N = N(d, p, \alpha, \beta, \delta, K, \omega)$ such that, for any $\lambda \geq \lambda_0$ and $u \in H_p^\alpha(\mathbb{R}^d)$ satisfying

$$-\mathcal{L}^a u + \lambda u = f \quad \text{in } \mathbb{R}^d, \quad (6.7)$$

we have

$$\|(-\Delta)^{\alpha/2} u\|_{L_p(\mathbb{R}^d)} + \lambda \|u\|_{L_p(\mathbb{R}^d)} \leq N \|f\|_{L_p(\mathbb{R}^d)}. \quad (6.8)$$

Furthermore, for any $f \in L_p(\mathbb{R}^d)$, there exists a unique $u \in H_p^\alpha(\mathbb{R}^d)$ satisfying (6.7) and (6.8).

Proof Let η be a smooth function in $\mathbb{R}$ with

$$\eta(0) = 0, \quad \int_0^1 |\eta(t)| dt > 0.$$

Set $\eta_n(t) = \eta(t/n)$, $t \in \mathbb{R}$, and consider $v_n = u\eta_n$. By applying Theorem 6.5 with $T = n$ to v_n , then letting n go to infinity, we obtain a priori estimate (6.8). Then, by the method of continuity and the unique solvability of equations with simple coefficients, for instance, in [7], one can finish the proof. $\square$

We will prove Theorem 6.5 by the standard freezing coefficient argument. One can find the argument in the proof of [12, Lemma 1.6.3] for second-order elliptic equations, and in the proof of [17, Lemma 8] for non-local equations. To show the exact parameters which the constants depend on in the estimates, we give a detailed proof of the following lemma.

Lemma 6.8 Assume that $\beta \in (0, 1)$, $\alpha \in (0, 2)$, and $p \in (1, \infty)$. Suppose that Assumption 6.2 holds. Then, there exist $\varepsilon = \varepsilon(d, p, \alpha, \beta, \delta, K, \omega)$, $N = N(d, p, \alpha, \delta, K)$, and $\lambda_0 = \lambda_0(d, p, \alpha, \beta, \delta, K, \omega)$ such that the following holds. If $\lambda \geq \lambda_0$, $u \in \mathcal{H}_p^{1,\alpha}(T)$ satisfies (6.2) and for every $t \in (0, T)$, $u(t, \cdot)$ has support in a ball of radius ε , then

$$\begin{aligned} \|u_t\|_{L_p(T)} + \|(-\Delta)^{\alpha/2} u\|_{L_p(T)} &\leq N \|f\|_{L_p(T)}, \\ \|u\|_{L_p(T)} &\leq N(T \wedge \lambda^{-1}) \|f\|_{L_p(T)}. \end{aligned}$$

Proof Without loss of generality, we assume that for all $t \in (0, T)$, $u(t, \cdot) \in C_0^\infty(B_\varepsilon)$. Let $\varphi \in C_0^\infty(B_{4\varepsilon})$, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ if $|x| \leq 2\varepsilon$, with

$$\|D\varphi\|_{L_\infty} \leq \frac{N}{\varepsilon}.$$

Set

$$\mathcal{L}_0 u(t, x) = \int_{\mathbb{R}^d} \nabla_y^\alpha u(t, x) a(t, 0, y) \frac{dy}{|y|^{d+\alpha}}.$$

In this proof we denote $\mathcal{L}^a$ by $\mathcal{L}$. Note that for any $\lambda > 0$, by Proposition 1 in [17], there is a constant $N_1 = N_1(d, p, \alpha, \delta, K)$ such that

$$\|u\|_{L_p(T)} \leq N_1(T \wedge \lambda^{-1}) \|u_t - \mathcal{L}_0 u + \lambda u\|_{L_p(T)}$$

and

$$\|u_t\|_{L_p(T)} + \|(-\Delta)^{\alpha/2} u\|_{L_p(T)} \leq N_1 \|u_t - \mathcal{L}_0 u + \lambda u\|_{L_p(T)}.$$

Then by the triangle inequality, we have

$$\begin{aligned}\|u\|_{L_p(T)} &\leq N_1(T \wedge \lambda^{-1}) (\|f\|_{L_p(T)} + \|\varphi(\mathcal{L} - \mathcal{L}_0)u\|_{L_p(T)} \\ &\quad + \|(\mathcal{L}u - \mathcal{L}_0u)(1 - \varphi)\|_{L_p(T)})\end{aligned}$$

and

$$\begin{aligned}\|u_t\|_{L_p(T)} + \|(-\Delta)^{\alpha/2}u\|_{L_p(T)} \\ \leq N_1 (\|f\|_{L_p(T)} + \|\varphi(\mathcal{L} - \mathcal{L}_0)u\|_{L_p(T)} + \|(\mathcal{L}u - \mathcal{L}_0u)(1 - \varphi)\|_{L_p(T)}).\end{aligned}$$

Observe that for any $t \in (0, T]$,

$$\sup_{z \in B_{8\varepsilon}, y \in \mathbb{R}^d} |a(t, z, y) - a(t, 0, y)|^p \leq \omega(8\varepsilon)^p.$$

Due to Lemma 5.1 with $a \rightarrow a(t, x, y) - a(t, 0, y)$ and $\varepsilon \rightarrow 4\varepsilon$, we see that

$$\|\varphi(\mathcal{L} - \mathcal{L}_0)u\|_{L_p(T)} \leq N_0 C(\varepsilon) \|(-\Delta)^{\alpha/2}u\|_{L_p(T)},$$

where

$$C(\varepsilon) = \omega(8\varepsilon) + \gamma(4\varepsilon)\varepsilon^\beta,$$

which goes to zero as $\varepsilon \rightarrow 0$. Since u has support in the ball of radius ε centered at 0, using the Minkowski inequality, we see that for each $t \in (0, T)$,

$$\begin{aligned}\|(\mathcal{L}u - \mathcal{L}_0u)(1 - \varphi)\|_{L_p} \\ = \left(\int_{B_{2\varepsilon}^c} \left| \int_{|y|>\varepsilon} \nabla_y^\alpha u(t, x) (a(t, x, y) - a(t, 0, y)) \frac{dy}{|y|^{d+\alpha}} \right|^p |1 - \varphi(x)|^p dx \right)^{1/p} \\ \leq \left(\int_{B_{2\varepsilon}^c} \left| \int_{|y|>\varepsilon} u(t, x+y) (a(t, x, y) - a(t, 0, y)) \frac{dy}{|y|^{d+\alpha}} \right|^p dx \right)^{1/p} \\ \leq N \int_{|y|>\varepsilon} \|u(t, \cdot + y)\|_{L_p} \frac{dy}{|y|^{d+\alpha}} \leq N_2 \varepsilon^{-\alpha} \|u(t, \cdot)\|_{L_p}.\end{aligned}$$

For a fixed $\varepsilon > 0$ satisfying $N_1 N_0 C(\varepsilon) \leq 1/2$, we have that

$$\|u_t\|_{L_p(T)} + \|(-\Delta)^{\alpha/2}u\|_{L_p(T)} \leq 2N_1 \|f\|_{L_p(T)} + 2N_1 N_2 \varepsilon^{-\alpha} \|u\|_{L_p(T)}$$

and

$$\|u\|_{L_p(T)} \leq 2N_1(T \wedge \lambda^{-1}) \|f\|_{L_p(T)} + 2N_1 N_2(T \wedge \lambda^{-1}) \varepsilon^{-\alpha} \|u\|_{L_p(T)}.$$

Taking $\lambda_0 = 4N_1 N_2 \varepsilon^{-\alpha}$, we obtain the desired estimates. $\square$

Proof of theorem 6.5

Using the partition of unity argument and standard freezing coefficient technique with Lemma 6.8, we have (6.3) and (6.4). The uniqueness comes from the estimates. Finally, using Theorem 2.5 and the method of continuity, we reach the existence of a solution.

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