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Calculus of Variations

Mixed boundary value problems for parabolic equations in Sobolev spaces with mixed-norms

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Abstract
We establish $L_{q,p}$-estimates and solvability for mixed Dirichlet–conormal problems for parabolic equations in a cylindrical Reifenberg-flat domain with a rough time-dependent separation.

Mathematics Subject Classification 35K20 · 35B65 · 35R05

1 Introduction

Let $\mathcal{Q}^T$ be a cylindrical domain in $\mathbb{R}^{d+1}$ of the form

$$\mathcal{Q}^T = (-\infty, T) \times \Omega,$$

where $T \in (-\infty, \infty]$ and Ω is either a bounded or unbounded domain in $\mathbb{R}^d$, $d \geq 2$. The lateral boundary of $\mathcal{Q}^T$ is divided into two components $\mathcal{D}^T$ and $\mathcal{N}^T$ separated by Γ^T, which is

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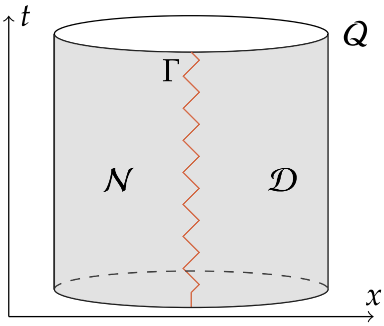

Fig. 1 Cylindrical domain with time-dependent separation

allowed to be time-dependent. See Fig. 1 below. We consider mixed boundary value problems for parabolic equations

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } \mathcal{Q}^T, \\ \mathcal{B}u = g_i n_i & \text{on } \mathcal{N}^T, \\ u = 0 & \text{on } \mathcal{D}^T, \end{cases} \tag{1.1}$$

where the operators $\mathcal{P}$ and $\mathcal{B}$ are defined by

$$\mathcal{P}u = -u_t + D_i(a^{ij} D_j u), \quad \mathcal{B}u = a^{ij} D_j u n_i,$$

and $n = (n_1, \dots, n_d)$ is the outward unit normal to $\partial\Omega$. The leading coefficients a^{ij} are assumed to be symmetric and satisfy the uniform ellipticity condition. The boundary value problem (1.1) arises naturally in mathematical physics and material science dealing with metallurgical melting, combustion, and wave phenomena, etc. We refer the reader to [1, 9, 19–22, 25] and references therein. It is also partly motivated by modeling exocytosis, which have a form of active transport mechanism. See [12].

In a recent paper [4], we proved the unique solvability in unmixed-norm Sobolev spaces $\mathcal{H}_p^1$ (see (2.1) and (2.2)) for the problem (1.1) when the coefficients a^{ij} have small bounded mean oscillation (BMO) with respect to all the variables (t, x), the base domain Ω is Reifenberg-flat, and the separation Γ is locally close to the graph of a Lipschitz function of m variables, where

$$m \in \{0, 1, \dots, d-2\} \quad \text{and} \quad p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1} \right).$$

For precise conditions on the domain and separation, see Assumption 2.2. Notice that if Γ is Reifenberg-flat of codimension 2 (i.e., $m = 0$), such range $p \in (4/3, 4)$ is optimal even in the stationary case, in view of the following classical example

$$u(x, y) = \operatorname{Im}(x + iy)^{1/2}$$

which is harmonic on the upper half space $\{y > 0\}$, and

$$u = 0 \text{ on } \{x > 0, y = 0\}, \quad \partial u / \partial \mathbf{n} = 0 \text{ on } \{x < 0, y = 0\}.$$

For other previous results on the mixed boundary value problems for parabolic equations in unmixed-norm spaces, we refer the reader to [2, 3, 14] and the references therein. We note that in these work, either p is assumed to be 2 or an implicit condition is imposed on the operator so that p needs to be sufficiently close to 2.

It is worth mentioning that here we focus on the case when $d \geq 2$ and the two types of boundary conditions touch at an nonempty Γ^T. This is the case when the difficulty of "optimal regularity" appears. When $\overline{\mathcal{D}} \cap \overline{\mathcal{N}} = \emptyset$, we can just apply the estimates for the pure Dirichlet and pure conormal problems and a partition of unity argument. In this case, the solutions are smooth given the domain, the operator, and the boundary data are all smooth. In particular, when $d = 1$, Ω is actually an interval (a, b). The mixed problems just mean that different boundary conditions are assigned at $x = a$ and $x = b$.

In this paper, we extend the result in [4] by proving $L_q^t(L_p^x)$ mixed-norm estimates for Du and the $\mathcal{H}_{q,p}^1$ (see (2.1)) solvability when

$$p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right) \quad \text{and} \quad q \in \left[p, \frac{2p}{(m+1)(p-2)_+}\right),$$

under the same smoothness assumptions on the coefficients, domain, and separation. In the special case when Γ is time-independent, we get the solvability for all

$$p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right) \quad \text{and} \quad q \in (1, \infty). \tag{1.2}$$

In particular, when $\partial\Omega$ and Γ^T are smooth enough, we can make a change of variables to locally flatten $\partial\Omega$ first and then make Γ^T to be time-independent. Hence, the full solvability range in (1.2) is achieved.

In [24] Savaré considered parabolic equations in a cylindrical domain with $C^{1,1}$ base domain and separation. Under a uniform linear bound condition on the excess of the separation with respect to t, he proved the unique solvability in L_2-based Sobolev spaces. We also refer the reader to Hieber–Rehberg [11] for quasilinear parabolic systems of reaction-diffusion equations in a cylindrical domain with Lipschitz base domain in $\mathbb{R}^d$ for $d = 2, 3$ and time-independent separation. Assuming an implicit topological isomorphism condition on the second-order operator, they established the solvability in mixed-norm spaces with $p = 2$ and $q > c$ for some c depending on the operator and dimension. To the best of the authors' knowledge, our results regarding the mixed boundary value problem are the first to deal with mixed-norm estimates for $p \neq 2$ even in the case when $\partial\Omega$ and Γ are smooth and Γ is time-independent. For other previous results on mixed-norm estimates for purely Dirichlet or conormal derivative boundary value problems, we refer the reader to [6, 8, 10, 13, 15–17] and references therein.

The proof in [4] relies on a decomposition argument using a carefully designed cut-off function near the separation Γ and on a level set method with the measure theoretic "crawling of ink spots" lemma originally due to Krylov and Safonov [18, 23]. While in [4] we used the decomposition argument and estimates in L_2-based spaces, in this paper, to prove our main result, we refine the decomposition argument in [4] in the setting of L_p-based spaces for $p < 2$ and exploit an idea of Krylov [17] to utilize the level set method in the t-variable only. Because the decomposition argument fails for $p > 2$, it remains open whether the mixed-norm estimates hold for

$$p \in \left(2, \frac{2(m+2)}{m+1}\right), \quad q \in \left(\frac{2p}{(m+1)(p-2)}, \infty\right)$$

when Γ is time-dependent. See the explanation after (4.12).

The remainder of the paper is organized as follows. In the next section, we introduce some notation and state our main result of the paper. In Sect. 3, we derive certain local estimates, which are used in Sect. 4 for the level set argument in the t-variable. Finally, we complete the proof of the main result in Sect. 5.

2 Notation and main result

We first introduce some notation used throughout the paper. We use $X = (t, x)$ to denote a generic point in the Euclidean space $\mathbb{R}^{d+1}$, where $d \geq 2$ and $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$. We also write $Y = (s, y)$ and $X_0 = (t_0, x_0)$, etc. Let $\mathcal{Q}$ be a cylindrical domain in $\mathbb{R}^{d+1}$ of the form

$$\mathcal{Q} = (-\infty, \infty) \times \Omega,$$

where Ω is a domain in $\mathbb{R}^d$. We assume that the lateral boundary of $\mathcal{Q}$, denoted by $\partial\mathcal{Q} = (-\infty, \infty) \times \partial\Omega$, is divided into two components $\mathcal{D}$ and $\mathcal{N}$ separated by Γ, i.e., as in [4], $\mathcal{D} \subset \partial\mathcal{Q}$ is an open set (relative to $\partial\mathcal{Q}$) and

$$\mathcal{N} = \partial\mathcal{Q} \setminus \mathcal{D}, \quad \Gamma = \overline{\mathcal{D}} \cap \overline{\mathcal{N}}.$$

Note that the separation Γ is allowed to be time-dependent. For $T \in (-\infty, \infty]$, we define

$$\mathcal{Q}^T = \{X \in \mathcal{Q} : t < T\}$$

and similarly define $\mathcal{D}^T$, $\mathcal{N}^T$, and Γ^T. For $R > 0$, we denote the parabolic cylinders by

$$\begin{aligned} Q_R(X) &= (t - R^2, t) \times B_R(x), \\ \mathbb{Q}_R(X) &= (t - R^2, t + R^2) \times B_R(x), \\ \mathcal{Q}_R(X) &= \mathcal{Q} \cap Q_R(X), \end{aligned}$$

where $B_R(x)$ is the usual Euclidean ball of radius R centered at x. The center will be omitted when it is the origin, i.e., for instance, we write Q_R for $Q_R(0)$.

For a function u on an open set $Q \subset \mathbb{R}^{d+1}$, we set

$$(u)_Q = \frac{1}{|Q|} \int_Q u \, dX = \fint_Q u \, dX,$$

where $|Q|$ is the $d + 1$-dimensional Lebesgue measure of Q. For $p, q \in [1, \infty)$, we define the mixed-norm on Q by

$$\|u\|_{L_q^t L_p^x(Q)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u|^p \mathbb{I}_Q \, dx \right)^{q/p} dt \right)^{1/q},$$

where $\mathbb{I}_Q$ is the usual characteristic function. Similarly, we define $L_q^t L_p^x$-norms with $p = \infty$ or $q = \infty$, and $L_p^x L_q^t$-norms. We often write $L_{q,p}$ and L_p for $L_q^t L_p^x$ and $L_{p,p}$.

We set

$$W^{0,1}_{q,p}(\mathcal{Q}^T) = \{u : u \in L_{q,p}(\mathcal{Q}^T), \ D_x u \in L_{q,p}(\mathcal{Q}^T)^d\}$$

with the norm

$$\|u\|_{W^{0,1}_{q,p}(\mathcal{Q}^T)} = \|u\|_{L_{q,p}(\mathcal{Q}^T)} + \|Du\|_{L_{q,p}(\mathcal{Q}^T)},$$

and we denote by $W^{0,1}_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)$ the closure of $C^\infty_{\mathcal{D}^T}(\mathcal{Q}^T)$ in $W^{0,1}_{q,p}(\mathcal{Q}^T)$, where $C^\infty_{\mathcal{D}^T}(\mathcal{Q}^T)$ is the set of all infinitely differentiable functions on $\mathbb{R}^{d+1}$ having a compact support in $\overline{\mathcal{Q}^T}$ and vanishing in a neighborhood of $\mathcal{D}^T$. We also set

$$\mathcal{H}^1_{q,p,\mathcal{D}^T}(\mathcal{Q}^T) = \left\{u : u \in W^{0,1}_{q,p,\mathcal{D}^T}(\mathcal{Q}^T), \ u_t \in \mathbb{H}^{-1}_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)\right\} \tag{2.1}$$

with the norm

$$\|u\|_{\mathcal{H}^1_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)} = \|u\|_{W^{0,1}_{q,p}(\mathcal{Q}^T)} + \|u_t\|_{\mathbb{H}^{-1}_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)},$$

where by $u_t \in \mathbb{H}^{-1}_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)$, we mean that there exist $g = (g_1, \dots, g_d) \in L_{q,p}(\mathcal{Q}^T)^d$ and $f \in L_{q,p}(\mathcal{Q}^T)$ satisfying

$$u_t = D_i g_i + f \quad \text{in } \mathcal{Q}^T, \quad g_i n_i = 0 \quad \text{on } \mathcal{N}^T$$

in the following distribution sense

$$(u_t, \varphi) := \int_{\mathcal{Q}^T} -u\varphi_t \, dX = \int_{\mathcal{Q}^T} (-g_i D_i \varphi + f\varphi) \, dX$$

for all $\varphi \in C^\infty_{\mathcal{D}^T}(\mathcal{Q}^T)$ vanishing at $t = T$, and that

$$\|u_t\|_{\mathbb{H}^{-1}_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)} = \inf\left\{\|g\|_{L_{q,p}(\mathcal{Q}^T)} + \|f\|_{L_{q,p}(\mathcal{Q}^T)} : u_t = D_i g_i + f \text{ in } \mathcal{Q}^T, \ g_i n_i = 0 \text{ on } \mathcal{N}^T\right\}$$

is finite. We abbreviate

$$\mathcal{H}^1_{p,p,\mathcal{D}^T}(\mathcal{Q}^T) = \mathcal{H}^1_{p,\mathcal{D}^T}(\mathcal{Q}^T). \tag{2.2}$$

Throughout this paper, we discuss weak solutions to the problem (1.1), which means the following integral identity holds for all $\varphi \in C^\infty_{\mathcal{D}^T}(\mathcal{Q}^T)$ vanishing at $t = T$,

$$\int_{\mathcal{Q}^T} u\varphi_t \, dX + \int_{\mathcal{Q}^T} (-a^{ij} D_j u D_i \varphi - \lambda u \varphi) \, dX = \int_{\mathcal{Q}^T} (-g_i D_i \varphi + f\varphi) \, dX. \tag{2.3}$$

We also discuss "local weak solutions" as, for instance, in (3.2), in which case, we mean that (2.3) holds with $f = 0$ for any test function $\varphi \in C^\infty(\overline{\mathcal{Q}_R})$ vanishing on $\partial \mathcal{Q}_R \setminus \mathcal{N}$.

2.1 Assumptions and main result

Throughout this paper, we assume that the leading coefficients a^{ij} of the operator $\mathcal{P}$ are symmetric and satisfy the uniform ellipticity condition

$$a^{ij}(X)\xi_j\xi_i \geq \Lambda|\xi|^2, \quad |a^{ij}(X)| \leq \Lambda^{-1}$$

for all $X \in \mathbb{R}^{d+1}$, $\xi \in \mathbb{R}^d$, and for some constant $\Lambda \in (0, 1]$. Regarding the symmetric condition, see Remark 2.6 for an explanation. We impose the following small BMO condition on the leading coefficients, where $\theta \in (0, 1)$ is a parameter to be specified.

Assumption 2.1 (θ) For any $X_0 \in \overline{\mathcal{Q}}$ and $R \in (0, R_0]$, we have

$$\fint_{\mathcal{Q}_R(X_0)} |a^{ij}(X) - (a^{ij})_{\mathcal{Q}_R(X_0)}| \, dX \leq \theta.$$

We also impose the following regularity assumptions on the boundary of the domain and separation, where $\gamma \in (0, 1)$ is a parameter to be specified.

Assumption 2.2 ($\gamma; m, M$) Let $m \in \{0, 1, \dots, d-2\}$ and $M \in (0, \infty)$.

(a) For any $x_0 \in \partial\Omega$ and $R \in (0, R_0]$, there is a coordinate system depending on x_0 and R such that in this coordinate system, we have

$$\{y : y^1 > x_0^1 + \gamma R\} \cap B_R(x_0) \subset \Omega \cap B_R(x_0) \subset \{y : y^1 > x_0^1 - \gamma R\} \cap B_R(x_0). \tag{2.4}$$

(b) For any $X_0 = (t_0, x_0) \in \Gamma$ and $R \in (0, R_0]$, there exist a spatial coordinate system and a Lipschitz function ϕ of m variables with Lipschitz constant M, such that in the new coordinate system, we have (2.4),

$$\big(\partial\mathcal{Q} \cap \mathbb{Q}_R(X_0) \cap \{(s, y) : y^2 > \phi(y^3, \ldots, y^{m+2}) + \gamma R\}\big) \subset \mathcal{D},$$
$$\big(\partial\mathcal{Q} \cap \mathbb{Q}_R(X_0) \cap \{(s, y) : y^2 < \phi(y^3, \ldots, y^{m+2}) - \gamma R\}\big) \subset \mathcal{N},$$

and

$$\phi(x_0^3, \ldots, x_0^{m+2}) = x_0^2.$$

Here, if $m = 0$, then the function ϕ is understood as the constant function $\phi \equiv x_0^2$.

Noting that Assumption 2.2 $(\gamma; 0, M)$ holds when $\partial\Omega$ is locally given by the graph $\{x^1 = \psi(x^2, \ldots, x^d)\}$ and Γ is locally given by its intersection with $\{x^2 = \tilde{\psi}(t, x^1, x^3, \ldots, x^d)\}$, where ψ and $\tilde{\psi}$ are Lipschitz functions ($\tilde{\psi}$ in the parabolic metric) with correspondingly small constants. The assumption also includes certain fractal structures.

The main result in the current paper reads as follows.

Theorem 2.3 *Let $R_0 \in (0, 1]$, $m \in \{0, 1, \ldots, d-2\}$, $M \in (0, \infty)$, and let*

$$p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right), \quad q \in \left(p, \frac{2p}{(m+1)(p-2)_+}\right). \tag{2.5}$$

There exist constants $\theta, \gamma \in (0, 1)$ and $\lambda_0 \in (0, \infty)$ with

$$(\theta, \gamma) = (\theta, \gamma)(d, \Lambda, M, p, q), \quad \lambda_0 = \lambda_0(d, \Lambda, M, p, q, R_0),$$

such that if Assumptions 2.1 (θ) and 2.2 $(\gamma; m, M)$ are satisfied, then we have the following. For any $\lambda \geq \lambda_0$, $g = (g_1, \ldots, g_d) \in L_{q,p}(\mathcal{Q}^T)^d$, and $f \in L_{q,p}(\mathcal{Q}^T)$, there exists a unique solution $u \in \mathcal{H}^1_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)$ to

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } \mathcal{Q}^T, \\ \mathcal{B}u = g_i n_i & \text{on } \mathcal{N}^T, \\ u = 0 & \text{on } \mathcal{D}^T \end{cases} \tag{2.6}$$

satisfying

$$\|Du\|_{L_{q,p}(\mathcal{Q}^T)} + \lambda^{1/2}\|u\|_{L_{q,p}(\mathcal{Q}^T)} \leq C\|g\|_{L_{q,p}(\mathcal{Q}^T)} + C\lambda^{-1/2}\|f\|_{L_{q,p}(\mathcal{Q}^T)}, \tag{2.7}$$

where $C = C(d, \Lambda, M, p, q)$. The same result holds for any p, q satisfying

$$p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right), \quad q \in (1, \infty)$$

instead of (2.5) when Γ is time-independent.

In Fig. 2, we draw a diagram to show the range of (p, q) in (2.5).

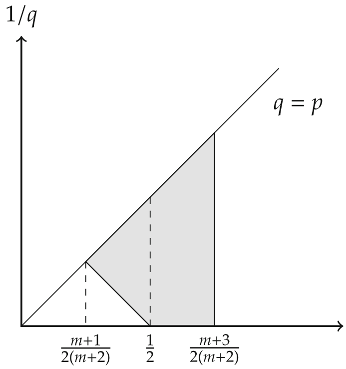

Fig. 2 Range of (p, q)

Remark 2.4 It is not clear to us if Theorem 2.3 still holds when $p \in \big(2, \frac{2(m+2)}{m+1}\big)$, $q \in \big(\frac{2p}{(m+1)(p-2)}, \infty\big)$, and Γ is time-dependent. In fact, the decomposition argument in Sect. 4.1 fails if $p > 2$.

Remark 2.5 (a) Based on the method of continuity, one can easily extend the results in Theorem 2.3 to parabolic operators with bounded lower-order terms

$$\mathcal{P}u = -u_t + D_i(a^{ij}D_j u + a^i u) + b^i D_i u + cu,$$
$$\mathcal{B}u = (a^{ij}D_j u + a^i u)n_i,$$

at the cost of possibly increasing the constant λ_0.

(b) From Theorem 2.3, we can also obtain the solvability of the initial boundary value problem on $(0, T) \times \Omega$ with the zero initial condition. In this case, we can take $\lambda_0 = 0$ with the help of the standard trick of considering $e^{-\lambda_0 t}u$, cf. [7, Theorem 8.2 (iii)].

Remark 2.6 In theorem 2.3 and throughout the paper, we require the symmetry of the coefficients a^{ij} for the optimal range $p \in (4/3, 4)$ when $m = 0$ for mixed boundary value problems. We refer the reader to [5, Theorem 4.1] and [4, Proposition 4.4] for the optimal estimates for model problems - Laplace and heat equations with flat boundary and separation. Notice that if a^{ij} is not symmetric, then the range of p can be more restrictive. See [5, Example 2.8].

3 Preliminary estimates

Hereafter in this paper, we use the following notation.

Notation 3.1 For nonnegative (variable) quantities A and B, we denote $A \lesssim B$ if there exists a generic positive constant C such that $A \le CB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant C on the parameters a and b.

Notation 3.2 For a given constant $\lambda \ge 0$ and functions u, f, and $g = (g_1, \ldots, g_d)$, we write

$$U = |Du| + \lambda^{1/2}|u|, \quad F = |g| + \lambda^{-1/2}|f|,$$

where we take $f = 0$ and $F = |g|$ whenever $\lambda = 0$.

The following constants in Assumptions 2.1 and 2.2 are fixed throughout the paper:

$$R_0 \in (0, 1], \quad m \in \{0, 1, \ldots, d-2\}, \quad M \in (0, \infty).$$

3.1 $\mathcal{H}_p^1$ solvability and localization

In this section, we derive some local estimates, in the proofs of which, we shall use the unmixed-norm L_p-estimates proved in [4]. For the reader's convenience, we present here the main result in [4].

Theorem 3.3 ([4, Theorem 2.4]) *Let $p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right)$. There exist constants $\theta, \gamma \in (0,1)$ and $\lambda_0 \in (0,\infty)$ with*

$$(\theta,\gamma) = (\theta,\gamma)(d,\Lambda,M,p), \quad \lambda_0 = \lambda_0(d,\Lambda,M,p,R_0),$$

such that if Assumptions 2.1 (θ) and 2.2 ($\gamma; m, M$) are satisfied, then the following assertions hold.

(a) For any $\lambda \geq \lambda_0$, $g = (g_1,\ldots,g_d) \in L_p(\mathcal{Q}^T)^d$, and $f \in L_p(\mathcal{Q}^T)$, there exists a unique solution $u \in \mathcal{H}^1_{p,\mathcal{D}^T}(\mathcal{Q}^T)$ to (2.6), which satisfies

$$\|U\|_{L_p(\mathcal{Q}^T)} \lesssim_{d,\Lambda,M,p} \|F\|_{L_p(\mathcal{Q}^T)}.$$

(b) Let $T \in (0,\infty)$. For any $g = (g_1,\ldots,g_d) \in L_p(\tilde{\mathcal{Q}})^d$ and $f \in L_p(\tilde{\mathcal{Q}})$, there exists a unique solution $u \in \mathcal{H}^1_{p,\tilde{\mathcal{D}}}(\tilde{\mathcal{Q}})$ to the initial boundary value problem

$$\begin{cases} \mathcal{P}u = D_i g_i + f & \text{in } \tilde{\mathcal{Q}} := (0,T)\times\Omega, \\ \mathcal{B}u = g_i n_i & \text{on } \tilde{\mathcal{N}} := ((0,T)\times\partial\Omega)\cap\mathcal{N}, \\ u = 0 & \text{on } \tilde{\mathcal{D}} := ((0,T)\times\partial\Omega)\cap\mathcal{D}, \\ u = 0 & \text{on } \{0\}\times\Omega, \end{cases}$$

which satisfies

$$\|u\|_{\mathcal{H}^1_{p,\tilde{\mathcal{D}}}(\tilde{\mathcal{Q}})} \lesssim_{d,\Lambda,M,p,R_0,T} \|g\|_{L_p(\tilde{\mathcal{Q}})} + \|f\|_{L_p(\tilde{\mathcal{Q}})}. \tag{3.1}$$

Remark 3.4 Theorem 3.3 (b) still holds with $f \in L_q(\tilde{\mathcal{Q}})$ and

$$\|u\|_{\mathcal{H}^1_{p,\tilde{\mathcal{D}}}(\tilde{\mathcal{Q}})} \lesssim_{d,\Lambda,M,p,R_0,T} \|g\|_{L_p(\tilde{\mathcal{Q}})} + \|f\|_{L_q(\tilde{\mathcal{Q}})}$$

instead of (3.1), if $q \in (1,p]$ is such that

$$\frac{d+2}{q} \leq 1 + \frac{d+2}{p}.$$

If the inequality above is strict, then we can take $q = 1$. The proof is based on a duality argument combined with embedding result for parabolic Sobolev spaces and the standard approximation argument. We refer the reader to [13, Lemma 3.1] for the duality argument and [13, Theorem 5.2] for the embedding result (with mixed-norms). Notice that in [13, Theorem 5.2], the boundedness of the domain is required. However, if unmixed-norms are considered, then by a covering argument we can remove the boundedness condition so that the result is applicable to our case.

From Theorem 3.3, we can obtain the following local estimates.

Lemma 3.5 *Let* $p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right)$ *and* θ, γ *be the constants from Theorem 3.3. If Assumptions 2.1* (θ) *and 2.2* $(\gamma; m, M)$ *are satisfied with these* θ *and* γ*, then the following assertions hold. Let* $(0,0) \in \overline{\mathcal{Q}}$*,* $R \in (0, R_0]$*, and* $u \in \mathcal{H}^1_p(\mathcal{Q}_R)$ *satisfy*

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i & \text{in } \mathcal{Q}_R, \\ \mathcal{B}u = g_i n_i & \text{on } Q_R \cap \mathcal{N}, \\ u = 0 & \text{on } Q_R \cap \mathcal{D}, \end{cases} \tag{3.2}$$

where $g = (g_1, \ldots, g_d) \in L_p(\mathcal{Q}_R)^d$.

(a) When $\lambda = 0$*, we have that*

$$\|Du\|_{L_p(\mathcal{Q}_{R/2})} \lesssim_{d,\Lambda,M,p} R^{-1}\|u\|_{L_p(\mathcal{Q}_R)} + \|g\|_{L_p(\mathcal{Q}_R)}.$$

(b) When $\lambda \geq 0$ *and* $g = 0$*, for any* $\hat{p} < 2(m+2)/(m+1)$*, we have* $U \in L_{\hat{p}}(\mathcal{Q}_{R/2})$ *with*

$$(U^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/2}} \lesssim_{d,\Lambda,M,p,\hat{p}} (U)_{\mathcal{Q}_R}. \tag{3.3}$$

Proof The result in (a) is obtained by localizing the estimate in Theorem 3.3 (a). The proof is the same as that of [4, Lemma 3.10], and hence is omitted.

To prove (b), we first deal with the case $\lambda = 0$. By a standard bootstrap argument with the solvability result in Remark 3.4, we see that $Du \in L_{\hat{p}}(\mathcal{Q}_\rho)$ for any $\hat{p} < 2(m+2)/(m+1)$ and $\rho \in (0, R)$. For the estimate (3.3) (with $\lambda = 0$), we prove that

$$(|Du|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{r/4}(X_0)} \lesssim (|Du|^{p_*})^{1/p_*}_{\mathcal{Q}_r(X_0)} \tag{3.4}$$

for $\hat{p} > \max\{p, 2\}$ and $p_* \in ((d+2)\hat{p}/(d+2+\hat{p}), \hat{p})$. Here $\mathcal{Q}_r(X_0) \subset \mathcal{Q}_{R/2}$ and we have one of the following four cases: $Q_r(X_0) \subset \mathcal{Q}$ (interior), or $X_0 \in \partial\mathcal{Q}$ with $Q_r(X_0) \cap \partial\mathcal{Q} \subset \mathcal{D}$ (Dirichlet), or $X_0 \in \partial\mathcal{Q}$ with $Q_r(X_0) \cap \partial\mathcal{Q} \subset \mathcal{N}$ (conormal), or $X_0 \in \Gamma$ (mixed).

The Dirichlet or mixed cases. From the result in (a) with $p = \hat{p}$, we have

$$(|Du|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{r/4}(X_0)} \lesssim r^{-1}(|u|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{r/2}(X_0)}.$$

Then (3.4) can be obtained by applying the following Sobolev-Poincaré inequality in [4, (3.9) in Lemma 3.8] with $p_0 = q_0 = \hat{p}$ and $p = q = p_*$:

$$(|u|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{r/2}(X_0)} \lesssim r(|Du|^{p_*})^{1/p_*}_{\mathcal{Q}_r(X_0)}.$$

The conormal or interior cases. Note that in either cases, if u is a solution and c is a constant, then $u - c$ is also a solution. Hence, the result in (a) with $p = \hat{p}$ and u being replaced with $u - (u)_{\mathcal{Q}_{r/2}(X_0)}$ yields

$$(|Du|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{r/4}(X_0)} \lesssim r^{-1}(|u - (u)_{\mathcal{Q}_{r/2}(X_0)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{r/2}(X_0)}.$$

Then (3.4) can be obtained by applying the embedding [4, (3.8) in Lemma 3.8].

From (3.4), the desired estimate (3.3) with $\lambda = 0$ can be proved in a standard way: rescaling, covering, and iteration.

The case when $\lambda > 0$ can be proved by using Agmon's idea of considering $u(t,x)\cos(\sqrt{\lambda}y + \pi/4)$ with an artificial variable $y \in \mathbb{R}$, noting

$$(\mathcal{P} + \partial_{yy})(u\cos(\sqrt{\lambda}y + \pi/4)) = (\mathcal{P} - \lambda)(u\cos(\sqrt{\lambda}y + \pi/4)),$$

cf. the proof of [4, Lemma 3.12]. The lemma is proved □

3.2 Equations with constant coefficients and a time-independent separation

In this section, we deal with

$$\begin{cases} -u_t + D_i(a_0^{ij} D_j u) - \lambda u = 0 & \text{in } \mathcal{Q}_R, \\ a_0^{ij} D_j u n_i = 0 & \text{on } Q_R \cap \mathcal{N}, \\ u = 0 & \text{on } Q_R \cap \mathcal{D}, \end{cases} \tag{3.5}$$

where $(a_0^{ij})_{i,j}$ is a constant symmetric matrix with the elliptic constant Λ and the interfacial boundary

$$\Gamma \cap Q_R = \overline{\mathcal{D}} \cap \overline{\mathcal{N}} \cap Q_R$$

is time-independent. In such a situation, we can differentiate both the equation and the boundary conditions in t. Furthermore, the usual time-average technique (the Steklov average) is available to build test functions.

Lemma 3.6 *Let* $p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right)$, $(0,0) \in \Gamma$, *and* $R \in (0, R_0]$. *For the constant* $\gamma \in (0,1)$ *from Theorem 3.3, if Assumption 2.2* $(\gamma; m, M)$ *is satisfied, then for any solution* $u \in \mathcal{H}_p^1(\mathcal{Q}_R)$ *to* (3.5) *with* $\lambda \geq 0$, *we have* $U \in L_p^x L_\infty^t(\mathcal{Q}_{R/4})$ *and*

$$\|U\|_{L_p^x L_\infty^t(\mathcal{Q}_{R/4})} \lesssim R^{-2/p} \|U\|_{L_p(\mathcal{Q}_R)}.$$

Proof Again, by Agmon's idea, we only deal with the case when $\lambda = 0$.

The case when $p = 2$. The lemma follows from [4, Proposition 4.1].

The case when $p \in \left(2, \frac{2(m+2)}{m+1}\right)$. Due to the time-independency, u_t satisfies the same equation and boundary conditions. Hence by Lemma 3.5 (a) with u_t in place of u,

$$\|Du_t\|_{L_p(\mathcal{Q}_{R/4})} \lesssim R^{-1} \|u_t\|_{L_p(\mathcal{Q}_{R/2})}. \tag{3.6}$$

Testing (3.5) by $u_t|u_t|^{p-2}$ and then applying Young's inequality, where we need $p \geq 2$, we obtain

$$\|u_t\|_{L_p(\mathcal{Q}_{R/2})} \lesssim R^{-1} \|Du\|_{L_p(\mathcal{Q}_R)}. \tag{3.7}$$

In this process, standard techniques including the mollification and iteration arguments as in the proof of [4, Lemma 4.3] are needed. Here we omit the details. From (3.6) and (3.7), we get

$$\|Du_t\|_{L_p(\mathcal{Q}_{R/4})} \lesssim R^{-2} \|Du\|_{L_p(\mathcal{Q}_R)}. \tag{3.8}$$

Now we use the Sobolev embedding (only in the t variable) to obtain

$$\begin{aligned} \|Du(\cdot, x)\|_{L_\infty^t((-(R/4)^2,0))} \lesssim\ & R^{2-2/p} \|Du_t(\cdot, x)\|_{L_p((-(R/4)^2,0))} \\ & + R^{-2/p} \|Du(\cdot, x)\|_{L_p((-(R/4)^2,0))}. \end{aligned}$$

Taking the L_p norm in x, and then using (3.8), we have

$$\|Du\|_{L_p^x L_\infty^t(\mathcal{Q}_{R/4})} \lesssim R^{-2/p} \|Du\|_{L_p(\mathcal{Q}_R)}.$$

The case when $p \in \left(\frac{2(m+2)}{m+3}, 2\right)$. In this case, we first see that $u \in \mathcal{H}_2^1(\mathcal{Q}_{R/2})$ by Lemma 3.5 (b). From Hölder's inequality and the estimate with $p = 2$, we obtain

$$\|U\|_{L_p^x L_\infty^t(\mathcal{Q}_{R/4})} \lesssim R^{d/p-d/2}\|U\|_{L_2^x L_\infty^t(\mathcal{Q}_{R/4})} \lesssim R^{d/p-d/2-1}\|U\|_{L_2(\mathcal{Q}_{R/2})} \lesssim R^{-2/p}\|U\|_{L_p(\mathcal{Q}_R)}.$$

In the last inequality, we applied (3.3) with $\hat{p} = 2$ and Hölder's inequality. The lemma is proved. □

4 Higher regularity of $\mathcal{H}_p^1$ solutions

In this section, we prove the following regularity result by a level set argument.

Proposition 4.1 *Let $p \in \big(\frac{2(m+2)}{m+3}, 2\big]$ and $q \in (p, \infty)$. There exist constants $\theta, \gamma \in (0, 1)$ and $\lambda_0 \in (0, \infty)$ with*

$$(\theta, \gamma) = (\theta, \gamma)(d, \Lambda, M, p, q), \quad \lambda_0 = \lambda_0(d, \Lambda, M, p, R_0)$$

such that if Assumptions 2.1 (θ) and 2.2 $(\gamma; m, M)$ are satisfied, then for any solution $u \in \mathcal{H}^1_{p,\mathcal{D}^T}(\mathcal{Q}^T)$ to (2.6), where $\lambda \geq \lambda_0$ and $g_i, f \in L_p(\mathcal{Q}^T) \cap L_{q,p}(\mathcal{Q}^T)$, we have $u \in \mathcal{H}^1_{q,p,\mathcal{D}^T}(\mathcal{Q}^T)$ satisfying

$$\|U\|_{L_{q,p}(\mathcal{Q}^T)} \lesssim_{d,\Lambda,M,p,q} \|F\|_{L_{q,p}(\mathcal{Q}^T)} + R_0^{2(1/q-1/p)}\|U\|_{L_p(\mathcal{Q}^T)}. \tag{4.1}$$

When Γ is time-independent, the same result is true for any $p \in \big(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\big)$.

The rest of Sect. 4 will be devoted to proving the proposition. Let us denote

$$\Phi_U(t) := \|U(t,\cdot)\|_{L_p(\Omega)} \quad \text{and} \quad \Phi_F(t) := \|F(t,\cdot)\|_{L_p(\Omega)}.$$

4.1 Decomposition

The key step in proving Proposition 4.1 is a suitable local decomposition of $\Phi_U(t)$.

Lemma 4.2 *Let $p \in \big(\frac{2(m+2)}{m+3}, 2\big]$ and $\hat{p} \in \big(2, \frac{2(m+2)}{m+1}\big)$. For the constants*

$$(\theta, \gamma) = (\theta, \gamma)(d, \Lambda, M, p), \quad \lambda_0 = \lambda_0(d, \Lambda, M, p, R_0)$$

in Theorem 3.3, if Assumptions 2.1 (θ) and 2.2 $(\gamma; m, M)$ are satisfied, then the following assertion holds. For any $t_0 \in (-\infty, T]$, $R \in (0, R_0]$, and $u \in \mathcal{H}^1_{p,\mathcal{D}^T}(\mathcal{Q}^T)$ satisfying (2.6) with $\lambda \geq \lambda_0$ and $g_i, f \in L_p(\mathcal{Q}^T)$, there exist nonnegative functions $\Phi_{W,R}(t)$ and $\Phi_{V,R}(t)$ defined on $(t_0 - (R/16)^2, t_0)$, such that

$$\begin{aligned} &\Phi_U \leq \Phi_{W,R} + \Phi_{V,R} \quad \textit{in } (t_0 - (R/16)^2, t_0), \\ &\int_{t_0-(R/16)^2}^{t_0} \Phi_{W,R}(t)^p\,dt \lesssim_{d,\Lambda,M,p,\hat{p}} (\theta+\gamma)^\tau \int_{t_0-R^2}^{t_0} \Phi_U(t)^p\,dt + \int_{t_0-R^2}^{t_0} \Phi_F(t)^p\,dt, \end{aligned} \tag{4.2}$$

where $\tau = p/2 - p/\hat{p}$, and

$$\sup_{t\in(t_0-(R/16)^2,t_0)} \Phi_{V,R}(t) \lesssim_{d,\Lambda,M,p,\hat{p}} \left(\fint_{t_0-R^2}^{t_0} \Phi_U(t)^p\,dt\right)^{1/p} + \left(\fint_{t_0-R^2}^{t_0} \Phi_F(t)^p\,dt\right)^{1/p}. \tag{4.3}$$

When Γ is time-independent, the same result holds with $\tau = 1 - p/\hat{p}$ for any $p \in \big(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\big)$ and $\hat{p} \in \big(p, \frac{2(m+2)}{m+1}\big)$.

Proof *Case 1: time-dependent* Γ, $p \in \left(\frac{2(m+2)}{m+3}, 2\right]$, and $\hat{p} \in \left(2, \frac{2(m+2)}{m+1}\right)$.

The first decomposition: source terms. By Theorem 3.3, there is a unique solution $u^{(1)} \in \mathcal{H}^1_{p,\mathcal{D}^T}(\mathcal{Q}^T)$ to (2.6) with $g_i \mathbb{I}_{(t_0-R^2,t_0)}$ and $f\mathbb{I}_{(t_0-R^2,t_0)}$ in place of g_i and f, which satisfies

$$\|U^{(1)}\|_{L_p(\mathcal{Q}^T)} \lesssim \left(\int_{t_0-R^2}^{t_0} \Phi_F(t)^p \, dt\right)^{1/p}. \tag{4.4}$$

Here $U^{(1)} = |Du^{(1)}| + \sqrt{\lambda}|u^{(1)}|$. Let $u^{(2)} := u - u^{(1)}$.

Next, for any point $X_0 = (x_0, t_0)$ with $x_0 \in \overline{\Omega}$, since $u^{(2)}$ satisfies a homogeneous equation in $\mathcal{Q}_R(X_0)$, by Lemma 3.5 (b) we see that $u^{(2)} \in \mathcal{H}^1_{\hat{p}}(\mathcal{Q}_{R/2}(X_0))$ and

$$(|U^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/2}(X_0)} \lesssim (U^{(2)})_{\mathcal{Q}_R(X_0)}, \tag{4.5}$$

where $U^{(2)} = |Du^{(2)}| + \sqrt{\lambda}|u^{(2)}|$. We claim that, for any $X_0 = (t_0, x_0)$, we can find positive functions W and V satisfying

$$U^{(2)} \le W + V \quad \text{in } \mathcal{Q}_{R/16}(X_0), \tag{4.6}$$

$$\|W\|_{L_p(\mathcal{Q}_{R/16}(X_0))} \lesssim (\theta + \gamma)^{1/2-1/\hat{p}} \|U^{(2)}\|_{L_p(\mathcal{Q}_R(X_0))}, \tag{4.7}$$

and

$$\|V\|_{L^t_\infty L^x_p(\mathcal{Q}_{R/16}(X_0))} \lesssim R^{-2/p} \|U^{(2)}\|_{L_p(\mathcal{Q}_R(X_0))}. \tag{4.8}$$

Let us first focus on the most complicated case – the mixed case, i.e., when $X_0 \in \Gamma$.

The second decomposition: approximating a_{ij} and Γ. In this case, we need to approximate Γ by a time-independent separation and a_{ij} by its average. Take the coordinate system in Assumption 2.2 $(\gamma; m, M)$, and by translation, we may assume that $X_0 = (0, 0)$. Let $\chi = \chi(x)$ be the cut-off function on $\mathbb{R}^d$ satisfying

$$0 \le \chi \le 1, \quad |D\chi| \lesssim_d \frac{1+M}{\gamma R},$$
$$\chi = 0 \quad \text{in } \{x : x^1 < \gamma R,\ x^2 > \phi - \gamma R\},$$
$$\chi = 1 \quad \text{in } \mathbb{R}^d \setminus \{x : x^1 < 2\gamma R,\ x^2 > \phi - 2\gamma R\}.$$

Then $\chi u^{(2)}$ satisfies

$$\begin{cases} \mathcal{P}_0(\chi u^{(2)}) - \lambda \chi u^{(2)} = D_i g_i^* + f^* & \text{in } \mathcal{Q}_{R/2}, \\ \mathcal{B}_0(\chi u^{(2)}) = g_i^* n_i & \text{on } (-(R/2)^2, 0) \times N_{R/2}, \\ \chi u^{(2)} = 0 & \text{on } (-(R/2)^2, 0) \times D_{R/2}, \end{cases}$$

where

$$\mathcal{P}_0 u := -u_t + D_i((a^{ij})_{\mathcal{Q}_R} D_j u) \quad \text{and} \quad \mathcal{B}_0 u := (a^{ij})_{\mathcal{Q}_R} D_j u n_i$$

have constant coefficients,

$$f^* = a^{ij} D_j u^{(2)} D_i \chi, \quad g_i^* = ((a^{ij})_{\mathcal{Q}_R} - a^{ij}) D_j(\chi u^{(2)}) + a^{ij} u^{(2)} D_j \chi,$$

and

$$\begin{aligned} D_{R/2} &:= \partial\Omega \cap B_{R/2} \cap \{x : x^2 > \phi - \gamma R\}, \\ N_{R/2} &:= \partial\Omega \cap B_{R/2} \cap \{x : x^2 < \phi - \gamma R\}. \end{aligned} \tag{4.9}$$

We first estimate $(1-\chi)u^{(2)}$. By Hölder's inequality and our construction of χ,

$$
\begin{aligned}
&(|(1-\chi)U^{(2)}|^p)^{1/p}_{\mathcal{Q}_{R/4}} + (|D\chi u^{(2)}|^p)^{1/p}_{\mathcal{Q}_{R/4}} \\
&\lesssim (|(1-\chi)U^{(2)}|^p)^{1/p}_{\mathcal{Q}_{R/4}} + \frac{1}{\gamma R}(|\mathbb{I}_{\operatorname{supp}(D\chi)}u^{(2)}|^p)^{1/p}_{\mathcal{Q}_{R/4}} \\
&\lesssim \gamma^{1/p-1/\hat{p}}\left((|U^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/4}} + \frac{1}{\gamma R}(|\mathbb{I}_{\operatorname{supp}(D\chi)}u^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/4}}\right) \\
&\lesssim \gamma^{1/p-1/\hat{p}}\left((|U^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/4}} + (|Du^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/2}}\right) \qquad (4.10)\\
&\lesssim \gamma^{1/p-1/\hat{p}}(|U^{(2)}|^p)^{1/p}_{\mathcal{Q}_R}. \qquad (4.11)
\end{aligned}
$$

Here, in (4.10), we applied the boundary Poincaré inequality in [4, Lemma 3.9] on narrow regions, noting

$$\operatorname{dist}(\operatorname{supp}(D\chi)\cap\mathcal{Q}_{R/4}, D_{R/2}\cap\mathcal{Q}_{R/4}) < C\gamma R.$$

In (4.11), we used (4.5) and Hölder's inequality.

The third decomposition. Next we decompose $\chi u^{(2)} = u^{(3)} + u^{(4)}$ with

$$
\begin{cases}
\mathcal{P}_0 u^{(3)} - \lambda u^{(3)} = D_i(g_i^*\mathbb{I}_{\mathcal{Q}_{R/4}}) + f^*\mathbb{I}_{\mathcal{Q}_{R/4}} & \text{in } \mathcal{Q}^0,\\
B_0 u^{(3)} = (g_i^*\mathbb{I}_{\mathcal{Q}_{R/4}})n_i & \text{on } (-\infty,0)\times N_{R/2},\\
u^{(3)} = 0 & \text{on } (-\infty,0)\times(\partial\Omega\setminus N_{R/2}).
\end{cases}
\qquad (4.12)
$$

Notice that the new separation in the above problem is time-independent but may not satisfy Assumption 2.2 (b). This is because the intersection of the boundary a small Reifenberg flat domain and a hyperplane might not be Reifenberg flat as the x^1-direction of the boundary (cf. Assumption 2.2) at small scales might be almost parallelled to the normal direction of the hyperplane. For such a reason, we apply [4, Lemma 3.5] which only requires the interfacial boundary to be time-independent to obtain the solution $u^{(3)}$ in $\mathcal{H}^1_2(\mathcal{Q}^0)$, whereas we are not able to utilize Theorem 3.3 to get the solution in $\mathcal{H}^1_p(\mathcal{Q}^0)$. This fact causes the restriction $p\le 2$. Clearly, $u^{(3)}=0$ for $t\le -(R/4)^2$. Moreover, we can test (4.12) by $u^{(3)}$ (with help of the usual Steklov average technique) to obtain

$$(|U^{(3)}|^2)_{\mathcal{Q}_{R/2}} \lesssim (|g_i^*|^2)^{1/2}_{\mathcal{Q}_{R/4}}(|D_i u^{(3)}|^2)^{1/2}_{\mathcal{Q}_{R/4}} + (|f^* u^{(3)}|)_{\mathcal{Q}_{R/4}}, \qquad (4.13)$$

where

$$U^{(3)} := |Du^{(3)}| + \sqrt{\lambda}|u^{(3)}|.$$

Furthermore, by Hölder's inequality, Assumption 2.1 (θ), and the boundary Poincaré inequality [4, Lemma 3.9] as above,

$$(|g_i^*|^2)^{1/2}_{\mathcal{Q}_{R/4}} \lesssim (|u^{(2)}D\chi|^2)^{1/2}_{\mathcal{Q}_{R/4}} + \theta^{1/2-1/\hat{p}}(|Du^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/4}} \lesssim (\theta+\gamma)^{1/2-1/\hat{p}}(|U^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/2}}. \qquad (4.14)$$

Similarly, by Hölder's inequality and [4, Lemma 3.9], we have

$$
\begin{aligned}
(|f^* u^{(3)}|^2)_{\mathcal{Q}_{R/4}} &\lesssim (|\mathbb{I}_{\operatorname{supp}(D\chi)}Du^{(2)}|^2)^{1/2}_{\mathcal{Q}_{R/4}}(|D\chi u^{(3)}|^2)^{1/2}_{\mathcal{Q}_{R/4}}\\
&\lesssim \gamma^{1/2-1/\hat{p}}(|U^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/2}}(|U^{(3)}|^2)^{1/2}_{\mathcal{Q}_{R/2}}. \qquad (4.15)
\end{aligned}
$$

From (4.13) to (4.15), canceling $(|U^{(3)}|^2)^{1/2}_{\mathcal{Q}_{R/2}}$ from both sides, we have

$$(|U^{(3)}|^2)^{1/2}_{\mathcal{Q}_{R/2}} \lesssim (\theta+\gamma)^{1/2-1/\hat{p}}(|U^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/2}}.$$

Recall that $p \le 2$. By Hölder's inequality, the inequality above, and (4.5), we have

$$(|U^{(3)}|^p)^{1/p}_{\mathcal{Q}_{R/2}} \lesssim (|U^{(3)}|^2)^{1/2}_{\mathcal{Q}_{R/2}} \lesssim (\theta+\gamma)^{1/2-1/\hat{p}}(|U^{(2)}|^{\hat{p}})^{1/\hat{p}}_{\mathcal{Q}_{R/2}}$$
$$\lesssim (\theta+\gamma)^{1/2-1/\hat{p}}(|U^{(2)}|^p)^{1/p}_{\mathcal{Q}_R}. \tag{4.16}$$

Let

$$W := |U^{(3)}| + |(1-\chi)U^{(2)}| + |D\chi u^{(2)}|.$$

From (4.11) and (4.16), we obtain

$$\|W\|_{L_p(\mathcal{Q}_{R/4})} \lesssim (\theta+\gamma)^{1/2-1/\hat{p}}\|U^{(2)}\|_{L_p(\mathcal{Q}_R)}. \tag{4.17}$$

Let $V := |Du^{(4)}| + \sqrt{\lambda}|u^{(4)}|$, where $u^{(4)} = \chi u^{(2)} - u^{(3)} \in \mathcal{H}^1_2(\mathcal{Q}_R)$ satisfies

$$\begin{cases} \mathcal{P}_0 u^{(4)} - \lambda u^{(4)} = 0 & \text{in } \mathcal{Q}_{R/4}, \\ B_0 u^{(4)} = 0 & \text{on } (-(R/4)^2, 0) \times N_{R/4}, \\ u^{(4)} = 0 & \text{on } (-(R/4)^2, 0) \times D_{R/4}, \end{cases}$$

where

$$D_{R/4} := \partial\Omega \cap B_{R/4} \cap \{x : x^2 > \phi - \gamma R\},$$
$$N_{R/4} := \partial\Omega \cap B_{R/4} \cap \{x : x^2 < \phi - \gamma R\}.$$

By [4, Proposition 4.1] (noting that $|\partial_t V| \le |Du_t^{(4)}| + \sqrt{\lambda}|u_t^{(4)}|$) and Hölder's inequality, we have

$$\|\partial_t V\|_{L_p(\mathcal{Q}_{R/8})} \lesssim R^{(d+2)/p-2-(d+2)}\|V\|_{L_1(\mathcal{Q}_{R/4})}$$
$$\lesssim R^{-2}\|V\|_{L_p(\mathcal{Q}_{R/4})}. \tag{4.18}$$

Now we use the Sobolev embedding (in t) to obtain, for any $x \in \Omega_{R/16}$,

$$\|V(\cdot,x)\|_{L_\infty((-(R/8)^2,0))} \lesssim R^{2-2/p}\|\partial_t V(\cdot,x)\|_{L_p((-(R/8)^2,0))}$$
$$+ R^{-2/p}\|V(\cdot,x)\|_{L_p((-(R/8)^2,0))}.$$

Taking the $L_p(\Omega_{R/8})$ norm in x, using (4.18), and noting

$$V \le |D(\chi u^{(2)})| + \sqrt{\lambda}|\chi u^{(2)}| + |U^{(3)}|,$$

we obtain

$$R^{2/p}\|V\|_{L^t_\infty L^x_p(\mathcal{Q}_{R/8})} \lesssim R^{2/p}\|V\|_{L^x_p L^t_\infty(\mathcal{Q}_{R/8})} \lesssim \|V\|_{L_p(\mathcal{Q}_{R/4})}$$
$$\lesssim \|D(\chi u^{(2)})\|_{L_p(\mathcal{Q}_{R/4})} + \sqrt{\lambda}\|\chi u^{(2)}\|_{L_p(\mathcal{Q}_{R/4})} + \|U^{(3)}\|_{L_p(\mathcal{Q}_{R/4})}$$
$$\lesssim \|U^{(2)}\|_{L_p(\mathcal{Q}_R)}. \tag{4.19}$$

In the last inequality, we used (4.11) for the estimate of $D\chi u^{(2)}$ and also (4.16). From the construction of W and V, (4.17), and (4.19), we complete the construction satisfying (4.6)–(4.8) when $X_0 \in \Gamma$.

Next, for X_0 having one of the following three positions: $\mathcal{Q}_R(X_0) \subset \mathcal{Q}$ (interior), $\mathcal{Q}_R(X_0) \cap \partial\mathcal{Q} \subset \mathcal{D}$ (pure Dirichlet), or $\mathcal{Q}_R(X_0) \cap \partial\mathcal{Q} \subset \mathcal{N}$ (pure Neumann), the construction is similar. Actually it is simpler since no approximation of Γ is presented. From these, the construction centered at any point X_0 can be achieved by a standard scaling and covering argument. The details are omitted.

Now we cover $(-(R/16)^2, 0) \times \Omega$ with $\bigcup_k \mathcal{Q}_{R/16}(X_0^{(k)})$ with the number of overlapping bounded by a number independent of R. On each $\mathcal{Q}_{R/16}(X_0^{(k)})$, we can define $W^{(k)}$ and $V^{(k)}$ as above. Then let

$$\Phi_{W,R}(t) := \left(\sum_k \| W^{(k)}(t,\cdot)\mathbb{I}_{\mathcal{Q}_{R/16}(X_0^{(k)})} \|_{L_p(\Omega)}^p + \| U^{(1)}(t,\cdot) \|_{L_p(\Omega)}^p \right)^{1/p} \tag{4.20}$$

and

$$\Phi_{V,R}(t) := \left(\sum_k \| V^{(k)}(t,\cdot)\mathbb{I}_{\mathcal{Q}_{R/16}(X_0^{(k)})} \|_{L_p(\Omega)}^p \right)^{1/p}. \tag{4.21}$$

We immediately have

$$\Phi_U(t) \le \Phi_{W,R}(t) + \Phi_{V,R}(t) \quad \forall t \in (-(R/16)^2, 0).$$

Furthermore, from (4.20), (4.7) with centers $X_0^{(k)}$, the fact $U^{(2)} \le U + U^{(1)}$, and (4.4),

$$\begin{aligned}
&\int_{-(R/16)^2}^0 \Phi_{W,R}(t)^p \, dt \\
&\lesssim (\theta+\gamma)^{p/2-p/\hat{p}} \int_{-R^2}^0 \|U^{(2)}(t,\cdot)\|_{L_p(\Omega)}^p \, dt + \int_{-R^2}^0 \Phi_F(t)^p \, dt \\
&\lesssim (\theta+\gamma)^{p/2-p/\hat{p}} \int_{-R^2}^0 \left(\|U^{(1)}(t,\cdot)\|_{L_p(\Omega)}^p + \|U(t,\cdot)\|_{L_p(\Omega)}^p \right) dt + \int_{-R^2}^0 \Phi_F(t)^p \, dt \\
&\lesssim (\theta+\gamma)^{p/2-p/\hat{p}} \int_{-R^2}^0 \Phi_U(t)^p \, dt + \int_{-R^2}^0 \Phi_F(t)^p \, dt.
\end{aligned}$$

This proves (4.2). Similarly, (4.3) can be obtained from (4.21), (4.8), $U^{(2)} \le U + U^{(1)}$, and (4.4). This finishes the proof of the time-dependent case.

Case 2: time-independent Γ, $p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right)$, and $\hat{p} \in \left(p, \frac{2(m+2)}{m+1}\right)$. We define $u^{(1)}$ as before and let $u^{(2)} = u - u^{(1)}$. For the local decomposition of $U^{(2)}$, in this case, we do not need to employ the cutoff argument with χ. Hence, the proof is simpler and the restriction $p \le 2$ is no longer needed. Let us give a sketch. We first freeze the coefficients by solving

$$\begin{cases} \mathcal{P}_0 u^{(3)} - \lambda u^{(3)} = D_i\left(((a^{ij})_{\mathcal{Q}_R} - a^{ij}) D_j u^{(2)} \mathbb{I}_{\mathcal{Q}_{R/4}} \right) & \text{in } \mathcal{Q}^0, \\ \mathcal{B}_0 u^{(3)} = ((a^{ij})_{\mathcal{Q}_R} - a^{ij}) D_j u^{(2)} \mathbb{I}_{\mathcal{Q}_{R/4}} n_i & \text{on } \mathcal{N}^0, \\ u^{(3)} = 0 & \text{on } \mathcal{D}^0. \end{cases}$$

By Theorem 3.3, the solution $u^{(3)} \in \mathcal{H}_p^1(\mathcal{Q}^0)$ exists and satisfies

$$\|U^{(3)}\|_{L_p(\mathcal{Q}_{R/4})} \lesssim \|((a^{ij})_{\mathcal{Q}_R} - a^{ij}) D_j u^{(2)}\|_{L_p(\mathcal{Q}_{R/4})} \lesssim (\theta+\gamma)^{1/p-1/\hat{p}} \|U^{(2)}\|_{L_p(\mathcal{Q}_R)}. \tag{4.22}$$

In the last inequality, we use Hölder's inequality and the reverse Hölder's inequality as in (4.16). We define

$$W := U^{(3)} \quad \text{and} \quad V := |Du^{(4)}| + \sqrt{\lambda}|Du^{(4)}|,$$

where $u^{(4)} := u^{(2)} - u^{(3)}$. Since v satisfies a homogeneous equation with time-independent separation satisfying the condition (b) of Assumption 2.2 $(\gamma; m, M)$ in $\mathcal{Q}_{R/4}$, by Lemma 3.6 and (4.22),

$$R^{2/p}\|V\|_{L_\infty^t L_p^x(\mathcal{Q}_{R/16})} \lesssim \|V\|_{L_p(\mathcal{Q}_{R/4})} \lesssim \|U^{(2)}\|_{L_p(\mathcal{Q}_{R/4})} + \|U^{(3)}\|_{L_p(\mathcal{Q}_{R/4})} \lesssim \|U^{(2)}\|_{L_p(\mathcal{Q}_R)}.$$

The rest of the proof remains the same as the time-dependent case. □

4.2 Level set estimates

In this section, we focus on the case when Γ is time-dependent and $p \in \big(\frac{2(m+2)}{m+3}, 2\big]$. When Γ is time-independent and $p \in \big(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\big)$, some minor changes are needed. See Sect. 4.3. For a function $h \in L_{1,\text{loc}}(-\infty, T)$, we define its (1 dimensional) maximal function by

$$\mathcal{M}(h)(t_0) := \sup_{(a,b)\ni t_0} \fint_a^b |h(t)|\mathbb{I}_{(-\infty,T)}\, dt.$$

We will estimate the following level sets

$$\begin{aligned} \mathcal{A}(s) &:= \big\{t \in (-\infty, T) : \mathcal{M}(\Phi_U^p)(t)^{1/p} > s\big\}, \\ \mathcal{B}(s) &:= \big\{t \in (-\infty, T) : \mathcal{M}(\Phi_U^p)(t)^{1/p} + (\theta+\gamma)^{1/\hat{p}-1/2}\mathcal{M}(\Phi_F^p)(t)^{1/p} > s\big\}. \end{aligned} \tag{4.23}$$

Lemma 4.3 *Let p, $\hat{p}$, and u be as in Lemma 4.2. There exists a constant $\kappa = \kappa(d, \lambda, M, p, \hat{p}) > 5$,*
such that for any interval $(a, b) \subset (-\infty, T)$ with $|b-a| \le (R_0/32)^2$ and

$$|(a,b) \cap \mathcal{A}(\kappa s)| > (\theta+\gamma)^{p/2-p/\hat{p}}|b-a|, \tag{4.24}$$

we must have

$$(a,b) \subset \mathcal{B}(s).$$

Proof We prove by contradiction. Suppose that for some interval (a, b) satisfying (4.24), there exists some $t_1 \in (a, b) \setminus \mathcal{B}(s)$, i.e.,

$$\mathcal{M}(\Phi_U^p)(t_1)^{1/p} + (\theta+\gamma)^{1/\hat{p}-1/2}\mathcal{M}(\Phi_F^p)(t_1)^{1/p} \le s. \tag{4.25}$$

Let

$$(a_1, b_1) := (a - |b-a|/2, \min\{b + |b-a|/2, T\}), \quad R = 16\sqrt{|b_1 - a_1|},$$

and observe that $(a_1, b_1) = (b_1 - (R/16)^2, b_1)$. By Lemma 4.2 with $t_0 = b_1$ and such R, we have the decomposition

$$\Phi_U \le \Phi_{W,R} + \Phi_{V,R} \quad \text{on } (a_1, b_1). \tag{4.26}$$

Moreover, from (4.2), (4.3), (4.25), and the fact that $t_1 \in (a, b) \subset (b_1 - R^2, b_1)$, we have

$$\left(\fint_{a_1}^{b_1} \Phi_{W,R}(t)^p\, dt\right)^{1/p} \le C_1(\theta+\gamma)^{1/2-1/\hat{p}} s \quad \text{and} \quad \sup_{t\in(a_1,b_1)} \Phi_{V,R}(t) \le C_1 s, \tag{4.27}$$

where $C_1 = C_1(d, \lambda, M, p, \hat{p})$. Now for any $\tilde{t} \in (a, b) \cap \mathcal{A}(\kappa s)$, by definition, there exists an interval $(\tilde{a}, \tilde{b})$ containing $\tilde{t}$, with $\tilde{b} \le T$ and

$$\fint_{\tilde{a}}^{\tilde{b}} \Phi_U(t)^p\, dt > (\kappa s)^p. \tag{4.28}$$

Actually, we must have $|\tilde{b}-\tilde{a}| \leq |b-a|/2$, since otherwise

$$\begin{aligned}\mathcal{M}(\Phi_U^p)(t_1) &\geq \fint_{\tilde{a}-|b-a|}^{\min\{\tilde{b}+|b-a|,T\}} \Phi_U(t)^p\,dt\\ &\geq \frac{|\tilde{b}-\tilde{a}|}{2|b-a|+|\tilde{b}-\tilde{a}|}\fint_{\tilde{a}}^{\tilde{b}} \Phi_U(t)^p\,dt > \frac{1}{5}(\kappa s)^p > s^p\end{aligned}$$

contradicting (4.25). Hence, $(\tilde{a},\tilde{b}) \subset (a_1, b_1)$. From this, (4.26)–(4.28), and the triangle inequality, we get for any $\tilde{t} \in (a,b) \cap \mathcal{A}(\kappa s)$,

$$\begin{aligned}\mathcal{M}\big(\Phi_{W,R}^p \mathbb{I}_{(a_1,b_1)}\big)(\tilde{t})^{1/p} &\geq \left(\fint_{\tilde{a}}^{\tilde{b}} \Phi_{W,R}(t)^p\,dt\right)^{1/p}\\ &\geq \left(\fint_{\tilde{a}}^{\tilde{b}} \Phi_U(t)^p\,dt\right)^{1/p} - \sup_{t\in(\tilde{a},\tilde{b})} \Phi_{V,R}(t) > \kappa s - C_1 s.\end{aligned}$$

Hence, by the Hardy–Littlewood maximal function theorem and (4.27),

$$\begin{aligned}|\mathcal{A}(\kappa s) \cap (a,b)| &\leq \left|\left\{\tilde{t} : \mathcal{M}\big(\Phi_{W,R}^p \mathbb{I}_{(a_1,b_1)}\big)(\tilde{t})^{1/p} > \kappa s - C_1 s\right\} \cap (a,b)\right|\\ &\leq C(\kappa - C_1)^{-p} C_1^p (\theta+\gamma)^{p/2-p/\hat{p}} |b-a|.\end{aligned}$$

Here we also used the fact that $R^2 \approx |b-a|$. Choosing κ large enough, we reach a contradiction with (4.24). The lemma is proved. □

From Lemma 4.3, the Hardy–Littlewood maximal function theorem

$$|\mathcal{A}(\kappa s)| \leq \|U\|_{L_p(\mathcal{Q}^T)}^p / (\kappa s)^p,$$

and a measure theoretic lemma called "crawling of the ink spot" in [18, 23], we have

Corollary 4.4 *For κ in Lemma 4.3, s satisfying*

$$s \geq s_0 := \kappa^{-1} \|U\|_{L_p(\mathcal{Q}^T)} \big((\theta+\gamma)^{p/2-p/\hat{p}} R_0^2/32^2\big)^{-1/p}, \tag{4.29}$$

and θ, γ satisfying

$$(\theta+\gamma)^{p/2-p/\hat{p}} < 1, \tag{4.30}$$

we have

$$|\mathcal{A}(\kappa s)| \leq C(\theta+\gamma)^{p/2-p/\hat{p}} |\mathcal{B}(s)|.$$

Here we omit the details, which can be found, for instance, in [4, Proof of Lemma 5.4]. The key idea here is a stopping time argument: for any $t \in \mathcal{A}(\kappa s)$, we shrink the interval (a,b) containing t until the first time (4.24) holds. The condition (4.29) guarantees that we can start this procedure with $|b-a| = (R_0/32)^2$. The condition (4.30) together with the Lebesgue differentiation theorem guarantees that such procedure will stop.

4.3 Proof of Proposition 4.1

Proof of Proposition 4.1 We mainly prove for general time-dependent Γ. For fixed $\hat{p} \in \left(p, \frac{2(m+2)}{m+1}\right)$, let κ and s_0 be the constants from Lemma 4.3 and Corollary 4.4, respectively. We also let θ and γ be small numbers satisfying (4.30) to be chosen below. For $S > s_0$,

$$\begin{aligned}
&\int_0^{\kappa S} |\mathcal{A}(s)| s^{q-1}\,ds = \kappa^q \int_0^S |\mathcal{A}(\kappa s)| s^{q-1}\,ds \\
&= \kappa^q \int_0^{s_0} |\mathcal{A}(\kappa s)| s^{q-1}\,ds + \kappa^q \int_{s_0}^S |\mathcal{A}(\kappa s)| s^{q-1}\,ds \\
&\le \kappa^q \|U\|_{L_p(Q^T)}^p \int_0^{s_0} (\kappa s)^{-p} s^{q-1}\,ds + C\kappa^q (\theta+\gamma)^{p/2-p/\hat{p}} \int_0^S |\mathcal{B}(s)| s^{q-1}\,ds. \quad (4.31)
\end{aligned}$$

Here in (4.31), we applied the Chebyshev inequality and Corollary 4.4 for the two terms, respectively. Noting $q > p$ and

$$\mathcal{B}(s) \subset \mathcal{A}(s/2) \cup \{t : \mathcal{M}(\Phi_F^p)(t) > 2^{-p} s^p (\theta+\gamma)^{p/2-p/\hat{p}}\},$$

using the integral formula for L_p norms in terms of level sets and the Hardy–Littlewood maximal function theorem, we have

$$\begin{aligned}
&\int_0^{\kappa S} |\mathcal{A}(s)| s^{q-1}\,ds \le C\kappa^{q-p} \|U\|_{L_p(Q^T)}^p s_0^{q-p} + C\kappa^q (\theta+\gamma)^{p/2-p/\hat{p}} \int_0^S |\mathcal{A}(s/2)| s^{q-1}\,ds \\
&\quad + C\kappa^q (\theta+\gamma)^{p/2-p/\hat{p}} \int_0^S \left|\left\{t : \mathcal{M}(\Phi_F^p)(t) > 2^{-p} s^p (\theta+\gamma)^{p/2-p/\hat{p}}\right\}\right| s^{q-1}\,ds \\
&\le C_{\theta,\gamma,\kappa} \left(\|U\|_{L_p(Q^T)}^q R_0^{2(p-q)/p} + \|\Phi_F\|_{L_q((-\infty,T))}^q \right) \\
&\quad + C\kappa^q (\theta+\gamma)^{p/2-p/\hat{p}} \int_0^{S/2} |\mathcal{A}(s)| s^{q-1}\,ds.
\end{aligned}$$

Absorbing the integral involving $\mathcal{A}(s)$ on the right-hand side by choosing θ and γ small enough, passing $S \to \infty$, we reach the desired estimate.

When Γ is time-independent, the proof is almost the same if we change the definition of $\mathcal{B}(s)$ in (4.23) to

$$\mathcal{B}(s) := \left\{t \in (-\infty, T) : \mathcal{M}(\Phi_U^p)(t)^{1/p} + (\theta+\gamma)^{1/\hat{p}-1/p} \mathcal{M}(\Phi_F^p)(t)^{1/p} > s\right\}.$$

The details are omitted. □

5 Proof of Theorem 2.3

With Proposition 4.1 at hand, now we prove Theorem 2.3.

Proof of Theorem 2.3 We consider the following three cases.

Case 1: time-dependent Γ, $p \le 2$. By approximation, we may assume that f and g_i have compact support in time, and hence $f, g_i \in L_q^t L_p^x \subset L_p^{t,x}$. By Theorem 3.3, we can find a solution $u \in \mathcal{H}_p^1$. Furthermore, by Proposition 4.1, $u \in \mathcal{H}_{q,p}^1$. To show (2.7), we are left with absorbing the U term on the right-hand side of (4.1). This step is standard, which can be done by multiplying a cut-off function in the t variable with sufficiently small support, using Hölder's inequality, and then choosing λ large enough. Such argument can be found in the proof of [5, Corollary 5.2]. The range of (p, q) corresponds to the shaded trapezoid area in Fig. 2.

Case 2: time-dependent Γ, $p > 2$. In this case, we interpolate the $\mathcal{H}^1_{\tilde{q},2}$ and $\mathcal{H}^1_{\tilde{p}}$ results, where $\tilde{q} > 2$ can be sufficiently large and $\tilde{p} \in (2, 2(m+2)/(m+1))$. To be more precise, let $\vartheta \in (0, 1)$ be the number such that

$$\frac{1}{p} = \frac{\vartheta}{2} + \frac{1-\vartheta}{\tilde{p}} \quad \text{and} \quad \frac{1}{q} = \frac{\vartheta}{\tilde{q}} + \frac{1-\vartheta}{\tilde{p}}.$$

Then the $\mathcal{H}^1_{q,p}$ solvability can be obtained from the $\mathcal{H}^1_{\tilde{p}}$ and $\mathcal{H}^1_{\tilde{q},2}$ results by applying the Riesz-Thorin interpolation theorem. Here we also used the following fact which can be found, for instance, in [26, Theorem 1.18.1]:

$$[L^t_{\tilde{q}}(L^x_2), L^t_{\tilde{p}}(L^x_{\tilde{p}})]_\vartheta = L^t_q([L^x_2, L^x_{\tilde{p}}]_\vartheta),$$

where $[\,\cdot\,,\,\cdot\,]_\vartheta$ represents the complex interpolation space. The range of (p, q) corresponds to the shaded triangle area in Fig. 2.

Case 3: time-independent Γ, $p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right)$. For $q > p$, the proof is exactly the same as the first case by using the last assertion of Proposition 4.1. For $q < p$, the result can be obtained by duality.

This finishes the proof of Theorem 2.3. □

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