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A Distributed Algorithm for Aggregative Games on Directed Communication Graphs

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Abstract— **This study focuses on aggregative games, a type of Nash games that is played over a network. In these games, the cost function of an agent is affected by its own choice and the sum of all decision variables of the players involved. We consider a distributed algorithm over a network, whereby to reach a Nash equilibrium point, each agent maintains a prediction of the aggregate decision variable and share it with its local neighbors over a strongly connected directed network. The existing literature provides such algorithms for undirected graphs which typically require the use doubly stochastic weight matrices. We consider a fixed directed communication network and investigate a synchronous distributed gradient-based method for computing a Nash equilibrium. We provide convergence analysis of the method showing that the algorithm converges to the Nash equilibrium of the game, under some standard conditions.**

Index Terms— **aggregative games; Nash equilibrium seeking; directed communication networks**

I. Introduction

In an aggregative game, each agent's utility (or cost) function is affected by the aggregate of all the other agents' decision variables in addition to the agent's own decision variable. References [1] and [2] are examples of studies focused on such games. These kind of games arise in a variety of applications, including demand side management for electric vehicles [3] and [4], demand response in marketplaces with congestion [5], and reducing network congestion [6]. From the standpoint of control theory, the purpose is to develop and demonstrate a distributed algorithm that assures convergence of agents' strategies to a non-cooperative game Nash equilibrium.

The aggregate decision variable is typically visible to all agents in a wide range of aggregative game theory studies on equilibrium computation. This assumption allows agents to adjust their decision based on their utility (or cost function) without having to communicate with each other [7]. A semi-decentralized communication system is often used to achieve this ideal condition, where a central unit collects and disseminates the aggregate decision variable to all agents in this system. Example of studies considering such a structure are [8], [9], [10]. In the absence of such a central entity, the aggregate of the decision variables is not immediately accessible to agents. We are interested in the case where the agents are connected over a communication network and may only receive a limited amount of information from their immediate neighbors. The search for a Nash equilibrium in network constrained games has received a lot of interest recently. The goal of this form of games is for players to minimize their own cost function in a selfish manner by making decisions based on an estimate of the other agents' decision variables. In this context, the agents are able to retain an estimate of decision variables of all the other agents and share them with their local neighbors to make up for the lack of immediate access to the aggregate decision [11].

Assuming that agents' immediate ability to obtain the aggregate decision variable is constrained, the paper [12] suggests a completely dispersed approach for figuring out a generalized Nash equilibrium in aggregative games with time-varying graphs of information exchange and coupling constraints. The work in [13] presents a unified convergence analysis for projected gradient algorithms used for computing a general Nash equilibrium in aggregative games. In particular, this study uses a comprehensive method based on the notion of monotone operators to demonstrate that sequential updates to projected gradient algorithms belong to the category of forward-backward splitting techniques with preconditioning [14], which was first proposed in [15] for multi agent games over a network. The work in [8] addresses the problem of guiding a population of non-cooperative heterogeneous agents in the direction of an aggregative equilibrium, where each agent has a cost function with a convex form based on the average population decision variable. In this study, it was assumed that a central coordinator exists who has access to the average population decision variable and can give control command to direct the agents' decentralized best responses. This study develops a dynamic control command based on notions from operator theory that guarantees global equilibrium convergence. Locating the zero point of the sum of monotone operators has been posed as global Nash equilibrium searching in network aggregative games in [16] using primal-dual analysis for the case when players have access only to their neighbors' information which is shared over an undirected communication graph.

Many studies in the context of finding Nash equilibrium in a networked game, including those listed above, assume that agents communicate over an undirected communication graph. This typically results in the use of a doubly stochastic weight matrix that agents use for ”tracking” the other agents decisions. However, symmetric communications are often not possible in some practical applications, such as sensor networks. The work in [11] has proposed an asynchronous random gossip-based approach for computing a Nash equilibrium of a network-constrained game in a directed commu-

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nication graph, which distinguishes it from other studies in the literature. However, the game setup in [11] is not fit for aggregative network games. The work in [17] is concerned with the problem of Nash equilibrium pursuit in aggregative games in directed communication graphs, and provides a distributed continuous-time algorithm that converges to the Nash equilibrium of the game.

Considering lack of studies focusing on Nash equilibrium seeking discrete algorithms in the network aggregative games over a directed communication network, this current paper proposes an extension to the algorithm introduced in [7] to bridge the gap in the literature in this regard. In this paper, for a fixed communication digraph, we propose and analyze an algorithm utilizing the tracking dynamic used in [7] combined with the weight matrices inspired by the work in [18]. Even-though the analysis is building on some related results in [7], our method and its analysis is different from that of the method in [7]. The main difference is that [7] deals with undirected networks and the weight matrices are doubly stochastic, whereas in this paper the graph is directed and the weight matrices are no longer balanced. Compared with [17], our proposed algorithm is a discrete-time one.

Organization of the paper: Section II provides problem formulation, notations, and some assumptions. In Section III, the algorithm is proposed and its convergence is analyzed. Section IV concludes the paper with some directions for further studies.

II. Formulations and Notations

We consider a game with N players, and we use $1, 2, \ldots, N$, to index the players, and the set $\mathcal{N} = \{1, 2, \ldots, N\}$ to denote the collection of all players. The ith player's decisions are restricted to its strategy set $K_i \subset \mathbb{R}^n$ and selected to minimize its cost function $f_i(x_i, \bar{x})$. In this cost function, the variables x_i and $\bar{x} = \sum_{i=1}^{N} x_i$ denote player i decision and the aggregate of all players decisions, respectively. We use $\bar{x}_{-i}$ to denote the aggregate of all players' decisions except for player i, i.e.,

$$\bar{x}_{-i} = \sum_{j=1, j\neq i}^{N} x_j.$$

Let us define the Minkowski sum of the sets K_i with $\bar{K}$ as follows:

$$\bar{K} \triangleq \sum_{i=1}^{N} K_i. \tag{1}$$

and let $\bar{x}$ be the aggregate of players decisions x_i, i.e.,

$$\bar{x} \triangleq \sum_{j=1}^{N} x_j = x_i + \bar{x}_{-i}, \qquad \bar{x} \in \bar{K}. \tag{2}$$

Having $\bar{x}_{-i}$, player i is confronted with the following optimization problem:

$$\begin{aligned} &\min f_i(x_i, \bar{x}) \triangleq f_i(x_i, x_i + \bar{x}_{-i}) \\ &s.t. \quad x_i \in K_i, \end{aligned} \tag{3}$$

where $K_i \subseteq \mathbb{R}^n$ and $f_i : K_i \times \bar{K} \to \mathbb{R}$ with the set $\bar{K} \subseteq \mathbb{R}^n$ as defined in (1). As quite natural in the game theoretic setup, the action set K_i and the cost function f_i are assumed to be known only by agent i. The following assumption on the constraint sets K_i and the cost functions f_i gives a sufficient condition to ensure the existence of a Nash equilibrium.

Assumption 1 (Assumption 1, [7]). *For each $i = 1, 2, \ldots, N$, the set $K_i \subset \mathbb{R}^n$ is compact and convex. Each function $f_i(x_i, y)$ is continuously differentiable in (x_i, y) over some open set containing the set $K_i \times \bar{K}$, while each function $x_i \mapsto f_i(x_i, \bar{x})$ is convex over the set K_i.*

Under Assumption 1, sufficient condition for existence of a Nash equilibrium of (3) can be investigated through a variational inequality problem $VI(K, \phi)$ [19] by determining if there is a point $x^* \in K$ such that

$$(x - x^*)^T \phi(x^*) \geq 0, \qquad \forall x \in K,$$

where

$$\phi(x) \triangleq \begin{pmatrix} \nabla_{x_1} f_1(x_1, \bar{x}) \\ \vdots \\ \nabla_{x_N} f_N(x_N, \bar{x}) \end{pmatrix}, \qquad K = \prod_{i=1}^{N} K_i \tag{4}$$

with $x \triangleq (x_1^T, \ldots, x_N^T)^T, x_i \in K_i$ for all i, and $\bar{x}$ defined by (2). Note that Assumption 1 guarantees that the set K is compact and convex in $\mathbb{R}^{nN}$ and that the mapping $\phi : K \to \mathbb{R}^{nN}$ is continuous. Let us define $F_i(x_i, \bar{x})$ to emphasize the special form of the mapping ϕ as follows:

$$F_i(x_i, \bar{x}) = \nabla_{x_i} f_i(x_i, \bar{x}), \qquad \forall i = 1, 2, \ldots, N. \tag{5}$$

The mapping $F(x, u)$ is given by

$$F(x, u) \triangleq \begin{pmatrix} F_1(x_1, u) \\ \vdots \\ F_N(x_N, u) \end{pmatrix}, \tag{6}$$

where maps $F_i : K_i \times \bar{K} \to \mathbb{R}^n$ are given by (5). Considering this notation we have

$$\phi(x) = F(x, \bar{x}), \qquad \forall x \in K. \tag{7}$$

To ensure the Nash equilibrium's uniqueness, the following assumption is made on the mapping $\phi(x)$.

Assumption 2 (Assumption 2, [7]). *The mapping $\phi(x)$ is strictly monotone over K, i.e.,*

$$\left(\phi(x) - \phi(x')\right)^T (x - x') > 0, \qquad \forall x, x' \in K, x \neq x'.$$

Proposition 1 (Proposition 1, [7]). *Suppose that Assumptions 1 and 2 for the game defined in (3) hold. Then, the game has a unique Nash equilibrium.*

To accommodate the analysis of the current paper, we make an additional assumption on the mappings F_i (the coordinate mappings of ϕ, see (4)–(7)), as follows.

Assumption 3 (Assumption 3, [7]). *Each mapping $F_i(x_i, u)$ is uniformly Lipschitz continuous in u over the set $\bar{K}$ for every fixed $x_i \in K_i$ i.e., for some $\bar{L}_i > 0$ and for all $z_1, z_2 \in \bar{K}$,*

$$\|F_i(x_i, z_1) - F_i(x_i, z_2)\| \leq \bar{L}_i \|z_1 - z_2\|,$$

where $\bar{K}$ is as defined in (1).

III. Proposed Method And its Convergence

We propose a distributed approach for determining the game's equilibrium in (3) in this section. This algorithm is based on each agent's estimation of the aggregate decision variable from the other agents perspective. After that update, the agents proceed with a projection-based gradient update.

Let E be the set of underlying directed edges among the agents and let $G = (\mathcal{N}, E)$ denote the fixed connectivity graph. Let $\mathcal{N}_i^{in}$ and $\mathcal{N}_i^{out}$ indicate set of immediate neighbors of agent i that this agent can pull data from and collection of agents who receive information from agent i, respectively. Mathematically, $\mathcal{N}_i^{in}$ can be expressed as: $\mathcal{N}_i^{in} = \{j : (j,i) \in E\}$ and $\mathcal{N}_i^{out}$ can be expressed as: $\mathcal{N}_i^{out} = \{j : (i,j) \in E\}$. The following assumption is made on the graph $G = (\mathcal{N}, E)$.

Assumption 4. *The directed graph $G = (\mathcal{N}, E)$ is strongly connected.*

At each time k, every agent maintains its own decision variable x_i^k and its own estimate $v_i^{i,k}$ of the network wide aggregate $\sum_{i=1}^N x_i^k$. Every agent i, also, maintains a variable $v_j^{i,k}$ for every other agent $j \neq i$ in the network, where $v_j^{i,k}$ is estimates the opinion of agent $j \in \mathcal{N}$ at time k about the aggregate $\sum_{i=1}^n x_i^k$. At time $k+1$, each agent l decides on the weights $w_{j,l}^i > 0$, for all $j \in \mathcal{N}$, for all of its out-neighbors $i \in \mathcal{N}_l^{out}$ and sends them the scaled values $w_{j,l}^i v_j^{l,k}$ for all $j \in \mathcal{N}$. Every agent i receives, from its in-neighbors $l \in \mathcal{N}_i^{in}$, the weighted estimates $w_{j,l}^i v_j^{l,k}$ of the aggregate decision variables for all $j \in \mathcal{N}$. Using its own decision x_i^k and estimate $v_i^{i,k}$, agent i updates its decision x_i^{k+1} based on its own cost function. This updated decision is used to compute $v_j^{i,k+1}$ for all $j \neq i$ in order to update the opinion estimate of the aggregate vector as seen by the other agents $j \neq i$. These updates of the aggregate estimates for the other agents are performed by taking an intermittent step, as follows: for all agents i,

$$\hat{v}_j^{i,k} = \sum_{l \in \mathcal{N}_i^{in} \cup \{i\}} w_{j,l}^i v_j^{l,k}, \qquad \forall j = 1, \ldots, N, \; j \neq i, \tag{8}$$

that is, for every $j \neq i$, agent i simply sums the scaled vectors $w_{j,l}^i v_j^{l,k}$ received from its in-neighbors and also includes its own scaled estimate $w_{j,i}^i v_j^{i,k}$. By defining $w_{j,l}^i = 0$ for $l \notin \mathcal{N}_i^{in} \cup \{i\}$, we can write

$$\hat{v}_j^{i,k} = \sum_{l=1}^N w_{j,l}^i v_j^{l,k},$$

At each time step $k+1$, every agent i updates its iterate and estimates of the aggregate decision variables for all other agents j, as follows:

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \alpha_k F_i(x_i^k, \hat{v}_i^{i,k})], \tag{9}$$

$$\hat{v}_j^{i,k} = \sum_{l=1}^N w_{j,l}^i v_j^{l,k}, \qquad \forall i, j \in \{1, 2, \ldots, N\}, \tag{10}$$

$$v_j^{i,k+1} = \hat{v}_j^{i,k} + x_i^{k+1} - x_i^k, \qquad \forall i, j \in \{1, 2, \ldots, N\}, \tag{11}$$

where $\alpha_k > 0$ is the stepsize, $\Pi_{K_i}[u]$ denotes the Euclidean projection of a vector u onto the set K_i, and F_i is as defined in (5). The term $\hat{v}_i^{i,k}$ in (9) is the estimate of agent i on aggregate decision variable $\sum_{j=1}^N x_j^k$ from its own point of view at time k. The algorithm is initialized arbitrarily with $x_i^0 \in K_i$ for all agents i, and with $v_j^{l,0} = x_l^0$, for all $j, l = 1, 2, \ldots, N$.

In the following, we discuss under what conditions on the agents' weights $w_{j,l}^i$ and the stepsize α_k, the iterate vector $(x_1^k, \ldots, x_N^k)$ and the estimate $\hat{v}_i^{i,k}$ converge to a Nash equilibrium $(x_1^*, \ldots, x_N^*)$ and $\sum_{j=1}^N x_j^*$, respectively.

To specify the conditions on the weights $w_{j,l}^i$ used in the updates in (10) and (11). For this, we define the matrix W_j for every $j \in \mathcal{N}$, as follows:

$$[W_j]_{i,l} = w_{j,l}^i \qquad \forall i, l = 1, \ldots, N. \tag{12}$$

We assume that the structure of each W_j is compliant with the graph connectivity structure $(\mathcal{N}, E)$ in the sense that positive entries in each W_j are associated with the links in the graph. We also require that each W_j is a column stochastic matrix. These conditions are given below.

Assumption 5. *For every $j \in \mathcal{N}$, we have*

(i) *For all $l \in \mathcal{N}$ with $l \neq j$, and for all $i \in \mathcal{N}_l^{out}$, we have $w_{j,l}^i > 0$, and otherwise $w_{j,l}^i = 0$.*
(ii) *For $\ell = j$, we have $w_{j,j}^j = 1$ and $w_{j,j}^i = 0$ for all $i \neq j$.*
(iii) $\sum_{i=1}^N w_{j,l}^i = 1$, $\forall l \in \mathcal{N}$.

Given an index $j \in \mathcal{N}$, Assumption 5(i) specifies the entries in each column $\ell = 1, \ldots, N$ of the matrix W_j except for the j-th column. Assumption 5(ii) states that the jth column of the matrix W_j consists of the vector e_j, where e_j is the j-th unit vector of the standard Euclidean basis. The condition in Assumption 5(iii) states that the entries of every column of each matrix W_j sum to 1. We note that since the weights $w_{j,l}^i$ do not vary with time, there exists a scalar $\eta \in (0,1)$ such that $w_{j,l}^i \geq \eta$ for all $l \in \mathcal{N}_i^{in} \bigcup \{i\}$ (η corresponds to the smallest positive entry in the matrices W_j, $j = 1, \ldots, N$).

The following assumption specifies the stepsize requirements that will be significant in the convergence analysis.

Assumption 6 (Assumption 6, [7])**.** *The stepsize sequence $\{\alpha_k\}$ is positive, monotonically nonincreasing i.e., $\alpha_{k+1} \leq \alpha_k$ for all k, and such that $\sum_{k=0}^\infty \alpha_k = \infty$, $\sum_{k=0}^\infty \alpha_k^2 < \infty$.*

We now consider the weight matrices W_j (see (12)). We define $\Phi_j(k,s) := W_j^{k-s+1}$ and denote the il-th element of this matrix by $[\Phi_j(k,s)]_{i,l}$. In the sequel, we state some convergence properties of the matrices $\Phi_j(k,s)$, $j \in \mathcal{N}$.

Lemma 1. *Let Assumptions 4 and 5 hold. Then, for every $j \in \mathcal{N}$ we have*

(a) $\lim_{k\to\infty} \Phi_j(k,s) = e_j 1^T$ *for any $s \in \mathbb{N}$, where e_j is the j-th unit vector of the standard Euclidean basis in $\mathbb{R}^n$.*
(b) $|[\Phi_j(k,s)]_{j,l} - 1| \leq \theta \beta^{t-s}$ *for $\theta = \frac{(N-1)}{(1-\eta^{(N-1)})} > 0$ and for some $0 < \beta = (1-\eta^{(N-1)})^{\frac{1}{N-1}} < 1$, where $\eta \in (0,1)$ is the smallest positive entry in any of the matrices W_j.*

Proof. The result can be obtained from Lemma 1 of [18]. Specifically, by considering a fixed connected communication network satisfying Assumptions 4 and 5 instead of time-varying setup in Lemma 1 of [18], and by transposing matrices in the proof of the latter lemma, the stated result follows. □

We define y_j^k for all $j \in \mathcal{N}$, as follows:

$$y_j^k = \sum_{i=1}^{N} v_j^{i,k}, \qquad \forall k \geq 0, \ \forall j \in \mathcal{N}. \tag{13}$$

These vectors track the aggregate decision variable $\sum_{i=1}^{N} x_i^k$ over time, as shown in the following lemma.

Lemma 2. *Let Assumptions 4 and 5 hold. Then, for each $j \in \mathcal{N}$, the vectors y_j^k, $k \geq 0$, defined by (13) are such that $y_j^k = \sum_{i=1}^{N} x_i^k$ for all $k \geq 0$.*

Proof. Let $j \in \mathcal{N}$ be arbitrary. The proof of the relation

$$\sum_{i=1}^{N} v_j^{i,k} = \sum_{i=1}^{N} x_i^k. \tag{14}$$

is by the induction on k. For $k = 0$ relation (14) holds since the method is initiated with $v_j^{l,0} = x_l^0$ for all $j, l = 1, 2, \ldots, N$. Assume that relation (14) holds for $k - 1$. For k, we have for all $j = 1, 2, \ldots, N$,

$$\begin{aligned}
\sum_{i=1}^{N} v_j^{i,k} &= \sum_{i=1}^{N} \left(\hat{v}_j^{i,k-1} + x_i^k - x_i^{k-1}\right) \\
&= \sum_{i=1}^{N} \sum_{l=1}^{N} w_{j,l}^{i} v_j^{l,k-1} + \sum_{i=1}^{N} x_i^k - \sum_{i=1}^{N} x_i^{k-1} \\
&= \sum_{l=1}^{N} v_j^{l,k-1} + \sum_{i=1}^{N} x_i^k - \sum_{i=1}^{N} x_i^{k-1},
\end{aligned}$$

where the first, second, and last equality follows from (11), (10), and Assumption 5(iii), respectively. The result follows by the induction hypothesis. □

The following result is a consequence of Lemma 2, and Assumptions 1 and 3.

Lemma 3. *Let Assumptions 1, 3, and 5 hold. Then, there is a constant C such that for all i and $k \geq 0$,*

$$\|F_i(x_i^k, y_i^k)\| \leq C, \qquad \|F_i(x_i^k, \hat{v}_i^{i,k})\| \leq C.$$

Proof. The proof is similar to that of Lemma 3 in [7]. □

The next lemma gives an upper bound on $\|y_i^k - \hat{v}_i^{i,k}\|$.

Lemma 4. *Let Assumptions 1-5 hold, and let y_i^k be defined by (13). Then, we have for all $i \in \mathcal{N}$ and all $k \geq 1$,*

$$\|y_i^k - \hat{v}_i^{i,k}\| \leq \theta\beta^k M + \theta N C \sum_{s=1}^{k} \beta^{k-s} \alpha_{s-1},$$

where $\hat{v}_i^{i,k}$ is defined in (10), θ and β are defined in Lemma 1, $M = \sum_{j=1}^{N} max_{x_j \in K_j} \|x_j\|$, and C is the bound from Lemma 3.

Proof. Using the definitions of the terms $v_i^{j,k+1}$ and $\hat{v}_i^{j,k}$ in (11) and (10), respectively, we have $v_i^{j,k+1} = \sum_{l=1}^{N} w_{i,l}^{j} v_i^{l,k} + x_j^{k+1} - x_j^k$. Using the preceding relation recursively, we obtain

$$\begin{aligned}
v_i^{j,k+1} &= \sum_{l=1}^{N} w_{i,l}^{j} \Big(\sum_{p=1}^{N} w_{i,p}^{l} v_i^{p,k} + x_l^k - x_l^{k-1} \Big) + x_j^{k+1} - x_j^k \\
&= \sum_{p=1}^{N} [\Phi_i(k, k-1)]_{j,p} v_i^{p,k-1} \\
&\quad + \sum_{l=1}^{N} [\Phi_i(k, k)]_{j,l} (x_l^k - x_l^{k-1}) + x_j^{k+1} - x_j^k \\
&= \ldots \\
&= \sum_{p=1}^{N} [\Phi_i(k, 0)]_{j,p} v_i^{p,0} \\
&\quad + \sum_{s=1}^{k} \sum_{l=1}^{N} [\Phi_i(k, s)]_{j,l} (x_l^s - x_l^{s-1}) + x_j^{k+1} - x_j^k.
\end{aligned}$$

Using relation (11) and the preceding equality, we have

$$\begin{aligned}
\hat{v}_i^{j,k} &= v_i^{j,k+1} - x_j^{k+1} + x_j^k \\
&= \sum_{p=1}^{N} [\Phi_i(k, 0)]_{j,p} v_i^{p,0} + \sum_{s=1}^{k} \sum_{l=1}^{N} [\Phi_i(k, s)]_{j,l} (x_l^s - x_l^{s-1})
\end{aligned} \tag{15}$$

By writing $y_i^k = y_i^0 + \sum_{s=1}^{k} (y_i^s - y_i^{s-1})$ and by using Lemma 2, we have $y_i^s = \sum_{l=1}^{N} x_l^s$ for all $s \geq 0$, implying that

$$\begin{aligned}
y_i^k &= y_i^0 + \sum_{s=1}^{k} \sum_{l=1}^{N} (x_l^s - x_l^{s-1}) \\
&= \sum_{p=1}^{N} v_i^{p,0} + \sum_{s=1}^{k} \sum_{l=1}^{N} (x_l^s - x_l^{s-1})
\end{aligned} \tag{16}$$

From (15) and (16) it follows that

$$\begin{aligned}
\|y_i^k - \hat{v}_i^{i,k}\| &= \| \sum_{p=1}^{N} \left(1 - [\Phi_i(k, 0)]_{i,p}\right) v_i^{p,0} \\
&\quad + \sum_{s=1}^{k} \sum_{l=1}^{N} \left(1 - [\Phi_i(k, s)]_{i,l}\right) (x_l^s - x_l^{s-1}) \| \\
&\leq \sum_{p=1}^{N} \left|1 - [\Phi_i(k, 0)]_{i,p}\right| \|v_i^{p,0}\| \\
&\quad + \sum_{s=1}^{k} \sum_{l=1}^{N} \left|1 - [\Phi_i(k, s)]_{i,l}\right| \|x_l^s - x_l^{s-1}\|
\end{aligned}$$

Using Lemma 1(b), we obtain for all i and $k \geq 1$,

$$\|y_i^k - \hat{v}_i^{i,k}\| \leq \theta \sum_{p=1}^{N} \beta^k \|v_i^{p,0}\| + \theta \sum_{s=1}^{k} \sum_{l=1}^{N} \beta^{k-s} \|x_l^s - x_l^{s-1}\|. \tag{17}$$

An estimate for $\|x_l^s - x_l^{s-1}\|$ can be obtained from (9), as follows: for all $s \geq 1$,

$$\begin{aligned}\|x_l^s - x_l^{s-1}\| &= \|\Pi_{K_l}[x_l^{s-1} - \alpha_{s-1} F_l(x_l^{s-1}, \hat{v}_l^{l,s-1})] - x_l^{s-1}\| \\ &\leq \|x_l^{s-1} - \alpha_{s-1} F_l(x_l^{s-1}, \hat{v}_l^{l,s-1}) - x_l^{s-1}\| \\ &= \alpha_{s-1} \|F_l(x_l^{s-1}, \hat{v}_l^{l,s-1})\| \\ &\leq C\alpha_{s-1}, \end{aligned} \tag{18}$$

where the first inequality is a result of the fact $x_l^{s-1} = \Pi_{K_l}[x_l^{s-1}]$ and the non-expansive property of projection maps, while the second one follows by Lemma 3. From (17) and (18) we have

$$\begin{aligned}\|y_i^k - \hat{v}_i^{i,k}\| &\leq \theta\beta^k \sum_{p=1}^N \|v_i^{p,0}\| + \theta N \sum_{s=1}^k \beta^{k-s}\alpha_{s-1} C \\ &\leq \theta\beta^k M + \theta N C \sum_{s=1}^k \beta^{k-s}\alpha_{s-1}\end{aligned}$$

where in the last inequality we utilized the fact that $v_i^{p,0} = x_p^0 \in K_p$ for all p, and $M = \sum_{p=1}^N \max_{x \in K_p} \|x\|$ which is finite since each set K_p is compact (see Assumption 1). ◻

The following theorem gives a convergence result for the iterate sequence $\{x^k\}$ of the method to the sole Nash equilibrium x^* of the underlying game, captured by the variational inequality problem $VI(K, \phi)$.

Theorem 1. *Let Assumptions 1-6 hold. Then, the sequence $\{x^k\}$ generated by the method (9-11) converges to the only solution x^* of the $VI(K, \phi)$, where the set K and the mapping ϕ are defined in* (4).

Proof. The proof of this theorem mainly relies on the deterministic version of Lemma 5 in [7]. Proposition 1 guarantees that there is a unique Nash point x^* solving the variational inequality $VI(K, \phi)$, which satisfies $x_i^* = \Pi_{K_i}[x_i^* - \alpha_k F_i(x_i^*, \bar{x}^*)]$ for all $i \in \mathcal{N}$ (see [19]). Using the preceding relation and the non-expansiveness property of projection operator, we obtain for all i and $k \geq 0$,

$$\begin{aligned}\|x_i^{k+1} - x_i^*\|^2 &= \|\Pi_{K_i}[x_i^k - \alpha_k F_i(x_i^k, \hat{v}_i^{i,k})] - x_i^*\|^2 \\ &\leq \|x_i^k - x_i^* - \alpha_k\big(F_i(x_i^k, \hat{v}_i^{i,k}) - F_i(x_i^*, \bar{x}^*)\big)\|^2\end{aligned}$$

By expanding the last item of the preceding inequality, we find that

$$\|x_i^{k+1} - x_i^*\|^2 \leq \|x_i^k - x_i^*\|^2 + \alpha_k^2 V_1 - 2\alpha_k V_2. \tag{19}$$

where

$$V_1 = \|F_i(x_i^k, \hat{v}_i^{i,k}) - F_i(x_i^*, \bar{x}^*)\|^2,$$

$$V_2 = \big(F_i(x_i^k, \hat{v}_i^{i,k}) - F_i(x_i^*, \bar{x}^*)\big)^T (x_i^k - x_i^*).$$

Taking inequality $(a+b)^2 \leq 2(a^2+b^2)$ in addition to triangle inequality into account we can write

$$V_1 \leq 2\|F_i(x_i^k, \hat{v}_i^{i,k})\|^2 + 2\|F_i(x_i^*, \bar{x}^*)\|^2 \leq B, \tag{20}$$

where

$$B = 2C^2 + 2 \max_{(x_i,\bar{x}) \in K_i \times \bar{K}} \|F_i(x_i, \bar{x})\|^2,$$

and C comes from Lemma 3. To estimate V_2 term, we add and subtract $F_i(x_i^k, y_i^k)$ and, thus, obtain

$$\begin{aligned}V_2 &= \big(F_i(x_i^k, \hat{v}_i^{i,k}) - F_i(x_i^k, y_i^k)\big)^T (x_i^k - x_i^*) \\ &\quad + \big(F_i(x_i^k, y_i^k) - F_i(x_i^*, \bar{x}^*)\big)^T (x_i^k - x_i^*).\end{aligned}$$

Applying the inequality $a^T b \geq -\|a\|\|b\|$ to the first term of the right hand side of the preceding equality, and using the Lipschitz continuity of $F_i(x_i, u)$ with respect to u, we have

$$\begin{aligned}&\big(F_i(x_i^k, \hat{v}_i^{i,k}) - F_i(x_i^k, y_i^k)\big)^T (x_i^k - x_i^*) \\ &\geq -\|F_i(x_i^k, \hat{v}_i^{i,k}) - F_i(x_i^k, y_i^k)\| \cdot \|x_i^k - x_i^*\| \\ &\geq -\bar{L}_i \|\hat{v}_i^{i,k} - y_i^k\| \cdot \|x_i^k - x_i^*\| \\ &\geq -2\bar{L}_i M \|\hat{v}_i^{i,k} - y_i^k\|,\end{aligned}$$

where M is such that $\max_{x_i \in K_i} \|x_i\| \leq M$, which is finite due to the compactness of K_i. Thus, it follows that

$$\begin{aligned}V_2 &\geq -2\bar{L}_i M \|\hat{v}_i^{i,k} - y_i^k\| \\ &\quad + \big(F_i(x_i^k, y_i^k) - F_i(x_i^*, \bar{x}^*)\big)^T (x_i^k - x_i^*).\end{aligned} \tag{21}$$

By substituting the estimates (20) and (21) for V_1 and V_2 back in relation (19), we obtain

$$\begin{aligned}\|x_i^{k+1} - x_i^*\|^2 &\leq \|x_i^k - x_i^*\|^2 + B\alpha_k^2 + 4\alpha_k \bar{L}_i M \|\hat{v}_i^{i,k} - y_i^k\| \\ &\quad - 2\alpha_k\big(F_i(x_i^k, y_i^k) - F_i(x_i^*, \bar{x}^*)\big)^T (x_i^k - x_i^*).\end{aligned}$$

Summing the preceding relations over $i = 1, \ldots, N$ and using $\sum_{i=1}^N \|x_i^k - x_i^*\|^2 = \|x^k - x^*\|^2$, we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 + NB\alpha_k^2 + 4\alpha_k M \sum_{i=1}^N \bar{L}_i \|\hat{v}_i^{i,k} - y_i^k\| \\ &\quad - 2\alpha_k \sum_{i=1}^N \big(F_i(x_i^k, y_i^k) - F_i(x_i^*, \bar{x}^*)\big)^T (x_i^k - x_i^*).\end{aligned}$$

Since $y_i^k = \sum_{i=1}^N x_i^k = \bar{x}^k$ and $F_i(x_i, \bar{x})$ is the ith coordinate map of the mapping $\phi(x) = F(x, \bar{x})$ (see (6) and (7)), it follows that

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 + NB\alpha_k^2 + 4\alpha_k M \sum_{i=1}^N \bar{L}_i \|\hat{v}_i^{i,k} - y_i^k\| \\ &\quad - 2\alpha_k\big(\phi(x^k) - \phi(x^*)\big)^T (x^k - x^*).\end{aligned}$$

We now apply the deterministic version of Lemma 5 in [7] to guarantee the convergence of $\{x^k\}$ to x^*. To apply this lemma we only need to prove the following inequality:

$$\sum_{k=0}^\infty \alpha_k \|\hat{v}_i^{i,k} - y_i^k\| < \infty, \qquad \forall i \in \mathcal{N}. \tag{22}$$

According to Lemma 4, we have

$$\|y_i^k - \hat{v}_i^{i,k}\| \leq \theta\beta^k M + \theta N C \sum_{s=1}^k \beta^{k-s}\alpha_{s-1},$$

for all $i \in \mathcal{N}$, and all $k \geq 1$, thus we only need to show that

$$\sum_{k=1}^\infty \alpha_k \Big(\sum_{s=1}^k \beta^{k-s}\alpha_{s-1}\Big) < \infty, \qquad \sum_{k=1}^\infty \alpha_k \beta^k < \infty.$$

Since $\{\alpha_k\}$ is monotonically non-increasing by Assumption 6, we can write

$$\sum_{k=1}^{\infty}\alpha_k\Big(\sum_{s=1}^{k}\beta^{k-s}\alpha_{s-1}\Big) = \sum_{k=1}^{\infty}\left(\sum_{s=1}^{k}\beta^{k-s}\alpha_k\alpha_{s-1}\right) \leq \sum_{k=1}^{\infty}\left(\sum_{s=1}^{k}\beta^{k-s}\alpha_{s-1}^2\right).$$

By applying Lemma 6 of [7], it follows that $\sum_{k=1}^{\infty}\alpha_k\big(\sum_{s=1}^{k}\beta^{k-s}\alpha_{s-1}\big) < \infty$. Convergence of $\sum_{k=1}^{\infty}\alpha_k\beta^k$ follows from $\sum_{k=1}^{\infty}\alpha_k\beta^k \leq \alpha_0\sum_{k=1}^{\infty}\beta^k < \infty$, where we use $\beta \in (0,1)$. Hence, inequality (22) is valid. As a result, the deterministic version of Lemma 5 in [7] yields the following statements:

(a) The sequence $\{\|x^k - x^*\|\}$ is convergent,
(b) $\sum_{k=0}^{\infty}\alpha_k\big(\phi(x^k) - \phi(x^*)\big)^T(x^k - x^*) < \infty$

Since $\{x^k\} \subset K$ and K is compact, the sequence $\{x_k\}$ has accumulation points. Due to $\sum_{k=0}^{\infty}\alpha_k = \infty$ and the strict monotonicity of ϕ, from the above statement (b) we have that $\big(\phi(x^k) - \phi(x^*)\big)^T(x^k - x^*) \to 0$ along a sub-sequence, say $\{x^{k_q}\}$. Due to the strict monotonicity of ϕ, it follows that $\{x^{k_q}\} \to x^*$ as $q \to \infty$. This and the above statement (a) imply that the entire sequence $\{x_k\}$ must converge to x^*. □

Remark. *Compared to the algorithm in [7] where the graphs are undirected and the agent estimates for $\bar{x}^k$ utilize doubly stochastic weights, our method* (9)–(11) *bypasses the use of such matrices by constructing different weights. This comes with the cost of every agent keeping track of "estimates of aggregates" for every other agent $j \neq i$ in the network.*

IV. Conclusions and Future Work

The current work examines a class of Nash games in which agents' cost functions are coupled through the aggregate of their decisions. Unlike the classic setup where the agents observe the other agents' decisions and, thus, possess instant access to the aggregate decision, we have considered the case where agents' interactions are limited by a digraph. We have developed a method which can be viewed as an extension to the synchronous algorithm introduced in [7]. Some directions for future work include relaxing the static communication network to a time-varying one, considering a constant stepsize for the case of the game with a strongly monotone mapping ϕ, and exploring more efficient options for estimating the aggregate decision (i.e., removing the need for every agent to track estimates of every other agent).

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