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representations of p -adic G_2*

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A LOCAL LANGLANDS PARAMETERIZATION FOR GENERIC SUPERCUSPIDAL REPRESENTATIONS OF p -ADIC G_2

BY MICHAEL HARRIS, CHANDRASHEKHAR B. KHARE
AND JACK A. THORNE
WITH AN APPENDIX BY GORDAN SAVIN

ABSTRACT. — We construct a Langlands parameterization of supercuspidal representations of G_2 over a p -adic field. More precisely, for any finite extension $K/\mathbb{Q}_p$ we will construct a bijection

$$\mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \longrightarrow \mathcal{G}^0(G_2, K)$$

from the set of *generic* supercuspidal representations of $G_2(K)$ to the set of irreducible continuous homomorphisms $\rho : W_K \rightarrow G_2(\mathbb{C})$ with W_K the Weil group of K . The construction of the map is simply a matter of assembling arguments that are already in the literature, together with a previously unpublished theorem of G. Savin on exceptional theta correspondences, included as an appendix. The proof that the map is a bijection is arithmetic in nature, and specifically uses automorphy lifting theorems. These can be applied thanks to a recent result of Hundley and Liu on automorphic descent from $\mathrm{GL}(7)$ to G_2 .

RÉSUMÉ. — Nous construisons une paramétrisation de Langlands des représentations supercuspidales de G_2 sur un corps p -adique. Plus précisément, pour chaque extension $K/\mathbb{Q}_p$ nous construisons une application bijective

$$\mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \longrightarrow \mathcal{G}^0(G_2, K)$$

de l'ensemble des représentations supercuspidales *génériques* de $G_2(K)$ vers l'ensemble des morphismes continus et irréductibles $\rho : W_K \rightarrow G_2(\mathbb{C})$, où W_K désigne le groupe de Weil de K . Pour construire cette application il suffit de combiner des arguments qui sont déjà dans la littérature, plus un théorème inédit de G. Savin sur les correspondances θ exceptionnelles, qui est démontré dans un appendice écrit par ce dernier. La démonstration de la bijectivité de l'application est de nature arithmétique, et utilise notamment des théorèmes de relèvement automorphes. Ceux-ci s'appliquent à notre problème grâce à un résultat récent de Hundley et Liu sur la descente automorphe de $\mathrm{GL}(7)$ vers G_2 .

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Introduction

The purpose of this article is to construct a Langlands parameterization of supercuspidal representations of G_2 over a p -adic field. More precisely, for any finite extension $K/\mathbb{Q}_p$ we will construct a bijection

$$\mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \rightarrow \mathcal{G}^0(G_2, K)$$

from the set of generic supercuspidal representations of $G_2(K)$ to the set of irreducible continuous homomorphisms $\rho : W_K \rightarrow G_2(\mathbb{C})$ with W_K the Weil group of K (more precisely, the bijection is between sets of equivalence classes). The construction of the map is simply a matter of assembling arguments that are already in the literature; the article [26] effectively contains the construction, although it doesn't specifically point out the application to supercuspidal representations. The proof of surjectivity is an application of a recent result of Hundley and Liu [24], which allows us to carry out a strategy, based on automorphy lifting theorems, that was initially developed in [4] as an application of Vincent Lafforgue's global parameterization of automorphic representations over function fields. The proof of injectivity also uses global arithmetic methods, including automorphy lifting theorems and the Ramanujan conjecture for self-dual, regular algebraic automorphic representations of $\mathrm{GL}(n)$, alongside known results on liftings (especially [36, 46]).

The parameterization is constructed in two steps. First, following [17, 11, 36], among other references, we use the exceptional dual reductive pair $(G_2, \mathrm{PGSp}(6))$ in E_7 to define local and global correspondences from representations of G_2 to representations of $\mathrm{PGSp}(6)$. We then lift to $\mathrm{Sp}(6)$ and use the functorial transfer of [8] to obtain an automorphic representation of $\mathrm{GL}(7)$. Using the local Langlands correspondence for $\mathrm{GL}(n)$, we can thus obtain a parameterization of supercuspidal representations of G_2 by Galois parameters with values in $\mathrm{GL}(7)$. We use a global argument and Chebotarev density (following [6]) to show that the parameter takes values in the image of G_2 under its 7-dimensional irreducible representation r_7 .

The proof of surjectivity is arithmetic. For the moment, let K be a p -adic field and let ρ be a continuous homomorphism

$$\rho : W_K \rightarrow G_2(\mathbb{C}).$$

We assume ρ is *irreducible*: that its image is contained in no proper parabolic subgroup. Since the image is finite, we may replace the coefficient field $\mathbb{C}$ by a sufficiently large finite field k of characteristic $\ell \neq p$. Following Moret-Bailly we show first that K may be viewed as the completion at a p -adic place v of a totally real field F , and that ρ can be extended to a surjective homomorphism $\mathrm{Gal}(\overline{F}/F) \rightarrow G_2(k)$ that is *odd*, in an appropriate sense. We then use the lifting method in [27] to lift ρ to a homomorphism $\tilde{\rho} : \mathrm{Gal}(\overline{F}/F) \rightarrow G_2(W(k))$ in such a way that $r_7 \circ \tilde{\rho}$ is geometric, in the sense of Fontaine-Mazur, and Hodge-Tate regular.

Now we can apply automorphy lifting theorems, as in [3], to show that $r_7 \circ \tilde{\rho}$ is potentially automorphic – that its restrictions to appropriate totally real Galois extensions F'/F are attached to a cuspidal cohomological self-dual automorphic representation π' of $\mathrm{GL}(7, \mathbf{A}_{F'})$. Choosing F' carefully, we can then descend π' to an automorphic representation π'' of $\mathrm{GL}(7, \mathbf{A}_{F''})$ over the fixed field F'' of a decomposition group $\mathrm{Gal}(F'_v/F_v) \subset \mathrm{Gal}(F'/F)$. At this point we apply the result of Hundley and Liu to show that π'' is in the image of the functorial transfer from $G_2(\mathbf{A}_{F''})$ to $\mathrm{GL}(7, \mathbf{A}_{F''})$ of an

automorphic representation Π of $G_2(\mathbf{A}_{F''})$, and we conclude by observing that the local component Π_v is supercuspidal and has parameter ρ . As a bonus, the construction of [24] provides a globally generic Π , so we see that ρ is the parameter of a generic supercuspidal representation.

There has been a good deal of work on the local representation theory as well as on the automorphic theory of G_2 . Notably, the articles [11], [37], and [36] come very close to establishing a complete local Langlands correspondence for G_2 and to relate the correspondence to the exceptional theta correspondence used here⁽¹⁾; the article [24] comes very close to characterizing the image of functoriality from G_2 to $\mathrm{GL}(7)$. The purpose of this article is not to replace the articles just cited – indeed, the results of these articles are used crucially in the proof of our main theorem – but rather to illustrate the possibility of applying a combination of arithmetic and automorphic methods to the local correspondence.

Acknowledgements

As mentioned above, this paper implements a strategy that was developed in our joint paper with Gebhard Böckle, and we are grateful to him for many discussions. Thanks are due to Joseph Hundley and Baiying Liu for bringing to our attention their recent result on descent for G_2 , on which our argument crucially depends. We also thank Jeff Adams, Wee Teck Gan, Dihua Jiang, Aaron Pollack, and Gordan Savin for help with references, and Freydoon Shahidi for pointing out an error in the L -function calculation in the original proof of Proposition 4.5. We thank Savin for agreeing to write the appendix that proves a global genericity result that allows us to define the local parameterization unambiguously.

We also thank the anonymous referees for their very careful reading, and for their many suggestions that, we hope, have made the text more readable.

Notation

If K is a perfect field, we will write Γ_K for its Galois group relative to a fixed algebraic closure. When K is a number field, we will fix an algebraic closure $\bar{K}/K$, algebraic closures $\bar{K}_v/K_v$ for each place v of K , and embeddings $\bar{K} \rightarrow \bar{K}_v$ extending $K \rightarrow K_v$. These choices determine embeddings $\Gamma_{K_v} \rightarrow \Gamma_K$ for each place v of K . If v is a finite place, then $I_{K_v} \subset \Gamma_{K_v}$ is the inertia group.

We write $\epsilon : \Gamma_K \rightarrow \mathbb{Z}_\ell^\times$ for the ℓ -adic cyclotomic character. By abuse of notation, we also write ϵ for the pushforward of this character to the group of units of any $\mathbb{Z}_\ell$ -algebra.

If F is a totally real number field, n is an *odd* integer, and π is a cuspidal, regular algebraic automorphic representation of $\mathrm{GL}(n, \mathbb{A}_F)$ which is self-dual, in the sense that $\pi \cong \pi^\vee$, then for any isomorphism $\iota : \bar{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ there is an associated semi-simple ℓ -adic Galois representation $r_\iota(\pi) : \Gamma_F \rightarrow \mathrm{GL}(n, \bar{\mathbb{Q}}_\ell)$. This is characterized, up to isomorphism, by

⁽¹⁾ While revising this article we learned of the new preprint [12] of Gan and Savin that establishes Howe duality as well as a dichotomy result for these exceptional theta correspondences. It is likely that some of the arguments in the present paper can now be simplified and many of the references can be consolidated. In the meantime, Gan and Savin have released a new paper [13] containing a complete proof of the local Langlands correspondence for G_2 over a p -adic field. The correspondence of [13], like the parametrization for generic supercuspidals given here, is compatible with global correspondences. The methods of [13] are quite different from the ones presented here.

its compatibility with the local Langlands correspondence for $\mathrm{GL}(n)$ at finite places. More precisely, if $v \nmid \ell$ is any finite place of F then there is an isomorphism

$$\mathrm{WD}(r_\iota(\pi)|_{\Gamma_{F_v}})^{\mathrm{F-ss}} \xrightarrow{\sim} \iota^{-1} \mathrm{rec}_{F_v}(\pi_v).$$

(The notation here is as defined in [3, pp. 509–510]: WD denotes the Weil-Deligne representation associated to an ℓ -adic representation, $\mathrm{F-ss}$ denotes the Frobenius-semisimplification of a Weil-Deligne representation, and rec_{F_v} is the local Langlands correspondence for the group $\mathrm{GL}(n, F_v)$.)

There is an isomorphism $r_\iota(\pi)^\vee \cong r_\iota(\pi)$. We note that our representation $r_\iota(\pi)$ differs by a Tate twist from the representation $r_{\iota, \iota}(\pi)$ defined in [3, Theorem 2.1.1]. Our normalization, which only makes sense when n is odd, suits our purposes here since we want representations π which arise as functorial lifts from G_2 to give rise to Galois representations which are pure of weight 0.

1. Galois parameterization of G_2

Let G_2 be the split group of that type over $\mathbb{Z}$. Let K be a local field of characteristic 0 and let $\mathcal{A}_g(G_2, K)$ denote the set of equivalence classes of generic irreducible admissible representations of $G_2(K)$ over $\mathbb{C}$. If K is non-archimedean, we let $\mathcal{A}_g^0(G_2, K) \subset \mathcal{A}_g(G_2, K)$ denote the subset of *supercuspidal* representations. Let $\mathcal{G}(G_2, K)$ denote the set of $G_2(\mathbb{C})$ -conjugacy classes of G_2 -completely reducible parameters

$$\rho : W_K \rightarrow G_2(\mathbb{C}),$$

and let $\mathcal{G}^0(G_2, K) \subset \mathcal{G}(G_2, K)$ denote the subset of classes of G_2 -irreducible parameters. We define $\mathcal{G}(\mathrm{GL}(7), K)$ and $\mathcal{G}^0(\mathrm{GL}(7), K)$ similarly.

The aim of §1 is to collect lifting results scattered in the literature (among them [11, 16, 17, 19, 20, 23, 24, 31, 32, 36, 38]) to construct a map

$$(1) \quad \mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \rightarrow \mathcal{G}^0(G_2, K).$$

We begin by constructing a map

$$(2) \quad \mathcal{L}'_g : \mathcal{A}_g(G_2, K) \rightarrow \mathcal{G}(\mathrm{GL}(7), K)$$

using purely local means. Conjugacy results for G_2 (see [18]) imply that the map

$$(3) \quad r_{7,*} : \mathcal{G}(G_2, K) \rightarrow \mathcal{G}(\mathrm{GL}(7), K)$$

determined by the standard representation r_7 of G_2 is injective, so the main problem is to show that $\mathcal{L}'_g(\mathcal{A}_g^0(G_2, K))$ lies in the image of $r_{7,*}$. This we achieve using a global argument.

We note that if K is a local field of positive characteristic, Genestier and Lafforgue construct the analogue of the map $\mathcal{L}'_g$ in [14] without reference to the theta lift, and at the same time show that its image is contained in the image of $r_{7,*}$.

1.1. Local generic theta lift

Let K be a local field of characteristic 0, and let E_i ($i = 6, 7$) denote the split adjoint reductive group of that type. We set $H_i = \mathrm{PGL}(3)$ (if $i = 6$) and $H_i = \mathrm{PGSp}(6)$ (if $i = 7$). Let $\theta_{i,K}$ be the minimal representation of $E_i(K)$, which we consider by restriction to be a representation of $G_2(K) \times H_i(K)$, $i = 6, 7$. When K is archimedean we work with the Harish-Chandra module of $\theta_{i,K}$ relative to a choice of maximal compact subgroup of $E_i(K)$ that contains a chosen product of maximal compact subgroups of $G_2(K)$ and $H_i(K)$. Let π be an irreducible admissible representation of $G_2(K)$, and let $\theta_{i,K,[\pi]}$ denote the maximal quotient of $\theta_{i,K}$ which is isotypic for π as representation of $G_2(K)$. Then we write

$$(4) \quad \theta_{i,K,[\pi]} = \pi \otimes \Theta_i(\pi),$$

where $\Theta_i(\pi)$ is a smooth representation of $H_i(K)$.

PROPOSITION 1.2. – (i) *Suppose that K is non-archimedean and that π is generic. Then $\Theta_7(\pi)$ admits a unique generic subquotient.*

(ii) *Suppose that $K = \mathbb{R}$ and π is a discrete series representation. If $\Theta_7(\pi)$ admits a generic constituent, then it is uniquely determined by π , up to isomorphism.*

Proof. – The first part is [11, Corollary 20]. This is based on Proposition 19 of [11], whose proof there is sketched. The proposition is restated, with a complete proof, as Theorem A.9 of Savin's appendix. The second follows from the fact that $\Theta_7(\pi)$ has regular integral infinitesimal character determined by that of π , hence any generic constituent of $\Theta_7(\pi)$ must be the unique discrete series representation with that infinitesimal character. We describe this in more detail below. $\square$

When $\Theta_7(\pi)$ has a unique generic subquotient, we denote it by $\theta_7(\pi)$. We can specify $\theta_7(\pi)$ precisely in two important special cases. First, the unramified, non-archimedean case:

PROPOSITION 1.3 ([16, Theorem 3.5], [37, Theorem 1.1]). – *Suppose that K is non-archimedean and that π is an unramified, generic representation of $G_2(K)$. Then $\theta_7(\pi)$ is the unramified representation of $\mathrm{PGSp}(6, K)$ determined by the embedding $G_2 \rightarrow \mathrm{Spin}(7)$ of L -groups.*

Proof. – The article [16] determines the Satake parameter of the theta lift $\Theta'(\pi)$ of π to $\mathrm{GSp}(6, K)$ by comparing the local Euler factors for the 8-dimensional Spin representation of the Langlands dual group $\mathrm{GSpin}(7)$ of $\mathrm{GSp}(6)$ (see the displayed formula at the bottom of p. 42 of [16]). The genericity of $\Theta'(\pi)$ is proved in the course of this comparison. The representation $\Theta'(\pi)$ is pulled back from the representation $\Theta_7(\pi)$ of $\mathrm{PGSp}(6, K)$, which implies that the Satake parameter of $\Theta'(\pi)$ lies in the subgroup $\mathrm{Spin}(7)$ of $\mathrm{GSpin}(7)$ and has the indicated form. See also the proof of [26, Proposition 5.2]. $\square$

Next, the real, discrete series case. We first recall an important fact. Let G be a quasi-split connected reductive group over $\mathbb{R}$ such that $G(\mathbb{R})$ admits discrete series representations, and let $K \subset G(\mathbb{R})$ denote a maximal compact subgroup. Let $B \subset G(\mathbb{R})$ be a Borel subgroup with unipotent radical N . Let $Z(\mathfrak{g})$ denote the center of the enveloping algebra $U(\mathfrak{g})$ of the complexified Lie algebra of G . If π is an irreducible representation of $G(\mathbb{R})$, we let $\xi_\pi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ denote its infinitesimal character. We recall

FACT 1.4. – *If W is an irreducible (algebraic) representation of G , then, up to infinitesimal equivalence, there is a bijection between Weyl chambers C for G , modulo the Weyl group of K , such that all simple roots in the chamber are non-compact, and unitary representations $\pi(C, W)$ of $G(\mathbb{R})$ such that*

- $\xi_W = \xi_{\pi(C, W)}$.
- $\pi(C, W)$ is generic.

Moreover, $\pi(C, W)$ is a discrete series representation.

The experts tell us that the literature contains no explicit statement of this well-known (and frequently cited) fact. At the request of the referee we provide a sketch. We thank Gordan Savin and Jeff Adams for pointing out errors in our first attempt and providing some missing references.

Salamanca-Riba proved in [35] that a unitary representation with infinitesimal character ξ_W is necessarily cohomological (an $A_{\mathfrak{q}}(\lambda)$). Now Theorem K of [29] asserts that a generic representation is “large” in the sense of [44], and it thus follows from Theorem 6.2 of [44] that a generic unitary representation of $G(\mathbb{R})$ can never be a Langlands quotient of a reducible principal series representation. But Theorem 6.16 of [45] implies that an $A_{\mathfrak{q}}(\lambda)$ is a (proper) Langlands quotient unless $\mathfrak{q}$ is a Borel subgroup. Thus a generic $A_{\mathfrak{q}}(\lambda)$ is in the discrete series, by our hypothesis on G . Then the bijection follows from [1, Lemma 5.7], which is a restatement in more contemporary language of [44, Theorem 6.2] as applied to discrete series.

One checks that when $G = G_2(\mathbb{R})$ or $G = \mathrm{PGSp}(6, \mathbb{R})$, there is a unique Weyl chamber C as in Fact 1.4; thus we may write $\pi(W)$ instead of $\pi(C, W)$. We follow the discussion in [30] of the Conjecture of Gross, which identifies the Harish-Chandra parameters of the generic discrete series of both groups explicitly. We fix normalizations, first for G_2 , and then for $\mathrm{PGSp}(6)$. Let ω_1 denote the highest weight of the irreducible 7-dimensional representation of G_2 , and let ω_2 be the other fundamental weight. Given non-negative integers a, b we let $W(a, b)$ denote the irreducible representation of G_2 with highest weight $a\omega_1 + b\omega_2$, and write $\pi_{a,b}$ for $\pi(W(a, b))$, a discrete series representation of $G_2(\mathbb{R})$. It follows from Fact 1.4 that the generic representation for $G_2(\mathbb{R})$ with given infinitesimal character is unique; it is the one with the marking (3) on p. 191 of [30]. For sufficiently regular infinitesimal character, $\pi_{a,b}$ belongs to the *integrable* discrete series [22]. Theorem 1.5 below is a partial confirmation of Gross’s conjecture for generic discrete series.

We denote characters $Z(\mathfrak{sp}(6)) \rightarrow \mathbb{C}$ in the standard way by triples (α, β, γ) of integers with $\alpha > \beta > \gamma > 0$ (cf. [30]), and let $W(\alpha, \beta, \gamma)$ be the irreducible algebraic representation of $\mathrm{PGSp}(6)$ with the corresponding infinitesimal character.

THEOREM 1.5. – *Let a, b be non-negative integers; we assume $\pi_{a,b}$ to be in the integrable discrete series. Then*

$$\theta_7(\pi_{a,b}) = \pi(W(a + 2b + 3, a + b + 2, b + 1)).$$

Proof. – See [23, Theorem 5.4]. In more detail, let us use the superscript $(?)^c$ to denote the compact form of a real reductive group. The article [23] treats the exceptional theta correspondence $(G_2(\mathbb{R}), \mathrm{PGSp}(6, \mathbb{R})^c)$ (among others), [20] treats (among others) the correspondence $(G_2(\mathbb{R})^c, \mathrm{PGSp}(6, \mathbb{R}))$, and [30] treats the split case $(G_2(\mathbb{R}), \mathrm{PGSp}(6, \mathbb{R}))$ but

only computes the correspondence for quaternionic discrete series of $G_2(\mathbb{R})$. However, as observed in [30, p. 204], the correspondence of infinitesimal characters is independent of real forms. The determination of the correspondence for infinitesimal characters of generic discrete series is completed in [31] (see Table 1 on p. 375 of that paper).

By Fact 1.4 this suffices to identify $\theta_7(\pi_{a,b})$, once we know that it is *unitary*. This is where we use the hypothesis that $\pi_{a,b}$ is in the integrable discrete series. Under that hypothesis, we can construct a globally generic cuspidal automorphic representation Π of $G_2(\mathbb{Q})$ with local component $\pi_{a,b}$ at the real place, as in the proof of Proposition 1.11 below. Theorem 1.7 then asserts that the global theta lift $\Theta_7(\Pi)$ of Π to an automorphic representation of $\mathrm{PGSp}(6, \mathbb{Q})$ does not vanish and is cuspidal. In particular, the archimedean component $\Theta_7(\Pi)_\infty$ is generic and unitary. By Proposition 1.2 and the compatibility of local and global correspondences, $\Theta_7(\Pi)_\infty = \theta_7(\pi_{a,b})$; this completes the proof. $\square$

REMARK 1.5.1. – Gross’s conjecture does not assume that $\pi_{a,b}$ is in the integrable discrete series and the hypothesis is certainly unnecessary. The argument above applies whenever $\pi_{a,b}$ can be realized as a local constituent of a globally generic cuspidal automorphic representation. Undoubtedly this is always possible, but the construction we use here requires the archimedean component to be in the integrable discrete series.

1.6. Global generic theta lift

Now let F be a totally real number field. When $G = E_i$, we let $\mathcal{A}(G)$ denote the space of automorphic forms on $G(F) \backslash G(\mathbf{A}_F)$, and we let $\theta_i := \theta_G \subset \mathcal{A}(G)$ denote the minimal automorphic representation, as described in the article [16] (see also [17]). Let π be a cuspidal automorphic representation of $G_2(\mathbf{A}_F)$. We define $\Theta_7(\pi)$ and $\Theta_6(\pi)$ to be the spaces of automorphic forms on $H_7 = \mathrm{PGSp}(6, \mathbf{A}_F)$ and $H_6 = \mathrm{PGL}(3, \mathbf{A}_F)$, respectively, defined to be the span of the functions $\Theta_i(f_\theta, \varphi)$, as $f_\theta \in \theta_G$ and φ runs through the automorphic forms in the space of the *contragredient* $\pi^\vee$ of π , and where

$$(5) \quad \Theta_i(f_\theta, \varphi)(h) = \int_{[G_2]} f_\theta(g, h) \varphi(g) dg, \quad h \in H_i(\mathbf{A}_F).$$

Here the notation $\int_{[G_2]}$ is the standard abbreviation of $\int_{G_2(F) \backslash G_2(\mathbf{A})}$, and (g, h) is a variable element of $G_2(\mathbf{A}_F) \times H_i(\mathbf{A}_F) \subset E_i(\mathbf{A}_F)$. In contrast to [17], we let φ to belong to $\pi^\vee$ (or equivalently to the complex conjugate of π) to guarantee compatibility with the local correspondence defined below.

THEOREM 1.7. – *Let π be a cuspidal automorphic representation of $G_2(\mathbf{A}_F)$. Then:*

- (i) *Let Π be an irreducible subquotient of $\Theta_i(\pi)$. Then for each place v of F , Π_v is an irreducible subquotient of $\Theta_i(\pi_v)$.*
- (ii) *Suppose that π is globally generic and that π_∞ is a discrete series representation. Then $\Theta_6(\pi) = 0$, and $\Theta_7(\pi)$ is cuspidal and globally generic. In particular, it is non-zero.*

Proof. – The first part is a formal consequence of the definition. The second part is a consequence of [17, Theorem B] (which shows that $\Theta_7(\pi)$ is non-zero), the vanishing of $\Theta_6(\pi)$ (which follows from the fact that π_∞ is discrete series, using the same argument as in [26, §6]), and Theorem 1.4 of Savin’s appendix to this paper (which shows that $\Theta_7(\pi)$ is globally generic). $\square$

We will only apply Theorem 1.7 in the case that π is a cuspidal, globally generic representation of $G_2(\mathbf{A}_F)$ such that π_∞ is discrete series. In this case, $\Theta_7(\pi)$ is a cuspidal, globally generic representation of $\mathrm{PGSp}(6, \mathbf{A}_F)$, and we write $\theta_7(\pi)$ for the unique globally generic cuspidal irreducible constituent of its restriction to $\mathrm{Sp}(6, F) \backslash \mathrm{Sp}(6, \mathbf{A}_F)$. We observe that for every place v of F , we have the equality $\theta_7(\pi_v) = \theta_7(\pi)_v$ (compatibility with local generic theta correspondence).

1.8. Lifting to $\mathrm{GL}(7)$

There are at least two ways of lifting from $\mathrm{Sp}(6)$ to $\mathrm{GL}(7)$ (see [8, 2]). We use the lifting constructed in [8].

THEOREM 1.9. – *Let K be a local field of characteristic 0. Then there exists a map Ψ from the set of (isomorphism classes of) generic irreducible admissible representations of $\mathrm{Sp}(6, K)$ to the set of (isomorphism classes of) generic irreducible admissible representations of $\mathrm{GL}(7, K)$. The map Ψ has the following properties:*

- (i) *If K is non-archimedean then Ψ is injective.*
- (ii) *If K is non-archimedean and π is an unramified representation of $\mathrm{Sp}(6, K)$, then $\Psi(\pi)$ is the unramified representation of $\mathrm{GL}(7, K)$ determined by the natural embedding $\mathrm{SO}(7) \rightarrow \mathrm{GL}(7)$ of L -groups.*
- (iii) *If $K = \mathbb{R}$ and π is the generic discrete series representation of $\mathrm{Sp}(6, K)$ with infinitesimal character (α, β, γ) , then $\Psi(\pi)$ is the unique generic representation of $\mathrm{GL}(7, \mathbb{R})$ which is cohomological for the irreducible algebraic representation of $\mathrm{GL}(7, \mathbb{R})$ of highest weight $(\alpha - 3, \beta - 2, \gamma - 1, 0, 1 - \gamma, 2 - \beta, 3 - \alpha)$.*
- (iv) *Let F be a number field, and let Π be a globally generic cuspidal automorphic representation of $\mathrm{Sp}(6, \mathbf{A}_F)$. Then there exist self-dual, cuspidal automorphic representations $\Psi_1, \dots, \Psi_r$ of $\mathrm{GL}(n_i, \mathbf{A}_F)$, where $\sum_{i=1}^r n_i = 7$, with the following property: let $\Psi = \Psi_1 \boxplus \dots \boxplus \Psi_r$. Then for any place v of F , $\Psi_v = \Psi(\Pi_v)$.*

Proof. – The paper [8] defines the local functorial lift of a generic irreducible admissible representation π of $\mathrm{Sp}(6, K)$ by agreement of L -functions and ϵ -factors of pairs, see [8, Definition 7.1]. This lift is uniquely characterized by the local converse theorem for $\mathrm{GL}(7, K)$, and [8, Proposition 7.5] asserts that it always exists. This defines the map Ψ . The local converse theorem for p -adic $\mathrm{Sp}(6, K)$, proved in [47], shows that Ψ is injective in this case. The final part of the theorem follows from [8, Theorem 7.2, Proposition 7.2]. $\square$

1.10. Definition of $\mathcal{L}'_g$ and $\mathcal{L}_g$

Let K be a non-archimedean local field of characteristic 0. We can now define the promised map $\mathcal{L}'_g : \mathcal{A}_g(G_2, K) \rightarrow \mathcal{G}(\mathrm{GL}(7), K)$. If $\pi \in \mathcal{A}_g(G_2, K)$, then we define $\mathcal{L}'_g(\pi) = \mathrm{rec}_K(\Psi(\sigma))^{ss}$, where σ is the unique irreducible constituent of $\theta_7(\pi)|_{\mathrm{Sp}(6, K)}$ which is generic (with respect to our fixed choice of Whittaker datum). This is independent of the choice of Whittaker datum on $\mathrm{Sp}(6)$. The effect of semisimplification here is to remove the nilpotent part of the Weil-Deligne representation $\mathrm{rec}_K(\Psi(\sigma))$, necessary because of our convention that $\mathcal{G}(\mathrm{GL}(7), K)$ consists of equivalence classes of semisimple representations of W_K .

The next step is to show that when π is supercuspidal, $\mathcal{L}'_g(\pi)$ can be conjugated to take values in $G_2(\mathbb{C})$. We will establish this using a global argument.

PROPOSITION 1.11. – *Let $\pi \in \mathcal{A}_g^0(G_2, K)$. Then $\mathcal{L}'_g(\pi)$ lies in the image of $r_{7,*}$.*

Proof. – Fix a totally real number field F with a finite place v such that $F_v \xrightarrow{\sim} K$. Let $\alpha_1, \dots, \alpha_r$ be the real places of F . For each $i = 1, \dots, r$, we choose a generic integrable discrete series representation π_i of the (split) group $G_2(\mathbb{R})$. Fix a finite place $w \neq v$ of F of residue characteristic different from 2 and a generic supercuspidal representation π_w of $G_2(F_w)$ with the property that $\theta_7(\pi_w)$ is a supercuspidal representation of $\mathrm{PGSp}(6, F_w)$. (Such representations are constructed explicitly in the proof of Theorem 5.2 of [26], at least when $F_w = \mathbb{Q}_p$, which we assume for convenience of reference.) By [25, Theorem 4.5] (see also [43, Theorem 2.2]), we can find a cuspidal, globally generic automorphic representation Π of $G_2(\mathbf{A}_F)$ such that $\Pi_v \cong \pi$, $\Pi_w \cong \pi_w$, and $\Pi_{\alpha_i} \cong \pi_i$ for each i .

Then $\theta_7(\Pi)$ is globally generic and cohomological, and its local component at w is supercuspidal, by construction. Applying Theorem 1.9 to the restriction of $\theta_7(\Pi)$ to $\mathrm{Sp}(6, \mathbf{A}_F)$, we can find cuspidal, self-dual automorphic representations $\Psi_1, \dots, \Psi_r$ of $\mathrm{GL}(7, \mathbf{A}_F)$ such that $\Psi(\theta_7(\Pi)) = \Psi_1 \boxplus \dots \boxplus \Psi_r$ is cohomological. In particular, each Ψ_i is cohomological up to twist. Fixing an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$, we get a compatible family $r_i(\Psi_i)$ of n_i -dimensional representations with values in $\mathrm{GL}(n_i, \overline{\mathbb{Q}}_\ell)$. We get a representation

$$(6) \quad r_i(\Pi) := r_i(\Psi(\theta_7(\Pi))) : \Gamma_F \rightarrow \mathrm{GL}(7, \overline{\mathbb{Q}}_\ell).$$

In fact, [6, Theorem 6.4] shows that the representation $r_i(\Pi)$ is conjugate to a representation contained in $r_7(G_2(\overline{\mathbb{Q}}_\ell))$, and such a representation is unique up to conjugation in $G_2(\overline{\mathbb{Q}}_\ell)$ (noting that the cuspidality of Ψ plays no role in the proof of that theorem). Our proof is now complete: we have $r_i(\Pi)|_{W_{F_v}}^{ss} \cong \iota^{-1} \mathrm{rec}_{F_v}(\Psi(\theta_7(\pi)))^{ss} \cong \iota^{-1} \mathcal{L}'_g(\pi)$, by definition, and this shows that $\mathcal{L}'_g(\pi)$ is conjugate to a representation valued in $G_2(\mathbb{C})$. $\square$

PROPOSITION 1.12. – *Let $\pi \in \mathcal{A}_g^0(G_2, K)$, and let $\rho : W_K \rightarrow G_2(\mathbb{C})$ be the unique parameter, up to $G_2(\mathbb{C})$ -conjugacy, with $r_7 \circ \rho = \mathcal{L}'_g(\pi)$. Then ρ is irreducible.*

Proof. – We split up into cases according to the dichotomy described in [36]. If $\theta_7(\pi)$ is supercuspidal then by [8, Theorem 7.2] there are self-dual, non-isomorphic supercuspidal representations $\psi_1, \dots, \psi_r$ of $\mathrm{GL}(n_i, K)$ with $\sum_{i=1}^r n_i = 7$ and $r_7 \circ \rho$ conjugate in $\mathrm{GL}(7, \mathbb{C})$ to $\bigoplus_{i=1}^r \mathrm{rec}_K(\psi_i)$. There is a unique way to conjugate such a parameter into $\mathrm{SO}(7, \mathbb{C})$, and it is irreducible there (indeed, its centralizer is finite). It follows that ρ must also be irreducible in this case.

If $\theta_7(\pi)$ is not supercuspidal, then [36, Proposition 3.6] shows that $\theta_7(\pi)$ is a subquotient of an unnormalized induction $\mathrm{Ind}_{Q_3}^{\mathrm{PGSp}(6)} \rho \otimes |\det|$, where ρ is a supercuspidal representation of $\mathrm{PGL}(3, K)$. By [8, Proposition 7.4], $r_7 \circ \rho$ is conjugate in $\mathrm{GL}(7, \mathbb{C})$ to $\mathrm{rec}_K(\rho) \oplus \mathbb{C} \oplus \mathrm{rec}_K(\rho)^\vee$. Since $\mathrm{rec}_K(\rho)$ is irreducible and 3-dimensional, [36, Proposition 1.5] shows that the image of this representation is not contained in the image of any Levi subgroup of G_2 , so once again ρ must be irreducible. $\square$

We may therefore complete our definition of $\mathcal{L}_g$ as follows: if $\pi \in \mathcal{A}_g^0(G_2, K)$ then $\mathcal{L}_g(\pi)$ is the unique parameter, up to conjugacy, with $r_7 \circ \mathcal{L}_g(\pi) = \mathcal{L}'_g(\pi)$. We record the following useful property of $\mathcal{L}_g$.

PROPOSITION 1.13. – *Let $\pi \in \mathcal{A}_g^0(G_2, K)$. and let $R : G_2 \rightarrow \mathrm{SO}(7)$ be the unique non-trivial homomorphism. Then $\theta_7(\pi)$ is supercuspidal if and only if $R \circ \mathcal{L}_g(\pi)$ is $\mathrm{SO}(7, \mathbb{C})$ -irreducible.*

Proof. – This is a corollary of the proof of Proposition 1.12. $\square$

2. Globalization of local G_2 parameters

Let $G = G_2$ be a split reductive group over $\mathbb{Z}$ of that type, let $\mathfrak{g}$ denote its Lie algebra, and let $r_7 : G \rightarrow \mathrm{GL}(7)$ denote the standard 7-dimensional representation. We fix a split maximal torus and Borel subgroup $T \subset B \subset G$. We may assume that $r_7(T)$ is diagonal and $r_7(B)$ is contained in the upper-triangular Borel subgroup of $\mathrm{GL}(7)$. Let $\Delta \subset \Phi = \Phi(G, T)$ denote the corresponding root basis, and $\Phi = \Phi^+ \sqcup \Phi^-$ the sets of positive and negative roots. We label $\Delta = \{\alpha_1, \alpha_2\}$ so that the fundamental weight ω_1 is the highest weight of r_7 . Let $\check{\omega}_1, \check{\omega}_2 \in X_*(T)$ denote the corresponding fundamental coweights, and let $\check{\delta} = \check{\omega}_1 + \check{\omega}_2$.

Let K be a finite extension of $\mathbb{Q}_p$, and let k be a finite field of characteristic $\ell \neq p$. We suppose given a continuous representation $\bar{\rho} : \Gamma_K \rightarrow G(k)$ such that $\bar{\rho}(I_K)$ has order prime to ℓ .

THEOREM 2.1. – *There exists a totally real field F and a continuous representation $\bar{\sigma} : \Gamma_F \rightarrow G(k)$ with the following properties:*

- (i) $\bar{\sigma}(\Gamma_F) = G(k)$.
- (ii) ℓ splits in F . For each place $v|\ell$ of F , $\bar{\sigma}|_{\Gamma_{F_v}}$ is $G(k)$ -conjugate to $\check{\delta}^2 \circ \epsilon$.
- (iii) For each place $v|p$ of F , there is an isomorphism $F_v \cong K$ such that $\bar{\sigma}|_{\Gamma_{F_v}}$ is $G(k)$ -conjugate to $\bar{\rho}$.
- (iv) If v is a finite place of F such that $v \nmid \ell p$, then $\bar{\sigma}|_{\Gamma_{F_v}}$ is unramified.
- (v) For each place $v|\infty$ of F (which determines a complex conjugation $c_v \in \Gamma_{F_v}$), $\bar{\sigma}(c_v)$ is $G(k)$ -conjugate to $\check{\delta}(-1)$.

Proof. – By weak approximation, we can find a totally real number field $E/\mathbb{Q}$ in which ℓ splits, and such that for each $v|p$ there is an isomorphism $E_v \cong K$. The existence of an extension F/E and a homomorphism $\bar{\sigma} : \Gamma_F \rightarrow G(k)$ with the claimed properties then follows from the main theorem of [33]. $\square$

THEOREM 2.2. – *Let $\bar{\sigma} : \Gamma_F \rightarrow G(k)$ be a representation satisfying the conclusion of Theorem 2.1, and suppose that $\ell > 28$. Then there exist a finite extension $E/\mathrm{Frac} W(k)$ with ring of integers $\mathcal{O}$ and a lift*

$$\sigma : \Gamma_F \rightarrow G(\mathcal{O})$$

of $\bar{\sigma}$ with the following properties:

- (i) $\sigma(\Gamma_F)$ contains a conjugate of $G(\mathbb{Z}_\ell)$.

- (ii) For each place $v|\ell$ of F , there exists $g \in G(\mathcal{O})$ such that $g\sigma|_{\Gamma_{F_v}} g^{-1}$ takes values in $B(\mathcal{O})$, and the projection of $g\sigma|_{\Gamma_{F_v}} g^{-1}$ to $T(\mathcal{O})$ equals $\check{\delta}^2 \circ \epsilon$. Consequently, $r_7 \circ \sigma|_{\Gamma_{F_v}}$ is crystalline ordinary of Hodge-Tate weights $\{6, 4, 2, 0, -2, -4, -6\}$ (with respect to any embedding $F_v \hookrightarrow \overline{\mathbb{Q}_\ell}$).
- (iii) For each place $v|p$ of F , reduction modulo $\mathfrak{m}_{\mathcal{O}}$ induces an isomorphism $\sigma(I_{F_v}) \xrightarrow{\sim} \overline{\sigma}(I_{F_v})$.
- (iv) For each finite place $v \nmid \ell p$ of F , $\sigma|_{\Gamma_{F_v}}$ is unramified.
- (v) For each place $v|\infty$ of F , $\sigma(c_v)$ is $G(\mathcal{O})$ -conjugate to $\check{\delta}(-1)$.

Proof. – We first remark that our hypotheses imply that $G(k)$ is its own derived group, hence $\overline{\sigma}(\Gamma_F(\xi_\ell)) = G(k)$, and $r_7 \circ \overline{\sigma}$ is absolutely irreducible (see [41]). This will be useful in applying the results of [3] cited below. We next observe that [4, Proposition 6.7] shows that any lift σ of $\overline{\sigma}$ to $G(\mathcal{O})$ has the property that $\sigma(\Gamma_F)$ contains a conjugate of $G(\mathbb{Z}_\ell)$.

To construct a lift, we use the Khare-Wintenberger method (cf. [27, Theorem 3.7]). Let $\mathcal{C}$ denote the category of complete local Noetherian $W(k)$ -algebras with residue field k , and let $\text{Def}_G : \mathcal{C} \rightarrow \text{Sets}$ denote the functor which assigns to any $A \in \mathcal{C}$ the set of $\ker(G(A) \rightarrow G(k))$ -conjugacy classes of homomorphisms $\sigma_A : \Gamma_F \rightarrow G(A)$ satisfying the following conditions:

- For each place $v|\ell$ of F , there exists $g \in G(A)$ such that $g\sigma_A|_{\Gamma_{F_v}} g^{-1}$ takes values in $B(A)$, and the projection of $g\sigma_A|_{\Gamma_{F_v}} g^{-1}$ to $T(A)$ equals $\check{\delta}^2 \circ \epsilon$.
- For each place $v|p$ of F , reduction modulo $\mathfrak{m}_A$ induces an isomorphism $\sigma_A(I_{F_v}) \xrightarrow{\sim} \overline{\sigma}(I_{F_v})$.
- For each finite place $v \nmid \ell p$ of F , $\sigma_A|_{\Gamma_{F_v}}$ is unramified.

Then ([34, Proposition 9.2]) Def_G is represented by an object $R_G \in \mathcal{C}$, and there exists an integer $g \geq 0$ such that R_G can be expressed as a quotient of $W(k)[[X_1, \dots, X_g]]$ by g relations. (Note that the local conditions are liftable local deformation conditions, in the sense of [34], which are analyzed in [34, §4.1] and [34, §4.4]. The conditions $\ell > 28$ and ℓ split in F imply that conditions (REG) and (REG*) of [34, §4.1] hold.) To apply the Khare-Wintenberger method, we must show that R_G is a finite $W(k)$ -algebra. This will imply that R_G is a finite flat complete intersection $W(k)$ -algebra, and therefore that there exists a finite extension $W(k) \rightarrow \mathcal{O}$ and a homomorphism $R_G \rightarrow \mathcal{O}$, corresponding to a lift $\sigma : \Gamma_F \rightarrow G(\mathcal{O})$ with the desired properties.

To prove the finiteness of R_G , we must compare it with another deformation ring. Let E/F be a totally imaginary quadratic extension in which the places of F above ℓ and p split; then E is a CM field. Let $\overline{\tau} = r_7 \circ \overline{\sigma}|_{\Gamma_E}$. Then $\overline{\tau}$ is absolutely irreducible and $\overline{\tau}^c \cong \overline{\tau}^\vee$ (where $c \in \text{Gal}(E/F)$ is the non-trivial element). Let $\mathcal{G}_7 = (\text{GL}(7) \times \text{GL}(1)) \rtimes \{\pm 1\}$ denote the group scheme defined in [3, §1.1]. It is equipped with a homomorphism $\nu : \mathcal{G}_7 \rightarrow \text{GL}(1)$ which is projection to $\text{GL}(1)$ on the connected component and which sends -1 to -1 . As described there, $\overline{\tau}$ extends to a homomorphism $\overline{\tau} : \Gamma_F \rightarrow \mathcal{G}_7(k)$ such that $\nu \circ \overline{\tau} = \delta_{E/F}$ and $\overline{\tau}(\Gamma_E) \subset \text{GL}(7, k)$. (We write $\delta_{E/F} : \text{Gal}(E/F) \rightarrow \{\pm 1\}$ for the unique non-trivial character.)

Let S denote the set of places of F dividing ℓp . Fix for each $v \in S$ a place $\tilde{v}$ of E lying above v . Say that a finite totally real extension F'/F is good if $\bar{\tau}(\Gamma_{F'}) = \bar{\tau}(\Gamma_F)$, as subgroups of $\mathcal{G}_7(k)$, and $\zeta_\ell \notin E' = EF'$. If F' is a good extension then we write $S_{F'}$ for the set of places of F' lying above a place of S , $E' = EF'$, and $\tilde{S}_{F'}$ for the set of places of E' lying above a place $\tilde{v}$ (some $v \in S$). If $v \in S_{F'}$ then we write $\tilde{v}$ for the unique place of $\tilde{S}_{F'}$ lying above v . We then write $\text{Def}_{F'} : \mathcal{C} \rightarrow \text{Sets}$ for the functor which assigns to any $A \in \mathcal{C}$ the set of $\ker(\text{GL}(7, A) \rightarrow \text{GL}(7, k))$ -conjugacy classes of homomorphisms $\tau_A : \Gamma_E \rightarrow \mathcal{G}_7(A)$ satisfying the following conditions:

- For each place $v|\ell$ of F' , $\tau_A|_{\Gamma_{E'_v}}$ defines an A -point of the ‘crystalline-ordinary’ lifting ring $R_{\bar{\sigma}|_{\Gamma_{E'_v}}, \{H_i\}, \text{cr-ord}}$ described in [3, §1.4], with sets

$$H_i = \{6, 4, 2, 0, -2, -4, -6\}$$
 of Hodge-Tate numbers ($i : E'_v \rightarrow \overline{\mathbb{Q}_\ell}$ any embedding).
- For each place $v|p$ of F' , reduction modulo ℓ induces an isomorphism $\tau_A(I_{E'_v}) \rightarrow \bar{\tau}(I_{E'_v})$.
- For each finite place $v \nmid \ell p$ of F' , $\tau_A|_{\Gamma_{F'_v}}$ is unramified.
- $v \circ \tau_A = \delta_{E'/F'}$.

The functor $\text{Def}_{F'}$ is represented by an object $R_{F'} \in \mathcal{C}$ (see [7, Proposition 2.2.9]). We claim that the representation r_7 determines a natural map $R_{F'} \rightarrow R_G$. The only point to check here is that if $\sigma_A : \Gamma_F \rightarrow G(A)$ arises from a homomorphism $R_G \rightarrow A$, then for any place $v|\ell$ of F' , $r_7 \circ \sigma_A|_{\Gamma_{E'_v}}$ defines a point of the ‘cr-ord’ lifting ring. This can be reduced to the universal case, meaning we must show that the classifying morphism

$$(7) \quad R_{r_7 \circ \bar{\sigma}|_{\Gamma_{E'_v}}}^\square \rightarrow R_{\bar{\sigma}|_{\Gamma_{F'_v}}}^{\square, \check{\delta}^2 \circ \epsilon}$$

associated to r_7 (where the first ring is the universal lifting ring for the $\text{GL}(7)$ -representation $r_7 \circ \bar{\sigma}|_{\Gamma_{E'_v}}$, as in [3, §1.2], and the second ring is the one defined in [34, Definition 4.1])

factors through the quotient $R_{r_7 \circ \bar{\sigma}|_{\Gamma_{E'_v}}}^\square \rightarrow R_{\bar{\sigma}|_{\Gamma_{E'_v}}, \{H_i\}, \text{cr-ord}}^\square$. The ring $R_{\bar{\sigma}|_{\Gamma_{F'_v}}}^{\square, \check{\delta}^2 \circ \epsilon}$ is formally smooth over $\mathcal{O}$, by [34, Proposition 4.4], so it suffices to show that the morphism (7) sends the $\overline{\mathbb{Q}_\ell}$ -points of $\text{Spec } R_{\bar{\sigma}|_{\Gamma_{F'_v}}}^{\square, \check{\delta}^2 \circ \epsilon}$ into $\text{Spec } R_{\bar{\sigma}|_{\Gamma_{E'_v}}, \{H_i\}, \text{cr-ord}}^\square$. This follows from [15, Lemma 3.9].

If $F = F'$, the map $R_F \rightarrow R_G$ is surjective (same proof as [4, Lemma 5.7]). For any F' , the map $R_{F'} \rightarrow R_F \rightarrow R_G$ is therefore a finite algebra homomorphism [3, Lemma 1.2.3]. To finish the proof of the theorem, we therefore just need to show there exists a good extension F'/F such that $R_{F'}$ is a finite $W(k)$ -algebra.

By [42, Theorem 10.2], this will follow if we can find a good extension F'/F and an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \rightarrow \mathbb{C}$ such that $\bar{\tau}|_{\Gamma_E}$ is the residual representation of an ι -ordinary, cuspidal, regular algebraic, polarizable automorphic representation of $\text{GL}(7, \mathbf{A}_{E'})$. The existence of such a representation follows from [3, Proposition 3.3.1]. This completes the proof. $\square$

THEOREM 2.3. – Let $\sigma : \Gamma_F \rightarrow G(\mathcal{O})$ be as in Theorem 2.2. Let

$$\sigma_7 = r_7 \circ \sigma : \Gamma_F \rightarrow \mathrm{GL}(7, \mathcal{O}) \hookrightarrow \mathrm{GL}(7, \overline{\mathbb{Q}}_\ell).$$

Then there exists a totally real Galois extension F'/F , a cuspidal, cohomological automorphic representation $\Psi(\sigma)$ of $\mathrm{GL}(7, \mathbb{A}_{F'})$, and an isomorphism $\overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ such that $r_\ell(\Psi(\sigma)) \cong \sigma_7|_{\Gamma_{F'}}$. Moreover, we can assume that $\Psi(\sigma)$ is everywhere unramified.

Proof. – It follows from [3, Corollary 4.5.2] that there exists a totally real Galois extension F'/F , a cuspidal, cohomological automorphic representation $\Psi(\sigma)$ of $\mathrm{GL}(7, \mathbb{A}_{F'})$, and an isomorphism $\overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ such that $r_\ell(\Psi(\sigma)) \cong \sigma_7|_{\Gamma_{F'}}$. Moreover, for each place $v'|\ell$ of F' , $\Psi(\sigma)_{v'}$ is unramified. We need to explain why $\Psi(\sigma)$ can be chosen to be everywhere unramified.

By weak approximation, we can find a totally real, Galois, soluble extension F''/F such that for each finite place $v'' \nmid \ell$ of F'' , $\sigma_7|_{\Gamma_{F''_{v''}}}$ is unramified. Let $F^1 = F' \cdot F''$. Then F^1/F' is a soluble, totally real extension, F^1/F is Galois, and $\sigma_7|_{\Gamma_{F^1}}$ is unramified at every finite place not dividing ℓ . We may therefore replace F' by F^1 and $\Psi(\sigma)$ by its base change to F^1 (which exists, by [3, Lemma 2.2.2], and which is everywhere unramified by local-global compatibility). $\square$

We now forget our existing assumptions and restate the above results in a form suitable for application in §3.

THEOREM 2.4. – Let K be a finite extension of $\mathbb{Q}_p$, and let k be a finite field of characteristic $\ell \neq p$. Let $\rho : W_K \rightarrow G_2(W(k))$ be a continuous homomorphism, irreducible over $\overline{\mathbb{Q}}_\ell$. Then $\rho(I_K)$ is finite; we assume that $\ell > 28$ and that ℓ does not divide the order of $\rho(I_K)$. Then we can find the following data:

- (i) A totally real number field L , together with a non-empty set Σ of p -adic places of L such that for each $v \in \Sigma$, $L_v \cong K$.
- (ii) A finite extension $E/\mathrm{Frac} W(k)$ with ring of integers $\mathcal{O}$ and a continuous representation $\sigma : \Gamma_L \rightarrow G_2(\mathcal{O})$ of Zariski dense image such that for each place $v \in \Sigma$, $\sigma|_{W_{L_v}}$ and ρ are $G_2(\mathcal{O})$ -conjugate.
- (iii) A cuspidal, tempered, regular algebraic, self-dual automorphic representation $\Psi(\sigma)$ of $\mathrm{GL}(7, \mathbb{A}_L)$, unramified outside Σ , and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ such that $r_\ell(\Psi(\sigma)) \cong r_7 \circ \sigma$.

Proof. – We begin with some remarks. First, ρ has finite image, and so extends to a representation of Γ_K . Indeed, $\rho(I_K)$ is finite since $r_7 \circ \rho$ is semisimple; the ℓ -adic monodromy theorem implies the existence of a finite extension K'/K such that $r_7 \circ \rho(I_{K'})$ is semisimple and unipotent, hence trivial. Let $\phi \in W_K$ be a Frobenius lift. Since $\rho(I_K)$ is finite, some power $\rho(\phi)^N$ centralizes $\rho(W_K)$. Since $\rho(W_K)$ is irreducible, this forces $\rho(\phi)^N$ to lie in the center of G , hence (since G is adjoint) to be trivial. Thus $\rho(W_K)$ is finite.

Second, let $\bar{\rho} : W_K \rightarrow G_2(k)$ denote the reduction of ρ modulo ℓ . Then any minimally ramified lift of $\bar{\rho}$ to $G_2(\mathcal{O})$ is $G_2(\mathcal{O})$ -conjugate to ρ . Indeed, [9, Lemma A.2] shows that $h^0(\Gamma_K, \mathfrak{g}_k) = 0$, hence (cf. [34, Lemma 4.17]) that the tangent space to the

minimal deformation functor is trivial. This implies that any minimally ramified lift is even $\ker(G_2(\mathcal{O}) \rightarrow G_2(k))$ -conjugate to ρ .

By Theorem 2.3, we can find the following objects:

- A totally real field F , and a totally real Galois extension F'/F .
- A continuous representation $\sigma : \Gamma_F \rightarrow G_2(\mathcal{O})$ such that $\sigma|_{\Gamma_{F'}}$ has Zariski dense image, σ is unramified outside ℓp , and for each place $v|\ell$ of F , $r_7 \circ \sigma|_{\Gamma_{F_v}}$ is crystalline with Hodge-Tate numbers $\{6, 4, 2, 0, -2, -4, -6\}$ (with respect to any embedding of F_v in $\overline{\mathbb{Q}_\ell}$).
- For each place $v|p$ of F , an isomorphism $F_v \cong K$ such that $\sigma|_{\Gamma_{F_v}}$ and ρ are $G_2(\mathcal{O})$ -conjugate.
- A cuspidal, regular algebraic, self-dual automorphic representation $\Psi(\sigma)'$ of $\mathrm{GL}(7, \mathbb{A}_{F'})$ which is everywhere unramified and an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \rightarrow \mathbb{C}$ such that $r_\iota(\Psi(\sigma)') \cong r_7 \circ \sigma|_{\Gamma_{F'}}$.

Let v' be a place of F' lying above v , with v dividing p , and let $D_{v'/v} \subset \mathrm{Gal}(F'/F)$ be the decomposition group. We set $F'' = (F')^{D_{v'/v}}$, and $v'' = v'|_{F''}$. Then $F_{v''}'' \cong K$ and $\sigma|_{\Gamma_{F_{v''}}''}$ and ρ are $G_2(\mathcal{O})$ -conjugate. Moreover, F'/F'' is a soluble extension. By soluble descent (see e.g., [3, Lemma 2.2.2]), there exists a cuspidal, regular algebraic, self-dual automorphic representation $\Psi(\sigma)''$ of $\mathrm{GL}(7, \mathbb{A}_{F''})$ such that $r_\iota(\Psi(\sigma)'') \cong r_7 \circ \sigma|_{\Gamma_{F''}}$. Finally, we once again use weak approximation, as in the proof of Theorem 2.3, to find a soluble totally real extension L/F'' in which v'' splits and such that the base change $\Psi(\sigma)$ of $\Psi(\sigma)''$ to L is unramified away from v'' . The proof is complete on taking Σ to be the set of places of L lying above v'' , and noting that [40, Corollary 1.3] shows that $\Psi(\sigma)$ is tempered. $\square$

3. Automorphic descent from $\mathrm{GL}(7)$ to G_2

The following theorem was recently proved by Hundley and Liu:

THEOREM 3.1. — *Let F be a number field, and let Ψ be a cuspidal automorphic representation of $\mathrm{GL}(7, \mathbb{A}_F)$. Suppose that the following conditions hold.*

- (i) *For almost all places v of F at which Ψ_v is unramified, the Satake parameter of the local component Ψ_v is conjugate, in $\mathrm{GL}(7, \mathbb{C})$, to an element of $r_7(G_2(\mathbb{C}))$.*
- (ii) *The partial L -function $L^S(s, \Psi, \wedge^3)$ has a pole at $s = 1$, for some finite set S .*

Then there exists a globally generic cuspidal automorphic representation Π of $G_2(\mathbb{A}_F)$ such that for all but finitely many places at which Π_v is unramified, the Satake parameter of Ψ_v is the image under r_7 of the Satake parameter of Π_v .

Moreover, if v is a finite place at which Ψ_v is both unramified and tempered, with Satake parameter conjugate to an element of $r_7(G_2(\mathbb{C}))$, then Π_v is unramified and the Satake parameter of Ψ_v is the image under r_7 of the Satake parameter of Π_v .

Proof. – The theorem up to “moreover” is contained in [24, Theorem 6.1.17] [24, Theorem 6.4.31] and [24, Theorem 6.5.6] (giving genericity, cuspidality and weak lifting, respectively). If v is any finite place at which Ψ_v is unramified with Satake parameter conjugate to $r_7(G_2(\mathbb{C}))$, then [24, Proposition 6.5.5] shows that Π_v is the generic subquotient of the unramified principal series representation of G_2 with Satake parameter corresponding to that of Ψ_v . If Ψ_v is tempered then this unramified principal series representation is also tempered and irreducible, implying the final claim. $\square$

We remark that if F is totally real and Ψ is cuspidal and regular algebraic with infinitesimal character “integral of G_2 type”, then Π_∞ must be discrete series. Indeed, the theta lift of Π is an automorphic representation of $\mathrm{PGSp}(6)$ whose restriction to $\mathrm{Sp}(6)$ gives rise by functorial transfer to an automorphic representation Ψ' of $\mathrm{GL}(7, \mathbf{A}_F)$ with the property that the infinitesimal character of Ψ'_∞ is obtained from that of Π_∞ by functoriality (cf. the discussion in §1.1). By strong multiplicity one, $\Psi = \Psi'$, and the infinitesimal character of Π_∞ is integral. Since Π is globally generic, fact 1.4 implies that Π_∞ is discrete series.

COROLLARY 3.2. – *Let L (a totally real number field), $\sigma : \Gamma_L \rightarrow G_2(\overline{\mathbb{Q}_\ell})$ (a continuous homomorphism) and $\Psi(\sigma)$ (a cuspidal tempered automorphic representation of $\mathrm{GL}(7, \mathbf{A}_L)$ with $r_7 \circ \sigma \cong r_1(\Psi(\sigma))$) be as in the statement of Theorem 2.4. Then there exists a globally generic, cuspidal automorphic representation $\Pi(\sigma)$ of $G_2(\mathbf{A}_L)$ such that $\Pi(\sigma)_\infty$ is discrete series and for every place v of L at which $\Psi(\sigma)$ is unramified, $\Pi(\sigma)_v$ is unramified and the Satake parameter of $\Psi(\sigma)_v$ is the image in $\mathrm{GL}(7, \mathbb{C})$ of the Satake parameter of $\Pi(\sigma)_v$.*

Proof. – Since the local parameter of $\Psi(\sigma)$ at every place at which σ is unramified factors through $r_7(G_2)$, $\Psi(\sigma)$ satisfies hypothesis (i) of Theorem 3.1. If hypothesis (ii) is satisfied, then $\Psi(\sigma)$ is the functorial lift of some cuspidal $\Pi(\sigma)$. Finally, the temperedness of $\Psi(\sigma)$ will imply that $\Psi(\sigma)_v$ is the lift of $\Pi(\sigma)_v$ for all unramified places v .

It remains to verify that (ii) is satisfied. Now $L^S(s, \Psi(\sigma), \wedge^3) = L^S(s, \wedge^3 \circ \sigma_7)$ if S contains all ramified places. Moreover, since $\sigma_7 = r_7 \circ \sigma$ factors through $r_7(G_2)$, it is well known that

$$(8) \quad \wedge^3 \circ \sigma_7 \xrightarrow{\sim} \sigma_7 \oplus \mathrm{Sym}^2 \circ \sigma_7.$$

Thus

$$L^S(s, \Psi(\sigma), \wedge^3) = L^S(s, \Psi(\sigma)) \cdot L^S(s, \Psi(\sigma), \mathrm{Sym}^2).$$

The first factor does not vanish at $s = 1$ by the theorem of Jacquet-Shalika and Shahidi, whereas the second factor has a simple pole at $s = 1$ because $\Psi(\sigma)$ is a self-contragredient representation of an odd general linear group (cf. the discussion of [5, p. 139]). $\square$

THEOREM 3.3. – *Let K be a p -adic local field. Then the map*

$$\mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \rightarrow \mathcal{G}^0(G_2, K)$$

is surjective.

Proof. – Let $\rho : W_K \rightarrow G_2(\mathbb{C})$ be an irreducible representation. We will construct $\pi \in \mathcal{A}_g^0(G_2, K)$ such that $\mathcal{L}_g(\pi) = \rho$. Fix a prime $\ell > 28$ which does not divide the (finite) order of $\rho(W_K)$, and let $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ be an isomorphism. We may assume that $\iota^{-1}\rho$ takes values in $G_2(\mathbb{Z}_\ell)$. We take the totally real number field L and homomorphism $\sigma : \Gamma_L \rightarrow G_2(\mathcal{O})$ as in Theorem 2.4, and apply Corollary 3.2. Thus there is a non-empty set Σ of p -adic places of L , and for each $v \in \Sigma$ an isomorphism $K \cong L_v$ such that $\iota^{-1}\rho, \sigma|_{\Gamma_{L_v}}$ are $G_2(\mathcal{O})$ -conjugate. Fix a choice of $v \in \Sigma$, and let $\pi = \Pi(\sigma)_v$. We must show that π is supercuspidal and that $\mathcal{L}_g(\pi) = \rho$.

We first note that $r_i(\Pi(\sigma))$ (defined as in the proof of Proposition 1.11) is $G_2(\overline{\mathbb{Q}}_\ell)$ -conjugate to σ ; indeed, this can be checked at unramified places, so follows from [18]. We therefore just need to check that π is supercuspidal. If $\theta_7(\pi)$ is supercuspidal, then Proposition 3.4 of [36] shows that π is also supercuspidal.

We can therefore assume that $\theta_7(\pi)$ is not supercuspidal. Thus its Jacquet module $J_P(\theta_7(\pi)) \neq 0$ for some maximal parabolic subgroup $P \subset \mathrm{PGSp}(6)$. It follows from [8, Proposition 7.5] that the image of the local parameter $r_7 \circ \rho$ is contained in a proper parabolic subgroup of $\mathrm{SO}(7)$. However, since the image of ρ is not contained in any proper parabolic of G_2 , it follows from [36, Proposition 1.1] that P must be the Siegel parabolic subgroup, with Levi quotient $\mathrm{GL}(3)$. In particular, $J_P(\theta_7(\pi))$ must be supercuspidal.

We now turn to the proof of [36, Proposition 3.6], specifically the discussion labeled (Siegel) on p. 759, which computes $J_P(\theta_7(\pi))$. Although the running hypothesis in that proposition is that π is supercuspidal, this hypothesis is only used in the first paragraph, which cites [32, Theorem 5.3] to prove that, (modifying the notation of [32]), $J_P(\theta_7(\pi)) \boxtimes \pi$ is a subrepresentation of $(\Pi_7)_{U_7}$. Here Π_7 is the minimal representation of the adjoint split group $\mathbf{E}_7$ over the field K , and U_7 is the unipotent radical of a specific maximal parabolic subgroup of $\mathbf{E}_7$ denoted $\mathbf{Q}_7$ (with Levi quotient isomorphic to $\mathbf{E}_6$).

Suppose $J_P(\theta_7(\pi)) \boxtimes \pi$ is *not* a subrepresentation of $(\Pi_7)_{U_7}$. Then the discussion on p. 759 of [36] implies that $J_P(\theta_7(\pi)) \boxtimes \pi$ intertwines with one of the terms (1) or (2) of [32, Theorem 5.3]. But we have seen that $J_P(\theta_7(\pi))$ is supercuspidal, and this is incompatible with the description of these terms, which are induced from proper parabolic subgroups of $\mathrm{GL}(3)$.

Thus $J_P(\theta_7(\pi)) \boxtimes \pi$ is a non-zero subrepresentation of $(\Pi_7)_{U_7}$. The second displayed formula on p. 759 of [36] then asserts that $\theta_6(\pi) \neq 0$, and moreover that $\theta_6(\pi)$ is a supercuspidal representation of $\mathrm{PGL}(3, K)$. But then [11, Theorem 21] implies that π is supercuspidal. $\square$

4. Injectivity

Let K be a p -adic local field. The final step in our argument is to prove that the parameterization

$$\mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \rightarrow \mathcal{G}^0(G_2, K)$$

is injective (hence bijective). We recall that if $\pi \in \mathcal{A}_g^0(G_2, K)$, then $\theta_7(\pi)$ is a representation of $\mathrm{PGSp}(6, K)$. Its restriction to $\mathrm{Sp}(6, K)$ contains a unique generic constituent σ , which lifts to $\mathrm{GL}(7, K)$ by Theorem 1.9, and $\mathcal{L}_g(\pi)$ is defined to be the unique G_2 -valued parameter which is conjugate in $\mathrm{GL}(7, \mathbb{C})$ to the Galois parameter of the lift $\Psi(\sigma)$ to $\mathrm{GL}(7, K)$.

The lifting from generic representations of $\mathrm{Sp}(6, K)$ to representations of $\mathrm{GL}(7, K)$ is injective. Savin-Weissman study the injectivity of the lift to $\mathrm{PGSp}(6, K)$ (see [36, Theorem 4.7]). The problematic step for us is therefore restriction from $\mathrm{PGSp}(6, K)$ to $\mathrm{Sp}(6, K)$. Fortunately this has been analyzed by Xu [46], and we will be able to prove injectivity by combining these results with a global argument.

LEMMA 4.1. – *Let $\pi, \pi' \in \mathcal{A}_g^0(G_2, K)$, and suppose that $\mathcal{L}_g(\pi) = \mathcal{L}_g(\pi')$. Then there exists a quadratic character $\omega : \mathrm{PGSp}(6, K) \rightarrow \mathbb{C}^\times$ such that $\theta_7(\pi) \cong \theta_7(\pi') \otimes \omega$.*

Proof. – Let σ be the unique generic constituent of the restriction of $\theta_7(\pi)$ to $\mathrm{Sp}(6, K)$, and define σ' similarly. Then our hypothesis implies $\Psi(\sigma) = \Psi(\sigma')$, hence (Theorem 1.9) $\sigma = \sigma'$. Then [46, Corollary 6.4] shows the existence of a character ω with the claimed property. $\square$

COROLLARY 4.2. – *Let $\rho \in \mathcal{G}^0(G_2, K)$. Then the following are equivalent:*

- (i) $\mathcal{L}_g^{-1}(\rho)$ is a singleton.
- (ii) *There exists $\pi \in \mathcal{A}_g^0(G_2, K)$ such that $\mathcal{L}_g(\pi) = \rho$, with the following property: for any non-trivial character $\omega : \mathrm{PGSp}(6, K) \rightarrow \mathbb{C}^\times$, either (a) $\theta_7(\pi) \cong \theta_7(\pi) \otimes \omega$ or (b) $\theta_7(\pi) \otimes \omega$ is not of the form $\theta_7(\pi')$ for any $\pi' \in \mathcal{A}_g^0(G_2, K)$.*

Proof. – Suppose that (i) holds, and choose any π with $\mathcal{L}_g(\pi) = \rho$. If $\pi' \in \mathcal{A}_g^0(G_2, K)$ and $\theta_7(\pi) \otimes \omega \cong \theta_7(\pi')$ then (as $\mathcal{L}_g$ is injective) we have $\pi = \pi'$, hence $\theta_7(\pi) \cong \theta_7(\pi) \otimes \omega$.

Suppose instead that (ii) holds, and let $\pi', \pi'' \in \mathcal{A}_g^0(G_2, K)$ be representations with $\mathcal{L}_g(\pi') = \mathcal{L}_g(\pi'') = \rho$. Let $\pi \in \mathcal{A}_g^0(G_2, K)$ be the given representation with $\mathcal{L}_g(\pi) = \rho$. Thus there exist characters ω', ω'' such that $\theta_7(\pi') \cong \theta_7(\pi) \otimes \omega'$, $\theta_7(\pi'') \cong \theta_7(\pi) \otimes \omega''$. Consideration of possibilities (a) and (b) shows that $\theta_7(\pi) = \theta_7(\pi')$. By symmetry, we also have $\theta_7(\pi) = \theta_7(\pi'')$, hence $\theta_7(\pi') = \theta_7(\pi'')$, hence $\pi' = \pi''$ by [36, Theorem 4.7]. $\square$

We first dispense with the simplest case.

PROPOSITION 4.3. – *Let $\pi, \pi' \in \mathcal{A}_g^0(G_2, K)$, and suppose that $\mathcal{L}_g(\pi) = \mathcal{L}_g(\pi')$. Suppose further that $\theta_7(\pi)$ is not supercuspidal. Then $\pi = \pi'$.*

Proof. – Our hypotheses imply that $\theta_7(\pi')$ is also not supercuspidal. By [36, Theorem 3.9] there exist supercuspidal representations τ, τ' of $\mathrm{GL}(3, K)$ of trivial central character such that $\theta_7(\pi)$ is the generic subquotient of $i_{Q_3}^{\mathrm{PGSp}(6)} \tau$, and similarly for $\theta_7(\pi')$ (and we use normalized induction). The existence of an isomorphism $\theta_7(\pi) \otimes \omega \cong \theta_7(\pi')$ implies that $\tau \otimes \omega, \tau'$ are conjugate under the stabilizer in the Weyl group of $\mathrm{PGSp}(6)$ of the Levi subgroup of Q_3 (uniqueness of supercuspidal support). This in turn implies that $\tau \otimes \omega$ is isomorphic to one of τ' or $(\tau')^\vee$. This is a contradiction, because the central character of $\tau \otimes \omega$ is non-trivial. $\square$

LEMMA 4.4. – *To show that $\mathcal{L}_g$ is injective, it is enough to prove the following statement:*

- *Let $\rho \in \mathcal{G}^0(G_2, K)$ be a parameter which remains irreducible in $\mathrm{SO}(7)$. Then there exists a representation $\pi \in \mathcal{A}_g^0(G_2, K)$ such that $\mathcal{L}_g(\pi) = \rho$ and for any character $\omega : \mathrm{PGSp}(6, K) \rightarrow \mathbb{C}^\times$ such that $\theta_7(\pi) \not\cong \theta_7(\pi) \otimes \omega$, the Shahidi L-function $L(\theta_7(\pi) \otimes \omega, \mathrm{Spin}, s)$ is holomorphic at $s = 0$.*

The Shahidi L -function is the one defined in [36, §5.1] using the realization of $\mathrm{GSp}(6)$ as a Levi subgroup of F_4 .

Proof. – By Proposition 4.3, it's enough to show that the condition (ii) of Corollary 4.2 holds for parameters ρ which are irreducible in $\mathrm{SO}(7)$. Then [36, Theorem 5.10] and [36, Proposition 4.6] together show that the holomorphy of $L(\theta_7(\pi) \otimes \omega, \mathrm{Spin}, s)$ at $s = 0$ implies that $\theta_7(\pi) \otimes \omega$ does not have the form $\theta_7(\pi')$ for any $\pi' \in \mathcal{A}_g^0(G_2, K)$. $\square$

PROPOSITION 4.5. – *To show that $\mathcal{L}_g$ is injective, it is enough to construct for each $\rho \in \mathcal{G}^0(G_2, K)$ which remains irreducible in $\mathrm{SO}(7)$ the following data:*

- (i) *a totally real field F , together with a non-empty set Σ of p -adic places of F and for each $v \in \Sigma$ an isomorphism $F_v \cong K$;*
- (ii) *a cuspidal, globally generic automorphic representation π of $G_2(\mathbf{A}_F)$ which is unramified outside Σ and discrete series at infinity;*
- (iii) *for each character $\omega : \mathrm{PGSp}(6, K) \rightarrow \mathbb{C}^\times$, a character*

$$\Omega : \mathrm{PGSp}(6, F) \backslash \mathrm{PGSp}(6, \mathbf{A}_F) \rightarrow \mathbb{C}^\times$$

which is unramified away from Σ and satisfies $\Omega|_{\mathrm{PGSp}(6, F_v)} = \omega$ for each $v \in \Sigma$;

with the following property:

- (iv) *Let Ψ be the lift of π to $\mathrm{GL}(7, \mathbf{A}_F)$. Then Ψ is cuspidal and there exists a prime $\ell \neq p$ and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ such that $r_\iota(\Psi)|_{W_{F_v}} \cong \iota^{-1} r_7 \circ \rho$ for all $v \in \Sigma$.*

Proof. – Recall that $\theta_7(\pi)$ is the globally generic, cuspidal lift of π to $\mathrm{PGSp}(6, \mathbf{A}_F)$, and Ψ is the lift of the globally generic constituent of $\theta_7(\pi)|_{\mathrm{Sp}(6, \mathbf{A}_F)}$ to $\mathrm{GL}(7, \mathbf{A}_F)$. Fix $v_0 \in \Sigma$, and let $\omega : \mathrm{PGSp}(6, F_{v_0}) \rightarrow \mathbb{C}^\times$ be a character such that $\theta_7(\pi_{v_0}) \not\cong \theta_7(\pi_{v_0}) \otimes \omega$. We must show that the Shahidi L -function $L(\theta_7(\pi_{v_0}) \otimes \omega, \mathrm{Spin}, s)$ is holomorphic at $s = 0$ (as then the criterion of Lemma 4.4 will be satisfied with $\mathcal{L}_g(\pi_{v_0}) = \rho$). We note that $\theta_7(\pi_v)$ is supercuspidal for each $v \in \Sigma$, by Proposition 1.13.

Invoking [39, Theorem 3.5], we have an identity

$$(9) \quad \gamma(\theta_7(\pi)_\infty \otimes \Omega_\infty, s) \prod_{v \in \Sigma} \gamma(\theta_7(\pi_v) \otimes \omega, s) = \frac{L^\Sigma(\theta_7(\pi) \otimes \Omega, \mathrm{Spin}, s)}{L^\Sigma(\theta_7(\pi) \otimes \Omega, \mathrm{Spin}, 1-s)},$$

where the γ -factors are those of *loc. cit.* and L^Σ denotes the prime-to- Σ -and- ∞ L -function, which is therefore a product of unramified local L -factors.

By [39, Proposition 7.3], the factors $\gamma(\theta_7(\pi_v) \otimes \omega, s)$ ($v \in \Sigma$) are rational functions of q_v^{-s} which are holomorphic at $s = 0$, and which vanish at $s = 0$ if and only if $L(\theta_7(\pi_v) \otimes \omega, \mathrm{Spin}, s)$ has a pole at $s = 0$. To prove the proposition, it will therefore be enough for us to show that $\prod_{v \in \Sigma} \gamma(\theta_7(\pi_v) \otimes \omega, s)$ does not vanish at the point $s = 0$. The zeroes and poles of $\gamma(\theta_7(\pi)_\infty \otimes \Omega_\infty, s)$ lie on finitely many lines parallel to the real axis, so it is even enough for us to show that the expression in (9) does not vanish at infinitely many points of the form $s = 2\pi i k / \log q_{v_0}$ ($k \in \mathbb{Z}$). This is what we will now do.

For any place v of F , we may identify the quotient $\mathrm{PGSp}(6, F_v)/\mathrm{Sp}(6, F_v)$ with $F_v^\times/(F_v^\times)^2$; we may further identify Ω with a quadratic character of $\mathrm{GL}_7(\mathbf{A}_F)$, by composition with the determinant. Using the computation of unramified L -functions, the right-hand side of (9) equals the quotient

$$\frac{L^\Sigma(\Psi \otimes \Omega, s) L^\Sigma(\Omega, s)}{L^\Sigma(\Psi \otimes \Omega, 1-s) L^\Sigma(\Omega, 1-s)}$$

of standard L -functions. Using the functional equation for these standard L -functions, we find that this equals

$$\prod_{v \in \Sigma} \frac{L_v(\Psi \otimes \Omega, 1-s) L_v(\Omega, 1-s)}{L_v(\Psi \otimes \Omega, s) L_v(\Omega, s)},$$

up to product with a meromorphic function with all of its zeroes and poles on the real axis. Since ω is non-trivial, each factor $L_v(\Omega, 1-s)/L_v(\Omega, s)$ is holomorphic on the line $\mathrm{Re}(s) = 0$, with zeroes only possible at the points of $(\log q_v)^{-1}(\pi i + 2\pi i \mathbb{Z})$. If $v \in \Sigma$ then the representation $\mathcal{L}_g(\pi_v)$ has finite image, so $L_v(\Psi \otimes \Omega, 1-s) = L_v((r_7 \circ \rho) \otimes \omega, 1-s)$ is holomorphic and non-vanishing on the line $\mathrm{Re}(s) = 0$. We are therefore reduced to showing that

$$\prod_{v \in \Sigma} L_v(\Psi \otimes \Omega, s) = L_{v_0}((r_7 \circ \rho) \otimes \omega, s)^{|\Sigma|}$$

does not have poles at infinitely many points of the form $s = 2\pi i k / \log q_v$ ($k \in \mathbb{Z}$). Equivalently, $(r_7 \circ \rho) \otimes \omega$ does not contain the trivial representation of W_K .

Now we make use of [46, Corollary 4.2]. Writing $R : G_2 \rightarrow \mathrm{Spin}(7)$ for the natural homomorphism, it states that $\theta_7(\pi_{v_0}) \cong \theta_7(\pi_{v_0}) \otimes \omega$ if and only if $R \circ \rho$, $(R \circ \rho) \otimes \omega$ are $\mathrm{Spin}(7)$ -conjugate. We are assuming that this is not the case. Lemma 4.6 below implies that $(r_7 \circ \rho) \otimes \omega$ does not contain the trivial representation; and this completes the proof of the proposition. $\square$

LEMMA 4.6. – *Let Γ be a group, and let $\rho : \Gamma \rightarrow G_2(\mathbb{C})$ be a completely reducible representation and $\omega : \Gamma \rightarrow \{\pm 1\}$ a non-trivial character such that $R \circ \rho$ and $(R \circ \rho) \otimes \omega$ are not $\mathrm{Spin}(7)$ -conjugate. Then $(r_7 \circ \rho) \otimes \omega$ does not contain the trivial representation.*

Proof. – Let $R' : \mathrm{Spin}(7) \rightarrow \mathrm{GL}(8)$ be the spin representation, $R'' : \mathrm{Spin}(7) \rightarrow \mathrm{GL}(7)$ the vector representation. Then $R' \circ R = r_7 \oplus \mathbb{C}$ and $R'' \circ R = r_7$. In particular, $R' \circ ((R \circ \rho) \otimes \omega) = (r_7 \circ \rho) \otimes \omega \oplus \omega$. We see that $(r_7 \circ \rho) \otimes \omega$ contains the trivial representation if and only if $(R \circ \rho) \otimes \omega$ fixes a non-zero vector in the spin representation.

Since the stabilizers of non-zero vectors in the spin representation are exactly the $\mathrm{Spin}(7)$ -conjugates of $R(G_2)$, we see that $(r_7 \circ \rho) \otimes \omega$ contains the trivial representation if and only if there exists $g \in \mathrm{Spin}(7, \mathbb{C})$ such that $\mathrm{Ad}(g) \circ ((R \circ \rho) \otimes \omega)$ is valued in $R(G_2(\mathbb{C}))$.

Suppose for contradiction that this is the case, and let $\rho' : \Gamma \rightarrow G_2(\mathbb{C})$ be the homomorphism such that $R \circ \rho' = \mathrm{Ad}(g) \circ ((R \circ \rho) \otimes \omega)$. Then $r_7 \circ \rho' = \mathrm{Ad}(R''(g)) \circ r_7 \circ \rho$. By [18], this implies that ρ, ρ' are themselves G_2 -conjugate, hence that $R \circ \rho$ and $R \circ \rho'$ are $R(G_2)$ -conjugate, hence that $R \circ \rho$ and $(R \circ \rho) \otimes \omega$ are $\mathrm{Spin}(7)$ -conjugate. This contradicts our hypothesis. $\square$

To complete the proof of injectivity, it remains to construct data as in Proposition 4.5. Given an irreducible representation $\rho : W_K \rightarrow G_2(\mathbb{C})$ which remains irreducible in $\mathrm{SO}(7)$, Theorem 2.4 implies the existence of the following data:

- (i) A totally real field F , together with a non-empty set Σ of p -adic places of F and for each $v \in \Sigma$ an isomorphism $F_v \cong K$.
- (ii) A prime $\ell \neq p$, an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$, and a homomorphism $\sigma : \Gamma_F \rightarrow G_2(\overline{\mathbb{Q}}_\ell)$ of Zariski dense image such that for each $v \in \Sigma$, $\sigma|_{W_{F_v}}$ is conjugate to $\iota^{-1}\rho$.
- (iii) A cuspidal, regular algebraic, self-dual automorphic representation Ψ of $\mathrm{GL}(7, \mathbf{A}_F)$ such that $r_\ell(\Psi) \cong r_7 \circ \sigma$ and Ψ is unramified outside Σ .

We choose for each character $\omega : \mathrm{PGSp}(6, K) \rightarrow \mathbb{C}^\times$ a globalization Ω such that for each $v \in \Sigma$, the restriction of Ω to $\mathrm{PGSp}(6, F_v)$ equals ω . After making a quadratic base change, split at Σ , we can assume moreover that each character Ω is unramified away from Σ .

Applying Theorem 3.1, we obtain a cuspidal, globally generic representation π of $G_2(\mathbf{A}_F)$, unramified outside Σ and discrete series at infinity. We have now constructed all of the required data.

5. Final remarks

It is possible to use the same strategy of passing to $\mathrm{GL}(7)$, combined with properties of L -functions, to show that no pure generic supercuspidal representation of G_2 is *incorrigible*, in the sense of [21]. But this can also be derived from the dichotomy of Savin and Weissman [36] and any proof using L -functions ultimately reduces to the dichotomy property.

Appendix

Genericity of a lift, by Gordan Savin

Let F be a global field and $\mathbb{A}$ its ring of adèles. The goal of the appendix is to show that any generic cuspidal automorphic form on $G_2(\mathbb{A})$ lifts to a generic automorphic form on $\mathrm{PGSp}_6(\mathbb{A})$.

A.1. Octonions and G_2

We follow the exposition [38] and work over $\mathbb{Q}$. Let $\mathbb{H}$ be the algebra of Hamilton quaternions, with the usual basis $\{1, i, j, k\}$. The split octonion algebra over $\mathbb{Q}$ is $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ with multiplication

$$(a, b) \cdot (c, d) = (ac + d\bar{b}, \bar{a}d + cb).$$

If $x = (a, b)$, let $\bar{x} = (\bar{a}, -b)$. Then $x \mapsto \bar{x}$ is a linear anti-involution of $\mathbb{O}$, defining norm and trace maps

$$\mathbb{N} : \mathbb{O} \rightarrow F, \quad x \mapsto x\bar{x} = \bar{x}x, \quad \mathrm{Tr} : \mathbb{O} \rightarrow F, \quad x \mapsto x + \bar{x}$$

satisfying

$$\mathbb{N}(x \cdot y) = \mathbb{N}(x)\mathbb{N}(y), \quad \mathrm{Tr}(x \cdot y) = \mathrm{Tr}(y \cdot x), \quad \mathrm{Tr}(x \cdot (y \cdot z)) = \mathrm{Tr}((x \cdot y) \cdot z).$$

Let $l = (0, 1)$, so $\mathbb{O}$ has a basis $\{1, i, j, k, l, li, lj, lk\}$. The following basis is particularly useful.

$$(10) \quad \begin{aligned} s_1 &= \frac{1}{2}(i + li), & s_2 &= \frac{1}{2}(j + lj), & s_3 &= \frac{1}{2}(k + lk), & s_4 &= \frac{1}{2}(1 + l), \\ t_1 &= \frac{1}{2}(i - li), & t_2 &= \frac{1}{2}(j - lj), & t_3 &= \frac{1}{2}(k - lk), & t_4 &= \frac{1}{2}(1 - l). \end{aligned}$$

The multiplication table for this basis is given in Table 1.

TABLE 1. Multiplication Table for Octonions

	s_1	s_2	s_3	t_1	t_2	t_3	s_4	t_4
s_1	0	$-t_3$	t_2	s_4	0	0	0	s_1
s_2	t_3	0	$-t_1$	0	s_4	0	0	s_2
s_3	$-t_2$	t_1	0	0	0	s_4	0	s_3
t_1	t_4	0	0	0	s_3	$-s_2$	t_1	0
t_2	0	t_4	0	$-s_3$	0	s_1	t_2	0
t_3	0	0	t_4	s_2	$-s_1$	0	t_3	0
s_4	s_1	s_2	s_3	0	0	0	s_4	0
t_4	0	0	0	t_1	t_2	t_3	0	t_4

Let $\mathcal{R} \subset \mathbb{O}$ be the $\mathbb{Z}$ -lattice spanned by s_i and t_i . It follows, from the multiplication table, that $\mathcal{R}$ is an order. It is maximal since the determinant of the trace pairing $\text{Tr}(x \cdot y)$ on $\mathcal{R}$ is 1. Let $\mathbb{O}^0$ be the subspace of trace zero elements. For every subspace $V \subset \mathbb{O}^0$, let V^Δ be the subspace of all $x \in \mathbb{O}^0$ such that $x \cdot y = 0$ for all $y \in V$. A subspace $V \subset \mathbb{O}^0$ on which multiplication is trivial is at most 2-dimensional. (We call such a subspace a *null space* or a *null subspace*.) Indeed, let (i, j, k) be a permutation of $(1, 2, 3)$. Then from the multiplication table we see that $\langle s_i \rangle^\Delta = \langle s_i, t_j, t_k \rangle$, and the null spaces of $\mathbb{O}^0$ which contain s_i are all of the form $\langle s_i, at_j + bt_k \rangle$ for fixed $a, b \in \mathbb{Q}$. Since G_2 , the group of automorphisms of $\mathbb{O}$, acts transitively on (nonzero) elements of trace zero and norm zero, this phenomenon is generic.

The group G_2 has two conjugacy classes of maximal parabolic subgroups, and they can be described as the stabilizers of null subspaces in $\mathbb{O}^0$. Let $V_1 \subset V_2$ be 1 and 2-dimensional null subspaces. Let $V_3 = V_1^\Delta \supset V_2$. Let $Q_1 = L_1 U_1$ and $Q_2 = L_2 U_2$ be the stabilizers of V_1 and V_2 , respectively. The Levi factors L_1 and L_2 are isomorphic to $\text{GL}(V_3/V_1)$ and $\text{GL}(V_2)$, respectively. The Borel subgroup $Q_0 = L_0 U_0 = Q_1 \cap Q_2$ stabilizes the full flag

$$V_1 \subset V_2 \subset V_3 \subset V_3^\perp \subset V_2^\perp \subset V_1^\perp$$

in $\mathbb{O}^0$, where $V^\perp$ stands for the orthogonal complement of V with respect to the trace pairing.

A.2. Albert algebra and E_7

This is an exceptional 27-dimensional Jordan algebra J over $\mathbb{Q}$. It is the set of matrices

$$A = \begin{pmatrix} \gamma & x & \bar{y} \\ \bar{x} & \beta & z \\ y & \bar{z} & \alpha \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{Q}$ and $x, y, z \in \mathbb{O}$. We have a cubic form (the determinant) on J

$$\det A = \alpha\beta\gamma - \alpha\mathbb{N}(x) - \beta\mathbb{N}(y) - \gamma\mathbb{N}(z) + \text{Tr}(xyz).$$

The group of similitudes of the cubic form is a reductive group of type E_6 . Its orbits on J are classified by the rank of the matrix A . If $A \neq 0$ then A has rank one if $A^2 = \text{Tr}(A) \cdot A$. Explicitly, this means that the entries of A satisfy the equalities

$$\mathbb{N}(x) = \beta\gamma, \mathbb{N}(y) = \gamma\alpha, \mathbb{N}(z) = \alpha\beta, \alpha\bar{x} = zy, \beta\bar{y} = xz, \gamma\bar{z} = yx.$$

Let G be the split, adjoint group of type E_7 . This group can be constructed from J by the Koecher-Tits construction, see Section 3 in [28]. In particular, G has a pair of opposite maximal parabolic subgroups $P = MN$ and $\bar{P} = M\bar{N}$, $N \cong J$ and $\bar{N} \cong J$, such that the conjugation action of M on N (this action is faithful since G is adjoint) gives an isomorphism of M and the group of similitudes of the cubic form on J :

$$M \cong \{g \in \text{GL}(J) \mid \text{for some } \lambda \in F^\times, \det(gA) = \lambda \det(A) \text{ for all } A \in J\}.$$

The action of M on J resulting from the isomorphism $\bar{N} \cong J$ is dual to the action arising from N . Observe that G_2 acts naturally on J (by acting on off-diagonal entries). This gives an embedding of G_2 into M . Let M_3 , N_3 and $\bar{N}_3$ be the centralizers of G_2 in M and N , respectively. It is clear that $N_3 \cong J_3$ and $\bar{N}_3 \cong J_3$, where J_3 is the Jordan algebra of symmetric 3×3 -matrices. The group M_3 is isomorphic to GL_3 . This isomorphism is realized by observing that GL_3 acts on J by similitudes

$$A \mapsto \det(g)^{-1} g A g^t$$

for all $g \in \text{GL}_3$. Thus the centralizer of G_2 in G is PGSp_6 with $P_3 = M_3 N_3$ and $\bar{P}_3 = M_3 \bar{N}_3$ a pair of opposite maximal parabolic subgroups.

A.3. Minimal representation of G

Let F be a local field of characteristic 0. In this section J , G etc stand for their sets of F -points. Fix $\psi : F \rightarrow \mathbb{C}^\times$, a non-trivial unitary additive character. Every $A \in J$ defines a character ψ_A of J

$$\psi_A(B) = \psi(\text{Tr}(A \circ B)) = \psi_B(A)$$

for all $B \in J$, where $A \circ B$ denotes the Jordan multiplication. Via the isomorphism $N \cong J$ we view ψ_A a character of N , and every unitary character of N is equal to ψ_A for some A . Let $\Omega \subseteq J$ be the set of rank one elements in J . We view $\Omega \subset \bar{N}$, where $\bar{N}$ is the opposite of N . A unitary model of the minimal representation is $L^2(\Omega)$. Here only the action of the maximal parabolic $P = MN$ is obvious: Let $n \in N$ correspond to $B \in J$ via the isomorphism $N \cong J$. Then, for $f \in L^2(\Omega)$,

$$\pi(n)f(A) = \psi_B(A) \cdot f(A).$$

Any $m \in M$ acts on $\bar{N}$ by conjugation, and therefore on $\Omega \subset J$ via the identification $\bar{N} \cong J$. Then, for $f \in L^2(\Omega)$,

$$\pi(m)f(A) = \chi(m)f(m^{-1}A),$$

where χ is a positive character of M that we shall not need. We have the following, see Propositions 7.2 and 8.3 in [28]:

THEOREM A.4. – *Let Π be the subspace of G -smooth vectors in the minimal representation. Then $C_c^\infty(\Omega) \subset \Pi \subset C^\infty(\Omega)$. If $A \in J$ is non-zero, then any continuous functional ℓ on Π such that $\ell(\pi(n)f) = \psi_A(n) \cdot \ell(f)$ for all $n \in N$ and $f \in \Pi$ is equal to a multiple of the delta functional*

$$f \mapsto f(A).$$

In particular, $\ell = 0$ if A is not rank one. If F is p -adic, we moreover have an exact sequence of P -modules

$$0 \rightarrow C_c^\infty(\Omega) \rightarrow \Pi \rightarrow \Pi_N \rightarrow 0.$$

The representation Π is spherical, and we describe a spherical vector in the p -adic case. Let O be the ring of integers in F , $\varpi \in O$ a uniformizing element and q the order of the residual field. The maximal order $\mathcal{R}$ in $\mathbb{O}$ defines an integral structure on J , let $J(O)$ be the lattice of O -points in J . The greatest common divisor of entries of $A \in J(O)$, is the largest power ϖ^n dividing A , that is, such that A/ϖ^n is in $J(O)$. We have the following Theorem 6.1 in [37]:

THEOREM A.5. – *Assume F is a p -adic field. Assume the conductor of ψ is O . Then the spherical vector in Π is a function $f^\circ \in C^\infty(\Omega)$ supported in $J(O)$. Its value at $A \in \Omega$ depends on the gcd of entries of A . More precisely, if the gcd of A is ϖ^n , and q is the order of the residual field, then*

$$f^\circ(A) = 1 + q^3 + \cdots + q^{3(n-1)}.$$

A.6. Local non-vanishing

In this section F is a p -adic field. Let $U_3 \subset M_3$ be the maximal unipotent subgroup that, via the isomorphism $M_3 \cong \mathrm{GL}_3$ given previously, is the group of matrices

$$u = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $U = U_3 N_3$ is a maximal unipotent subgroup of PGSp_6 . We define a Whittaker character ψ_U on U as follows. For every $u \in U_3$, written as above, $\psi_U(u) = \psi(a + b)$. For every $u \in N_3$, which we identify with $B \in J_3 \cong N_3$, $\psi_U(u) = \psi_E(B)$ where

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The following result was stated in Proposition 17 of [11] without a proof. We give details following Theorem 7.1 in [15], where a version of this result was proved for groups over finite fields.

THEOREM A.7. – *Let Π be the minimal representation of G , U the maximal unipotent subgroup of PGSp_6 and ψ_U the Whittaker character of U . Then there exists a maximal unipotent subgroup U_0 of G_2 and a Whittaker character ψ_0 of U_0 such that*

$$\Pi_{U, \psi_U} \cong \mathrm{ind}_{U_0}^{G_2} \psi_0.$$

Proof. – We shall compute (U, ψ_U) -coinvariants in two steps, (N_3, ψ_U) -coinvariants, followed by (U_3, ψ_U) -coinvariants. From Theorem A.4 it follows that

$$\Pi_{N_3, \psi_U} \cong C_c^\infty(\Omega)_{N_3, \psi_U} \cong C_c^\infty(\omega)$$

where ω is the set of all rank one elements A such that $\psi_A = \psi_E$ on $N_3 \cong J_3$. This works out to all

$$A = \begin{pmatrix} 0 & x & -y \\ -x & 0 & z \\ y & -z & 1 \end{pmatrix} \in J,$$

where $x, y, z \in \mathbb{O}$ satisfying

$$\mathrm{Tr}(x) = \mathrm{Tr}(y) = \mathrm{Tr}(z) = 0,$$

$$x^2 = y^2 = z^2 = 0,$$

$$xz = xy = 0, yz = x.$$

It is now useful to write $\omega = \omega' \cup \omega''$ where ω' is the open subset given by $x \neq 0$. We claim that G_2 acts transitively on ω' . To that end, we shall prove that x, y, z can be G_2 -conjugated to the elements s_1, t_2, t_3 . Conjugate by an element in G_2 so that $x = s_1$. Since $\langle s_1 \rangle^\Delta = \langle s_1, t_2, t_3 \rangle$, the elements y, z must be contained in this 3-dimensional space. Let $Q_1 = L_1 U_1$ be the maximal parabolic subgroup of G_2 stabilizing the line $\langle s_1 \rangle$, with the Levi subgroup $L_1 \cong \mathrm{GL}(\langle t_2, t_3 \rangle)$. The stabilizer of s_1 in G_2 is $L_1^{\mathrm{der}} U_1$ where $L_1^{\mathrm{der}} \cong \mathrm{SL}(\langle t_2, t_3 \rangle)$. The octonion multiplication gives a skew-symmetric form on $\langle s_1, t_2, t_3 \rangle / \langle s_1 \rangle$ valued in $\langle s_1 \rangle$. Thus we can conjugate y, z by an element in L_1^{der} to

$$y = t_2 + as_1 \text{ and } z = t_3 + bs_1$$

for $a, b \in F$. We still have the unipotent radical U_1 to work with. The derived subgroup $[U_1, U_1]$ acts trivially on $\langle s_1, t_2, t_3 \rangle$ and the 2-dimensional quotient $U_1/[U_1, U_1]$ acts simply transitively on the pairs $(a, b) \in F^2$ as above. This proves the claim. Note that we have also proved that x, y, z are linearly independent if $x \neq 0$. The group $U_3 \subset M_3$ acts on ω , and hence on the triples (x, y, z) of off-diagonal terms. Explicitly this action is

$$u^{-1} \cdot (x, y, z) = (x, y + bx, z + ay + cx).$$

Hence U_3 acts freely on ω' , by the linear independence of x, y, z , but with large stabilizers on ω'' , assuring that

$$\Pi_{U, \psi_U} \cong C_c^\infty(\omega')_{U_3, \psi_U}.$$

Summarizing, the situation is very pleasant: G_2 acts transitively on ω' , while U_3 acts freely. View ω' as the G_2 -orbit of A_0 corresponding to the triple $(x, y, z) = (s_1, t_2, t_3)$. The triple defines a partial flag

$$\langle s_1 \rangle \subset \langle s_1, t_2 \rangle \subset \langle s_1, t_2, t_3 \rangle.$$

Let U_0 be the unipotent radical of the Borel subgroup in G_2 stabilizing this flag. Observe that any $u_0 \in U_0$ necessarily acts on the triple (s_1, t_2, t_3) as an element $u \in U_3$. In fact, this action can be made explicit for elements in simple root spaces of U_0 since they are contained in the Levi factors of the two maximal parabolic subgroups ($\mathrm{GL}(\langle s_1, t_2 \rangle)$ and $\mathrm{GL}(\langle t_2, t_3 \rangle)$):

$$(s_1, t_2, t_3) \mapsto (s_1, t_2 + \lambda \cdot s_1, t_3) \text{ and } (s_1, t_2, t_3) \mapsto (s_1, t_2, t_3 + \lambda \cdot t_2)$$

where $\lambda \in F$. Hence we have a surjective homomorphism $\varphi : U_0 \rightarrow U_3$, such that the pullback $\tilde{\psi}_0$ of ψ_U is a Whittaker functional on U_0 . It follows that

$$\Pi_{U, \psi_U} \cong \mathrm{ind}_{U_0}^{G_2}(\psi_0). \quad \square$$

A.8. Global non-vanishing

Assume now that F is a global field, and let $\mathbb{A}$ be the ring of adèles over F . Let $\Pi = \bigotimes \Pi_v$ be the restricted tensor product of minimal representations over all local places v of F . Every element in Π is a finite linear combination of pure tensors $f = \bigotimes f_v$, where $f_v = f_v^\circ$ for almost all places v . There is a unique, up to a non-zero scalar, embedding

$$\theta : \Pi \rightarrow \mathcal{A}(G(F) \backslash G(\mathbb{A}))$$

of Π into the space of automorphic functions of uniform moderate growth. We Fourier expand $\theta(f)$ along N . More precisely, fix a non-trivial character $\psi : \mathbb{A}/F \rightarrow \mathbb{C}^\times$. Then, as in the local case, any $A \in J(F)$ defines a character ψ_A of $N(F) \backslash N(\mathbb{A})$ by the isomorphism $N(\mathbb{A}) \cong J(\mathbb{A})$. Let

$$\theta(f)_A(g) = \int_{N(F) \backslash N(\mathbb{A})} \theta(f)(ng) \bar{\psi}_A(n) \, dn.$$

We have a Fourier expansion

$$\theta(f)(g) = \theta(f)_0(g) + \sum_{A \in \Omega(F)} \theta(f)_A(g)$$

supported on the set of rank one elements. Let $A \in \Omega(F)$. Observe that $f \mapsto \theta(f)_A(1)$ is a continuous, $(N(\mathbb{A}), \psi_A)$ -equivariant functional on Π . By uniqueness of local functionals, Theorem A.4, there exists a non-zero scalar c_A such that $\theta_A(f)(1) = c_A \cdot f(A)$, for all $f \in \Pi$. Hence

$$\theta(f)_A(g) = c_A \cdot (\pi(g)f)(A)$$

for all $f \in \Pi$ and all $g \in G(\mathbb{A})$, where π denotes the action of $g \in G(\mathbb{A})$ on $f \in \Pi$. This formula is particularly useful if $g \in G_2(\mathbb{A})$. Then $(\pi(g)f)(A) = f(g^{-1}A)$, where $g^{-1}A$ is the result of the natural action of g^{-1} on the off-diagonal entries of A .

For every cusp form $h \in \mathcal{A}(G_2(F) \backslash G_2(\mathbb{A}))$ consider the function $\Theta(f, h)$ on $\mathrm{PGSp}_6(\mathbb{A})$ defined by

$$\Theta(f, h)(g_1) = \int_{G_2(F) \backslash G_2(\mathbb{A})} \theta(f)(g_1 g) h(g) \, dg.$$

The function $\theta(f)$ is of moderate growth on G , hence it is also on $G_2(\mathbb{A}) \times \mathrm{PGSp}_6(\mathbb{A})$. In particular, the integral converges absolutely, since h is a cusp form, and the output $\Theta(f, h)$ is a function of uniform moderate growth on $\mathrm{PGSp}_6(\mathbb{A})$. Let U_0 be the maximal unipotent subgroup stabilizing the partial flag $\langle s_1 \rangle \subset \langle s_1, t_2 \rangle \subset \langle s_1, t_2, t_3 \rangle$. Let $U'_0 \subset U_0$ be the stabilizer

of the triple (s_1, t_2, t_3) . Then U_0/U'_0 is isomorphic to U_3 and, via this isomorphism, the Whittaker character ψ_U of $U(F)\backslash U(\mathbb{A})$ transfers to a Whittaker character ψ_0 of $U_0(F)\backslash U_0(\mathbb{A})$ as in the local case. Let

$$h_{U_0, \psi_0}(g) = \int_{U_0(F)\backslash U_0(\mathbb{A})} h(ug) \bar{\psi}_0(u) du.$$

THEOREM A.9. – *Let ψ_U be the Whittaker character of $U(F)\backslash U(\mathbb{A})$. If $h_{U_0, \psi_0} \neq 0$ then, for some choice of $f \in \Pi$,*

$$\Theta(f, h)_{U, \psi_U}(1) := \int_{U(F)\backslash U(\mathbb{A})} \Theta(f, h)(u) \bar{\psi}_U(u) du \neq 0.$$

Proof. – The first part of the proof involves using the Fourier expansion of $\theta(f)$ and unfolding the integral. This follows closely the proof of the local Theorem A.7, and we shall be brief. Firstly we integrate over $N_3(F)\backslash N_3(\mathbb{A})$. This reduces the Fourier sum to over the subset $\omega(F)$

$$\Theta(f, h)_{N_3, \psi_U}(1) = \int_{G_2(F)\backslash G_2(\mathbb{A})} \sum_{A \in \omega(F)} \theta(f)_A(g) h(g) dg.$$

Write $\omega = \omega' \cup \omega''$ as in the local case. We can ignore the sum over ω'' since it will vanish after integrating over $U_3(F)\backslash U_3(\mathbb{A})$. Also, $\omega'(F)$ is the $G_2(F)$ -orbit of A_0 corresponding to the triple $(x, y, z) = (s_1, t_2, t_3)$ with the stabilizer $U'_0(F)$. Then

$$\int_{G_2(F)\backslash G_2(\mathbb{A})} \sum_{A \in \omega'(F)} \theta(f)_A(g) h(g) dg = c_{A_0} \int_{U'_0(F)\backslash G_2(\mathbb{A})} f(g^{-1}A_0) h(g) dg.$$

We integrate over $U'_0(F)\backslash U'_0(\mathbb{A})$, and use that the function $g \mapsto f(g^{-1}A_0)$ is left $U'_0(\mathbb{A})$ -invariant, hence

$$c_{A_0} \int_{U'_0(F)\backslash G_2(\mathbb{A})} f(g^{-1}A_0) h(g) dg = c_{A_0} \int_{U'_0(\mathbb{A})\backslash G_2(\mathbb{A})} f(g^{-1}A_0) h_{U'_0}(g) dg,$$

where $h_{U'_0}$ is the constant term of h along U'_0 . Finally, we integrate over $U_3(F)\backslash U_3(\mathbb{A})$ against the character $\bar{\psi}_U$. Using the isomorphism $U_3 \cong U_0/U'_0$, we obtain

$$\Theta(f, h)_{U, \psi_U}(1) = c_{A_0} \int_{U'_0(\mathbb{A})\backslash G_2(\mathbb{A})} \tilde{f}(g) h_{U'_0}(g) dg,$$

where $\tilde{f}$ is the product of

$$\tilde{f}_v(g) = \int_{U'_0(F_v)\backslash U_0(F_v)} f_v((ug)^{-1}A_0) \psi_0(u) du.$$

Observe that $\tilde{f}$ is a Whittaker function i.e., $\tilde{f}(ug) = \bar{\psi}_0(u) \tilde{f}(g)$, for all $u \in U_0(\mathbb{A})$. Hence

$$\Theta(f, h)_{U, \psi_U}(1) = c_{A_0} \int_{U_0(\mathbb{A})\backslash G_2(\mathbb{A})} \tilde{f}(g) h_{U_0, \psi_0}(g) dg.$$

The next step is to show that this integral reduces to a finite number of places S , as in Section 7 of [10]. Assume that S contains all archimedean places and, if $v \notin S$, then

- f_v is the spherical vector f_v° ,
- the cusp form h is $G_2(O_v)$ -invariant,

— the conductor of ψ restricted to F_v is O_v .

Let B_0 be the Borel subgroup containing U_0 . We fix a Levi factor L_0 (a torus) so that it stabilizes the lines through s_1, t_2, t_3 . Thus, if $l_v \in L_0(F_v)$ then

$$l_v t_2 = \lambda_2 \cdot t_2, \quad l_v t_3 = \lambda_3 \cdot t_3, \quad l_v s_1 = (\lambda_2 \lambda_3) \cdot s_1,$$

for some non-zero scalars λ_2 and λ_3 , where the last identity follows from $s_1 = t_2 t_3$.

LEMMA A.10. — *If $v \notin S$ then the Whittaker function $\tilde{f}_v^\circ$ is supported on $U_0(F_v)G_2(O_v)$ and $\tilde{f}_v^\circ(1) = 1$.*

Proof. — Let ord_v denote the valuation on $F_v^\times$. Since $\tilde{f}_v^\circ$ is a spherical Whittaker function, it is determined by its restriction to $L_0(F_v)$ and there it is supported on the cone consisting of l_v such that $\text{ord}_v(\alpha_1(l_v)) \geq 0$ and $\text{ord}_v(\alpha_2(l_v)) \geq 0$ where α_1 and α_2 are the simple roots. In terms of the explicit action of l_v on s_1, t_2, t_3 described above, these inequalities are

$$\text{ord}_v(\lambda_2) \geq \text{ord}_v(\lambda_3) \geq 0.$$

Since f_v° is supported on the lattice $J(O_v)$, it is easily seen that, for any $u_v \in U_0(F_v)$, the function $l_v \mapsto f_v^\circ((u_v l_v)^{-1} A_0)$ is supported on the cone

$$\text{ord}_v(\lambda_2), \text{ord}_v(\lambda_3) \leq 0.$$

This completes the support part of the statement. Using again that f_v° is supported on the lattice $J(O_v)$, the function $u_v \mapsto f_v^\circ(u_v^{-1} A_0)$, is supported on $U'_0(F_v)U_0(O_v)$ and this implies that $\tilde{f}_v^\circ(1) = 1$, since the character ψ_0 is trivial on $U_0(O_v)$. $\square$

Let $\mathbb{A}_S$ be the product of F_v over all $v \in S$. The previous lemma implies that

$$\Theta(f, h)_{U, \psi_U}(1) = c_{A_0} \int_{U_0(\mathbb{A}_S) \backslash G_2(\mathbb{A}_S)} \tilde{f}_S(g) h_{U_0, \psi_0}(g) dg.$$

Since, by Theorem A.4, f_v can be any smooth, compactly supported function on $\Omega(F_v)$, for every $v \in S$, the integral can be arranged to be non-zero. This completes the proof of the theorem. $\square$

REMARK. — *Note that $\Theta(f, h)$ is unramified for all $v \notin S$.*

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