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General Section

Modular forms on indefinite orthogonal groups of rank three ☆

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with appendix "Next to minimal representation" by
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ABSTRACT

We develop a theory of modular forms on the groups $\mathrm{SO}(3, n+1)$, $n \geq 3$. This is very similar to, but simpler, than the notion of modular forms on quaternionic exceptional groups, which was initiated by Gross-Wallach and Gan-Gross-Savin. We prove the results analogous to those of earlier papers of the author on modular forms on exceptional groups, except now in the familiar setting of classical groups. Moreover, in the setting of $\mathrm{SO}(3, n+1)$, there is a family of absolutely convergent Eisenstein series, which are modular forms. We prove that these Eisenstein series have algebraic Fourier coefficients, like the classical holomorphic Eisenstein series on $\mathrm{SO}(2, n)$. As an application, using a local result of Savin, we prove that the so-called "next-to-minimal" modular form on quaternionic E_8 has rational Fourier expansion.

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1. Introduction

That there is a notion of *modular forms* on the quaternionic exceptional groups goes back to Gross-Wallach and Gan-Gross-Savin. This theory is based on the so-called quaternionic discrete series, whose study was initiated by Gross-Wallach [GW94], [GW96]. Here, by a "modular form", we mean loosely a special automorphic form that possesses some sort of robust Fourier expansion, similar to the (holomorphic) Siegel modular forms on Sp_{2n}. The modular forms on the quaternionic exceptional groups have been the subject of the papers [GGS02,Wei06,Pol20a,Pol20b,Pol19].

It turns out that there is a completely analogous but much simpler theory of "modular forms" on the classical groups $\mathrm{SO}(3, n+1)$. (Note that when n is even, these groups do not have discrete series.) The purpose of this paper is to write down this notion of modular forms, and prove a few of the basic theorems. In particular, we

(1) find the explicit form of the Fourier expansion of such modular forms, in complete analogy with the results of [Pol20a];
(2) prove that certain absolutely convergent degenerate Eisenstein series that are modular forms have algebraic Fourier coefficients, in a precise sense.

While the result (1) is analogous to the results of [Pol20a], the result (2)–which is the main result of the paper–goes beyond what is known for exceptional groups.

One source of examples of these special automorphic forms comes from certain constant terms of modular forms on the quaternionic exceptional groups. More precisely, suppose G_J is[1] a quaternionic exceptional group as in [Pol20a] with rational root type F_4, so that G_J has absolute Dynkin type F_4, E_6, E_7 or E_8. Then G_J possesses a maximal parabolic $Q_J = L_J V_J$ with L_J having rational root type B_3. Up to anisotropic factors, L_J is isogenous to a group $\mathrm{SO}(3, n+1)$ where $n = 3, 4, 6, 10$ if G_J has type F_4, E_6, E_7, E_8, respectively. One can take the constant term of a modular form of weight ℓ on G_J down to L_J, and we prove in section 5 that these constant terms are modular forms of weight ℓ on L_J. Combining the above facts with a p-adic result of Savin (Theorem B.1.1 proved in Appendix B) and an analysis of certain degenerate Eisenstein series on $E_{8,4}$, we prove that the so-called "next-to-minimal" modular form on quaternionic E_8 has rational Fourier expansion.

The result (2) on the Fourier coefficients of Eisenstein series is the analogue of the fact that on $\mathrm{SO}(2, n)$ or another hermitian tube domain, the absolutely convergent holomorphic Eisenstein series have algebraic Fourier expansions. As this paper shows, the notion of modular forms on $\mathrm{SO}(3, n+1)$ is very similar to that of modular forms on the quaternionic exceptional groups, such as $E_{8,4}$. However, because $\mathrm{SO}(3, n+1)$ is a classical group, and more importantly because the natural Fourier expansion of modular

[1] The subscript "J" comes from the fact that these groups are associated to certain cubic Jordan algebras J.

forms on $\mathrm{SO}(3, n+1)$ takes place along an *abelian* unipotent group, the notion of these modular forms is substantially simpler than that on the exceptional groups. Thus we hope that $\mathrm{SO}(3, n+1)$ can be used as a test case for developing analogous results on the quaternionic exceptional groups. In particular, the algebraicity of the Fourier coefficients of the degenerate Eisenstein series from [Pol20b] appears difficult. Part of the motivation for writing this paper was to get closer to proving that the Fourier coefficients of these Eisenstein series on exceptional groups are algebraic. It follows from the proof of Theorem 1.1.2 that the absolutely convergent degenerate Eisenstein series on quaternionic E_8 studied in [Pol20b] have algebraic rank $0, 1$, and 2 Fourier coefficients; the algebraicity of the rank 3 and 4 Fourier coefficients remains open.

1.1. Statement of theorems

The definition of the modular forms on $\mathrm{SO}(3, n+1)$ is very similar to that of the modular forms on the exceptional groups from [Pol20a]. In particular, if V is a rational quadratic space of signature $(3, n+1)$, then the maximal compact subgroup K of $\mathrm{SO}(V)(\mathbf{R})$ is $S(O(3) \times O(n+1))$. This group maps to $O(3) = (\mathrm{SU}(2)/\mu_2) \rtimes \langle \pm 1 \rangle$. Denote by $\mathbb{V}_\ell = Sym^{2\ell}(\mathbf{C}^2)$ the $(2\ell+1)$-dimensional representation of K that factors through $O(3)$. *Modular forms* on $\mathrm{SO}(V)$ of weight ℓ are then $\mathbb{V}_\ell$-valued automorphic functions φ on $\mathrm{SO}(V)(\mathbf{A})$ that

(1) satisfy $\varphi(gk) = k^{-1}\varphi(g)$ for all $g \in \mathrm{SO}(V)(\mathbf{A})$ and $k \in K$
(2) and are annihilated by a special linear differential operator $\mathcal{D}_\ell$.

The precise definition of modular forms, and in particular of the operator $\mathcal{D}_\ell$, is given in section 3 below. Throughout the paper, (V, q) is rational quadratic space of Witt rank three and signature $(3, n+1)$ over $\mathbf{R}$. Denote by $(x, y) = q(x+y) - q(x) - q(y)$ the associated bilinear form. We write $V = \mathbf{Q}e \oplus V' \oplus \mathbf{Q}f$ with V' a non-degenerate quadratic space of signature $(2, n)$ and e, f isotropic vectors in $(V')^\perp$ with $(e, f) = 1$.

The first result is the Fourier expansion of modular forms on $G = \mathrm{SO}(V)$, in complete analogy to Theorem 1.2.1 of [Pol20a]. Denote by $P = MN$ the parabolic subgroup of $\mathrm{SO}(V)$ that stabilizes the isotropic line $\mathbf{Q}e$, so that $M \simeq \mathrm{GL}_1 \times \mathrm{SO}(V')$ and $N \simeq V'$ is abelian. Let $n : V' \to N$ denote this identification, which is specified in section 2 below. If φ is an automorphic form on G, then one has

$$\varphi(g) = \sum_{\eta \in V'(\mathbf{Q})} \varphi_\eta(g) \tag{1}$$

where

$$\varphi_\eta(g) = \int\limits_{V'(\mathbf{Q})\backslash V'(\mathbf{A})} \psi^{-1}((\eta, x))\varphi(n(x)g)\, dx$$

and $\psi : \mathbf{Q}\backslash\mathbf{A} \to \mathbf{C}^\times$ is our fixed standard additive character.

The first result Theorem 1.1.1 is a refinement of the expansion (1) when φ is a modular form of weight ℓ on G. See Definition 3.2.2 below for the precise definition of the functions $\mathcal{W}_\eta : G \to \mathbb{V}_\ell$ that appear in Theorem 1.1.1. They are defined in terms of K-Bessel functions, exactly as in the Fourier expansion of the modular forms on the quaternionic exceptional groups in [Pol20a]. In section 2 we specify a basis $\{x^{2\ell}, x^{2\ell-1}y, \dots, xy^{2\ell-1}, y^{2\ell}\}$ of $\mathbb{V}_\ell$.

Theorem 1.1.1. *Suppose φ is a modular form of weight $\ell \geq 1$ on G. Then for $\eta \in V'(\mathbf{Q})$ with $q(\eta) \geq 0$, there are locally constant functions $a_\varphi(\eta) : G(\mathbf{A}_f) \to \mathbf{C}$ so that*

$$\varphi(g) = \varphi_0(g) + \sum_{0 \neq \eta \in V'(\mathbf{Q}), q(\eta) \geq 0} a_\varphi(\eta)(g_f)\mathcal{W}_{2\pi\eta}(g_\infty) \tag{2}$$

for every $g = g_f g_\infty$ in $G(\mathbf{A}_f) \times G(\mathbf{R})$. Moreover, for $m \in M$, the constant term φ_0 is of the form

$$\varphi_0(m) = t^\ell |t| \left(\Phi(m)x^{2\ell} + \beta(m_f)x^\ell y^\ell + \Phi'(m)y^{2\ell}\right)$$

where Φ is an automorphic function associated to a holomorphic modular form of weight ℓ on M, β is a locally constant function on $M(\mathbf{A}_f)$, and Φ' is a certain $(K \cap M)$-right translate of Φ.

The second theorem concerns the Fourier expansion of degenerate Eisenstein series on G. More precisely, if $\ell > n+1$ is even and n is even then there is (a family of) absolutely convergent Eisenstein series $E_\ell(g)$, which are modular forms of weight ℓ on G. These Eisenstein series are associated to the induction space $Ind_P^G(\delta_P^{(\ell+1)/(n+2)})$. See section 4 for the precise definition. These degenerate Eisenstein series are the analogues of the degenerate Heisenberg Eisenstein series considered in [Pol20b] or the classical degenerate holomorphic Siegel Eisenstein series on Sp_{2n}. Theorem 1.1.2 below states that the Fourier coefficients of $E_\ell(g)$ are algebraic numbers.

To set up the result, suppose that $\ell > 0$ is even, and φ a modular form on G of weight ℓ. Let the Fourier expansion of φ be as in (2). We say that φ has Fourier coefficients in a field E if

(1) The locally constant functions $a_\varphi(\eta)$, when restricted to $M(\mathbf{A}_f)$, are valued in E;
(2) The holomorphic modular form associated to Φ has Fourier coefficients in E;
(3) The locally constant function β, when restricted to $M(\mathbf{A}_f)$, is valued in $E \cdot \frac{\zeta(\ell+1)}{(2\pi)^\ell}$.

The perhaps unusual-looking normalization of the constant β is dictated, similar to the results of [Pol20b], by the fact that the modular forms one constructs in practice have Fourier coefficients valued in some fixed field E as in the above definition.

Theorem 1.1.2. *Suppose that* $\dim(V') = n+2$ *is a multiple of* 4 *and that* $\ell > n+1$ *is even. Then the Eisenstein series* $E_\ell(g)$ *has* $\overline{\mathbf{Q}}$*-valued Fourier coefficients.*

Note that under the assumptions of Theorem 1.1.2, the group $G(\mathbf{R}) = \mathrm{SO}(3, n+1)$ does not possess discrete series. Nevertheless, the modular forms exist and one can prove that the most basic modular forms–the degenerate Eisenstein series–have algebraic Fourier coefficients. See Theorem 4.5.10 for the precise statement. The key step in the proof of Theorem 1.1.2 is the evaluation of a certain Archimedean Jacquet integral, which is Theorem 4.5.9 below and might be of independent interest.

1.2. Applications

The main application of the above results is to the rationality of the Fourier expansion of the next-to-minimal modular form θ_{ntm} on quaternionic E_8, which is realized as a special value of a degenerate Heisenberg Eisenstein series. Recall from [Pol20a] or [Pol20b] that modular forms on quaternionic E_8 have Fourier coefficients of various *ranks*, between 0 and 4 inclusive, with rank four Fourier coefficients being non-degenerate and rank 0 and rank 1 Fourier coefficients the most degenerate ones. We prove directly that θ_{ntm} has rational rank 0, rank 1, and rank 2 Fourier coefficients. The p-adic result Theorem B.1.1 of Savin implies that the rank 3 and rank 4 Fourier coefficients of θ_{ntm} vanish, giving the full rationality.

Here is the precise result. In the statement of the theorem, the group G_J is the $\mathbf{Q}$-group of type E_8 from, e.g., [Pol20a] or [Pol20b], that has rational root system of type F_4.

Theorem 1.2.1. *Let* $E_J(g, s; 8)$ *denote the degenerate Heisenberg Eisenstein series on* G_J *that is spherical at every finite place and "weight* 8*" at infinity. Then* $E_J(g, s; 8)$ *is regular at* $s = 9$ *and defines a square integrable modular form of weight* 8 *at this point. Moreover, the modular form* $\theta_{ntm}(g) = E_J(g, s = 9; 8)$ *has rational Fourier expansion.*

See Theorem 6.0.2 below for the precise statement. The Eisenstein series $E_J(g, s; \ell)$ are the subject of [Pol20b]. The modular form $E_J(g, s = 9; 8)$ is expected to be the next-to-minimal modular form on G_J. In the case of split E_8, the next-to-minimal automorphic representation has been considered in [GMV15] and more recently in [GGK+19].

The *minimal* modular form θ_{Gan} on quaternionic E_8 was considered in [Gan00a, Gan00b,Pol20b]; it is of weight 4. The first part of Theorem 1.2.1 on the regularity of $E_J(g, s; 8)$ at $s = 9$ is analogous to some results of Gan from [Gan00a]. The weight four modular form θ_{Gan} is the E_8-analogue of Kim's weight 4 exceptional modular form [Kim93] on $GE_{7,3}$, and in fact Kim's weight 4 modular form appears in the constant term of θ_{Gan} along the unipotent radical of the Heisenberg parabolic. The *next-to-minimal* modular form θ_{ntm} that is the subject of Theorem 1.2.1 is weight 8 and is the analogue of

Kim's weight 8 singular modular form on $GE_{7,3}$ from [Kim93]. Moreover, Kim's singular modular form shows up in the constant term of θ_{ntm} along the same Heisenberg parabolic.

The final result we give is to the minimal modular form on the groups $\mathrm{SO}(3, 8k+3)$ and to a so-called distinguished modular form on $\mathrm{SO}(3, 8k+2)$. This is done in section 7, and is the analogue of the results in [Pol20b] to the classical groups of type D_{4k+3}. Specifically, we prove the following theorem; see Theorem 7.0.1 below.

Theorem 1.2.2. *Let $k \geq 1$ be an integer, and let G be the $\mathbf{Q}$-group of type D_{4k+3} that is split at every finite place and $\mathrm{SO}(3, 8k+3)$ at infinity. The Eisenstein series $E_{4k}(g, s)$ is regular at $s = 4k+1$. The value $\theta(g) = E_{4k}(g, s = 4k+1)$ is a modular form on G of weight $4k$, having rational Fourier expansion with all non-degenerate Fourier coefficients equal to 0. Its restriction θ' to groups $G' = \mathrm{SO}(3, 8k+2) \subseteq G$ is a modular form of weight $4k$ that is distinguished.*

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2. Notation

In this section we define the notation that we will use throughout the paper. Let (V_2, q_2) denote a two-dimensional rational quadratic space with positive definite quadratic form. Similarly, let (V_n, q_n) denote an n-dimensional rational quadratic space with positive definite quadratic form. Set $V' = V_2 \oplus V_n$ with quadratic form $q(x, y) = q_2(x) - q_n(y)$, so that V' has signature $(2, n)$. We set $V = \mathbf{Q}e \oplus V' \oplus \mathbf{Q}f$ with quadratic form $q(\alpha e + v' + \beta f) = \alpha\beta + q'(v')$. Thus V has signature $(3, n+1)$. For some of the results below, we will assume V' has Witt rank two, although this is not necessary everywhere.

Let ι be the involution on V given by $\iota(\alpha e + x + y + \beta f) = \beta e + x - y + \alpha f$, where $x \in V_2$, $y \in V_n$. Then $(v, \iota(v)) \geq 0$, and conjugation by ι is a Cartan involution θ_ι on $\mathrm{SO}(V)(\mathbf{R})$. We set $u_+ = e + f$ and $u_- = e - f$ so that $q(u_+) = 1$ and $q(u_-) = -1$. We let v_1, v_2 be an orthonormal basis of $V_2(\mathbf{R})$ so that $(v_i, v_j) = \delta_{ij}$, and $\{u_1, u_2, \dots, u_n\}$ be a basis of V_n.

We set $V_3 = V_2 \oplus \mathbf{R}u_+$ and $V_{n+1} = V_n \oplus \mathbf{R}u_-$. The induced Cartan involution on the Lie algebra $\mathfrak{g}_0 = \mathfrak{so}(V)$ produces the decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ with $\mathfrak{k}_0 = \mathfrak{g}_0^{\theta_\iota = 1}$ and $\mathfrak{p}_0 = \mathfrak{g}_0^{\theta_\iota = -1}$. Under the isomorphism $\mathfrak{g}_0 \simeq \wedge^2 V$, one has $\mathfrak{p}_0 = V_3 \otimes V_{n+1} \subseteq \wedge^2 V$ and $\mathfrak{k}_0 = \wedge^2 V_3 \oplus \wedge^2 V_{n+1} \subseteq \wedge^2 V$. We set $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbf{C}$ and $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbf{C}$.

The Lie algebra $\wedge^2 V_3 \otimes \mathbf{C} \subseteq \mathfrak{k}$ is isomorphic to $\mathfrak{sl}_2(\mathbf{C})$. For an $\mathfrak{sl}_2$-triple (E, H, F) in $\wedge^2 V \otimes \mathbf{C}$, one can take $E = (iv_1 - v_2) \wedge u_+/\sqrt{2}$, $H = -2iv_1 \wedge v_2$, $F = (iv_1 + v_2) \wedge u_+/\sqrt{2}$. Then $[E, F] = H$, $[H, E] = 2E$ and $[H, F] = -2F$, so that indeed (E, H, F) is an $\mathfrak{sl}_2$-triple.

Denote by $P = MN$ the parabolic subgroup of G that fixes the line $\mathbf{Q}e$. We are letting G act on the left of V. Denote by $\nu : P \to \mathrm{GL}_1$ the character so that $pe = \nu(p)e$. We let M be the Levi subgroup that also fixes the line $\mathbf{Q}f$. Denote by N the unipotent radical of P. Then $N \simeq V'$ is abelian, and for $x \in V'$, we set $n(x) = \exp(e \wedge x)$. Thus $n : V' \to N$ is an isomorphism. One has $n(x) = \exp(e \wedge x)$ takes $e \mapsto e$, $v \mapsto v + (x, v)e$ if $v \in V'$, and $f \mapsto f - x - \frac{1}{2}(x, x)e$. The matrix corresponding to $n(x)$ is $\begin{pmatrix} 1 & {}^t x & -(x,x)/2 \\ & 1 & -x \\ & & 1 \end{pmatrix}$.

As mentioned in the introduction, we let $\mathbb{V}_\ell$ denote the $(2\ell + 1)$-dimensional representation of $K \subseteq \mathrm{SO}(3, n + 1)$ that factors through $O(3)$. Let x, y be a fixed weight basis of the two-dimensional representation $Y_2 \simeq \mathbf{C}^2$ of $\wedge^2 V_3 \otimes \mathbf{C} \simeq \mathfrak{sl}_2(\mathbf{C})$. We may identify $V_3 \otimes \mathbf{C}$ with the symmetric square representation $S^2(Y_2)$ of this two-dimensional representation, which has basis $\{x^2, xy, y^2\}$. We choose this weight basis x, y and the identification $S^2(Y_2) \simeq V_3 \otimes \mathbf{C}$ so that x^2 corresponds to $iv_1 - v_2$, xy corresponds to $u_+/\sqrt{2}$, and y^2 corresponds to $iv_1 + v_2$.

Throughout the paper, the letter H denotes a hyperbolic plane. Moreover, we frequently use the subscript 0 to denote an integral lattice inside a rational quadratic space. Thus, for example $H_0 \cong \mathbf{Z} \oplus \mathbf{Z}$.

3. Modular forms and their Fourier expansion

In this section we define the modular forms on $G = \mathrm{SO}(V)$, and give the explicit form of their Fourier expansion. That is, we prove Theorem 1.1.1 of the introduction.

3.1. Definition of modular forms

We now define modular forms on $G = \mathrm{SO}(V)$. As mentioned in the introduction, a modular form on G of weight ℓ is an automorphic function $\varphi : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbb{V}_\ell$ of moderate growth satisfying

(1) $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in G(\mathbf{A})$ and $k \in K$
(2) $D_\ell \varphi \equiv 0$ for a certain linear differential operator D_ℓ defined below.

To define the differential operator D_ℓ, let X_γ be a basis of $\mathfrak{p}$ and $X_\gamma^\vee$ be the dual basis of $\mathfrak{p}^\vee$. Suppose $\varphi : G(\mathbf{A}) \to \mathbb{V}_\ell$ satisfies $\varphi(gk) = k^{-1}\varphi(g)$. Define $\widetilde{D_\ell}\varphi = \sum_\gamma X_\gamma \phi \otimes X_\gamma^\vee$, which is valued in $\mathbb{V}_\ell \otimes \mathfrak{p}^\vee$. Here $X_\gamma \varphi$ denotes the right-regular action of $\mathfrak{p}$ on φ. Note that

$$\mathbb{V}_\ell \otimes \mathfrak{p}^\vee = (S^{2\ell}(Y_2) \otimes S^2(Y_2)) \boxtimes V_{n+1} = (S^{2\ell+2}(Y_2) \oplus S^{2\ell}(Y_2) \oplus S^{2\ell-2}(Y_2)) \boxtimes V_{n+1}.$$

Denote by pr the K-equivariant projection $\mathbb{V}_\ell \otimes \mathfrak{p}^\vee \to (S^{2\ell}(Y_2) \oplus S^{2\ell-2}(Y_2)) \boxtimes V_{n+1}$. We define $D_\ell = \mathrm{pr} \circ \widetilde{D}$.

Note that $S^2(Y_2) \subseteq S^1(Y_2) \otimes S^1(Y_2)$, and thus pr is also the composition

$$\begin{aligned}
&\mathbb{V}_\ell \otimes \mathfrak{p}^\vee \subseteq (S^{2\ell}(Y_2) \otimes S^1(Y_2) \otimes S^1(Y_2)) \boxtimes V_{n+1} \\
&= (S^{2\ell+1}(Y_2) \oplus S^{2\ell-1}(Y_2)) \otimes S^1(Y_2) \boxtimes V_{n+1} \\
&\to S^{2\ell-1}(Y_2) \otimes (S^1(Y_2) \boxtimes V_{n+1}).
\end{aligned}$$

This last line makes clear the analogy between modular forms on $\mathrm{SO}(3, n+1)$ and modular forms in the sense of [Pol20a].

3.2. The Fourier expansion of modular forms

In this subsection we give the precise Fourier expansion of modular forms on G. More precisely, suppose $\ell \geq 1$, $\eta \in V'(\mathbf{R})$. We say that a function $F : G(\mathbf{R}) \to \mathbb{V}_\ell$ is a generalized Whittaker function of type η if F is of moderate growth and satisfies

(1) $F(n(x)g) = e^{i(\eta,x)}F(g)$
(2) $F(gk) = k^{-1} \cdot F(g)$
(3) $D_\ell F(g) = 0$

for all $g \in G(\mathbf{R})$, $k \in K$ and $x \in V'(\mathbf{R})$. In this subsection, we completely characterize the generalized Whittaker functions of type η, for all $\eta \in V'(\mathbf{R})$. In particular, we prove that if $q(\eta) < 0$, the only such function is the 0 function, while if $\eta \neq 0$ and $q(\eta) \geq 0$ then all such functions are scalar multiples of the function $\mathcal{W}_\eta$ mentioned in the introduction.

In order to understand these generalized Whittaker functions, we make relatively explicit the differential equation $D_\ell F = 0$ in coordinates. To do this, we begin by making an explicit Iwasawa decomposition of some elements of the Lie algebra of G. In more detail, let $\mathfrak{n}, \mathfrak{m}$ denote the complexified Lie algebras of N, M; one has a decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{m} + \mathfrak{k}$. We have

$$\mathfrak{p} = (\mathbf{R}u_+ \oplus V_2) \wedge (\mathbf{R}u_- \oplus V_n) = \mathbf{R}u_+ \wedge u_- \oplus u_+ \wedge V_n \oplus V_2 \wedge u_- \oplus V_2 \wedge V_n.$$

In $\mathfrak{n} + \mathfrak{m} + \mathfrak{k}$ coordinates, a basis of $\mathfrak{p}$ decomposes as follows:

- $u_+ \wedge u_- = (e+f) \wedge (e-f) = -2e \wedge f \in \mathfrak{m}$.
- $u_+ \wedge u_j = (e+f) \wedge u_j = (2e - u_-) \wedge u_j = 2e \wedge u_j - u_- \wedge u_j \in \mathfrak{n} + \mathfrak{k}$. (Recall that the u_j are a basis of V_n.)
- $v_i \wedge u_j \in \mathfrak{m}$. (Recall that v_1, v_2 is a basis of V_2.)
- $v_i \wedge u_- = v_i \wedge (e-f) = v_i \wedge (2e - u_+) = -2e \wedge v_i + u_+ \wedge v_i \in \mathfrak{n} + \mathfrak{k}$.

For ease of notation, let $[x^j] = \frac{x^j}{j!}$ and similarly $[y^j] = \frac{y^j}{j!}$. Let F_v denote the components of the $\mathbb{V}_\ell$-valued function F; that is

$$F = \sum_{-\ell \leq v \leq \ell} F_v [x^{\ell+v}][y^{\ell-v}].$$

Let $\{u_1^\vee, \dots, u_n^\vee\}$ be the basis dual to the basis $\{u_1, \dots, u_n\}$ and $u_-^\vee$ dual to u_-. Denote by $D^M_{iv_1 \pm v_2, u_j}$ the differential operator on functions on M corresponding to the (differential right-regular) action of $(iv_1 \pm v_2) \wedge u_j$ on F. For future reference, note that $(iv_1 - v_2, iv_1 + v_2) = -2$.

Suppose $t \in \mathbf{R}^\times$, $m \in \mathrm{SO}(2,n)$ and $x \in V'(\mathbf{R})$ so that $n(x)\,\mathrm{diag}(t, m, t^{-1}) \in N(\mathbf{R})M(\mathbf{R}) = P(\mathbf{R})$. Restricting the function F to P, we write $F(x,t,m) := F(n(x)\,\mathrm{diag}(t,m,t^{-1}))$. For $w \in V'$, denote

$$D^{V'}_w F(x,t,m) = \frac{d}{d\lambda} F(x + \lambda w, t, m)|_{\lambda=0}$$

the partial derivative in the w-direction. Also, note that $(e \wedge f)F = t\partial_t F$.

Suppose $F : G(\mathbf{R}) \to \mathbb{V}_\ell$ is a function satisfying $F(gk) = k^{-1}F(g)$ for all $g \in G(\mathbf{R})$ and $k \in K$. The following proposition computes $D_\ell F(x,t,m)$ explicitly in coordinates, in terms of the differential operators D^M, $D^{V'}$ and $t\partial_t$. To state the result, note that the operator D_ℓ is valued in $S^{2\ell-1}(Y_2) \otimes (Y_2 \boxtimes V_{n+1})$, which has a basis consisting of elements $[x^{\ell+v-1}][y^{\ell-v}] \otimes y \otimes u_-^\vee$, $[x^{\ell+v-1}][y^{\ell-v}] \otimes x \otimes u_-^\vee$, $[x^{\ell+v-1}][y^{\ell-v}] \otimes y \otimes u_j^\vee$, $[x^{\ell+v-1}][y^{\ell-v}] \otimes x \otimes u_j^\vee$.

Proposition 3.2.1. *Suppose $F : G(\mathbf{R}) \to \mathbb{V}_\ell$ is a function satisfying $F(gk) = k^{-1}F(g)$ for all $g \in G(\mathbf{R})$ and $k \in K$. The coefficients of linear independent terms in $2D_\ell F$ are as follows:*

(1) $[x^{\ell+v-1}][y^{\ell-v}] \otimes y \otimes u_-^\vee$:

$$2D^{V'}_{tm(iv_1 - v_2)} F_v - \sqrt{2}(\ell + v) F_{v-1} + \sqrt{2} t\partial_t F_{v-1}$$

(2) $[x^{\ell+v-1}][y^{\ell-v}] \otimes x \otimes u_-^\vee$:

$$-\sqrt{2} t\partial_t F_v - 2D^{V'}_{tm(iv_1 + v_2)} F_{v-1} + \sqrt{2}(\ell - v + 1) F_v$$

(3) $[x^{\ell+v-1}][y^{\ell-v}] \otimes y \otimes u_j^\vee$:

$$-D^M_{iv_1 - v_2, u_j} F_v - \sqrt{2} D^{V'}_{tmu_j} F_{v-1}$$

(4) $[x^{\ell+v-1}][y^{\ell-v}] \otimes x \otimes u_j^\vee$:

$$\sqrt{2} D^{V'}_{tmu_j} F_v + D^M_{iv_1 + v_2, u_j} F_{v-1}.$$

Proof. As the computation is straightforward, we relegate the details to the appendix. See subsection A.1. □

As a corollary of the above proposition, we obtain the complete description of the generalized Whittaker functions of type η. Thus suppose $\eta \in V'(\mathbf{R})$ and F is a generalized Whittaker function of type η. That is, assume F is of moderate growth and $F(x,t,m)$ satisfies $F(x+w,t,m) = e^{i(\eta,w)}F(x,t,m)$ for all $w \in V'$, so that $D_w^{V'}F = i(\eta,w)F$.

To state the theorem, we first define the function $\mathcal{W}_\eta$ that plays a crucial role in this paper.

Definition 3.2.2. Suppose $\eta \in V'(\mathbf{R})$, $\eta \neq 0$, and $(\eta,\eta) \geq 0$. For $t \in \mathrm{GL}_1(\mathbf{R})$, $m \in \mathrm{SO}(V')(\mathbf{R})$ set

$$u_\eta(t,m) = \sqrt{2}ti(\eta, m(iv_1 - v_2)).$$

Define

$$\mathcal{W}_\eta(t,m) = t^\ell|t| \sum_{-\ell \leq v \leq \ell} \left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)}\right)^v K_v(|u_\eta(t,m)|)[x^{\ell+v}][y^{\ell-v}].$$

Here recall the K-Bessel function $K_v(y)$ is defined as

$$K_v(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^v\,\frac{dt}{t}.$$

It satisfies the differential equation $(y\partial_y)^2K_v(y) = (v^2+y^2)K_v(y)$, diverges at $y \to 0$ and is of rapid decay as $y \to \infty$. As $K_v(y)$ diverges for $y \to 0$, Definition 3.2.2 only makes sense because of the following lemma.

Lemma 3.2.3. *Suppose $\eta \in V'(\mathbf{R})$ is such that $(\eta, m(iv_1 - v_2)) \neq 0$ for all $m \in \mathrm{SO}(V')(\mathbf{R})$. Then $(\eta,\eta) \geq 0$. Conversely, if $\eta \neq 0$ and $(\eta,\eta) \geq 0$ then $(\eta, m(iv_1 - v_2)) \neq 0$ for every $m \in \mathrm{SO}(V')(\mathbf{R})$.*

Proof. The hypothesis $(\eta, m(iv_1 - v_2)) \neq 0$ for every $m \in \mathrm{SO}(V')(\mathbf{R})$ is equivalent to the statement that the projection of η to every positive definite 2-subspace of V' is nonzero. Suppose first that $(\eta,\eta) \geq 0$. Set $\eta' = m^{-1}\eta$. Then $(\eta',\eta') \geq 0$. Thus the projection of η' to $V_2 = \mathrm{Span}\{v_1, v_2\}$ is not 0, because otherwise η' would lie in $V_n = (V_2)^\perp$ which would imply $(\eta',\eta') < 0$.

Conversely, suppose that $(\eta,\eta) < 0$. Then $(\mathbf{R}\eta)^\perp$ contains a positive definite 2-plane $V_2' = mV_2$ for some $m \in \mathrm{SO}(V')(\mathbf{R})$. Then $(\eta, m(iv_1 - v_2)) = 0$, as desired. □

With the above notation, we have the following result.

Theorem 3.2.4. *Suppose that F is a generalized Whittaker function of type η as above. Assume $\eta \neq 0$. If $(\eta, \eta) < 0$, then F is identically 0. Conversely, if $(\eta, \eta) \geq 0$, then $F(t, m) = C\mathcal{W}_\eta(t, m)$ for a constant $C \in \mathbf{C}$.*

Proof. We explain here that the function $\mathcal{W}_\eta$ has the correct $(K \cap M)$-equivariance property. See section A.1 for the rest of the proof.

We have $K \cap M = \mu_2 \times S(O(2) \times O(n))$. Consider the element $\epsilon = \mathrm{diag}(-1, 1, -1)$ in $M \cap K$. Then $\epsilon u_+ = -u_+$ while ϵ acts as the identity on $V_2 = \mathbf{R}v_1 \oplus \mathbf{R}v_2$. Thus ϵ acts on $V_3 \simeq S^2(Y_2)$ as $\epsilon x^2 = x^2$, $\epsilon xy = -xy$ and $\epsilon y^2 = y^2$. It follows that on $\mathbb{V}_\ell = S^{2\ell}(Y_2)$ one has $\epsilon x^{\ell+v} y^{\ell-v} = (-1)^{\ell+v} x^{\ell+v} y^{\ell-v}$. Thus $F_v(x, -t, m) = (-1)^{\ell+v} F_v(x, t, m)$, from which the formula $t^\ell |t|$ follows.

Let us consider the equivariance for the SO(2) part. Normalize the isomorphism $z : \mathrm{SO}(V_2) \simeq S^1$ by $k(v_1 + iv_2) = z(k)(v_1 + iv_2)$. Then $k(v_1 - iv_2) = z(k)^{-1}(v_1 - iv_2)$ and we have $k(x^{\ell+v} y^{\ell-v}) = z(k)^v x^{\ell+v} y^{\ell-v}$. As $F_v(t, mk) = z(k)^{-v} F_v(t, m)$, the $(K \cap M)$-equivariance follows for $k \in \mathrm{SO}(2) \times SO(n)$. For the nontrivial element of $\pi_0(S(O(2) \times O(n)))$, set ϵ' to be any element of $S(O(2) \times O(n))$ with $\epsilon' v_1 = v_1$ and $\epsilon' v_2 = -v_2$. Then, on the one hand, $\epsilon'(x^2) = y^2$, $\epsilon'(y^2) = x^2$ and $\epsilon'(xy) = xy$, from which it follows that $\epsilon'(x^{n+v} y^{n-v}) = x^{n-v} y^{n+v}$. On the other hand,

$$F_v(t, m\epsilon') = t^\ell |t| \left(\frac{|u_\eta(t, m)|}{u_\eta(t, m)^*} \right)^v K_v(|u_\eta(t, m)|) = F_{-v}(t, m),$$

from which the $(K \cap M)$-equivariance follows for this element. □

We now spell out what the generalized Whittaker functions of type η look like when $\eta = 0$. For $k \in \mathrm{SO}(2) \times \mathrm{SO}(n)$, recall that $z(k) \in S^1 \subseteq \mathbf{C}^\times$ is defined by the equality $k(v_1 + iv_2) = z(k)(v_1 + iv_2)$. Additionally, denote by ϵ' an element of $S(O(2) \times O(n)) \subseteq K \cap M$ with $\epsilon' v_1 = v_1$ and $\epsilon' v_2 = -v_2$.

Corollary 3.2.5. *Suppose F is a generalized Whittaker function of type $\eta = 0$. Then $F_v(t, m) = 0$ if $v \notin \{-\ell, 0, \ell\}$. On $M(\mathbf{R})$, one has $F_0(t, m) = \beta t^\ell |t|$, $F_{\pm\ell}(t, m) = |t| F'_{\pm\ell}(m)$ for some constant $\beta \in \mathbf{C}$ and functions $F'_{\pm\ell}(m)$ that are independent of t. The functions $F'_{\pm\ell}(m)$ satisfy $D^M_{iv_1 - v_2, u} F'_\ell(m) = 0$ and $D^M_{iv_1 + v_2, u} F'_{-\ell}(m) = 0$ for all $u \in V_n$. Moreover, $F'_{\pm\ell}(mk) = z(k)^{\mp\ell} F'_{\pm\ell}(m)$ for all $k \in \mathrm{SO}(2) \times \mathrm{SO}(n)$ and $F_{-\ell}(t, m) = F_\ell(t, m\epsilon')$.*

Conversely, suppose $F'_\ell(m)$ satisfies $F'_\ell(mk) = z(k)^{-\ell} F'_\ell(m)$ for all $k \in \mathrm{SO}(2) \times \mathrm{SO}(n)$ and $D^M_{iv_1 + v_2, u} F'_\ell(m) = 0$ for all $u \in V_n$. Define $F_\ell(t, m) = |t| F'_\ell(m)$, $F_{-\ell}(t, m) = F(t, m\epsilon')$, and $F_0(t, m) = \beta t^\ell |t|$ for any constant $\beta \in \mathbf{C}$. Then $F(t, m) = \sum_{-\ell \leq v \leq \ell} F_v(t, m)[x^{\ell+v}][y^{\ell-v}]$ is $(K \cap M)$-equivariant and satisfies the differential equations of Proposition 3.2.1.

Proof. First suppose that $(t, m) \in M(\mathbf{R})^0$, the connected component of the identity. If $-\ell + 1 \leq v \leq \ell - 1$, then we have $(t\partial_t - (\ell + v + 1))F_v = 0$ and $(-t\partial_t + (\ell - v + 1))F_v = 0$.

Adding the equations gives $-2vF_v = 0$, so $F_v = 0$ unless $v = -\ell, 0$, or ℓ. Because $\eta = 0$, we obtain $D^M_{iv_1 \pm v_2, u_j} F_0 = 0$. As in the proof of Theorem 3.2.4, the $(K \cap M)$-equivariance now implies $F_0(t, m) = t^{\ell+1}$ on $M(\mathbf{R})^0$. The formulas for $F_{\pm\ell}(t, m)$ on $M(\mathbf{R})^0$ follow easily. Additionally, the absolute values $|t|$ and the relationship between $F_\ell(t, m)$ and $F_{-\ell}(t, m)$ follow from the $(K \cap M)$-equivariance as in the proof of Theorem 3.2.4.

The converse follows easily, using the formulas for the $(K \cap M)$-action on $\mathbb{V}_\ell$ from the proof of Theorem 3.2.4. □

Below, we will require the following lemma. Denote by $f^1_\ell(g, s)$ the $\mathbb{V}_\ell$-valued, K-equivariant inducing section in $Ind^{G(\mathbf{R})}_{P(\mathbf{R})}(|\nu|^s)$, whose restriction to $M(\mathbf{R})$ is $f^1_\ell((t, m, t^{-1}), s) = |t|^s [x^\ell][y^\ell]$.

Lemma 3.2.6. *Denote by $f^2_\ell(g, s)$ the $\mathbb{V}_\ell$-valued, K-equivariant inducing section in $Ind^{G(\mathbf{R})}_{P(\mathbf{R})}(|\nu|^s)$, whose restriction to $M(\mathbf{R})$ is $|t|^s \left([x^\ell][y^{\ell-1}] \otimes y - [x^{\ell-1}][y^\ell] \otimes x\right) \otimes u^\vee_-$. Then*

$$\sqrt{2} D_\ell f^1_\ell(g, s) = (s - \ell - 1) f^2_\ell(g, s).$$

Proof. From Proposition 3.2.1, on $M(\mathbf{R})^0$ one has

$$\begin{aligned}\sqrt{2} D_\ell f^1_\ell((t, m, t^{-1}), s) &= (t\partial_t - (\ell + 1))(t^s) \left([x^\ell][y^{\ell-1}] \otimes y - [x^{\ell-1}][y^\ell] \otimes x\right) \otimes u^\vee_- \\ &= (s - \ell - 1) t^s \left([x^\ell][y^{\ell-1}] \otimes y - [x^{\ell-1}][y^\ell] \otimes x\right) \otimes u^\vee_-\end{aligned}$$

using that $D^M_{iv_1 \pm v_2, u_j} f^1_\ell((t, m, t^{-1}), s) = 0$ because f^1_ℓ is independent of the variable $m \in \mathrm{SO}(V')(\mathbf{R})$. The lemma follows from the $(K \cap M)$-equivariance. □

4. The Fourier expansion of Eisenstein series

There is a $\mathbb{V}_\ell$-valued degenerate Eisenstein series on G, $E_\ell(g, s)$ associated to the (non-normalized) induction $Ind^G_P(|\nu|^s)$. If ℓ is even, then at $s = \ell + 1$ and for appropriate inducing data, this Eisenstein series is a modular form in sense of subsection 3.1. The purpose of this section is to prove that indeed we get a modular form as above, and to compute the Fourier expansion of this Eisenstein series $E_\ell(g, s = \ell + 1)$ along the unipotent radical N, at least when $\dim(V')$ is a multiple of four and the Eisenstein series is absolutely convergent.

The Eisenstein series $E_\ell(g, s)$ is defined using the inducing section $f_{\ell;\infty}(g, s) := f^1_\ell(g, s)$ of Lemma 3.2.6 at the archimedean place. The computation of its Fourier expansion consists of various parts, which we break into subsections. Let us describe these parts now, before getting into the computation.

To define some terminology, note that the non-constant Fourier coefficients of a modular form φ of weight ℓ are parametrized by $\eta \in V'(\mathbf{R})$, which can be either isotropic or anisotropic. We call the Fourier coefficients corresponding to the nonzero isotropic

η *rank one* Fourier coefficients, while those corresponding to the anisotropic η the *rank two* Fourier coefficients.

(1) By applying Lemma 3.2.6, it is immediate to see that if the Eisenstein series is absolutely convergent at $s = \ell + 1$ (which occurs if $\ell + 1 > \dim(V') = n + 2$), then $E_\ell(g, s = \ell + 1)$ is a modular form of weight ℓ for G.
(2) If the Eisenstein series is not absolutely convergent, then it is not clear–and not necessarily true–that $E_\ell(g, s)$ is a modular form at $s = \ell + 1$. To see when it is, we make various archimedean intertwiner computations in subsection 4.1. Although this is not needed for the Fourier expansion of the absolutely convergent Eisenstein series, it is useful for other applications.
(3) We then compute the constant term of the absolutely convergent Eisenstein series $E_\ell(g, s = \ell + 1)$ in subsection 4.2. Similar to what occurs with the degenerate Heisenberg Eisenstein series considered in [Pol20b], this constant term is a sum of a holomorphic weight ℓ degenerate Eisenstein series on $\mathrm{SO}(V')$ and a constant function.
(4) The rank one Fourier coefficients of the Eisenstein series $E_\ell(g, s = \ell + 1)$ are computed exactly as are the rank one Fourier coefficients of the degenerate Heisenberg Eisenstein series of [Pol20b]. We state the results in subsection 4.3.
(5) The computation of the rank two Fourier coefficients of $E_\ell(g, s = \ell + 1)$ splits into two parts, a finite part and an archimedean part. The finite part can be extracted from the literature (e.g. [Shu95]). For the convenience of the reader, we give this computation in subsection 4.4.
(6) The archimedean part of the computation of the rank two Fourier coefficients of the Eisenstein series $E_\ell(g, s = \ell+1)$ is the main theorem of the paper. This computation is done in subsection 4.5. Denote by w the Weyl group element of G that exchanges the parabolic P with its opposite. Then one has a function on $V'(\mathbf{R})$ given by

$$x \mapsto f_\ell(wn(x); s = \ell + 1).$$

What is computed in subsection 4.5 is the Fourier transform of this function.

We now define the Eisenstein series $E_\ell(g, \Phi_f, s)$ that is the object of what follows. Specifically, suppose Φ_f is a Schwartz-Bruhat function on $V(\mathbf{A}_f)$. For $g_f \in \mathrm{SO}(V)(\mathbf{A}_f)$, define

$$f_{fte}(g_f, \Phi_f, s) = \int\limits_{\mathrm{GL}_1(\mathbf{A}_f)} |t|^s \Phi_f(tg_f^{-1}e)\, dt.$$

Now for $g = g_f g_\infty \in G(\mathbf{A}_f) \times G(\mathbf{R})$, let $f_\ell(g, \Phi_f, s) = f_{fte}(g_f, \Phi_f, s) f_{\ell;\infty}(g_\infty, s)$ and set $E_\ell(g, \Phi_f, s) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f(\gamma g, \Phi_f, s)$ the Eisenstein series. Here recall that $f_{\ell;\infty}(g_\infty, s) := f_\ell^1(g_\infty, s)$ and this latter function was defined before Lemma 3.2.6. When

the Schwartz-Bruhat function Φ_f is $\mathbf{Q}$-valued (or $\overline{\mathbf{Q}}$-valued), these are the Eisenstein series that are the subject of Theorem 1.1.1 and we will prove that the Fourier coefficients of $(2\pi)^{-\ell}E_\ell(g,\Phi_f,s=\ell+1)$ are $\overline{\mathbf{Q}}$-valued.

4.1. Archimedean intertwiners

In this subsection we compute some archimedean intertwining operators. Specifically we compute the intertwining operator

$$M_\infty(w,s)f_{\ell,\infty}(g,s)=\int\limits_{V'(\mathbf{R})} f_{\ell,\infty}(wn(x)g,s)\,dx.$$

This is the content of Proposition 4.1.2 below.

We begin with the following well-known lemma, which computes a spherical Archimedean intertwiner on the groups $\mathrm{SO}(N,1)$.

Lemma 4.1.1. *Suppose U is a positive definite quadratic space, and $V_1=H\oplus U=\mathbf{R}e_1\oplus U\oplus\mathbf{R}f_1$ is the orthogonal direct sum of U and a hyperbolic plane $H=\mathbf{R}e_1\oplus\mathbf{R}f_1$. Denote by ι_1 the involution on V_1 defined as $\iota_1(\alpha e_1+v+\beta f_1)=\beta e_1+v+\alpha f_1$, and K_1 the maximal compact subgroup of $G_1=\mathrm{SO}(V_1)$ that commutes with ι_1. Set $P_1=M_1N_1$ the parabolic subgroup of $\mathrm{SO}(V_1)$ that fixes the line $\mathbf{R}f_1$ via a right-action of $\mathrm{SO}(V_1)$ on V_1 and define $\nu:P_1\to\mathrm{GL}_1$ as $f_1p=\nu(p)^{-1}f_1$. Let $f_A(g,s)\in Ind_{P_1}^{G_1}(|\nu|^s)$ be the K_1-spherical inducing section and $n_1:U\simeq N_1$ the identification of U with the unipotent radical of P_1. Then the intertwiner*

$$\int\limits_{U(\mathbf{R})} f_A(\iota_1 n_1(x)g,s)\,dx=c(s)f_A(g,\dim U-s)$$

where $c(s)$ is a nonzero constant times $\frac{\Gamma(s-\dim(U)/2)}{\Gamma(s)}$.

Proof. Although, as mentioned, this lemma is surely well-known, we sketch a proof for the convenience of the reader. Let $(\cdot,\cdot)_1$ denote the quadratic form on V_1. Define $||v||^2=(v,\iota_1(v))_1$ for $v\in V_1$, and set $\Phi_\infty(v)=e^{-||v||^2}$ a Schwartz function on V_1. Now,

$$f_A(g,s)=\int\limits_{\mathrm{GL}_1(\mathbf{R})}|t|^s\Phi_\infty(t(0,0,1)g)\,dt$$

defines a K_1-spherical section in $Ind_{P_1(\mathbf{R})}^{G_1(\mathbf{R})}(|\nu|^s)$ with $f_A(1,s)=\Gamma(s/2)$. Thus we can compute the c-function using this section $f_A(g,s)$.

We obtain

$$\int\limits_{U(\mathbf{R})} f_A(\iota_1 n_1(x)1, s)\,dx = \int\limits_{\mathrm{GL}_1(\mathbf{R})} \int\limits_{U(\mathbf{R})} |t|^s e^{-t^2||(1,x,-||x||^2/2)||^2}\,dt\,dx$$
$$= \Gamma(s/2) \int\limits_{U(\mathbf{R})} \frac{dx}{(1+||x||^2/2)^s}.$$

Thus

$$c(s) = \int\limits_{U(\mathbf{R})} \frac{dx}{(1+||x||^2/2)^s}$$
$$\doteq \int\limits_0^\infty u^{\dim U/2}(1+u)^{-s}\frac{du}{u}$$

where the $\doteq$ means up to a nonzero constant. This last integral is easily computed to be a nonzero constant times $\frac{\Gamma(s-\dim(U)/2)}{\Gamma(s)}$. $\square$

Applying Lemma 4.1.1, we can now compute $M(w,s)f_{\ell,\infty}$. For $z \in \mathbf{C}$ and $k \geq 0$ an integer, let $(z)_k = (z)(z+1)\cdots(z+k-1)$ denote the Pochhammer symbol.

Proposition 4.1.2. *Suppose*

$$V = \mathbf{R}e_1 \oplus \mathbf{R}e_2 \oplus \mathbf{R}e_3 \oplus U \oplus \mathbf{R}f_3 \oplus \mathbf{R}f_2 \oplus \mathbf{R}f_1 = \mathbf{R}e_1 \oplus V' \oplus \mathbf{R}f_1$$

with U negative definite of dimension m and e_i, f_j isotropic with $(e_i, f_j) = \delta_{ij}$. Denote by $P = MN$ the parabolic stabilizing $\mathbf{R}e_1$ for the left action of $\mathrm{SO}(V)$ on V and $\nu : P \to \mathrm{GL}_1$ the character defined by $pe_1 = \nu(p)e_1$. Suppose $w \in \mathrm{SO}(V)$ is defined by $we_1 = f_1$, $wf_1 = e_1$ and w is the identity of V'. Denote by K the maximal compact subgroup of $G = \mathrm{SO}(V)$ that commutes with the involution ι that exchanges e_i with f_i and is the identity on U. Suppose $f_{\ell,\infty}(g,s)$ is the K-equivariant, $\mathbb{V}_\ell$-valued section in $Ind_{P(\mathbf{R})}^{G(\mathbf{R})}(|\nu|^s)$ with $f_{\ell,\infty}((t,m,t^{-1}),s) = |t|^s x^\ell y^\ell$. Then

$$\int\limits_{N(\mathbf{R})} f_{\ell,\infty}(wng, s)\,dn = c_\ell^{B_3}(s) f_{\ell,\infty}(g, 4+m-s)$$

where

$$c_\ell^{B_3}(s) = \frac{\left(\frac{s-\ell-1}{2}\right)_{\ell/2}}{\left(\frac{s-2}{2}\right)_{\ell/2+1}} \cdot \frac{\Gamma\left(s-2-\frac{m}{2}\right)}{\Gamma(s-2)} \cdot \frac{\left(\frac{s-2-m-\ell}{2}\right)_{\ell/2}}{\left(\frac{s-3-m}{2}\right)_{\ell/2+1}}$$

up to exponential factors and nonzero constants. Consequently, when $\ell > 2+m$, $c_\ell^{B_3}(s)$ is finite and 0 at $s = \ell+1$.

Proof. Let w_{12} denote the element of $\mathrm{SO}(V)$ that exchanges e_1 with e_2, f_1 with f_2 and is the identity on $\mathrm{Span}(e_1, e_2, f_1, f_2)^{\perp}$. Similarly define w_{23}, and let w_3 denote the Weyl element that exchanges e_3 with f_3 is the identity on $\mathrm{Span}(e_3, f_3)^{\perp}$. With this notation, the element w factorizes as $w_{12}w_{23}w_3w_{23}w_{12}$.

Denote by r_1, r_2, r_3 the absolute values of the characters of the split torus, so that

$$r_j(\mathrm{diag}(t_1, t_2, t_3, 1, t_3^{-1}, t_2^{-1}, t_1^{-1})) = |t_j|.$$

With P_0 the upper-triangular minimal parabolic, we have $\delta_{P_0} = (m+4)r_1 + (m+2)r_2 + mr_3$, so that $f_{\ell,\infty}(g,s) \in Ind_P^G(\delta_{P_0}^{1/2}\lambda_s)$ with $\lambda_s = (s-2-m/2)r_1 - (1+m/2)r_2 - (m/2)r_3$.

The intertwining operator $M(w) = M(w_{12})M(w_{23})M(w_3)M(w_{23})M(w_{12})$ moves around the induction spaces as follows:

- $\lambda_s = (s-2-m/2)r_1 - (1+m/2)r_2 - (m/2)r_3$
- $\overset{w_{12}}{\mapsto} -((2+m)/2)r_1 + (s-(4+m)/2)r_2 + -(m/2)r_3$
- $\overset{w_{23}}{\mapsto} -((2+m)/2)r_1 - (m/2)r_2 + (s-(4+m)/2)r_3$
- $\overset{w_{e_3}}{\mapsto} -((2+m)/2)r_1 - (m/2)r_2 + ((4+m)/2-s)r_3$
- $\overset{w_{23}}{\mapsto} -((2+m)/2)r_1 + ((4+m)/2-s)r_2 - (m/2)r_3$
- $\overset{w_{12}}{\mapsto} ((4+m)/2-s)r_1 - ((2+m)/2)r_2 - (m/2)r_3$
- $= \lambda_{4+m-s}$.

Now applying $M(w,s)$ to the section $f_{\ell,\infty}(g,s)$ one obtains

$$M(w,s)f_{\ell,\infty}(g,s) = M(w_{12}w_{23}) \circ M(w_3) \circ M(w_{23}w_{12})f_{\ell,\infty}(g,s).$$

Proposition 4.1.4 below computes the two outer intertwining operators $M(w_{12}w_{23})$ and $M(w_{23}w_{12})$. Lemma 4.1.1 computes the inner intertwining operator $M(w_3)$. Putting these results together gives that, up to exponential factors and nonzero constants,

$$c_\ell^{B_3}(s) = \frac{\left(\frac{s-\ell-1}{2}\right)_{\ell/2}}{\left(\frac{s-2}{2}\right)_{\ell/2+1}} \cdot \frac{\Gamma\left(s-2-\frac{m}{2}\right)}{\Gamma(s-2)} \cdot \frac{\left(\frac{s-2-m-\ell}{2}\right)_{\ell/2}}{\left(\frac{s-3-m}{2}\right)_{\ell/2+1}}.$$

The proposition follows. □

Remark 4.1.3. In section 6 below, we will apply Proposition 4.1.2 in the following special case: $\ell = 8$ and $m = 8$. We note now that for these values, $c(s)$ is finite and nonzero at $s = 9$:

$$c_\ell^{B_3}(s) \overset{\ell=8,m=8}{=} \frac{\left(\frac{s-9}{2}\right)_4}{\left(\frac{s-2}{2}\right)_5} \cdot \frac{\Gamma(s-6)}{\Gamma(s-2)} \cdot \frac{\left(\frac{s-18}{2}\right)_4}{\left(\frac{s-11}{2}\right)_5}.$$

As used in the proof of the above proposition, we require the computation of a certain length two intertwiner of an archimedean inducing section on SL_3. This computation is

done in Proposition 4.1.4 below. To set up the proposition, let b_1, b_2, b_3 be the standard basis of $\mathbf{R}^3$, thought of as column vectors. Let x, y be the standard basis of the two-dimensional representation of $\mathrm{SL}_2(\mathbf{C})$, so that x^2, xy, y^2 are a basis of the 3-dimensional representation of $K' = \mathrm{SO}(3)$. We identify $b_2 + ib_3 = x^2$, $ib_1 = xy$ and $b_2 - ib_3 = y^2$. Set $f_1 = x + y$ and $f_2 = x - y$; this abuse of notation should not cause the reader any confusion.

Now suppose $\ell \geq 0$ is even and $f'_\ell(g, s) : \mathrm{SL}_3(\mathbf{R}) \to \mathbb{V}_\ell$ is the section satisfying

(1) $f'_\ell(gk, s) = k^{-1} \cdot f'_\ell(g, s)$ for all $k \in K' = \mathrm{SO}(3)$, $g \in \mathrm{SL}_3(\mathbf{R})$;
(2) $f'_\ell(pg, s) = \chi_s(p) f'_\ell(g, s)$, where $p = \left(\begin{smallmatrix} m_1 & * \\ 0 & m_2 \end{smallmatrix}\right) \in P_{1,2}$ in $(1, 2)$ block form and

$$\chi_s(p) = |m_1|^s = |\det(m_2)|^{-s} = |m_1^2/\det(m_2)|^{s/3};$$

(3) $f'_\ell(1, s) = x^\ell y^\ell$.

Let s_{12} and s_{23} in SL_3 be the Weyl group elements corresponding to the two simple roots, in obvious notation. We compute the intertwiner $M(s_{23}) \circ M(s_{12}) f'_\ell(g, s)$.

Proposition 4.1.4. *Denote by $f''_\ell(g, s)$ the inducing section satisfying the first two enumerated properties above, but with P_{12} replaced with P_{21} and $f''_\ell(1, s) = f_1^\ell f_2^\ell$. Then*

$$M(s_{23}) \circ M(s_{12}) f'_\ell(g, s) = C_\ell(s) f''_\ell(g, s - 3) \tag{3}$$

with

$$C_\ell(s) = \frac{(\frac{s-\ell-1}{2})_{\ell/2}}{(s/2 - 1)_{\ell/2+1}} = \frac{\Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2} - 1\right)}{\Gamma\left(\frac{s-\ell-1}{2}\right) \Gamma\left(\frac{s+\ell}{2}\right)}$$

up to exponential factors and nonzero constants.

Proof. We begin by constructing the inducing section $f'_\ell(g, s)$ explicitly. Throughout, we compute up to nonzero scalars.

Let $\Phi_\ell : \mathbf{R}^3 \to \mathbb{V}_\ell$ be given by $\Phi_\ell(v) = v^\ell e^{-||v||^2}$, where we consider $v \in \mathbb{V}_1$ and v^ℓ in the quotient $\mathbb{V}_\ell$ of the ℓ-th symmetric power of $\mathbb{V}_1$. Then

$$f'_\ell(g, s) = \frac{1}{\Gamma((s + \ell)/2)} \int\limits_{\mathrm{GL}_1(\mathbf{R})} |t|^s \Phi_\ell(tg^{-1} b_1)\, dt.$$

One checks easily that $M(s_{23}) \circ M(s_{12}) f'_\ell(g, s)$ is K'-equivariant and lands in the induction space as specified in the statement of the proposition. Thus it suffices to compute this intertwiner when $g = 1$. This, then, is computed by

$$\Gamma((s+\ell)/2) M(s_{23}) \circ M(s_{12}) f'_\ell(1, s) = \int\limits_{\mathbf{R}^\times \times \mathbf{R}^2} |t|^{s+\ell} (ub_1 + vb_2 + b_3)^\ell e^{-(u^2+v^2+1)t^2}\, dt\, du\, dv.$$

Under the change of variables indicated above,

$$\begin{aligned} ub_1 + vb_2 + b_3 &= \frac{1}{2}\left((v-i)x^2 + 2iuxy + (v+i)y^2\right) \\ &= zf_1^2 - 2if_1f_2 + z^*f_2^2 \end{aligned}$$

where $z = v + iu$. One obtains that

$$C_\ell(s) = \int\limits_{\mathbf{C}} \frac{(zf_1^2 - 2if_1f_2 + z^*f_2^2)^\ell}{(|z|^2+1)^{(\ell+s)/2}}\, dz.$$

Because of the $S^1 \subseteq \mathbf{C}^\times$ symmetry of the domain of integration, only the coefficient of $f_1^\ell f_2^\ell$ contributes. This coefficient is immediately seen to be

$$\sum_{0 \le k \le \ell/2} \frac{\ell!}{k!k!(\ell-2k)!} z^k (z^*)^k (-2i)^{\ell-2k}.$$

Now

$$\begin{aligned} \int\limits_{\mathbf{C}} \frac{|z|^{2k}}{(|z|^2+1)^{(\ell+s)/2}}\, dz &= 2\pi \int\limits_0^\infty \frac{r^{2k+1}}{(r^2+1)^{(\ell+s)/2}}\, dr \\ &\doteq \frac{\Gamma(k+1)\Gamma((\ell+s)/2 - k - 1)}{\Gamma((\ell+s)/2)} \end{aligned}$$

where the implied constant in the $\doteq$ is independent of k.

Summing up, we have proved (3) with

$$\begin{aligned} C_\ell(s) &= \sum_{0 \le k \le \ell/2} 2^{\ell-2k}(-1)^k \frac{\ell!}{k!k!(\ell-2k)!} \frac{\Gamma(k+1)\Gamma((\ell+s)/2-k-1)}{\Gamma((\ell+s)/2)} \\ &= \sum_{0 \le k \le \ell/2} 2^{\ell-2k}(-1)^k \frac{\ell!}{k!(\ell-2k)!} \frac{1}{\left(\frac{\ell+s}{2} - k - 1\right)_{k+1}} \\ &= \frac{1}{(s/2-1)_{\ell/2+1}} \sum_{0 \le j \le \ell/2} (-4)^j \frac{\ell!}{(2j)!(\ell/2-j)!} (s/2-1)_j \end{aligned}$$

making the substitution $j = \ell/2 - k$. The proposition thus follows from the following lemma. □

Lemma 4.1.5. *One has*

$$D_\ell(s) := \sum_{0 \le j \le \ell/2} (-4)^j \frac{\ell!}{(2j)!(\ell/2-j)!} (s/2-1)_j = 2^\ell \left(\frac{s-\ell-1}{2}\right)_{\ell/2}.$$

Proof. First,

$$\sum_{0\leq j\leq \ell/2}(-4)^j\frac{\ell!}{(2j)!(\ell/2-j)!}(s/2-1)_j$$
$$=(-4)^{\ell/2}\sum_{0\leq j\leq \ell/2}\binom{\ell}{2j}\left(\frac{1}{2}\right)_{\ell/2-j}(-1)^{\ell/2-j}(s/2-1)_j.$$

Now

$$\binom{\ell}{2j}=\binom{\ell/2}{j}\frac{\left(\frac{1}{2}\right)_{\ell/2}}{\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_{\ell/2-j}}.$$

Moreover

$$\frac{\left(\frac{1}{2}\right)_{\ell/2}(-1)^{\ell/2-j}}{\left(\frac{1}{2}\right)_j}=\left(\frac{1-\ell}{2}\right)_{\ell/2-j}.$$

Thus

$$D_\ell(s)=2^\ell\sum_{0\leq j\leq \ell/2}\binom{\ell/2}{j}(s/2-1)_j((1-\ell)/2)_{\ell/2-j}.$$

By the binomial property $(a+b)_n=\sum_{0\leq k\leq n}\binom{n}{k}(a)_k(b)_{n-k}$ of the Pochhammer symbol, the lemma follows. □

4.2. Constant term

As mentioned above, we begin with the computation of the constant term of $E_\ell(g,\Phi_f,s)$ along N. For general s, there are three terms: $f_\ell(g,\Phi_f,s)$, an Eisenstein series $E_\ell^M(g,\Phi_f,s)$ on the Levi subgroup M, and an intertwined inducing section $M(w,s)f_\ell(g,\Phi_f,s)$.

We will see that at $s=\ell+1$ (in the range of absolute convergence) the intertwined inducing section $M(w,s)f_\ell$ vanishes and that the Eisenstein series $E_\ell^M(g,\Phi_f,s=\ell+1)$ is the automorphic function associated to a holomorphic weight ℓ modular form on $\mathrm{SO}(V')$.

Let us first handle the intertwining operator. Denote by w the element of $\mathrm{SO}(V)$ that exchanges e with f and is the identity on V'. The intertwining operator is

$$M(w,s)f_\ell(g,s)=\int\limits_{V'(\mathbf{A})}f_\ell(wn(x)g,s)\,dx.$$

Lemma 4.2.1. *Suppose ℓ is even and $\ell>n+1$. Then $M(w,s)f_\ell(g,s)$ vanishes at $s=\ell+1$.*

Proof. The integral is absolutely convergent, so it suffices to see that the archimedean intertwiner

$$M_\infty(w,s)f_{\ell,\infty}(g,s) = \int\limits_{V'(\mathbf{R})} f_{\ell,\infty}(wn(x)g,s)\,dx$$

vanishes at $s=\ell+1$. This follows from Proposition 4.1.2. □

The other nontrivial piece of the constant term is an Eisenstein series on the Levi subgroup M, $E_\ell^M(g,\Phi_f,s=\ell+1)$ associated to a new inducing section $f'_M(g,\Phi_f,s)$ on M. This inducing section is defined as follows. Let b_0 be an isotropic vector in V'. The section $f'_M(g,\Phi_f,s)$ is given by an integral

$$f'_M(g,\Phi_f,s) = \int\limits_{((b_0)^\perp\backslash V')(\mathbf{A})} f_\ell(w_0n(x)g,s)\,dx$$

where $fw_0=b_0$ using the right action of G on V. Write $f'_{M,\infty}(g,s)$ for the corresponding archimedean inducing section, so that

$$f'_{M,\infty}(g,s) = \int\limits_{((b_0)^\perp\backslash V')(\mathbf{R})} f_{\ell,\infty}(w_0n(x)g,s)\,dx.$$

Denote by P' the parabolic subgroup of M that fixes $\mathbf{Q}b_0$ via this right action. With $f'_M(g,\Phi_f,s)$ defined as above, $E_\ell^M(g,\Phi_f,s)=\sum_{\gamma\in P'(\mathbf{Q})\backslash M(\mathbf{Q})} f'_M(\gamma g,\Phi_f,s)$.

Regarding this Eisenstein series, one has the following proposition. For $m\in \mathrm{SO}(V')(\mathbf{R})$, set

$$f_{hol,\ell}(m,s) = \frac{(b_0m, v_1+iv_2)^\ell}{||b_0m||^{s+\ell}}.$$

Note that at $s=\ell$, $f_{hol,\ell}$ is the inducing section for the automorphic function associated to a holomorphic weight ℓ Eisenstein series on $\mathrm{SO}(V')$.

Proposition 4.2.2. *Let the notation be as above.*

(1) *One has*

$$f'_{M,\infty}(\mathrm{diag}(t,m,t^{-1}),s=\ell+1) \doteq |t|\left(f_{hol,\ell}(m,s=\ell)x^{2\ell} + f_{hol,\ell}(m\epsilon',s=\ell)y^{2\ell}\right).$$

(2) *Suppose that Φ_f is $\mathbf{Q}$-valued. The automorphic function $\pi^{-\ell}E_\ell^M(g,\Phi_f,s=\ell+1)$ corresponds to a holomorphic modular form on $\mathrm{SO}(V')$ of weight ℓ with algebraic Fourier coefficients.*

Proof. The proof of the first part follows exactly as the proof of Proposition 3.3.2 of [Pol20b]. Note that the equality here is only true at the special values of s indicated; it is not true for general s.

Keeping track of the constants, the second part follows from the first, using the fact the absolutely convergent holomorphic Eisenstein series associated to a $\overline{\mathbf{Q}}$-valued inducing section has algebraic Fourier coefficients. See, e.g., [Shi82] or [Shu95]. □

4.3. Rank one Fourier coefficients

In this subsection, we prove that the rank one Fourier coefficients of $\pi^{-\ell}E_\ell(g, \Phi_f, s = \ell+1)$ are algebraic numbers. We also prove that certain of these Eisenstein series have rational Fourier coefficients. As the argument and computation is identical to the calculation of the rank one Fourier coefficients of the degenerate Heisenberg Eisenstein series of [Pol20b], we are very brief.

We require the following definition. Recall the M-invariant decomposition $V = \mathbf{Q}e \oplus V' \oplus \mathbf{Q}f$. We say a Schwartz-Bruhat function Φ_f on $V(\mathbf{A}_f)$ is *block-tensorial* if the following conditions are satisfied:

(1) There exists Schwartz-Bruhat functions Φ^e, Φ^f on $\mathbf{A}_f$ and Φ' on $V'(\mathbf{A}_f)$ so that $\Phi_f(\alpha e + v' + \beta f) = \Phi^e(\alpha)\Phi'(v')\Phi^f(\beta)$, where $v' \in V'(\mathbf{A}_f)$ and $\alpha, \beta \in \mathbf{A}_f$.
(2) The functions $\Phi^?$ with $? \in \{e, f, '\}$ satisfy $\Phi^?(\mu x) = \Phi^?(x)$ for all $\mu \in \widehat{\mathbf{Z}}^\times$.

Note that the condition is invariant under translation by $M(\mathbf{A}_f)$.

Here is the result.

Proposition 4.3.1. *Suppose $\ell > n+1$ so that the Eisenstein series $E_\ell(g, \Phi_f, s)$ is absolutely convergent at $s = \ell+1$ and that the Schwartz-Bruhat function Φ_f is $\overline{\mathbf{Q}}$-valued. Then the rank one Fourier coefficients of $\pi^{-\ell}E_\ell(g, \Phi_f, s = \ell+1)$ are $\overline{\mathbf{Q}}$-valued. If moreover Φ_f is $\mathbf{Q}$-valued and block-tensorial, then the rank one Fourier coefficients of $\pi^{-\ell}E_\ell(g, \Phi_f, s = \ell+1)$ are $\mathbf{Q}$-valued.*

Proof. The first step of the proof is the fact that these rank one Fourier coefficients of $E_\ell(g, \Phi_f, s = \ell+1)$ are Euler products, given by an integral

$$\int\limits_{V'(\mathbf{Q})\backslash V'(\mathbf{A})} \psi((\eta, x))E_\ell(g, \Phi_f, s = \ell+1)\, dx$$
$$= \int\limits_{(\eta)^\perp(\mathbf{A})\backslash V'(\mathbf{A})} \psi((\eta, x))f(\gamma_\eta n(x)g, \Phi_f, s = \ell+1)\, dx \tag{4}$$

if $\eta \neq 0$ is isotropic. Here $\gamma_\eta \in G(\mathbf{Q})$ satisfies $f\gamma_\eta = \eta \in V'(\mathbf{Q})$.

To prove (4), one computes that in the range of absolute convergence, the left-hand side is equal to a sum of two terms: the term appearing on the right of (4) and an integral

$$\int\limits_{V'(\mathbf{A})} \psi((\eta, x)) f(wn(x)g, \Phi_f, s)\, dx. \tag{5}$$

The content of (4) is that (5) vanishes at $s = \ell + 1$ if η is isotropic. To see this, note that the integral is absolutely convergent, so it suffices to see that the archimedean integral vanishes at $s = \ell + 1$ for such an η. This vanishing could be obtained by the arguments used to prove Theorem 3.2.5 in [Pol20b]. However, it also follows immediately from Corollary 4.5.8 below, so we omit this argument.

The archimedean and unramified local integrals that arise from the right-hand side of (4) are computed just as in section 3.4 of [Pol20b]. In this case, the unramified finite adelic integral gives a rational number, equal to 1 almost everywhere, and the archimedean integral using $f_{\ell,\infty}(g, \ell; s)$ gives the π^ℓ. Finally, the finitely many "bad" local integrals at the finite places give algebraic numbers.

We now sketch how to prove these facts about the local integrals

$$\int\limits_{(\eta)^\perp(\mathbf{Q}_p)\backslash V'(\mathbf{Q}_p)} \psi((\eta, x)) f(\gamma_\eta n(x)g, \Phi_p, s = \ell + 1)\, dx. \tag{6}$$

Plugging in the definition of $f(g, \Phi_p, s)$, we obtain that (6) is equal to

$$\int\limits_{\mathbf{Q}_p^\times} \int\limits_{(\eta)^\perp(\mathbf{Q}_p)\backslash V'(\mathbf{Q}_p)} \psi((\eta, x))|t|^{\ell+1} \Phi_p(t\eta + t(\eta, x)f)\, dx\, dt. \tag{7}$$

We can identify $(\eta)^\perp(\mathbf{Q}_p)\backslash V'(\mathbf{Q}_p)$ with $\mathbf{Q}_p$ via the map $x \mapsto (\eta, x)$. Thus (7) is equal to

$$\int\limits_{\mathbf{Q}_p^\times} \int\limits_{\mathbf{Q}_p} \psi(y/t)\Phi_p(t\eta + yf)|t|^\ell\, dt\, dy. \tag{8}$$

The function $(t, y) \mapsto \Phi_p(t\eta + yf)$ is a Schwartz-Bruhat function $\Phi'_p(t, y)$ on $\mathbf{Q}_p^2$, so the evaluation of (8) is the same calculation that one does to compute of the Fourier coefficients of Eisenstein series on GL_2. For the reader's convenience, we now explain how to analyze (8). First, without loss of generality, we can assume $\Phi'_p(t, y)$ is a pure tensor, so that $\Phi'_p(t, y) = \Phi_1(t)\Phi_2(y)$, with Φ_1, Φ_2 $\mathbf{Q}$-valued if Φ_p is $\mathbf{Q}$-valued. Then (8) becomes

$$\int\limits_{\mathbf{Q}_p^\times} |t|^\ell \Phi_1(t)\widehat{\Phi_2}(1/t)\, dt, \tag{9}$$

where $\widehat{\Phi_2}$ denotes the Fourier transform of Φ_2; note that it is valued in $\overline{\mathbf{Q}}$ if Φ_2 is $\overline{\mathbf{Q}}$-valued, and valued in the cyclotomic extension of $\mathbf{Q}$ if Φ_2 is $\mathbf{Q}$-valued. Because $\widehat{\Phi_2}$ is a Schwartz-Bruhat function, the integral (9) is actually a finite sum, and equal to 1 almost everywhere. This proves the desired facts about the local integral (6).

For the rationality statement, note that we know from above that the local integrals (6) are equal to 1 almost everywhere are valued in the cyclotomic extension of $\mathbf{Q}$ at finitely many places, using that Φ_p is $\mathbf{Q}$-valued. We claim that these local integrals are in fact valued in $\mathbf{Q}$. To see this, one applies an element σ of the Galois group $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ to (6). One obtains an integral

$$\int\limits_{(\eta)^{\perp}(\mathbf{Q}_p)\backslash V'(\mathbf{Q}_p)} \psi(\mu(\eta,x))f(\gamma_\eta n(x)g,\Phi_p,s=\ell+1)\,dx \tag{10}$$

where μ is the value of the p-adic cyclotomic character of σ. Now, making a change of variables $x \mapsto \mu^{-1}x$ in (10), one arrives at (6) because Φ_f is block-tensorial. This completes the proof. □

4.4. Rank two Fourier coefficients: finite part

In this subsection we do the finite part of the calculation of the rank two Fourier coefficients of our Eisenstein series. The result of this calculation is well-known; it can be extracted from [Shu95]. We briefly give the computation for the convenience of the reader. Throughout this subsection, F is a local non-archimedean field with ring of integers $\mathcal{O}$, uniformizer p, and $|\mathcal{O}/p| = q$.

The local section for the Eisenstein series is

$$\int\limits_{\mathrm{GL}_1(F)} |t|^s \Phi_p(t(0,0,1)g)\,dt.$$

Here Φ_p is a Schwartz-Bruhat function on $V(F)$ and $V(F) = Fe \oplus V' \oplus Ff$. For the rank two Fourier coefficients, the integral that must be calculated is

$$J(s,\eta,\Phi_p) := \int\limits_{\mathrm{GL}_1(F)} \int\limits_{V'(F)} \psi((\eta,x))\Phi_p(t(1,x,-q'(x))|t|^s\,dt\,dx.$$

Here η is a rank two element of V', i.e. $(\eta,\eta) \neq 0$.

Assume that Φ_p is unramified, i.e., that Φ_p is the characteristic function of the lattice $\mathcal{O}e \oplus V'(\mathcal{O}) \oplus \mathcal{O}f$, where $V'(\mathcal{O})$ is such that $V'(\mathcal{O}/p)$ is a non-degenerate split quadratic space over $\mathcal{O}/p$. Breaking into pieces as determined by the valuation of t, one obtains

$$J(s,\eta,\Phi_p) = \sum_{r\geq 0} |p|^{rs} \left(\int\limits_{p^{-r}V'(\mathcal{O})} \psi((\eta,x))\,\mathrm{char}(p^r q'(x) \in \mathcal{O})\,dx \right).$$

In this unramified case, we will check that the terms with $r \geq 2$ vanish and will calculate the $r = 1$ term explicitly.

The vanishing of the terms with $r \geq 2$ follows from the following lemma. If $\eta \in V'(F)$, we say that η is unramified if $\eta \in V'(\mathcal{O})$ and $q'(\eta) \in \mathcal{O}^\times$.

Lemma 4.4.1. *Suppose $r \geq 2$, $V'(\mathcal{O}/p)$ is a non-degenerate quadratic space and η is unramified. Then*

$$\sum_{x \in V'(\mathcal{O}/p^r), q'(x) \equiv 0 \pmod{p^r}} \psi\left(\frac{(\eta, x)}{p^r}\right) = 0.$$

Proof. The idea is to consider together all the x with fixed reduction in $V'(\mathcal{O}/p^{r-1})$. Specifically, suppose $x \in V'(\mathcal{O}/p^r)$, $q'(x) \equiv 0 \pmod{p^r}$. Consider $x + p^{r-1}\epsilon$ for some $\epsilon \in V'(\mathcal{O})$. Then $\frac{1}{p^r} q'(x + p^{r-1}\epsilon) = \frac{1}{p^r}(q'(x) + p^{r-1}(x, \epsilon) + p^{2r-2}q'(\epsilon))$. As $r \geq 2$, $2r - 2 \geq r$ so this is in $\mathcal{O}$ if and only if $(x, \epsilon) \in p\mathcal{O}$. The point is that for $x \in V'(\mathcal{O}/p)$ fixed with $q'(x) \equiv 0$, there is ϵ with $(x, \epsilon) \in p\mathcal{O}$ and $(\eta, \epsilon) \in \mathcal{O}^\times$. Indeed, if $(\eta, \epsilon) \equiv 0$ whenever $(x, \epsilon) \equiv 0$, then x and η would be $\mathcal{O}^\times$ proportional in $V'(\mathcal{O}/p)$. But $q'(\eta) \not\equiv 0$ while $q'(x) \equiv 0$ by assumption, so such an ϵ can be found. Perturbing the sum of those terms with reduction x by ϵ, one gets 0, as desired. □

For the $r = 1$ term, we begin with the following lemma. Let U_n be the split quadratic space over $\mathcal{O}$, i.e., $U_n = \mathcal{O}^{2n}$ with quadratic form $q_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n$.

Lemma 4.4.2. *Denote by $C(n)$ the number of elements u of $U_n(\mathcal{O})/p$ with $q_n(u) \equiv 0$. Then $C(1) = 2q - 1$ and $C(n + 1) = qC(n) + (q - 1)q^{2n}$.*

Proof. The formula for $C(1)$ is clear. As for the recurrence relation, note that the elements in $U_{n+1}(\mathcal{O}/p)$ with $q_{n+1} \equiv 0$ are either of the form $(0, u_n, *)$ with $q_n(u_n) \equiv 0$ or $(*, u_n, y_1)$ with y_1 determined. The recurrence follows. □

We can now calculate the $r = 1$ term in case V' is split, even dimensional by induction on n. Define

$$S_n = \sum_{v \in U_n(\mathcal{O}/p), q(v) \equiv 0 \pmod{p}} \psi((\eta, v)/p).$$

Lemma 4.4.3. *Suppose $\eta = (1, 0, \ldots, 0, 1)$. Then $S_{n+1} = -q^n$.*

Proof. First we claim that

$$S_{n+1} = -\sum_{v \in U_n(\mathcal{O}/p)} \psi(-q(v)/p). \tag{11}$$

To see this, note that if $v \in U_{n+1}(\mathcal{O}/p)$ with $q(v) \equiv 0$, then either $v = (0, v', *)$ with $q'(v') \equiv 0$ or $v = (x_1, v', -x_1^{-1}q(v'))$ with $x_1 \in \mathcal{O}^\times$. Summing over the first set of v's gives 0, because $\sum_{y_1 \in \mathcal{O}/p} \psi(y_1/p) = 0$. Summing over the second set of v's gives

$$\sum_{x_1 \in (\mathcal{O}/p)^\times, v' \in U_n} \psi((x_1 - x_1^{-1}q(v'))/p).$$

But note that

$$\sum_{v' \in U_n(\mathcal{O}/p)} \psi(q(v')/p) = \sum_{v' \in U_n(\mathcal{O}/p)} \psi(\alpha q(v')/p)$$

for any $\alpha \in \mathcal{O}^\times$, because the split quadratic form q takes all values. Thus

$$S_{n+1} = \left(\sum_{x_1 \in (\mathcal{O}/p)^\times} \psi(x_1/p) \right) \left(\sum_{v' \in U_n(\mathcal{O}/p)} \psi(-q(v')/p) \right)$$

which gives (11).

From (11), one can calculate S_{n+1} in terms of $C(n)$, by breaking the sum up into those v with $q(v) \equiv 0$ and those v with $q(v) \not\equiv 0$. One obtains

$$S_{n+1} = C(n) + (-1)\left(\frac{q^{2n} - C(n)}{q-1}\right) = -\frac{q}{q-1}(C(n) - q^{2n-1}).$$

But now by Lemma 4.4.2 one obtains

$$\frac{C(n+1) - q^{2n+1}}{q-1} = q\left(\frac{C(n) - q^{2n-1}}{q-1}\right)$$

so that $S_{n+1} = qS_n$. The lemma follows. □

Putting everything together, we arrive at the following proposition.

Proposition 4.4.4. *Suppose* $\dim(V)$ *is even,* η *is unramified,* $V'(\mathcal{O}/p)$ *is a non-degenerate quadratic space, and* Φ_p *is unramified. Then*

$$J(s, \eta, \Phi_p) = 1 - |p|^{s - \frac{\dim V'}{2} + 1} = \frac{1}{\zeta_p(s - \dim(V')/2 + 1)}.$$

Consequently, if ℓ is even and 4 divides $\dim(V')$ then the product of the unramified factors at $s = \ell + 1$ gives $\pi^{-(\ell + 2 - \dim(V')/2)}$ times a rational number.

We also must understand what happens at the bad finite places:

Lemma 4.4.5. *Suppose that Φ_p is $\mathbf{Q}$-valued, and $f(g,\Phi_p,s)=\int_{\mathrm{GL}_1(\mathbf{Q}_p)}|t|^s\Phi_p(t(0,0,1)g)\,dt$. Then*

$$\int\limits_{V'(\mathbf{Q}_p)} f(w_\ell n(x)g,\Phi_p,s)\psi((\eta,x))\,dx$$

is finite and valued in $\mathbf{Q}(\psi_p)$ at $s=n$ a positive integer. If moreover Φ_p is block-tensorial, then the integral is valued in $\mathbf{Q}$.

Proof. First, changing Φ_p to Φ_p^g, $\Phi_p^g(v)=\Phi_p(vg)$, one can assume that $g=1$. Now, from [Kar79, Theorem 3.6], one obtains that there is a compact set U of $V'(\mathbf{Q}_p)$ so that

$$\int\limits_{V'(\mathbf{Q}_p)} f(w_\ell n(x),\Phi_p,s)\psi((\eta,x))\,dx=\int\limits_{U} f(w_\ell n(x),\Phi_p,s)\psi((\eta,x))\,dx. \tag{12}$$

But now, because Φ_p is Schwartz, $f(\cdot,\Phi_p,s)$ is right invariant under a compact open, so that the right-hand side of (12) is a finite sum. The lemma follows because Φ_p being R-valued (for some ring R) implies $f(g,\Phi_p,s)$ is equal to $P(q^{-s})(1-q^{-s})^{-1}$ for some R-valued polynomial $P(X)$.

The rationality claim of the lemma follows just as in the proof of Proposition 4.3.1. □

4.5. Rank two Fourier coefficients: archimedean part

In this subsection, we calculate the archimedean contribution to the rank two Fourier coefficients of the Eisenstein series $E_\ell(g,\Phi_f,s)$. More precisely, let $f_\ell(g,s)$ denote the $\mathbb{V}_\ell$-valued section of the Eisenstein series on G. The main result of this subsection is the computation of the Fourier transform

$$I(\omega,\ell)=\int\limits_{V'(\mathbf{R})} e^{-i(\omega,x)}f_\ell(wn(x),s=\ell+1)\,dx. \tag{13}$$

For the involution ι on V that gives rise to the Cartan involution, write $||v||^2:=(v,\iota(v))$ for the associated positive-definite norm. Before beginning this computation, let us note that $f_\ell(wn(x),s)$ is the function

$$x\mapsto\frac{(p_{V_3}((1,x,-q(x)))^\ell}{||(1,x,-q(x)||^{s+\ell}}.$$

Following [KO03, (2.8.1)] define

$$\tau(x_n,x_2)^2=\left(1+\frac{||x_n||^2-||x_2||^2}{4}\right)^2+||x_2||^2.$$

One has the following simple lemma. Denote by $p_{V_3} : V = V_3 \oplus V_{n+1} \to V_3$ the orthogonal projection.

Lemma 4.5.1. *Suppose $w \in V'(\mathbf{R})$ and $w = w_2 + w_n$ is the decomposition of w into $V_2 \oplus V_n$, and $v = (1, w, -q'(w))$ so that v is isotropic. Then*

$$p_{V_3}(v) = -\frac{1}{2\sqrt{2}}\left((\sqrt{2}w_2, iv_1 + v_2)x^2 + (||w_2||^2 - ||w_n||^2 - 2)xy + (\sqrt{2}w_2, iv_1 - v_2)y^2\right)$$

and

$$||v||^2 = \tau(\sqrt{2}w_n, \sqrt{2}w_2)^2.$$

Proof. With notation as above, we have

$$v = \left(w_2 + \frac{1}{2}(1 - q'(w))(e + f)\right) + \left(w_n + \frac{1}{2}(1 + q'(w))(e - f)\right).$$

Thus

$$\begin{aligned} p_{V_3}(v) &= w_2 + \frac{1}{2}(1 - q'(w))(e + f) \\ &= -\frac{1}{2}\left((w_2, iv_1 + v_2)x^2 + \sqrt{2}(q_2(w_2) - q_n(w_n) - 1)xy + (w_2, iv_1 - v_2)y^2\right) \\ &= -\frac{1}{2\sqrt{2}}\left((\sqrt{2}w_2, iv_1 + v_2)x^2 + (||w_2||^2 - ||w_n||^2 - 2)xy + (\sqrt{2}w_2, iv_1 - v_2)y^2\right) \end{aligned}$$

as claimed.

One computes

$$\begin{aligned} ||(1, w, -q_{V'}(w))||^2 &= ((1, w, -q_{V'}(w)), (-q_{V'}(w), \iota(w), 1)) \\ &= 1 + (q_{V'}(w))^2 + ||w_2||^2 + ||w_n||^2 \\ &= 1 + \left(\frac{||w_2||^2 - ||w_n||^2}{2}\right)^2 + ||w_2||^2 + ||w_n||^2 \\ &= \left(\frac{||w_2||^2 - ||w_n||^2}{2} - 1\right)^2 + 2||w_2||^2 \\ &= \tau(\sqrt{2}w_n, \sqrt{2}w_2)^2. \end{aligned}$$

This gives the lemma. □

To compute (13), we start with the answer, and compute the inverse Fourier transform. This strategy is only possible because the unipotent group N is abelian, and this is why modular forms on G are much easier than modular forms on the quaternionic exceptional groups.

Thus we wish to compute

$$I_v(x;\ell) = \int\limits_{V'(\mathbf{R})} e^{i(\omega,x)} \operatorname{char}(q(\omega)>0) q(\omega)^A \left(-\frac{|(\omega, v_1+iv_2)|}{(\omega, v_1+iv_2)}\right)^v K_v(\sqrt{2}|(\omega, v_1+iv_2)|)\, d\omega. \tag{14}$$

Eventually, we will substitute $A = \ell - n/2$. The computations are inspired by, and use results from [KO03] and [KM11]. Compare also [Shi82].

Let us first explain that the integral $I_v(x;\ell)$ is absolutely convergent if $A = \ell - n/2 \geq 0$ and $n \geq 1$.

Lemma 4.5.2. *Suppose $A = \ell - n/2 \geq 0$ and $n \geq 1$. Then the integral $I_v(x;\ell)$ is absolutely convergent.*

Proof. Taking absolute values, one obtains

$$\begin{aligned}
&\int\limits_{V'(\mathbf{R})} \operatorname{char}(q(w)>0) q(\omega)^A K_v(\sqrt{2}|(\omega, v_1+iv_2)|)\, d\omega \\
&\qquad = C \int\limits_{t_2, t_n} \operatorname{char}(t_2 > t_n)(t_2^2 - t_n^2)^A K_v(\sqrt{2}t_2) t_2 t_n^{n-1}\, dt_2\, dt_n \\
&\qquad = C \int\limits_{t_2=0}^{\infty} \int\limits_{0\leq w\leq 1} t_2^{2A+n+1}(1-w^2)^A K_v(\sqrt{2}t_2) w^{n-1}\, dw\, dt_2 \\
&\qquad = C' \int\limits_0^{\infty} t_2^{2A+n+1} K_v(\sqrt{2}t_2)\, dt_2
\end{aligned}$$

for positive constants C, C'. Here we have made the variable change $t_n = wt_2$, and because $A \geq 0$ and $n \geq 1$ the integral over w is finite. Because $A = \ell - n/2$, $2A+n+1 = 2\ell+1$. Thus because $|v| \leq 2\ell$, $t_2^{2\ell+1} K_v(\sqrt{2}t_2)$ is 0 at $t_2 = 0$ so the integral over t_2 in the final line above is finite. □

As the integral defining $I_v(x;\ell)$ is absolutely convergent, we may apply the Fourier inversion theorem, as mentioned above.

The computation of $I_v(x;\ell)$ is given in the following proposition. We will assume $v \geq 0$ in this proposition. Because $K_{-v}(y) = K_v(y)$, we can obtain the case $I_v(x;\ell)$ for $v < 0$ by the case of $v > 0$ by exchanging v_2 with its negative. Recall the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ and Appell's hypergeometric function $F_4(a,b;c;d;x;y)$.

Proposition 4.5.3. *Suppose $v \geq 0$. One has*

$$I_v(x;\ell) = (2\pi)^{(n+2)/2} 2^{\ell-(n+v+2)/2} (-x_2, iv_1+v_2)^v \frac{\Gamma(\ell+v+1)\Gamma(\ell-n/2+1)}{\Gamma(v+1)}$$

$$\times F_4(\ell+1, \ell+1+v; \ell+1; v+1; -||x_n||^2/2; -||x_2||^2/2).$$

Proof. From [KM11, (3.3.4) page 55] (which cites [Hel84, Lemma 3.6, Introduction]), one has

$$\int\limits_{S^{m-1}} e^{i\lambda(\eta,\omega)}\phi(\omega)\,d\omega = (2\pi)^{m/2} i^{\ell}\phi(\eta)\lambda^{1-m/2} J_{m/2-1+\ell}(\lambda) \tag{15}$$

if ϕ is Harmonic of degree ℓ, $\lambda > 0$, and η is in the sphere S^{m-1}. Here $J_\bullet$ is the J-Bessel function.

Let $S(V_n) = \{x \in V_n : ||x||^2 = 1\}$ be the sphere of radius one in V_n, and similarly let $S(V_2)$ be the sphere of radius one in V_2. We write $\omega = t_2\sigma_2 + t_n\sigma_n$, with $t_2, t_n \in \mathbf{R}_{>0}$, $\sigma_2 \in S(V_2)$ and $\sigma_n \in S(V_n)$. Let

$$\phi_v(\omega_2) = \left(-\frac{|(\omega, v_1+iv_2)|}{(\omega, v_1+iv_2)}\right)^v K_v(\sqrt{2}|(\omega, v_1+iv_2)|).$$

Define $x_2 \in V_2$ and $x_n \in V_n$ so that $x = x_2 + x_n$. Then we compute

$$\begin{aligned} I_v(x;\ell) = &\int\limits_{t_2,t_n,\sigma_2,\sigma_n} \mathrm{char}(t_2 > t_n)(t_2^2 - t_n^2)^A e^{it_2(\sigma_2,x_2)} e^{it_n(\sigma_n,x_n)} \\ &\times \phi(t_2\sigma_2) t_2 t_n^{n-1}\,dt_2\,dt_n\,d\sigma_2\,d\sigma_n \\ = (2\pi)^{n/2} &\int\limits_{t_2,t_n,\sigma_2} \mathrm{char}(t_2 > t_n)(t_2^2 - t_n^2)^A e^{it_2(\sigma_2,x_2)} \phi(t_2\sigma_2)(t_n||x_n||)^{1-n/2} \\ &\times J_{n/2-1}(t_n||x_n||) t_2 t_n^{n-1}\,dt_2\,dt_n\,d\sigma_2. \end{aligned}$$

Now by (15) we have

$$\int\limits_{\sigma_2 \in S(V_2)} e^{it_2(\sigma_2,x_2)}\phi_v(t_2\sigma_2)\,d\sigma_2 = (2\pi)\left(\frac{|(x_2, iv_1 - v_2)|}{(x_2, iv_1 - v_2)}\right)^v J_v(||x_2||t_2) K_v(\sqrt{2}t_2).$$

Thus

$$\begin{aligned} I_v(x;\ell) = (2\pi)^{(n+2)/2}&\left(\frac{|(x_2, iv_1 - v_2)|}{(x_2, iv_1 - v_2)}\right)^v ||x_n||^{1-n/2} \\ &\times \int\limits_{t_2,t_n} \mathrm{char}(t_2 > t_n)(t_2^2 - t_n^2)^A t_2 t_n^{n/2} K_v(\sqrt{2}t_2) J_v(||x_2||t_2) J_{n/2-1}(||x_n||t_n)\,dt_n\,dt_2. \end{aligned}$$

We now compute the integral over t_n. One has

$$\int\limits_0^{t_2} (t_2^2 - t_n^2)^A t_n^{n/2} J_{n/2-1}(||x_n||t_n)\,dt_n = (t_2)^{2A+n/2+1} \int\limits_0^1 (1-w^2)^A w^{n/2} J_{n/2-1}(t_2||x_n||w)\,dw$$

$$= 2^A\Gamma(A+1)||x_n||^{-(A+1)}(t_2)^{A+n/2}J_{n/2+A}(||x_n||t_2)$$

where the last line is by [GR07, 6.567.1].

Combining, we obtain

$$I_v(x;\ell) = (2\pi)^{(n+2)/2}\left(\frac{|(x_2, iv_1 - v_2)|}{(x_2, iv_1 - v_2)}\right)^v ||x_n||^{-(A+n/2)}2^A\Gamma(A+1)$$
$$\times \int_0^\infty t_2^{A+n/2+1}J_{n/2+A}(||x_n||t_2)J_v(||x_2||t_2)K_v(\sqrt{2}t_2)\,dt_2.$$

This last integral over t_2 is worked out in [KO03, page 586]. Following [KO03], by e.g., [GR07, 6.578.2], one obtains

$$\int_0^\infty t_2^{A+n/2+1}J_{n/2+A}(||x_n||t_2)J_v(||x_2||t_2)K_v(\sqrt{2}t_2)\,dt_2$$
$$= 2^{-(1+v/2)}||x_n||^{(n/2+A)}||x_2||^v\frac{\Gamma(A+n/2+1+v)}{\Gamma(v+1)}$$
$$\times F_4(A+n/2+1, A+n/2+1+v; A+n/2+1; v+1; -||x_n||^2/2, -||x_2||^2/2).$$

Set $\ell = A + n/2$, and note that

$$||x_2||\left(\frac{|(x_2, iv_1 - v_2)|}{(x_2, iv_1 - v_2)}\right) = -(x_2, iv_1 + v_2). \tag{16}$$

Taking (16) into account, the proposition follows. □

Corollary 4.5.4. *Suppose $v \geq 0$ and $||x_2|| + ||x_n|| < \sqrt{2}$. Then*

$$\tau(\sqrt{2}x_n, \sqrt{2}x_2)^{2\ell+1}I_v(x;\ell)$$
$$= (2\pi)^{(n+2)/2}2^{\ell-(n+v+2)/2}(-x_2, iv_1 + v_2)^v\frac{\Gamma(\ell+v+1)\Gamma(\ell-n/2+1)}{\Gamma(v+1)}$$
$$\times \tau(\sqrt{2}x_n, \sqrt{2}x_2)^{\ell-v}\,{}_2F_1\left(\frac{v-\ell}{2}, \frac{v+\ell+1}{2}; v+1; \frac{2||x_2||^2}{\tau(\sqrt{2}x_n, \sqrt{2}x_2)^2}\right).$$

Proof. By [KO03, page 586, Lemma 5.7], if $||x_n|| + ||x_2|| < \sqrt{2}$,

$$F_4(\ell+1, \ell+1+v; \ell+1; v+1; -||x_n||^2/2, -||x_2||^2/2) = \tau(\sqrt{2}x_2, \sqrt{2}x_n)^{-(\ell+v+1)}$$
$$\times\ {}_2F_1\left(\frac{v-\ell}{2}, \frac{\ell+v+1}{2}; \frac{v+1}{2}; \frac{2||x_2||^2}{\tau(\sqrt{2}x_2, \sqrt{2}x_n)^2}\right).$$

The corollary follows. □

We now restate Corollary 4.5.4 in a slightly different form. Define

$$J_v(x;\ell) = \frac{\Gamma(\ell+1)}{\Gamma(\ell-n/2+1)}\pi^{-(n+2)/2}\tau(\sqrt{2}x_n,\sqrt{2}x_2)^{2\ell+1}\frac{I_v(x;\ell)}{(\ell+v)!(\ell-v)!}.$$

Corollary 4.5.5. *For $v \geq 0$ and $||x_n|| + ||x_2|| < \sqrt{2}$, one has* $J_v(x;\ell)$

$$= 2^{\ell-v}\binom{\ell}{v}(-\sqrt{2}x_2, iv_1+v_2)^v\tau(\sqrt{2}x_n,\sqrt{2}x_2)^{\ell-v}\,{}_2F_1$$
$$\times\left(\frac{v-\ell}{2},\frac{v+\ell+1}{2};v+1;\frac{2||x_2||^2}{\tau(\sqrt{2}x_n,\sqrt{2}x_2)^2}\right).$$

For $v \leq 0$, one has

$$J_v(x;\ell) = \left(-\frac{a^*}{a}\right)^{|v|} J_{|v|}(x;\ell),$$

where $a = \sqrt{2}(x_2, iv_1+v_2)$.

Proof. The first part of the corollary has already been proved. The second follows immediately from the first by replacing v_2 with $-v_2$ and noting that $\sqrt{2}(x_2, iv_1-v_2) = -a^*$ if $a = \sqrt{2}(x_2, iv_1+v_2)$. □

We now use the following lemma. We will apply it with

$$a = -\sqrt{2}(x_2, iv_1+v_2),\ \ b = \left|1+\frac{2||x_n||^2-2||x_2||^2}{4}\right|.$$

Lemma 4.5.6. *For $\ell \geq 0$ even, $a \in \mathbf{C}^\times$ and $b > 0$ with $|a| < b$, one has*

$$(ax^2+2bxy-a^*y^2)^\ell = \sum_{0\leq v\leq\ell}\binom{\ell}{v}2^{\ell-v}\delta_{v,0}^{1/2}(xy)^{\ell-v}x^{2v}a^v(|a|^2+b^2)^{(\ell-v)/2}$$
$$\times\ {}_2F_1\left(\frac{v-\ell}{2},\frac{v+\ell+1}{2};v+1;\frac{|a|^2}{|a|^2+b^2}\right)$$
$$+\sum_{0\leq v\leq\ell}\binom{\ell}{v}2^{\ell-v}\delta_{v,0}^{1/2}(xy)^{\ell-v}y^{2v}(-a^*)^v(|a|^2+b^2)^{(\ell-v)/2}$$
$$\times\ {}_2F_1\left(\frac{v-\ell}{2},\frac{v+\ell+1}{2};v+1;\frac{|a|^2}{|a|^2+b^2}\right).$$

Here $\delta_{v,0}^{1/2}$ is equal to $1/2$ if $v=0$ and equal to 1 otherwise.

Proof. Denote by S_ℓ^a the first sum on the right-hand side of the displayed equation in the statement of the lemma and $S_\ell^{a^*}$ the second sum on the right-hand side this equation.

One has the well-known identity

$$ {}_2F_1(a', b'; c'; z) = (1-z)^{-a'} \, {}_2F_1\left(a', c'-b'; c'; \frac{z}{z-1}\right). $$

This follows from the integral representation

$$ {}_2F_1(a', b'; c'; z) = \frac{\Gamma(c')}{\Gamma(b')\Gamma(c'-b')} \int_0^1 t^{b'-1}(1-t)^{c'-b'-1}(1-zt)^{-a'} \, dt, $$

valid for $Re(c') > Re(b') > 0$, by making the substitution $t \mapsto 1-t$. Thus

$$ \begin{aligned} &(|a|^2+b^2)^{(\ell-v)/2} \, {}_2F_1\left(\frac{v-\ell}{2}, \frac{v+\ell+1}{2}; v+1; \frac{|a|^2}{|a|^2+b^2}\right) \\ &\quad = b^{\ell-v} \, {}_2F_1\left(\frac{v-\ell}{2}, \frac{v-\ell}{2}+\frac{1}{2}; v+1; -\frac{|a|^2}{b^2}\right). \end{aligned} $$

We thus must evaluate the sum

$$ \sum_{0 \le v \le \ell} \binom{\ell}{v} (2bxy)^{\ell-v} a^v \delta_{v,0}^{1/2} x^{2v} \, {}_2F_1\left(\frac{v-\ell}{2}, \frac{v-\ell}{2}+\frac{1}{2}; v+1; -\frac{|a|^2}{b^2}\right) $$

Now, note that if $\delta \ge 0$ is an integer then $\left(-\frac{\delta}{2}\right)_m \left(-\frac{\delta}{2}+\frac{1}{2}\right)_m = \left(\frac{1}{4}\right)^m \frac{\delta!}{(\delta-2m)!}$. Thus, plugging in the definition of ${}_2F_1$, we obtain

$$ \begin{aligned} S_\ell^a &= \sum_{0 \le v \le \ell, 0 \le m} \binom{\ell}{v} (2bxy)^{\ell-v} a^v \delta_{v,0}^{1/2} x^{2v} (-1)^m \frac{\binom{\ell-v}{2m}(2m)!}{(v+1)_m m!} \left(\frac{|a|}{2b}\right)^{2m} \\ &= \sum_{0 \le v \le \ell, 0 \le m} \frac{\ell!}{(\ell-v-2m)!(v+m)!m!} (-1)^m (2bxy)^{\ell-v-2m} a^v |a|^{2m} (xy)^{2m} \delta_{v,0}^{1/2} x^{2v}. \end{aligned} $$

Similarly,

$$ S_\ell^{a^*} = \sum_{0 \le v \le \ell, 0 \le m} \frac{\ell!}{(\ell-v-2m)!(v+m)!m!} (-1)^m (2bxy)^{\ell-v-2m} (-a^*)^v |a|^{2m} (xy)^{2m} \delta_{v,0}^{1/2} y^{2v}. $$

The lemma now follows easily. □

The condition $|a| < b$ in Lemma 4.5.6 can now be removed:

Corollary 4.5.7. *The statement of Lemma 4.5.6 holds under the condition $b > 0$, and not just $b > |a|$.*

Proof. Suppose $b > 0$ and set $t = a/b$, so that $t \in \mathbf{C}$. Dividing both sides of the statement of Lemma 4.5.6 by b^ℓ, one obtains

$$\begin{aligned}(tx^2 + 2xy - t^*y^2)^\ell = &\sum_{0 \le v \le \ell} \binom{\ell}{v} 2^{\ell-v} \delta_{v,0}^{1/2} (xy)^{\ell-v} x^{2v} t^v (|t|^2+1)^{(\ell-v)/2} \\ &\times {}_2F_1\left(\frac{v-\ell}{2}, \frac{v+\ell+1}{2}; v+1; \frac{|t|^2}{|t|^2+1}\right) \\ + &\sum_{0 \le v \le \ell} \binom{\ell}{v} 2^{\ell-v} \delta_{v,0}^{1/2} (xy)^{\ell-v} y^{2v} (-t^*)^v (|t|^2+1)^{(\ell-v)/2} \\ &\times {}_2F_1\left(\frac{v-\ell}{2}, \frac{v+\ell+1}{2}; v+1; \frac{|t|^2}{|t|^2+1}\right).\end{aligned}$$

From Lemma 4.5.6, the above equality holds for $t \in \mathbf{C}$ with $|t| < 1$. However, both sides are real analytic functions of t, so their equality for $|t| < 1$ implies their equality for all $t \in \mathbf{C}$. The corollary follows. □

Combining Corollary 4.5.7 with Corollary 4.5.5, we obtain:

Corollary 4.5.8. *Set* $a_1 = \sqrt{2}(x_2, iv_1 + v_2)$ *and* $b_1 = (||x_2||^2 - ||x_n||^2 - 2)/2$. *Then*

$$\frac{\Gamma(\ell+1)}{\Gamma(\ell - n/2 + 1)} \pi^{-(n+2)/2} \sum_{-\ell \le v \le \ell} I_v(x;\ell) \frac{x^{\ell+v} y^{\ell-v}}{(\ell+v)!(\ell-v)!} = \frac{(a_1 x^2 + 2b_1 xy - a_1^* y^2)^\ell}{\tau(\sqrt{2}x_n, \sqrt{2}x_2)^{2\ell+1}}. \tag{17}$$

Proof. First suppose $||x_2|| + ||x_n|| < \sqrt{2}$. Note that the assumption $||x_2|| + ||x_n|| < \sqrt{2}$ implies that b_1 is negative, and thus $b = |b_1| = -b_1$. Therefore, from Corollaries 4.5.7 and 4.5.5 the equality above holds so long as $||x_2|| \neq 0$, $||x_n|| \neq 0$, and $b = |1 + \frac{||x_n||^2 - ||x_2||^2}{2}| \neq 0$. The conditions $||x_2|| \neq 0$ and $||x_n|| \neq 0$ are used in the manipulations used to prove Proposition 4.5.3. But now the absolute convergence computations for $I_v(x;\ell)$ prove that $I_v(x;\ell)$ is a continuous function of x. As both sides of (17) are continuous in x, the corollary follows in this case.

For the general case, it follows from Proposition 4.5.3 that $I_v(x;\ell)$ is an analytic function of x_2 and $||x_n||^2$. Therefore, the equality (17) for $||x_2|| + ||x_n|| < \sqrt{2}$ implies the equality for all x_2, x_n. □

We arrive at the main archimedean theorem of this subsection.

Theorem 4.5.9. *The Fourier transform*

$$\int_{V'(\mathbf{R})} e^{-2\pi i(\omega,x)} f_\ell(wn(x), s = \ell+1)\, dx = C_{\ell,n} (2\pi)^{2\ell+2-\dim(V')/2} q(\omega)^{\ell - n/2} \mathcal{W}_{2\pi\omega}(1)$$

for a nonzero rational number $C_{\ell,n}$.

Proof. Making the change of variable $\omega \mapsto 2\pi\omega$ in the integral $I_v(x;\ell)$ of (14), one gets

$$(2\pi)^{-(2\ell+2)} I_v(x;\ell) = \int\limits_{V'(\mathbf{R})} e^{2\pi i(\omega,x)} q(\omega)^{\ell-n/2} \left(-\frac{|(2\pi\omega, v_1+iv_2)|}{(2\pi\omega, v_1+iv_2)} \right)^v \times K_v(\sqrt{2}|(2\pi\omega, v_1+iv_2)|)\, d\omega. \tag{18}$$

From Corollary 4.5.8, one obtains

$$\pi^{-(2\ell+2)} \sum_{-\ell\leq v\leq \ell} I_v(x;\ell) \frac{x^{\ell+v} y^{\ell-v}}{(\ell+v)!(\ell-v)!} \doteq \pi^{(n+2)/2-(2\ell+2)} \frac{(a_1x^2 + 2b_1xy - a_1^* y^2)^\ell}{\tau(\sqrt{2}x_n, \sqrt{2}x_2)^{2\ell+1}} \tag{19}$$

in the notation of that corollary, where the $\doteq$ means that the two sides are equal up to a nonzero rational number. By applying Lemma 4.5.1, the right-hand side of (19) is $\pi^{(n+2)/2-(2\ell+2)} f_\ell(wn(x), s=\ell+1)$. Thus from (18), one gets

$$\pi^{2\ell+2-\dim(V')/2} \int\limits_{V'(\mathbf{R})} e^{2\pi i(\omega,x)} q(\omega)^{\ell-n/2} \mathcal{W}_{2\pi\omega}(1)\, d\omega \doteq f_\ell(wn(x), s=\ell+1).$$

The theorem follows by Fourier inversion. □

Combining Theorem 4.5.9 with Proposition 4.2.2 and Proposition 4.3.1 proves the first main theorem of this paper:

Theorem 4.5.10. *Suppose $\ell > n+1$ is even, $\dim(V')$ is a multiple of 4 and Φ_f is valued in $\overline{\mathbf{Q}}$. Then the Fourier coefficients of the Eisenstein series $\pi^{-\ell}E(g,\Phi_f,\ell;s)$ at $s=\ell+1$ are algebraic numbers. If moreover Φ_f is $\mathbf{Q}$-valued and block-tensorial, then the Fourier coefficients of the Eisenstein series $\pi^{-\ell}E(g,\Phi_f,\ell;s)$ at $s=\ell+1$ are rational numbers.*

5. Constant terms

In this section, we show that the constant terms of modular forms on $\mathrm{SO}(4,n+2)$ (in the sense of [Pol20a]) to $\mathrm{SO}(3,n+1)$ are modular forms in the sense of section 3.1. Moreover, the quaternionic exceptional groups of type F_4, E_6, E_7, E_8 have Levi subgroups L of type $B_{3,3}, D_{4,3}, \mathrm{SU}(2)\times D_{5,3}$ and $D_{7,3}$ respectively. We also check that the constant terms of modular forms on these exceptional groups to the above L are modular forms on L. More precisely, in section 3 we defined modular forms on groups $\mathrm{SO}(V)$, but the definition extends immediately to groups L isogenous to these $\mathrm{SO}(V)$, which is what occurs for the above exceptional groups.

5.1. Orthogonal groups of rank four

Let $V_{4,n+2} = H \oplus V$ be the rational quadratic space that is the orthogonal direct sum of V and a hyperbolic plane. The signature of V is $(4, n+2)$. Denote by e_0, f_0 a standard basis of the hyperbolic plane, so that the pairing $(e_0, f_0) = 1$. Denote by $P_0 = M_0 N_0$ the parabolic subgroup of $\mathrm{SO}(V_{4,n+2})$ that stabilizes the line spanned by e_0 for the left action of $\mathrm{SO}(V_{4,n+2})$ on $V_{4,n+2}$. The Levi subgroup M_0 is defined to be the one that stabilizes both $\mathrm{Span}(e_0)$ and $\mathrm{Span}(f_0)$. Extend the involution ι on V to $V_{4,n+2}$ by defining it to exchange e_0 and f_0. We take as a Cartan involution on $\mathrm{SO}(V_{4,n+2})(\mathbf{R})$ conjugation by ι. Let $K_{4,n+2}$ be the maximal compact subgroup that is the fixed points of this involution.

In [Pol20a] we defined and considered modular forms on the group $\mathrm{SO}(V_{4,n+2})$. If φ is a modular form of weight $\ell \geq 1$ on $\mathrm{SO}(V_{4,n+2})$, one can take the constant term of φ along N_0 to obtain an automorphic function φ_{N_0} on M_0. The purpose of this subsection is to prove that this constant term is a modular form on M_0 of weight ℓ, in the sense of section 3. This fact follows immediately from the following proposition.

To set up the proposition precisely and to prove it, we make a few notations. Let $\{X_\gamma\}_\gamma$ be a basis of $\mathfrak{p} = \mathfrak{p}_{3,n+1} = V_3 \otimes V_{n+1}$, $\{u_1, \dots, u_n, u_{n+1}\}$ be a basis of V_{n+1}, and w_1, w_2, w_3 a basis of V_3. Write $y_+ = e_0 + f_0$ and $y_- = e_0 - f_0$. Recall the operator $\widetilde{D}_\ell$ from subsection 3.1, and the analogous operator from [Pol20a, subsection 7.1]. To distinguish these two operators, we write $\widetilde{D}_{4,n+2}$ for the one that acts on $\mathbb{V}_\ell$-valued automorphic functions on $\mathrm{SO}(V_{4,n+2})$ and similarly $\widetilde{D}_{3,n+1}$ for the one that acts on $\mathbb{V}_\ell$-valued automorphic functions on $\mathrm{SO}(V)$. Analogously, we write $D_{4,n+2}$, respectively $D_{3,n+1}$, for the so-called Schmid operators, which by definition are the $\widetilde{D}$'s followed by an appropriate $\mathrm{SU}(2)$-contraction $\mathrm{pr}_- : Y_2 \otimes Sym^{2\ell}(Y_2) \to Sym^{2\ell-1}(Y_2)$.

Proposition 5.1.1. *Let the notation be as above. Suppose $\varphi' : \mathrm{SO}(V_{4,n+2})(\mathbf{R}) \to \mathbb{V}_\ell$ is left $N_0(\mathbf{R})$-invariant, $\varphi'(gk) = k^{-1} \cdot \varphi'(g)$ for all $k \in K_{4,n+2}$ and $g \in \mathrm{SO}(V_{4,n+2})(\mathbf{R})$, and $D_{4,n+2}\varphi'(g) = 0$. Denote by φ_{M_0} the restriction of φ' to $\mathrm{SO}(V)(\mathbf{R}) \subseteq M_0(\mathbf{R})$. Then $D_{3,n+1}\varphi_{M_0} = 0$.*

Proof. The proof follows without much difficulty, directly from the definitions.

With the above notation, we have

$$\widetilde{D}_{4,n+2}\varphi' = \sum_\gamma X_\gamma \varphi' \otimes X_\gamma^\vee + \sum_{1 \leq j \leq n+1} (y_+ \wedge u_j)\varphi' \otimes (y_+ \wedge u_j)^\vee + (y_+ \wedge y_-)\varphi' \otimes (y_+ \wedge y_-)^\vee + \sum_{1 \leq k \leq 3} (w_k \wedge y_-)\varphi' \otimes (w_k \wedge y_-)^\vee.$$

Note that restricting to M_0 and applying pr_- to the first term gives $D_{3,n+1}\varphi_{M_0}$. Moreover, restricting the second term to M_0 gives 0 because $y_+ \wedge u_j = 2e_0 \wedge u_j - y_- \wedge u_j \in \mathfrak{n}_0 \oplus Lie(\mathrm{SO}(n+2))$, and φ' is invariant on the left under N_0 and on the right under $\mathrm{SO}(n+2)$. Thus we obtain

$$D_{4,n+2}\varphi_{M_0} = D_{3,n+1}\varphi_{M_0} + \mathrm{pr}_- \left(\sum_{1\leq j\leq 4} (h_j \wedge y_-)\varphi' \otimes (h_j \wedge y_-)^\vee \right)\Bigg|_{M_0}$$

where $\{h_1, h_2, h_3, h_4\} = \{y_+, w_1, w_2, w_3\}$ is a basis of the four-dimensional space $V_{4,n+2}^{\iota=1}$. Note that the pr_--term is linearly independent from the $D_{3,n+1}$ term, because it contains a $y_-^\vee$. This proves that $D_{3,n+1}\varphi_{M_0} = 0$, as desired. □

5.2. Exceptional groups

Suppose C is a rational composition algebra, with $C \otimes \mathbf{R}$ positive definite. Set $J = H_3(C)$ the Hermitian 3×3 matrices with coefficients in C and G_J the quaternionic exceptional group associated to J as in [Pol20a]. Thus G_J has rational root type F_4 and is of Dynkin type F_4, E_6, E_7 or E_8 depending on if $\dim C$ is $1, 2, 4$ or 8. Let $Q_J = L_J V_J$ be the standard maximal parabolic subgroup of G_J with Levi subgroup L_J of rational type B_3. In this subsection, we prove that the constant term φ_{V_J} of a modular form φ of weight ℓ on G_J is a modular form of weight ℓ on L_J. Moreover, we prove that the rank one and rank two Fourier coefficients of φ_{V_J} are the rank one and rank two Fourier coefficients of φ.

To state precisely these results and prove them, we now make some definitions. Let the simple roots of F_4 be α_j with $1 \leq j \leq 4$. We label the simple roots so that α_j is connected to α_{j+1} in the Dynkin diagram, for $j = 1, 2, 3$, and with α_1, α_2 the long roots:

$$\circ - - - - \circ \; == > == \; \circ - - - - \circ;$$

the roots are labeled $1, 2, 3, 4$ from left to right. Write a positive root as a four-tuple $[n_1, n_2, n_3, n_4]$, which corresponds to $\sum_j n_j \alpha_j$. The rational root spaces corresponding to long roots of F_4 are one-dimensional while the rational root spaces corresponding to short roots spaces of F_4 can be identified with the composition algebra C.

The Heisenberg parabolic of G_J that is central to [Pol20a] is the maximal parabolic with simple root α_1 in its unipotent radical. We define $Q_J = L_J V_J$ to be the standard maximal parabolic subgroup of G_J with simple root α_4 in its unipotent radical V_J. Thus L_J has rational root type B_3. The parabolic subgroup Q_J of G_J defines a 5-step $\mathbf{Z}$-grading on the Lie algebra $\mathfrak{g}(J) = Lie(G_J)$. Specifically, for $j = -2, -1, 1, 2$, set V_J^j the subspace of $\mathfrak{g}(J)$ consisting of those rational roots spaces $[n_1, n_2, n_3, n_4]$ where $n_4 = j$. Then $V_J^{\pm 2}$ are each a direct sum of 6 long root spaces and one short root space, while $V_J^{\pm 1}$ is a direct sum of 8 short root spaces. One has a direct sum decomposition

$$V_J^{-2} \oplus V_J^{-1} \oplus Lie(L_J) \oplus V_J^1 \oplus V_J^2.$$

See also [SW11, section 2] for more on this Lie algebra decomposition.

As mentioned, the group L_J is, up to anisotropic factors, isogenous to $\mathrm{SO}(H^3 \oplus C) = \mathrm{SO}(V_J^2)$. We now write down an explicit map $L_J \to \mathrm{GSO}(V_J^2)$. More specifically, L_J acts

on V_J^2 by the adjoint action, and we write down the non-degenerate rational quadratic form on V_J^2 that L_J preserves up to scalar. To be completely concrete, any element v of V_J^2 is of the form

$$v = b_1 E_{13} + v_1 \otimes \mathrm{diag}(b_2, 0, 0) + \delta_3 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_3 & b_0 \\ 0 & b_0^* & b_{-3} \end{pmatrix} + b_{-2} E_{23} + v_2 \otimes \mathrm{diag}(b_{-1}, 0, 0)$$

in the notation of [Pol20a, section 4.2]. In this notation, the quadratic form on V_J^2 is given by

$$\frac{1}{2}(v, v)_J = b_1 b_{-1} - b_2 b_{-2} + b_3 b_{-3} - n_C(b_0),$$

where n_C is the norm on the composition algebra C.

Proposition 5.2.1. *This quadratic form is fixed up to scalar multiple by the Levi subgroup L_J.*

Proof. Because L_J is connected, by virtue of being a reductive quotient of a parabolic subgroup of a connected group, to prove the proposition, it suffices to check it on the Lie algebra level. To prove it on the Lie algebra level, we check the semi-invariance of the quadratic form on V_J^2 for certain group and Lie algebra elements, and piece together the results. Because L_J is defined to be the parabolic subgroup of type B_3 in the F_4 root system, we work in this root system.

First, consider the diagonal torus $T = \mathrm{diag}(t_1, t_2, t_3) \in \mathrm{SL}_3 \subseteq G_J$. The Lie algebra of this torus is in $Lie(L_J)$. This torus acts on v above to give

$$\begin{aligned} v' = {} & \frac{t_1}{t_3} b_1 E_{13} + t_1 v_1 \otimes \mathrm{diag}(b_2, 0, 0) \\ & + t_3^{-1} \delta_3 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_3 & b_0 \\ 0 & b_0^* & b_{-3} \end{pmatrix} + \frac{t_2}{t_3} b_{-2} E_{23} + t_2 v_2 \otimes \mathrm{diag}(b_{-1}, 0, 0). \end{aligned}$$

Using that $t_1 t_2 t_3 = 1$, one sees that the quadratic form is scaled by $\frac{t_1 t_2}{t_3} = t_3^{-2}$.

Next, consider the subgroup P_{11} of $M_J^1 \subseteq G_J$ that fixes the line spanned by $e_{11} := \mathrm{diag}(1, 0, 0) \in J$. The group M_J^1 acts on the E_{ij} as the identity. Suppose P_{11} satisfies $pe_{11} = \alpha e_{11}$. Then, in its action on $J^\vee$, P_{11} stabilizes the subspace $\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ of $J^\vee$, because this subspace is $e_{11} \times J$. Moreover, because

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & b_3 & b_0 \\ 0 & b_0^* & b_{-3} \end{pmatrix}^{\#} = (b_3 b_{-3} - n(b_0))e_{11},$$

one has the $p \in P_{11}$ scales the quantity $b_3 b_{-3} - n(b_0)$ by α. It follows that p scales the quadratic form by α.

The rest of $Lie(L_J)$ is made up of nilpotent elements, and it is easy to check that they preserve the quadratic form on V_J^2. We give one such computation, and leave the rest to the reader. Consider $n = v_2 \otimes X$ with $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_2 & x_1 \\ 0 & x_1^* & c_3 \end{pmatrix}$. We check that n preserves the quadratic form. To see this, we compute $(v, [n, v])_J$. Set $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_3 & b_0 \\ 0 & b_0^* & b_{-3} \end{pmatrix}$. Then

$$[v_2 \otimes X, v] = -\delta_3 \otimes (\mathrm{diag}(b_2, 0, 0) \times X) - (X, Y)E_{23}.$$

Consequently, $-(v, [v_2 \otimes X, v])_J = (e_{11} \times Y, \mathrm{diag}(b_2, 0, 0) \times X) + (-b_2)(X, Y) = 0$, as desired. □

That the constant term φ_{V_J} is a modular form of weight ℓ on L_J follows immediately from the following proposition. Similar to subsection 5.1, let $\widetilde{D}_J$, $D_J = \mathrm{pr}_- \circ \widetilde{D}_J$ denote the differential operators used to define modular forms on G_J and $\widetilde{D}$, $D = \mathrm{pr}_- \circ \widetilde{D}$ denote the differential operators used to define modular forms on L_J.

Proposition 5.2.2. *Let the notation be as above. Suppose $\varphi' : G_J(\mathbf{R}) \to \mathbb{V}_\ell$ is left $V_J(\mathbf{R})$-invariant, $\varphi'(gk) = k^{-1}\varphi'(g)$ for all $k \in K_J$ and $g \in G_J(\mathbf{R})$, and $D_J\varphi'(g) = 0$. Denote by φ_{L_J} the restriction of φ' to $L_J(\mathbf{R})$. Then $D\varphi_{L_J} = 0$.*

Proof. Set L' to be a subgroup of $G_J(\mathbf{R})$ that contains $L_J(\mathbf{R})$ and has Lie algebra $V_J^{-2} \oplus Lie(L_J) \oplus V_J^2$. Denote by $Q' = L' \cap Q_J(\mathbf{R})$; Q' is a parabolic subgroup of L' with Levi subgroup $L_J(\mathbf{R})$. Then L' has real root type B_4; in particular, it is isogenous to $\mathrm{SO}(4, 4 + \dim(C))$.

The idea of the proof is simple. We check that φ' restricted to L' satisfies the assumptions of the φ' of Proposition 5.1.1. That is, denoting $\varphi'' := \varphi'|_{L'}$, we check that $D_{4,4+\dim(C)}\varphi'' = 0$. The other assumptions on the φ' of Proposition 5.1.1 are immediately verified. Then, applying Proposition 5.1.1 to φ'', one concludes that

$$\varphi''|_{L_J} = (\varphi'|_{L'})\,|_{L_J} = \varphi_{L_J}$$

is annihilated by D.

So, it remains to check that $D_{4,4+\dim(C)}\varphi'' = 0$. We use the notation of [Pol20a, section 6.3]. For an element r in the composition algebra C, and an integer $j \in \{1, 2, 3\}$

let $x_j(r)$ be the corresponding element of $H_3(C) = J$. Note that if $v \in \mathrm{Span}\{v_1, v_2, v_3\}$, so that $v \otimes x_j(r) \in \mathfrak{g}(J)$, then

$$\frac{1}{2}\left(v \otimes x_j(r) + \iota(v) \otimes x_j(r)\right) = n_v(x_j(r))$$

is in the compact Lie algebra[2] $Lie(L^0(J))$. Again, see [Pol20a, section 6.3].

The vector space V_J^1 of $\mathfrak{g}(J)$ is spanned by elements of the form $v \otimes x_j(r)$ and $\delta \otimes x_j(r)$. Suppose $X \in V_1$ so that $X - \theta(X) \in \mathfrak{p}_J$. We have $X - \theta(X) = 2X - (X + \theta(X))$, with $X + \theta(X) \in Lie(L^0(J))$. Consequently, if $g \in Q'$, then $((X - \theta(X))\varphi')(g) = 0$ because φ' is left $V_J(\mathbf{R})$-invariant and right L_J^0-invariant. It follows formally that one has an equality $D_J\varphi'(g) = D_{4,4+\dim(C)}\varphi'(g)$. As $D_J\varphi' = 0$ by assumption, the proposition follows. □

In the above results, we checked that the constant term φ_{V_J} of a modular form φ on G_J satisfies a particular differential equation. In order to incorporate results about the Fourier expansion of φ_{V_J}, we will–for convenience–make an additional assumption on these constant terms. Namely, we will assume that these constant terms are pullbacks from $\mathrm{GSO}(V_J^2)$ of modular forms on this latter group. It is clear from the definition of the Eisenstein series $E_8^{L_J}(g, s)$ after the statement of Theorem 6.0.2 that this assumption holds in our case of interest, namely, for $\varphi = \theta_{ntm}$. Here is the definition of modular forms on $\mathrm{GSO}(V_J^2)$.

Definition 5.2.3. We say an automorphic form $\xi : \mathrm{GSO}(V_J^2)(\mathbf{Q})\backslash\mathrm{GSO}(V_J^2)(\mathbf{A}) \to \mathbb{V}_\ell$ is a modular form of weight ℓ if the pullbacks $(g_f \cdot \xi)|_{\mathrm{SO}(V_J^2)}$ are modular forms of weight ℓ for $g_f \in \mathrm{GSO}(V_J^2)(\mathbf{A}_f)$. The Fourier coefficients of such a modular form are the Fourier coefficients of the restriction $(m_f \cdot \xi)|_{\mathrm{SO}(V_J^2)}$ for all finite-adelic points m_f in the rational B_2-type Levi subgroup of $\mathrm{GSO}(V_J^2)$.

We end this subsection by comparing the Fourier coefficients of a modular form of weight ℓ on G_J to those of its constant term along V_J, in the case when G_J is of type E_8. For $J = H_3(\Theta)$ with Θ the non-split octonions, recall the maximal compact subgroup K_f of $G_J(\mathbf{A}_f)$ specified in [Gan00b, section 6].

Proposition 5.2.4. *Set $J = H_3(\Theta)$ with Θ the non-split octonions, so that G_J is of type E_8. Suppose φ is a modular form of weight ℓ on G_J with constant term φ_{V_J} to L_J. Suppose that all for all K_f-translates $k_f \cdot \varphi$ with $k_f \in K_f$, the constant term $(k_f \cdot \varphi)_{V_J}$ is the pullback from $\mathrm{GSO}(V_J^2)$ of a modular form of weight ℓ on $\mathrm{GSO}(V_J^2)$ with Fourier coefficients in some field $E \subseteq \mathbf{C}$. Then all K_f-translates of φ have rank one and rank two Fourier coefficients in E.*

[2] The author apologizes for the similar-looking notations L_J and $L^0(J)$. The group L_J is the Levi of a maximal parabolic subgroup of G_J, while $L^0(J) \subseteq G_J(\mathbf{R})$ is compact. In case $J = H_3(\Theta)$ so that G_J is of type E_8, L_J is a rational reductive group of type D_7, while $L(J)^0$ is a compact form of E_7.

Before giving the proof of Proposition 5.2.4, let us note that it is not a statement that has an analogue for general automorphic functions; although, the analogous statement *is* true for holomorphic Siegel modular forms. Rather, the truth of Proposition 5.2.4 crucially uses the robust Fourier expansion ([Pol20a, Corollary 1.2.3]) of modular forms on G_J and the Fourier expansion of modular forms on L_J (Theorem 1.1.1). Specifically, the crux of the matter is that the generalized Whittaker functions $\mathcal{W}_\omega(g)$ of [Pol20a, Theorem 1.2.1] have the following extra invariance property:

$$\omega \in W_J(\mathbf{R}), m \in H_J^1(\mathbf{R}), \text{ and } m \cdot \omega = \omega \text{ implies } \mathcal{W}_\omega(mg) = \mathcal{W}_\omega(g). \tag{20}$$

Here the notation is from [Pol20a] so that H_J^1 is the similitude-equal-one part of the Levi of the Heisenberg parabolic $P_J = H_J N_J$ of G_J and $W_J = N_J/[N_J, N_J]$ is the defining representation of H_J.

Proof of Proposition 5.2.4. Identify W_J with the degree 1 part of the 5-step $\mathbf{Z}$-grading on $\mathfrak{g}(J)$ determined by the Heisenberg parabolic P_J of G_J. Now, set $W_0' = W_J \cap Lie(L_J)$ and $W_2' = W_J \cap V_J^2$. Then W_0' and W_2' are paired nontrivially under the symplectic form $\langle \cdot, \cdot \rangle$ on W_J and both can be identified with $H^2 \oplus C$ for a hyperbolic plane H.

Now, suppose $\omega \in W_J(\mathbf{Q})$ has rank at most two, and we wish to compute the Fourier coefficient

$$\varphi_\omega(g) = \int\limits_{N_J(\mathbf{Q})\backslash N_J(\mathbf{A})} \psi^{-1}(\langle \omega, \overline{n} \rangle) \varphi(ng)\, dn.$$

Here $\overline{n}$ denotes the image of n in W_J. Because ω has rank at most two, there is $m \in H_J^1(\mathbf{Q})$ so that $m\omega \in W_2'$. Consequently, by automorphy of φ, we can assume $\omega \in W_2'$.

The $\omega \in W_2'$ determines an $\eta \in W_0'$ so that the Fourier coefficient $\varphi_{V_J,\eta}$ of φ_{V_J} defined by η can be written as an integral of φ_ω. Specifically, one has formally an equality

$$\varphi_{V_J,\eta}(g) = \int\limits_{(V_J \cap H_J)(\mathbf{Q})\backslash (V_J \cap H_J)(\mathbf{A})} \varphi_\omega(ng)\, dn. \tag{21}$$

But the elements of $V_J \cap H_J$ have similitude equal to 1 and act as the identity on W_2'. Consequently, applying approximation for the unipotent group $(V_J \cap H_J)(\mathbf{A}_f)$ and the invariance property (20), the integral in (21) becomes $\varphi_\omega(g)$. We therefore obtain

$$\varphi_{V_J,\eta}(g) = \varphi_\omega(g) \tag{22}$$

for all $g \in G_J(\mathbf{A})$.

Our assumption on the constant term φ_{V_J} implies that we can make precise sense out of its Fourier coefficients. To compare these Fourier coefficients with those of φ, it suffices to consider $g = g_f g_\infty$ in (22) with $g_f \in H_J(\mathbf{A}_f)$ and g_∞ in the intersection $L_J'(\mathbf{R}) \cap H_J(\mathbf{R})$; here L_J' is the derived group of L_J. Comparing the generalized Whittaker

functions of Definition 3.2.2 with those of [Pol20a, Theorem 1.2.1], and taking note of the rational quadratic form $(\ ,\ )_J$ on V_J^2 defined above Proposition 5.2.2, one sees that these two sets of generalized Whittaker functions agree for g_∞ in $L'_J(\mathbf{R}) \cap H_J(\mathbf{R})$. One now obtains the proposition using the Iwasawa decomposition on g_f. □

6. The next-to-minimal modular form

In this section, we give an application of all of the above results, and prove that the so-called next-to-minimal modular form on $E_{8,4}$ has rational Fourier coefficients. On the split form of E_8, next-to-minimal representations and some results about their Fourier coefficients have appeared in [KS15], [GGK+19].

More precisely, we prove the following result. Let Θ be the positive definite octonions, $J = H_3(\Theta)$, and $E_J(g, s; n) = \sum_{\gamma \in P_J(\mathbf{Q}) \backslash G_J(\mathbf{Q})} f_J(\gamma g, s, n)$ the Eisenstein series of [Pol20b] with *spherical* inducing data at every finite place, normalized so that the inducing section $f_J(g, s, n)$ takes the value $\frac{\zeta(n+1)}{(2\pi)^n} x^n y^n$ at $s = n + 1$, $g = 1$.

Theorem 6.0.2 states that $\theta_{ntm}(g) := E_J(g, s = 9; 8)$ is a modular form on G_J of weight 8. The following result of Savin proved in Appendix B implies that θ_{ntm} has vanishing rank 3 and rank 4 Fourier coefficients.

Theorem 6.0.1 *(Savin, Theorem B.1.1). Suppose p is odd and denote by Π the spherical constituent of the induced representation $Ind_{P_J(\mathbf{Q}_p)}^{G_J(\mathbf{Q}_p)}(\delta_P^{s_1})$, for $s_1 = \frac{20}{29} = \frac{1}{2} + \frac{11}{58}$. Then the twisted Jacquet module $\Pi_{N_J, \chi}$ is 0 for every unitary character χ of N_J that is rank three or rank four.*

Applying Theorem 6.0.1, the purpose of this section is to prove

Theorem 6.0.2. *The Eisenstein series $E_J(g, s; 8)$ is regular at $s = 9$ and defines a square integrable modular form of weight 8 at this point. Its rank zero, rank one, and rank two Fourier coefficients are all rational numbers. Consequently, $\theta_{ntm}(g) := E_J(g, s = 9; 8)$ has rational Fourier expansion.*

The value $E_J(g, s = 9; 8)$ is expected to be the "next-to-minimal" modular form on $E_{8,4}$. Theorem 6.0.2 is the analogue for the "next-to-minimal" modular form on quaternionic E_8 of results proved about the minimal modular form in [Gan00a] and [Pol20b]. The proof of Theorem 6.0.2 consists of a few steps, which we now outline.

(1) First, we analyze a certain spherical Eisenstein series $E_{V',8}(g, s)$ on the group $\mathrm{SO}(H^2 \oplus \Theta)$, which has signature $(2, 10)$. With our normalizations, the point $s = 8$ is outside the range of absolute convergence for this Eisenstein series, but we check that at $s = 8$ $E_{V',8}(g, s)$ is regular and defines a holomorphic modular form. Moreover, this Eisenstein series has rational Fourier coefficients. These facts are likely well-known, but we briefly prove them because the author is unaware of a suitable reference.

(2) Second, we analyze the spherical Eisenstein series $E_8(g, s = 9)$ on the group $\mathrm{SO}(H^3 \oplus \Theta)$. The point $s = 9$ is outside the range of absolute convergence, but by doing the appropriate intertwining operator calculations and using the first step, one can show that $E_8(g, s)$ is regular at $s = 9$ and defines a modular form of weight 8 at this point. Moreover, via calculations as above, we know that $\pi^{-8}E_8(g, s = 9)$ has rational Fourier coefficients.

(3) Thirdly, we show that the Eisenstein series $E_J(g, s; 8)$ is regular at $s = 9$ and defines a modular form of weight 8 at this point. The proof proceeds similarly to the proof of Corollary 4.1.2 of [Pol20b]. In particular, by doing many intertwining operator calculations, we compute the constant term of $E_J(g, s; 8)$ at $s = 9$ along the minimal parabolic P_0 of G_J. From this calculation, we deduce that the differential operator D_8 annihilates the constant term of $E_J(g, s = 9; 8)$ and then consequently the entire Eisenstein series: $D_8 E_J(g; s = 9; 8) = 0$.

(4) Fourthly, we prove that the constant term $E_J^V(g; s = 9; 8)$ of $E_J(g; s = 9; 9)$ to the $D_{7,3}$ Levi subgroup L_J is essentially the Eisenstein series $E_8(g, s = 9)$. More specifically, we prove that the constant term $E_J^V(g; s = 9; 8)$ is equal to the value at $s = 9$ of the Eisenstein series $E_8^{L_J}(g, s) := \sum_{(L_J \cap P)(\mathbf{Q}) \backslash L_J(\mathbf{Q})} f_J(\gamma g, s; 8)$. To do this, one considers the difference $E_J^V(g; s = 9; 8) - E_8^{L_J}(g, s = 9)$, and shows using the third step that this difference has constant term 0 to the minimal parabolic $P_0 \cap L_J$ of L_J. It follows easily from this fact that $E_J^V(g; s = 9, 8) = E_8^{L_J}(g, s = 9)$.

(5) The constant term of $E_J(g, s = 9; 8)$ along the unipotent radical of the Heisenberg parabolic yields Kim's weight 8 singular modular form on $GE_{7,3}$, which has rational Fourier coefficients [Kim93]. By applying Proposition 5.2.4, one obtains that $E_J(g, s = 9; 8)$ has rational rank one and rank two Fourier coefficients as well. Thus, $E_J(g, s = 9; 8)$ has rational rank zero, rank one, and rank two Fourier coefficients.

Before proceeding, we explain why Theorem 6.0.1 of Savin implies that the rank three and rank four Fourier coefficients of θ_{ntm} are 0:

Lemma 6.0.3. *The rank three and rank four Fourier coefficients of θ_{ntm} are 0.*

Proof. Let Π' denote the representation of $G_J(\mathbf{Q}_p)$ generated by θ_{ntm} in the space of automorphic forms on $G_J(\mathbf{A})$, and let Π be as in Theorem 6.0.1. Note that it is not clear that the Eisenstein series map is $G_J(\mathbf{Q}_p)$-equivariant, so one does not know *a priori* that Π' is a quotient of Π; we must make an argument. However, one does know that the spherical vector in Π has the same Hecke eigenvalues as that of θ_{ntm}, and we will leverage this fact to prove the lemma. Indeed, to see this fact about the Hecke eigenvalues, one notes that they can be computed by analytic continuation in s from the eigenvalues when $Re(s) >> 0$, and when $Re(s) >> 0$ the Eisenstein series map is indeed intertwining.

In more detail, let $\eta = \mathrm{char}(K_p g K_p)$ be in the Hecke algebra at p. We can write $K_p g K_p = \sqcup_i p_i K_p$ with $p_i \in P_J(\mathbf{Q}_p)$. Let $f(g, s)$ be the flat spherical section in $Ind_{P_J(\mathbf{Q}_p)}^{G_J(\mathbf{Q}_p)}(|\nu|^s)$ with $f(1, s) = 1$. We have $\eta * f(g, s) = \lambda(s) f(g, s)$, where $\lambda(s) =$

$\sum_i |\nu(p_i)|^s$ is an entire function of s. The Hecke eigenvalue of η on the spherical vector in Π is thus $\lambda(9)$. On the other hand, let $E_J(g,s;8)$ be the spherical Eisenstein series that we use to define θ_{ntm}. We have

$$\eta * \theta_{ntm}(g) = \sum_i \theta_{ntm}(gp_i) = \sum_i E_J(gp_i, s=9;8)$$
$$= \lim_{s\to 9} \sum_i E_J(gp_i, s; 8) = \lim_{s\to 9} \eta * E_J(g,s;8).$$

When $Re(s) >> 0$, the Eisenstein series map is intertwining, so $\eta * E_J(g,s;8) = \lambda(s)E_J(g,s;8)$. Because $\lambda(s)$ is entire, we deduce $\eta * \theta_{ntm}(g) = \lambda(9)\theta_{ntm}(g)$, as desired.

Now, consider again the representation Π'. Because θ_{ntm} is square integrable, Π' is semisimple. By virtue of being semisimple and generated by a single eigenvector for the bi-K_p-invariant Hecke algebra, Π' is irreducible. Indeed, write $\Pi' = \pi_1 \oplus \cdots \oplus \pi_r$ with each π_i irreducible, and correspondingly $\theta_{ntm} = v_1 + \cdots + v_r$. Because Π' is generated by θ_{ntm}, each $v_j \neq 0$. Because θ_{ntm} is K_p-invariant, so is each v_j. Thus all π_j are spherical. Because θ_{ntm} is an eigenvector for the bi-K_p-invariant Hecke algebra, the Hecke eigenvalues of each v_j are equal. Consequently, the π_j are all isomorphic. Because Π' is generated by a single vector, we now must have $r = 1$ as desired.

Let Π'' denote a nonzero irreducible quotient of Π. Because Π is (by definition) generated by a single spherical vector, so is Π''. Moreover, the Hecke eigenvalues on the spherical vector of Π'' are the same as on Π', i.e., on θ_{ntm}. Consequently, Π' is abstractly isomorphic to Π''. In other words, Π' is a quotient of Π.

Because Π' is a quotient of Π, the Fourier coefficient map on Π' factors through the corresponding twisted Jacquet module of Π. The lemma follows. □

We now split up the various pieces of the outline above into subsections. Throughout this section, set $\zeta_\Theta(s) = \zeta(s)\zeta(s-3)$ [Gan00a].

6.1. The holomorphic Eisenstein series

Set $V' = H^2 \oplus \Theta$, a quadratic space of signature $(2,10)$ that comes equipped with an integral lattice $V_0' = H_0^2 \oplus \Theta_0$. In this subsection we construct and analyze a certain holomorphic spherical Eisenstein series on $\mathrm{SO}(V')$. Let the bases of the two copies of H be e_1, f_1 and e_2, f_2.

To define this Eisenstein series, we proceed as follows. First, denote by v_1, v_2 an orthonormal basis of $V_2 = V'(\mathbf{R})^+$. For an even positive integer ℓ, define the Schwartz function $\Phi_{\infty,\ell}$ on $V'(\mathbf{R})$ as $\Phi_{\infty,\ell}(v) = (v_1 + iv_2, v)^\ell e^{-\pi||v||^2}$. Let Φ_f be the characteristic function of $V_0'(\widehat{\mathbf{Z}})$ and set $\Phi = \Phi_f \otimes \Phi_{\infty,\ell}$, a Schwartz-Bruhat function on $V'(\mathbf{A})$.

We now set

$$f_{V',\ell}(g,\Phi,s) = \int_{\mathrm{GL}_1(\mathbf{A})} |t|^s \Phi(tf_2 g)\, dt.$$

Denote by $P_{V'}$ the parabolic subgroup of $\mathrm{SO}(V')$ that stabilizes the line spanned by f_2. The Eisenstein series $E_{V',\ell}(g,s)$ is defined as

$$E_{V',\ell}(g,s) = \sum_{\gamma \in P'(\mathbf{Q}) \backslash \mathrm{SO}(V')(\mathbf{Q})} f_{V',\ell}(\gamma g, \Phi, s).$$

The sum converges absolutely when $Re(s) > 10$. The purpose of this subsection is to prove the following proposition.

Proposition 6.1.1. *The Eisenstein series $E_{V',8}(g,s)$ is regular at $s = 8$, and is the automorphic function associated to a holomorphic weight 8 modular form on $\mathrm{SO}(V')$ with rational Fourier coefficients.*

As mentioned above, Proposition 6.1.1 is likely well-known; as we do not know of a precise reference, we give a brief sketch of the proof.

Proof. Denote by $P_{0,V'}$ the minimal parabolic of $\mathrm{SO}(V')$ that stabilizes the flag

$$V' \supseteq \mathrm{Span}(e_2, \Theta, f_2, f_1) \supseteq \mathrm{Span}(\Theta, f_2, f_1) \supseteq \mathrm{Span}(f_2, f_1) \supseteq \mathrm{Span}(f_1) \supseteq 0.$$

We begin by computing the constant term of $E_{V',8}(g,s)$ to $P_{0,V'}$. Denote by r_1, r_2 characters of the diagonal split torus of $\mathrm{SO}(V')$, so that the positive roots associated to $P_{0,V'}$ are $\{r_1 - r_2, r_1, r_2, r_1 + r_2\}$. Denote the simple reflections associated to these positive roots by w_{12} and w_2, in obvious notation. The constant term of $E_{V',8}(g,s)$ along $P_{0,V'}$ is the sum of four terms of the form $M(w,s) f_{V',8}(g, \Phi, s)$, for $w = 1, w_{12}, w_2 w_{12}, w_{12} w_2 w_{12}$.

Define $\lambda_s = (s-5)r_1 + (-4)r_2$. Then the inducing section $f_{V',\ell}(g,\Phi,s)$ defines an element of $Ind_{P_{0,V'}}^{\mathrm{SO}(V')}(\delta_{P_{0,V'}}^{1/2} \lambda_s)$ that is spherical at every finite place, but not spherical at infinity. Applying the Weyl group elements w, the character λ_s is moved as follows:

- $\lambda_s = (s-5)r_1 + (-4)r_2$
- $\overset{w_{12}}{\mapsto} (-4)r_1 + (s-5)r_2$, $\frac{\zeta(s-1)}{\zeta(s)} \frac{\Gamma_{\mathbf{R}}(s-1)}{\Gamma_{\mathbf{R}}(s)} \frac{\left(\frac{2-s}{2}\right)_{\ell/2}}{\left(\frac{s}{2}\right)_{\ell/2}}$
- $\overset{w_2}{\mapsto} (-4)r_1 + (5-s)r_2$, $\frac{\zeta_\Theta(s-5)}{\zeta_\Theta(s-1)} \frac{\Gamma(s-5)}{\Gamma(s-1)} = \frac{\zeta(s-5)\zeta(s-8)}{\zeta(s-1)\zeta(s-4)} \frac{\Gamma(s-5)}{\Gamma(s-1)}$
- $\overset{w_{12}}{\mapsto} (5-s)r_1 + (-4)r_2$, $\frac{\zeta(s-9)}{\zeta(s-8)} \frac{\Gamma_{\mathbf{R}}(s-9)}{\Gamma_{\mathbf{R}}(s-8)} \frac{\left(\frac{10-s}{2}\right)_{\ell/2}}{\left(\frac{s-8}{2}\right)_{\ell/2}}$
- $= \lambda_{10-s}$.

We have also included above the $c(w,s)$-factors introduced by the intertwining operators. Plugging in $\ell = 8$, one finds that the above $c(w,s)$-functions are 0 at $s = 8$, using that $\zeta(0)$ is finite and nonzero.

It follows that the constant term of $E_{V',8}(g, s = 8)$ to $P_{V',0}$ consists only of $f_{V',8}(g, \Phi, s = 8)$. Moreover, for $g = g_\infty \in \mathrm{SO}(V')(\mathbf{R})$, one has

$$f_{V',8}(g,\Phi,s=8)=\pi^{-8}\zeta(8)\Gamma(8)\frac{(f_2g,v_1+iv_2)^8}{||f_2g||^{16}}=\frac{C_8}{(f_2g,v_1-iv_2)^8}$$

for a nonzero rational number C_8.

From the above facts one can deduce that $E_{V',8}(g,\Phi,s=8)$ corresponds to a holomorphic modular form on $\mathrm{SO}(V')$ of weight 8 with rational Fourier coefficients. □

6.2. *The Eisenstein series on* $\mathrm{SO}(H^3\oplus\Theta)$

Set $V=H^3\oplus\Theta=H\oplus V'$, with H spanned by e_1,f_1 and $V'=\mathrm{Span}\{e_2,e_3,\Theta,f_3,f_2\}$. The fixed integral lattice in V is $V_0=H_0^3\oplus\Theta_0$. Denote by Φ_f the characteristic function of the lattice $V_0(\widehat{\mathbf{Z}})$ in $V(\mathbf{A}_f)$. In this subsection, we analyze the Eisenstein series $E_8(g,\Phi_f,s)$ on $\mathrm{SO}(V)$ at $s=9$, which is outside the range of absolute convergence. In particular, we prove the following proposition.

Proposition 6.2.1. *The Eisenstein series $E_8(g,\Phi_f,s)$ is regular at $s=9$ and defines a modular form of weight 8 on $\mathrm{SO}(V)$ at this point. Moreover, $\pi^{-8}E_8(g,\Phi_f,s=9)$ has rational Fourier expansion.*

Proof. We consider the constant term of $E_8(g,\Phi_f,s)$ to the Levi subgroup $\mathrm{GL}_1\times\mathrm{SO}(V')$. There are three terms: The inducing section (supported on x^8y^8), the Eisenstein series $E_{V',8}(g,s-1;8)$ (supported on $x^{16}+y^{16}$; see [Pol20b, Proposition 3.3.2]), and an intertwining operator $M(w_0,s)$ applied to the inducing section.

As the inducing section is spherical at every finite place, the finite part of the intertwining operator $M(w_0,s)$ is computed easily. For the finite places, one obtains for the function $c(w_0,s)$

$$\begin{aligned}c(w_0,s)&=\frac{\zeta(s-1)}{\zeta(s)}\frac{\zeta(s-2)}{\zeta(s-1)}\frac{\zeta_\Theta(s-6)}{\zeta_\Theta(s-2)}\frac{\zeta(s-10)}{\zeta(s-9)}\frac{\zeta(s-11)}{\zeta(s-10)}\\&=\frac{\zeta(s-6)\zeta(s-11)}{\zeta(s)\zeta(s-5)}.\end{aligned}$$

Consequently, $c(w_0,s)$ vanishes at $s=9$. Moreover, by Remark 4.1.3, the archimedean intertwiner is finite and nonzero at $s=9$. Therefore, the intertwined inducing section vanishes at $s=9$.

Applying Proposition 6.1.1, one obtains that $E_8(g,s=9;8)$ has constant term a sum of the inducing section $f(g,\Phi_f,s=9)$ and the holomorphic Eisenstein series $E_{V',8}(g,s=8;8)$. It now follows from Corollary 3.2.5 that D_8 annihilates this constant term. From Lemma 3.2.6 one then concludes that $D_8E_8(g,s=9;8)=0$, proving that this Eisenstein series is a modular form of weight 8 on $\mathrm{SO}(V)$.

To prove that the Fourier expansion of $E_8(g,s=9;8)$ is rational, we use Corollary 4.5.8. In particular, in Corollary 4.5.8, one only needs the inequality $\ell\geq n/2$, not $\ell>n+1$. In our case of interest, $\ell=8$ and $n=10$, so we may apply this result. We

claim that the rank one Fourier coefficients of $E_8(g, s = 9; 8)$ consists only of a single term, as in (4); the integral in (5) again vanishes.

To see that the integral (5) vanishes, one proceeds as follows. First, because η is isotropic, we assume without loss of generality that $\eta = rf_2$. Because our inducing section is spherical, we can assume r is an integer. Now by Corollary 4.5.8–which we may apply as remarked above–the archimedean part of (5) vanishes at $s = 9$. Thus, we must show that the finite adelic part does not give rise to a pole at this value of s. To see this, for this particular $\eta = rf_2$, one can compute the integral (5) directly, by factoring it as a spherical intertwining operator and then a one-dimensional character integral. The spherical intertwining operator is $M(w, s) = M(w_{23}w_{e_3}w_{23}w_{12}, s)$, in the notation of Proposition 4.1.2. This intertwining operator gives a function $c(w, s)$ as

$$\begin{aligned} c(w,s) &= \frac{\zeta(s-1)}{\zeta(s)} \frac{\zeta(s-2)}{\zeta(s-1)} \frac{\zeta_\Theta(s-6)}{\zeta_\Theta(s-2)} \frac{\zeta(s-10)}{\zeta(s-9)} \\ &= \frac{\zeta(s-6)\zeta(s-9)}{\zeta(s)\zeta(s-5)} \end{aligned}$$

which is finite and nonzero at $s = 9$. The one-dimensional character integral produces a factor of $\frac{\sigma_{s-11}(n)}{\zeta(s-10)}$, which again is finite and non-zero at $s = 9$. Altogether, one sees that the integral (5) vanishes at $s = 9$, as desired.

Because of this vanishing, and again because Corollary 4.5.8 applies in the case $\ell = 8$, $n = 10$, the calculation of the Fourier coefficients of $E_8(g, s = 9)$ now proceeds exactly as in Theorem 4.5.10, using Proposition 6.1.1 to treat the constant term. Because the inducing section is spherical at every finite place, the Fourier coefficients are valued in $\mathbf{Q} \subseteq \overline{\mathbf{Q}}$. This completes the proof of the proposition. □

6.3. Intertwining operators and the modular form of weight 8 on G_J

In this subsection we analyze the Eisenstein $E_J(g, s; 8)$ that is spherical at every finite place. See [Pol20b, section 2.2] for this Eisenstein series. Let P_0 denote the minimal parabolic of G_J. The purpose of this subsection is to prove the following proposition.

Proposition 6.3.1. *The Eisenstein series $E_J(g, s; 8)$ is regular at $s = 9$ and defines a square integrable modular form of weight 8 at this point. Moreover, its constant term along P_0 is a sum of two terms.*

Our proof of Proposition 6.3.1 rests on the computation of several intertwining operators. The rational root system of G_J is of type F_4; let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the simple roots:

$$\circ - - - - \circ \Longrightarrow \circ - - - - \circ;$$

the roots are labeled $1, 2, 3, 4$ from left to right. Let Φ_+ denote the positive roots for this root system and Φ_{C_3} the roots inside the Levi of type C_3. As is standard, let

$$[W_{F_4}/W_{C_3}] = \{w \in W_{F_4} : w(\Phi_{C_3} \cap \Phi_+) \subseteq \Phi_+\}$$

be the set of minimal length coset representatives. Here W_{F_4} is the Weyl group of the F_4 root system, and W_{C_3} is the subgroup of W_{F_4} generated by the simple reflections corresponding to the roots $\alpha_2, \alpha_3, \alpha_4$. The set $[W_{F_4}/W_{C_3}]$ has 24 elements.

As in [Gan00a], we single out two special elements of $[W_{F_4}/W_{C_3}]$:

$$w_0 = [123214323412321]$$
$$w_{-1} = [23214323412321],$$

of length 15 and 14 respectively. Here the indices indicate how w_0, w_1 are expressed as a product of simple reflections. All other elements of $[W_{F_4}/W_{C_3}]$ have length less than 14.

Denote by $f_J(g, s; 8)$ the inducing section used to define the Eisenstein series $E_J(g, s; 8)$ that is spherical at every finite place. For $w \in [W_{F_4}/W_{C_3}]$, we consider the intertwining operator

$$M(w, s)f(g, s; 8) = \int\limits_{U_w(\mathbf{A})} f(w^{-1}ng, s; 8)\, dn.$$

Here U_w is the unipotent group defined as

$$U_w = \prod_{\alpha>0: w^{-1}(\alpha)<0} U_\alpha$$

and U_α is the unipotent group associated to the rational root α. Note that if α is a long root, then $\dim U_\alpha = 1$, whereas if α is a short root then $\dim U_\alpha = 8$.

With notation as above, the content of this subsection is to prove the following proposition, which will be the main step in proving Proposition 6.3.1.

Proposition 6.3.2. *Suppose $w \in [W_{F_4}/W_{C_3}]$. Then*

(1) *If $w \neq w_0, w \neq w_{-1}$, then $M(w, s)f(g, s; 8)$ is finite at $s = 20$.*
(2) *If $w = w_{-1}$, then $M(w, s)f(g, s; 8)$ has a simple pole at $s = 20$.*
(3) *If $w = w_0$, then $M(w, s)f(g, s; 8)$ has a simple pole at $s = 20$ and vanishes at $s = 9$.*

Proof. Let us first write down the long intertwiner $M(w_0, s)$. At the finite places, one obtains [GRS97]

$$c(w_0, s) = \frac{\zeta(2s-29)\zeta(s-28)\zeta(s-23)\zeta(s-19)}{\zeta(2s-28)\zeta(s)\zeta(s-5)\zeta(s-9)}.$$

The function $c(w_0, s)$ is finite nonzero at $s = 20$ and 0 at $s = 9$.

At the archimedean place, one can compute $c_\infty(w_0, s)$ by factorizing $w_0 = [12321] \circ [43234] \circ [12321]$, and then using Proposition 4.1.2 to compute the [12321] factors. From now on, all archimedean intertwining operators are calculated up to exponential factors and nonzero constants. The middle [43234] intertwiner turns out to be spherical. One obtains

$$c_\infty(w_0, s) = c_8^{B_3}(s - 17) c_{mid}^{C_3}(s) c_8^{B_3}(s)$$

where

$$c_{mid}^{C_3}(s) = \frac{\Gamma(s-10)}{\Gamma(s-6)} \frac{\Gamma(s-14)}{\Gamma(s-10)} \frac{\Gamma_{\mathbf{R}}(2s-29)}{\Gamma_{\mathbf{R}}(2s-28)} \frac{\Gamma(s-15)}{\Gamma(s-11)} \frac{\Gamma(s-19)}{\Gamma(s-15)} = \frac{\Gamma(s-\frac{29}{2})\Gamma(s-19)}{\Gamma(s-6)\Gamma(s-11)}$$

and $c_\ell^{B_3}(s)$ is from Proposition 4.1.2. In this case,

$$c_8^{B_3}(s) = \frac{\left(\frac{s-9}{2}\right)_4}{\left(\frac{s-2}{2}\right)_5} \cdot \frac{\Gamma(s-6)}{\Gamma(s-2)} \cdot \frac{\left(\frac{s-18}{2}\right)_4}{\left(\frac{s-11}{2}\right)_5}.$$

Simplifying,

$$c_\infty(w_0, s) = \frac{\Gamma(s-6)\Gamma(s-23)}{\Gamma(s-2)\Gamma(s-19)} \cdot \frac{\left(\frac{s}{2}-9\right)_4 \left(\frac{s-17}{2}-9\right)_4}{\left(\frac{s}{2}-1\right)_5 \left(\frac{s-11}{2}\right) \left(\frac{s-19}{2}\right)_5 \left(\frac{s}{2}-14\right)}.$$

This function is immediately checked to be finite and nonzero at $s = 9$ and has a pole at $s = 20$. Combining with properties of $c(w_0, s)$, this gives part (3) of the proposition.

Most of the $w \in [W_{F_4}/W_{C_3}]$ give absolutely convergent adelic integrals at $s = 20$. There are 7 that do not, and these 7 have the following factorizations:

- [4323412321]
- [3214323412321]
- $w_{-1} = [23214323412321]$
- [214323412321]
- [21323412321]
- [14323412321]
- $w_0 = [123214323412321]$

We will explain in a bit of detail the computation of $M(w_{-1}, s)$. The computation of the other intertwining operators is completely analogous or simpler.

To record the computations, we use the standard Euclidean model of the F_4-root system. Specifically, consider $\mathbf{Z}^4$, with inner product $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$, where $x = (x_1, x_2, x_3, x_4)_E$ and $y = (y_1, y_2, y_3, y_4)_E$. We write the subscript 'E' to indicate the implicit Euclidean inner product. Now, set

- $\alpha_1 = (0, 1, -1, 0)_E$
- $\alpha_2 = (0, 0, 1, -1)_E$
- $\alpha_3 = (0, 0, 0, 1)_E$
- $\alpha_4 = \frac{1}{2}(1, -1, -1, -1)_E$.

With the α_i as the simple roots, this gives a model of the F_4-root system.

Let $\lambda_s = (s - 23, s - 6, -5, -4)$, which we think of as an unramified character of P_0. For this λ_s, one has $f(g, s; 8) \in Ind_{P_0}^{G_J}(\lambda_s \delta_{P_0}^{1/2})$. Although it is a bit more than is necessary to compute $M(w_{-1}, s)$, we record how the long element w_0 moves around λ_s:

- $\lambda_s = (s - 23, s - 6, -5, -4)_E$
- $[1]; (s - 23, -5, s - 6, -4)_E; s - 1$
- $[2]; (s - 23, -5, -4, s - 6)_E; s - 2$
- $[3]; (s - 23, -5, -4, 6 - s)_E; s - 6$
- $[2]; (s - 23, -5, 6 - s, -4)_E; s - 10$
- $[1]; (s - 23, 6 - s, -5, -4)_E; s - 11$
- $[4]; (-13, -4, s - 15, s - 14)_E; s - 10$
- $[3]; (-13, -4, s - 15, 14 - s)_E; s - 14$
- $[2]; (-13, -4, 14 - s, s - 15)_E; 2s - 19$
- $[3]; (-13, -4, 14 - s, 15 - s); s - 15$
- $[4]; (6 - s, s - 23, -5, -4)_E; s - 19$
- $[1]; (6 - s, -5, s - 23, -4)_E; s - 18$
- $[2]; (6 - s, -5, -4, s - 23)_E; s - 19$
- $[3]; (6 - s, -5, -4, 23 - s)_E; s - 23$
- $[2]; (6 - s, -5, 23 - s, -4)_E; s - 27$
- $[1]; (6 - s, 23 - s, -5, -4)_E; s - 28$
- $= \lambda_{29-s}$.

In each line, the $[j]$ indicates that a simple reflection corresponding to the root j has been performed, to get from the previous line to the current line. The final parameter $s - k$ is the parameter needed to calculate a rational-rank-one intertwining operator, and is given as follows: If one has $[j]\mu' = \mu$, then the final parameter is the Euclidean inner product $\alpha_j \cdot \mu'$. For example, in the line

$$\bullet [2]; (s - 23, -5, -4, s - 6)_E; s - 2,$$

one has $j = 2, \mu = (s - 23, -5, -4, s - 6)_E$, $\mu' = (s - 23, -5, s - 6, -4)_E$, and

$$\alpha_2 \cdot \mu' = (0, 0, 1, -1)_E \cdot (s - 23, -5, s - 6, -4)_E = (1)(s - 6) + (-1)(-4) = s - 2.$$

With the above data, and combining Proposition 4.1.4, Proposition 4.1.2, and the technique of [Pol20b, page 26], one can compute the intertwining operators $M(w, s)$ without too much difficulty. As mentioned, we will now detail the computation of $M(w_{-1}, s)$.

First, the finite, spherical part of $M(w_{-1}, s)$ is computed immediately from the terms $s-k$ of the above itemized data. One gets

$$c(w_{-1}, s) = \frac{\zeta(2s-29)\zeta(s-27)\zeta(s-23)\zeta(s-19)}{\zeta(2s-28)\zeta(s)\zeta(s-5)\zeta(s-9)}.$$

The function $c(w_{-1}, s)$ has a simple pole at $s = 20$.

The archimedean calculation is, of course, more involved. First, applying the first 10 elements in the factorization of w_{-1} gives an archimedean factor of $c_{mid}^{C_3}(s)c_8^{B_3}(s)$, in the notation above. This product is

$$c_{mid}^{C_3}(s)c_8^{B_3}(s) = \frac{\Gamma(s-\frac{29}{2})\Gamma(s-19)}{\Gamma(s-6)\Gamma(s-11)} \frac{\left(\frac{s-9}{2}\right)_4}{\left(\frac{s-2}{2}\right)_5} \cdot \frac{\Gamma(s-6)}{\Gamma(s-2)} \cdot \frac{\left(\frac{s-18}{2}\right)_4}{\left(\frac{s-11}{2}\right)_5},$$

which is finite and nonzero at $s = 20$.

Set $f_1 = x + y$, $f_2 = x - y$, as before Proposition 4.1.4. The archimedean intertwiner $M_\infty([21], s)$ that comes after the [4323412321] can now be computed by Proposition 4.1.4. One obtains a factor of $C_8(s-17)$, in the notation of that proposition, and the resulting inducing section is supported on $f_1^8 f_2^8$ at $g_\infty = 1$. The function $C_8(s-17)$ is[3]

$$C_8(s-17) = \frac{\left(\frac{s-26}{2}\right)_4}{\left(\frac{s-17}{2}-1\right)_5}.$$

The next intertwining operator, an application of $M([3], s)$, is spherical, and gives a factor of $\frac{\Gamma(s-23)}{\Gamma(s-19)}$, which has a simple pole at $s = 20$. Thus, the archimedean intertwiner $M_\infty([3214323412321], s)$ is finite and nonzero at $s = 20$.

To do the final application of an intertwining operator, the $M_\infty([2], s)$, we use the technique of [Pol20b, page 26]. Expressing $f_1^8 f_2^8$ in the x, y basis, one gets

$$f_1^8 f_2^8 = C_4(x^{16} + y^{16}) + C_3(x^{14}y^2 + x^2y^{14}) + C_2(x^{12}y^4 + x^4y^{12}) + C_1(x^{10}y^6 + x^6y^{10}) + C_0(x^8y^8)$$

for nonzero constants C_k, $0 \le k \le 4$. For $0 \le k \le 4$, on the term multiplying C_k the operator $M_\infty([2], s)$ now produces factors of the form

$$\frac{\Gamma_{\mathbf{R}}(s-27)}{\Gamma_{\mathbf{R}}(s-26)} \frac{\left(\frac{28-s}{2}\right)_k}{\left(\frac{s-26}{2}\right)_k}.$$

For $0 \le k \le 3$ this is zero, while for $k = 4$ this is finite and nonzero. This completes our analysis of $M(w_{-1}, s)$, and with it, the proposition. □

[3] Although this function vanishes to first order $s = 20$, the function $c([214323412321], s)$ has a simple pole at $s = 20$, so there is no contradiction to the statement of the proposition.

We now complete the proof of Proposition 6.3.1.

Proof of Proposition 6.3.1. Using Proposition 6.3.2, the proof of Proposition 6.3.1 proceeds exactly as the proof of Corollary 4.1.2 of [Pol20b]. The only thing left to remark upon is the square integrability of $E_J(g, s = 9; 8)$. For this, we apply Jacquet's criterion [MgW95, I.4.11 Lemma]. Writing in terms of the simple rational roots, one has

$$\lambda_s = (s-23, s-6, -5, -4)_E = (2s-29)\alpha_1 + (3s-57)\alpha_2 + (4s-84)\alpha_3 + (2s-46)\alpha_4$$

and

$$[1]\lambda_s = (s-23, -5, s-6, -4)_E = (s-28)\alpha_1 + (3s-57)\alpha_2 + (4s-84)\alpha_3 + (2s-46)\alpha_4.$$

Plugging in $s = 9$, one sees that all the exponents are negative in these characters, and thus $E_J(g, s = 9; 8)$ is square integrable. □

6.4. Proof of Theorem 6.0.2

The proof of this theorem, in its entirety, was outlined at the beginning of section 6. The only thing left to prove is step (4) of this outline and to discuss the constant term $E_J^{N_J}(g, s = 9; 8)$ of $E_J(g, s = 9; 8)$ along the unipotent radical of the Heisenberg parabolic.

For step (4), note that the constant term $E_J^V(g, s; 8)$ for g in the Levi subgroup $L_J(\mathbf{A})$ is a sum of Eisenstein series on L_J, one for each element of the double coset

$$P_J(\mathbf{Q})\backslash G_J(\mathbf{Q})/Q_J(\mathbf{Q}) = W_{C_3}\backslash W_{F_4}/W_{B_3}.$$

The Eisenstein series associated to the double coset $P_J(\mathbf{Q})1Q_J(\mathbf{Q})$ is $E_8^{L_J}(g, s)$.

At $s = 9$, we have computed the constant term of each of these Eisenstein series down to P_0, and it is clear that they are identified, because the two terms contributing to the constant term of $E_J(g, s = 9; 8)$ along P_0 are those that come from the elements of length 0 and 1 of $[W_{C_3}\backslash W_{F_4}]$. Consequently, at $s = 9$, the difference $E_J^V(g, s; 8) - E_8^{L_J}(g, s)$ has vanishing constant term along P_0. Because the difference $E_J^V(g, s; 8) - E_8^{L_J}(g, s)$ is a sum of Eisenstein series on L_J, we conclude $E_J^V(g, s = 9; 8) = E_8^{L_J}(g, s = 9)$. This proves step (4) of the outline above.

The constant term $E_J^{N_J}(g, s = 9; 8)$ is analyzed in [Pol20b, Corollary 3.5.1]. The holomorphic weight 8 Siegel Eisenstein on $H_J = GE_{7,3}$ appears, along with the constant $\frac{\zeta(9)}{(2\pi)^8}$. As mentioned previously, this weight 8 Eisenstein series is analyzed in [Kim93], who proves that it has rational Fourier coefficients. This completes the proof of Theorem 6.0.2.

7. The minimal modular form

In this final section, we discuss the minimal modular form on $G = \mathrm{SO}(3, 8k+3)$ in certain special cases, and an application to the construction of a distinguished modular form on $G' = \mathrm{SO}(3, 8k+2)$. The results of this section are, in a sense, the analogues of the results in [Pol20b] with quaternionic E_8 replaced by the classical group $D_{4k+3,3}$.

In more detail, the group G supports a minimal modular form θ, that is spherical at all finite places. The purpose of this section is to recognize θ as the value of an Eisenstein series on G, to compute its Fourier expansion, and to show that the restriction $\theta' = \theta|_{G'}$ of θ to G' is a distinguished modular form on G'. The fact that θ' is distinguished is an example of the simplest "lifting law" from [Pol18], the one in section 2.2 of [Pol18], whereas the distinguished and singular modular forms constructed in [Pol20b] use the (more complicated) lifting laws considered in section 7 and section 8 of [Pol18].

Let Θ_0 denote the ring of Coxeter's octonions, so that Θ_0 is the even unimodular quadratic lattice of dimension 8. For an integer $k \geq 1$, let $V_0 = H_0^3 \oplus \Theta_0^k$ be the fixed lattice inside $V = H^3 \oplus \Theta^k$. Set $V' = H^2 \oplus \Theta^k$ and set $G = \mathrm{SO}(V)$. Fix a vector $\omega \in V'$ with $q'(\omega) < 0$. Denote by $V_\omega = (\mathbf{Q}\omega)^\perp$ the orthogonal complement of $\mathbf{Q}\omega$, so that $V = \mathbf{Q}\omega \oplus V_\omega$. Set $G' = \mathrm{SO}(V_\omega) \simeq \mathrm{SO}(3, 8k+2)$ and define $V'_\omega = V_\omega \cap V'$.

Let Φ_f be the characteristic function of $V_0 \otimes \widehat{\mathbf{Z}} \subseteq V \otimes \mathbf{A}_f$ and $E(g,s) = \pi^{-4k} E_{4k}(g, \Phi_f, s)$ the Eisenstein series which was studied in detail in section 4. The purpose of this section is to prove the following theorem.

Theorem 7.0.1. *The Eisenstein series $E(g,s)$ is regular at $s = 4k+1$ and $\theta(g) := E(g, s = 4k+1)$ is a modular form of weight $\ell = 4k$ at this point. The modular form θ has rational Fourier expansion with all rank two Fourier coefficients equal to 0. The restriction $\theta' = \theta|_{G'}$ is a modular form on G' of weight ℓ. It is distinguished in the sense that if $\eta \in V'_\omega$ has $q'(\eta) \neq 0$, then the Fourier coefficient $a_{\theta'}(\eta) \neq 0$ implies $q'(\eta) \in (\mathbf{Q}^\times)^2(-q(\omega))$.*

Proof. Most of the work is to check the relevant intertwining operators, which show that $E(g,s)$ is regular at $s = \ell + 1$ and defines a modular form of weight ℓ at this point. To do this, we proceed as in section 6 and consider first the case of the Eisenstein series $E_{V',\ell}(g,s)$ on $\mathrm{SO}(V')$ evaluated at $s = \ell = 4k$.

The constant term of the Eisenstein series $E_{V',\ell}(g,s)$ along the minimal rational parabolic consists of four terms: the inducing section, and intertwining operators $M(w,s)$ applied to the inducing section, where $w = w_{12}, w_2 w_{12}, w_{12} w_2 w_{12}$ in the notation of Proposition 6.1.1. For ease of notation, set $m_0 = 4k$ and $\zeta_{an}(s) = \zeta(s)\zeta(s - m_0 + 1)$. These intertwining operators produce functions $c(w,s)$ as follows:

(1) $c(w_{12}, s) = \frac{\zeta(s-1)}{\zeta(s)} \frac{\Gamma_{\mathbf{R}}(s)}{\Gamma_{\mathbf{R}}(s)} \frac{\left(\frac{2-s}{2}\right)_{\ell/2}}{\left(\frac{s}{2}\right)_{\ell/2}}$

(2) $c(w_2 w_{12}, s) = c(w_{12}, s) \frac{\zeta_{an}(s - m_0 - 1)}{\zeta_{an}(s-1)} \frac{\Gamma(s - m_0 - 1)}{\Gamma(s-1)}$

$$(3)\quad c(w_{12}w_2w_{12},s)=c(w_2w_{12},s)\frac{\zeta(s-2m_0-1)}{\zeta(s-2m_0)}\frac{\Gamma_{\mathbf{R}}(s-2m_0-1)}{\Gamma_{\mathbf{R}}(s-2m_0)}\frac{\left(\frac{2m_0+2-s}{2}\right)_{\ell/2}}{\left(\frac{s-2m_0}{2}\right)_{\ell/2}}.$$

One checks easily that these functions $c(w_{12},s)$, $c(w_2w_{12},s)$ and $c(w_{12}w_2w_{12},s)$ vanish at $s=\ell=m_0$. Consequently, as in Proposition 6.1.1, $E_{V',\ell}(g,s)$ is regular at $s=\ell$ and is the automorphic function associated to a holomorphic modular form of weight ℓ on $\mathrm{SO}(V')$ with rational Fourier coefficients.

Next, to see that $E_\ell(g,\Phi_f,s)$ is regular at $s=\ell+1=4k+1$, we consider its constant term along the parabolic $P=MN$, just as in Proposition 6.2.1. Just as in the proof of this proposition, to see that $E_\ell(g,\Phi_f,s)$ is regular at $s=\ell+1$ and defines a modular form of weight ℓ at this point, it suffices to check that the c-function associated to the long intertwiner $M(w_0,s)$ vanishes at $s=\ell+1$. The finite part of this intertwiner gives

$$\begin{aligned}c_f(w_0,s)&=\frac{\zeta(s-1)\zeta(s-2)\zeta_{an}(s-2-m_0)\zeta(s-2-2m_0)\zeta(s-3-2m_0)}{\zeta(s)\zeta(s-1)\zeta_{an}(s-2)\zeta(s-1-2m_0)\zeta(s-2-2m_0)}\\&=\frac{\zeta(s-2-m_0)\zeta(s-3-2m_0)}{\zeta(s)\zeta(s-1-m_0)}.\end{aligned}$$

This function vanishes at $s=\ell+1=m_0+1=4k+1$. The archimedean part of this intertwining operator was computed in Proposition 4.1.2. One obtains

$$c_\infty(w_0,s)=\frac{\left(\frac{s-\ell-1}{2}\right)_{\ell/2}}{\left(\frac{s-2}{2}\right)_{\ell/2+1}}\frac{\Gamma(s-2-m_0)}{\Gamma(s-2)}\frac{\left(\frac{s}{2}-1-m_0-\ell/2\right)_{\ell/2}}{\left(\frac{s-3}{2}-m_0\right)_{\ell/2+1}}.$$

This function is finite and nonzero at $s=m_0+1=\ell+1$. Consequently, $c(w_0,s)=c_f(w_0,s)c_\infty(w_0,s)$ vanishes at $s=\ell+1$, so that $E_\ell(g,\Phi_f,s)$ is regular at $s=\ell+1$ and defines a modular form of weight ℓ at this point.

The function $\theta(g)$ is defined to be the value $\pi^{-\ell}E_\ell(g,\Phi_f,s=\ell+1)=\pi^{-4k}E_{4k}(g,\Phi_f,s=4k+1)$. By [MS97, Theorem 1.1] and [Sav94, Proposition 4.1, Corollary 4.2], the minimal representation of split p-adic D_{4k+3} occurs as the spherical sub in $Ind_P^G(|\nu|^{4k+1})=Ind_P^G(\delta_P^{1/2}|\nu|^{-1})$. By these cited results, it follows that the modular form $\theta(g)$ has vanishing rank two Fourier coefficients.

The rationality of the rank one Fourier coefficients of $\theta(g)$ is treated similarly to the rationality of the rank one Fourier coefficients of the Eisenstein series considered in 6.2.1, but with a little more work. Specifically, the rationality follows from the vanishing of the integral (5). As in the proof of Proposition 6.2.1, to see that this integral vanishes, we factorize it into an intertwining operator and a one-dimensional character integral. The intertwining operator is associated to the element $w'=w_{23}w_3w_{23}w_{12}$ of the Weyl group, in the notation of Proposition 4.1.2. The finite part $M_f(w',s)$ produces a c-function

$$c_f(w',s)=\frac{\zeta(s-1)\zeta(s-2)\zeta_{an}(s-2-m_0)\zeta(s-2-2m_0)}{\zeta(s)\zeta(s-1)\zeta_{an}(s-2)\zeta(s-1-2m_0)}$$

$$= \frac{\zeta(s-2-m_0)\zeta(s-2-2m_0)}{\zeta(s)\zeta(s-1-m_0)},$$

which is finite and nonzero at $s = m_0 + 1$. Similar to the evaluation of $M(w_{-1}, s)$ in the proof of Proposition 6.3.2, the archimedean part of the intertwiner $M(w', s)$ produces c-functions

$$c_j(w', s) = \frac{\left(\frac{s-\ell-1}{2}\right)_{\ell/2}}{\left(\frac{s-2}{2}\right)_{\ell/2+1}} \frac{\Gamma(s-2-m_0)}{\Gamma(s-2)} \frac{\Gamma_{\mathbf{R}}(s-2-2m_0)}{\Gamma_{\mathbf{R}}(s-1-2m_0)} \frac{\left(\frac{3+2m_0-s}{2}\right)_j}{\left(\frac{s-1-2m_0}{2}\right)_j}$$

for integers j with $0 \leq j \leq \ell/2 = m_0/2$. All of these functions $c_j(w', s)$ vanish at $s = m_0+1 = \ell+1$. Finally, the one-dimensional character integrals produce holomorphic functions of s divided by $\zeta(s-2-2m_0)\Gamma_{\mathbf{R}}(s-2-2m_0)$. This latter function is finite at $s = m_0 + 1$. Combining this with the calculation of the functions $c(w', s)$, one sees that the integral (5) vanishes at $s = m_0 + 1$, as desired. The rationality of the Fourier expansion of $\theta(g)$ follows.

The fact that θ' is a modular form of weight ℓ follows by a simple analysis of the differential operators D_ℓ on G and on G' as in Proposition 5.1.1. Finally, that θ' is distinguished follows from the discussion in [Pol18, section 2.2]. This completes the proof of the theorem. □

Appendix A. Proofs of selected results

This appendix contains the proofs of some of the results stated in the main body of the text but not proved there.

A.1. Proofs from section 3

Proof of Proposition 3.2.1. We first write out $\widetilde{D_\ell}$ in coordinates. We obtain

$$\begin{aligned}
\widetilde{D}F = \sum_{j=1}^{n} & D^M_{iv_1-v_2,u_j} F \otimes ((-1/2)y \otimes y \otimes u_j^\vee) \\
& + (u_+ \wedge u_j)F \otimes ((\sqrt{2}/4)(x \otimes y + y \otimes x) \otimes u_j^\vee) \\
& + D^M_{iv_1+v_2,u_j} F \otimes ((-1/2)x \otimes x \otimes u_j^\vee) \\
& + ((iv_1 - v_2) \wedge u_-)F \otimes ((-1/2)y \otimes y \otimes u_-^\vee) \\
& + (u_+ \wedge u_- F) \otimes ((\sqrt{2}/4)(x \otimes y + y \otimes x) \otimes u_-^\vee) \\
& + ((iv_1 + v_2) \wedge u_- F)((-1/2)x \otimes x \otimes u_-^\vee).
\end{aligned}$$

To compute the operator D_ℓ in coordinates, we must apply the contraction pr : $\mathbb{V}_\ell \otimes \mathfrak{p}^\vee \to (S^{2\ell}(Y_2) \oplus S^{2\ell-2}(Y_2)) \boxtimes V_{n+1}$. With our coordinates x, y this contraction

is $[x^{\ell+v}][y^{\ell-v}] \otimes y \mapsto [x^{\ell+v-1}][y^{\ell-v}]$ and $[x^{\ell+v}][y^{\ell-v}] \otimes x \mapsto -[x^{\ell+v}][y^{\ell-v-1}]$. For D_ℓ we therefore obtain

$$\begin{aligned}
2D_\ell F = \sum_{j=1}^{n} u_j^\vee \otimes \Bigg(& \sum_{-\ell\le v\le \ell} D^M_{iv_1-v_2,u_j} F_v(-[x^{\ell+v-1}][y^{\ell-v}] \otimes y) \\
& + \sum_{-\ell\le v\le \ell} \frac{\sqrt{2}}{2}(u_+ \wedge u_j) F_v([x^{\ell+v-1}][y^{\ell-v}] \otimes x - [x^{\ell+v}][y^{\ell-v-1}] \otimes y) \\
& + \sum_{-\ell\le v\le \ell} D^M_{iv_1+v_2,u_j} F_v([x^{\ell+v}][y^{\ell-v-1}] \otimes x) \Bigg) \\
& + u_-^\vee \otimes \Bigg(\sum_{-\ell\le v\le \ell} ((iv_1 - v_2) \wedge u_- F)_v(-[x^{\ell+v-1}][y^{\ell-v}] \otimes y) \\
& + \sum_{-\ell\le v\le \ell} \frac{\sqrt{2}}{2}(u_+ \wedge u_-) F_v([x^{\ell+v-1}][y^{\ell-v}] \otimes x - [x^{\ell+v}][y^{\ell-v-1}] \otimes y) \\
& + \sum_{-\ell\le v\le \ell} ((iv_1 + v_2) \wedge u_- F)_v([x^{\ell+v}][y^{\ell-v-1}] \otimes x). \Bigg)
\end{aligned}$$

From the fact that $E \cdot ([x^{\ell+v}][y^{\ell-v}]) = (\ell + v + 1)[x^{\ell+v+1}][y^{\ell-v-1}]$ and $F \cdot ([x^{\ell+v}][y^{\ell-v}]) = (\ell - v + 1)[x^{\ell+v-1}][y^{\ell-v+1}]$ we get

$$((iv_1 - v_2) \wedge u_-)\, F = \sum_{-\ell\le v\le \ell} (-2D^{V'}_{tm(iv_1-v_2)} F_v + \sqrt{2}(\ell + v) F_{v-1})[x^{\ell+v}][y^{\ell-v}]$$

and

$$((iv_1 + v_2) \wedge u_-)\, F = \sum_{-\ell\le v\le \ell} (-2D^{V'}_{tm(iv_1+v_2)} F_v + \sqrt{2}(\ell - v) F_{v+1})[x^{\ell+v}][y^{\ell-v}].$$

The proposition follows. □

Conclusion of proof of Theorem 3.2.4. In subsection 3.2 we proved that the functions $\mathcal{W}_\eta$ of Definition 3.2.2 satisfied the correct $(K \cap M)$-equivariance property. We now complete the rest of the proof of Theorem 3.2.4.

We begin by considering the differential equations of Proposition 3.2.1. Assume for now that $t > 0$. From Proposition 3.2.1, we obtain that the F_v satisfy the following differential-difference equations:

(1) $(t\partial_t - (\ell + v))F_{v-1} = -u_\eta(t, m)F_v$
(2) $(t\partial_t - (\ell - v + 1))F_v = -u_\eta(t, m)^* F_{v-1}$

(3) $D^M_{iv_1-v_2,u_j}F_v = -i\sqrt{2}t(\eta, mu_j)F_{v-1}$
(4) $D^M_{iv_1+v_2,u_j}F_{v-1} = -i\sqrt{2}t(\eta, mu_j)F_v$.

Define F^0_v by the equality $F_v = t^{\ell+1}F^0_v$. Then we obtain

$$\begin{aligned}((t\partial_t)^2 - v^2)F^0_v &= (t\partial_t - v)(t\partial_t + v)F^0_v \\ &= (t\partial_t - v)\left(-u_\eta(t,m)^* F^0_{v-1}\right) \\ &= -u_\eta(t,m)^*(t\partial_t - v + 1)F^0_{v-1} \\ &= |u_\eta(t,m)|^2 F^0_v.\end{aligned}$$

Because $F_v(t,m)$ is of moderate growth at $t \to \infty$, we deduce that $F^0_v(t,m) = C_v(m)K_v(|u_\eta(t,m)|)$ for some function $C_v(m)$ of m.

Now, because $(y\partial_y + v)K_v(y) = -yK_{v-1}(y)$, $(t\partial_t + v)K_v(t|\mu|) = -|\mu|tK_{v-1}(|\mu|t)$ if μ is independent of t. Set $\mu = \sqrt{2}i(\eta, m(iv_1 - v_2))$. Thus

$$\begin{aligned}-(|\mu|t)C_v(m)K_{v-1}(|\mu|t) &= (t\partial_t + v)F^0_v \\ &= -u_\eta(t,m)^* F^0_{v-1} \\ &= -u_\eta(t,m)^* C_{v-1}(m)K_{v-1}(|u_\eta(t,m)|).\end{aligned}$$

Thus

$$C_v(m) = \left(\frac{u_\eta(t,m)^*}{|u_\eta(t,m)|}\right) C_{v-1}(m) = \left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)}\right) C_{v-1}(m).$$

We conclude that $C_v(m) = C_0(m)\left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)}\right)^v$ for some function $C_0(m)$ that does not depend on t.

To see that $C_0(m) = C$ is a constant, independent of m, we use the final two differential equations involving $D^M_{iv_1\pm v_2,u_j}$. One verifies immediately from the definitions that

$$D^M_{iv_1\pm v_2,u_j}\{m \mapsto (\eta, m(iv_1 - v_2))\} = -(iv_1 \pm v_2, iv_1 - v_2)(\eta, mu_j). \tag{23}$$

This is 0 for $D^M_{iv_1-v_2,u_j}$ and $2(\eta, mu_j)$ for $D^M_{iv_1+v_2,u_j}$. It follows that

$$D^M_{iv_1+v_2,u_j}(|u_\eta(t,m)|) = \frac{|u_\eta(t,m)|}{u_\eta(t,m)}\sqrt{2}ti(\eta, mu_j) \tag{24}$$

and

$$D^M_{iv_1-v_2,u_j}(|u_\eta(t,m)|) = \left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)}\right)^{-1}\sqrt{2}ti(\eta, mu_j). \tag{25}$$

From (24) and (25) and the third and fourth enumerated equations applied to the case $v = 0$, resp. $v = 1$, one obtains that $D^M_{iv_1\pm v_2,u_j}C_0(m) = 0$. By the $(K \cap M)$-equivariance

proved above, we know that $C_0(mk) = C_0(m)$ for all $k \in K \cap M$. Combined with the differential equations $D^M_{iv_1 \pm v_2, u_j} C_0(m) = 0$ for all j, this gives that $C_0(m) = C$ is a constant, as desired.

Let us now check that $F(t,m) = \sum_{\ell \leq v \leq \ell} F_v(t,m) \frac{x^{\ell+v} y^{\ell-v}}{(\ell+v)!(\ell-v)!}$ with

$$F_v(t,m) = t^{\ell+1} \left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)} \right)^v K_v(|u_\eta(t,m)|)$$

satisfies the above differential equations on the connected component of the identity. To see this, first note that the identity $(y\partial_y + v)K_v(y) = -yK_{v-1}(y)$ implies that the F_v satisfy the second difference-differential equation. Similarly, the identity $(y\partial_y - v)K_v(y) = -yK_{v+1}(y)$ implies that the F_v satisfy the first difference-differential equation. To see that this $F(t,m)$ satisfies the third and fourth difference-differential equations, first note that

$$D^M_{iv_1+v_2,u_j} \left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)} \right) = -\frac{|u_\eta(t,m)|}{u_\eta(t,m)^2} \sqrt{2} t i(\eta, mu_j).$$

Moreover, from $\partial_y K_v(y) = \frac{v}{y} K_v(y) - K_{v+1}(y)$, one obtains

$$\begin{aligned} &D^M_{iv_1+v_2,u_j} K_v(|u_\eta(t,m)|) \\ &\quad = \left(\frac{v}{|u_\eta(t,m)|} K_v(|u_\eta(t,m)|) - K_{v+1}(|u_\eta(t,m)|) \right) \left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)} \right) \sqrt{2} t i(\eta, mu_j). \end{aligned}$$

Combining these two equations gives

$$\begin{aligned} &D^M_{iv_1+v_2,u_j} \left(\left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)} \right)^v K_v(|u_\eta(t,m)|) \right) \\ &\quad = -\sqrt{2} t i(\eta, mu_j) \left(\left(\frac{|u_\eta(t,m)|}{u_\eta(t,m)} \right)^{v+1} K_{v+1}(|u_\eta(t,m)|) \right) \end{aligned}$$

which shows that the $F(t,m)$ satisfies the fourth enumerated differential equation. The case of the third equation is similar.

Finally, we consider the condition $(\eta, \eta) \geq 0$. By Lemma 3.2.3, we must check that $F(t,m) \equiv 0$ if there exists $m \in \mathrm{SO}(V')(\mathbf{R})$ so that $(\eta, m(iv_1 - v_2)) = 0$. This follows by the argument of [Pol20a, Proposition 8.2.4]. □

Appendix B. "Next to minimal representation" by Gordan Savin

Abstract. Let F be a p-adic field. We show that the spherical representation of the exceptional E_8 whose Satake parameter corresponds to the sub-sub regular unipotent orbit has its wave-front set contained in the closure of the orbit $2A_1$.

B.1. Notation and statement

Let G be the group of F-points of the exceptional group of type E_8, and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{s} \subset \mathfrak{g}$ be a rank 2 split Cartan subalgebra spanned by two adjacent co-roots. Then $\mathfrak{s}$-grading of $\mathfrak{g}$ gives a restricted root system Φ of type G_2, and we can write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha \in \Phi}\ \mathfrak{g}_\alpha),$$

where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. Then $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{s}$, and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a simple Lie algebra of type E_6. The dimension of $\mathfrak{g}_\alpha$ is 1 for long roots and 27 for short roots. As a helpful convention, we shall denote elements in long root spaces with lower case letters, and elements in short root spaces with upper case letters. Short root spaces have a cubic form defined as follows. Let β be a short root. Then there exists a unique long root α so that $\Pi = \{\alpha, \beta\}$ is a set of simple roots. In particular, $\gamma = \alpha + 3\beta$ is a long root. Fix non-zero elements $x \in \mathfrak{g}_\alpha$ and $z \in \mathfrak{g}_\gamma$. For every $Y \in \mathfrak{g}_\beta$ define $\det(Y) \in F$ by

$$[Y, [Y, [Y, x]]] = \det(Y) \cdot z.$$

The root system Φ, and the choice of simple roots $\Pi = \{\alpha, \beta\}$ defines a pair maximal parabolic subgroups $P = MN$ and $Q = LU$ of G with Levi factors of type E_7 and $E_6 \times A_1$, respectively. The unipotent radical N is a Heisenberg group with one dimensional center N_1. We have

$$\mathrm{Lie}(N_1) \cong \mathfrak{g}_{2\alpha+3\beta}$$

and

$$\mathrm{Lie}(N/N_1) \cong \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha+2\beta} \oplus \mathfrak{g}_{\alpha+3\beta}.$$

Let $\psi : F \to \mathbb{C}^\times$ be a non-zero character. Any character of N is of the form

$$\psi_{\bar{n}}(n) = \psi(\langle \log n, \bar{n} \rangle),$$

for some

$$\bar{n} \in \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\alpha-\beta} \oplus \mathfrak{g}_{-\alpha-2\beta} \oplus \mathfrak{g}_{-\alpha-3\beta} \cong \mathrm{Lie}(N/N_1)^*$$

where $\langle \log n, \bar{n} \rangle$ is the Killing form pairing on $\mathfrak{g}$. Open M-orbits on $\mathrm{Lie}(N/N_1)^*$ are parameterized by $F^\times/(F^\times)^2$. To see this, the stabilizer C of $(1, 0, 0, 1)$ in M is a semi-direct product of a simply connected E_6 and S_2. Since the Galois cohomology of p-adic simply connected groups is trivial, it follows that the Galois cohomology of C is equal to $H^1(F, S_2) \cong F^\times/(F^\times)^2$. More precisely, there is a quartic homogeneous polynomial q on

$\mathrm{Lie}(N/N_1)^*$ and a character ν of M, such that $q(m(\bar{n})) = \nu(m)^2 q(\bar{n})$, for all $\bar{n}$. Thus an open orbit consists of all $\bar{n} \in \mathrm{Lie}(N/N_1)^*$ such that $q(\bar{n})$ is in a fixed class of squares in $F^\times$. The reader can find a general formula for q in Section 4.3 [Po]. We shall only need the case $\bar{n} = (a, 0, A, 0)$ when $q(\bar{n})$ is a multiple of $a \cdot \det(A)$. In particular, every open orbit has an element $\bar{n}$ of this form.

Let $I(s)$, $s \in \mathbb{R}$ be the degenerate principal series representation of G obtained by inducing unramified, $\mathbb{R}^+$-valued, characters of P. The induction is normalized, and the parameter s is chosen so that the trivial representation is a quotient of $I(29/2)$ and a submodule of $I(-29/2)$. More generally, we have $I(s)^\vee \cong I(-s)$.

Theorem B.1.1. *Let $V_0 \subseteq I(-11/2)$ be a submodule generated by a spherical vector. Then V_0 is small, i.e.*

$$(V_0)_{N,\psi_{\bar{n}}} = 0$$

for all $\bar{n} = (a, 0, A, 0)$ with $\det(A) \neq 0$. In particular, its Wave-Front set [JLS] is contained in the closure of the nilpotent orbit whose Bala-Carter notation is $2A_1$.

We observe that, since V_0 is a submodule of $I(-11/2)$, its Wave-Front set is not larger than the closure of the nilpotent orbit whose Bala-Carter notation is A_2. The vanishing given by Theorem B.1.1 guarantees that the Wave-Front is small as stated.

B.2. Fourier Jacobi functor

Now we turn our attention to the other maximal parabolic subgroup $Q = LU$. The unipotent subgroup U has a filtration $U \supset U_1 \supset U_2$ where $U_1 = [U, U]$ and U_2 is the center of U. We have

$$\mathrm{Lie}(U_2) \cong \mathfrak{g}_{\alpha+3\beta} \oplus \mathfrak{g}_{2\alpha+3\beta},$$
$$\mathrm{Lie}(U_1/U_2) \cong \mathfrak{g}_{\alpha+2\beta}$$

and

$$\mathrm{Lie}(U/U_1) \cong \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta}.$$

Any $A \in \mathfrak{g}_{-\alpha-2\beta} \cong \mathrm{Lie}(U_1/U_2)^*$ defines a character ψ_A of U_1 by

$$\psi_A(u) = \psi(\langle \log u, A \rangle).$$

Assume that $\det A \neq 0$. (We note that the Levi factor L acts transitively on the set of A such that $\det(A) \neq 0$ [SW].) The character ψ_A is trivial on U_2, and there exists a unique irreducible (Heisenberg) representation ρ_A of U/U_2 such that U_1/U_2 acts on it as ψ_A. This representation can be realized as follows: Let $U' = U \cap N$. Let ψ'_A be the

restriction of $\psi_{\bar{n}}$ to U', where $\bar{n} = (a, 0, A, 0)$. The notation reflects the fact that ψ'_A depends only on A. The subgroup U' is a polarization of U needed to write down the Heisenberg representation:

$$\rho_A \cong \mathrm{Ind}_{U'}^{U}(\psi'_A).$$

Let SL_2 be the factor of $[L, L]$ corresponding to the long root α. Since α is perpendicular to $\alpha + 2\beta$, SL_2 acts trivially by conjugation on U_1/U_2 and hence on the character ψ_A. By the general theory of Weil representation, ρ_A extends to a representation of a two-fold cover $\widetilde{\mathrm{SL}}_2$ (ρ_A is essentially a tensor product of 27 Weil representations of $\widetilde{\mathrm{SL}}_2$). Thus ρ_A is a representation of (a two-fold cover) of the Jacobi group

$$J = \mathrm{SL}_2 \ltimes U.$$

Let (π, V) be a smooth J-module. Then SL_2 naturally acts on the quotient V_{U_1,ψ_A}. Let

$$FJ_A(V) = \mathrm{Hom}_U(\rho_A, V_{U_1,\psi_A}).$$

On $FJ_A(V)$ we define an action of $\widetilde{\mathrm{SL}}_2$ by

$$T \mapsto \pi(g) \circ T \circ \rho_A(g^{-1}),$$

for $g \in \widetilde{\mathrm{SL}}_2$. By Proposition 3.1 in [We], the functor $V \mapsto FJ_A(V)$ is exact and, by Corollary 2.4 in [We], we have a natural isomorphism of J-modules

$$FJ_A(V) \otimes \rho_A \cong V_{U_1,\psi_A}.$$

Let $U_0 = \mathrm{SL}_2 \cap N$, so that $\mathrm{Lie}(U_0) \cong \mathfrak{g}_\alpha$. Recall that $\bar{n} = (a, 0, A, 0)$ and let $\psi_a : U_0 \to \mathbb{C}^\times$ be the restriction of $\psi_{\bar{n}}$ to $U_0 \subset N$.

Lemma B.2.1. *Let V be a smooth J-module. We have an isomorphism*

$$V_{N,\psi_{\bar{n}}} \cong FJ_A(V)_{U_0,\psi_a}.$$

Proof. Recall that $U' = U \cap N$ and ψ'_A is the restriction of $\psi_{\bar{n}}$ to U'. Since $N = U'U_0$, we can write

$$V_{N,\psi_{\bar{n}}} = (V_{U',\psi'_A})_{U_0,\psi_a}.$$

Since $U_1 \subset U'$ and the restriction of ψ'_A to U_1 is ψ_A, we can redundantly write

$$(V_{U',\psi'_A}) = (V_{U_1,\psi_A})_{U',\psi'_A}.$$

Substituting $V_{U_1,\psi_A} \cong FJ_A(V) \otimes \rho_A$,

$$V_{U',\psi'_A} \cong FJ_A(V) \otimes (\rho_A)_{U',\psi'_A}.$$

Using $\rho_A \cong \mathrm{Ind}_{U'}^{U}(\psi'_A)$ it is well known that

$$(\rho_A)_{U',\psi'_A} \cong \mathbb{C}$$

and this isomorphism is given by $f \mapsto f(1)$, evaluating $f \in \mathrm{Ind}_{U'}^{U}(\psi'_A)$ at 1. The normalizer $\widetilde{B}_0$ of U_0 in $\widetilde{\mathrm{SL}}_2$ acts naturally on this line, hence U_0 must act trivially on this line. Now completing the proof is trivial. □

Theorem B.2.2. *We have an isomorphism of $\widetilde{\mathrm{SL}}_2$-modules*

$$FJ_A(I(s)) \cong i(s) := \mathrm{Ind}_{\widetilde{B}_0}^{\widetilde{\mathrm{SL}}_2} \widetilde{\chi}_s$$

where $\widetilde{\chi}_s$ a Weil index twisted by $|\cdot|^s$. The induction is normalized so it is irreducible for $s \neq \pm 1/2$.

Proof. We have an isomorphism of J-modules

$$(\mathrm{Ind}_{\widetilde{B}_0}^{\widetilde{\mathrm{SL}}_2} \widetilde{\chi}_s) \otimes \rho_A \cong \mathrm{Ind}_{\widetilde{B}_0}^{\widetilde{\mathrm{SL}}_2}(\rho_A \otimes \widetilde{\chi}_s)$$

given by $f \otimes v \mapsto F$,

$$F(g) = f(g) \cdot \rho_A(g)(v),\ g \in \widetilde{\mathrm{SL}}_2.$$

T he Heisenberg representation ρ_A, when restricted to $\widetilde{B}_0 U$, is induced from the character of $\widetilde{B}_0 U'$ obtained by action of this group on the line $(\rho_A)_{U',\psi'_A}$. Using transitivity of induction, it follows that the J-module $i(s) \otimes \rho_A$ is induced from a character of $B_0 U'$. Call μ_s that character.

Let W be the Weyl group of G, and $w_0 \in W$ the longest element. Since $N \subset J$, Pw_0J is an open subset of G. Let $I_0(s) \subset I(s)$ be the J-submodule consisting of functions in $I(s)$ supported on Pw_0J. It is fairly straightforward to check that $(I_0(s))_{U_1,\psi_A}$ is induced from the character μ_s of B_0U' (see Theorem 4.3.1 in [We] for a similar computation). Thus

$$(I_0(s))_{U_1,\psi_A} \cong i(s) \otimes \rho_A$$

and $i(s) \subset FJ_A(I(s))$. It remains to show that $FJ_A(I(s))$ is not larger. So assume that $FJ_A(I(s))/i(s) \neq 0$. Since any genuine representation of $\widetilde{\mathrm{SL}}_2$ is Whittaker generic (i.e. (U_0, ψ_a)-coinvariants are non-trivial for some $a \in F^\times$) and $i(s)$ is Whittaker generic for every $a \in F^\times$, it follows that the dimension of $FJ_A(I(s))_{U_0,\psi_a}$ is at least 2 for some $a \neq 0$. Hence, by Lemma B.2.1,

$$\dim(I(s)_{N,\psi_{\bar{n}}}) \geq 2.$$

This is a contradiction since this space is one-dimensional. □

Observe that, if the statement of Theorem B.1.1 fails for V, then $FJ_A(V) \neq 0$. Thus, exactness of the FJ functor, combined with Theorem B.2.2, implies that $I(s)$, for $s \neq \pm 1/2$ contains only one big irreducible subquotient. Thus, in order to prove Theorem B.1.1, it suffices to show that the quotient $I(-11/2)/V_0$ is not small. We shall execute this strategy in the next section.

B.3. Finishing the proof

We have the standard intertwining operator $A_s : I(s) \to I(-s)$. This intertwining operator is non-zero for every s, although it can have poles: for every constant section $f_s \in I(s)$, $A_s(f_s)$ is a rational function in q^s. In particular, it has finitely many poles on $\mathbb{R}$. Let f_s° be a non-zero, constant, spherical, section. Then $A_s(f_s^\circ) = c(s)f_{-s}^\circ$ where

$$c(s) = \frac{\zeta(2s)\zeta(s-27/2)\zeta(s-17/2)\zeta(s-9/2)}{\zeta(2s+1)\zeta(s+29/2)\zeta(s+19/2)\zeta(s+11/2)}$$

and $\zeta(s) = 1/(1-q^{-s})$. Note that $c(s)$ vanishes for $s = -29/2, -19, 2, -11/2$ and $-1/2$. In particular, the intertwining operator kills $V_0 \subset I(-11/2)$ and hence maps a proper quotient of $I(-11/2)/V_0$ into $I(11/2)$. Thus it suffices to prove that any non-trivial submodule of $I(11/2)$ is not small.

Let $B \subset G$ be a Borel subgroup. Let $T \subset B$ be a maximal split. This gives a root system with a choice of simple roots, and we use the standard realization of the E_8 root system in $E = \mathbb{R}^8$. For convenience we write down simple roots:

$$\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$$

$$\alpha_2 = e_2 - e_1,\ \alpha_3 = e_3 - e_2,\ \alpha_4 = e_1 + e_2,\ \alpha_5 = e_4 - e_3,\ \alpha_6 = e_5 - e_4,\ \alpha_7 = e_6 - e_5,$$

$$\alpha_8 = e_7 - e_6$$

Any unramified character χ of the maximal torus T with values in $\mathbb{R}^+$ (hence forth a real character) can be identified with an element in E, denoted by the same symbol such that

$$\chi \circ \alpha^\vee(t) = |t|^{\langle \chi, \alpha \rangle}$$

for all $t \in F^\times$ where $\alpha^\vee : F^\times \to T$ is the co-root of α and $\langle \chi, \alpha \rangle$ is the usual dot product in E.

Let V be a smooth G-module. Let $r_B(V)$ denote the normalized Jacquet functor with respect to B. Any irreducible character of T that appears as a quotient of V is called an

exponent of V. If $V \subseteq \mathrm{Ind}_B^G(\chi)$ then χ is an exponent of V by the Frobenius reciprocity. Assume that χ is real, and $\langle \chi, \alpha \rangle \neq \pm 1$. Then $\mathrm{Ind}_B^G(\chi) \cong \mathrm{Ind}_B^G(\chi^s)$, where s is the simple reflection given by α. This is a simple consequence of induction in stages, to the parabolic $B \cup BsB$, and using the knowledge of real principal series of $\mathrm{SL}_2(F)$, that is, $\langle \chi, \alpha \rangle \neq \pm 1$ guarantees that we are staying away from reducibility points. Thus, if $\langle \chi, \alpha \rangle \neq \pm 1$, then χ^s is also an exponent of V. We shall use this observation to write down some exponents of $V \subseteq I(s)$, where $s \in \mathbb{R}$. We observe that $I(s) \subset \mathrm{Ind}_B^G(\chi)$ where

$$\chi = (0, -1, -2, -3, -4, -5, s + \frac{17}{2}, s - \frac{17}{2}).$$

For example, if $s = -29/2$, then $\chi = -\rho$, the exponent of the trivial representation, and indeed the trivial representation is a submodule of $I(-29/2)$. Consider the case $s = 11/2$, where we have a proper spherical quotient and a non-spherical submodule V. Then

$$\chi = (0, -1, -2, -3, -4, -5, 14, -3).$$

Using simple reflections we can move 14 all the way to the left, to obtain another exponent

$$(14, -1, -2, -3, -4, -5, -3)$$

of V. Using the reflection s_1 we get

$$\frac{1}{2}(15, 13, 11, 9, 7, 5, 3, -19)$$

and then followed by s_4,

$$\frac{1}{2}(-13, -15, 11, 9, 7, 5, 3, -19).$$

Now we move -15 and -13 to the right to arrive at

$$\chi' = \frac{1}{2}(11, 9, -13, 7, -15, 5, 3, -19).$$

Let $R \supset B$ be the parabolic group, whose Levi is of the type A_1^3, and the simple factors correspond to simple roots α_4, α_5 and α_7. Observe that

$$\langle \chi', \alpha_4 \rangle = \langle \chi', \alpha_5 \rangle = \langle \chi', \alpha_7 \rangle = 10.$$

Since $\langle \chi', \alpha_i \rangle \neq \pm 1$, for $i = 4, 5, 7$, χ' is an exponent of an irreducible principal series representation of R. It follows that $r_R(V)$ is a Whittaker generic representation of the Levi factor of R. This implies that V has a non-zero degenerate Whittaker model corresponding to the orbit $3A_1$. By Theorem A in [GGS] non-vanishing of any degenerate Whittaker model corresponding to the orbit $3A_1$ implies non-vanishing of the generalized model corresponding to $3A_1$. In the language of this paper this means that $FJ(V) \neq 0$. Hence V is not small, and this completes the proof of Theorem B.1.1.

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