ON THE STABLE REDUCTION OF HYPERELLIPTIC CURVES

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Abstract

Let $f : S \to B$ be a surface fibration of genus $g \geq 2$ over $\mathbb{C}$. The semistable reduction theorem asserts there is a finite base change $\pi : B' \to B$ such that the fibration $S \times_B B' \to B'$ admits a semistable model. An interesting invariant of f, denoted by $N(f)$, is the minimum of $\deg(\pi)$ for all such π. In an early paper of Xiao, he gives a uniform multiplicative upper bound N_g for $N(f)$ depending only on the fibre genus g. However, it is not known whether Xiao's bound is sharp or not. In this paper, we give another uniform upper bound N'_g for $N(f)$ when f is hyperelliptic. Our N'_g is optimal in the sense that for every $g \geq 2$ there is a hyperelliptic fibration f of genus g so that $N(f) = N'_g$. In particular, Xiao's upper bound N_g is optimal when $N_g = N'_g$. We show that this last equation $N_g = N'_g$ holds for infinitely many g.

1 Introduction

We work over $\mathbb{C}$ throughout this paper. By a surface fibration $f : S \to B$ we mean a flat proper morphism from a normal projective surface S to a smooth projective curve B such that a general fibre of f is smooth and connected. We have the following well known theorem.

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Theorem 1.1 (Semistable reduction theorem of curves, [1] or [6]). *Assume the fibre genus of f is $g \geq 2$, then there is a finite morphism of smooth projective curves $\pi : B' \to B$ such that the relative minimal model of $f' : S \times_B B' \to B'$ is semistable.*

We call any such base change $\pi : B' \to B$ in the above theorem a *stabilizing* base change. The minimal degree of a stabilizing base change π is an interesting invariant of f.

Definition 1.2 We define

$$N(f) := \mathrm{Min}\{ \deg(\pi) \mid \pi : B' \to B \text{ is stabilizing } \}.$$

We shall see in § 3.3 that $N(f)$ is actually the G.C.D. of $\deg(\pi)$ for all stabilizing base change π. An interesting question associated is to figure out a uniform (multiplicative) upper bound of $N(f)$ in terms of g when f varies. In [10], Xiao gives such an upper bound.

Theorem 1.3 ([10, Thm. 1]). *For any $g \geq 2$ there is a constant N_g such that $N(f) \mid N_g$ for any surface fibration $f : S \to B$ of genus g.*

The equivalence of the interpretation of [10, Thm. 1] there and here is explained in § 3.3. The explicit formula of N_g (cf. [10, Thm. 2]) is

$$N_g = \prod_{\text{prime } p \leq 2g+1} p^{\mu_p}, \tag{1.1}$$

where μ_p is the largest integer such that

$$2g \geq p^{\mu_p} - p^{\mu_p - 1}. \tag{1.2}$$

However, whether this uniform upper bound N_g is optimal is not yet known. Xiao then asked the following question in the same paper.

Question 1.4 ([10, pp 387, remark]) For any $g \geq 2$, is there a genus g surface fibration $f : S \to B$ such that $N(f) = N_g$?

Instead of considering all fibrations of genus g, in this paper we concentrate on the upper bound of $N(f)$ for **a HYPERELLIPTIC** fibration f.

We define

$$N'_g = \prod_{\text{prime } p \leq 2g+1} p^{\nu_p}, \tag{1.3}$$

where ν_p is the largest integer such that

$$2g \geq p^{\nu_p} - 1. \tag{1.4}$$

if p is odd and $\nu_2 = \mu_2$. It is clear that $N'_g \mid N_g$ and note that it is possible $N'_g = N_g$ for some $g \geq 2$.

The main result of the paper is the following.

Theorem 1.5 (Main Theorem). *(1). We have $N(f) \mid N'_g$ for any hyperelliptic surface fibration $f : S \to B$ of fibre genus $g \geq 2$.*

(2). For each $g \geq 2$, there exists a hyperelliptic surface fibration $h : X \to \mathbb{P}^1$ of genus g such that $N(h) = N'_g$. In particular, the upper bound N'_g is optimal.

The first statement of this theorem is a direct consequence of Proposition 3.9 and Theorem 4.4 below and the second statement of this theorem follows immediately from Theorem 5.1, Example 4.5 and Example 4.6 (cf. Remark 5.2).

As a direct consequence of Theorem 1.5, we obtain an affirmative answer to Question 1.4 for those g such that $N_g = N'_g$. In Section 6, we prove that such g are in fact abundant.

Theorem 1.6. *There are infinitely many integers $g \geq 2$ such that $N_g = N'_g$.*

This theorem is proved using purely number theoretic methods and seems of independent interest. We actually prove that the set of such g has a positive *lower logarithmic density.*

2 Preliminaries

We shall use the following conventions in this paper:

- $e(n), n \in \mathrm{N}_+$ is Euler's totient function, namely

$$e(n) = n \cdot \prod_{\text{prime } p|n} (1 - \frac{1}{p}).$$

- $\Delta := \{t \in \mathbb{C} | |t| < 1\}$ is the unit disc, we always denote by t the canonical parameter of Δ and for a proper fibration $f : \mathscr{X} \to \Delta$, the central fibre means the fibre $f^{-1}(0)$ as a divisor.

2.1 Canonical resolution of flat double covers

We recall the theory of flat doubles, One can also consult [5, § 0] and [3, Ch.III,§ 6-7]. For simplicity, we will only mention the flat double cover theory for schemes and note that parallel results also holds for complex manifolds (cf. [3, Ch.III,§ 6-7]).

Let $\mathscr{Y}$ be a connected regular noetherian scheme of dimension 2 over $\mathbb{C}$. A flat double cover of $\mathscr{Y}$ is a finite flat morphism $\mu : \mathscr{X} \to \mathscr{Y}$ of degree 2. It is well known that such μ is given by data $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible sheaf on $\mathscr{Y}$ and $s \in H^0(\mathscr{Y}, \mathcal{L}^2)$. In fact, given data $(\mathcal{L}, s)$, we can first endow an $\mathcal{O}_{\mathscr{Y}}$-algebra structure on the locally free $\mathcal{O}_{\mathscr{Y}}$-sheaf $\mathcal{A} := \mathcal{O}_{\mathscr{Y}} \oplus \mathcal{L}^{-1}$ by defining the multiplication map via $\mathcal{L}^{-1} \otimes \mathcal{L}^{-1} = \mathcal{L}^{-2} \xrightarrow{s} \mathcal{O}_{\mathscr{Y}}$. Then $\mu : \mathscr{X} \to \mathscr{Y}$ is taken as $\mathrm{Spec}(\mathcal{A}) \to \mathscr{Y}$. When $s \neq 0$, the divisor $B := \mathrm{div}(s) \subseteq \mathscr{Y}$ is called the branch divisor associated to this flat double cover.

Proposition 2.1 ([5, Chap. 0])**.** *If $\mathscr{Y}$ is regular, then:*

(i) $\mathscr{X}$ is reduced if and only if $s \neq 0$;

(ii) when $s \neq 0$, $\mathscr{X}$ is regular if and only if B is regular.

In the following, we consider the case where $\mathscr{X}$ is normal (we call such flat double cover as *a normal flat double cover* in this paper). By Proposition 2.1, to resolve the singularities of $\mathscr{X}$, it suffices to resolve the singularity of the branch divisor B. We then recall the following so-called canonical resolution of singularity of $\mathscr{X}$ for normal flat double. We start by taking $\mu_0 = \mu : \mathscr{X}_0 = \mathscr{X} \to \mathscr{Y}_0 := \mathscr{Y}$. Blowing up a singular point $y_1 \in B_0$ by $\rho_1 : \mathscr{Y}_1 \to \mathscr{Y}_0$, the normalisation $\mu_1 : \mathscr{X}_1 \to \mathscr{Y}_1$ of $\mathscr{Y}_1$ in the fractional field of $\mathscr{X}$ is again a flat double with branch divisor B_1 such that $\rho_1^* B_0 = B_1 + 2l_1 \cdot E_1$ for a positive integer l_1. Here E_1 is the exceptional divisor for ρ_1. Continue this process by keeping blowing up a singularity y_i of the branch divisor B_{i-1} of each $\mu_{i-1} : \mathscr{X}_{i-1} \to \mathscr{Y}_{i-1}$, we shall finally stop at some $\mu_n : \mathscr{X}_n \to \mathscr{Y}_n$ such that its branch divisor B_n is regular. In particular, $\mathscr{X}_n$ is regular by Proposition 2.1.

$$
\begin{array}{ccccccccc}
\mathscr{X}_n & \longrightarrow & \mathscr{X}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \mathscr{X}_1 & \longrightarrow & \mathscr{X}_0 \\
\downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \mu_{n-1}} & & & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} \\
\mathscr{Y}_n & \xrightarrow{\rho_n} & \mathscr{Y}_{n-1} & \xrightarrow{\rho_{n-1}} & \cdots & \longrightarrow & \mathscr{Y}_1 & \xrightarrow{\rho_1} & \mathscr{Y}_0
\end{array}
\qquad \rho : \mathscr{Y}_n \to \mathscr{Y}_0
$$

By abuse of language, starting from a normal flat double cover $\mu : \mathscr{X} \to \mathscr{Y}$, we call the map $\rho : \mathscr{Y}_n \to \mathscr{Y}$ the canonical resolution of μ and $\mathscr{X}_n$ the birational model obtained from the canonical resolution. Note by construction, $\mathscr{X}_n$ is regular.

2.2 Refined canonical resolution of flat double covers in fibration

Let us consider a special case of normal flat double cover $\mu : \mathscr{X} \to \mathscr{Y}$ where $\mathscr{Y} = \mathbb{P}^1 \times \Delta$ carrying a natural fibration onto Δ. After the canonical resolution $\rho : \mathscr{Y}_n \to \mathscr{Y}$, the branch divisor $B_n \subseteq \mathscr{Y}_n$ is regular. But B_n can fail to perform well with respect to the fibration $\mathscr{Y} \to \Delta$. For example, let F_n be the central fibre of $\mathscr{Y}_n \to \Delta$, the divisor $B_n + F_n$ may not have simple normal crossings. As a result, some further blowing-ups are needed. By a sequence of suitable further blowing-ups after the canonical resolution, we can find some model $\rho' : \mathscr{Y}_m \to \mathscr{Y}, m \geq n$ so the divisor $B_m + F_m$ has normal crossings and B_m itself is regular. Here B_m and F_m are the branch divisor and the central fibre of $\mathscr{Y}_m \to \Delta$ associated. The map $\rho' : \mathscr{Y}_m \to \mathscr{Y}$ is then called the refined canonical resolution. And we call the flat double model $\mathscr{X}_m$ the birational model associated to the refined canonical resolution.

Remark 2.2 In a particular case, where the horizontal part B_h of the branch divisor B is consisting of sections, then the canonical resolution $\mathscr{Y}_n$ is already the refined canonical resolution. This follows from the fact the central fibre of $\mathscr{Y}_n$ has normal crossings automatically since it has genus zero and each section of the fibration must intersect the central fibre transversely.

Proposition 2.3. *The birational model $\mathscr{X}_m$ associated to the refined canonical resolution has a normal crossing central fibre over Δ.*

PROOF. Now take an arbitrary point x on the central fibre of $\mathscr{X}_m$ and write $y = \mu_m(x) \in \mathscr{Y}_m$.

- If y is not contained in the branch locus, then $\mu_m : \mathscr{X}_m \to \mathscr{Y}_m$ is locally étale at x and therefore the central fibre of $\mathscr{X}_m$ has normal crossing at x as $\mathscr{Y}_m$ is so at y by assumption.

- If y is contained in the horizontal branch locus, then there is only one irreducible component E in the central fibre of $\mathscr{Y}_m$ passing through y. By assumption E is not contained in the branch locus. Then locally, we can find local parameters u, v at y such that $u = 0, v = 0$ gives the divisor E and the horizontal branch divisor and $t = u^s$ for some s, recall here t is the canonical parameter of the base Δ. As a result, the local functions defining x and the fibrations are

$$\begin{cases} z^2 = v; \\ t = u^s. \end{cases}$$

  As a result, the local parameter of x can be chosen as u, z and $t = u^s$. The central fibre of $\mathscr{X}_m$ has normal crossing at x.

- If y is contained in a vertical branch irreducible divisor E_1.

  - If y is not the intersection of E_1 with another vertical irreducible component, we can choose a local parameter u at y such that $u = 0$ gives E_1 and $t = u^s$

for some s. Then the local functions defining x and the fibration are

$$\begin{cases} z^2 = u; \\ t = u^s. \end{cases}$$

As a result, the central fibre of $\mathscr{X}_m$ has normal crossing at x.

– If y is the intersection of E_1 with another vertical irreducible component E_2. Then E_2 is not contained in the branch locus and E_1, E_2 intersect transversely. We can then choose a local parameter u, v at y such that $u = 0, v = 0$ gives E_1, E_2 respectively and $t = u^s v^r$ for some $s, r \in \mathrm{N}_+$. Then the local functions defining x and the fibrations are

$$\begin{cases} z^2 = u; \\ t = u^s v^r. \end{cases}$$

As a result, the local parameter of x can be chosen as z, v and $t = z^{2s} v^r$. So the central fibre of $\mathscr{X}_m$ has normal crossing at x.

□

3 The Picard-Lefschetz Monodromy and Xiao's upper bound

3.1 The Picard-Lefschetz monodromy and semistable reduction

By a local holomorphic surface fibration we mean a flat projective holomorphic map $\varphi : \mathscr{X} \to \Delta$ from a 2-dimensional irreducible and reduced complex analytic variety such that φ is smooth over $\Delta^* = \Delta \backslash \{0\}$.

$$\begin{array}{ccc} \mathscr{X}^* := \mathscr{X} \times_\Delta \Delta^* & \hookrightarrow & \mathscr{X} \\ \downarrow{\scriptstyle \varphi|_{\Delta^*}} & & \downarrow{\scriptstyle \varphi} \\ \Delta^* & \hookrightarrow & \Delta \end{array}$$

By restricting to Δ^*, the smooth fibration $\mathscr{X}^* \to \Delta^*$ is a fibre bundle of curves of genus g and thus provides a monodromy representation

$$\Psi : \pi_1(\Delta^*, \gamma) \to \mathrm{GL}(H^1(F_\gamma, \mathbb{Z})) \simeq \mathrm{GL}_{2g}(\mathbb{Z})$$

for a fixed base point $\gamma \in \Delta^*$. This representation is known as the Picard-Lefschetz monodromy representation. As $\pi_1(\Delta^*, \gamma)$ is canonically identified with the cyclic group $\mathbb{Z}$, this representation is given by a matrix $M(\varphi) \in \mathrm{GL}_{2g}(\mathbb{Z})$ unique up to conjugate: $M(\varphi)$ is the image of the canonical generator of $\pi_1(\Delta^*, \gamma)$ under Ψ. We also call this $M(\varphi)$ the Picard-Lefschetz monodromy matrix at $0 \in \Delta$.

Theorem 3.1 ([8, Appendix] & [7] and [6])**.** *If the fibre genus of f is at least 2, then*

(i) the matrix $M(\varphi)$ is quasi-unipotent, namely $M(\varphi)^n$ is a unipotent matrix for some $n \in \mathrm{N}_+$;

(ii) the fibration $\mathscr{X}^* \to \Delta^*$ admits at worst semistable reduction at $0 \in \Delta$ if and only if $M(\varphi)$ is unipotent.

In other words, the eigenvalues of $M(\varphi)$ are all roots of unity and f admits semi-stable reduction at 0 if and only if the eigenvalues of $M(\varphi)$ are all equal to 1.

Definition 3.2 We define $\delta(\varphi) := \min\{n \in \mathrm{N}_+ \mid M(\varphi)^n \text{ is unipotent}\}$. We shall call this number the local stabilizing index (l.s.i. for short) for φ.

Remark 3.3 By definition, this number $\delta(\varphi)$ is a birational invariant: it depends only on $f^* = f|_{\Delta^*} : \mathscr{X}^* \to \Delta^*$.

The name l.s.i. makes sense for the following reason. For a base change $\pi_n : \Delta_n = \Delta \xrightarrow{z \mapsto z^n} \Delta$ the associated monodromy matrix $M(\varphi_n)$ of $\varphi_n : X \times_\Delta \Delta_n \to \Delta_n$ is nothing but $M(\varphi_n) = M(\varphi)^n$. By Theorem 3.1, we immediately have the following.

Corollary 3.4. *The base change π_n is stabilizing if and only if $\delta(\varphi) \mid n$.*

3.2 A characterization of the constant N_g

Now for the Picard-Lefschetz monodromy matrix $M(\varphi)$ associated to a local fibration $\varphi : \mathscr{X} \to \Delta$, let $\xi_1, \cdots, \xi_{2g}$ be all the eigenvalues of $M(\varphi)$. Then by Definition 3.2, we have $\delta(\varphi) = \mathrm{Min}\{n \in \mathrm{N}_+ \mid \xi_i^n = 1, \forall\, i\}$. It then remains to study the eigenvalue of $M(\varphi)$.

Lemma 3.5. *Let a root of unity $\xi \in \mu_\infty(\mathbb{C})$ be an eigenvalue of a matrix $M \in \mathrm{GL}_{2g}(\mathbb{Z})$ and let n be the order of $\xi \in \mu_\infty(\mathbb{C})$, then $e(n) \le 2g$.*

Proof. Note that M has integral coefficients, so any other primitive root of unity of the same order n is also an eigenvalue of M. As there are exactly $e(n)$ such primitive roots of unity, we have $e(n) \le 2g$. □

Lemma 3.6. *Xiao's constant N_g (cf. (1.1) & (1.2)) is the least common multiple of all positive integer n such that $e(n) \le 2g$.*

Proof. Let $N(g)$ be the least common multiple of all positive integer n such that $e(n) \le 2g$. We shall write $N(g) = \prod\limits_{\text{prime } p} p^{l_p}$. For a fixed p we need to show that $l_p = \mu_p$

(cf. (1.2)). By construction, there is some $n \in \mathrm{N}_+$ such that $p^{l_p} \mid n$ and $e(n) \leq 2g$. By definition, we have

$$2g \geq e(n) \geq e(p^{l_p}) = p^{l_p}(1 - \frac{1}{p}) = p^{l_p} - p^{l_p - 1}.$$

So $\mu_p \geq l_p$ by (1.2). Conversely, we have $e(p^{\mu_p}) = p^{\mu_p} - p^{\mu_p - 1} \leq 2g$ and thus $l_p \geq \mu_p$. We are done. □

As a consequence of Lemma 3.5 and Lemma 3.6, we have the next corollary.

Corollary 3.7. *For any local holomorphic fibration $\varphi : \mathscr{X} \to \Delta$ of fibre genus $g \geq 2$, we have $\delta(\varphi) \mid N_g$.*

In [10], Xiao gives another proof of this corollary by a careful study of the configuration the central fibres. Though our proof is simpler, his result is more powerful in the sense that for a specific φ, he can tell the precise value of $\delta(\varphi)$ as the following proposition.

Proposition 3.8 ([10, Prop. 1])**.** *If the central fibre F_0 of $\varphi : \mathscr{X} \to \Delta$ has simple normal crossings, then the l.s.i. $\delta(\varphi)$ is the least common multiple of the multiplicities of principal components (cf. [10, pp. 383]) of F_0 .*

3.3 The value of $N(f)$

Now let $f : S \to C$ be a fibration of fibre genus $g \geq 2$ from a surface S onto a curve C. We have defined the constant $N(f)$ as the minimal degree of a stabilizing base change (cf. Definition 1.2). This constant is actually contributed by all local factors explained below. Let $b_1, \cdots, b_s$ be all the images of non-semistable singular fibres of f. At each b_i, we can choose a small open disc $\Delta \subseteq C$ containing b_i as the origin and therefore obtain a local fibration $\varphi_i : \mathscr{X}_i \to \Delta$ by base change for each i. As in § 3.1, we have the local stabilizing index $\delta(\varphi_i)$ for each φ_i. Let $\pi : C' \to C$ be a finite covering of curves and b'_i be a point lying above b_i. Then by Corollary 3.4, $X \times_C C' \to C'$ admits a semistable reduction at b'_i if and only if the local multiplicity of π at b'_i is divided by $\delta(\varphi_i)$. As a consequence, if $\pi : C' \to C$ is stabilizing, then $\delta(\varphi_i) | \deg(\pi)$. Let

$$\delta := \text{least common multiple of all } \delta(\varphi_i),$$

then $\delta \mid \deg(\pi)$. Conversely, we can easily construct a cyclic cover $\pi_0 : C'_0 \to C$ of order δ totally ramified at $b_1, \cdots, b_s$ (and possibly other points). Such a morphism is stabilizing as we have analysed and hence we have the following.

Proposition 3.9. *The number $N(f)$ is the least common multiple of all $\delta(\varphi_i)$. Moreover $N(f) \mid \deg(\pi)$ for all stabilizing base change $\pi : C' \to C$.*

By Corollary 3.7 and the above proposition, we have the next corollary.

Corollary 3.10 (Xiao's bound). *We have $N(f) \mid N_g$.*

Now let us return to Question 1.4. From the above explanation of $N(f)$, Question 1.4 is difficult in two parts.

(i) It is not known whether there are local holomorphic fibrations $\varphi_i : \mathscr{X}_i \to \Delta, i = 1, \cdots, s$, such that the least common multiple of $\delta(\varphi_i)$ is equal to N_g.

(ii) It is not known how to glue local holomorphic fibrations $\varphi_i : \mathscr{X}_i \to \Delta, i = 1, \cdots, s$, into a single global surface fibration $f : S \to B$ without losing l.s.i..

In the next section, we shall overcome these two difficulties for hyperelliptic fibrations.

4 Upper bound for hyperelliptic fibrations

We study the optimal upper bound of $N(f)$ for hyperelliptic fibrations.

4.1 Upper bound for l.s.i. of hyperelliptic fibrations

Let $\varphi : \mathscr{X} \to \Delta$ be a hyperelliptic local holomorphic surface fibration of genus g. Besides the Picard-Lefschetz monodromy, we have another monodromy coming from the Weierstrass multisection. In fact, denote by σ the hyperelliptic involution ded on $\mathscr{X}^*$ over Δ^*. Then we obtain a flat double cover: $\pi : \mathscr{X}^* \to \mathscr{X}^*/\sigma \simeq \mathbb{P}^1 \times \Delta^*$. The branch divisor $W^* \subseteq \mathbb{P}^1 \times \Delta^*$ is a horizontal divisor of degree $2g+2$ finite étale over the base Δ^*. So a homomorphism $\Phi : \pi_1(\Delta^*, \gamma) \to S_{2g+2}$ is given to characterize this étale cover $W^* \to \Delta^*$. We shall call this homomorphism the Weierstrass monodromy homomorphism. Again, as $\pi_1(\Delta^*, \gamma) \simeq \mathbb{Z}$ canonically, we call the image $M'(\varphi) \in S_{2g+2}$ of the canonical generator of $\pi_1(\Delta^*, \gamma)$ under Φ the Weierstrass monodromy permutation.

Lemma 4.1. *If the Weierstrass monodromy homomorphism Φ is trivial, then $\delta(\varphi) = 1$ or 2.*

Proof. Since the invariant $\delta(\varphi)$ is a birational invariant on $\mathscr{X}$, we may assume that $\mu : \mathscr{X} \to \mathbb{P}^1 \times \Delta$ is a normal flat double cover(cf. § 2.1). Denote by $B \subseteq \mathbb{P}^1 \times \Delta$ the associated branch divisor. Then $B^* := B \times_\Delta \Delta^*$ is isomorphic to the Weierstrass multisection W^*, which by assumption consists of $2g+2$ different sections (of Δ^*) since the Weierstrass monodromy is trivial. To resolve the singularity of $\mathscr{X}$, we run the canonical resolution of the flat double μ (cf. § 2.1). Denote by $\rho : \widetilde{P} \to \mathbb{P}^1 \times \Delta$ the canonical resolution and $\mathscr{X}'$ the birational model obtained from the canonical resolution (cf. § 2.1).

Denote by $\varphi' : \mathscr{X}' \to \Delta$ the associated fibration. By Remark 2.2 and Proposition 2.3, $\mathscr{X}'$ has a normal crossing central fibre. A key observation in this case is that since each irreducible component in B is itself regular, the central fibre of the canonical resolution model $\nu : \widetilde{P} \to \Delta$ is reduced. Namely, in each blowing-up step of the canonical resolution, the center is always a smooth point with respect to the fibration to Δ. Then we immediately proved that $\delta(\varphi) = \delta(\varphi') = 1$ or 2 by Proposition 3.8 since the irreducible components of the central fibre of $\mathscr{X}'$ can have multiplicity at most 2 times that of $\widetilde{P}$. □

As a result, we have the next corollary.

Corollary 4.2. *We have* $\delta(\varphi) \mid 2 \cdot \mathrm{ord}(M'(\varphi))$.

Proof. Let $n := \mathrm{ord}(M'(\varphi))$, then the Weierstrass monodromy

$$M'(\varphi_n) = M'(\varphi)^n = \mathrm{id} \in S_{2g+2}$$

for $\varphi_n : \mathscr{X} \times_\Delta \Delta_n \to \Delta_n$, here as before $\pi_n : \Delta_n := \Delta \to \Delta, t \mapsto t^n$. By Lemma 4.1, we have $\delta(\varphi_n) = 1$ or 2. As a result, by construction we have $\delta(\varphi)|n \cdot \delta(\varphi_n)$ and hence $\delta(\varphi) \mid 2 \cdot \mathrm{ord}(M'(\varphi))$. □

In fact, we have a partial strengthening of the above corollary:

Proposition 4.3. *Suppose the hyperelliptic fibration* $\varphi : \mathscr{X} \to \Delta$ *admits potentially good reduction, then* $\mathrm{ord}(M'(\varphi)) \mid \delta(\varphi)$.

Proof. Let $n := \delta(\varphi)$. Then by assumption, $\varphi_n : \mathscr{X} \times_\Delta \Delta_n \to \Delta_n$ admits a proper birational model $\widetilde{\varphi}_n : \widetilde{\mathscr{X}_n} \to \Delta_n$ which is smooth. As a result, the Weierstrass monodromy of $\widetilde{\varphi}_n$, which is nothing but M^n, is trivial. We are done. □

Theorem 4.4. *We have* $\delta(\varphi) \mid N'_g$ *for any hyperelliptic local fibration* $\varphi : \mathscr{X} \to \Delta$ *of genus* $g \geq 2$.

Proof. We need to show for each p, the largest integer l_p such that $p^{l_p} \mid \delta(\varphi)$ is bounded above by ν_p (cf. (1.4)). If p is odd, we see that the largest l'_p such that $p^{l'_p} \mid \mathrm{ord}(M'(\varphi))$ is bounded above by ν_p since $M'(\varphi) \in S_{2g+2}$. Corollary 4.2 then asserts that $l'_p \geq l_p$ and we are done for odd primes. For prime 2, since by definition $\nu_2 = \mu_2$ and we already proved that $\delta(\varphi) \mid N_g = 2^{\mu_2} \cdot$ an odd number, we are done. □

This theorem then implies Theorem 1.5(1) by Proposition 3.9.

4.2 Example of local hyperelliptic fibrations

For any fixed $g \geq 2$ and prime $p \leq 2g+1$, we present examples of hyperelliptic fibrations $\varphi_p : \mathscr{X}_p \to \Delta$ with $p^{\nu_p} \mid \delta(\varphi_p)$.

Example 4.5 (Odd prime case)**.** *Fixing any odd prime $p \leq 2g$, we consider the local hyperelliptic fibration $\varphi_p : \mathscr{X}_p \to \Delta$ given by the following equation:*

$$y^2 = (x^{2r+1} - t)(x - a_1)(x - a_2) \cdots (x - a_i) \cdots (x - a_{2(g-r)+1})$$

where r is such that $p^{\nu_p} = 2r+1$, t is the parameter of Δ and the numbers $a_i \in \mathbb{C} \backslash \Delta, i = 1, \cdots, 2(g-r)+1$, are all different. Then $p^{\nu_p} \mid \delta(\varphi_p)$.

Proof. We see that the Weierstrass monodromy of φ_p is presented by a cyclic permutation $M'(\varphi_p) \in S_{2g+2}$ of order $p^{\nu_p} = 2r+1$. As a result, we have $n := \delta(\varphi_p) \mid 4r+2$ by Corollary 4.2. We claim that

- the relative minimal model of $\mathscr{X}_p \times_\Delta \Delta_{4r+2} \to \Delta_{4r+2}$ is either stable or smooth, where
$$\pi_{4r+2} : \Delta_{4r+2} = \Delta \to \Delta, t \mapsto t^{4r+2}$$
is the cyclic cover of degree $4r+2$.

Once the claim is true, we have $p^{\nu_p} = 2r+1 \mid \delta(\varphi_p)$ if the relative minimal model is smooth by Proposition 4.3. On the other hand, when φ_p admits potentially bad reduction, then it is well known that relatively minimal model of $\mathscr{X}_p \times_\Delta \Delta_m$ is semi-stable but not stable for all m divided by n but is not equal to n. This implies $n = 4r+2$ by our claim. So it remains to prove our claim.

By construction the hyperelliptic defining equation of $\mathscr{X}_p \times_\Delta \Delta_{4r+2} \to \Delta_{4r+2}$ is:

$$y^2 = (\prod_{j=0}^{2r}(x - \xi_{2r+1}^j t^2)) \cdot (x - a_1)(x - a_2) \cdots (x - a_{2(g-r)+1}), \ \ \xi_{2r+1} = \exp(\frac{2\pi\sqrt{-1}}{2\text{r}+1}).$$

With this hyperelliptic equation, we see that $\mathscr{X}_p \times_\Delta \Delta_{4r+2}$ admits a flat double cover $\nu : \mathscr{X}_p \times_\Delta \Delta_{4r+2} \to P_0 := \mathbb{P}^1_{\mathbb{C}} \times \Delta_{4r+2}$, in which x is the standard affine coordinate of $\mathbb{A}^1_{\mathbb{C}} \subseteq \mathbb{P}^1_{\mathbb{C}}$. To obtain the relative minimal model, we run the canonical resolution of the hyperelliptic double covering (cf. § 2.1). We show the process of the canonical resolution in the first row of Figure 1 below.

- In the first picture, we draw the configuration of the branch divisor and the central fibre on $P_0 = \mathbb{P}^1_{\mathbb{C}} \times \Delta_{4r+2}$. The bend lines are the branch divisors defined by $x - \xi^j_{2r+1}t^2, j = 0, \cdots, 2r$ and the horizontal ones below are defined by $x - a_i, i = 1, \cdots, 2(g-r)+1$, the central vertical dotted line is the central fibre of P_0.

- Blowing up the unique intersection point Q_0 in the first picture we obtain $P_1 \to P_0$. The configuration of branch divisors on P_1 is shown in the second picture. Note as the branch divisors has multiplicity $2r+1$ at Q_0, the exceptional divisor E is contained in the branch locus.

- Then blowing up the unique multiple point Q_1 in the second picture we obtain $P_2 \to P_1$. The configuration of branch divisors on P_1 is shown in the third picture. This time, the exceptional divisor E_1 is not contained in the branch locus.

Then in the second row of Figure 1, we present the configuration of the central fibre of the canonical resolution.

- In middle we draw the configuration of the central fibre of the birational model obtained from the canonical resolution, that is the flat double cover of P_2. There are three components of the central fibre: A, B, C are preimages of $\overline{E}$, $\overline{F}$ and E_1 respectively. The multiplicities of A, B, C in the fibre are $2, 1, 1$ as only $\overline{E}$ is contained in the branch locus. Following [2, Prpostion 1.8], we have $4A^2 = 2\overline{E}^2 = -4$. Then note $A \to \overline{E}$ is birational, A must be isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ and hence is an (-1)-curve.

- In the right picture, we contract the (-1)-curve A. As A has multiplicity 2 in the fibre, the rest two components intersect transversely.

- In case $g \neq r$, we see from the last picture, the associated relatively minimal model is stable. On the other hand, if $g = r$, then B' is a (-1)-curve. A further contraction show that the relative minimal model is in fact smooth.

As a result, the minimal model of $\mathscr{X}_p \times_\Delta \Delta_{4r+2} \to \Delta_{4r+2}$ is either stable $(r < g)$ or smooth $(r = g)$. We are done. □

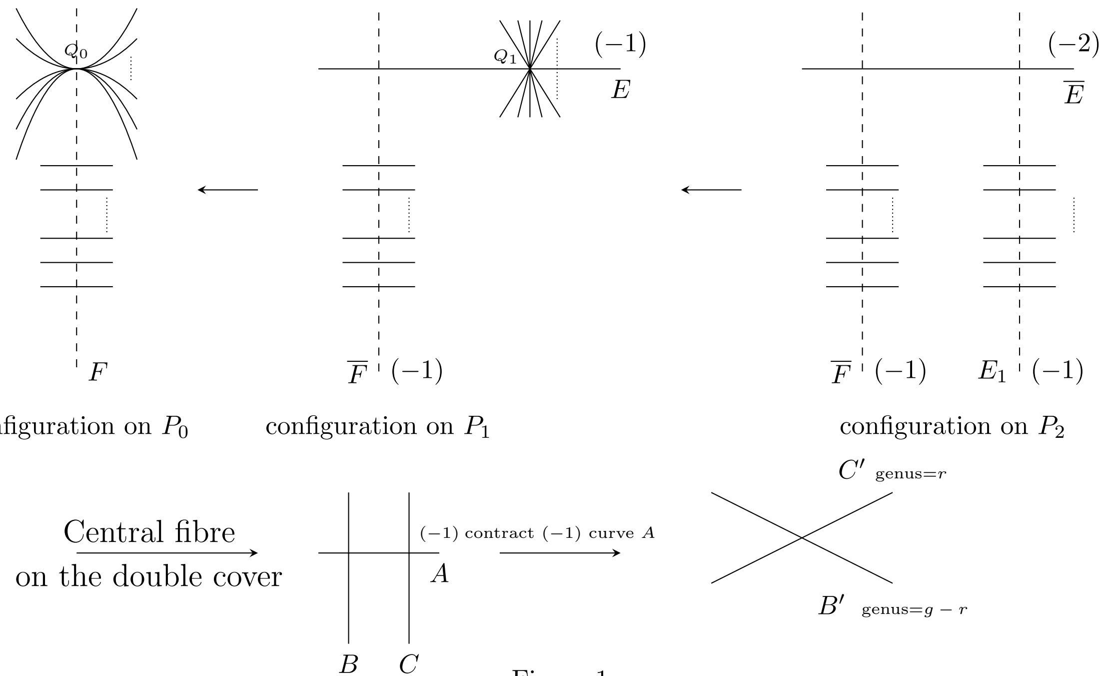

Figure 1

Example 4.6 (Prime 2 case)**.** *Now, we consider the local hyperelliptic fibration* $\varphi_2 : \mathscr{X}_2 \to \Delta$ *given by the following equation:*

$$y^2 = x(x^{2k} - t)(x - a_1)(x - a_2) \cdots (x - a_i) \cdots (x - a_{2(g-k)+1})$$

where $k = 2^l$ *is the largest* 2*-power that is smaller or equal to* g*,* t *is the parameter of* Δ *and* $a_i \in \mathbb{C}\backslash\Delta, i = 1, ..., 2(g-k)+1,$ *are all different. Then* $\delta(\varphi_2) = 4k$*.*

Proof. By construction, the Weierstrass monodromy of φ_2 is represented by a cyclic permutation $M'(\varphi_2) \in S_{2g+2}$ of order $2k$. As a result, we have $\delta(\varphi_2) \mid 4k$. It suffices to show that $\mathscr{X}_2$ does not admit a semi-stable reduction after the cyclic cover $\pi_{2k} : \Delta_{2k} = \Delta \to \Delta, t \mapsto t^{2k}$. In fact after this cyclic base change of degree $2k$, the hyperelliptic equation becomes:

$$y^2 = x \cdot \prod_{i=0}^{2k-1} (x - \xi_{2k}^i t) \cdot (x - a_1) \cdots (x - a_i) \cdots (x - a_{2(g-k)+1}), \;\; \xi_{2k} = \exp(\frac{\pi\sqrt{-1}}{\mathrm{k}}).$$

With this hyperelliptic equation, there defines a flat double cover $\mathscr{X}_2 \times_\Delta \Delta_{2k} \to P_0 := \mathbb{P}^1 \times \Delta_{2k}$. The canonical resolution of this flat double cover is given in Figure 2 below.

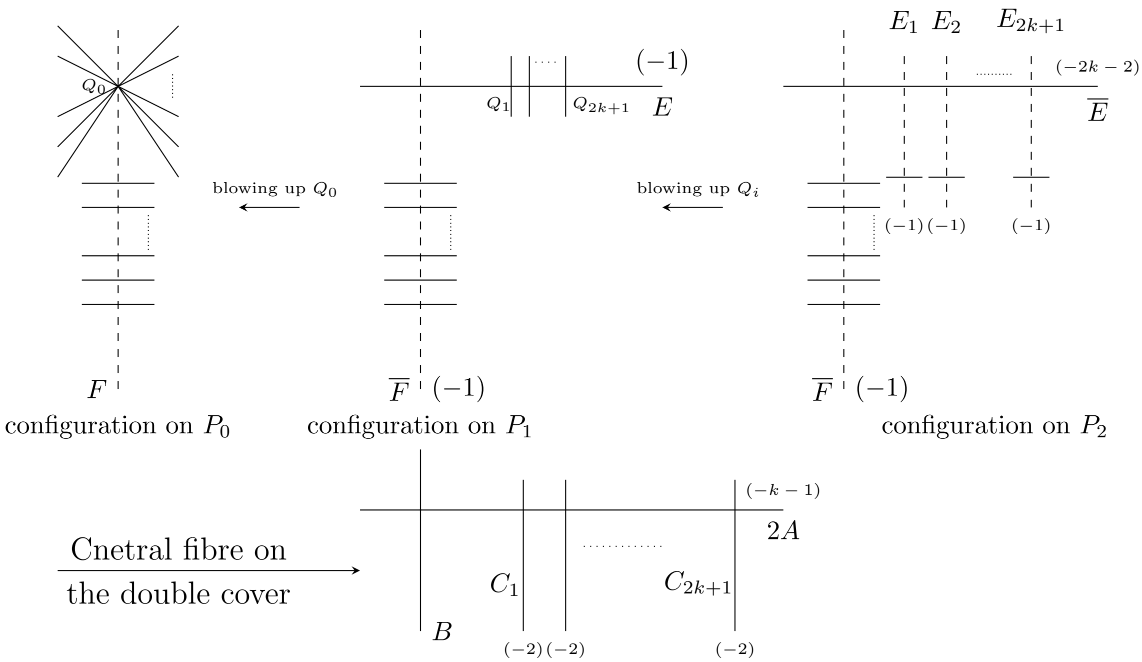

Figure 2

Let us briefly explain this Figure 2.

- The first row is to show the configuration of the branch divisor and the central fibre. In the first picture, the family of slashes are those branch divisors defined by x or $x-\xi_{2k}^j t, j=0,1,\cdots,2k-1$. The rest part is similar to Figure 1, except that in the second step, we have to blow up all the intersection points $Q_i, i=1,\cdots,2k+1$.

- In the second picture, we show the configuration of the central fibre of the birational model obtained from the canonical resolution, that is the model obtained from the flat double cover of P_2. We see that all the components of the central fibre of this model but A which lies above $\overline{E}$ has multiplicity 1. As a result, the components C_i above $E_i, i=1,\cdots,2k+1$, all has self-intersection number $2\cdot(E_i^2)=-2$ and B has even self intersection number $B^2=-2A\cdot B$ and thus there is no (-1)-curve on the fibre. So this model is already minimal, but as A has multiplicity 2, it is not semi-stable.

We are done. □

Remark 4.7 When the prime p ranges over all primes in $[2,2g+1]$, the $\delta(\varphi_p)$ given in Example 4.5 and Example 4.6 has a least common multiple equal to N_g'.

5 Global hyperelliptic fibration with large $N(f)$

We shall prove the following theorem in this section.

Theorem 5.1 (Gluing Theorem)**.** *Suppose $\varphi_i : \mathscr{X}_i \to \Delta, i = 1, \cdots, s$ are finitely many hyperelliptic local holomorphic surface fibrations of fibre genus $g \geq 2$, then there is a hyperelliptic surface fibration $f : S \to \mathbb{P}^1_{\mathbb{C}}$ of genus g such that $N(f)$ is divided by $\delta(\varphi_i)$ for each i.*

Remark 5.2 This theorem along Remark 4.7 proves Theorem 1.5(2).

5.1 Strictly hyperelliptic local fibrations

When studying hyperelliptic surface fibrations, we can always reduce to the study of flat double cover of a $\mathbb{P}^1$ bundle.

Definition 5.3 A local holomorphic surface fibration $\varphi : \mathscr{X} \to \Delta$ of genus $g \geq 2$ is called *strictly hyperelliptic* if there is a normal flat double cover $\pi : \mathscr{X} \to \mathbb{P}^1 \times \Delta$ relative to Δ. A strictly hyperelliptic pair is the pair of the form (φ, π) as above.

Note that

(i) every hyperelliptic local fibration $\varphi : \mathscr{X} \to \Delta$ is birational to a strict hyperelliptic local fibration. In particular their l.s.i. are the same (cf. Remark 3.3);

(ii) if $(\varphi : \mathscr{X} \to \Delta, \pi : \mathscr{X} \to \mathbb{P}^1 \times \Delta)$ is a strictly hyperelliptic pair, we denote by $\mathscr{B}(\varphi, \pi) \subseteq \mathbb{P}^1 \times \Delta$ the branch divisor of π. This divisor can be uniquely written as

$$\mathscr{B}(\varphi, \pi) = \mathscr{B}'(\varphi, \pi) + \iota(\varphi, \pi) \cdot F_0$$

with $\mathscr{B}'(\varphi, \pi)$ horizontal, $\iota(\varphi, \pi) = 0$ or 1 and F_0 is the central $\mathbb{P}^1$ of $\mathbb{P}^1 \times \Delta$. The data $\mathscr{B}'(\varphi, \pi) \subseteq \mathbb{P}^1 \times \Delta$ induces a holomorphic map $(\varphi, \pi)^* : \Delta \to \mathrm{Hilb}^{2g+2}_{\mathbb{P}^1}$ to the Hilbert scheme of points of degree $2g + 2$ on $\mathbb{P}^1$.

Definition 5.4 Two strictly hyperelliptic pairs (φ, π) and (ϕ, τ) are called equivalent up to level n if $\iota(\varphi, \pi) = \iota(\phi, \tau)$ and $(\varphi, \pi)^*$ is equivalent to $(\phi, \tau)^*$ up to n-th formal neighbourhood at the origin. Namely,

$$(\varphi, \pi)^*_n : \mathrm{Spec}(\mathbb{C}[[t]]/t^n) \to \Delta \xrightarrow{(\varphi,\pi)^*} \mathrm{Hilb}^{2g+2}_{\mathbb{P}^1}$$

coincides with $(\phi, \tau)^*_n$ defined similarly. Equivalently, we have

$$\mathscr{B}'(\varphi, \pi)|_{\mathbb{P}^1 \times_{\mathbb{C}} \mathrm{Spec}(\mathbb{C}[[t]]/t^n)} = \mathscr{B}'(\phi, \tau)|_{\mathbb{P}^1 \times_{\mathbb{C}} \mathrm{Spec}(\mathbb{C}[[t]]/t^n)}.$$

Two strictly hyperelliptic local fibrations φ, ϕ are called equivalent up to level n if there are strict hyperelliptic pairs (φ, π) and (ϕ, τ) equivalent up to level n.

Theorem 5.5. *For any genus $g \geq 2$ strictly hyperelliptic local holomorphic surface fibration $\varphi : \mathscr{X} \to \Delta$, there is an integer n such that we have an equality of l.s.i. $\delta(\varphi) = \delta(\phi)$ for any other strictly hyperelliptic local holomorphic surface fibration $\phi : \mathscr{Y} \to \Delta$ equivalent to φ up to level n.*

Proof. Fix an arbitrary strictly hyperelliptic pair (φ, π) and take $\alpha : \mathscr{T} \to \mathbb{P}^1 \times \Delta$ to be the refined canonical resolution of the flat double cover π(cf. § 2.2). Namely, we can write

$$\alpha^*\mathscr{B}(\varphi, \pi) = \widetilde{\mathscr{B}}_0(\varphi, \pi) + \mathcal{V}_1 + 2\mathcal{V}_2$$

where $\widetilde{\mathscr{B}}_0(\varphi, \pi)$ is the strict transform of $\mathscr{B}'(\varphi, \pi)$, $\mathcal{V}_i, i = 1, 2$, are vertical effective divisors and $\widetilde{\mathscr{B}}_0(\varphi, \pi) + F'$ has normal crossings, here F' is the central fibre of $\mathscr{T} \to \Delta$. Let n_0 be the largest multiplicity of irreducible components in $\alpha^*\mathscr{B}(\varphi, \pi)$ and $n = \max\{n_0, 2\}$. We shall show that n is the desired one.

Let (ϕ, τ) be another strictly hyperelliptic pair equivalent to (φ, π) up to level n. Then by construction, for each irreducible central vertical component D of $\mathscr{T} \to \Delta$, its coefficients in both $\alpha^*\mathscr{B}(\varphi, \pi)$ and $\alpha^*\mathscr{B}(\phi, \tau)$ are the same. In fact, for each exceptional irreducible component $D \subseteq \mathscr{T}$, let $y \in \mathbb{P}^1 \times \Delta$ be its image. Denote by f_1, f_2 the local functions defining $\mathscr{B}'(\varphi, \pi)$ and $\mathscr{B}'(\phi, \tau)$ at y respectively. Then up to a suitable choice, we can assume $f_1 - f_2 = t^n \cdot h$ for some $h \in \mathcal{O}_y$. Denote by v_D the associated normalized valuation of D, then by construction we have $v_D(f_1) < n$ and $v_D(t) \geq 1$. As a result, we have $v_D(f_2) = v_D(f_1)$ and we are done for this claim.

So we have

$$\alpha^*\mathscr{B}(\phi, \tau) = \widetilde{\mathscr{B}}_0(\phi, \tau) + \mathcal{V}_1 + 2\mathcal{V}_2.$$

In particular, by restricting both $\alpha^*\mathscr{B}(\varphi, \pi)$ and $\alpha^*\mathscr{B}(\phi, \tau)$ to the n-th formal neighbourhood of the central fibre of $\mathscr{T} \to \Delta$, we obtain the same divisor $\mathscr{B}_n = \alpha^*\mathscr{B}(\varphi, \pi)|_{\mathscr{T}_n} = \alpha^*\mathscr{B}(\phi, \tau)|_{\mathscr{T}_n}$ on $\mathscr{T}_n := \mathscr{T} \times_\Delta \operatorname{Spec}(\mathbb{C}[[t]]/t^n)$. As a consequence, the divisor $\widetilde{\mathscr{B}}_0(\phi, \tau) + F'$ also has normal crossings since $n \geq 2$.

It remains to show that we can work out the l.s.i. of both φ and ϕ from $\mathscr{B}_n$. Denote by $\pi' : \mathscr{X}' \to \mathscr{T}$ (resp. $\tau' : \mathscr{Y}' \to \mathscr{T}$) the associated birational model obtained by this refined resolution (cf. § 2.2). Then the local fibration $\varphi' : \mathscr{X}' \to \mathscr{T} \to \Delta$ (resp. $\phi' : \mathscr{Y}' \to \mathscr{T} \to \Delta$) is birational to φ and hence it suffices to work out the l.s.i. for φ' (resp. ϕ'). Note both central fibres of $\mathscr{X}', \mathscr{Y}'$ have simple normal crossings by Proposition 2.3. And since their branch divisors on $\mathscr{T}$ are equivalent up to level $n \geq 2$, Proposition 3.8 applies to give $\delta(\varphi) = \delta(\phi)$. □

5.2 Proof of Theorem 5.1

Given finitely many hyperelliptic local surface fibrations $\varphi_i : \mathscr{X}_i \to \Delta$. By replacing each $\mathscr{X}_i$ by a suitable birational model (so the l.s.i. δ is preserved), we can assume all there fibrations are strictly hyperelliptic. Then we equip each φ_i with an associated flat double cover $\pi_i : \mathscr{X}_i \to \mathbb{P}^1 \times \Delta$. Denote by $\theta_i := (\varphi_i, \pi_i)^* : \Delta \to \mathrm{Hilb}^{2g+2}_{\mathbb{P}^1}$ the associated morphism induced by the horizontal branch divisor in π_i.

We first note that $\mathrm{Hilb}^{2g+2}_{\mathbb{P}^1}$ is actually isomorphic to a projective space $\mathbb{P}^{2g+2}_{\mathbb{C}}$. We fix one such isomorphism $\mathrm{Hilb}^{2g+2}_{\mathbb{P}^1} \simeq \mathbb{P}^{2g+2}_{\mathbb{C}}$ so that all images of θ_i are contained in $\mathbb{A}^{2g+2}_{\mathbb{C}} \subseteq \mathbb{P}^{2g+2}$ (if necessary, shrinking φ_i). Fixing another point $Q \in \mathbb{A}^{2g+2}_{\mathbb{C}} \subseteq \mathrm{Hilb}^{2g+2}_{\mathbb{P}^1}$ whose associated degree $2g+2$ point in $\mathbb{P}^1$ is smooth, denote by $\theta_0 : \Delta \to \mathbb{A}^{2g+2}_{\mathbb{C}}$ the constant map mapping all points to Q.

Fixing any $s+1$ distinct points $\xi_0, \xi_1, \cdots, \xi_s \in \mathbb{C}$, we can then approximate the along $s+1$ maps $\theta_i : \Delta \to \mathbb{A}^{2g+2}_{\mathbb{C}}$ at their respective points ξ_i simultaneously.

Proposition 5.6. *For any n, there is a polynomial map*

$$\mu_n : \mathbb{C} \to \mathbb{A}^{2g+2}_{\mathbb{C}} : \lambda \mapsto (f_1(\lambda), \cdots, f_{2g+2}(\lambda)), f_i \in \mathbb{C}[t]$$

such that the associated maps $\Delta \overset{+\xi_i}{\to} \mathbb{C} \overset{\mu_n}{\to} \mathbb{A}^{2g+2}_{\mathbb{C}}, t \mapsto t + \xi_i \mapsto \mu_n(t+\xi_i)$ approximates θ_i up to to level n.

Proof. For any $j = 1, \cdots, 2g+2$, we write h_{ij} for the pull back of the j-th coordinate function of $\mathbb{A}^{2g+2}_{\mathbb{C}}$ by θ_i. Then it suffices to find a polynomial $f_j(t) \in \mathbb{C}[t]$ such that $f_j(t-\xi_i) - h_{ij}(t)$ vanishes at $0 \in \mathbb{C}$ up to order at least n. The following lemma asserts this fact. □

Lemma 5.7. *For $s+1$ distinct points $\xi_0, \xi_1, \cdots, \xi_s \in \mathbb{C}$, $s+1$ arbitrary holomorphic functions $h_0(t), \cdots, h_s(t)$ on Δ and any positive integer $n \in \mathrm{N}$, there is a polynomial $f \in \mathbb{C}[t]$ such that $f(t-\xi_i) - h_i(t)$ has a zero of order at least n at the origin.*

Proof. It is an easy exercise of interpolation. □

Proof of Theorem 5.1. Take $n \gg 0$ and let $\mu_n : \mathbb{C} \to \mathbb{A}^{2g+2}_{\mathbb{C}}$ be as above. It then extends to a holomorphic map $\widetilde{\mu}_n : \mathbb{P}^1_{\mathbb{C}} \to \mathrm{Hilb}^{2g+2}_{\mathbb{P}^1}$. As a result, it gives a horizontal divisor $\mathscr{B} := \mathcal{U} \times_{\mathrm{Hilb}^{2g+2}_{\mathbb{P}^1}, \widetilde{\mu}_n} \mathbb{P}^1_{\mathbb{C}} \subseteq \mathbb{P}^1 \times \mathbb{P}^1_{\mathbb{C}}$. Here $\mathcal{U}$ is the universal family of degree $2g+2$ points.

$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & \mathbb{P}^1 \times \mathrm{Hilb}^{2g+2}_{\mathbb{P}^1} \\ & \searrow \quad \swarrow_{p_2} & \\ & \mathrm{Hilb}^{2g+2}_{\mathbb{P}^1} & \end{array}$$

We then construct a flat double cover $\pi : S \to \mathbb{P}^1 \times \mathbb{P}^1_{\mathbb{C}}$ with branch divisor $\mathscr{B}$ plus fibres $\mathbb{P}^1 \times \xi_i$ for those i such that $\iota(\varphi_i, \pi_i) = 1$ and plus another fibre $\mathbb{P}^1 \times \xi$ for a general ξ if the previous divisor is not an even divisor. Then via the second projection, we obtain a fibration $f : S \xrightarrow{\pi} \mathbb{P}^1 \times \mathbb{P}^1_{\mathbb{C}} \xrightarrow{p_2} \mathbb{P}^1_{\mathbb{C}}$. By construction, we have:

- f has a connected smooth fibre over ξ_0 and hence all general fibres are connected and smooth;
- for each i, the inclusion $v_i : \Delta \xrightarrow{+\xi_i} \mathbb{C} \subseteq \mathbb{P}^1_C$ gives a fibration $\phi_i : \mathscr{Y}_i := S \times_{\mathbb{P}^1_{\mathbb{C}}, v_i} \Delta \to \Delta$. We have $\delta(\phi_i) = \delta(\varphi_i)$ by our construction and Theorem 5.5.

So we are done by Proposition 3.9 and Corollary 3.7. □

6 An equidistribution theorem and the proof of Theorem 1.6

We call a natural number number *p-leading* if the leftmost digit in its base p expansion is $p-1$. Thus, m is p-leading precisely when $p^k - p^{k-1} \leq m < p^k$ for some positive integer k. It is straightforward to check, by comparing (1.3) and (1.6), that Theorem 1.6 is equivalent to the following proposition (where m plays the role of $2g+1$).

Proposition 6.1. *There are infinitely many odd natural numbers m that are not p-leading for any prime $p \geq 3$.*

For the proof of Proposition 6.1, it is convenient to introduce the notion of *logarithmic density*. If S is a set of natural numbers, by the logarithmic density of S we mean the value of the limit

$$\lim_{N\to\infty} \frac{\sum_{n\leq N,\, n\in S} 1/n}{\sum_{n\leq N} 1/n},$$

if the limit exists. The *upper* and *lower* logarithmic densities are defined similarly, but with lim sup and lim inf replacing lim, respectively. Since the difference

$$\sum_{n\leq N} \frac{1}{n} - \log N \qquad \text{is a bounded function of } N,$$

in all of these definitions the denominator $\sum_{n\leq N} 1/n$ can be replaced with $\log N$ without any change in meaning. This explains the term "logarithmic density."

Let $\{x_n\}$ be a sequence of a points in $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. We say $\{x_n\}$ is *logarithmically equidistributed* if

$$\lim_{N\to\infty} \frac{\sum_{n\leq N} f(x_n)/n}{\sum_{n\leq N} 1/n} = \int_{\mathbb{T}^d} f(x)\, dx. \tag{6.1}$$

for every Riemann integrable function f on $\mathbb{T}^d$. Note that if f is the characteristic function of a subset S of $\mathbb{T}^d$, then the left-hand side of (6.1) is precisely the logarithmic density of those n for which $x_n \in S$.

We require the following logarithmic variant of the well-known Weyl criterion for equidistribution mod 1. For real numbers t, we use $\mathbb{E}(t)$ to denote $\exp(2\pi i t)$.

Weyl's criterion for logarithmic equidistribution. *Let $\{x_n\}$ be a sequence of points in $\mathbb{T}^d$. Then $\{x_n\}$ is logarithmically equidistributed if and only if*

$$\lim_{N\to\infty} \frac{\sum_{n\le N} \mathbb{E}(k\cdot x_n)/n}{\sum_{n\le N} 1/n} = 0$$

for each nonzero $k \in \mathbb{Z}^d$.

Proof. The proof is nearly identical to that of the usual Weyl criterion (as given in detail in Chapter 6 of [9], for example) and so we content ourselves with a sketch. The "only if" half is immediate, taking $f(x) = \mathbb{E}(k\cdot x)$ in our definition of logarithmic equidistribution. So we need only discuss the "if" direction.

By assumption, the relation (6.1) holds for the functions $\mathbb{E}(k\cdot x)$ when $k \neq 0$; it also holds trivially when $k = 0$. Since every continuous function on $\mathbb{T}^d$ can be uniformly approximated by a finite linear combination of the functions $\mathbb{E}(k\cdot x)$ (see, for instance, Theorem 6.13 of [9]), we easily deduce that (6.1) holds for all continuous functions f.

It remains to prove (6.1) for the wider class of Riemann integrable functions. If f is an arbitrary Riemann integrable function, then for every $\epsilon > 0$ one can find step functions f^- and f^+ with $f^- \le f \le f^+$ and $\int_{\mathbb{T}^d}(f^+(x) - f^-(x))\,dx < \epsilon$. Exploiting linearity, this reduces the proof of (6.1) for integrable f to the proof of (6.1) for characteristic functions of intervals χ_I. This in turn is easily reduced to the continuous case, established above: We use that for any interval I of $\mathbb{T}^d$, and any $\epsilon > 0$, there are continuous functions χ_I^-, χ_I^+ with $\chi_I^- \le \chi \le \chi_I^+$ and $\int_{\mathbb{T}^d}(\chi_I^+(x) - \chi_I^-(x))\,dx < \epsilon$. □

The following result seems perhaps of independent interest (for instance, in the study of Benford's law).

Theorem 6.2. *Suppose $g_1, g_2 \ge 2$ are integers for which $\log g_1$ and $\log g_2$ are linearly independent over $\mathbb{Q}$. Let q, a be integers with $q \ge 1$ and $0 \le a < q$. The sequence of ordered pairs $(\log(qn+a)/\log g_1, \log(qn+a)/\log g_2)$, reduced mod $\mathbb{Z}^2$, is equidistributed in $\mathbb{T}^2$.*

Proof. By our version of Weyl's criterion, it is enough to show that if $(0,0) \neq (k_1, k_2) \in \mathbb{Z}^2$, then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n\le N} \frac{\mathbb{E}(k_1 \frac{\log(qn+a)}{\log g_1} + k_2 \frac{\log(qn+a)}{\log g_2})}{n} = 0. \tag{6.2}$$

For each natural number n, we can write

$$k_1 \frac{\log(qn+a)}{\log g_1} + k_2 \frac{\log(qn+a)}{\log g_2} = k_1 \frac{\log n}{\log g_1} + k_2 \frac{\log n}{\log g_2} + \kappa + \epsilon(n),$$

where

$$\kappa = k_1 \frac{\log q}{\log g_1} + k_2 \frac{\log q}{\log g_2}, \qquad \text{and} \qquad \epsilon(n) = k_1 \frac{\log(1+\frac{a}{qn})}{\log g_1} + k_2 \frac{\log(1+\frac{a}{qn})}{\log g_2}.$$

Notice that $|\epsilon(n)| \leq C/n$, where $C = |k_1|/\log g_1 + |k_2|/\log g_2$. Since $|\mathbb{E}(t)| = 1$ for all t, while $|\mathbb{E}(t) - 1| = |\int_0^t \mathbb{E}'(t)\, dt| \leq 2\pi|t|$, it follows that

$$\left| \frac{\mathbb{E}(k_1 \frac{\log(qn+a)}{\log g_1} + k_2 \frac{\log(qn+a)}{\log g_2})}{n} - \mathbb{E}(\kappa) \frac{\mathbb{E}(k_1 \frac{\log n}{\log g_1} + k_2 \frac{\log n}{\log g_2})}{n} \right| \leq \frac{2\pi C}{n^2}.$$

As $\sum_{n\geq 1} 1/n^2 < \infty$, we deduce that to prove (6.2), it is enough to show that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n\leq N} \frac{\mathbb{E}(k_1 \frac{\log n}{\log g_1} + k_2 \frac{\log n}{\log g_2})}{n} = 0. \tag{6.3}$$

Continuing, observe that $\frac{\mathbb{E}(k_1 \frac{\log n}{\log g_1} + k_2 \frac{\log n}{\log g_2})}{n} = n^{i\eta - 1}$, where $\eta = 2\pi(k_1/\log g_1 + k_2/\log g_2)$, and that that $\eta \neq 0$ since $\log g_1, \log g_2$ are $\mathbb{Q}$-linearly independent. By the Euler–Maclaurin summation formula,

$$\sum_{n\leq N} n^{i\eta-1} = \int_1^N x^{i\eta-1}\, dx + \frac{1}{2}(N^{i\eta-1} + 1) + \int_1^N (i\eta - 1) x^{i\eta-2}(x - \lfloor x \rfloor - \frac{1}{2})\, dx.$$

We now show that each of the three right-hand summands is bounded independently of N, from which (6.3) follows immediately. We have

$$\left| \int_1^N x^{i\eta-1}\, dx \right| = \left| \frac{1}{i\eta}(N^{i\eta} - 1) \right| \leq \frac{2}{|\eta|},$$

$$\left| \frac{1}{2}(N^{i\eta-1} + 1) \right| \leq 1,$$

and

$$\left| \int_1^N (i\eta - 1) x^{i\eta-2}(x - \lfloor x \rfloor - 1/2)\, dx \right| \leq \frac{1}{2}|i\eta - 1| \int_1^N x^{-2}\, dx \leq \frac{1}{2}|i\eta - 1|. \qquad \square$$

Corollary 6.3. *The set of n for which $2n+1$ is neither 3-leading nor 5-leading has logarithmic density* $\frac{\log 2}{\log 3} \cdot \frac{\log 4}{\log 5}$.

PROOF. Observe that n is p-leading in base p precisely when the fractional part of $\log n/\log p$ belongs to the interval $[\log(p-1)/\log p, 1)$. To deduce Corollary 6.3 from Theorem 6.2, we let $q=2$, $a=1$, $g_1=3$, $g_2=5$, and we let f be the characteristic function of $[0,\log 2/\log 3)\times[0,\log 4/\log 5) \pmod{\mathbb{Z}^2}$. Note that $\log 3$ and $\log 5$ are certainly linearly independent over $\mathbb{Q}$: Otherwise, after clearing denominators, we would find positive integers m, n with $m\log 3 - n\log 5 = 0$; but then $3^m = 5^n$, which is absurd. □

Proof of Proposition 6.1. Let S be the set of n for which $2n+1$ is p-leading for some prime $p>5$. We will argue that S has upper logarithmic density smaller than 0.52. Since

$$\frac{\log 2}{\log 3}\cdot\frac{\log 4}{\log 5} = 0.543\ldots,$$

we deduce from Corollary 6.3 that the set of n for which $2n+1$ is not p-leading for any $p\geq 3$ has lower logarithmic density larger than $0.54-0.52=0.02$. As a set of positive lower logarithmic density is necessarily infinite, we obtain Proposition 6.1.

If $2n+1$ is p-leading, with p odd, then $p^{k-1}(p-1) < 2n+1 < p^k$ for some positive integer k. Given p and k, there are $\frac{1}{2}(p^{k-1}-1) < \frac{1}{2}p^{k-1}$ integers n satisfying this inequality, and each of these n has size at least $\frac{1}{2}p^{k-1}(p-1)$. Hence, if we sum the reciprocals of all solutions n to this inequality (for a given p, k), then

$$\sum \frac{1}{n} < \frac{1}{2}p^{k-1}\frac{1}{\frac{1}{2}p^{k-1}(p-1)} = \frac{1}{p-1}.$$

Suppose that $n\leq N$ and $n\in S$. We can choose a prime $p>5$ and a positive integer k with $p^{k-1}(p-1) < 2n+1 < p^k$. Then $p^k \leq 2p^{k-1}(p-1)\leq 4n\leq 4N$, and therefore $k\leq \log(4N)/\log p$. So by the result of the last paragraph,

$$\sum_{n\leq N,\ n\in S}\frac{1}{n} \leq \sum_{p>5}\sum_{1\leq k\leq \frac{\log(4N)}{\log p}}\frac{1}{p-1} \leq \sum_{p>5}\frac{\log(4N)}{(p-1)\log p}.$$

Dividing by $\log N$ and sending N to infinity reveals that S has upper logarithmic density at most

$$\sum_{p>5}\frac{1}{(p-1)\log p}.$$

H. Cohen [4] has shown that $\sum_p \frac{1}{p\log p} = 1.6366163\ldots$, where the sum ranges over all primes p, so that

$$\sum_{p>5}\frac{1}{p\log p} = \sum_p \frac{1}{p\log p} - \left(\frac{1}{2\log 2}+\frac{1}{3\log 3}+\frac{1}{5\log 5}\right) < 0.488.$$

Clearly,

$$\sum_{p>5}\frac{1}{(p-1)\log p} = \sum_{p>5}\frac{1}{p\log p} + \sum_{p>5}\frac{1}{p(p-1)\log p}.$$

A short and direct computation with `PARI/GP` shows that $\sum_{5<p\le 10^5} \frac{1}{p(p-1)\log p} < 0.02361$. Also, $\sum_{p>10^5} \frac{1}{p(p-1)\log p} < \sum_{m>10^5} \frac{1}{m(m-1)} = \frac{1}{10^5}$. Hence, $\sum_{p>5} \frac{1}{p(p-1)\log p} < 0.024$ and

$$\sum_{p>5} \frac{1}{(p-1)\log p} < 0.488 + 0.024 < 0.52.$$

This completes the proof. □

Remark. The argument given for Theorem 6.2 is easily adapted to prove the logarithmic equidistribution of the d-tuples $(\log(qn+a)/\log g_1, \ldots, \log(qn+a)/\log g_d) \bmod \mathbb{Z}^d$ in $\mathbb{T}^d$, under the hypothesis that $1/\log g_1, \ldots, 1/\log g_d$ are $\mathbb{Q}$-linearly independent. (Note that when $d = 2$, this condition reduces to the $\mathbb{Q}$-linear independence of $\log g_1$ and $\log g_2$.) Now suppose that the numbers $1/\log p$, for primes $p \ge 3$, were known to be $\mathbb{Q}$-linearly independent. Using this generalization of Theorem 6.2, we could modify our arguments to show that the set of n for which $2n+1$ is not p-leading for any prime $p \ge 3$ has logarithmic density given exactly by the infinite product $\prod_{p\ge 3} \frac{\log(p-1)}{\log p}$. Unfortunately, whether or not the numbers $1/\log p$, for primes $p \ge 3$, are $\mathbb{Q}$-linearly independent appears to be an open question. An affirmative answer would follow from Schanuel's conjecture in transcendence theory: *If* $z_1, \ldots, z_n \in \mathbb{C}$ *are* $\mathbb{Q}$*-linearly independent, then the transcendence degree of* $\mathbb{Q}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$ *over* $\mathbb{Q}$ *is at least* n.

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