POWERFREE SUMS OF PROPER DIVISORS

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Abstract. Let $s(n) := \sum_{d|n,\ d<n} d$ denote the sum of the proper divisors of n. It is natural to conjecture that for each integer $k \geq 2$, the equivalence

$$n \text{ is } k\text{th powerfree} \Longleftrightarrow s(n) \text{ is } k\text{th powerfree}$$

holds almost always (meaning, on a set of asymptotic density 1). We prove this for $k \geq 4$.

1. Introduction

A 19th century theorem of Gegenbauer asserts that for each fixed k, the set of positive integers not divisible by the kth power of an integer larger than 1 has asymptotic density $\zeta(k)^{-1}$, where $\zeta(s)$ is the familiar Riemann zeta function. Recall that the asymptotic density of a set $\mathcal{A}$ of positive integers is the limiting proportion of the elements of $\mathcal{A}$ up to x, more precisely the limit as $x \to \infty$ of the quantity $\frac{1}{x}\#\{a \leq x : a \in \mathcal{A}\}$, subject to existence.

Call an integer kth-power-free, or k-free for short when it is not divisible by the kth power of an integer larger than 1. In this note we investigate the frequency with which the sum-of-proper-divisors function $s(n) := \sum_{d|n,\ d<n} d$ assumes k-free values. As we proceed to explain, there is a natural guess to make here, formulated below as Conjecture 1.1.

Fix $k \geq 2$. If n is not k-free, then $p^k \mid n$ for some prime p. Moreover, if $y = y(x)$ is any function tending to infinity, the the upper density of n divisible by p^k for some $p > y^{1/k}$ is at most $\sum_{p>y^{1/k}} p^{-k} = o(1)$. Hence, almost always a non k-free number n is divisible by p^k for some $p^k \leq y$. To be precise, when we say a statement about positive integers n holds almost always, we mean that it holds for all $n \leq x$ with $o(x)$ exceptions, as $x \to \infty$. (Importantly, we allow the statement itself to involve the growing upper bound x.)

It was noticed by Alaoglu and Erdős [AE44] that whenever $y = y(x)$ tends to infinity with x slowly enough, $\sigma(n)$ is divisible by all of the integers in $[1, y]$ almost always. (We give a proof below with $y := (\log\log x)^{1-\epsilon}$; see Lemma 2.2.) Hence, almost always n and $s(n) = \sigma(n) - n$ share the same set of divisors up to y. Putting this together with the observations of the last paragraph, we see that if n is not k-free, then $s(n)$ is not k-free, almost always. The same reasoning shows that if n is k-free, then $s(n)$ is not divisible by p^k for any $p \leq y^{1/k}$, almost always. Thus, if it could be shown that almost always $s(n)$ is not divisible by p^k for any prime $p > y^{1/k}$, then we would have established the following conjecture.

Conjecture 1.1. *Fix $k \geq 2$. On a set of integers n of asymptotic density 1,*

$$n \text{ is } k\text{-free} \Longleftrightarrow s(n) \text{ is } k\text{-free}.$$

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The case $k = 2$ of Conjecture 1.1 is alluded to by Luca and Pomerance in [LP15] (see Lemma 2.2 there and the discussion following). Their arguments show that $s(n)$ is squarefree on a set of positive lower density (in fact, of lower density at least $\zeta(2)^{-1}\log 2$). Conjecture 1.1, for every $k \geq 2$, would then be a consequence of the following general conjecture of Erdős–Granville–Pomerance–Spiro [EGPS90] (see Remark 3.5 below).

Conjecture 1.2. *If $\mathcal{A}$ is a set of natural numbers of positive upper density, then $s(\mathcal{A}) := \{s(n) : n \in \mathcal{A}\}$ also has positive upper density.*

We recall that the upper and lower densities of a set of positive integers are defined in the exact same way as the asymptotic density, but with lim sup and lim inf replacing the limit respectively (so that these always exist).

Our result is as follows.

Theorem 1.3. *Conjecture 1.1 holds for each $k \geq 4$.*

To prove Conjecture 1.1 for a given k, it is enough (by the above discussion) to show that almost always $s(n)$ is not divisible by p^k for any $p^k > (\log\log x)^{0.9}$. The range $p \leq x^{o(1)}$ can be treated quickly using familiar arguments (versions of which appear, e.g., in [Pol14]). The main innovation in our argument — and the source of the restriction to $k \geq 4$ — is the handling of larger p using a theorem of Wirsing [Wir59] that bounds the "popularity" of values of the function $\sigma(n)/n$.

The reader interested in other work on powerfree values of arithmetic functions may consult [Pap03, PSS03, BL05, BP06] as well as the survey [Pap05].

Notation and conventions. We reserve the letters p, q, P, with or without subscripts, for primes and we write $\log_k$ for the kth iterate of the natural logarithm. We write $P^+(n)$ and $P^-(n)$ for the largest and smallest prime factors of n, with the conventions that $P^+(1) = 1$ and $P^-(1) = \infty$. We adopt the Landau–Bachmann–Vinogradov notation from asymptotic analysis, with all implied constants being absolute unless specified otherwise.

2. Preliminaries

The following lemma is due to Pomerance (see [Pom77, Theorem 2]).

Lemma 2.1. *Let a, k be integers with $\gcd(a,k) = 1$ and $k > 0$. Let $x \geq 3$. The number of $n \leq x$ for which there does not exist a prime $p \equiv a \pmod{k}$ for which $p \parallel n$ is $O(x(\log x)^{-1/\varphi(k)})$.*

The next lemma justifies the claim in the introduction that $\sigma(n)$ is usually divisible by all small primes. It is well-known (see, e.g., Lemma 2.1 of [LP15]) but we include the short proof.

Lemma 2.2. *Fix $\epsilon > 0$. Almost always, the number $\sigma(n)$ is divisible by every positive integer $d \leq (\log_2 x)^{1-\epsilon}$.*

Proof. Notice that $d \mid \sigma(n)$ whenever there is a prime $p \equiv -1 \pmod{d}$ such that $p \parallel n$. For each $d \leq (\log_2 x)^{1-\epsilon}$, the number of $n \leq x$ for which there is no such p is $O(x\exp(-(\log_2 x)^{\epsilon}))$, by Lemma 2.1. Now sum on $d \leq (\log_2 x)^{1-\epsilon}$. $\square$

Our next lemma bounds the number of $n \le x$ for which n and $\sigma(n)$ possess a large common prime divisor. In what follows, we say that a positive integer a is squarefull if no prime appears only to the first power in a; or, in other words, if p^2 divides a for every prime p dividing a. By the squarefull part of a natural number, we shall mean its largest squarefull divisor.

Lemma 2.3. *Almost always, the greatest common divisor of n and $\sigma(n)$ has no prime divisor exceeding $\log_2 x$.*

With more effort, it could be shown that $\gcd(n, \sigma(n))$ is almost always the largest divisor of n supported on primes not exceeding $\log_2 x$. Compare with Theorem 8 in [ELP08], which is the corresponding assertion with $\sigma(n)$ replaced by $\varphi(n)$.

Proof. Put $y := \log_2 x$. We start by removing those $n \le x$ with squarefull part exceeding $\frac{1}{2}y$. The number of these n is $O(xy^{-1/2})$, which is $o(x)$ and hence negligible.

Suppose that n survives and there is a prime $p > y$ dividing n and $\sigma(n)$. Since $p \mid \sigma(n)$, we can choose a prime power $q^e \parallel n$ for which $p \mid \sigma(q^e)$. Then $y < p \le \sigma(q^e) < 2q^e$, forcing $e = 1$. Hence, $p \mid \sigma(q) = q + 1$ and $q \equiv -1 \pmod p$. Since $pq \mid n$, we deduce that the number of n belonging to this case is at most

$$\sum_{p>y} \sum_{\substack{q \equiv -1 \pmod p \\ q \le x}} \frac{x}{pq} \ll x \sum_{p>y} \frac{1}{p} \sum_{\substack{q \le x \\ q \equiv -1 \pmod p}} \frac{1}{q} \ll x \log_2 x \sum_{p>y} \frac{1}{p^2} \ll \frac{x \log_2 x}{y \log y} = \frac{x}{\log_3 x},$$

which is again $o(x)$. Here the sum on q has been estimated by the Brun–Titchmarsh inequality (see, e.g., Theorem 416 on p. 83 of [Ten15]) and partial summation. □

The next lemma bounds the number of $n \le x$ with two large prime factors that are multiplicatively close.

Lemma 2.4. *For all large x, the number of $n \le x$ divisible by a pair of primes q_1, q_2 with*

$$x^{1/10 \log_3 x} < q_1 \le x \quad \text{and} \quad q_1 x^{-1/(\log_3 x)^2} \le q_2 \le q_1$$

is $O(x/\log_3 x)$.

Proof. The number of such n is at most $x \sum_{x^{1/10 \log_3 x} < q_1 \le x} \frac{1}{q_1} \sum_{q_1 x^{-1/(\log_3 x)^2} \le q_2 \le q_1} \frac{1}{q_2}$. By Mertens' theorem, the inner sum is

$$\ll \log\left(\frac{\log(q_1)}{\log(q_1 x^{-1/(\log_3 x)^2})}\right) + \frac{1}{\log(q_1 x^{-1/(\log_3 x)^2})} \ll \frac{\log x}{(\log q_1)(\log_3 x)^2},$$

leading to an upper bound for our count of n of

$$\ll \frac{x \log x}{(\log_3 x)^2} \sum_{x^{1/10 \log_3 x} < q_1 \le x} \frac{1}{q_1 \log q_1} \ll \frac{x \log x}{(\log_3 x)^2} \cdot \frac{\log_3 x}{\log x} = \frac{x}{\log_3 x}.$$

Here the final sum has been estimated by the prime number theorem and partial summation. □

We conclude this section by quoting the main result of [Wir59].

Lemma 2.5 (Wirsing). *There exists an absolute constant $\lambda_0 > 0$ such that*

$$\#\left\{m \le x : \frac{\sigma(m)}{m} = \alpha\right\} \le \exp\left(\lambda_0 \frac{\log x}{\log_2 x}\right)$$

for all $x \ge 3$ and all real numbers α.

3. Proof of Theorem 1.3

As discussed in the introduction, it is enough to establish the following proposition. From now on, $y := (\log_2 x)^{0.9}$.

Proposition 3.1. *Fix $k \ge 4$. Almost always, $s(n)$ is not divisible by p^k for any $p^k > y$.*

We split the proof of Proposition 3.1 into two parts, according to the size of p.

3.1. **... when $y < p^k \le x^{1/2 \log_3 x}$.** The following is a weakened form of Lemma 2.8 from [Pol14].

Lemma 3.2. *For all large x, there is a set $\mathcal{E}(x)$ having size $o(x)$, as $x \to \infty$, such that the following holds. For all $d \le x^{1/2 \log_3 x}$, the number of $n \le x$ not belonging to $\mathcal{E}(x)$ for which $d \mid s(n)$ is $O(x/d^{0.9})$.*

Summing the bound of Lemma 3.2 over $d = p^k$ with $y < p^k \le x^{1/2 \log_3 x}$ gives $o(x)$. It follows that almost always, $s(n)$ is not divisible by p^k for any $p^k \in (y, x^{1/2 \log_3 x}]$.

3.2. **... when $p^k > x^{1/2 \log_3 x}$.** The treatment of this range of p is based on the following result, which may be of independent interest.

Theorem 3.3. *For all large x, there is a set $\mathcal{E}(x)$ having size $o(x)$, as $x \to \infty$, such that the following holds. The number of $n \le x$ not belonging to $\mathcal{E}(x)$ for which $d \mid s(n)$ is*

$$\ll \frac{x}{d^{1/4} \log x}$$

uniformly for positive integers $d > x^{1/2 \log_3 x}$ satisfying $P^-(d) > \log_2 x$.

The crucial advantage of Theorem 3.3 over Lemma 3.2 is the lack of any restriction on the size of d. Since $k \ge 4$, when we sum the bound of Theorem 3.3 over $d = p^k$ with $x^{1/2 \log_3 x} < p^k < x^2$, the result is $O(x \log_2 x / \log x)$, which is $o(x)$. So the proof of Theorem 1.3 will be completed once Theorem 3.3 is established.

Turning to the proof of Theorem 3.3, let $\mathcal{E}(x)$ denote the collection of $n \le x$ for which at least one of the following fails:

(1) $n > x/\log x$,

(2) the largest squarefull divisor of n is at most $\log_2 x$,

(3) $P^+(n) > x^{1/10 \log_3 x}$,

(4) $P^+(n)^2 \nmid n$,

(5) $P^+(\gcd(n, \sigma(n))) \le \log_2 x$,

(6) $P^+(n) > P_2^+(n)x^{1/(\log_3 x)^2}$, where $P_2^+(n) := P^+(n/P^+(n))$ is the second-largest prime factor of n.

Let us show that only $o(x)$ integers $n \le x$ fail one of (1)–(6). This is obvious for (1). The count of $n \le x$ failing (2) is $\ll x \sum_{r > \log_2 x,\ r \text{ squarefull}} 1/r \ll x/\sqrt{\log_2 x}$, and thus is $o(x)$. That the count of $n \le x$ failing (3) is $o(x)$ follows from standard bounds on the counting function of smooth (friable) numbers (e.g., Theorem 5.1 on p. 512 of [Ten15]), or Brun's sieve. The set of $n \le x$ passing (3) but failing (4) has cardinality $\ll x \sum_{r > x^{1/10 \log_3 x}} 1/r^2 = o(x)$. Condition (5) is handled by Lemma 2.3. That the count of $n \le x$ satisfying (1)–(5) and failing (6) is $o(x)$ follows from Lemma 2.4.

Let d be as in Theorem 3.3. We separate the count of $n \notin \mathcal{E}(x)$ for which $d \mid s(n)$ according to whether $P^+(n) < d^{1/4}(\log x)^2$ or $P^+(n) \ge d^{1/4}(\log x)^2$.

We first consider $n \notin \mathcal{E}(x)$ with $P^+(n) \ge d^{1/4}(\log x)^2$. Write $n = mP$, where $P := P^+(n)$. Then $\gcd(m, P) = 1$, and

$$x/m \ge d^{1/4}(\log x)^2.$$

We can rewrite the condition $d \mid s(n)$ as

$$Ps(m) \equiv -\sigma(m) \pmod{d}.$$

For this congruence to have solutions, we must have $\gcd(s(m)\sigma(m), d) = 1$. Indeed, if there exists a prime q dividing both $\sigma(m)$ and d, then from $q \mid d$, we have $q > \log_2 x$, whereas since $d \mid s(n)$, we also have $q \mid s(n)$. But then the divisibility $q \mid \sigma(m) \mid \sigma(n)$ leads to $q \mid \gcd(n, \sigma(n))$, contradicting condition (5) above. Since any common prime divisor of $s(m)$ and d would, by the congruence, have to divide $\sigma(m)$ as well, we must indeed have $\gcd(s(m)\sigma(m), d) = 1$.

As such, the above congruence condition on P places it in a unique coprime residue class modulo d. Hence, given m, the number of possible P (and hence possible $n = mP$) is

$$\ll \frac{x}{md} + 1 \ll \frac{x}{md} + \frac{x}{md^{1/4}(\log x)^2},$$

which when summed over $m \le x$ is $\ll x/d^{1/4} \log x$, consistent with Theorem 3.3. (We use here the lower bound on d.)

It remains to count $n \le x$, $n \notin \mathcal{E}(x)$ where $d \mid s(n)$ and $P^+(n) < d^{1/4}(\log x)^2$. For this case, we fix a constant

$$\lambda > 2\lambda_0,$$

where λ_0 is the constant appearing in Wirsing's bound (Lemma 2.5). We will assume that $d \le x^{3/2}$, since $s(n) \le \sigma(n) < x^{3/2}$ for all $n \le x$, once x is sufficiently large (e.g., as a consequence of the bound $\sigma(n) \ll n \log_2(3n)$; see Theorem 323 in [HW08]).

We write $n = AB$, where A is the least unitary squarefree divisor of $n/P^+(n)$ exceeding $d^{1/4} \exp\left(\frac{\lambda}{2} \frac{\log x}{\log_2 x}\right)$. Such a divisor exists as $n > x/\log x$ has maximal squarefull divisor at most $\log_2 x$, whereupon its largest unitary squarefree divisor coprime to $P^+(n)$ must be no less than

$$\frac{1}{d^{1/4}(\log x)^2 \log_2 x} \cdot \frac{x}{\log x} > d^{1/4} \exp\left(\frac{\lambda}{2} \frac{\log x}{\log_2 x}\right).$$

(We assume throughout this argument that x is sufficiently large.) Then

$$B \leq \frac{x}{A} \leq \frac{x}{d^{1/4}} \exp\left(-\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right). \tag{1}$$

Furthermore,

$$P^+(A) \leq P_2^+(n) < P^+(n)x^{-1/(\log_3 x)^2} < d^{1/4}(\log x)^2 x^{-1/(\log_3 x)^2} < d^{1/4}x^{-\lambda/\log_2 x}.$$

Since $A/P^+(A)$ is a unitary squarefree divisor of $n/P^+(n)$, to avoid contradicting the choice of A, we must have $A \leq d^{1/2}\exp\left(-\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right)$. Then $\sigma(A) \ll A\log_2 A \ll A\log_2 x$, so that (for large x) $\sigma(A) < d^{1/2}$.

For each B as above, we bound the number of corresponding A. First of all, since $\gcd(A, B) = 1$, the divisibility $d \mid s(n)$ translates to the congruence $\sigma(A)\sigma(B) \equiv AB \pmod{d}$. Now, we claim that $\gcd(A\sigma(B), d) = 1$: indeed, for any prime q dividing both A and d, we must have, on one hand, $q \geq P^-(d) > \log_2 x$, while on the other, $q \mid d \mid s(n)$ and $q \mid A \mid n$ imply $q \mid \gcd(n, \sigma(n))$. This contradicts (5). It follows by an analogous argument that $\gcd(\sigma(B), d) = 1$, thus proving our claim. Consequently, the above congruence may be rewritten as

$$\frac{\sigma(A)}{A} \equiv \frac{B}{\sigma(B)} \pmod{d}.$$

Now for some B, consider any pair of squarefree integers A_1 and A_2 satisfying the above congruence along with the conditions $\sigma(A_1), \sigma(A_2) < d^{1/2}$. Then $\sigma(A_1)/A_1 \equiv \sigma(A_2)/A_2 \pmod{d}$, leading to $\sigma(A_1)A_2 \equiv A_1\sigma(A_2) \pmod{d}$. But also

$$|\sigma(A_1)A_2 - A_1\sigma(A_2)| \leq \max\{\sigma(A_1)A_2, A_1\sigma(A_2)\} < d,$$

thereby forcing $\sigma(A_1)/A_1 = \sigma(A_2)/A_2$. This shows that for each B, all corresponding A have $\sigma(A)/A$ assume the same value, whereupon Lemma 2.5 bounds the number of possible A by $\exp\left(\lambda_0\frac{\log x}{\log_2 x}\right)$. Keeping in mind the upper bound (1) on B, we deduce that the number of n falling into this case is at most

$$\frac{x}{d^{1/4}}\exp\left(-\frac{\lambda}{2}\frac{\log x}{\log_2 x}\right)\cdot\exp\left(\lambda_0\frac{\log x}{\log_2 x}\right) = \frac{x}{d^{1/4}}\exp\left(\left(\lambda_0 - \frac{\lambda}{2}\right)\frac{\log x}{\log_2 x}\right).$$

Since $\lambda > 2\lambda_0$, this final quantity is $\ll x/d^{1/4}\log x$. This completes the proof of Theorem 3.3, and so also that of Theorem 1.3.

Remark 3.4. It is to be noted that one needs the condition $k \geq 4$ in order to have the sum

$$\sum_{x^{1/2\log_3 x} < p^k < x^2} \frac{x}{p^{k/4}\log x},$$

which arises from summing our upper bound in Theorem 3.3 over all $d := p^k > x^{1/2\log_3 x}$, be $o(x)$. Indeed, if $k \leq 3$, then this sum would be $\gg x^{7/6}/(\log x)^2$.

Remark 3.5. The conjecture of Erdős, Granville, Pomerance, and Spiro (quoted above as Conjecture 1.2) can be restated as saying that $s^{-1}(\mathcal{A})$ has density 0 whenever $\mathcal{A}$ has density 0. If this holds, then the conclusion of Proposition 3.1 follows for each $k \geq 2$: take

$$\mathcal{A} = \{n \text{ divisible by } p^k \text{ for some } p^k > \log_3(100n)\}.$$

Unfortunately, very little is known in the direction of the EGPS conjecture. The record result (still quite weak) seems to be that of [PPT18], where it is shown that $s^{-1}(\mathcal{A})$ has density 0 whenever $\mathcal{A}$ has counting function bounded by $x^{1/2+o(1)}$, as $x \to \infty$.

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References

[AE44] L. Alaoglu and P. Erdős, *A conjecture in elementary number theory*, Bull. Amer. Math. Soc. **50** (1944), 881–882.

[BL05] W.D. Banks and F. Luca, *Roughly squarefree values of the Euler and Carmichael functions*, Acta Arith. **120** (2005), 211–230.

[BP06] W.D. Banks and F. Pappalardi, *Values of the Euler function free of kth powers*, J. Number Theory **120** (2006), 326–348.

[EGPS90] P. Erdős, A. Granville, C. Pomerance, and C. Spiro, *On the normal behavior of the iterates of some arithmetic functions*, Analytic number theory (Allerton Park, IL, 1989), Progr. Math., vol. 85, Birkhäuser Boston, Boston, MA, 1990, pp. 165–204.

[ELP08] P. Erdős, F. Luca, and C. Pomerance, *On the proportion of numbers coprime to a given integer*, Anatomy of integers, CRM Proc. Lecture Notes, vol. 46, Amer. Math. Soc., Providence, RI, 2008, pp. 47–64.

[HW08] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, sixth ed., Oxford University Press, Oxford, 2008.

[LP15] F. Luca and C. Pomerance, *The range of the sum-of-proper-divisors function*, Acta Arith. **168** (2015), 187–199.

[Pap03] F. Pappalardi, *Square free values of the order function*, New York J. Math. **9** (2003), 331–344.

[Pap05] ———, *A survey on k-freeness*, Number theory, Ramanujan Math. Soc. Lect. Notes Ser., vol. 1, Ramanujan Math. Soc., Mysore, 2005, pp. 71–88.

[Pol14] P. Pollack, *Some arithmetic properties of the sum of proper divisors and the sum of prime divisors*, Illinois J. Math. **58** (2014), 125–147.

[Pom77] C. Pomerance, *On the distribution of amicable numbers*, J. Reine Angew. Math. **293(294)** (1977), 217–222.

[PPT18] P. Pollack, C. Pomerance, and L. Thompson, *Divisor-sum fibers*, Mathematika **64** (2018), 330–342.

[PSS03] F. Pappalardi, F. Saidak, and I.E. Shparlinski, *Square-free values of the Carmichael function*, J. Number Theory **103** (2003), 122–131.

[Ten15] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015.

[Wir59] E. Wirsing, *Bemerkung zu der Arbeit über vollkommene Zahlen*, Math. Ann. **137** (1959), 316–318.

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