

SUMS OF PROPER DIVISORS FOLLOW THE ERDŐS–KAC LAW

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ABSTRACT. Let $s(n) = \sum_{d|n, d < n} d$ denote the sum of the proper divisors of n . The second-named author proved that $\omega(s(n))$ has normal order $\log \log n$, the analogue for s -values of a classical result of Hardy and Ramanujan. We establish the corresponding Erdős–Kac theorem: $\omega(s(n))$ is asymptotically normally distributed with mean and variance $\log \log n$. The same method applies with $s(n)$ replaced by any of several other unconventional arithmetic functions, such as $\beta(n) := \sum_{p|n} p$, $n - \varphi(n)$, and $n + \tau(n)$ (τ being the divisor function).

1. INTRODUCTION

Let $s(n) = \sum_{d|n, d < n} d$ denote the sum of the proper divisors of the positive integer n , so that $s(n) = \sigma(n) - n$. Interest in the value distribution of $s(n)$ traces back to the ancient Greeks, but the modern study of $s(n)$ could be considered to begin with Davenport [Dav33], who showed that $s(n)/n$ has a continuous distribution function $D(u)$. Precisely: For each real number u , the set of n with $s(n) \leq un$ has an asymptotic density $D(u)$ which varies continuously with u . Moreover, $D(0) = 0$ and $\lim_{u \rightarrow \infty} D(u) = 1$.

While the values of $\sigma(n) = \prod_{p^e \| n} \frac{p^{e+1}-1}{p-1}$ are multiplicatively special, we expect shifting by $-n$ to rub out the peculiarities. That is, we expect the multiplicative statistics of $s(n)$ to resemble those of numbers of comparable size. By Davenport’s theorem, it is usually safe to interpret “of comparable size” to mean “of the same order of magnitude as n itself”.

Various results in the literature validate this expectation about $s(n)$. For example, the first author has shown [Pol14] that $s(n)$ is prime for $O(x/\log x)$ values of $n \leq x$ (and he conjectures that the true count is $\sim x/\log x$, in analogy with the prime number theorem). The second author [Tro20] has proved, in analogy with a classical result of Landau and Ramanujan, that there are $\asymp x/\sqrt{\log x}$ values of $n \leq x$ for which $s(n)$ is a sum of two squares. Writing $\omega(n)$ for the number of distinct prime factors of n , he also showed [Tro15] that $\omega(s(n))$ has normal order $\log \log n$. This is in harmony with the classical theorem of Hardy and Ramanujan [HR00] that $\omega(n)$ itself has normal order $\log \log n$.

In this note, we pick back up the study of $\omega(s(n))$. Strengthening the result of [Tro15], we prove that $\omega(s(n))$ satisfies the conclusion of the Erdős–Kac theorem [EK40].

Theorem 1. *Fix a real number u . As $x \rightarrow \infty$,*

$$\frac{1}{x} \#\{1 < n \leq x : \omega(s(n)) - \log \log x \leq u \sqrt{\log \log x}\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^u e^{-\frac{1}{2}t^2} dt.$$

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To prove Theorem 1, we adapt a simple and elegant proof of the Erdős–Kac theorem due to Billingsley ([Bil69], or [Bil95, pp. 395–397]). Making this go requires estimating, for squarefree d , the number of $n \leq x$ for which $d \mid s(n)$. A natural attack on this latter problem is to break off the largest prime factor P of n , say $n = mP$. Most of the time, P does not divide m , so that $\sigma(n) = \sigma(m)(P + 1)$. Then asking for d to divide $s(n)$ amounts to imposing the congruence $s(m)P \equiv -\sigma(m) \pmod{d}$. For a given m , the corresponding P are precisely those that lie in a certain interval I_m and a certain (possibly empty) set of arithmetic progressions. At this point we adopt (and adapt) a strategy of Banks, Harman, and Shparlinski [BHS05]. Rather than analytically estimate the number of these P , we relate the count of such P back to the total number of primes in the interval I_m , which we leave unestimated! This allows one to avoid certain losses of precision in comparison with [Tro15]. A similar strategy was used recently in [LLPSR] to show that $s(n)$, for composite $n \leq x$, is asymptotically uniformly distributed modulo primes $p \leq (\log x)^A$ (with $A > 0$ arbitrary but fixed).

Our proof of Theorem 1 is fairly robust. In the final section, we describe the modifications necessary to prove the corresponding result with $s(n)$ replaced by $\beta(n) := \sum_{p \mid n} p$, $n - \varphi(n)$, or $n + \tau(n)$, where $\tau(n)$ is the usual divisor-counting function.

For other recent work on the value distribution of $s(n)$, see [LP15, PP16, Pom18, PPT18].

Notation. Throughout, the letters p and P are reserved for primes. We write (a, b) for the greatest common divisor of a, b . We let $P^+(n)$ denote the largest prime factor of n , with the convention that $P^+(1) = 1$. We write \log_k for the k th iterate of the natural logarithm. We use \mathbb{E} for expectation and \mathbb{V} for variance.

2. OUTLINE

We let x be a large real number and we work on the probability space

$$\Omega := \{n \leq x : n \text{ composite}, P^+(n) > x^{1/\log_4 x}, \text{ and } P^+(n)^2 \nmid n\},$$

equipped with the uniform measure. Standard arguments (compare with the proof of Lemma 2.2 in [Tro15]) show that as $x \rightarrow \infty$,

$$\#\Omega = (1 + o(1))x.$$

We let $y = (\log x)^2$ and $z = x^{1/\log_3 x}$, and we define

$$\mathcal{P} = \{\text{primes } p \text{ with } y < p \leq z\}.$$

It turns out that counting prime factors of $s(n)$ from this “truncated” set of primes is sufficient (cf. equation (3) and the surrounding discussion). Our choice of z as a function of x for which $\log z / \log x$ decays to zero sufficiently slowly will be familiar to students of Billingsley’s proof. The need to introduce y is less apparent. Shortly we will need to bound the frequency with which d divides $s(n)$ for certain products d of primes from \mathcal{P} . Perhaps surprisingly, our method below only gives good control of this proportion when d has no small prime factors. There is some flexibility in what we choose to count as “small”; $y = (\log x)^2$ turns out to be a convenient threshold for bounding the error terms that appear.

For each prime $p \leq x^2$, we introduce the random variable X_p on Ω defined by

$$X_p(n) = \begin{cases} 1 & \text{if } p \mid s(n), \\ 0 & \text{otherwise.} \end{cases}$$

We let Y_p be Bernoulli random variables which take the value 1 with probability $1/p$. We define

$$X = \sum_{p \in \mathcal{P}} X_p \quad \text{and} \quad Y = \sum_{p \in \mathcal{P}} Y_p;$$

we think of Y as an idealized model of X .

Observe that

$$\begin{aligned} \mu &:= \mathbb{E}[Y] = \sum_{p \in \mathcal{P}} \frac{1}{p} = \log \log z - \log \log y + o(1) \\ (1) \quad &= \log \log x + o(\sqrt{\log \log x}) \end{aligned}$$

while

$$(2) \quad \sigma^2 := \mathbb{V}[Y] = \sum_{p \in \mathcal{P}} \frac{1}{p} \left(1 - \frac{1}{p}\right) \sim \log \log x.$$

We renormalize Y to have mean 0 and variance 1 by defining

$$\tilde{Y} = \frac{Y - \mu}{\sigma}.$$

Lemma 2. *\tilde{Y} converges in distribution to the standard normal \mathcal{N} , as $x \rightarrow \infty$. Moreover, $\mathbb{E}[\tilde{Y}^k] \rightarrow \mathbb{E}[\mathcal{N}^k]$ for each fixed positive integer k .*

Proof (sketch). Both claims follow from the proof in [Bil95, pp. 391–392] of the central limit theorem through the method of moments. One needs only that the recentered variables $Y'_p := Y_p - \frac{1}{p}$, for $p \in \mathcal{P}$, are independent mean 0 variables of finite variance, bounded by 1 in absolute value, with $\sum_{p \in \mathcal{P}} \mathbb{V}[Y'_p] \rightarrow \infty$ as $x \rightarrow \infty$. (Note that $\sum_{p \in \mathcal{P}} \mathbb{V}[Y'_p] = \sum_{p \in \mathcal{P}} \mathbb{V}[Y_p] = \mathbb{V}[Y] = \sigma^2$ in our above notation.) \square

Let $\tilde{X} = \frac{X - \mu}{\sigma}$. The next section is devoted to the proof of the following proposition.

Proposition 3. *For each fixed positive integer k ,*

$$\mathbb{E}[\tilde{X}^k] - \mathbb{E}[\tilde{Y}^k] \rightarrow 0.$$

Lemma 2 and Proposition 3 imply that $\mathbb{E}[\tilde{X}^k] \rightarrow \mathbb{E}[\mathcal{N}^k]$, for each k . So by the method of moments [Bil95, Theorem 30.2, p. 390], $\tilde{X} = \frac{X - \mu}{\sigma}$ converges in distribution to the standard normal.

This is most of the way towards Theorem 1. Since $\#\Omega = (1 + o(1))x$ and μ, σ satisfy the estimates (1), (2), Theorem 1 will follow if we show that $\frac{\omega(s(\cdot)) - \mu}{\sigma}$ (viewed as a random variable on Ω) converges in distribution to the standard normal. Observe that $s(n) \leq \sum_{d < n} d < n^2 \leq x^2$

for every $n \leq x$. So defining $X^{(s)} = \sum_{p \leq y} X_p$ and $X^{(\ell)} = \sum_{z < p \leq x^2} X_p$, we have $\omega(s(\cdot)) = X^{(s)} + X + X^{(\ell)}$ on Ω and

$$(3) \quad \frac{\omega(s(\cdot)) - \mu}{\sigma} = \tilde{X} + \frac{X^{(s)}}{\sigma} + \frac{X^{(\ell)}}{\sigma}.$$

Since \tilde{X} converges to the standard normal, to complete the proof of Theorem 1 it suffices to show that $\frac{X^{(s)}}{\sigma}$ and $\frac{X^{(\ell)}}{\sigma}$ converge to 0 in probability (see [Bil95, Theorem 25.4, p. 332]). Convergence to 0 in probability is obvious for $X^{(\ell)}/\sigma$: A positive integer not exceeding x^2 has at most $\frac{\log(x^2)}{\log z} = 2 \log_3 x$ prime divisors exceeding z , so that

$$|X^{(\ell)}/\sigma| \leq 2 \log_3 x / \sigma = o(1)$$

on the entire space Ω . Since $\sigma \sim \sqrt{\log \log x}$, that $X^{(s)}/\sigma$ tends to 0 in probability follows from the next lemma together with Markov's inequality.

Lemma 4. $\mathbb{E}[X^{(s)}] \ll \log_3 x \log_4 x$ for all large x .

Proof. Put $L = x^{1/\log_4 x}$, and for each $m \leq x$, let $L_m = \max\{x^{1/\log_4 x}, P^+(m)\}$. The n belonging to Ω are precisely the positive integers n that admit a decomposition $n = mP$, where $m > 1$ and $L_m < P \leq x/m$. Note that this decomposition of n is unique whenever it exists, since one can recover P from n as $P^+(n)$.

Let $n \in \Omega$ and write $n = mP$ as above. Then $s(mP) = \sigma(m)(P+1) - mP = Ps(m) + \sigma(m)$. Hence, for each $p \leq y$,

$$\sum_{n \in \Omega} X_p(n) = \sum_{\substack{n \in \Omega \\ p | s(n)}} 1 = \sum_{1 < m < x/L} \sum_{\substack{L_m < P \leq x/m \\ Ps(m) \equiv -\sigma(m) \pmod{p}}} 1.$$

If $p \nmid s(m)$, then the congruence $Ps(m) \equiv -\sigma(m) \pmod{p}$ puts P in a determined congruence class mod p (possibly 0 mod p). By Brun–Titchmarsh, the number of such $P \leq x/m$ is

$$\ll \frac{x}{mp \log(x/mp)} \ll \frac{x \log_4 x}{mp \log x}.$$

(We use here that $x/mp > L/p > L^{1/2}$ and $\log(L^{1/2}) \gg \log x / \log_4 x$.) On the other hand, if $p \mid s(m)$, then the congruence $Ps(m) \equiv -\sigma(m) \pmod{p}$ has integer solutions P only when $p \mid \sigma(m)$, in which case $p \mid \sigma(m) - s(m) = m$. In that scenario, every prime P satisfies $Ps(m) \equiv -\sigma(m) \pmod{p}$. Since there are $\ll \frac{x \log_4 x}{m \log x}$ primes $P \leq x/m$, we conclude that

$$\sum_{1 < m < x/L} \sum_{\substack{L_m < P \leq x/m \\ Ps(m) \equiv -\sigma(m) \pmod{p}}} 1 \ll \sum_{\substack{m \leq x \\ p | m}} \frac{x \log_4 x}{m \log x} + \sum_{m \leq x} \frac{x \log_4 x}{mp \log x} \ll \frac{x \log_4 x}{p}.$$

Keeping in mind that $|\Omega| \sim x$,

$$\mathbb{E}[X^{(s)}] \ll \frac{1}{x} \sum_{p \leq y} \frac{x \log_4 x}{p} \ll \log_4 x \log_2 y \ll \log_4 x \log_3 x. \quad \square$$

3. COMPLETION OF THE PROOF OF THEOREM 1: PROOF OF PROPOSITION 3

Throughout this section, k is a fixed positive integer. All estimates are to be understood as holding for x large enough, allowed to depend on k , and implied constants in Big-oh relations and \ll symbols may depend on k .

Recalling the definitions of \tilde{X} , \tilde{Y} and expanding,

$$\begin{aligned}\mathbb{E}[\tilde{X}^k] - \mathbb{E}[\tilde{Y}^k] &= \frac{1}{\sigma^k} \sum_{j=1}^k \binom{k}{j} (-\mu)^{k-j} (\mathbb{E}[X^j] - \mathbb{E}[Y^j]) \\ &\ll (\log_2 x)^{O(1)} \sum_{j=1}^k |\mathbb{E}[X^j] - \mathbb{E}[Y^j]|.\end{aligned}$$

For each $j = 1, 2, \dots, k$,

$$\mathbb{E}[X^j] - \mathbb{E}[Y^j] = \sum_{p_1, \dots, p_j \in \mathcal{P}} (\mathbb{E}[X_{p_1} \cdots X_{p_j}] - \mathbb{E}[Y_{p_1} \cdots Y_{p_j}]).$$

Writing d for the product of the distinct primes from the list p_1, \dots, p_j , we have $X_{p_1} \cdots X_{p_j} = \prod_{p|d} X_p$, $Y_{p_1} \cdots Y_{p_j} = \prod_{p|d} Y_p$, and

$$\mathbb{E}[X_{p_1} \cdots X_{p_j}] - \mathbb{E}[Y_{p_1} \cdots Y_{p_j}] = \frac{1}{|\Omega|} \sum_{\substack{n \in \Omega \\ d|s(n)}} 1 - \frac{1}{d}.$$

Observe that given d and j , there are only $O(1)$ possibilities for the original list p_1, \dots, p_j . Since there are $O(1)$ possibilities for j , we conclude that

$$\mathbb{E}[\tilde{X}^k] - \mathbb{E}[\tilde{Y}^k] \ll (\log_2 x)^{O(1)} \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \left| \frac{1}{|\Omega|} \sum_{\substack{n \in \Omega \\ d|s(n)}} 1 - \frac{1}{d} \right|.$$

We will show that

$$(4) \quad \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \left| \frac{1}{|\Omega|} \sum_{\substack{n \in \Omega \\ d|s(n)}} 1 - \frac{1}{d} \right| \ll \frac{(\log_2 x)^{O(1)}}{\log x}.$$

Hence, $\mathbb{E}[\tilde{X}^k] - \mathbb{E}[\tilde{Y}^k] \rightarrow 0$ as claimed.

Let d be a product of at most k distinct primes from \mathcal{P} . It will be useful in subsequent arguments to keep in mind that $d = x^{O(1/\log_3 x)}$, and so is of size $L^{o(1)}$. Decomposing each $n \in \Omega$ in the form mP , as in the proof of Lemma 4, we see that

$$(5) \quad \sum_{\substack{n \in \Omega \\ d|s(n)}} 1 = \sum_{1 < m < x/L} \sum_{\substack{L_m < P \leq x/m \\ Ps(m) + \sigma(m) \equiv 0 \pmod{d}}} 1,$$

where as before $L = x^{1/\log_4 x}$ and $L_m = \max\{x^{1/\log_4 x}, P^+(m)\}$. To analyze the right-hand double sum, we consider various cases for m .

Say that m is d -compatible if for every prime p dividing d , either p divides both $s(m)$ and $\sigma(m)$ or p divides neither. Then m is d -compatible precisely when the congruence $us(m) + \sigma(m) \equiv 0 \pmod{d}$ has a solution u coprime to d ; in this case, the primes P with $Ps(m) + \sigma(m) \equiv 0 \pmod{d}$ are precisely those belonging to a certain coprime residue class modulo $d/(d, s(m))$. We call m d -ideal if $\gcd(d, s(m)\sigma(m)) = 1$; equivalently, m is d -ideal if m is d -compatible and $\gcd(d, s(m)) = 1$. Note that only d -compatible values of m contribute to the right side of (5).

When m is d -ideal,

$$\sum_{\substack{L_m < P \leq x/m \\ Ps(m) + \sigma(m) \equiv 0 \pmod{d}}} 1 = \frac{1}{\varphi(d)} \sum_{L_m < P \leq x/m} 1 + O(E(x/m; d)),$$

where

$$E(T; q) := \max_{2 \leq t \leq T} \max_{\gcd(a, q)=1} \left| \pi(t; q, a) - \frac{\pi(t)}{\varphi(q)} \right|.$$

So the contribution to the right-hand side of (5) from d -ideal m is

$$\begin{aligned} (6) \quad & \frac{1}{\varphi(d)} \sum_{1 < m < x/L} \sum_{L_m < P \leq x/m} 1 - \frac{1}{\varphi(d)} \sum_{\substack{1 < m < x/L \\ \text{not } d\text{-ideal}}} \sum_{L_m < P \leq x/m} 1 + O\left(\sum_{m < x/L} E(x/m; d)\right) \\ &= \frac{|\Omega|}{\varphi(d)} - \frac{1}{\varphi(d)} \sum_{\substack{1 < m < x/L \\ \text{not } d\text{-ideal}}} \sum_{L_m < P \leq x/m} 1 + O\left(\sum_{m < x/L} E(x/m; d)\right). \end{aligned}$$

Since d is a product of $O(1)$ primes all of which exceed y , the first main term here admits the estimate

$$(7) \quad \frac{|\Omega|}{\varphi(d)} = \frac{|\Omega|}{d} (1 + O(1/y)) = \frac{|\Omega|}{d} + O(x/dy).$$

We bound the second main term, involving the double sum on m, P , from above. The inner sum is no more than $\pi(x/m) \ll \frac{x}{m \log(x/m)} \ll \frac{x \log_4 x}{m \log x}$, so that

$$(8) \quad \frac{1}{\varphi(d)} \sum_{\substack{1 < m < x/L \\ \text{not } d\text{-ideal}}} \sum_{L_m < P \leq x/m} 1 \ll \frac{x \log_4 x}{\log x} \sum_{\substack{1 < m < x/L \\ \text{not } d\text{-ideal}}} \frac{1}{md}.$$

Next, we investigate the contribution to the right-hand side of (5) from m that are d -compatible but not d -ideal. For these m , the corresponding primes P are restricted to a progression mod $d/(d, s(m))$, and so by the Brun–Titchmarsh inequality these m contribute

$$(9) \quad \ll \sum_{\substack{1 < m < x/L \\ d\text{-compat} \\ \text{not } d\text{-ideal}}} \frac{x}{m \cdot \varphi(d/(d, s(m))) \log(x(d, s(m))/md)} \ll \frac{x \log_4 x}{\log x} \sum_{\substack{1 < m < x/L \\ d\text{-compat} \\ \text{not } d\text{-ideal}}} \frac{(d, s(m))}{md}.$$

We derive from (6), (7), (8), and (9) that

$$\left| \frac{1}{|\Omega|} \sum_{\substack{n \in \Omega \\ d|s(n)}} 1 - \frac{1}{d} \right| \ll \frac{1}{x} \sum_{m < x/L} |E(x/m; d)| + \frac{1}{dy} + \frac{\log_4 x}{\log x} \sum_{\substack{1 < m < x/L \\ \text{not } d\text{-ideal}}} \frac{1}{md} + \frac{\log_4 x}{\log x} \sum_{\substack{1 < m < x/L \\ d\text{-compat} \\ \text{not } d\text{-ideal}}} \frac{(d, s(m))}{md}.$$

Now we sum on d .

First off, the Bombieri–Vinogradov theorem implies that

$$\begin{aligned} \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \left(\frac{1}{x} \sum_{m < x/L} |E(x/m; d)| \right) &\leq \frac{1}{x} \sum_{m < x/L} \sum_{d \leq (x/m)^{1/3}} |E(x/m; d)| \\ &\ll \frac{1}{x} \sum_{m < x/L} \frac{x/m}{(\log(x/m))^2} \ll \frac{(\log_4 x)^2}{(\log x)^2} \sum_{m < x/L} \frac{1}{m} \ll \frac{(\log_4 x)^2}{\log x}. \end{aligned}$$

Next,

$$\sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \frac{1}{dy} \leq \frac{1}{y} \sum_{j=0}^k \frac{1}{j!} \left(\sum_{p \in \mathcal{P}} \frac{1}{p} \right)^j \ll \frac{(\log_2 x)^k}{(\log x)^2}.$$

Continuing, note that if m is not d -ideal, then there is a prime $p \mid d$ with $p \mid s(m)\sigma(m)$. Hence,

$$\begin{aligned} \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \left(\frac{\log_4 x}{\log x} \sum_{\substack{1 < m < x/L \\ \text{not } d\text{-ideal}}} \frac{1}{md} \right) &\leq \frac{\log_4 x}{\log x} \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \frac{1}{d} \sum_{p|d} \sum_{\substack{1 < m < x/L \\ p|s(m)\sigma(m)}} \frac{1}{m} \\ &\leq \frac{\log_4 x}{\log x} \sum_{1 < m < x/L} \frac{1}{m} \sum_{\substack{p|s(m)\sigma(m) \\ p \in \mathcal{P}}} \sum_{\substack{d \leq x \text{ squarefree} \\ p|d \\ \omega(d) \leq k}} \frac{1}{d} \ll \frac{(\log_2 x)^{O(1)}}{\log x} \sum_{1 < m < x/L} \frac{1}{m} \sum_{\substack{p|s(m)\sigma(m) \\ p \in \mathcal{P}}} \frac{1}{p}. \end{aligned}$$

Since each $p \in \mathcal{P}$ exceeds y , the final sum on p is $\ll \omega(s(m)\sigma(m))/y \ll \log x/y = 1/\log x$, and so the last displayed expression is

$$\ll \frac{(\log_2 x)^{O(1)}}{(\log x)^2} \sum_{1 < m < x/L} \frac{1}{m} \ll \frac{(\log_2 x)^{O(1)}}{\log x}.$$

Finally, suppose m is d -compatible but not d -ideal. Then $(d, s(m)) > 1$, $(d, s(m)) \mid \sigma(m)$, and $(d, s(m)) \mid \sigma(m) - s(m) = m$. Hence, thinking of d' as $(d, s(m))$,

$$(10) \quad \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \left(\frac{\log_4 x}{\log x} \sum_{\substack{1 < m < x/L \\ d\text{-compat} \\ \text{not } d\text{-ideal}}} \frac{(d, s(m))}{md} \right) \leq \frac{\log_4 x}{\log x} \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \frac{1}{d} \sum_{\substack{d'|d \\ d' > 1}} d' \sum_{\substack{1 < m < x/L \\ d'|m, d'|\sigma(m)}} \frac{1}{m}.$$

Let us estimate the inner sum on m . Write $m = d'm'$. The contribution to that sum from m with $(d', m') > 1$ is at most

$$\sum_{p|d'} \frac{1}{d'} \sum_{\substack{m' < x \\ p|m'}} \frac{1}{m'} \ll \frac{1}{d'} \log x \sum_{p|d'} \frac{1}{p} \ll \frac{\log x}{d'y} \omega(d') \ll \frac{1}{d' \log x}.$$

Suppose now that $\gcd(d', m') = 1$. If $d' \mid \sigma(m)$, then $P^+(d') \mid \sigma(d')\sigma(m')$, while $P^+(d') > P^+(\sigma(d'))$ (since d' is a squarefree product of odd primes and $d' > 1$). Thus, $P^+(d') \mid \sigma(m')$. Choose a prime power $q^e \parallel m'$ with $P^+(d') \mid \sigma(q^e)$. If $e \geq 2$, then $y < P^+(d') \leq \sigma(q^e) < 2q^e$, and so m' has squarefull part exceeding $y/2$. If $e = 1$, then $q \parallel m'$ with $q \equiv -1 \pmod{P^+(d')}$. Hence, these m make a contribution to the inner sum bounded by

$$\begin{aligned} \frac{1}{d'} \left(\sum_{\substack{r > y/2 \\ \text{squarefull}}} \sum_{\substack{m' < x \\ r|m'}} \frac{1}{m'} + \sum_{p|d'} \sum_{\substack{q < x \\ q \equiv -1 \pmod{p}}} \sum_{\substack{m' < x \\ q|m'}} \frac{1}{m'} \right) &\ll \frac{\log x}{d'} \left(\sum_{\substack{r > y/2 \\ \text{squarefull}}} \frac{1}{r} + \sum_{p|d'} \sum_{\substack{q < x \\ q \equiv -1 \pmod{p}}} \frac{1}{q} \right) \\ &\ll \frac{\log x}{d'} \left(\frac{1}{\log x} + \sum_{p|d'} \frac{\log x}{p} \right) \ll \frac{1}{d'} + \frac{(\log x)^2}{d'} \sum_{p|d'} \frac{1}{p} \ll \frac{1}{d'} + \frac{(\log x)^2}{d'} \frac{1}{y} \ll \frac{1}{d'}. \end{aligned}$$

Inserting these estimates back above, the right-hand side of (10) is seen to be

$$\ll \frac{\log_4 x}{\log x} \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \frac{1}{d} \sum_{\substack{d'|d \\ d' > 1}} 1 \ll \frac{\log_4 x}{\log x} \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \frac{1}{d} \ll \frac{(\log_2 x)^{O(1)}}{\log x}.$$

Assembling the last several estimates yields (4), which completes the proof of Theorem 1.

Remark. As with most variants of Erdős–Kac, Theorem 1 remains valid if we count prime factors with multiplicity. Define $\omega'(n) = \sum_{p^k \parallel n} k$. (We avoid the more familiar notation $\Omega(n)$, since Ω denotes our sample space.) It is shown in [Tro15] that, for a certain subset Ω' of $(1, x]$ containing $(1 + o(1))x$ elements,

$$\frac{1}{x} \sum_{n \in \Omega'(x)} (\omega'(s(n)) - \omega(s(n))) \ll (\log_3 x)^2.$$

(See p. 133 of [Tro15].) It follows that away from a set of $o(x)$ elements of $(1, x]$, we have $\omega'(s(n)) - \omega(s(n)) < (\log \log x)^{0.49}$ (say). Hence, the Erdős–Kac theorem for $\omega'(s(n))$ is a consequence of the corresponding theorem for $\omega(s(n))$.

4. OTHER ARITHMETIC FUNCTIONS

The astute reader will observe that many of the calculations above do not depend on properties specific to $s(n)$. In this section, we discuss how to adapt the previous argument for other arithmetic functions.

Let f be an integer-valued arithmetic function with $f(n)$ nonzero for $n > 1$ and $|f(n)| \leq x^{O(1)}$ for all $n \leq x$. Assume that for all positive integers m and all primes P not dividing m , there are integers $a(m)$ and $b(m)$ such that $f(mP) = Pa(m) + b(m)$, with $a(m), b(m)$ nonzero for $m > 1$. Finally, assume that $|a(m)|, |b(m)| \leq x^{O(1)}$ for all $1 < m \leq x$. (For $f(n) = s(n)$, we

have $0 < s(n) \leq x^2$ when $1 < n \leq x$, and $s(mP) = Ps(m) + \sigma(m)$ for any positive integer m and any prime $P \nmid m$.) All symbols are defined as in Section 2, except that the random variable X_p is now equal to 1 if $p \mid f(n)$ and is 0 otherwise.

To obtain an Erdős–Kac-type result for $\omega(f(n))$, we follow the same general strategy as in the case $f(n) = s(n)$. By the method of moments, Lemma 2 and the analogue of Proposition 3 (once shown) will establish that $\tilde{X} = \frac{X - \mu}{\sigma}$ converges in distribution to the standard normal. Recall that $y = (\log x)^2$ and $z = x^{1/\log_3 x}$; then

$$\frac{\omega(f(\cdot)) - \mu}{\sigma} = \tilde{X} + \frac{X^{(s)}}{\sigma} + \frac{X^{(l)}}{\sigma},$$

where $X^{(s)} = \sum_{p \leq y} X_p$ and $X^{(l)} = \sum_{z < p \leq x^c} X_p$, where $c > 0$ is a constant such that $|f(n)| \leq x^c$ for all $n \leq x$.

As before, our task is to show that $\frac{X^{(s)}}{\sigma}$ and $\frac{X^{(l)}}{\sigma}$ converge to 0 in probability. The argument for $X^{(l)}$ is the same, with the exponent 2 replaced by c . For $X^{(s)}$, we again hope to use Markov's inequality coupled with an upper bound for $\mathbb{E}[X^{(s)}]$ of size $o(\sqrt{\log_2 x})$, analogous to Lemma 4. The argument there yields, in this case,

$$\mathbb{E}[X^{(s)}] \ll \log_3 x \log_4 x + \frac{\log_4 x}{\log x} \sum_{p \leq y} \sum_{\substack{m \leq x \\ p|a(m) \text{ and } p|b(m)}} \frac{1}{m}.$$

Thus, the aim is to show that

$$\sum_{p \leq y} \sum_{\substack{m \leq x \\ p|a(m) \text{ and } p|b(m)}} \frac{1}{m} = o\left(\frac{\sqrt{\log_2 x}}{\log_4 x} \log x\right).$$

We now turn our attention to the analogue of Proposition 3. Say that m is d -compatible if for every $p \mid d$, either p divides both $a(m)$ and $b(m)$ or p divides neither; and m is d -ideal if $\gcd(d, a(m)b(m)) = 1$. Equivalently, m is d -ideal if m is d -compatible and $\gcd(d, a(m)) = 1$. Tracing through the argument in Section 3, we see that few of the calculations depend on specific properties of $f(n)$; in fact, the analogue of Proposition 3 is established if

$$\sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \sum_{\substack{1 < m < x \\ d\text{-compat} \\ \text{not } d\text{-ideal}}} \frac{(d, a(m))}{md} \ll (\log_2 x)^{O(1)}.$$

We summarize the above discussion in the following proposition.

Proposition 5. *Suppose $f(n)$ is an integer-valued arithmetic function with $f(n)$ nonzero when $n > 1$ and $|f(n)| \leq x^{O(1)}$ for all $n \leq x$. Suppose also that for every positive integer m , there are $a(m)$ and $b(m)$ such that*

$$f(mP) = Pa(m) + b(m) \text{ for all primes } P \text{ not dividing } m,$$

and that

$$|a(m)|, |b(m)| \leq x^{O(1)} \text{ whenever } m \leq x.$$

Suppose also that $a(m), b(m)$ are nonzero whenever $m > 1$. Then, if

$$(11) \quad \sum_{p \leq y} \sum_{\substack{m \leq x \\ p|a(m) \text{ and } p|b(m)}} \frac{1}{m} = o\left(\frac{\sqrt{\log_2 x}}{\log_4 x} \log x\right)$$

and

$$(12) \quad \sum_{\substack{d \text{ squarefree} \\ p|d \Rightarrow p \in \mathcal{P} \\ \omega(d) \leq k}} \sum_{\substack{1 < m < x \\ d\text{-compat} \\ \text{not } d\text{-ideal}}} \frac{(d, a(m))}{md} \ll (\log_2 x)^{O(1)},$$

Theorem 1 is true with $f(n)$ in place of $s(n)$.

4.1. The sum of prime divisors. For each positive integer n , let $\beta(n) := \sum_{p|n} p$ denote the sum of the prime divisors of n . If $1 < n \leq x$, then $0 < \beta(n) \leq n \leq x$. If P is a prime not dividing the integer m , then

$$\beta(mP) = P + \beta(m).$$

We apply Proposition 5, with $f(n) = \beta(n)$, $a(m) = 1$, and $b(m) = \beta(m)$. Since $a(m) = 1$, one quickly observes that the sums on the left-hand sides of (11) and (12) are empty. Thus, Theorem 1 holds with $\beta(n)$ in place of $s(n)$. The same argument applies, verbatim, with $\beta(n)$ replaced by $A(n) = \sum_{p^k || n} kp$, where prime factors are summed with multiplicity. For other work on the value distribution of $\beta(n)$ and $A(n)$, see [Hal70, Hal71, Hal72, AE77, Pol14, Gol17].

4.2. A shifted divisor function. Let $f(n) = n + \tau(n)$, where $\tau(n)$ denotes the number of divisors of n . Then if $n \leq x$, $f(n) < x^{O(1)}$ trivially. If P is a prime not dividing the positive integer m , then

$$f(mP) = mP + \tau(mP) = Pm + 2\tau(m),$$

so $a(m) = m$ and $b(m) = 2\tau(m)$ in this case. For $m \leq x$, the largest exponent appearing in the prime factorization of m , and hence the largest prime divisor of $\tau(m)$, is $\ll \log x$. This means there is no value of m that is d -compatible but not d -ideal, since every prime $p | d$ satisfies $p > (\log x)^2$. Equation (12) is therefore satisfied, since the sum is empty.

Equation (11) is handled nearly as easily. Ignoring the condition $p | b(m)$ in the inner sum, the left-hand side of (11) is at most

$$(13) \quad \sum_{p \leq y} \sum_{\substack{m \leq x \\ p|m}} \frac{1}{m} \ll \log x \sum_{p \leq y} \frac{1}{p} \ll \log x \log_3 x = o\left(\frac{\sqrt{\log_2 x}}{\log_4 x} \log x\right),$$

as desired. Thus, by Proposition 5, Theorem 1 holds with $f(n) = n + \tau(n)$ in place of $s(n)$. Similar arguments apply to $n - \tau(n)$ and $n \pm \omega(n)$. The functions $n - \tau(n)$ and $n - \omega(n)$ appear in work of Luca [Luc05]; for each of these two functions, he shows that the range is missing infinitely many positive integers.

4.3. The cototient function. Let $f(n) = n - \varphi(n)$, where (as above) $\varphi(n)$ is Euler's totient function. (See [Erd73, BS95, FL00, GM05, PY14, PP16] for studies of the range of $n - \varphi(n)$.) Note that $0 < f(n) < n$ for $n > 1$ and, if P is a prime not dividing m ,

$$f(mP) = Pm - \varphi(Pm) = Pm - (P - 1)\varphi(m) = P(m - \varphi(m)) + \varphi(m).$$

We apply Proposition 5 with $f(n) = n - \varphi(n)$, $a(m) = m - \varphi(m)$, and $b(m) = \varphi(m)$. We first observe that equation (11) can be established as in (13), noting that if $p \mid a(m)$ and $p \mid b(m)$, then $p \mid a(m) + b(m) = m$. To show (12), use the argument surrounding (10), replacing $s(m)$ by $a(m) = m - \varphi(m)$ and $\sigma(m)$ by $b(m) = \varphi(m)$. The argument carries through with only the slightest of modifications. By Proposition 5, Theorem 1 holds with $f(n) = n - \varphi(n)$ in place of $s(n)$.

Several other applications of our method could be given, although some require slight changes to the framework. For example, fix an integer $a \neq 0$ and consider the shifted totient function $f(n) = \varphi(n) + a$. It is not hard to prove that the hypotheses of Proposition 5 are satisfied with $a(m) = \varphi(m)$ and $b(m) = -\varphi(m) + a$, with one exception: If $a > 0$ is in the range of φ , then $b(m)$ will vanish at some $m > 1$. However, it is still true that $b(m)$ is nonvanishing for all $m > m_0(a)$, and one can simply run our argument with the condition $n/P^+(n) > m_0(a)$ added to the definition of Ω .

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