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Abstract

In this paper we study the $\mathbb{C}^*$-fixed points in moduli spaces of Higgs bundles over a compact Riemann surface for a complex semisimple Lie group and its real forms. These fixed points are called Hodge bundles and correspond to complex variations of Hodge structure. We introduce a topological invariant for Hodge bundles that generalizes the Toledo invariant appearing for Hermitian Lie groups. An important result of this paper is a bound on this invariant which generalizes the Milnor–Wood inequality for a Hodge bundle in the Hermitian case, and is analogous to the Arakelov inequalities of classical variations of Hodge structure. When the generalized Toledo invariant is maximal, we establish rigidity results for the associated variations of Hodge structure which generalize known rigidity results for maximal Higgs bundles and their associated maximal representations in the Hermitian case.

1. Introduction

Since their introduction in Hitchin's seminal paper [Hit87], Higgs bundles over a compact Riemann surface have been of tremendous interest in geometry, topology and theoretical physics. Within the moduli space of Higgs bundles there is a special subvariety determined by the fixed points of a natural $\mathbb{C}^*$-action. These fixed points are called Hodge bundles and correspond to holonomies of complex variations of Hodge structure. They are part of the global nilpotent cone, and coincide with critical points of a natural energy function on the moduli space. The importance of the $\mathbb{C}^*$-fixed points also stems from the fact that, roughly speaking, the subvariety of Hodge bundles determines the topology of the moduli space of Higgs bundles (see [Hit87, GHS14, GH13, Got94]). In this paper we investigate some basic properties of Hodge bundles and their moduli.

To describe our results, let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and let X be a compact Riemann surface with genus $g \geqslant 2$ and canonical bundle K. A G-Higgs bundle on X is a pair (E, φ), where E is a holomorphic principal G-bundle on X and $\varphi \in H^0(E(\mathfrak{g}) \otimes K)$

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is a holomorphic section of the Lie algebra bundle twisted by K. The moduli space of polystable G-Higgs bundles will be denoted by $\mathcal{M}(\mathrm{G})$. By the nonabelian Hodge correspondence, the moduli space $\mathcal{M}(\mathrm{G})$ is homeomorphic to the character variety $\mathcal{R}(\mathrm{G})$ of conjugacy classes of reductive representations $\rho : \pi_1(X) \to \mathrm{G}$.

The $\mathbb{C}^*$-action on the Higgs bundle moduli space is defined by $\lambda \cdot (E, \varphi) = (E, \lambda\varphi)$. To describe the $\mathbb{C}^*$-fixed points, fix a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$. Let $\zeta \in \mathfrak{g}_0$ be the grading element, that is, $[\zeta, x] = jx$ for all $x \in \mathfrak{g}_j$, and let $\mathrm{G}_0 < \mathrm{G}$ be the centralizer of ζ. A Higgs bundle (E, φ) is said to be a Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_k)$ if E reduces to a holomorphic G_0-bundle E_{G_0}, and $\varphi \in H^0(E_{\mathrm{G}_0}(\mathfrak{g}_k) \otimes K)$. A polystable Higgs bundle is a fixed point if and only if it is a Hodge bundle [Sim92]. In fact it suffices to consider Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ (see § 4.2 for more details).

The representations $\rho : \pi_1(X) \to \mathrm{G}$ associated to Hodge bundles factor through a real form $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}} \to \mathrm{G}$ of Hodge type, that is, a real form such that the rank of its maximal compact subgroup is equal to the rank of G_0. If $\mathrm{H}_0^{\mathbb{R}} < \mathrm{G}_0$ is a maximal compact subgroup, then the homogeneous space $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ has a natural complex structure and is called a period domain. The holomorphic tangent bundle of $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ decomposes as $\bigoplus_{j\geqslant 0} \mathrm{G}^{\mathbb{R}} \times_{\mathrm{H}_0^{\mathbb{R}}} \mathfrak{g}_j$. In [Sim92], Simpson showed that the representations $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}} < \mathrm{G}$ arising from Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ define ρ-equivariant holomorphic maps

$$f_\rho : \widetilde{X} \to \mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$$

such that ∂f_ρ is valued in the first graded piece $\mathrm{G}^{\mathbb{R}} \times_{\mathrm{H}_0^{\mathbb{R}}} \mathfrak{g}_1$ of the holomorphic tangent bundle. Such pairs (ρ, f_ρ) are called variations of Hodge structure.

When the grading in the above discussion is $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the real form $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ is a group of Hermitian type and the period domain $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ is the Riemannian symmetric space of $\mathrm{G}^{\mathbb{R}}$. In this case, the symmetric space is Kähler and Hodge bundles define representations $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$ and ρ-equivariant holomorphic maps to the symmetric space of $\mathrm{G}^{\mathbb{R}}$. Pulling back the Kähler form and integrating it over X defines an invariant which is usually called the Toledo invariant $\tau(\rho)$. The Toledo invariant is defined for all representations into Hermitian Lie groups and satisfies the Milnor–Wood inequality

$$|\tau(\rho)| \leqslant (2g-2)\,\mathrm{rk}(\mathrm{G}^{\mathbb{R}}/\mathrm{H}^{\mathbb{R}}).$$

This inequality was first proven by Milnor [Mil58] for $\mathrm{PSL}_2\mathbb{R}$, and more generally in [DT87, BIW03]. Using Higgs bundles, one can also define the Toledo invariant and obtain the Milnor–Wood inequality [Hit87, BGG03, BGR17].

The set of representations which maximize the invariant are called maximal representations and have many interesting geometric features. For example, maximal representations define connected components of the character variety which consist entirely of discrete and faithful representations [BIW10]. For $\mathrm{PSL}_2\mathbb{R}$, maximal representations correspond to Fuchsian representations and the representation uniformizing the Riemann surface X defines the unique maximal Hodge bundle. There are two classes of Hermitian groups, those of tube type and those of nontube type. Generalizing the complex hyperbolic geometry results of [Tol89], maximal representations into nontube-type groups always factor through a maximal subtube up to a compact factor [BGG03, BIW03, BGR17]. For example $\mathrm{SU}(p, p)$ is the maximal subtube of $\mathrm{SU}(p, q)$ when $q \geqslant p$.

Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ form their own moduli space $\mathcal{M}(\mathrm{G}_0, \mathfrak{g}_1)$. Consider a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with grading element ζ. Define a character of the Lie algebra $\mathfrak{g}_0$,

$$\chi_T : \mathfrak{g}_0 \to \mathbb{C}, \quad \chi_T(x) = B(x, \zeta)B(\gamma, \gamma),$$

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where B is the Killing form of $\mathfrak{g}$ and γ is the longest root such that the root space $\mathfrak{g}_\gamma \subset \mathfrak{g}_1$. As explained in § 3.3, the normalization constant $B(\gamma,\gamma)$ normalizes the minimum of the holomorphic sectional curvature of the period domain to be -1. Recall that a Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_1)$ is a pair (E,φ), where E is a holomorphic G_0-bundle and $\varphi \in H^0(E(\mathfrak{g}_1)\otimes K)$. For some positive integer q, the multiple $q\cdot\chi_T$ exponentiates to a character $\chi : \mathrm{G}_0 \to \mathbb{C}^*$ of the Lie group G_0 and defines a line bundle $E(\chi)$. We define the Toledo invariant of a Hodge bundle (E,φ) by

$$\tau(E,\varphi) = \frac{1}{q}\deg E(\chi).$$

For gradings $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, the Lie algebra character χ_T and the invariant τ agree with the definitions of the Toledo character and invariant for Hermitian groups in [BGR17].

To generalize the notion of rank, we use the fact that the space $\mathfrak{g}_1$ is a prehomogeneous vector space (PHVS) for the action of G_0. That is, $\mathfrak{g}_1$ has a unique open dense G_0-orbit $\Omega \subset \mathfrak{g}_1$. The theory of PHVSs was introduced by Sato (see [SK77, Kim03, Mor91, Man13]) and provides an ideal set of tools to study our problem. Let $e \in \Omega \subset \mathfrak{g}_1$ be any point in the open orbit and complete it to an $\mathfrak{sl}_2$-triple $\{f,h,e\}$ with $h\in\mathfrak{g}_0$. We define the rank of $(\mathrm{G}_0,\mathfrak{g}_1)$ by

$$\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1) = \tfrac{1}{2}\chi_T(h).$$

In § 3 we show that this is independent of the choices made. Again, for Hermitian groups this definition recovers the rank of the symmetric space.

Let us state our first main result.

Theorem A (Corollary 5.5). *A polystable Hodge bundle (E,φ) of type $(\mathrm{G}_0,\mathfrak{g}_1)$ satisfies the inequality*

$$|\tau(E,\varphi)| \leqslant (2g-2)\,\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1). \tag{1.1}$$

Remark 1.1. In terms of variations of Hodge structure, the Toledo invariant is the degree of the pullback of a natural line bundle on the period domain $\mathrm{G}^{\mathbb{R}}/(\mathrm{G}_0\cap\mathrm{H}^{\mathbb{R}})$. Inequality (1.1) is simply saying that this is bounded by the Euler characteristic times a number (the rank) which is determined by the Lie theory of the period domain. In Proposition 3.12, we prove that the rank is the negative inverse of the maximal value of the holomorphic sectional curvature (in the $+1$ space) of the period domain, which coincides with the usual rank in the case of Hermitian symmetric spaces.

Remark 1.2. We refer to inequality (1.1) as the Arakelov–Milnor inequality as it generalizes the Milnor–Wood inequality when the Higgs bundle defines a variation of Hodge structure in a Hermitian symmetric space, and is analogous to the Arakelov inequalities for variations of Hodge structure. These are generalizations of the classical Arakelov inequality for the degree of the relative canonical bundle of a family of curves over another curve (see, for example, [Pet90, JZ02, Vie09, VZ03, MVZ12]). As in the Milnor–Wood case, maximality of the bound in the Arakelov inequalities is accompanied by some rigidity phenomena. As already mentioned, in the Hermitian case for the grading $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, the character χ_T and the invariant τ agree with the definition of the Toledo character and the Toledo invariant for Hermitian groups in [BGR17], and the rank of $(\mathrm{G}_0,\mathfrak{g}_1)$ coincides, as mentioned above, with the rank of the symmetric space.

We refer to Hodge bundles with $|\tau(E,\varphi)| = \mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)(2g-2)$ as maximal Hodge bundles of type $(\mathrm{G}_0,\mathfrak{g}_1)$. As in the Hermitian case of [BGR17], the Arakelov–Milnor inequality follows from a more general and more precise inequality established in Theorem 5.3. Indeed, the stability of a Hodge bundle of type $(\mathrm{G}_0,\mathfrak{g}_1)$ depends on a parameter $\alpha=\lambda\zeta$ with $\lambda\in\mathbb{R}$. In Theorem 5.3 we give an inequality for an α-semistable Hodge bundle (E,φ). Moreover, we give a refined

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inequality which depends on the orbit in $\mathfrak{g}_1$ which contains the generic value of φ. While we are mostly interested in the case $\alpha = 0$, since this relates to the stability of the Higgs bundle obtained from (E, φ) by extension of structure group, considering the arbitrary value of the α case has proven to be a powerful tool to study the moduli space for $\alpha = 0$ (see [BGG03, GHS14]). We believe that the same principle will apply in this general situation.

Example 1.3. The inequality is usually easy to write down in concrete cases. For example, for $\mathrm{G} = \mathrm{SO}_{2p+q}\mathbb{C}$ and $\mathrm{G}_0 = \mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}$, one finds that $\mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1) = 2\min(p, q)$ if $q > 1$ and 1 if $q = 1$. Let us take $q > 1$. A bundle (E, φ) of type $(\mathrm{G}_0, \mathfrak{g}_1)$ can be written as $E = V \oplus W \oplus V^*$, where V is a $\mathrm{GL}_p\mathbb{C}$-bundle and W is a $\mathrm{SO}_q\mathbb{C}$-bundle, and the Higgs field has the form

$$\varphi = \begin{pmatrix} \begin{pmatrix} 0 & \theta & 0 \\ 0 & 0 & -\theta^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

where $\theta : W \to V \otimes K$. One obtains $\tau(E, \varphi) = 2 \deg V$, and it follows that inequality (1.1) takes the form (actually $\deg V \leqslant 0$)

$$\deg V \geqslant -\min(p, q)(2g - 2).$$

As mentioned above, we actually prove a more precise inequality, which depends on the orbit which contains the generic value of φ. In our case, if the image of θ^{T} in W is the sum of a nondegenerate subspace of dimension r_1 and of a totally isotropic subspace of dimension r_2, then the inequality can be refined as

$$\deg V \geqslant -\left(r_1 + \frac{r_2}{2}\right)(2g - 2). \tag{1.2}$$

These values are calculated in Example 3.7.

Fundamental in the proof of the Arakelov–Milnor inequality (1.1) is the construction of a maximal Jacobson–Morozov (JM) regular prehomogeneous vector subspace (PHVSS) of $(\mathrm{G}_0, \mathfrak{g}_1)$ and the existence of a polynomial relative invariant for this subspace. This is the analogue of a maximal subtube in the Hermitian case. To explain this, let $\{f, h, e\} \subset \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple such that $[h, e] = 2e$ and $[e, f] = h$. The weights of ad_h are integral. When the weights of ad_h are all even, $\{f, h, e\}$ is called an even $\mathfrak{sl}_2$-triple and defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with grading element $h/2$ and $e \in \mathfrak{g}_1$. In fact $e \in \Omega \subset \mathfrak{g}_1$ is in the open G_0-orbit (see § 2.2). A PHVS $(\mathrm{G}_0, \mathfrak{g}_1)$ is called regular if the G_0-stabilizer of a point in the open orbit Ω is reductive. For gradings arising from even $\mathfrak{sl}_2$-triples $\{f, h, e\}$, $(\mathrm{G}_0, \mathfrak{g}_1)$ is regular since the G_0-centralizer of $e \in \Omega$ coincides with the G-centralizer of the $\mathfrak{sl}_2\mathbb{C}$-subalgebra. In this case, we refer to $(\mathrm{G}_0, \mathfrak{g}_1)$ as a JM-regular PHVS.

Fix a basis $\{f, h, e\}$ of $\mathfrak{sl}_2\mathbb{C}$ and let $\mathrm{T} < \mathrm{PSL}_2\mathbb{C}$ be the connected subgroup with Lie algebra $\langle h \rangle$. The Hodge bundle associated to the representation uniformizing the Riemann surface X is given by (E_{T}, e), where E_{T} is the holomorphic frame bundle of K^{-1} and $e \in H^0(E_{\mathrm{T}}(\langle e \rangle) \otimes K)$. For $\mathrm{SL}_2\mathbb{C}$, we take E_{T} to be the frame bundle of a square root of K^{-1}. Suppose $\{f, h, e\} \subset \mathfrak{g}$ is an even $\mathfrak{sl}_2$-triple and consider the associated JM-regular PHVS $(\mathrm{G}_0, \mathfrak{g}_1)$. From our setup, it follows that extending structure group $(E_{\mathrm{T}}(\mathrm{G}_0), e)$ defines a maximal Hodge bundle,

$$\tau(E_{\mathrm{T}}(\mathrm{G}_0), e) = \mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)(2g - 2).$$

Let $\mathrm{C} < \mathrm{G}_0$ be the G-centralizer of the $\mathfrak{sl}_2\mathbb{C}$-subalgebra. Since T and C are commuting subgroups of G_0, we can form a G_0-bundle $E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}$ out of E_{T} and a holomorphic C-bundle E_{C}. Since C acts trivially on $\langle e \rangle$, we have $e \in H^0(E_{\mathrm{C}} \otimes_{\mathrm{G}_0} E_{\mathrm{T}}(\langle e \rangle) \otimes K)$. Moreover, this process does not

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change the Toledo invariant and defines a map

$$\begin{aligned}\Psi_e :\ \mathcal{N}(\mathrm{C}) &\longrightarrow \mathcal{M}^{\max}(\mathrm{G}_0, \mathfrak{g}_1)\\ E_{\mathrm{C}} &\longmapsto (E_{\mathrm{C}} \otimes_{\mathrm{G}_0} E_{\mathrm{T}}, e)\end{aligned}$$

from the moduli space of degree zero polystable C-bundles to the moduli space of maximal Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$.

Theorem B (Theorem 6.2). *If $(\mathrm{G}_0, \mathfrak{g}_1)$ is a JM-regular prehomogeneous vector space, then the map Ψ_e defines an isomorphism between the moduli space of degree zero polystable* C*-bundles and the moduli space of maximal Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$.*

Remark 1.4. We note that the map Ψ_e is a moduli space version of a restriction of the so-called global Slodowy slice map of [CS21]. In [BCGGO21], an extension of the map Ψ_e to the entire Slodowy slice gives rise to the Cayley correspondence used to describe certain components of the moduli space of $\mathrm{G}^{\mathbb{R}}$-Higgs bundles related to higher Teichmüller theory, generalizing the Cayley correspondence for the Hermitian group case given in [BGR17]. It would be interesting to extend these results to the full Slodowy slice for every even $\mathfrak{sl}_2$-triple.

When $(\mathrm{G}_0, \mathfrak{g}_1)$ is JM-regular, the representations $\rho : \pi_1(X) \to \mathrm{G}$ associated to maximal Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ are described by the following theorem.

Theorem C (Theorem 6.5). *Fix a Riemann surface X of genus $g \geqslant 2$ and suppose $(\mathrm{G}_0, \mathfrak{g}_1)$ is a JM-regular PHVS associated to an even $\mathfrak{sl}_2$-triple $\{f, h, e\}$. Let $\mathrm{S} < \mathrm{G}$ be the associated connected subgroup and let* C *be the* G*-centralizer of $\{f, h, e\}$. The Higgs bundle associated to a reductive representation $\rho : \pi_1(X) \to \mathrm{G}$ is a maximal Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_1)$ if and only if ρ is a product $\rho = \rho_u * \rho_{\mathrm{C}}$, where*

- *$\rho_u : \pi_1(X) \to \mathrm{S}^{\mathbb{R}} < \mathrm{S}$ is the uniformizing $\mathrm{PSL}_2\mathbb{R}$-representation of X if $\mathrm{S} \cong \mathrm{PSL}_2\mathbb{C}$ and a lift of the uniformizing representation to $\mathrm{SL}_2\mathbb{R}$ if $\mathrm{S} \cong \mathrm{SL}_2\mathbb{C}$, and*
- *$\rho_{\mathrm{C}} : \pi_1(X) \to \mathrm{C}^{\mathbb{R}} < \mathrm{C}$ is any representation into the compact real form of* C*.*

The representations $\rho = \rho_u * \rho_{\mathrm{C}} : \pi_1(X) \to \mathrm{G}$ all factor through a real form of Hodge type $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ which is canonically associated to the grading $\bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ (see Proposition 3.8). Moreover, the $\mathrm{G}^{\mathbb{R}}$-centralizer of any such representation is compact, and hence these representations do not factor through any proper parabolic subgroups $\mathrm{P}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$.

From the above results, it follows that the equivariant holomorphic map $f_\rho : \widetilde{X} \to \mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ associated to such a maximal variation of Hodge structure (ρ, f_ρ) is a totally geodesic embedding which maximizes the holomorphic sectional curvature. In this paper we discuss how the choice of the Toledo character is related to a metric of minimal holomorphic sectional curvature -1 on $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$, providing an alternative proof of Theorem A. Our bounds for the sectional curvature give the bounds found in [Li22] in the case of $\mathrm{SL}_n\mathbb{C}$.

Now consider a general grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with grading element ζ, not necessarily coming from an $\mathfrak{sl}_2$-triple. Pick a point $e \in \Omega \subset \mathfrak{g}_1$ in the open G_0-orbit, and complete e to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$ with $h \in \mathfrak{g}_0$. If $(\mathrm{G}_0, \mathfrak{g}_1)$ is not a JM-regular PHVS, then $s = \zeta - h/2$ is nonzero and we define $\hat{\mathrm{G}}_0 < \mathrm{G}_0$ to be the G_0 centralizer of s and $\hat{\mathfrak{g}}_1 = \{x \in \mathfrak{g}_1 \mid [s, x]\} = 0$. With this setup, $\hat{\mathrm{G}}_0$ is reductive and $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ defines a PHVSS of $(\mathrm{G}_0, \mathfrak{g}_1)$ which contains $e \in \hat{\Omega} \subset \hat{\mathfrak{g}}_1$ in the open orbit (see § 2.4). Moreover, the G_0-centralizer $\hat{\mathrm{C}}$ of the $\mathfrak{sl}_2$-triple $\{f, h, e\}$ satisfies $\hat{\mathrm{C}} < \hat{\mathrm{G}}_0$. We call $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ a maximal JM-regular PHVSS of $(\mathrm{G}_0, \mathfrak{g}_1)$.

In § 6.3, we show that all maximal Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ reduce to a maximal Hodge bundles of type $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ (see Proposition 6.15). We then show that the moduli space

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$\mathcal{M}^{\max}(\mathrm{G}_0, \mathfrak{g}_1)$ of maximal Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ is isomorphic to the moduli space of polystable $\hat{\mathrm{C}}$-bundles and deduce rigidity results for maximal variations of Hodge structure (see Theorems 6.16 and 6.17). This recovers rigidity results for maximal variations of Hodge structure in the Hermitian case.

2. Prehomogeneous vector spaces, $\mathbb{Z}$-gradings and the Toledo character

For this section, let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$. Throughout the paper, the word *character* will be used to refer to a Lie group morphism $\chi : \mathrm{G} \to \mathbb{C}^*$ and also a Lie algebra morphism $\chi : \mathfrak{g} \to \mathbb{C}$, what type of object is being considered should be clear from the context.

2.1 Prehomogeneous vector spaces

We gather together some basic facts about PHVSs. The main references are [Kim03, Kna02, Mor91, SK77].

A *prehomogeneous vector space* for G is a finite-dimensional complex vector space V together with a holomorphic representation $\rho : \mathrm{G} \to \mathrm{GL}(V)$ such that V has an open G-orbit. Such an open orbit is necessarily unique and dense. If V is a PHVS, let Ω denote the *open orbit* in V and $S = V \setminus \Omega$ be the *singular set*. For $x \in V$, denote the G-stabilizer of x by G^x. A PHVS vector space V is called *regular* if G^x is reductive for $x \in \Omega$, otherwise it is called *nonregular*.

We say that (H, W) is a *PHVSS* of (G, V) if (H, W) is a PHVS, $\mathrm{H} \subset \mathrm{G}$ is a subgroup, $W \subset V$ is a vector subspace, and the action of H is the restriction of the action of G.

Example 2.1.

(1) The vector space $\mathbb{C}^n$ is a PHVS for the standard representation of $\mathrm{GL}_n\mathbb{C}$. For this example, $\Omega = \mathbb{C}^n \setminus \{0\}$, and it is regular only when $n = 1$.
(2) The vector space $M_{p,q}$ of $p \times q$ matrices is a PHVS for the action of $\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C})$ given by $(A, B) \cdot M = AMB^{-1}$. Here, $\Omega = \{M \in M_{p,q} \mid \mathrm{rk}(M) = \min(p, q)\}$. This example is regular only when $p = q$.
(3) The vector space $M_{p,q}$ is also a PHVS for the action of $\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}$ given by $(A, B) \cdot M = AMB^{-1}$. Here $\Omega = \{M \in M_{p,q} \mid \mathrm{rk}(M \cdot M^{\mathrm{T}}) = \min(p, q)\}$. Also, the vector space $M_{p,q} \oplus M_{q,r}$ is a PHVS for the action of $\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_r\mathbb{C})$ given by $(A, B, C) \cdot (M, N) = (AMB^{-1}, BNC^{-1})$. Here $\Omega = \{(M, N) \in M_{p,q} \oplus M_{q,r} \mid \mathrm{rk}(MN) = \min(p, q, r)\}$.

The first example is regular when $p \leqslant q$ and the second example is regular when $p = r$ and $p \leqslant q$. Note that the inclusions

$$\begin{aligned} M_{p,q} &\longrightarrow M_{p,q} \oplus M_{q,p} & \mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C} &\longrightarrow \mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_p\mathbb{C}) \\ M &\longmapsto (M, -M^{\mathrm{T}}), & (A, B) &\longmapsto (A, B, (A^{\mathrm{T}})^{-1}) \end{aligned}$$

make $(\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}, M_{p,q})$ a PHVSS of $(\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_p\mathbb{C}), M_{p,q} \oplus M_{q,p})$.

Let V be a PHVS for G with representation ρ. A nonconstant rational function $F : V \to \mathbb{C}$ is called a *relative invariant* if there exists a Lie group character $\chi : \mathrm{G} \to \mathbb{C}^*$ such that

$$F(\rho(g) \cdot x) = \chi(g)F(x) \quad \text{for all } g \in \mathrm{G} \text{ and } x \in V.$$

Here are some fundamental facts appearing in [SK77, §4].

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Proposition 2.2. *Let V be a PHVS for G with representation ρ.*

(i) *Up to a constant, a relative invariant is uniquely determined by its corresponding character. In particular, any relative invariant is a homogeneous function.*
(ii) *V is regular if and only if the singular set S is a hypersurface.*
(iii) *Let $\chi : \mathrm{G} \to \mathbb{C}^*$ be a character. Then there is a relative invariant for χ if and only if χ is trivial on the stabilizers of points in Ω, that is, $\chi|_{\mathrm{G}^x} = 1$ for all $x \in \Omega$.*

Example 2.3. The regular PHVS $M_{p,p}$ from Example 2.1(2) has a relative invariant $F : M_{p,p} \to \mathbb{C}$ given by $F(M) = \det(M)$. The associated character $\chi : \mathrm{G} \to \mathbb{C}^*$ is given by $\chi(A, B) = \det(A)\det(B)^{-1}$ since

$$F((A, B) \cdot M) = \det(AMB^{-1}) = \chi(A, B)F(M).$$

The regular PHVS $M_{p,q} \oplus M_{q,p}$ with $p \leqslant q$ from Example 2.1(3) has a relative-invariant given by $F(M, N) = \det(MN)$. The associated character is $\chi(A, B, C) = \det(A)\det(C^{-1})$ since

$$F((A, B, C) \cdot (M, N)) = \det(AMB^{-1}BNC^{-1}) = \chi(A, B, C)F(M, N).$$

This relative invariant also defines a relative invariant for the $\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}$ PHVSS given by $(M, N) = (M, -M^{\mathrm{T}})$. For $(A, B) \in \mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}$, the associated character is $\chi(A, B) = \det(AA^{\mathrm{T}}) = \det(A)^2$.

2.2 Prehomogeneous vector spaces associated to $\mathbb{Z}$-gradings

A $\mathbb{Z}$-grading of a semisimple Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j \quad \text{such that } [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

The subalgebra $\mathfrak{p} = \bigoplus_{j \geqslant 0} \mathfrak{g}_j$ is a parabolic subalgebra with Levi subalgebra $\mathfrak{g}_0 \subset \mathfrak{p}$. There is an element $\zeta \in \mathfrak{g}_0$ such that $\mathfrak{g}_j = \{X \in \mathfrak{g} \mid [\zeta, x] = jx\}$; the element ζ is called the *grading element* of the $\mathbb{Z}$-grading.

Given a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, let $\mathrm{G}_0 < \mathrm{G}$ be the centralizer of ζ; G_0 acts on each factor $\mathfrak{g}_j$. The relation between $\mathbb{Z}$-gradings and PHVSs is given by the following theorem of Vinberg (see [Kna02, Theorem 10.19]).

Proposition 2.4. *For each $j \neq 0$, $\mathfrak{g}_j$ is a PHVS for G_0.*

Prehomogeneous vector spaces arising from $\mathbb{Z}$-gradings of $\mathfrak{g}$ are said to be of *parabolic type*. From now on, we will consider only PHVSs of parabolic type.

Example 2.5. All of the PHVSs from Example 2.1 are of parabolic type. Below the parabolic refers to the subgroup with Lie algebra $\mathfrak{p} = \bigoplus_{j \geqslant 0} \mathfrak{g}_j$.

– For $(\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C}), M_{p,q})$, $\mathrm{G} = \mathrm{SL}_{p+q}\mathbb{C}$ and the parabolic is the stabilizer a p-plane in $\mathbb{C}^{p+q}$.
– For $(\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}, M_{p,q})$, $\mathrm{G} = \mathrm{SO}_{2p+q}\mathbb{C}$ and the parabolic is the stabilizer of an isotropic p-plane in $\mathbb{C}^{2p+q}$.
– For $(\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_r\mathbb{C}), M_{p,q} \oplus M_{q,r})$, $\mathrm{G} = \mathrm{SL}_{p+q+r}\mathbb{C}$ and the parabolic is the stabilizer of a flag $\mathbb{C}^p \subset \mathbb{C}^{p+q} \subset \mathbb{C}^{p+q+r}$.

Consider a nonzero nilpotent element $e \in \mathfrak{g}$. By the Jacobson–Morozov theorem, e can be completed to an $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$, that is, a triple satisfying the bracket relations of $\mathfrak{sl}_2\mathbb{C}$:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Moreover, given $\{h, e\}$ so that $[h, e] = 2e$ and $h \in \mathrm{ad}_e(\mathfrak{g})$, there is a unique $f \in \mathfrak{g}$ such that $\{f, h, e\}$ is an $\mathfrak{sl}_2$-triple.

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Given an $\mathfrak{sl}_2$-triple $\{f,h,e\}$, the semisimple element h acts on $\mathfrak{g}$ with integral weights and thus defines a $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, where $\mathfrak{g}_j=\{x\in\mathfrak{g}\,|\,\mathrm{ad}_h(x)=jx\}$. Note that $e\in\mathfrak{g}_2$. We have the following result of Kostant and Malcev (see [Kna02, Theorem 10.10]).

Proposition 2.6. *Let $\{f,h,e\}\subset\mathfrak{g}$ be an $\mathfrak{sl}_2$-triple with associated $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, and let $\mathrm{G}_0<\mathrm{G}$ be the associated analytic subgroup with Lie algebra $\mathfrak{g}_0$. Then e is in the open orbit Ω of the PHVS $(\mathrm{G}_0,\mathfrak{g}_2)$.*

Corollary 2.7. *The PHVS $(\mathrm{G}_0,\mathfrak{g}_2)$ arising from an $\mathfrak{sl}_2$-triple $\{f,h,e\}$ is regular.*

Proof. Since $e\in\Omega\subset\mathfrak{g}_2$, we need to show that the stabilizer G_0^e is reductive. The group G_0 centralizes h, so the stabilizer G_0^e of e stabilizes both h and e. By the uniqueness part of the Jacobson–Morozov theorem, G_0^e stabilizes the $\mathfrak{sl}_2$-triple. As the centralizer of an $\mathfrak{sl}_2$-subalgebra is reductive (see [CM93, Lemma 3.7.3]), we conclude that G_0^e is reductive. □

Let $B:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ denote the Killing form of $\mathfrak{g}$. Given an $\mathfrak{sl}_2$-triple, $\{f,h,e\}\subset\mathfrak{g}$ with associated $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, define the Lie algebra character $\chi_h:\mathfrak{g}_0\to\mathbb{C}$ by

$$\chi_h(x)=\tfrac{1}{2}B(h,x). \tag{2.1}$$

Proposition 2.8. *There is a positive integer q such that $q\cdot\chi_h$ exponentiates to a character*

$$\chi_{h,q}:\mathrm{G}_0\to\mathbb{C}^*$$

which has a relative invariant of degree $q\cdot B(h/2,h/2)$.

Remark 2.9. In fact, we will prove that the associated relative invariant is a polynomial in Proposition 3.16.

Proof. Let $\mathfrak{g}_0^e\subset\mathfrak{g}_0$ be the Lie algebra of the G_0-stabilizer of $e\in\mathfrak{g}_2$. For $x\in\mathfrak{g}_0^e$, we have

$$B(h,x)=B([e,f],x)=B(f,[e,x])=0.$$

Thus, $\chi_h(\mathfrak{g}_0^e)=0$, and there is a positive integer q so that $q\cdot\chi_h$ exponentiates to a character which is trivial on the identity component of G_0^e. Since G_0^e has a finite number of components, we can choose q so that $q\cdot\chi_h$ exponentiates to a character $\chi_{h,q}:\mathrm{G}_0\to\mathbb{C}^*$ whose restriction to G_0^e is trivial. Hence, by part (iii) of Proposition 2.2, the character $\chi_{h,q}$ has a relative invariant $F:\mathfrak{g}_2\to\mathbb{C}$ such that $F(e)\neq0$. For the degree, we have $\mathrm{Ad}_{\exp(th)}(e)=\exp(2t)e$. Thus,

$$F(\exp(t)e)=F\left(\exp\left(t\frac{h}{2}\right)\cdot e\right)=\chi_{h,q}\left(\exp\left(t\frac{h}{2}\right)\right)F(e)=\exp\left(tq\cdot B\left(\frac{h}{2},\frac{h}{2}\right)\right)F(e). \tag{2.2}$$

By Proposition 2.2 F is homogeneous, and so has degree $q\cdot B(h/2,h/2)$. □

Remark 2.10. Not every regular PHVS of the form $(\mathrm{G}_0,\mathfrak{g}_2)$ arises from an $\mathfrak{sl}_2$-triple. For examples in E_6, E_7 and E_8 see [Rub11, Remark 4.18]. In fact, the B_3 example in [Rub11, Table 1] gives a very simple example of a regular PHVS which does not arise from an $\mathfrak{sl}_2$-triple. For such regular PHVSs the grading element ζ does not have a relative invariant.

2.3 Canonical $\mathbb{Z}$-gradings associated to parabolics

In this section we fix some normalizations for the $\mathbb{Z}$-gradings we will consider.

Up to conjugation, all $\mathbb{Z}$-gradings of $\mathfrak{g}$ arise from labeling the nodes of the Dynkin diagram of $\mathfrak{g}$ with nonnegative integers [Vin94, Chapter 3.5]. This works as follows. Let $\mathfrak{t}\subset\mathfrak{g}$ be a Cartan subalgebra and $\Delta=\Delta(\mathfrak{g},\mathfrak{t})$ be the corresponding set of roots. Pick a set of simple roots $\Pi\subset\Delta$ and let $\Delta^+\subset\Delta$ be the set of positive roots. Every root $\alpha\in\Delta^+$ can be written as $\alpha=\sum_{\alpha_i\in\Pi}n_i\alpha_i$, where all n_i are nonnegative integers. For each $\alpha_i\in\Pi$, choose nonnegative integers p_i,

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that is, label the nodes of the Dynkin diagram of Π with nonnegative integers p_i. This choice defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$, where $\mathfrak{g}_j$ is the direct sum of all root spaces $\mathfrak{g}_\alpha$ such that $\alpha = \sum_{\alpha_i\in\Pi} n_i\alpha_i$ and $j = \sum_i n_i p_i$.

In the above construction, the associated parabolic $\mathfrak{p} = \bigoplus_{j\geqslant 0} \mathfrak{g}_j$ and the Levi subalgebra $\mathfrak{g}_0$ depend only on the labels p_i mod 2. Namely, up to conjugation, parabolic subalgebras $\mathfrak{p}$ are determined by subsets $\Theta \subset \Pi$, where

$$\mathfrak{p}_\Theta = \mathfrak{t} \oplus \bigoplus_{\alpha\in span(\Theta)\cap\Delta^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha.$$

We define the *canonical $\mathbb{Z}$-grading of* $\mathfrak{p}_\Theta$ by labeling the α_i-node of the Dynkin diagram 0 if $\alpha_i \in \Theta$ and 1 if $\alpha_i \in \Pi \setminus \Theta$. That is, $\mathfrak{g}_j$ is given by

$$\mathfrak{g}_j = \bigoplus \mathfrak{g}_\alpha, \quad \text{where } \alpha = \sum_{\alpha_i\in\Pi} n_i\alpha_i \text{ and } j = \sum_{\alpha_i\in\Pi\setminus\Theta} n_i.$$

Remark 2.11. We will often consider PHVSs of parabolic type and of the form $(G_0, \mathfrak{g}_1)$ where $\mathfrak{g}_1$ is the weight-1 piece of the canonical grading of a parabolic subalgebra of $\mathfrak{g}$. This is not a major restriction since, given a general PHVS $(G_0, \mathfrak{g}_k)$ of parabolic type, the subalgebra consisting of graded pieces $\mathfrak{g}_j$ with $j = 0 \bmod k$ is reductive and $\mathfrak{g}_k$ is the weight 1 piece of the canonical grading of a parabolic in this subalgebra.

An $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}$ is called *even* if ad_h has only even eigenvalues. Parabolic subalgebras arising from even $\mathfrak{sl}_2$-triples are called *even Jacobson–Morozov parabolics*. For such parabolics, the canonical grading is given by $\mathrm{ad}_{h/2}$; the PHVS $(G_0, \mathfrak{g}_1)$ is regular by Corollary 2.7 and, by Proposition 2.8, there is a relative invariant associated to the character χ_h from (2.1). As a result we will call a PHVS $(G_0, \mathfrak{g}_1)$ which arises from an even $\mathfrak{sl}_2$-triple a *JM-regular PHVS*.

Example 2.12. When $\Theta = \emptyset$, G_0 is Cartan subgroup and $\mathfrak{g}_1$ is the direct sum of simple root spaces. Such a PHVS $(G_0, \mathfrak{g}_1)$ is always JM-regular and arises from a principal $\mathfrak{sl}_2$-triple. Here $\Omega \subset \mathfrak{g}$ consists of vectors with nonzero projection onto each simple root space and the stabilizer of such a point is the center of G.

Example 2.13. Let $\eta = \sum_{\alpha_i\in\Pi} n_i\alpha_i$ be the longest root of Δ^+. For $\mathfrak{g}$ not of type E_8, F_4, G_2, there is at least one simple root α_i with $n_i = 1$. For such a root set $\Theta = \Pi \setminus \{\alpha_i\}$. In this case, the canonical $\mathbb{Z}$-grading of the parabolic $\mathfrak{p}_\Theta$ is given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

For these examples, $(G_0, \mathfrak{g}_1)$ is regular if and only if it is JM-regular. The associated flag variety G/P_Θ is the compact dual of a Hermitian symmetric space and $(G_0, \mathfrak{g}_1)$ is regular if and only if the associated Hermitian symmetric space is of tube type. An example of this is given in Example 2.1, in which case the symmetric space of $SU(p, q)$ is the relevant Hermitian symmetric space; it is of tube type only when $q = p$.

Remark 2.14. In Example 2.13, the space $\mathfrak{g}_1$ is an irreducible G_0-representation. In general, if $(G_0, \mathfrak{g}_1)$ is a PHVS such that G_0 acts irreducibly on $\mathfrak{g}_1$, then $(G_0, \mathfrak{g}_1)$ is regular if and only if it is JM-regular. For example, this implies that the PHVS $(GL_p\mathbb{C} \times SO_q\mathbb{C}, M_{p,q})$ from Examples 2.1 and 2.5 is JM-regular whenever $p \leqslant q$. The regular PHVS $(S(GL_p\mathbb{C} \times GL_q\mathbb{C} \times GL_p\mathbb{C}), M_{p,q} \oplus M_{q,p})$ from Examples 2.1 and 2.5 does not come from a maximal parabolic of $SL_{2p+q}\mathbb{C}$ but it is still a JM-regular PHVS when $p \leqslant q$.

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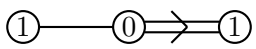
Example 2.15. The simplest example of a PHVS $(\mathrm{G}_0, \mathfrak{g}_1)$ which is regular but not JM-regular occurs in $\mathrm{SO}_7\mathbb{C}$ with $\Theta = \{\alpha_2\}$, that is, with labeled Dynkin diagram



Here $(\mathrm{G}_0, \mathfrak{g}_1) \cong (\mathrm{GL}_1\mathbb{C} \times \mathrm{GL}_2\mathbb{C}, \mathbb{C}^2 \oplus \mathbb{C}^2)$ for the action $(\lambda, A) \cdot (v, w) = (\lambda v A^{-1}, A w^{\mathrm{T}})$. A point in the open orbit is given by $v = (1, 0)$ and $w = (1, 0)$. For this point, the G_0-stabilizer is given by $\lambda = 1$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \xi \end{smallmatrix}\right)$ for $\xi \in \mathbb{C}^*$. Geometrically, this parabolic stabilizes an isotropic flag of the form $\mathbb{C} \subset \mathbb{C}^3 \subset \mathbb{C}^7$.

2.4 Jacobson–Morozov regular prehomogeneous vector subspaces

Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with grading element ζ and consider a PHVS $(\mathrm{G}_0, \mathfrak{g}_1)$. Following the proof [CM93, Theorem 3.3.1] of the Jacobson–Morozov theorem, one can show that any nonzero element $e \in \mathfrak{g}_1$ can be completed to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$ with $f \in \mathfrak{g}_{-1}$ and $h \in \mathfrak{g}_0$.

The semisimple element h defines a new grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \tilde{\mathfrak{g}}_j$ where

$$\tilde{\mathfrak{g}}_j = \{x \in \mathfrak{g} \,|\, \mathrm{ad}_h(x) = jx\}.$$

Define $\hat{\mathfrak{g}}_j = \mathfrak{g}_j \cap \tilde{\mathfrak{g}}_{2j}$ and the subalgebra $\hat{\mathfrak{g}} \subset \mathfrak{g}$ given by

$$\hat{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}} \hat{\mathfrak{g}}_j. \tag{2.3}$$

Note that h and ζ are both in $\hat{\mathfrak{g}}_0$ and $e \in \hat{\mathfrak{g}}_1$. The difference $s = \zeta - h/2$ is semisimple or zero since ζ and $h/2$ are semisimple and $[\zeta, h/2] = 0$. Thus, $\hat{\mathfrak{g}}$ is reductive since $\hat{\mathfrak{g}} = \mathfrak{g}^s$ is the centralizer of s. The following proposition is immediate.

Proposition 2.16. *Let $\hat{\mathrm{G}}_0 < \mathrm{G}$ be the G_0-centralizer of h. Then $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ is a PHVSS of $(\mathrm{G}_0, \mathfrak{g}_1)$ which is JM-regular and e is in the open $\hat{\mathrm{G}}_0$-orbit $\hat{\Omega} \subset \hat{\mathfrak{g}}_1$.*

Definition 2.17. If $e \in \Omega$ is in the open G_0-orbit, then $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ will be called a maximal JM-regular PHVSS of $(\mathrm{G}_0, \mathfrak{g}_1)$. For any $e \in \mathfrak{g}_1 \setminus \{0\}$, we will call $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ a maximal JM-regular PHVSS for e.

Remark 2.18. If $(\mathrm{G}_0, \mathfrak{g}_1)$ is JM-regular, then for any $e \in \Omega$ an associated $\mathfrak{sl}_2$-triple has the form $\{f, 2\zeta, e\}$. In this case, $(\mathrm{G}_0, \mathfrak{g}_1) = (\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$.

Example 2.19. Consider the PHVS $(\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C}), M_{p,q})$ from Example 2.1(2). If $p \neq q$, then the PHVS is not JM-regular. For $p > q$, a maximal JM-regular PHVSS is isomorphic to $(\mathrm{S}(\mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_{p-q}\mathbb{C}), M_{q,q})$, for the action $(A, B, C) \cdot M = AMB^{-1}$. In general, the maximal JM-regular PHVSS of the PHVSs $(\mathrm{G}_0, \mathfrak{g}_1)$ from Example 2.13 is related to the maximal subtube of the associated Hermitian symmetric space.

For the PHVS $(\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}, M_{p,q})$ from the first example of Example 2.1(3) with $p > q$, the PHVS is not JM-regular and a maximal JM-regular PHVSS is isomorphic to $(\mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_{p-q}\mathbb{C} \times \mathrm{SO}_q\mathbb{C}, M_{q,q})$, where the action is given by $(A, B, C) \cdot M = AMC^{-1}$. The maximal JM-regular PHVSS of $(\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_r\mathbb{C}), M_{p,q} \oplus M_{q,r})$ can be described similarly. For the regular (non-JM-regular) PHVS $(\mathrm{G}_0, \mathfrak{g}_1) = (\mathrm{GL}_1\mathbb{C} \times \mathrm{GL}_2\mathbb{C}, \mathbb{C}^2 \oplus \mathbb{C}^2)$ from Example 2.15, the maximal JM-regular PHVSS $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ containing $(v, w) = ((1, 0), (1, 0))$ is

$$\hat{\mathrm{G}}_0 = \left\{ (\lambda, A) \in \mathrm{GL}_1\mathbb{C} \times \mathrm{GL}_2\mathbb{C} \,\middle|\, A = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) \right\} \quad \text{and} \quad \hat{\mathfrak{g}}_1 = \langle v \rangle \oplus \langle w \rangle.$$

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The construction of a maximal JM-regular PHVSS of $(\mathrm{G}_0, \mathfrak{g}_1)$ containing $e \in \mathfrak{g}_1$ depends on e and a choice of $\mathfrak{sl}_2$-triple $\{f, h, e\}$. However, if $e, e' \in \mathfrak{g}_1$ are in the same G_0-orbit then any two maximal JM-regular PHVSSs containing e and e' are G_0-conjugate.

Proposition 2.20. *Let $e, e' \in \mathfrak{g}_1$ so that $e' \in \mathrm{G}_0 \cdot e$, and let $\{f, h, e\}, \{f', h', e'\}$ be two $\mathfrak{sl}_2$-triples with $h, h' \in \mathfrak{g}_0$. Then there is $g \in \mathrm{G}_0$ so that*

$$\{f', h', e'\} = \{\mathrm{Ad}_g f, \mathrm{Ad}_g h, \mathrm{Ad}_g e\}.$$

In particular, the associated maximal JM-regular PHVSSs $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ and $(\hat{\mathrm{G}}'_0, \hat{\mathfrak{g}}'_1)$ containing e and e' respectively are G_0-conjugate.

Proof. We may assume $e = e'$. Let $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \tilde{\mathfrak{g}}_j$ be the $\mathbb{Z}$-grading associated to the $\mathfrak{sl}_2$-triple $\{f, h, e\}$ and $\mathfrak{g}^e$ be the centralizer of e and $\mathfrak{u}^e = [e, \mathfrak{g}] \cap \mathfrak{g}^e = \mathfrak{g}^e \cap \bigoplus_{j>0} \tilde{\mathfrak{g}}_j$. We have $h' - h \in \mathfrak{u}^e$ since $[h - h', e] = 0$ and $[e, f - f'] = h' - h$. Since $h, h' \in \mathfrak{g}_0$, following the proof [CM93, Theorem 3.4.7], we can inductively construct $Z \in \mathfrak{g}_0 \cap \mathfrak{u}^e$ so that $\mathrm{Ad}_{\exp(Z)}(h) = h + (h' - h) = h'$. Note that $\mathrm{Ad}_{\exp(Z)} e = e$, and, by the uniqueness part of the Jacobson–Morozov theorem, we have $\mathrm{Ad}_{\exp(Z)} f = f'$. □

Finally, we note that every $e \in \mathfrak{g}_1$ defines a parabolic subgroup of $\mathrm{P}_{0,e} < \mathrm{G}_0$. For $e \in \mathfrak{g}_1 \setminus \{0\}$, let $\{f, h, e\}$ be an associated $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$. If $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \tilde{\mathfrak{g}}_j$ is the $\mathbb{Z}$-grading with grading element h, then $\tilde{\mathfrak{p}} = \bigoplus_{j\geqslant 0} \tilde{\mathfrak{g}}_j$ is a parabolic subalgebra of $\mathfrak{g}$. Moreover, $\tilde{\mathfrak{p}}$ depends only on $e \in \mathfrak{g}_1 \setminus \{0\}$ and not on the $\mathfrak{sl}_2$-triple $\{f, h, e\}$ (see, for example, [CM93, Remark 3.8.5]). Note that the parabolic $\tilde{\mathfrak{p}}$ can also be defined by

$$\tilde{\mathfrak{p}} = \{x \in \mathfrak{g} \mid \mathrm{Ad}_{\exp(-t(h/2))} \text{ is bounded as } t \to \infty\}. \tag{2.4}$$

The parabolic subalgebra $\mathfrak{p}_0(e) \subset \mathfrak{g}_0$ is defined by

$$\mathfrak{p}_{0,e} = \mathfrak{g}_0 \cap \tilde{\mathfrak{p}}.$$

We will denote the associated parabolic subgroup by $\mathrm{P}_{0,e} < \mathrm{G}_0$. Note that $\hat{\mathfrak{g}}_0 \subset \mathfrak{p}_{0,e}$ and $\hat{\mathrm{G}}_0 < \mathrm{P}_{0,e}$ define a Levi subalgebra and subgroup, respectively.

Proposition 2.21. *Let $e \in \mathfrak{g}_1 \setminus \{0\}$ and $\{f, h, e\}$ be an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$. If $s = \zeta - h/2$, then*

$$\mathfrak{p}_{0,e} = \{x \in \mathfrak{g}_0 \mid \mathrm{Ad}_{\exp(ts)} x \text{ is bounded as } t \to \infty\}.$$

Proof. The proposition follows immediately from the description of $\tilde{\mathfrak{p}}$ in (2.4) and the fact that $\mathrm{Ad}_{\exp(t\zeta)} x = x$ for all $x \in \mathfrak{g}_0$. □

3. The Toledo character and period domains

In this section we generalize notions from [BGR17], which concerned $\mathbb{Z}$-gradings $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, to arbitrary $\mathbb{Z}$-gradings.

3.1 The Toledo character and rank

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and Killing form B. Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with grading element ζ. Recall that $(\mathrm{G}_0, \mathfrak{g}_1)$ is a PHVS, and let $\Omega \subset \mathfrak{g}_1$ be the open G_0-orbit. Since $\mathfrak{g}_0$ is the centralizer of ζ, $B(\zeta, -) : \mathfrak{g}_0 \to \mathbb{C}$ defines a character.

Definition 3.1. The Toledo character $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is defined by

$$\chi_T(x) = B(\zeta, x)B(\gamma, \gamma),$$

where γ is the longest root such that $\mathfrak{g}_\gamma \subset \mathfrak{g}_1$.

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Remark 3.2. The normalization factor $B(\gamma,\gamma)$ guarantees that the Toledo character is independent of the choice of invariant bilinear form B.

Let $e, e' \in \mathfrak{g}_1 \setminus \{0\}$ such that $e' \in \mathrm{G}_0 \cdot e$ and let $\{f,h,e\}$, $\{f',h',e'\}$ be two $\mathfrak{sl}_2$-triples with $h, h' \in \mathfrak{g}_0$. By Proposition 2.20, there is $g \in \mathrm{G}_0$ such that $\{\mathrm{Ad}_g f, \mathrm{Ad}_g h, \mathrm{Ad}_g e\} = \{f',h',e'\}$. Since $\mathrm{Ad}_g \zeta = \zeta$, we have

$$\chi_T(h) = B(\zeta,h)B(\gamma,\gamma) = B(\mathrm{Ad}_g \zeta, \mathrm{Ad}_g h)B(\gamma,\gamma) = \chi_T(h'). \tag{3.1}$$

As a result, we make the following definition.

Definition 3.3. Let $e \in \mathfrak{g}_1$ and $\{f,h,e\}$ be an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$. Define the Toledo rank of e by

$$\mathrm{rk}_T(e) = \tfrac{1}{2}\chi_T(h).$$

Define the Toledo rank of the PHVS $(\mathrm{G}_0, \mathfrak{g}_1)$ by

$$\mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1) = \mathrm{rk}_T(e) \quad \text{for } e \in \Omega.$$

Remark 3.4. Note that if $(\mathrm{G}_0, \mathfrak{g}_1)$ is a JM-regular PHVS, then $\mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1) = B(\zeta,\zeta)B(\gamma,\gamma)$.

Example 3.5. For the PHVSs $(\mathrm{G}_0, \mathfrak{g}_1)$ from Example 2.13, the Toledo rank of $(\mathrm{G}_0, \mathfrak{g}_1)$ agrees with the rank of the associated Hermitian symmetric space.

Proposition 3.6. *Let $e \in \mathfrak{g}_1$ and $\{f,h,e\}$ be an associated $\mathfrak{sl}_2$-triples with $h \in \mathfrak{g}_0$. Then*

$$\mathrm{rk}_T(e) = B\left(\frac{h}{2}, \frac{h}{2}\right) B(\gamma,\gamma). \tag{3.2}$$

Proof. Write $\zeta = h/2 + s$. Note that $s \in \mathfrak{g}^e$ and $h/2 \in \mathrm{ad}_e(\mathfrak{g})$. Thus, $B(h/2, s) = 0$ and

$$\mathrm{rk}_T(e) = B\left(\zeta, \frac{h}{2}\right) B(\gamma,\gamma) = B\left(\frac{h}{2} + s, \frac{h}{2}\right) B(\gamma,\gamma) = B\left(\frac{h}{2}, \frac{h}{2}\right) B(\gamma,\gamma). \qquad \square$$

Example 3.7. We illustrate these notions in the case of Example 1.3, with $G = \mathrm{SO}_{2p+q}\mathbb{C}$ and $G_0 = \mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}$. Taking an isotropic basis of $\mathbb{C}^{2p+q} = \mathbb{C}^p \oplus \mathbb{C}^q \oplus \mathbb{C}^p$, such that $\langle e_i, e_{2p+q+1-i}\rangle = 1$, we have the grading element

$$\zeta = \begin{pmatrix} \mathrm{Id}_p & & \\ & 0 & \\ & & -\mathrm{Id}_p \end{pmatrix}.$$

Then $\mathfrak{g}_1 = \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$, where $u \in \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$ represents the matrix

$$e = \left(\begin{array}{c|c|c} & -u^{\mathrm{T}} & \\ \hline & & u \\ \hline & & \end{array}\right), \quad \text{in which } u^{\mathrm{T}}_{ij} = u_{p+1-j,q+1-i}.$$

The orbits are classified by two integers (r_1, r_2), such that the image of $u \in \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$ is the sum of a nondegenerate subspace of dimension r_1 and a totally isotropic subspace of dimension r_2. Let J_r be the rank r matrix

$$J_r = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

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Then one can take

$$u = \begin{pmatrix} \mathrm{Id}_{r_2} & & & \\ & \mathrm{Id}_{r_1} & & \\ & 0 & & \\ & J_{r_1} & & \\ & & & 0 \end{pmatrix}, \quad h = \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & 2\,\mathrm{Id}_{r_1} & & & & & & & \\ & & \mathrm{Id}_{r_2} & & & & & & \\ \hline & & & \mathrm{Id}_{r_2} & & & & & \\ & & & & 0 & & & & \\ & & & & & -\mathrm{Id}_{r_2} & & & \\ \hline & & & & & & -\mathrm{Id}_{r_2} & & \\ & & & & & & & -2\,\mathrm{Id}_{r_1} & \\ & & & & & & & & 0 \end{array}\right).$$

(The formula for u is valid if $r_2 + r_1 \leqslant q/2$; the reader will modify u accordingly if $r_2 + r_1 > q/2$. The formula for h remains the same.) Take the standard invariant form $B(X, Y) = \mathrm{tr}(XY)$. If $q > 1$ we have $B(\gamma, \gamma) = 1$ and therefore $\mathrm{rk}_T\, e = \frac{1}{4}B(h, h) = 2r_1 + r_2$. On the other hand $\chi_T(x) = B(\zeta, x)$, so that we will have $\tau(E, \varphi) = 2 \deg V$ if the Higgs bundle (E, φ) is given as $V \oplus W \oplus V^*$ with V a $\mathrm{GL}_p\mathbb{C}$-bundle and W a $\mathrm{SO}_q\mathbb{C}$-bundle. This justifies inequality (1.2) as a consequence of Corollary 5.5.

3.2 Real forms and period domains

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and consider a conjugate linear involution $\sigma : \mathfrak{g} \to \mathfrak{g}$. The fixed point set $\mathfrak{g}^\sigma \subset \mathfrak{g}$ is a real subalgebra such that $\mathfrak{g}^\sigma \otimes \mathbb{C} = \mathfrak{g}$. Such a subalgebra is called a real form of $\mathfrak{g}$. On the level of groups, a real form $\mathrm{G}^\sigma < \mathrm{G}$ is the fixed point set of an anti-holomorphic involution $\sigma : \mathrm{G} \to \mathrm{G}$. A real form is called compact if G^σ is compact or $\mathfrak{g}^\sigma$ is the Lie algebra of a maximal compact subgroup. Compact real forms exist and are unique up to conjugation.

Real forms of $\mathfrak{g}$ can be equivalently defined in terms of complex linear involutions $\theta : \mathfrak{g} \to \mathfrak{g}$. Namely, Cartan proved that, given a real form σ, there is a compact real form $\tau : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma \circ \tau = \tau \circ \sigma$, and that given a complex linear involution θ, there is a compact real form $\tau : \mathfrak{g} \to \mathfrak{g}$ such that $\theta \circ \tau = \tau \circ \theta$. The correspondence is then given by setting $\theta = \sigma \circ \tau$. A real subalgebra $\sigma : \mathfrak{g} \to \mathfrak{g}$ is said to be of *Hodge type* if σ is an inner automorphism of $\mathfrak{g}$. If τ is a compact real form such that $\theta = \tau \circ \sigma = \sigma \circ \tau$, then σ is of Hodge type if and only if there is a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that $\theta|_{\mathfrak{t}} = \mathrm{Id}$.

Given a complex linear involution $\theta : \mathfrak{g} \to \mathfrak{g}$, we will write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for ± 1-eigenspaces of θ, namely $\theta|_{\mathfrak{h}} = \mathrm{Id}$ and $\theta_{\mathfrak{m}} = -\mathrm{Id}$. We will call the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the *complexified Cartan decomposition* of a real form $\mathfrak{g}^{\mathbb{R}} \subset \mathfrak{g}$. There is a real form of Hodge type associated to every $\mathbb{Z}$-grading of $\mathfrak{g}$.

Proposition 3.8. *Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a $\mathbb{Z}$-grading, and define $\theta : \mathfrak{g} \to \mathfrak{g}$ by*

$$\theta|_{\mathfrak{g}_j} = (-1)^j\, \mathrm{Id}\,.$$

Then θ is a Lie algebra involution which defines a real form of Hodge type.

Proof. It is clear that θ is a Lie algebra involution. Moreover, from the constructions in § 2.3, $\mathfrak{g}_0$ contains a Cartan subalgebra of $\mathfrak{g}$. Hence, the real form associated to the involution θ is of Hodge type. □

Example 3.9. For gradings of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the canonical real forms are exactly the set of real forms of Hermitian type. For example, the real form of $\mathrm{SL}_{p+q}\mathbb{C}$ associated to the PHVS $(\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C}, M_{p,q})$ from Examples 2.1 and 2.5 is $\mathrm{SU}(p, q)$.

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The real form of $\mathrm{SO}_{2p+q}\mathbb{C}$ associated to the PHVS $(\mathrm{GL}_p\mathbb{C}\times\mathrm{SO}_q\mathbb{C}, M_{p,q})$ from Examples 2.1 and 2.5 is $\mathrm{SO}(2p,q)$. Similarly, the real form of $\mathrm{SL}_{p+q+r}\mathbb{C}$ associated to the PHVS $(\mathrm{S}(\mathrm{GL}_p\mathbb{C}\times\mathrm{GL}_q\mathbb{C}\times\mathrm{GL}_r\mathbb{C}), M_{p,q}\oplus M_{q,r})$ is $\mathrm{SU}(p+r,q)$. For the regular (non JM-regular) PHVS from Example 2.15, the associated real form of $\mathrm{SO}_7\mathbb{C}$ is $\mathrm{SO}(3,4)$. For the JM-regular PHVS associated to a principal $\mathfrak{sl}_2$-triple (Example 2.12), the associated real form is the split real form if the split real form is of Hodge type, otherwise it is the quasisplit real form which is nonsplit. For example, for $\mathrm{G}=\mathrm{SL}_{2p}\mathbb{C}$, the split real form is $\mathrm{SL}_{2p}\mathbb{R}$ but the canonical real form associated to the grading is $\mathrm{SU}(p,p)$.

Remark 3.10. Given a compact real form τ, $B(-\tau(x),y)$ defines a nondegenerate positive definite Hermitian inner product on $\mathfrak{g}$. Thus, $\tau(\mathfrak{g}_j)=\mathfrak{g}_{-j}$ for any $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_j\mathfrak{g}_j$.

Let θ be the involution from Proposition 3.8 and τ be a compact real form such that $\sigma=\theta\circ\tau:\mathfrak{g}\to\mathfrak{g}$ is the associated real form of Hodge type. Let $\mathrm{G}^{\mathbb{R}}<\mathrm{G}$ be the associated real form of G and let $\mathrm{H}^{\mathbb{R}}<\mathrm{G}^{\mathbb{R}}$ be the associated maximal compact subgroup. Note that the real form σ restricts to a compact real form on $\mathfrak{g}_0$. Set $\mathrm{H}_0^{\mathbb{R}}=\mathrm{G}_0\cap\mathrm{G}^{\mathbb{R}}$.

Consider the homogeneous space

$$D=\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}.$$

Note that there is a fibration $D\to\mathrm{G}^{\mathbb{R}}/\mathrm{H}^{\mathbb{R}}$ over the Riemannian symmetric space of $\mathrm{G}^{\mathbb{R}}$. In fact, the homogeneous space D has a natural homogeneous complex structure. Indeed, the tangent bundle of $D=\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ is isomorphic to the associated bundle

$$TD\cong\mathrm{G}^{\mathbb{R}}\times_{\mathrm{H}_0^{\mathbb{R}}}\mathfrak{g}^{\mathbb{R}}/\mathfrak{h}_0^{\mathbb{R}}.$$

Let $\mathfrak{g}^{\mathbb{R}}=\mathfrak{h}_0^{\mathbb{R}}\oplus\mathfrak{q}^{\mathbb{R}}$ be an orthogonal decomposition and $\mathfrak{q}=\mathfrak{q}^{\mathbb{R}}\otimes\mathbb{C}$. Then the complexified tangent bundle is $T_{\mathbb{C}}D\cong\mathrm{G}^{\mathbb{R}}\times_{\mathrm{H}_0^{\mathbb{R}}}\mathfrak{q}$, and a complex structure on D is equivalent to an $\mathrm{H}_0^{\mathbb{R}}$-invariant decomposition $\mathfrak{q}=\mathfrak{q}_-\oplus\mathfrak{q}_+$ such that $\sigma(\mathfrak{q}_+)=\mathfrak{q}_-$ and $[\mathfrak{q}_\pm,\mathfrak{q}_\pm]\subset\mathfrak{q}_\pm$. The decomposition $\mathfrak{q}=\mathfrak{q}_-\oplus\mathfrak{q}_+$ is given by setting

$$\mathfrak{q}_+=\bigoplus_{j>0}\mathfrak{g}_j\quad\text{and}\quad\mathfrak{q}_-=\bigoplus_{j<0}\mathfrak{g}_j.$$

The complex manifold D is called a *period domain* for $\mathrm{G}^{\mathbb{R}}$. Note that the holomorphic tangent bundle of a period domain decomposes as

$$T^{1,0}_{\mathbb{C}}D\cong\mathrm{G}^{\mathbb{R}}\times_{\mathrm{H}_0^{\mathbb{R}}}\mathfrak{q}_+=\bigoplus_{j>0}\mathrm{G}^{\mathbb{R}}\times_{\mathrm{H}_0^{\mathbb{R}}}\mathfrak{g}_j.$$

Example 3.11. For the PHVSs from Example 2.13, $\mathrm{G}^{\mathbb{R}}<\mathrm{G}$ is a Hermitian real form and $\mathrm{H}_0^{\mathbb{R}}<\mathrm{G}^{\mathbb{R}}$ is a maximal compact subgroup. Hence, the period domain $D=\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ is the Riemannian symmetric space for $\mathrm{G}^{\mathbb{R}}$. For the PHVS $(\mathrm{G}_0,\mathfrak{g}_1)$ from Example 2.12, $\mathrm{G}_0<\mathrm{G}$ is a Cartan subgroup, $\mathrm{G}^{\mathbb{R}}$ is the quasisplit real form of Hodge type and the period domain is $D=\mathrm{G}^{\mathbb{R}}/\mathrm{U}(1)^{\mathrm{rk}(\mathfrak{g})}$. For the PHVS $(\mathrm{GL}_p\mathbb{C}\times\mathrm{SO}(p,\mathbb{C}),M_{p,q})$ from Examples 2.1 and 3.9, $\mathrm{G}^{\mathbb{R}}=\mathrm{SO}(2p,q)$ and the period domain is $D=\mathrm{SO}(2p,q)/(\mathrm{U}_p\times\mathrm{SO}_q)$.

3.3 The Toledo character and holomorphic sectional curvature

Here we discuss how the choice of the Toledo character is related to a metric of minimal holomorphic sectional curvature -1 on $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$.

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As is well known, one can define an invariant Hermitian metric on $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ by using the Hermitian scalar product

$$\langle x, y\rangle = B(x^*, y) \quad \text{if } x, y \in \mathfrak{g}_k,$$

where $x^* = -\tau(x)$ and τ is the compact conjugation fixed on $\mathfrak{g}$. This does not give a Kähler metric (the corresponding 2-form is not closed) but one can still define the holomorphic sectional curvature K of its Chern connection. We have the well-known formula (see, for example, [CMSP17])

$$K(x) = -\frac{|[x, x^*]|^2}{|x|^4}.$$

We are interested in the function K on $\mathfrak{g}_1$. It is known (and re-proved quickly below) that critical points of K are obtained on elements e such that $\{e^*, h = [e, e^*], e\}$ is an $\mathfrak{sl}_2$-triple. For such e, we have $|[e, e^*]|^2 = \langle [[e, e^*], e], e\rangle = 2|e|^2$, and $|e|^2 = \langle [\zeta, e], e\rangle = \langle \zeta, [e, e^*]\rangle$, therefore

$$K(e) = -\frac{2}{B(\zeta, h)},$$

which is, up to a constant, the inverse of the Toledo rank defined earlier. Then we will prove the following proposition.

Proposition 3.12. *The maximum of K on $\mathfrak{g}_1$ is attained on the open orbit, and is equal to $-2/B(\zeta, h_{\mathrm{reg}})$, where e is in the open orbit and $\{e^*, h_{\mathrm{reg}} = [e, e^*], e\}$ is an $\mathfrak{sl}_2$-triple.*

The minimum of K on $\mathfrak{g}_1$ is attained on a minimal orbit, and is equal to $-B(\gamma, \gamma)$ for a long root $\gamma \in \Delta_1$ (the minimum is attained on e_γ, which belongs to a minimal orbit).

Therefore, after multiplying the Hermitian metric by $B(\gamma, \gamma)$, we obtain a normalized metric with normalized holomorphic sectional curvature

$$-1 \leqslant K_{\mathrm{norm}} \leqslant -\frac{1}{\mathrm{rk}_T(G_0, \mathfrak{g}_1)}.$$

At a general critical point $e \in \mathfrak{g}_1$ of the curvature (therefore $\{e^*, [e, e^*], e\}$ is an $\mathfrak{sl}_2$-triple), we have the normalized value $K_{\mathrm{norm}}(e) = -1/\mathrm{rk}_T(e)$.

From the Toledo character we obtain a $\mathrm{G}^{\mathbb{R}}$-invariant 2-form on $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ by

$$\omega(x, y) = i\chi_T([x, y]_0), \quad \text{for } x, y \in \mathfrak{q}^{\mathbb{R}},$$

where $[x, y]_0$ is the projection of the bracket onto $\mathfrak{g}_0$. This actually defines a pseudo-Hermitian metric, which coincides with the previous normalized metric in the horizontal directions (i.e., in $\mathfrak{g}_1$). This explains the choice of the normalization for the Toledo character.

Proof of Proposition 3.12. For completeness, we begin by proving that the critical points come from $\mathfrak{sl}_2$-triples $\{e^*, [e, e^*], e\}$. Since $K(e)$ is invariant by homothety, we can restrict to variations $\dot{e} \perp e$. Then

$$\dot{K} = -\frac{2}{|e|^4}\langle [e, e^*], [\dot{e}, e^*] + [e, \dot{e}^*]\rangle = -\frac{4}{|e|^4}\Re\langle [[e, e^*], e], \dot{e}\rangle$$

It follows that for a critical point, $[[e, e^*], e] = \lambda e$, and up to renormalizing e one can suppose $\lambda = 2$.

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We will now establish a second variation formula. Start from an $\mathfrak{sl}_2$-triple $(h = [e, e^*], e, e^*)$ and take a vector $x \perp e$ such that $|x| = |e|$. Consider

$$e(t) = \cos(t)e + \sin(t)x = \left(1 - \frac{t^2}{2}\right)e + tx + o(t^2).$$

Then $e(t)$ has constant norm, and, up to order 2,

$$[e(t), e(t)^*] = [e, e^*] + t([x, e^*] + [e, x^*]) + t^2([x, x^*] - [e, e^*]).$$

Therefore, again up to order 2,

$$\begin{aligned} -|e|^4 K(e(t)) &= |[e(t), e(t)^*]|^2 \\ &= |[e, e^*]|^2 + t^2(|[x, e^*] + [e, x^*]|^2 + 2\Re\langle [e, e^*], [x, x^*] - [e, e^*]\rangle). \end{aligned}$$

We have already noticed above that $|[e, e^*]|^2 = 2|e|^2 = 2|x|^2$, and therefore

$$\begin{aligned} \left.\frac{d^2}{dt^2}\right|_{t=0} |e|^4 K(e(t)) &= 2(4|x|^2 - 2\langle [h, x], x\rangle - |[x, e^*] + [e, x^*]|^2) \\ &= 4\langle (2 - \mathrm{ad}_h)x, x\rangle - 2|[x, e^*] + [e, x^*]|^2. \end{aligned} \tag{3.3}$$

We first deduce that the maximum is attained on the open orbit. Observe that e is in the open orbit if and only if $\mathfrak{g}_1 \cap \ker \mathrm{ad}_{e^*} = 0$. Therefore, if e is not in the open orbit, we can take $x \in \ker \mathrm{ad}_{e^*}$. Since $\mathrm{ad}_h \leqslant 0$ on $\ker \mathrm{ad}_{e^*}$, it follows from (3.3) that in the direction of x we have

$$\left.\frac{d^2}{dt^2}\right|_{t=0} |e|^4 K(e(t)) \geqslant 8|x|^2,$$

and therefore the maximum cannot be attained at e. This proves the claim that the maximum is obtained on the open orbit.

To find the minimum, we use the following fact [Man13, Corollary 7, p. 100]. Denote $\mathfrak{u}$ the subspace of $\mathfrak{g}_1$ given as the sum of the eigenspaces of ad_h for the eigenvalues at least 2. Then all orbits in the closure of the orbit of e meet $\mathfrak{u}$. Therefore, suppose that the orbit of e is not minimal. It follows that we can take $x \in \mathfrak{u}$, and (3.3) gives us, in the direction of x,

$$\left.\frac{d^2}{dt^2}\right|_{t=0} |e|^4 K(e(t)) \leqslant -2|[x, e^*] + [e, x^*]|^2.$$

Such x cannot be in the kernel of ad_{e^*} ($\mathrm{ad}_h \leqslant 0$ on this kernel). Therefore, the right-hand side is nonzero, so that the minimum cannot be attained at e. Therefore, the minimum can be attained only at a minimal orbit.

There can be several minimal orbits if the prehomogeneous space is not irreducible. They are the orbits of elements e_γ for $\gamma \in \Delta$ the longest root. As we have seen, $K(e_\gamma) = -2/|e_\gamma|^2 = -4/|h_\gamma|^2 = -B(\gamma, \gamma)$. The result follows. □

Remark 3.13. If we fix an orbit $\mathcal{O}$, then K_{norm} must have a maximum on $\overline{\mathcal{O}}$. It follows from the proof of the proposition, and in particular from the description of the orbits in the closure of $\mathcal{O}$, that the maximum on $\overline{\mathcal{O}}$ is achieved at some element $e \in \mathcal{O}$ such that $(e^*, [e, e^*], e)$ is an $\mathfrak{sl}_2$-triple; it is therefore equal to $K_{\mathrm{norm}}(e) = -1/\mathrm{rk}_T(e)$. In particular, we obtain that for any $x \in \overline{\mathcal{O}}$,

$$-1 \leqslant K_{\mathrm{norm}}(x) \leqslant -\frac{1}{\mathrm{rk}_T(e)}. \tag{3.4}$$

Example 3.14. The case $\mathrm{G} = \mathrm{SL}_n\mathbb{C}$ is particularly simple to calculate. A first observation is that any nilpotent element $e \in \mathfrak{sl}_n\mathbb{C}$ belongs to the open orbit of the nilpotent subalgebra of a

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parabolic algebra of $\mathfrak{sl}_n\mathbb{C}$. It follows that e can be considered as a point in the open orbit of a $\mathfrak{g}_1$ for some grading of $\mathfrak{sl}_n\mathbb{C}$. This observation is not essential for our calculation but relates every nilpotent e to variations of Hodge structures.

We now calculate the corresponding bounds for the holomorphic sectional curvature,

$$-1 \leqslant K \leqslant -\frac{1}{\mathrm{rk}_T\, e},$$

where, by (3.2), we have $\mathrm{rk}_T\, e = \frac{1}{4}B(h,h)B(\gamma,\gamma) = \frac{1}{2}B(h,h)$ since $B(\gamma,\gamma) = 2$ in $\mathfrak{sl}_n$ for the standard invariant form $\mathrm{tr}(XY)$. If we have a Jordan block of size k, the eigenvalues of h are $k-1, k-3, \ldots, -(k-1)$, therefore $B(h,h) = (k-1)^2 + (k-3)^2 + \cdots + (-(k-1))^2 = \frac{1}{3}k(k^2-1)$. Finally, if e has Jordan blocks of size $k_1, k_2, \ldots, k_j$ we obtain

$$\mathrm{rk}_T\, e = \frac{1}{6}\sum_1^j k_i(k_i^2 - 1).$$

This gives immediately some curvature bounds in [Li22].

Proposition 3.15. *Let $e \in \mathfrak{g}_1$ and let $\Omega \subset \mathfrak{g}_1$ be the open G_0-orbit of the PHVS $(\mathrm{G}_0, \mathfrak{g}_1)$. Then*

$$0 \leqslant \mathrm{rk}_T(e) \leqslant \mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1),$$

with equality if and only if $e \in \Omega$.

Proof. We give an indirect proof applying Proposition 3.12 and Remark 3.13. The Toledo rank is related to the normalized holomorphic sectional curvature K_{norm} of the corresponding period domain by $K_{\mathrm{norm}}(e) = -1/\mathrm{rk}_T(e)$ if (f,h,e) is a *real* $\mathfrak{sl}_2$-triple (in a given orbit one can always find an e which is part of such a triple; see Remark 3.13). Then it is proved that $K_{\mathrm{norm}}(e) \leqslant K_{\mathrm{norm}}(e')$ if the orbit of e is included in the closure of the orbit of e'. Moreover, the maximum of K_{norm} is attained only on the open orbit. The proposition follows. □

We end this section with a proposition which will be used often in subsequent sections.

Proposition 3.16. *If $(\mathrm{G}_0, \mathfrak{g}_1)$ is a JM-regular PHVS, then there is positive integer multiple $q \cdot \chi_T$ of the Toledo character which exponentiates to a character $\chi_{T,q} : \mathrm{G}_0 \to \mathbb{C}^*$ which has a polynomial relative invariant of degree $q \cdot \mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)$.*

Proof. By Proposition 2.8, there is a positive integer multiple $q \cdot \chi_T$ of the Toledo character which exponentiates to a character $\chi_{T,q} : \mathrm{G}_0 \to \mathbb{C}^*$ which has a relative invariant F of degree $q \cdot \mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)$. We prove that F is a polynomial by proving that it attains a finite value on any element $e' \in \mathfrak{g}_1$.

First note that if $e' = 0$, then (2.2) implies $F(0) = 0$. For $e' \neq 0$, complete e' into an $\mathfrak{sl}_2$-triple $\{f', h', e'\}$ with $f' \in \mathfrak{g}_{-1}$ and $h' \in \mathfrak{g}_0$. Then $e' + \ker(\mathrm{ad}_{f'})$ cuts all other G-orbits, but this is not true in general for G_0-orbits. Nevertheless, it remains true that $e' + \ker(\mathrm{ad}_{f'})$ cuts the open orbit, because $\mathfrak{g}_0 \cdot e' + (\ker(\mathrm{ad}_{f'}) \cap \mathfrak{g}_1) = \mathfrak{g}_1$, which implies that $\mathrm{G}_0 \cdot (e' + \ker(\mathrm{ad}_{f'}))$ contains an open neighborhood of e'. So up to conjugating by some element of G_0 we can suppose that $e = e' + X$ with $[f', X] = 0$. Since X belongs to the nonpositive eigenspaces of $\mathrm{ad}_{h'}$, it follows that $e' = \lim_{t\to+\infty} e^{t(h'-h)} \cdot e$. Therefore,

$$F(e') = \lim_{t\to+\infty} \chi_{h,q}(e^{t(h'-h)})F(e) = \lim_{t\to+\infty} e^{tq\chi_h(h'-h)}F(e).$$

This is finite if $\chi_h(h'-h) \leqslant 0$, that is, if $B(h,h') \leqslant B(h,h)$. Since $h' = [e', f']$ and $[h, e'] = 2e'$, we have $B(h,h') = 2B(e',f')$ and $B(h,h) = 2B(e,f)$, so the inequality to prove is

$$B(e', f') \leqslant B(e, f).$$

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As before, we may use the Hermitian scalar product from the associate real form and assume that $(e')^* = f'$. We claim that we can also assume $e^* = f$ while keeping the condition $e \in e' + \ker(\mathrm{ad}_{f'})$. Indeed, since $[f', X] = 0$ we have $[e, e^*] = [e', f'] + [X, X^*]$ so we want $[X, X^*] = h - h'$; but $h - h'$ centralizes the triple $\{e', f', h'\}$ and preserves the slice $e' + \ker(\mathrm{ad}_{f'})$. Thus, it is sufficient to conjugate e by an element of the centralizer of $\{e', f', h'\}$ in G_0 to obtain $[e, e^*] = h$.

Then $\langle e', X\rangle = \frac{1}{2}\langle [h', e'], X\rangle = \frac{1}{2}\langle h', [X, f']\rangle = 0$. It follows that $B(e', f') = |e'|^2$ and $B(e, f) = |e|^2 = |e'|^2 + |X|^2 \geqslant B(e', f')$. The proposition follows. □

4. Higgs bundles and variations of Hodge structure

For this section let X be a compact Riemann surface of genus $g \geqslant 2$ and let K be its canonical bundle.

4.1 Higgs bundles and Hodge bundles

Let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$ and nondegenerate G-invariant bilinear $\langle \cdot, \cdot \rangle$. Let $\rho : \mathrm{G} \to \mathrm{GL}(V)$ be a holomorphic representation. If $E \to X$ is a G-bundle, we will denote the V-bundle $E \times_{\mathrm{G}} V$ associated to E via the representation ρ by $E(V)$.

Definition 4.1. A (G, V)-Higgs pair is a pair (E, φ) where E is a holomorphic principal G-bundle on X and φ is a holomorphic section of $E(V) \otimes K$.

When $V = \mathfrak{g}$ and the representation ρ is the adjoint representation, a $(\mathrm{G}, \mathfrak{g})$-Higgs pair is called a G-Higgs bundle. Suppose $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ is a real form with complexified maximal compact $\mathrm{H} < \mathrm{G}$ and complexified Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. When $\rho : \mathrm{H} \to \mathrm{GL}(\mathfrak{m})$ is the restriction of the adjoint representation of G, then an $(\mathrm{H}, \mathfrak{m})$-Higgs pair is called a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle.

Remark 4.2. When $\mathrm{G}^{\mathbb{R}}$ is compact, a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle is just a holomorphic G-bundle. When $\mathrm{G}^{\mathbb{R}} = \mathrm{G}$ (viewed as a real form of $\mathrm{G} \times \mathrm{G}$), a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle is a G-Higgs bundle.

If E is a principal G-bundle and $\hat{\mathrm{G}} < \mathrm{G}$ is a subgroup, then a structure group reduction of E to $\hat{\mathrm{G}}$ is a section σ of the bundle $E(\mathrm{G}/\hat{\mathrm{G}})$. Associated to such a reduction is a principal $\hat{\mathrm{G}}$-subbundle $E_\sigma \subset E$ such that $E_\sigma(\mathrm{G})$ is canonically isomorphic to E.

Definition 4.3. Let G be a complex reductive Lie group and $\rho : \mathrm{G} \to \mathrm{GL}(V)$ be a holomorphic representation. Let $\hat{\mathrm{G}} < \mathrm{G}$ and $\hat{V} \subset V$ be a $\rho(\hat{\mathrm{G}})$-invariant subspace. We say that a (G, V)-Higgs pair (E, φ) reduces to a $(\hat{\mathrm{G}}, \hat{V})$-Higgs pair, if there is a holomorphic reduction $E_{\hat{\mathrm{G}}}$ of E to $\hat{\mathrm{G}}$ such that $\varphi \in H^0(E_{\hat{\mathrm{G}}}(\hat{V}) \otimes K) \subset H^0(E_{\hat{\mathrm{G}}}(V) \otimes K)$.

For example, if $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ is a real form, $\mathrm{H} < \mathrm{G}$ is the complexification of a maximal compact of $\mathrm{G}^{\mathbb{R}}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is complexified Cartan decomposition, then a G-Higgs bundle (E, φ) reduces to a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle if there is a holomorphic reduction E_{H} of E to H such that $\varphi \in H^0(E_{\mathrm{H}}(\mathfrak{m}) \otimes K)$.

Example 4.4. Let $\{f, h, e\}$ be a basis for $\mathfrak{sl}_2\mathbb{C}$, and let $\mathrm{T} < \mathrm{PSL}_2\mathbb{C}$ be the subgroup with Lie algebra $\langle h \rangle$. Note that $\mathrm{T} \cong \mathbb{C}^*$ and the adjoint action of T on $\langle e \rangle$ is given by $\lambda \cdot e = \lambda e$. So, if E_{T} is the frame bundle of K^{-1}, then the associated bundle $E_{\mathrm{T}}(\langle e \rangle) \otimes K \cong \mathcal{O}_X$. Hence, we have a $\mathrm{PSL}_2\mathbb{C}$-Higgs bundle

$$(E_{\mathrm{T}}(\mathrm{PSL}_2\mathbb{C}), e).$$

Since $\deg(K)$ is even, this example can be lifted to $\mathrm{SL}_2\mathbb{C}$. The lifted action of the subgroup $\mathbb{C}^* \cong \hat{\mathrm{T}} < \mathrm{SL}_2\mathbb{C}$ with Lie algebra $\langle h \rangle$ is given by $\lambda \cdot e = \lambda^2 e$. As a result, if $E_{\hat{\mathrm{T}}}$ is the frame

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bundle of a square root $K^{-1/2}$ of K^{-1}, then $(E_{\hat{\mathrm{T}}}(\mathrm{SL}_2\mathbb{C}), e)$ defines an $\mathrm{SL}_2\mathbb{C}$-Higgs bundle. These Higgs bundles will be the fundamental building blocks for the results of § 6.

Now suppose G is a complex reductive Lie group and $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ is a $\mathbb{Z}$-grading of its Lie algebra with grading element $\zeta \in \mathfrak{g}_0$. Let $\mathrm{G}_0 < \mathrm{G}$ be the centralizer of ζ. Note that $\exp(\lambda\zeta)$ is in the center of G_0 and $\mathrm{Ad}(\exp(\lambda\zeta))$ acts on each $\mathfrak{g}_j$ by $\lambda^j \cdot \mathrm{Id}$. Let $(E_{\mathrm{G}_0}, \varphi)$ be a $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pair. Note that the central element $\exp((\lambda/k)\zeta) \in \mathrm{G}_0$ defines a holomorphic automorphism of E_{G_0} and acts on φ by multiplication by λ. As a result we have an isomorphism of $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pairs

$$(E_{\mathrm{G}_0}, \varphi) \cong (E_{\mathrm{G}_0}, \lambda\varphi) \quad \text{for all } \lambda \in \mathbb{C}^*. \tag{4.1}$$

Extending the structure group defines a G-Higgs bundle $(E_{\mathrm{G}}, \varphi)$,

$$(E_{\mathrm{G}}, \varphi) = (E_{\mathrm{G}_0}(\mathrm{G}), \varphi),$$

since $E_{\mathrm{G}_0}(\mathfrak{g}_k) \subset E_{\mathrm{G}_0}[\mathfrak{g}] \cong E_{\mathrm{G}}(\mathfrak{g})$. Moreover, $(E_{\mathrm{G}}, \varphi) \cong (E_{\mathrm{G}}, \lambda\varphi)$ for all $\lambda \in \mathbb{C}^*$.

Definition 4.5. A G-Higgs bundle (E, φ) is called a *Hodge bundle of type* $(\mathrm{G}_0, \mathfrak{g}_k)$ if it reduces to a $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pair.

Example 4.6. The Higgs bundles from Example 4.4 are Hodge bundles for the grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \langle f\rangle \oplus \langle h\rangle \oplus \langle e\rangle$. In these cases, G_0 is a Cartan subgroup of G.

Given a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$, we can define a subalgebra $\tilde{\mathfrak{g}}$ consisting of summands $\mathfrak{g}_j$ with $j = 0 \bmod k$. Note that $\tilde{\mathfrak{g}}$ has an associated $\mathbb{Z}$-grading with $\mathfrak{g}_k = \tilde{\mathfrak{g}}_1$. Let $\tilde{\mathrm{G}} < \mathrm{G}$ be the associated reductive subgroup. The following proposition is immediate.

Proposition 4.7. *Let (E, φ) be a G-Higgs bundle which is a Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_k)$. Then (E, φ) reduces to a $\tilde{\mathrm{G}}$-Higgs bundle and, as a $\tilde{\mathrm{G}}$-Higgs bundle, it is a Hodge bundle of type $(\mathrm{G}_0, \tilde{\mathfrak{g}}_1)$.*

As a result, we will usually consider Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$. Recall from Proposition 3.8 that there is a canonical real form $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ of Hodge type associated to a $\mathbb{Z}$-grading. The complexified Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfies $\mathfrak{g}_{2j} \subset \mathfrak{h}$ and $\mathfrak{g}_{2j+1} \subset \mathfrak{m}$, in particular $\mathrm{G}_0 < \mathrm{H}$. As a result, a G-Higgs bundle which is a Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_1)$ reduces to a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle. Combining this observation with Proposition 4.7 gives the following result which was first observed by Simpson [Sim92, § 4].

Proposition 4.8. *Let (E, φ) be a G-Higgs bundle which is a Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_k)$. Then (E, φ)-reduces to a $\tilde{\mathrm{G}}^{\mathbb{R}}$-Higgs bundle, where $\tilde{\mathrm{G}}^{\mathbb{R}} < \mathrm{G}$ is a real group of Hodge type. In fact, $\tilde{\mathrm{G}}^{\mathbb{R}}$ is the real form of the subgroup $\tilde{\mathrm{G}} < \mathrm{G}$ from Proposition 4.7 associated to the grading of $\tilde{\mathfrak{g}}$.*

Example 4.9. The Hodge bundles in Example 4.4 thus reduce to the real forms $\mathrm{PSL}_2\mathbb{R} < \mathrm{PSL}_2\mathbb{C}$ and $\mathrm{SL}_2\mathbb{R} < \mathrm{SL}_2\mathbb{C}$.

The above results also apply to $\mathrm{G}^{\mathbb{R}}$-Higgs bundles. Namely, fix a real form $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ with a maximal compact subgroup $\mathrm{H}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$. Let $\mathrm{H} < \mathrm{G}$ be the complexification of $\mathrm{H}^{\mathbb{R}}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a complexified Cartan decomposition. Consider a $\mathbb{Z}$-grading of $\mathfrak{g}$ given by $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{h}_j \oplus \mathfrak{m}_j$ with grading element $\zeta \in \mathfrak{h}_0$. Let $\mathrm{H}_0 < \mathrm{H}$ be the centralizer of ζ. Given an $(\mathrm{H}_0, \mathfrak{m}_k)$-Higgs pair $(E_{\mathrm{H}_0}, \varphi)$, extending the structure group to H defines a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle $(E_{\mathrm{H}_0}(\mathrm{H}), \varphi)$ such that $(E_{\mathrm{H}_0}(\mathrm{H}), \varphi) \cong (E_{\mathrm{H}_0}(\mathrm{H}), \lambda\varphi)$ for all $\lambda \in \mathbb{C}^*$.

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A $\mathrm{G}^{\mathbb{R}}$-Higgs bundle (E,φ) is called a *Hodge bundle of type* $(\mathrm{H}_0,\mathfrak{m}_k)$ if it reduces to a $(\mathrm{H}_0,\mathfrak{m}_k)$-Higgs pair. Consider the subalgebra $\tilde{\mathfrak{g}}=\tilde{\mathfrak{h}}\oplus\tilde{\mathfrak{m}}$ given by

$$\tilde{\mathfrak{h}}=\bigoplus_{j=0 \bmod 2k}\mathfrak{h}_j \quad\text{and}\quad \tilde{\mathfrak{m}}=\bigoplus_{j=k \bmod 2k}\mathfrak{m}_j.$$

Let $\tilde{\mathfrak{g}}^{\mathbb{R}}\subset\tilde{\mathfrak{g}}$ be the associated real form of Hodge type with complexified Cartan decomposition $\tilde{\mathfrak{g}}=\tilde{\mathfrak{h}}\oplus\tilde{\mathfrak{m}}$. As in the complex case, we have the following result.

Proposition 4.10. *Let (E,φ) be a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle which is a Hodge bundle of type $(\mathrm{H}_0,\mathfrak{m}_k)$. Then there is a reductive subgroup $\tilde{\mathrm{G}}^{\mathbb{R}}<\mathrm{G}^{\mathbb{R}}$ of Hodge type such that (E,φ) reduces to a $\tilde{\mathrm{G}}^{\mathbb{R}}$-Higgs bundle. Moreover, the resulting $\tilde{\mathrm{G}}^{\mathbb{R}}$-Higgs bundle is a $(\mathrm{H}_0,\tilde{\mathfrak{m}}_1)=(\tilde{\mathrm{G}}_0,\tilde{\mathfrak{g}}_1)$-Hodge bundle.*

As a result, we will mostly consider Hodge bundles of type $(\mathrm{G}_0,\mathfrak{g}_1)$.

4.2 Moduli spaces and fixed points

To form a moduli space of Higgs pairs, we need to define suitable notions of stability. We describe this below and refer to [GGM09, BGM03] for more details. Let G be a complex reductive Lie group and $\rho:\mathrm{G}\to\mathrm{GL}(V)$ be a holomorphic representation. Fix a maximal compact subgroup $\mathrm{K}^{\mathbb{R}}<\mathrm{G}$ and let $\mathfrak{k}^{\mathbb{R}}$ be its Lie algebra. An element $s\in i\mathfrak{k}^{\mathbb{R}}$ defines subspaces of V via the representation ρ:

$$V_s^0=\{v\in V\,|\,\rho(e^{ts})v=v\}\quad\text{and}\quad V_s=\{v\in V\,|\,\rho(e^{ts})(v)\text{ is bounded as }t\to\infty\}.$$

When $\rho:\mathrm{G}\to\mathrm{GL}(\mathfrak{g})$ is the adjoint representation, $\mathfrak{g}_s=\mathfrak{p}_s\subset\mathfrak{g}$ is a parabolic subalgebra with Levi subalgebra $\mathfrak{g}_s^0=\mathfrak{l}_s\subset\mathfrak{p}_s$. The associated subgroups $\mathrm{L}_s<\mathrm{P}_s$ are given by

$$\mathrm{L}_s=\{g\in\mathrm{G}\,|\,\mathrm{Ad}(g)s=s\}\quad\text{and}\quad\mathrm{P}_s=\{g\in\mathrm{G}\,|\,\mathrm{Ad}(e^{ts})(g)\text{ is bounded as }t\to\infty\}.$$

Moreover, s defines a character $\chi_s:\mathfrak{p}_s\to\mathbb{C}$ by

$$\chi_s(x)=\langle s,x\rangle\quad\text{for } x\in\mathfrak{p}_s.$$

Let E be a G-bundle, $s\in i\mathfrak{k}^{\mathbb{R}}$ and $\mathrm{P}_s<\mathrm{G}$ be the associated parabolic subgroup. A reduction of structure group of E to P_s is a P_s-subbundle $E_{\mathrm{P}_s}\subset E$; this is equivalent to a section $\sigma\in\Gamma(E(\mathrm{G}/\mathrm{P}_s))$ of the associated bundle. We will denote the associated P_s-subbundle by E_σ. The degree of such a reduction will be defined using Chern–Weil theory. Since P_s is homotopy equivalent to the maximal compact $\mathrm{K}_s^{\mathbb{R}}=\mathrm{K}^{\mathbb{R}}\cap\mathrm{L}_s$ of L_s, given a reduction of structure σ of E to P_s, there is a further reduction of E to $\mathrm{K}_s^{\mathbb{R}}$ which is unique up to homotopy. Let $E_{\sigma'}\subset E$ be the resulting $\mathrm{K}_s^{\mathbb{R}}$ principal bundle. The curvature F_A of a connection A on $E_{\sigma'}$ satisfies $F_A\in\Omega^2(X,E_{\sigma'}(\mathfrak{k}_s^{\mathbb{R}}))$. Thus, evaluating the character χ_s on the curvature, we have $\chi_s(F_A)\in\Omega(X,i\mathbb{R})$, and we define the degree of σ as

$$\deg E(\sigma,s)=\frac{i}{2\pi}\int_X\chi_s(F_A).\tag{4.2}$$

If a multiple of $q\cdot\chi_s$ exponentiates to a character $\tilde{\chi}_s:\mathrm{P}_s\to\mathbb{C}^*$ and $\sigma\in\Gamma(E(\mathrm{G}/\mathrm{P}_s))$ is a reduction, then $E_\sigma\times_{\chi_s}\mathbb{C}^*=E_\sigma(\tilde{\chi}_s)$ is a line bundle and $\deg E(\sigma,s)=(1/q)\deg E_\sigma(\tilde{\chi}_s)$. When s is in the center $\mathfrak{z}$ of $\mathfrak{g}$ then $\mathrm{P}_s=\mathrm{G}$. In this case, the degree given in (4.2) is simply the degree of E with respect to χ_s, and will be denoted by $\deg_\chi(E)$. Again, if a multiple $q\cdot\chi_s$ exponentiates to a character $\tilde{\chi}:\mathrm{G}\to\mathbb{C}^*$ we have

$$\deg_\chi(E)=\frac{1}{q}\deg E(\tilde{\chi}).\tag{4.3}$$

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Let $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the differential of ρ and let $\mathfrak{z}^{\mathbb{R}}$ be the center of $\mathfrak{k}^{\mathbb{R}}$ and $\mathfrak{k}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}}_{ss} = \mathfrak{z}^{\perp}$. Define

$$\mathfrak{k}^{\mathbb{R}}_{\rho} = \mathfrak{k}^{\mathbb{R}}_{ss} \oplus (\ker(d\rho|_{\mathfrak{z}^{\mathbb{R}}}))^{\perp}.$$

We are now ready to define α-stability notions for $\alpha \in i\mathfrak{z}^{\mathbb{R}}$.

Definition 4.11. Let $\alpha \in i\mathfrak{z}^{\mathbb{R}}$. A (G, V)-Higgs pair (E, φ) is:

- *α-semistable* if for any $s \in i\mathfrak{k}^{\mathbb{R}}$ and any holomorphic reduction $\sigma \in H^0(E(\mathrm{G}/\mathrm{P}_s))$ such that $\varphi \in H^0(E_\sigma(V_s) \otimes K)$, we have $\deg E(\sigma, s) \geqslant \langle \alpha, s \rangle$;
- *α-stable* if it is α-semistable and for any $s \in i\mathfrak{k}^{\mathbb{R}}_{\rho}$ and any holomorphic reduction $\sigma \in H^0(E(\mathrm{G}/\mathrm{P}_s)$ such that $\varphi \in H^0(E_\sigma(V_s) \otimes K)$, we have $\deg E(\sigma, s) > \langle \alpha, s \rangle$;
- *α-polystable* if it is α-semistable and whenever $s \in i\mathfrak{k}^{\mathbb{R}}$ and $\sigma \in H^0(E(\mathrm{G}/\mathrm{P}_s))$ satisfy $\varphi \in H^0(E_\sigma(V_s) \otimes K)$ and $\deg E(\sigma, s) = \langle \alpha, s \rangle$, there is a further holomorphic reduction $\sigma' \in H^0(E_\sigma(\mathrm{P}_s/\mathrm{L}_s))$ such that $\varphi \in H^0(E_{\sigma'}(V_s^0) \otimes K)$.

The *moduli space of α-polystable (G, V)-Higgs pairs* over X is defined as the set of isomorphism classes of α-polystable (G, V)-Higgs pairs and will be denoted by $\mathcal{M}^{\alpha}(\mathrm{G}, V)$. A GIT construction of these spaces is given by Schmitt in [Sch08] and by Simpson for the moduli space of 0-polystable G-Higgs bundles [Sim94].

Remark 4.12. When $\alpha = 0$, we refer to 0-stability simply as stability (similarly for semistability and polystability), and denote the moduli space by $\mathcal{M}(\mathrm{G}, V)$. The *moduli space of polystable* G*-Higgs bundles* will be denoted by $\mathcal{M}(\mathrm{G})$, and, for a real form $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$, the *moduli space of polystable* $\mathrm{G}^{\mathbb{R}}$*-Higgs bundles* will be denoted by $\mathcal{M}(\mathrm{G}^{\mathbb{R}})$.

There is a natural $\mathbb{C}^*$-action on the moduli spaces of α-polystable Higgs pairs given by $\lambda \cdot (E, \varphi) = (E, \lambda\varphi)$. If $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is a $\mathbb{Z}$-grading, then the $\mathbb{C}^*$-action is trivial on the moduli space of $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pairs by (4.1). As a result, polystable Higgs bundles which are Hodge bundles define fixed points of the $\mathbb{C}^*$-action on the moduli space of Higgs bundles. Simpson proved the converse [Sim88, Sim92]. Namely, all $\mathbb{C}^*$-fixed points in the moduli space of Higgs bundles are Hodge bundles. In fact, if a Higgs bundle is a Hodge bundle, then polystability of the Higgs bundle is equivalent to polystability of the associated pair. To explain why this is true, we need to use the correspondence between stability and solutions to gauge-theoretic equations.

We first describe this for $\mathrm{G}^{\mathbb{R}}$-Higgs bundles. Fix a maximal compact subgroup $\mathrm{K}^{\mathbb{R}} < \mathrm{G}$ and let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the resulting conjugate linear involution. Now fix a real form $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ such that $\mathrm{K}^{\mathbb{R}} \cap \mathrm{G}^{\mathbb{R}} = \mathrm{H}^{\mathbb{R}}$ is a maximal compact subgroup of $\mathrm{G}^{\mathbb{R}}$ and let $\mathrm{H} < \mathrm{G}$ be the complexification of $\mathrm{H}^{\mathbb{R}}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the complexified Cartan decomposition of the real form. Consider a $\mathrm{G}^{\mathbb{R}}$-Higgs bundle (E, φ). A metric on E is by definition a structure group reduction $h \in \Gamma(E(\mathrm{H}/\mathrm{H}^{\mathbb{R}}))$ of E to $\mathrm{H}^{\mathbb{R}}$. Associated to a metric h, there is a unique connection A_h (the Chern connection) which is compatible with the reduction and the holomorphic structure. Let E_h be the $\mathrm{H}^{\mathbb{R}}$-subbundle associated to a metric h. Then $E(\mathfrak{m})$ is canonically identified with $E_h(\mathfrak{m})$. The compact real form τ defines an anti-holomorphic involution of $E_h(\mathfrak{m})$. Combining this with conjugation of 1-forms defines an involution

$$\tau : \Omega^{0,1}(E_h(\mathfrak{m})) \to \Omega^{0,1}(E_h(\mathfrak{m})).$$

The Higgs field $\varphi \in H^0(E(\mathfrak{m}) \otimes K)$ defines a $(1,0)$-form valued in $E(\mathfrak{m})$. Thus $[\varphi, -\tau(\varphi)]$ defines a 2-form valued in $E_h[\mathfrak{h}^{\mathbb{R}}]$, and we can make sense of the equation

$$F_h + [\varphi, -\tau(\varphi)] = 0, \tag{4.4}$$

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where F_h is the curvature of the Chern connection. These are the *Hitchin equations*. The following theorem relates solutions of the Hitchin equations with stability; it was proven by Hitchin [Hit87] for $\mathrm{SL}_2\mathbb{C}$ and by Simpson [Sim88] for complex reductive Lie groups (see [GGM09] for the general statement).

Theorem 4.13. *A $\mathrm{G}^{\mathbb{R}}$-Higgs bundle (E,φ) is polystable if and only if there exists a metric h on E which solves the Hitchin equations* (4.4).

Remark 4.14. When the group $\mathrm{G}^{\mathbb{R}}$ is compact, then the Hitchin equations are just $F_h = 0$. That is, there is a flat metric on the bundle, and the classical results of Narasimhan and Seshadri [NS65] and of Ramanathan [Ram75].

For general (G,V)-Higgs pairs, we fix a maximal compact $\mathrm{K}^{\mathbb{R}} < \mathrm{G}$ and a $\mathrm{K}^{\mathbb{R}}$ invariant Hermitian inner product on V. Now the action of $\mathrm{K}^{\mathbb{R}}$ on V is Hamiltonian and has an associated moment map $\mu : V \to (\mathfrak{k}^{\mathbb{R}})^* \to \mathfrak{k}^{\mathbb{R}}$, where we use the inner product on $\mathfrak{k}^{\mathbb{R}}$ to identify $(\mathfrak{k}^{\mathbb{R}})^*$ with $\mathfrak{k}^{\mathbb{R}}$. If $\varphi \in \Omega^{1,0}(E(V))$, then one can define a bundle version of the moment map so that $\mu(\varphi) \in \Omega^{1,1}(E(\mathfrak{k}^{\mathbb{R}}))$, and polystability of (G,V)-Higgs pairs is equivalent the existence of a metric h solving the equation $F_h + \mu(\varphi) = 0$ (see [GGM09] for more details). For G-Higgs bundles, we fix a compact real form $\mathrm{K}^{\mathbb{R}} < \mathrm{G}$ and let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the associated involution. The Hermitian inner product on $\mathfrak{g}$ is given by $B(\cdot, -\tau(\cdot))$ and the moment map $\mu_{\mathrm{G}} : \mathfrak{g} \to \mathfrak{k}^{\mathbb{R}}$ is given by $\mu_{\mathrm{G}}(x) = [x, -\tau(x)]$.

For $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pairs, we choose a compact real form $\mathrm{K}^{\mathbb{R}} < \mathrm{G}$ such that $\tau|_{\mathfrak{g}_0} = \mathrm{Id}$. Recall from Remark 3.10 that $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for all j. As a result, $\mathrm{K}_0^{\mathbb{R}} = \mathrm{G}_0 \cap \mathrm{K}^{\mathbb{R}}$ is a compact form of G_0 and $B(x, -\tau(y))$ defines a $\mathrm{K}_0^{\mathbb{R}}$-invariant Hermitian form on $\mathfrak{g}_k$. The moment map $\mu_{\mathrm{G}_0} : \mathfrak{g}_k \to \mathfrak{k}_0^{\mathbb{R}}$ is then given by restricting the moment map $\mu_{\mathrm{G}} : \mathfrak{g} \to \mathfrak{k}^{\mathbb{R}}$ and orthogonally projecting onto $\mathfrak{k}_0^{\mathbb{R}}$. That is, $\mu_{\mathrm{G}_0}(x)$ is the orthogonal projection of $[x, -\tau(x)]$ onto $\mathfrak{k}_0^{\mathbb{R}}$. But for $x \in \mathfrak{g}_k$, orthogonal projection is unnecessary since $[x, -\tau(x)] \in \mathfrak{k}_0^{\mathbb{R}}$. Thus, a $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pair (E,φ) is polystable if and only if there is a metric $h \in \Gamma(E/\mathrm{K}_0^{\mathbb{R}})$ such that $F_h + [\varphi, -\tau(\varphi)] = 0$. Such a metric solves the Hitchin equations (4.4) for the associated G-Higgs bundle $(E(\mathrm{G}),\varphi)$. Conversely, using an averaging argument, Simpson [Sim88] showed that, if (E,φ) is a polystable G-Higgs bundle which is a Hodge bundle, then the metric solving the Hitchin equations is compatible with the Hodge bundle reduction. We summarize this in a proposition.

Proposition 4.15 [Sim88]. *A $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pair (E,φ) is polystable as a $(\mathrm{G}_0, \mathfrak{g}_k)$-Higgs pair if and only if the associated* G*-Higgs bundle $(E(\mathrm{G}),\varphi)$ is polystable as a* G*-Higgs bundle. In particular, there is a well-defined map of moduli spaces*

$$\mathcal{M}(\mathrm{G}_0, \mathfrak{g}_k) \to \mathcal{M}(\mathrm{G})$$

whose image consists of $\mathbb{C}^*$-fixed points, and every $\mathbb{C}^*$-fixed point in $\mathcal{M}(\mathrm{G})$ is in the image of such a map for some $(\mathrm{G}_0, \mathfrak{g}_k)$.

Remark 4.16. For stable and simple Higgs bundles, the associated type of Hodge bundle is unique.

Remark 4.17. Note that for gradings $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $(\mathrm{G}_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}_1)$-pair stability agrees with Higgs bundle stability. This is not true for other gradings. Namely, $(\mathrm{G}_0, \bigoplus_{j\neq 0} \mathfrak{g}_j)$ or $(\mathrm{G}_0, \mathfrak{g}_{-1} \oplus \bigoplus_{j>0} \mathfrak{g}_j)$-pair stability does not imply Higgs bundle stability.

Example 4.18. The $\mathrm{PSL}_2\mathbb{C}$-Higgs bundle (E_{T}, e) from Example 4.4 is polystable. Since E_{T} is the frame bundle of K^{-1}, a metric on E_{T} defines a metric on the Riemann surface X. A solution to the

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Hitchin equations in this case is equivalent to a metric of constant curvature on X (see [Hit87]). As a result, the Higgs bundle (E_{T}, e) will be referred to as the uniformizing Higgs bundle for X.

4.3 Character varieties and variations of Hodge structure

Given a polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle (E, φ), there is a metric h on E solving the Hitchin equations (4.4). For such a metric h, the connection

$$D = A_h + \varphi - \tau(\varphi)$$

defines a flat connection on the $\mathrm{G}^{\mathbb{R}}$-bundle $E_h(\mathrm{G}^{\mathbb{R}})$, where E_h is the $\mathrm{H}^{\mathbb{R}}$-bundle associated to the metric h. As a result, a polystable $\mathrm{G}^{\mathbb{R}}$-Higgs bundle (E, φ) defines representations $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$ such that $E_h(\mathrm{G}^{\mathbb{R}}) \cong \widetilde{X} \times_\rho \mathrm{G}^{\mathbb{R}}$.

Given a representation $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$, a metric h_ρ on a flat bundle $\widetilde{X} \times_\rho \mathrm{G}^{\mathbb{R}}$ can be interpreted as a ρ-equivariant map to the Riemannian symmetric space of $\mathrm{G}^{\mathbb{R}}$:

$$h_\rho : \widetilde{X} \to \mathrm{G}^{\mathbb{R}}/\mathrm{H}^{\mathbb{R}}.$$

A metric h_ρ is called harmonic if it is a critical point of the energy function

$$\mathcal{E}(h_\rho) = \frac{1}{2}\int_X |dh_\rho|^2.$$

This makes sense since, for two-dimensional domains, the energy only depends on the conformal structure of the domain. It turns out that a metric h solves the Hitchin equations (4.4) if and only if the ρ-equivariant map $h_\rho : \tilde{X} \to \mathrm{G}^{\mathbb{R}}/\mathrm{H}^{\mathbb{R}}$ is harmonic.

Remark 4.19. In this correspondence, the Higgs field φ is identified with the $(1,0)$-part of the differential of the map h_ρ. As a result, for the uniformizing Higgs bundle (E_{T}, e), the harmonic metric $h_\rho : \widetilde{X} \to \mathbb{H}^2$ is a ρ-equivariant biholomorphism. Thus, $X = \mathbb{H}^2/\rho(\pi_1(X))$ and ρ is the uniformizing representation of the Riemann surface.

A representation $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$ is called reductive if postcomposing ρ with the adjoint representation of $\mathrm{G}^{\mathbb{R}}$ decomposes as a direct sum of irreducible representations. Corlette's theorem [Cor88] (proven by Donaldson [Don87] for $\mathrm{SL}_2\mathbb{C}$) asserts that given a representation $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$, there exists a ρ-equivariant harmonic map $h_\rho : \widetilde{X} \to \mathrm{G}^{\mathbb{R}}/\mathrm{H}^{\mathbb{R}}$ if and only if ρ is reductive. Denote the set of reductive representations $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$ by $\mathrm{Hom}^+(\pi_1(X), \mathrm{G}^{\mathbb{R}})$. The moduli space of $\mathrm{G}^{\mathbb{R}}$-conjugacy classes of representations $\pi_1(X)$ in $\mathrm{G}^{\mathbb{R}}$ is called the *character variety* and denoted by

$$\mathcal{R}(\mathrm{G}^{\mathbb{R}}) = \mathrm{Hom}^+(\pi_1(X), \mathrm{G}^{\mathbb{R}})/\mathrm{G}^{\mathbb{R}}.$$

Combining Corlette's theorem with the Hitchin–Kobayashi correspondence defines a homeomorphism $\mathcal{M}(\mathrm{G}^{\mathbb{R}}) \cong \mathcal{R}(\mathrm{G}^{\mathbb{R}})$ between the moduli space of $\mathrm{G}^{\mathbb{R}}$-Higgs bundles on X and the $\mathrm{G}^{\mathbb{R}}$-character variety. This is called the *nonabelian Hodge correspondence*.

Let $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$ be a representation and $\tilde{\mathrm{G}}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$ be a reductive subgroup. We say that ρ factors through $\tilde{\mathrm{G}}^{\mathbb{R}}$ if ρ can be written as $\rho : \pi_1(X) \to \tilde{\mathrm{G}}^{\mathbb{R}} \hookrightarrow \mathrm{G}^{\mathbb{R}}$. The following proposition is immediate from the nonabelian Hodge correspondence.

Proposition 4.20. *Let $\tilde{\mathrm{G}}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$ be a reductive subgroup with maximal compact $\tilde{\mathrm{H}}^{\mathbb{R}} < \mathrm{H}^{\mathbb{R}}$ and let $\rho : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}$ be a reductive representation. The following assertions are equivalent.*

- *ρ factors through $\tilde{\mathrm{G}}^{\mathbb{R}}$.*
- *The ρ-equivariant harmonic map h_ρ factors as $h_\rho : \widetilde{X} \to \tilde{\mathrm{G}}^{\mathbb{R}}/\tilde{\mathrm{H}}^{\mathbb{R}} \hookrightarrow \mathrm{G}^{\mathbb{R}}/\mathrm{H}^{\mathbb{R}}$.*
- *The associated $\mathrm{G}^{\mathbb{R}}$-Higgs bundle reduces to a polystable $\tilde{\mathrm{G}}^{\mathbb{R}}$-Higgs bundle.*

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We now describe the special properties of the representations and harmonic maps arising from $\mathbb{C}^*$-fixed points in the Higgs bundle moduli space. Let $G^{\mathbb{R}} < G$ be a real form of Hodge type with maximal compact $H^{\mathbb{R}}$ and complexified Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ such that $\mathfrak{g}_{2j} \subset \mathfrak{h}$ and $\mathfrak{g}_{2j+1} \subset \mathfrak{m}$ for all j. Recall that the homogeneous space $D = G^{\mathbb{R}}/H_0^{\mathbb{R}}$ is a period domain. The holomorphic tangent bundle decomposes as

$$T^{1,0}_{\mathbb{C}} D = \bigoplus_{j>0} (T^{1,0}_{\mathbb{C}} D)_j = \bigoplus_{j>0} G^{\mathbb{R}} \times_{H_0^{\mathbb{R}}} \mathfrak{g}_j.$$

Definition 4.21. Fix a period domain $D = G^{\mathbb{R}}/H_0^{\mathbb{R}}$. A variation of Hodge structure over X is a pair (ρ, f_ρ), where $\rho : \pi_1(X) \to G^{\mathbb{R}}$ is a representation and $f_\rho : \widetilde{X} \to D$ is a ρ-equivariant holomorphic map such that

$$\partial f_\rho(K^{-1}) \subset (T^{1,0}_{\mathbb{C}} D)_1.$$

The following proposition describes the representations and harmonic maps associated to polystable Higgs bundles fixed by the $\mathbb{C}^*$-action.

Proposition 4.22. *Let $\rho : \pi_1(X) \to G^{\mathbb{R}}$ be a reductive representation. Then the Higgs bundle associated to ρ is a $\mathbb{C}^*$-fixed point if and only if ρ factors through a reductive subgroup $\tilde{G}^{\mathbb{R}} < G^{\mathbb{R}}$ of Hodge type, and there is a period domain $\tilde{G}^{\mathbb{R}}/\tilde{H}_0^{\mathbb{R}}$ such that the associated ρ-equivariant harmonic map $h_\rho : \widetilde{X} \to \tilde{G}^{\mathbb{R}}/H^{\mathbb{R}}$ lifts to a variation of Hodge structure*

$$\begin{array}{ccccc} & & \tilde{G}^{\mathbb{R}}/\tilde{H}_0^{\mathbb{R}} & & \\ & \overset{f_\rho}{\nearrow} & \downarrow & & \\ \widetilde{X} & \xrightarrow[h_\rho]{} & \tilde{G}^{\mathbb{R}}/\tilde{H}^{\mathbb{R}} & \longrightarrow & G^{\mathbb{R}}/H^{\mathbb{R}} \end{array}$$

5. Toledo invariant and Arakelov–Milnor inequality

Let G be a complex semisimple Lie group and $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ be a $\mathbb{Z}$-grading with grading element $\zeta \in \mathfrak{g}_0$, and let $G_0 < G$ be the centralizer of ζ. Consider the PHVS $(G_0, \mathfrak{g}_1)$ and recall from (3.1) that the Toledo character $\chi_T : \mathfrak{g}_0 \to \mathbb{C}$ is given by $\chi_T(x) = B(\zeta, x)B(\gamma, \gamma)$. Recall that the degree of a bundle with respect to a character is defined by (4.3).

Definition 5.1. Let (E, φ) be a $(G_0, \mathfrak{g}_1)$-Higgs pair and $\chi_T : G_0 \to \mathbb{C}^*$ be the Toledo character. Then the Toledo invariant $\tau(E, \varphi)$ is defined by

$$\tau(E, \varphi) = \deg_{\chi_T}(E).$$

Remark 5.2. For the grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the canonical real form $G^{\mathbb{R}} < G$ from Proposition (3.8) is a Hermitian Lie group. For such real forms, a $G^{\mathbb{R}}$-Higgs bundle is a $(G_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}_1)$-Higgs pair. The Toledo invariant of such a Higgs bundle agrees with the Toledo invariant defined above [BGR17].

Recall Definition 3.3 of the Toledo rank $\mathrm{rk}_{\chi_T}(v)$ of a point $v \in \mathfrak{g}_1$. We define the *Toledo rank* $\mathrm{rk}_T(\varphi)$ of a $(G_0, \mathfrak{g}_1)$-Higgs pair (E, φ) to be

$$\mathrm{rk}_T(\varphi) = \mathrm{rk}_T(\varphi(x)) \quad \text{for a generic } x \in X.$$

Theorem 5.3. *Let G be a complex semisimple Lie group and $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ be a $\mathbb{Z}$-grading with grading element $\zeta \in \mathfrak{g}_0$. Let $G_0 < G$ be the centralizer of ζ and $\alpha = \lambda\zeta$ for $\lambda \in \mathbb{R}$. If (E, φ) is an*

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α-semistable $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pair, then the Toledo invariant $\tau(E,\varphi)$ satisfies the inequality

$$-\operatorname{rk}_T(\varphi)(2g-2) + \lambda(B(\gamma,\gamma)B(\zeta,\zeta) - \operatorname{rk}_T(\varphi)) \leqslant \tau(E,\varphi) \leqslant \lambda B(\gamma,\gamma)B(\zeta,\zeta).$$

Before proving the theorem we list some immediate consequences.

Corollary 5.4. *Assume $(\mathrm{G}_0, \mathfrak{g}_1)$ is a JM-regular PHVS and set $\alpha = \lambda\zeta$. Then the Toledo invariant of a α-semistable $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pair (E,φ) satisfies*

$$-\operatorname{rk}_T(\varphi)(2g-2) + \lambda(\operatorname{rk}_T(\mathrm{G}_0, \mathfrak{g}_1) - \operatorname{rk}_T(\varphi)) \leqslant \tau(E,\varphi) \leqslant \lambda \operatorname{rk}(\mathrm{G}_0, \mathfrak{g}_1).$$

For the application to Higgs bundles fixed by the $\mathbb{C}^*$-action we are interested in the case $\alpha = 0$.

Corollary 5.5. *For $\alpha = 0$, the Toledo invariant $\tau(E,\varphi)$ of a polystable $(\mathrm{G}_0, \mathfrak{g}_1)$-pair satisfies the inequality $-\operatorname{rk}_T(\varphi)(2g-2) \leqslant \tau(E,\varphi) \leqslant 0$. In particular,*

$$|\tau(E,\varphi)| \leqslant \operatorname{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)(2g-2).$$

Proof. This is a direct application of Theorem 5.3 in the case $\lambda = 0$, for which in fact it is enough to assume the semistability of (E,φ). However, with the extra assumption of polystability, the result can also be deduced from our holomorphic sectional curvature computations in Proposition 3.12, as we shall now explain. By Theorem 4.13 such (E,φ) admits a solution h to the Hitchin equations (4.4), so we have a variation of Hodge structure as studied in § 4.3. In particular, we obtain an equivariant horizontal holomorphic map $f : \tilde{X} \to \mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$. Denote by ω the 2-form associated to the Hermitian metric on $\mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ considered in § 3.3, normalized to have horizontal holomorphic sectional curvature $-1 \leqslant K \leqslant -1/\operatorname{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)$. By the Schwarz lemma, one has

$$\frac{1}{2\pi}\int_X f^*\omega \leqslant \operatorname{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)(2g-2).$$

The result will then follow from the identification

$$f^*\omega = -\chi_T(iF_h),$$

where F_h is the curvature as in (4.4). From (4.4) we have $\chi_T(F_h) = -\chi_T([\varphi,\varphi^*])$; locally writing $\varphi = a\,dz$, we obtain

$$\begin{aligned}\chi_T(F_h) &= -B(\zeta,[a,a^*])B(\gamma,\gamma)dz\wedge d\bar{z}\\ &= -B([\zeta,a],a^*)B(\gamma,\gamma)dz\wedge d\bar{z}\\ &= -B(a,a^*)B(\gamma,\gamma)dz\wedge d\bar{z}\\ &= -\langle a,a\rangle dz\wedge d\bar{z}.\end{aligned}$$

On the other hand, $f^*\omega = \omega(a,\bar{a})dz\wedge d\bar{z} = i\langle a,a\rangle dz\wedge d\bar{z}$. Therefore $f^*\omega = -\chi_T(iF_h)$, which completes the proof.

The same argument gives the more precise inequality $\tau(E,\varphi) \geqslant -\operatorname{rk}_T(\varphi)(2g-2)$, starting from the holomorphic curvature estimate (3.4). □

Since the Toledo invariant has discrete values, it is constant on the connected components of the moduli space of polystable $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pairs $\mathcal{M}(\mathrm{G}_0, \mathfrak{g}_1)$. As a result, we have the following decomposition of $\mathcal{M}(\mathrm{G}_0, \mathfrak{g}_1)$.

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Corollary 5.6. *The moduli space of polystable* $(\mathrm{G}_0, \mathfrak{g}_1)$*-Higgs pairs decomposes as*

$$\mathcal{M}(\mathrm{G}_0, \mathfrak{g}_1) = \coprod_{-\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)(2g-2)\leqslant\tau\leqslant 0} \mathcal{M}^\tau(\mathrm{G}_0, \mathfrak{g}_1),$$

where $(E, \varphi) \in \mathcal{M}^\tau(\mathrm{G}_0, \mathfrak{g}_1)$ *if and only if* $\tau(E, \varphi) = \tau$.

Remark 5.7. We will discuss the special properties of the space $\mathcal{M}^\tau(\mathrm{G}_0, \mathfrak{g}_1)$ for the minimal value $\tau = -(2g-2)\,\mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)$ in the next section.

Proof of Theorem 5.3. The upper bound follows immediately from stability. Note that $s = -B(\gamma,\gamma)\zeta$ is in the center of $\mathfrak{g}_0$ and has the property that $\mathrm{Ad}(e^{ts})(x)$ is bounded as $t \to \infty$ for all $x \in \mathfrak{g}_1$. Since s is in the center of $\mathfrak{g}_0$, the associated parabolic is all of G_0. Hence, α-semistability of (E, φ) implies

$$\deg E(-B(\gamma,\gamma)\zeta) \geqslant B(\alpha, -B(\gamma,\gamma)\zeta).$$

Since $-\tau(E,\varphi) = \deg E(-B(\gamma,\gamma)\zeta)$ and $\alpha = \lambda\zeta$ we have $\tau(E,\varphi) \leqslant \lambda B(\gamma,\gamma)B(\zeta,\zeta)$.

The lower bound is more interesting. For generic $x \in X$, $\varphi(x) \in \mathfrak{g}_1$ is in a fixed G_0-orbit. Let e be any element in this orbit and $\{f, h, e\}$ be an associated $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$. Let $(\hat{\mathrm{G}}_0, \hat{\mathfrak{g}}_1)$ be the associated maximal JM-regular PHVSS of $(\mathrm{G}_0, \mathfrak{g}_1)$ for e (see Definition 2.17). By Proposition 2.16, $e \in \hat{\Omega} \subset \mathfrak{g}_1$ is in the open $\hat{\mathrm{G}}_0$-orbit. Recall from Proposition 2.21 that e defines a parabolic subalgebra $\mathfrak{p}_{0,e} \subset \mathfrak{g}_0$ and a corresponding parabolic subgroup $\mathrm{P}_{0,e} < \mathrm{G}_0$. Since the construction of $\mathrm{P}_{0,e}$ is canonical, we obtain a holomorphic reduction of structure group of E to $\mathrm{P}_{0,e}$ on the open subset of X where $\mathrm{rk}_T(\varphi(x)) = \mathrm{rk}_T(e)$. This can be extended to a reduction over all of X. Let E_σ be the associated $\mathrm{P}_{0,e}$-bundle.

Set $\hat{\zeta} = h/2$ and write $\zeta = \hat{\zeta} + s$. Recall from Proposition 2.21 that the parabolic subalgebra $\mathfrak{p}_{0,e}$ is given by

$$\mathfrak{p}_{0,e} = \{x \in \mathfrak{g}_0 \mid \mathrm{Ad}(e^{ts})x \text{ is bounded as } t \to \infty\}.$$

By definition of the Toledo character, we have

$$\chi_T(x) = B(\gamma,\gamma)(B(\hat{\zeta}, x) + B(s, x)).$$

Thus,

$$\deg E(\sigma, B(\gamma,\gamma)s) = \tau(E,\varphi) - \deg_{\hat{\chi}_T}(E_\sigma), \tag{5.1}$$

where $\hat{\chi}_T : \mathfrak{p}_{0,e} \to \mathbb{C}$ is the character defined by $\hat{\chi}_T(x) = B(\gamma,\gamma)B(\hat{\zeta}, x)$.

The Levi factor of $P_{0,e}$ is isomorphic to $\hat{\mathrm{G}}_0$. Hence, projecting onto the Levi factor defines a holomorphic reduction $E_\sigma(\hat{\mathrm{G}}_0)$ with $\varphi \in H^0(E_\sigma(\hat{\mathrm{G}}_0)(\hat{\mathfrak{g}}_1) \otimes K)$. Recall from Proposition 3.16 that there is a positive integer q such that the character $q \cdot \hat{\chi}_T$ exponentiates to a character of $\hat{\chi}_{T,q} : \hat{\mathrm{G}}_0 \to \mathbb{C}^*$ which has a polynomial relative invariant $F : \hat{\mathfrak{g}}_1 \to \mathbb{C}$ of degree $q \cdot \hat{\chi}_T(\hat{\zeta})$. Since φ is generically in the open $\hat{\mathrm{G}}_0$-orbit of $\hat{\mathfrak{g}}_1$, applying the relative invariant to the Higgs field defines a nonzero holomorphic section of a line bundle $E_\sigma(\hat{\chi}_{T,q}) \otimes K^{q\cdot\hat{\chi}_T(\hat{\zeta})}$:

$$F(\varphi) \in H^0(E_\sigma(\hat{\chi}_{T,q}) \otimes K^{q\cdot\hat{\chi}_T(\hat{\zeta})}) \setminus \{0\}.$$

Note that $\deg_{\hat{\chi}_T}(E_\sigma) = (1/q)\deg(E_\sigma(\hat{\chi}_{T,q}))$. Thus, we have

$$\deg_{\hat{\chi}_T}(E_\sigma) \geqslant -\hat{\chi}_T(\hat{\zeta})(2g-2). \tag{5.2}$$

We now use the assumption that (E, φ) is α-semistable for $\alpha = \lambda\zeta$. Since $[s, \hat{\mathfrak{g}}_1] = 0$, we have $\hat{\mathfrak{g}}_1 \subset \mathfrak{g}_1^s$. Thus, we have $\varphi \in H^0(E_\sigma(\hat{\mathfrak{g}}_1^s) \otimes K)$. By the definition of the stability for $\alpha = \lambda\zeta$,

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we have

$$B(\lambda\zeta, B(\gamma,\gamma)s) \leqslant \deg E(\sigma, B(\gamma,\gamma)s).$$

Combining this with (5.1) and (5.2) gives an inequality for the Toledo invariant:

$$-\hat{\chi}_T(\hat{\zeta})(2g-2) + \lambda B(\gamma,\gamma)B(\zeta,s) \leqslant \tau(E,\varphi).$$

To finish the proof, recall that $B(\hat{\zeta},s)=0$ and $B(\gamma,\gamma)B(\zeta,\hat{\zeta}) = \mathrm{rk}_T(\varphi)$. Thus,

$$-\hat{\chi}_T(\hat{\zeta}) = B(\gamma,\gamma)B(\hat{\zeta},\hat{\zeta}) = B(\gamma,\gamma)B(\zeta,\hat{\zeta}) = \mathrm{rk}_T(\varphi)$$

and

$$\lambda B(\gamma,\gamma)B(\zeta,s) = \lambda(B(\gamma,\gamma)B(\zeta,\zeta) - B(\gamma,\gamma)B(\zeta,\hat{\zeta})) = \lambda(B(\gamma,\gamma)B(\zeta,\zeta) - \mathrm{rk}_T(\varphi)).$$

Hence, we obtain the desired inequality

$$-\mathrm{rk}_T(\varphi)(2g-2) + \lambda(B(\gamma,\gamma)B(\zeta,\zeta) - \mathrm{rk}_T(\varphi)) \leqslant \tau(E,\varphi). \qquad \square$$

6. Rigidity results for maximal Hodge bundles and variations of Hodge structure

For this section let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and let X be a compact Riemann surface of genus $g \geqslant 2$ and canonical bundle K. Fix a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ with grading element ζ and let $\mathrm{G}_0 < \mathrm{G}$ be the centralizer of ζ.

We will call a polystable $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pair (E,φ) *maximal* if the absolute value of the Toledo invariant is maximized. By Corollary 5.5, the Toledo invariant $\tau(E,\varphi)$ of a polystable $(\mathrm{G}_0,\mathfrak{g}_1)$-Higgs pair on X satisfies the inequality

$$-\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)(2g-2) \leqslant \tau(E,\varphi) \leqslant 0.$$

Thus, (E,φ) is maximal if and only if $\tau(E,\varphi) = -\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)(2g-2)$.

6.1 The JM-regular case

We start with the JM-regular case. Assume $(\mathrm{G}_0,\mathfrak{g}_1)$ is a JM-regular PHVS. Fix $e \in \Omega$ and let $\{f,h,e\} = \{f,2\zeta,e\}$ be the associated $\mathfrak{sl}_2$-triple. Let $\mathrm{S} < \mathrm{G}$ be the connected subgroup with Lie algebra spanned by $\{f,h,e\}$. Note that S is isomorphic to $\mathrm{PSL}_2\mathbb{C}$ or $\mathrm{SL}_2\mathbb{C}$ depending on G and the $\mathfrak{sl}_2$-triple. Finally, let $\mathrm{C} < \mathrm{G}$ be the reductive group which centralizes the $\mathfrak{sl}_2$-triple $\{f,h,e\}$. Note that $\mathrm{C} = \mathrm{G}_0^e < \mathrm{G}_0$ is also the G_0-stabilizer of e.

Let $\mathrm{T} < \mathrm{S}$ be the connected subgroup with Lie algebra $\langle h\rangle$, and note that $\mathrm{T} < \mathrm{G}_0$. Recall the uniformizing Hodge bundle (E_T, e) for X from Examples 4.4 and 4.18. Here E_T is the holomorphic frame bundle of K^{-1} if $\mathrm{S} = \mathrm{PSL}_2\mathbb{C}$ and the holomorphic frame bundle of $K^{-1/2}$ if $\mathrm{S} = \mathrm{SL}_2\mathbb{C}$. Since $E_\mathrm{T}(\langle e\rangle) \otimes K \cong \mathcal{O}$, the Lie algebra element e defines a holomorphic section of $E_\mathrm{T}(\langle e\rangle)\otimes K$. Moreover, (E_T, e) is polystable, and a solution to the Hitchin equations is equivalent to a metric of constant curvature on X. Extending the structure group to G_0 defines a maximal polystable $(\mathrm{G}_0,\mathfrak{g}_1)$-Higgs pair.

Proposition 6.1. *The $(\mathrm{G}_0,\mathfrak{g}_1)$-Higgs pair $(E_\mathrm{T}(\mathrm{G}_0), e)$ is polystable and maximal.*

Proof. First note that $(E,\varphi) = (E_\mathrm{T}(\mathrm{G}_0), e)$ is polystable since (E_T, e) polystable. By construction, we have $\tau(E,\varphi) = -\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)(2g-2)$. $\square$

Other examples of maximal $(\mathrm{G}_0,\mathfrak{g}_1)$-Higgs pairs are given by twisting the above uniformizing Higgs bundle by a holomorphic C-bundle. Let E_C be a holomorphic C-bundle on X. Since T and C are commuting subgroups of G_0, the multiplication map $m : \mathrm{T}\times\mathrm{C} \to \mathrm{G}_0$ is a

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group homomorphism. Thus, we can form a G_0-bundle

$$E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}} = (E_{\mathrm{T}} \times E_{\mathrm{C}}) \times_m \mathrm{G}_0.$$

This is the principal bundle version of the tensor product of vector bundles, where $\mathrm{GL}_m\mathbb{C}$ and $\mathrm{GL}_n\mathbb{C}$ are commuting subgroups of $\mathrm{GL}_{mn}\mathbb{C}$.

As with vector bundles, given metrics h_{T} and h_{C} on E_{T} and E_{C} respectively, there is a metric $h_{\mathrm{T}} \otimes h_{\mathrm{C}}$ on $E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}$. More specifically, metrics on E_{T} and E_{C} are reductions of structure group to the maximal compact subgroups, say $E_{\mathrm{T}^{\mathbb{R}}} \subset E_{\mathrm{T}}$ and $E_{\mathrm{C}^{\mathbb{R}}} \subset E_{\mathrm{C}}$. The metric $h_{\mathrm{T}} \otimes h_{\mathrm{C}}$ on $E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}$ is the reduction of structure group given by $E_{\mathrm{T}^{\mathbb{R}}} \otimes_{\mathrm{G}_0^{\mathbb{R}}} E_{\mathrm{C}^{\mathbb{R}}} \subset E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}$, where $\mathrm{G}_0^{\mathbb{R}} \subset \mathrm{G}_0$ is a maximal compact subgroup containing $\mathrm{C}^{\mathbb{R}}$ and $\mathrm{T}^{\mathbb{R}}$. Similarly, given connections A_{T} and A_{C} on E_{T} and E_{C} respectively, $A_{\mathrm{T}} + A_{\mathrm{C}}$ defines a connection on $E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}$ and the curvature satisfies

$$F_{A_{\mathrm{T}}+A_{\mathrm{C}}} = F_{A_{\mathrm{T}}} + F_{A_{\mathrm{C}}}.$$

These formulas make sense because the tensor product is made from the $\mathrm{T} \times \mathrm{C}$-principal bundle $E_{\mathrm{T}} \times E_{\mathrm{C}}$ by applying the multiplication $m : \mathrm{T} \times \mathrm{C} \to \mathrm{G}_0$, with corresponding Lie algebra morphism $\mathfrak{t} \oplus \mathfrak{c} \to \mathfrak{t} + \mathfrak{c} \subset \mathfrak{g}_0$. In the case of the general linear group, this corresponds to first forming the product bundle as a $\mathrm{GL}_m\mathbb{C} \times \mathrm{GL}_n\mathbb{C}$-principal bundle and then producing the tensor product as a $\mathrm{GL}_{mn}\mathbb{C}$-principal bundle by applying the multiplication map $\mathrm{GL}_m\mathbb{C} \times \mathrm{GL}_n\mathbb{C} \to \mathrm{GL}_{mn}\mathbb{C}$ given by $(g, g') \mapsto g \otimes g'$.

Since C acts trivially on $\langle e \rangle$, we have

$$e \in H^0((E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}})(\mathfrak{g}_1) \otimes K).$$

Also, since the Lie algebra $\mathfrak{c}$ is perpendicular to ζ, the Toledo invariant is unchanged:

$$\tau((E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}, e)) = \tau(E_{\mathrm{T}}(\mathrm{G}_0), e).$$

Moreover, if E_{C} is a polystable C-bundle[1], then there is a metric h_{C} on E_{C} whose associated Chern connection is flat (see Remark 4.14). As a result, if h_{T} is a metric on E_{T} solving the Hitchin equations for (E_{T}, e), then $h_{\mathrm{T}} \otimes h_{\mathrm{C}}$ solves the Hitchin equations for the Higgs bundle $(E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}(\mathrm{G}), e)$. Hence, if E_{C} is a polystable C-bundle, then $(E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}, e)$ is a polystable $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pair.

So far, we have shown that there is a well-defined map

$$\begin{aligned} \Psi_e : \ \mathcal{N}(\mathrm{C}) &\longrightarrow \mathcal{M}^{\max}(\mathrm{G}_0, \mathfrak{g}_1) \\ E_{\mathrm{C}} &\longmapsto (E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}, e) \end{aligned} \tag{6.1}$$

from the moduli space $\mathcal{N}(\mathrm{C})$ of polystable C-bundles to the moduli space of polystable $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pairs which are maximal. The map Ψ_e from (6.1) is a moduli space version of a restriction of the global Slodowy slice map from [CS21] and generalizes a restriction of the Cayley correspondence in the case of a $\mathbb{Z}$-grading defining a Hermitian group [BGR17].

Theorem 6.2. *The map* $\Psi_e : \mathcal{N}(\mathrm{C}) \to \mathcal{M}^{\max}(\mathrm{G}_0, \mathfrak{g}_1)$ *from* (6.1) *defines an isomorphism between the moduli space of polystable* C*-bundles and the moduli space of polystable maximal* $(\mathrm{G}_0, \mathfrak{g}_1)$*-Higgs pairs.*

Remark 6.3. For classical Lie groups, the centralizers $\mathrm{C} < \mathrm{G}$ of $\mathfrak{sl}_2$-triples are easily described in terms of partitions [SS70] (see [CM93, Theorem 6.1.3]).

[1] In particular, E_{C} must have degree zero with respect to any character $\chi : \mathfrak{c} \to \mathbb{C}$.

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Proof. We first show that Ψ_e is surjective. Let (E,φ) be a polystable maximal $(\mathrm{G}_0,\mathfrak{g}_1)$-Higgs pair. As above, let E_{T} be the holomorphic frame bundle of K^{-1} or $K^{-1/2}$ if S is isomorphic to $\mathrm{PSL}_2\mathbb{C}$ or $\mathrm{SL}_2\mathbb{C}$, respectively. Note that

$$E(\mathfrak{g}_1)\otimes K\cong (E_{\mathrm{T}}^{-1}\otimes_{\mathrm{G}_0}E)(\mathfrak{g}_1).$$

Since (G_0,φ) is JM-regular, we have $\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)=\chi_T(\zeta)$, where $\zeta=h/2$. Also, by Proposition 3.16, a positive multiple of the Toledo character $q\cdot\chi_T(x)=q\cdot B(\gamma,\gamma)B(\zeta,x)$ exponentiates to a character $\chi_{T,q}:\mathrm{G}_0\to\mathbb{C}^*$ which has a polynomial relative invariant $F:\mathfrak{g}_1\to\mathbb{C}$ of degree $q\cdot\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)$. As in the proof of Theorem 5.3, applying the relative invariant F to the Higgs field defines a nonzero holomorphic section

$$F(\varphi)\in H^0(E(\chi_{T,q})\otimes K^{q\,\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)}).$$

The degree of $E(\chi_{T,q})\otimes K^{q\,\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)}$ is $q(\tau(E,\varphi)+\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)(2g-2))=0$. Thus, $F(\varphi)$ is nowhere vanishing, and $\mathrm{rk}_T(\varphi(x))=\mathrm{rk}_T(\mathrm{G}_0,\mathfrak{g}_1)$ for all $x\in X$. That is, $\varphi(x)$ is in the open orbit Ω for all $x\in X$.

Thus the Higgs field φ is defines a holomorphic section of $E(\Omega)\otimes K$. But $\Omega=\mathrm{G}_0/\mathrm{C}$ and $E(\Omega)\otimes K\cong(E_{\mathrm{T}}^{-1}\otimes_{\mathrm{G}_0}E)(\mathrm{G}_0/\mathrm{C})$. Thus, the Higgs field defines a holomorphic reduction of structure group of $(E_{\mathrm{T}}^{-1}\otimes_{\mathrm{G}_0}E)$ to C. Let E_{C} be the resulting holomorphic C-bundle. Twisting both sides by E_{T}, we obtain an isomorphism

$$E\cong E_{\mathrm{T}}\otimes_{\mathrm{G}_0}E_{\mathrm{C}}.$$

By construction, $\varphi\in H^0((E_{\mathrm{T}}\otimes_{\mathrm{G}_0}E_{\mathrm{C}})(\langle e\rangle)\otimes K)$. Since a $(\mathrm{G}_0,\mathfrak{g}_1)$-Higgs pairs (E,φ) is isomorphic to $(E,\lambda\varphi)$ for all $\lambda\in\mathbb{C}^*$, we can take $\varphi=e$.

To complete the proof of surjectivity, we show that if $(E,\varphi)=(E_{\mathrm{T}}\otimes_{\mathrm{G}_0}E_{\mathrm{C}}),e)$ is a polystable (semistable) $(\mathrm{G}_0,\mathfrak{g}_1)$-Higgs pair, then E_{C} is a polystable (semistable) bundle. Suppose that we have a reduction $(E_{\mathrm{P}'_s},\sigma')$ of the structure group of E_{C} to the parabolic subgroup $\mathrm{P}'_s\subset\mathrm{C}$ defined by an $s\in i\mathfrak{c}^{\mathbb{R}}$, where $\mathfrak{c}^{\mathbb{R}}$ is the Lie algebra of the maximal compact subgroup $\mathrm{C}^{\mathbb{R}}$ of C. The element s defines a parabolic subgroup $\mathrm{P}_s\subset\mathrm{G}_0$ as well, and, using the map $\mathrm{C}/\mathrm{P}'_s\to\mathrm{G}_0/\mathrm{P}_s$, we obtain from σ' a reduction σ of the structure group of E to P_s, resulting in a P_s-bundle E_{P_s}. Since C stabilizes $e\in\mathfrak{g}_1$, we have $e\in\mathfrak{g}_{1,s}^0$. Since $\langle s,h\rangle=0$, there is no contribution coming from the twisting $E\otimes_{\mathrm{G}_0}E_{\mathrm{T}}^{-1}$ in the computation of $\deg E(\sigma,s)$. Hence, $\deg(E)(\sigma,s)=\deg(E_{\mathrm{C}})(\sigma',s)$, and (semi)stability of (E,φ) implies the (semi)stability of E_{C}. For polystability, one must just check additionally that in the equality case, reduction for the Levi subgroup $\mathrm{L}_s\subset\mathrm{P}_s$ to an $E_{\mathrm{L}_s}\subset E$ implies reduction for the Levi subgroup $\mathrm{L}'_s\subset\mathrm{P}'_s$, but it is sufficient to take $E_{\mathrm{L}'_s}=(E_{\mathrm{L}_s}\otimes_{\mathrm{G}_0}E_{\mathrm{T}}^{-1})\cap E_{\mathrm{C}}$.

For injectivity, note that two polystable C-bundles E_{C} and E'_{C} define the same point in $\mathcal{M}^{\max}(\mathrm{G}_0,\mathfrak{g}_1)$ if there is a holomorphic isomorphism of G_0-bundles

$$\Phi:E_{\mathrm{C}}\otimes_{\mathrm{G}_0}E_{\mathrm{T}}\to E'_{\mathrm{C}}\otimes_{\mathrm{G}_0}E_{\mathrm{T}}$$

such that $\Phi^*e=e$. Since the G_0 stabilizer of $e\in\mathfrak{g}_1$ is C, we conclude that Φ induces an isomorphism between E_{C} and E'_{C}. Note that injectivity implies that the automorphism group of $\Psi_e(E_{\mathrm{C}})$ is equal to the automorphism group of the polystable C-bundle E_{C}. □

The component count for the moduli space of polystable C-Higgs bundles is given the number of topologically distinct C-bundles on X which have degree zero with respect to any character $\chi:\mathfrak{c}\to\mathbb{C}$. For example, when C is connected and $\Gamma<\pi_1(\mathrm{C})$ is the torsion subgroup, $\mathcal{N}(\mathrm{C})$ has $|\Gamma|$ connected components. As an immediate corollary, we have the following component count.

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Corollary 6.4. *We have* $\pi_0(\mathcal{M}^{\max}(\mathrm{G}_0, \mathfrak{g}_1)) = \pi_0(\mathcal{N}(\mathrm{C}))$. *In particular, the component count is given by the number of topological degree zero* C*-bundles on* X.

We now use Theorem 6.2 to deduce rigidity results for variations of Hodge structure. Consider the complex linear involution $\theta : \mathfrak{g} \to \mathfrak{g}$ which is $(-1)^j$ Id on each graded factor $\mathfrak{g}_j$. Recall from § 3.2 that a compact real form $\tau : \mathfrak{g} \to \mathfrak{g}$ satisfies $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$. Moreover, we can choose such a τ so that the $\mathfrak{sl}_2$-triple $\{f, h, e\}$ satisfies $f = -\tau(e)$. Let $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ be the associated real form of Hodge type. By construction, the subgroup $\mathrm{S} < \mathrm{G}$ defines a subgroup $\mathrm{S}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$ which is isomorphic to $\mathrm{PSL}_2\mathbb{R}$ if $\mathrm{S} \cong \mathrm{PSL}_2\mathbb{C}$ and isomorphic to $\mathrm{SL}_2\mathbb{R}$ if $\mathrm{S} \cong \mathrm{SL}_2\mathbb{C}$. Also by construction, the $\mathrm{G}^{\mathbb{R}}$-centralizer of $\mathrm{S}^{\mathbb{R}}$ is the compact real form $\mathrm{C}^{\mathbb{R}} < \mathrm{C}$ of the centralizer of S.

Since $\mathrm{C}^{\mathbb{R}}$ and $\mathrm{S}^{\mathbb{R}}$ are commuting subgroups of $\mathrm{G}^{\mathbb{R}}$, given representations $\rho_1 : \pi_1(X) \to \mathrm{S}^{\mathbb{R}}$ and $\rho_2 : \pi_1(X) \to \mathrm{C}^{\mathbb{R}}$, we can form a new representation, multiplying the images

$$\rho_1 * \rho_2 : \pi_1(X) \to \mathrm{G}^{\mathbb{R}}, \quad (\rho_1 * \rho_2)(\gamma) = \rho_1(\gamma) \cdot \rho_2(\gamma).$$

Recall that $(\mathrm{G}_0, \mathfrak{g}_1)$ is JM-regular if and only if it arises from an even $\mathfrak{sl}_2$-triple.

Theorem 6.5. *Let* G *be a complex semisimple Lie group with Lie algebra* $\mathfrak{g}$. *Let* $\{f, h, e\} \subset \mathfrak{g}$ *be an even* $\mathfrak{sl}_2$*-triple and* $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$ *be the associated* $\mathbb{Z}$*-grading with grading element* $h/2$, *let* $\mathrm{S} < \mathrm{G}$ *be the associated connected subgroup and let* $\mathrm{C} < \mathrm{G}$ *be the centralizer of* $\{f, h, e\}$. *Suppose* $\rho : \pi_1(X) \to \mathrm{G}$ *is a reductive representation. The Higgs bundle associated to* ρ *is a maximal Hodge bundle of type* $(\mathrm{G}_0, \mathfrak{g}_1)$ *if and only if, up to conjugation,* $\rho = \rho_u * \rho_{\mathrm{C}} : \pi_1(X) \to \mathrm{G}$, *where*

– $\rho_u : \pi_1(X) \to \mathrm{S}^{\mathbb{R}} < \mathrm{S}$ *is (a lift of) the uniformizing representation of* X, *and*
– $\rho_{\mathrm{C}} : \pi_1(X) \to \mathrm{C}^{\mathbb{R}} < \mathrm{C}$ *is any representation into the compact real form of* C.

Remark 6.6. Note that the representations $\rho = \rho_u * \rho_{\mathrm{C}} : \pi_1(X) \to \mathrm{G}$ all factor through the canonical real form of Hodge type $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$ associated to the grading $\bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$. Moreover, the $\mathrm{G}^{\mathbb{R}}$-centralizer of any such representations is compact since the $\mathrm{G}^{\mathbb{R}}$-centralizer of $\mathrm{S}^{\mathbb{R}}$ is $\mathrm{C}^{\mathbb{R}}$, which is compact. As a result, representations associated to maximal Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ do not factor through proper parabolic subgroups $\mathrm{P}^{\mathbb{R}} < \mathrm{G}^{\mathbb{R}}$.

Proof. The proof is immediate from the nonabelian Hodge correspondence and Theorem 6.2. Namely, given an even $\mathfrak{sl}_2$-triple $\{f, h, e\}$ with associated grading $\bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$, every polystable $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pair (E, φ) with Toledo invariant $-\mathrm{rk}_T(\mathrm{G}_0, \mathfrak{g}_1)(2g-2)$ is isomorphic to $(E_{\mathrm{T}} \otimes_{\mathrm{G}_0} E_{\mathrm{C}}, e)$, where (E_{T}, e) is a uniformizing Higgs bundle of X and E_{C} is a polystable C-bundle. A solution to the Hitchin equations is given by $h_{\mathrm{T}} \otimes h_{\mathrm{C}}$, where h_{T} solves the Hitchin equations for the uniformizing Higgs bundle (E_{T}, e) and the Chern connection of h_{C} is flat. The associated flat connection is then

$$A_{h_{\mathrm{T}}} + e - \tau(e) + A_{h_{\mathrm{C}}}.$$

As a result, the associated representation $\rho : \pi_1(X) \to \mathrm{G}$ is given by $\rho_u * \rho_{\mathrm{C}}$, where $\rho_u : \pi_1(X) \to \mathrm{S}^{\mathbb{R}} < \mathrm{S}$ is the uniformizing representation of X if $\mathrm{S} \cong \mathrm{PSL}_2\mathbb{C}$ and a lift of the uniformizing representation if $\mathrm{S} \cong \mathrm{SL}_2\mathbb{C}$ and $\rho_{\mathrm{C}} : \pi_1(X) \to \mathrm{C}^{\mathbb{R}} < \mathrm{C}$ is a representation into the compact real form of C. □

Let Σ be a closed topological surface of genus $g \geqslant 2$. For $\mathrm{S}^{\mathbb{R}}$ isomorphic to $\mathrm{PSL}_2\mathbb{R}$ or $\mathrm{SL}_2\mathbb{R}$, injective representations $\rho : \pi_1(\Sigma) \to \mathrm{S}^{\mathbb{R}}$ with discrete image are called Fuchsian representations. The set of conjugacy classes of Fuchsian representations defines an open and closed set in the character variety $\mathcal{R}(\mathrm{S}^{\mathbb{R}})$, each connected component of which is identified with the Teichmüller space of the surface Σ by the uniformization theorem.

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Corollary 6.7. *Suppose* $(G_0, \mathfrak{g}_1)$ *is JM-regular and let* $\rho : \pi_1(\Sigma) \to G$ *be a reductive representation. There exists a Riemann surface structure* X_ρ *on* Σ *such that the Higgs bundle associated to* ρ *is a maximal Hodge bundle of type* $(G_0, \mathfrak{g}_1)$ *on* X_ρ *if and only if*

$$\rho = \rho_{\mathrm{Fuch}} * \rho_{C^{\mathbb{R}}},$$

where $\rho_{\mathrm{Fuch}} : \pi_1(\Sigma) \to S^{\mathbb{R}}$ *is a Fuchsian representation and* $\rho_{C^{\mathbb{R}}} : \pi_1(\Sigma) \to C^{\mathbb{R}}$ *is a representation into the compact real form of* C. *In particular, for any such representations, the associated Riemann surface* X_ρ *is unique.*

The following corollary is also immediate from Theorem 6.5.

Corollary 6.8. *Suppose* $(G_0, \mathfrak{g}_1)$ *is JM-regular. A variation of Hodge structure* (ρ, f_ρ) *associated to a Hodge bundle of type* $(G_0, \mathfrak{g}_1)$ *is maximal if and only if* $f_\rho : \widetilde{X} \to G^{\mathbb{R}}/H_0^{\mathbb{R}}$ *is a totally geodesic embedding which maximizes the holomorphic sectional curvature. In particular, the image of* f_ρ *is independent of* ρ *and rigid.*

6.2 Some explicit examples

We list some explicit examples for $(G_0, \mathfrak{g}_1)$ JM-regular.

Example 6.9. When $G_0 < G$ is a Cartan subgroup, $(G_0, \mathfrak{g}_1)$ is a JM-regular PHVS associated to a principal $\mathfrak{sl}_2$-triple. In this case, the group C is the center of G. The Higgs bundles $(E_C \otimes_{G_0} E_T, e)$ associated to maximal $(G_0, \mathfrak{g}_1)$-Higgs pairs are the image of 0 under the Hitchin section [Hit92].

Example 6.10. For JM-regular PHVSs $(G_0, \mathfrak{g}_1)$ associated to gradings $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the real form $G^{\mathbb{R}} < G$ is a Hermitian Lie group of tube type (see Example 3.9). In this case, the above results recover the subset of the results in [BGR17]. In this case, there is a particular stratum of the boundary of the noncompact Riemannian symmetric space $G^{\mathbb{R}}/H^{\mathbb{R}}$ called the Shilov boundary. The group $C^{\mathbb{R}}$ is isomorphic to the stabilizer of a generic point in the Shilov boundary.

Example 6.11. We now describe the JM-regular $(\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}, M_{p,q})$ with $q \geqslant p$ from Examples 2.1, 2.5 and 3.9 in more detail. Fix a decomposition $\mathbb{C}^{2p+q} = \mathbb{C}^p \oplus \mathbb{C}^q \oplus \mathbb{C}^p$ with orthogonal structure $Q(x, y, z) = xz^{\mathrm{T}} + x^{\mathrm{T}}z + yy^{\mathrm{T}}$. Then we may write $\mathfrak{so}_{2p+q}\mathbb{C}$ as

$$\mathfrak{so}_{2p+q}\mathbb{C} = \left\{ \begin{pmatrix} A & W & X \\ Y & B & -W^{\mathrm{T}} \\ Z & -Y^{\mathrm{T}} & -A^{\mathrm{T}} \end{pmatrix} \middle| \begin{array}{c} A \in M_{p,p},\ B = -B^{\mathrm{T}} \in M_{q,q},\ W \in M_{p,q} \\ X = -X^{\mathrm{T}} \in M_{p,p},\ Y \in M_{q,p},\ Z = -Z^{\mathrm{T}} \in M_{p,p} \end{array} \right\}.$$

The even $\mathfrak{sl}_2$-triple is given by

$$f = \begin{pmatrix} 0 & 0 & 0 \\ \begin{pmatrix} 2\,\mathrm{Id}_p \\ 0 \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} -2\,\mathrm{Id}_p & 0 \end{pmatrix} & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 2\,\mathrm{Id}_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\,\mathrm{Id}_p \end{pmatrix}, \quad e = \begin{pmatrix} 0 & \begin{pmatrix} \mathrm{Id}_p & 0 \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} -\mathrm{Id}_p \\ 0 \end{pmatrix} \\ 0 & 0 & 0 \end{pmatrix},$$

where Id_p is the $p \times p$ identity matrix. The $\mathbb{Z}$-grading is $\mathfrak{g}_{-2} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where the space $\mathfrak{g}_j$ is the jth super block diagonal.

The subgroup $G_0 < \mathrm{SO}_{2p+q}\mathbb{C}$ is $\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}$ and the G_0-stabilizer of e is given by

$$C = G_0^e = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) & 0 \\ 0 & 0 & (A^{-1})^{\mathrm{T}} \end{pmatrix} \middle| AA^{\mathrm{T}} = \mathrm{Id} \text{ and } BB^{\mathrm{T}} = \mathrm{Id} \right\} \cong \mathrm{S}(\mathrm{O}_p\mathbb{C} \times \mathrm{O}_{q-p}\mathbb{C}).$$

Using the standard representation of $\mathrm{SO}_{2p+q}\mathbb{C}$, an $\mathrm{SO}_{2p+q}\mathbb{C}$-Higgs bundle is a triple (E, Q, Φ), where E is a rank $2p+q$ holomorphic bundle with trivialized determinant bundle, Q is

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an everywhere nondegenerate symmetric bilinear form on E and $\Phi \in H^0(\mathrm{End}(E) \otimes K)$ is a K-twisted endomorphism of E which is skew symmetric with respect to Q. An $\mathrm{SO}_{2p+q}\mathbb{C}$ is a Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_1)$ if E splits holomorphically as $E = V \oplus W \oplus V^*$, where V is a rank-p isotropic subbundle and W is a rank-q orthogonal subbundle, and with respect to this splitting we have

$$\Phi = \begin{pmatrix} 0 & \theta & 0 \\ 0 & 0 & -\theta^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} : V \oplus W \oplus V^* \to (V \oplus W \oplus V^*) \otimes K. \tag{6.2}$$

Such a Higgs bundle is a maximal $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pair if and only if W decomposes holomorphically and orthogonally as $W = U_p \oplus U_{q-p}$ and $V = U_p \otimes K^{-1}$ and

$$\theta = \begin{pmatrix} \mathrm{Id}_{U_p} & 0 \end{pmatrix} : W = U_p \oplus U_{q-p} \to U_p = V \otimes K.$$

In this example, the group $\mathrm{C} = \mathrm{S}(\mathrm{O}_p\mathbb{C} \times \mathrm{O}_{q-p}\mathbb{C})$ and the space of such Higgs bundles is parameterized by the polystable $\mathrm{S}(\mathrm{O}_p\mathbb{C} \times \mathrm{O}_{q-p}\mathbb{C})$ bundle $U_p \oplus U_{q-p}$. These Higgs bundles reduce to $\mathrm{SO}(2p, q)$-Higgs bundles and the resulting representations $\rho : \pi_1(X) \to \mathrm{SO}(2p, q)$ are conjugate to $\rho_u * \rho_{\mathrm{C}^{\mathbb{R}}}$,where $\rho_u : \pi_1(X) \to \mathrm{PSL}_2\mathbb{R}$ is the uniformizing representation of X and $\rho_{\mathrm{C}^{\mathbb{R}}} : \pi_1(X) \to \mathrm{S}(\mathrm{O}_p \times \mathrm{O}_{q-p})$ is any representation. When $p = 1$, this is agrees with Example 6.10 for the Hermitian Lie group $\mathrm{SO}(2, q)$.

Remark 6.12. For the JM-regular PHVS $(\mathrm{S}(\mathrm{GL}_p\mathbb{C} \times \mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_p\mathbb{C}), M_{p,q} \oplus M_{q,p})$ with $q \geqslant p$ from Examples 2.1, 2.5 and 3.9, the moduli space $\mathrm{SL}_{2p+q}\mathbb{C}$-Higgs bundles which are maximal Hodge bundles of type $(\mathrm{G}_0, \mathfrak{g}_1)$ can be described similarly. Namely, they are given by pairs (E, Φ) where E is a holomorphic rank-$(2p + q)$ vector bundle with trivialized determinant and Φ is a traceless K-twisted endomorphism of E such that $E = (U_p \otimes K^{-1}) \oplus U_p \oplus U_{q-p} \oplus (U_p \otimes K)$, $\det(U_p)^3 = \det(U_{q-p})$ and

$$\Phi = \begin{pmatrix} 0 & \mathrm{Id}_p & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : E \to E \otimes K.$$

Example 6.13. There is a class of nilpotent elements called distinguished nilpotents which play an important role in the Bala–Carter theory classification of nilpotent orbits in $\mathfrak{g}$. A nilpotent $e \in \mathfrak{g}$ is distinguished if and only if the centralizer of an associated $\mathfrak{sl}_2$-triple $\mathrm{C} < \mathrm{G}$ is discrete. Such $\mathfrak{sl}_2$-triples are necessarily even, and hence the resulting moduli space of maximal JM-regular $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pairs is discrete. In particular, the associated $\mathbb{C}^*$-fixed points in the G-Higgs bundle moduli space are isolated.

The principal $\mathfrak{sl}_2$-triples are always distinguished and give rise to the isolated fixed points from Example 6.9. In type A, this is the only distinguished nilpotent. However, in all other types there are many distinguished nilpotents. For example, for $\mathfrak{g} = \mathfrak{so}_n\mathbb{C}$ a nilpotent is distinguished if and only if it is associated to a partition $n = 1 \cdot n_1 + \cdots + 1 \cdot n_k$, where each n_j is odd. For $\mathfrak{g} = \mathfrak{sp}_{2n}\mathbb{C}$ a nilpotent is distinguished if and only if it is associated to a partition $2n = 1 \cdot n_1 + \cdots + 1 \cdot n_k$, with each n_j even. Moreover, for the Lie group G_2 there are two even $\mathfrak{sl}_2$-triples and both are distinguished. See [CM93, §§ 6 & 8] for more details. This means that the moduli space of G-Higgs bundles has isolated fixed points which do not arise from the Hitchin section when $\mathfrak{g}$ is not of type A.

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Example 6.14. In [BCGGO21], a class of even nilpotents is identified called magical nilpotents. For such even $\mathfrak{sl}_2$-triples, there is a real form $G^{\mathbb{R}} < G$ such that the associated maximal $(G_0, \mathfrak{g}_1)$-Higgs pairs define $G^{\mathbb{R}}$-Higgs bundles which are local minima of the energy function on the moduli space $G^{\mathbb{R}}$-Higgs bundles. For such Hodge bundles, the global Slodowy slice map from [CS21] descends to moduli spaces and describes connected components of the moduli space $\mathcal{M}(G^{\mathbb{R}})$ with many interesting properties. In particular, the associated components of the character variety $\mathcal{R}(G^{\mathbb{R}})$ define the higher Teichmüller components conjectured by Guichard, Labourie and Wienhard [GLW21, GW18]. There are four families of magical nilpotents two of which appear in Examples 6.9 and 6.10.

6.3 The non-JM-regular case

We now describe the moduli space of maximal $(G_0, \mathfrak{g}_1)$-Higgs pairs when $(G_0, \mathfrak{g}_1)$ is not JM-regular. Assume $(G_0, \mathfrak{g}_1)$ is not a JM-regular PHVS. Let $\Omega \subset \mathfrak{g}_1$ be the open G_0-orbit. For $e \in \Omega$ we complete it to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$ with $h \in \mathfrak{g}_0$ and set $s = \zeta - h/2$. Recall $s \neq 0$ exactly when $(G_0, \mathfrak{g}_1)$ is not JM-regular. A maximal JM-regular PHVSS $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ for $(G_0, \mathfrak{g}_1)$ is given by letting $\hat{G}_0 < G_0$ be the G_0-centralizer of s and $\hat{\mathfrak{g}}_1 = \{x \in \mathfrak{g}_1 \,|\, [s, x] = 0\}$. Recall also that s determines a parabolic subgroup $P_s < G_0$ and $\hat{G}_0 < P_s$ is the Levi subgroup.

Proposition 6.15. *Assume $(G_0, \mathfrak{g}_1)$ is not JM-regular and let $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ be a maximal JM-regular PHVSS. Then there are no stable $(G_0, \mathfrak{g}_1)$-Higgs pairs which are maximal. Moreover, every maximal polystable $(G_0, \mathfrak{g}_1)$-Higgs pair with Toledo invariant reduces to a polystable maximal $(\hat{G}_0, \hat{\mathfrak{g}}_1)$-Higgs pair.*

Proof. Let (E, φ) be a polystable maximal $(G_0, \mathfrak{g}_1)$-Higgs pair. Since (E, φ) is maximal, $\varphi(x)$ is in the open orbit Ω for generic $x \in X$. Let $e \in \Omega$, $\{f, h, e\}$ be an $\mathfrak{sl}_2$-triple with $h \in \mathfrak{g}_0$ and $(\hat{G}_0, \hat{\mathfrak{g}}_1)$ be the associated maximal JM-regular PHVSS of $(G_0, \mathfrak{g}_1)$. Let $s = \zeta - h/2$, and $P_s < G_0$ be the associated parabolic subgroup. As in the proof of Theorem 5.3, the Higgs field defines a reduction of structure group σ of E to a P_s-bundle E_σ such that $\varphi \in H^0(E_\sigma(\mathfrak{g}_{1,s}) \otimes K)$. Since (E, φ) is maximal, we have $\deg E(\sigma, s) = 0$. Hence, (E, φ) is not stable and there is a holomorphic reduction $\hat{\sigma}$ to the Levi subgroup $\hat{G}_0 < P_s$ with $\varphi \in H^0(E_{\hat{\sigma}}(\hat{\mathfrak{g}}_1) \otimes K$. Thus, (E, φ) reduces to a polystable maximal $(\hat{G}_0, \hat{\mathfrak{g}}_1)$-Higgs pair. □

Let $\hat{C} < \hat{G}_0$ be the $\hat{G}_0$-stabilizer of the $\mathfrak{sl}_2$-triple $\{f, h, e\}$. Note that $\{f, h, e, s\} \subset \mathfrak{g}$ generates a subalgebra isomorphic to $\mathfrak{gl}_2\mathbb{C}$, and $\hat{C}$ is the G-centralizer of the $\{f, h, e, s\}$. As in the JM-regular case, extending the structure group of the uniformizing Higgs bundle (E_T, e) to G_0 defines a polystable maximal $(G_0, \mathfrak{g}_1)$-Higgs pair $(E_T(G_0), e)$. Moreover, if $E_{\hat{C}}$ is a polystable $\hat{C}$-bundle, then $(E_T \otimes_{G_0} E_{\hat{C}}, e)$ is a polystable maximal $(G_0, \mathfrak{g}_1)$-Higgs pair. Thus we have a map

$$\begin{aligned} \hat{\Psi}_e : \ \mathcal{N}(\hat{C}) &\longrightarrow \mathcal{M}^{\max}(G_0, \mathfrak{g}_1) \\ E_{\hat{C}} &\longmapsto (E_T \otimes_{G_0} E_{\hat{C}}, e) \end{aligned} \tag{6.3}$$

from the moduli space of polystable $\hat{C}$-bundles on X to the moduli space of maximal $(G_0, \mathfrak{g}_1)$-Higgs pairs.

Theorem 6.16. *If $(G_0, \mathfrak{g}_1)$ is not JM-regular, then map $\hat{\Psi}_e : \mathcal{N}(\hat{C}) \to \mathcal{M}^{\max}(G_0, \mathfrak{g}_1)$ from (6.3) defines an isomorphism between the moduli space of polystable $\hat{C}$-bundles and the moduli space of polystable maximal $(G_0, \mathfrak{g}_1)$-Higgs pairs.*

Proof. Surjectivity of the map $\hat{\Psi}_e$ follows from Proposition 6.15 and the proof of surjectivity in the JM-regular case. For injectivity, note that two polystable $\hat{C}$-bundles $E_{\hat{C}}$ and $E'_{\hat{C}}$ define the

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same point in $\mathcal{M}^{\max}(\mathrm{G}_0, \mathfrak{g}_1)$ if there is a holomorphic isomorphism of G_0-bundles

$$\Phi : E_{\hat{\mathrm{C}}} \otimes_{\mathrm{G}_0} E_{\mathrm{T}} \to E'_{\hat{\mathrm{C}}} \otimes_{\mathrm{G}_0} E_{\mathrm{T}}$$

such that $\Phi^* e = e$. Unlike the JM-regular case, the G_0-stabilizer of e is not $\hat{\mathrm{C}}$. However, $E_{\hat{\mathrm{C}}} \otimes_{\mathrm{G}_0} E_{\mathrm{T}} \cong E'_{\hat{\mathrm{C}}} \otimes_{\mathrm{G}_0} E_{\mathrm{T}}$ implies that Φ acts trivially on E_{T}. In particular, Φ is valued in the G_0-centralizer of $\{h, e\}$. By the uniqueness part of the Jacobson–Morozov theorem, this is the G_0-centralizer of the $\mathfrak{sl}_2$-triple $\{f, h, e\}$ which is $\hat{\mathrm{C}}$. Hence, Φ induces a holomorphic isomorphism of the $\hat{\mathrm{C}}$-bundles. □

The rigidity results for variations of Hodge structure in the JM-regular case have analogues in the non-JM-regular setting. Namely, Consider the Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ which is $(-1)^j \mathrm{Id}$ on $\mathfrak{g}_j$ and choose the compact real form $\tau : \mathfrak{g} \to \mathfrak{g}$ such that $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ and $\tau(e) = -f$. Let $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$, $\mathrm{S}^{\mathbb{R}} < \mathrm{S}$, $\hat{\mathrm{C}}^{\mathbb{R}} < \hat{\mathrm{C}}$ be the associated real forms. The proof of the following theorem is immediate.

Theorem 6.17. *Suppose* $(\mathrm{G}_0, \mathfrak{g}_1)$ *is not JM-regular, and* $\rho : \pi_1(X) \to \mathrm{G}$ *is a reductive representation. The Higgs bundle associated to* ρ *is a maximal Hodge bundle of type* $(\mathrm{G}_0, \mathfrak{g}_1)$ *if and only if, up to conjugation,* $\rho = \rho_u * \rho_{\hat{\mathrm{C}}} : \pi_1(X) \to \mathrm{G}$, *where*

- $\rho_u : \pi_1(X) \to \mathrm{S}^{\mathbb{R}} < \mathrm{S}$ *is (a lift of) the uniformizing representation of* X, *and*
- $\rho_{\hat{\mathrm{C}}} : \pi_1(X) \to \hat{\mathrm{C}}^{\mathbb{R}} < \hat{\mathrm{C}}$ *is any representation into the compact real form of* $\hat{\mathrm{C}}$.

Remark 6.18. As in the JM-regular case, such representations factor through the real form $\mathrm{G}^{\mathbb{R}} < \mathrm{G}$. However, unlike the JM-regular setting, the $\mathrm{G}^{\mathbb{R}}$-centralizer of these representations is not necessarily compact since $\hat{\mathrm{C}}$ is not the full G-centralizer of the $\mathfrak{sl}_2$-triple $\{f, h, e\}$.

Note that Corollary 6.8 now holds without the JM-regular assumption.

Corollary 6.19. *Consider a PHVS* $(\mathrm{G}_0, \mathfrak{g}_1)$. *A variation of Hodge structure* (ρ, f_ρ) *associated to a maximal Hodge bundle of type* $(\mathrm{G}_0, \mathfrak{g}_1)$ *if and only if* $f_\rho : \widetilde{X} \to \mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}$ *is a totally geodesic embedding which maximizes the holomorphic sectional curvature. If* $(\mathrm{G}_0, \mathfrak{g}_1)$ *is not JM-regular then* f_ρ *factors through a maximal JM-regular subdomain*

$$f_\rho : \widetilde{X} \to \hat{\mathrm{G}}^{\mathbb{R}}/\hat{\mathrm{H}}_0^{\mathbb{R}} \hookrightarrow \mathrm{G}^{\mathbb{R}}/\mathrm{H}_0^{\mathbb{R}}.$$

Example 6.20. Recall from Examples 2.1 and 2.19 that the PHVS $(\mathrm{GL}_p\mathbb{C} \times \mathrm{SO}_q\mathbb{C}, M_{p,q})$ with $q < p$ is not JM-regular and a maximal JM-regular PHVSS is isomorphic to $(\mathrm{GL}_q\mathbb{C} \times \mathrm{GL}_{p-q}\mathbb{C} \times \mathrm{SO}_q\mathbb{C}, M_{q,q})$. Following Example 6.11, an $\mathrm{SO}_{2p+q}\mathbb{C}$-Higgs bundle (E, Q, Φ) is a Hodge bundle of type $(\mathrm{G}_0, \mathfrak{g}_1)$ if and only if E splits holomorphically as $V \oplus W \oplus V^*$, where V is a rank-p isotropic subbundle and the Higgs field is determined by a holomorphic map $\theta : W \to V \otimes K$ as in (6.2). Such a Higgs bundle is a maximal $(\mathrm{G}_0, \mathfrak{g}_1)$-Higgs pair if and only if V decomposes holomorphically as $V = WK^{-1} \oplus V_{q-p}$, where $\mathrm{rk}(V_q - p)$ is a rank-$(q-p)$ degree zero polystable vector bundle and

$$\theta = \begin{pmatrix} \mathrm{Id}_W \\ 0 \end{pmatrix} W \to (WK^{-1} \oplus V_{q-p}) \otimes K = V \otimes K.$$

The resulting representation $\rho : \pi_1(X) \to \mathrm{SO}_{2p+q}\mathbb{C}$ factors through $\mathrm{SO}(2q, q) \times \mathrm{U}_{p-q} < \mathrm{SO}(2p, q) < \mathrm{SO}_{2p+q}\mathbb{C}$.

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