Journal of Algebra 629 (2023) 307–357

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Openness of splinter loci in prime characteristic ☆

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ARTICLE INFO

Article history:
Received 23 October 2022
Available online 3 April 2023
Communicated by Claudia Polini

Keywords:
Splinter
Open loci
Trace ideals
F-purity
Frobenius splitting
Uniformly F-compatible ideals
Algebra closures
Absolute integral closure
Plus closure
Approximately Gorenstein rings

ABSTRACT

A splinter is a notion of singularity that has seen numerous recent applications, especially in connection with the direct summand theorem, the mixed characteristic minimal model program, Cohen–Macaulayness of absolute integral closures and cohomology vanishing theorems. Nevertheless, many basic questions about these singularities remain elusive. One outstanding problem is whether the splinter property spreads from a point to an open neighborhood of a noetherian scheme. Our paper addresses this problem in prime characteristic, where we show that a locally noetherian scheme that has finite Frobenius or that is locally essentially of finite type over a quasi-excellent local ring has an open splinter locus. In particular, all varieties over fields of positive characteristic have open splinter loci. Intimate connections are established between the openness of splinter loci and F-compatible ideals, which are prime characteristic analogues of log canonical centers. We prove the surprising fact that for a large class of noetherian rings with pure (aka universally injective) Frobenius, the splinter condition is detected by the splitting of a single generically étale finite extension. We also show that

☆ The first author was supported in part by an AMS-Simons travel grant, and the second author in part by NSF Grants DMS #1602070, #1707661, #1246844, and #2200716.
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https://doi.org/10.1016/j.jalgebra.2023.03.025

for a noetherian **N**-graded ring over a field, the homogeneous maximal ideal detects the splinter property.

1. Introduction

A noetherian ring is a splinter if it is a direct summand of every finite cover. A splinter is a notion of singularity since we now know that regular rings satisfy this property by the celebrated direct summand theorem [32,6,9]. For any notion of singularity, or more generally, a property $\mathcal{P}$ of noetherian local rings, it is natural to ask if

$$\{x \in X \colon \mathcal{O}_{X,x} \text{ has } \mathcal{P}\}$$

is an open subset of a locally noetherian scheme X. Such openness of loci questions were perhaps first considered systematically by Grothendieck in [24]. Many fundamental local properties such as R_n, S_n, reduced, normal, Gorenstein, complete intersection, Cohen–Macaulay, among others, are known to have open loci for *most* locally noetherian schemes that one encounters in arithmetic or geometry [24,22,74].

In this paper we consider the question of the openness of the splinter locus of a locally noetherian scheme. As a preliminary observation, the splinter condition for noetherian local **Q**-algebras is equivalent to normality, and the normal loci is open for locally noetherian schemes that have open regular loci [24, Cor. (6.13.5)]. In particular, the splinter locus of any quasi-excellent **Q**-scheme is open because the normal locus of such a scheme is open. Our main result illustrates that a similar result holds for some large classes of locally noetherian schemes over $\mathbf{F}_p$.

Theorem 1.0.1. (see Theorem 4.3.7) *Let X be a scheme of prime characteristic $p > 0$ that satisfies any of the following conditions:*

(i) X is locally noetherian and F-finite.
(ii) X is locally essentially of finite type over a noetherian local ring $(A, \mathfrak{m})$ of prime characteristic $p > 0$ with geometrically regular formal fibers.

Then $\{x \in X \colon \mathcal{O}_{X,x}$ is a splinter$\}$ is open in X.

In particular, the splinter locus of any scheme of finite type over a field, or more generally, a complete local ring of positive prime characteristic is open.

Showing the openness of the splinter locus has proved to be a challenging problem in prime and mixed characteristics because splinters are far more mysterious away from equal characteristic 0. For example, in prime and mixed characteristics, splinters surprisingly coincide with a derived counterpart called a derived splinter [8,10], and excellent

local splinters are Cohen–Macaulay and pseudo-rational [62,64,10] (in prime characteristic they are also F-rational [62]). Furthermore, prime characteristic splinters are conjecturally equivalent to F-regular singularities [60,14], which are analogues of Kawamata log terminal singularities that arise in the birational classification of algebraic varieties over the complex numbers [64,53,26]. This last conjectural equivalence implies one of the outstanding problems in tight closure theory, namely that weak F-regularity is preserved under localization. The notion of a splinter globalizes, and in a non-affine setting in mixed and positive characteristics they have been recently called *globally +-regular schemes* [12]. This is in part because of their similarities with globally F-regular varieties in positive characteristic [66,67].

By working on an affine cover, the openness of loci for a local property reduces to a question about affine schemes, and Theorem 1.0.1 follows from the following, more refined, affine result.

Theorem 1.0.2. (see Theorem 4.3.1) *Let R be a noetherian F-pure domain of prime characteristic $p > 0$ and assume that R satisfies any of the following conditions:*

(i) *R is F-finite.*
(ii) *R is local (not necessarily excellent).*
(iii) *$(A, \mathfrak{m})$ is a noetherian local ring of prime characteristic $p > 0$ with geometrically regular formal fibers and R is essentially of finite type over A.*

Let $\mathcal{C}$ be the collection of finite R-subalgebras of R^+ and for an R-algebra S, let

$$\tau_{S/R} \coloneqq \operatorname{im}(\operatorname{Hom}_R(S, R) \xrightarrow{\text{eval @1}} R).$$

Then we have the following:

(1) *$\{\tau_{S/R} \colon S \in \mathcal{C}\}$ is a finite set of radical ideals of R.*
(2) *The splinter locus of $\operatorname{Spec}(R)$ is open and its complement is $\mathbf{V}(\tau_R)$, where*

$$\tau_R \coloneqq \bigcap_{S \in \mathcal{C}} \tau_{S/R}.$$

(3) *There exists $S \in \mathcal{C}$ such that $R \subseteq S$ is generically étale and if $R \hookrightarrow S$ splits, then R is a splinter.*

Here R^+ denotes the absolute integral closure of R, that is, R^+ is the integral closure of R in an algebraic closure of its fraction field. F-purity is the universal injectivity of the Frobenius map on R. It is a mild assumption when discussing questions pertaining to splinters in prime characteristic because splinters are automatically F-pure. A surprising aspect of Theorem 1.0.2 is perhaps the fact that for most F-pure noetherian domains that arise in arithmetic and geometry, the splinter property is determined by the splitting of

a *single* generically étale finite extension domain, even though the definition of a splinter a priori requires the splitting of all finite extension domains. Theorem 1.0.1 is a formal consequence of Theorem 1.0.2 because the splinter locus is contained in the F-pure locus of any locally noetherian $\mathbf{F}_p$-scheme, and the F-pure locus is known to be open when X satisfies the assumptions of Theorem 1.0.1.

The proof of Theorem 1.0.2 is not particularly involved in the noetherian F-finite setting, so we briefly discuss our strategy for the expert. The ideal $\tau_{S/R}$ of Theorem 1.0.2 is called the *trace of S over R*, and the content of the Theorem is that these traces stabilize as S ranges over the finite R-subalgebras of R^+ with appropriate purity assumptions on R. The stabilization of traces follows from the fact that trace ideals satisfy the property of being *uniformly F-compatible*, which is a prime characteristic analogue of the notion of a center of log canonicity [57,58,70]. For us, the key fact about uniformly F-compatible ideals is their finiteness under appropriate assumptions. Namely, Schwede showed that an F-pure noetherian F-finite ring has only finitely many uniformly F-compatible ideals [57]. In fact, an explicit bound on the Hilbert–Samuel multiplicity of an F-pure noetherian local ring [41] allows one to obtain explicit bounds for the number of uniformly F-compatible ideals of a given coheight in the local setting (see also [70] and Proposition 3.4.1). Thus, the finiteness of uniformly F-compatible ideals for a noetherian F-finite Frobenius split domain R readily implies the finiteness of the set of trace ideals of finite extensions R. One then shows that the stable trace ideal has to define the non-splinter locus.

The drawback of the above approach is that there is typically no control over when the trace ideals of finite extensions of a noetherian Frobenius split ring R stabilize. Thus, it feels hopeless to obtain a more explicit description of the ideal τ_R that defines the closed non-splinter locus of R via the approach of uniformly F-compatible ideals. We devote a significant portion of our paper to obtaining a better understanding of τ_R. Our strategy involves looking at the plus closure operation. Just as tight closure detects weak F-regularity, *plus closure* detects the splinter property in the sense that a noetherian domain is a splinter precisely when all ideals of the domain are plus closed. By analyzing closure operations associated with R-algebras [55,36], we show that the ideal τ_R that defines the non-splinter locus of R in Theorem 1.0.2 is the *big test ideal* of plus closure. Said differently, τ_R coincides with the ideal that one morally expects to define the non-splinter locus of a noetherian domain.

Proposition 1.0.3. (see Propositions 3.3.1, 4.1.4 and Corollary 3.3.3) *Let R be an approximately Gorenstein noetherian domain of arbitrary characteristic (i.e. without any restrictions on characteristic) and let $\mathcal{C}$ be the collection of finite R-subalgebras of R^+. Then we have the following:*

(1) The ideal $\tau_R \coloneqq \bigcap_{S\in\mathcal{C}} \tau_{S/R}$ equals the big plus closure test ideal $\bigcap_I (I : IR^+ \cap R)$. Here the latter intersection ranges over all ideals of R.
(2) If R is complete local and B is an R-algebra, then $\tau_{B/R} = \bigcap_I (I : IB \cap R)$.

The class of approximately Gorenstein rings is fairly broad and includes noetherian normal rings and reduced locally excellent rings. Taking $B = R^+$ in Proposition 1.0.3(2), we see that when R is a complete local domain, the image of the map $\mathrm{Hom}_R(R^+, R) \xrightarrow{\mathrm{eval}\,@1} R$ equals $\bigcap_I (I : IR^+ \cap R)$, which in turn equals τ_R. This observation recovers a result of Hochster and Zhang that to the best of our knowledge has not appeared in print. We refer the reader to Subsection 3.3 for further details on test ideals of algebra closures, where among other things, we partially answer a question raised by Pérez and R.G. [55] in the affirmative about the equality of big and finitistic test ideals of closure operations associated with algebras and certain modules.

Proposition 1.0.3(1) and ideal theoretic results from our work on permanence properties of splinters [19] allow us to obtain some transformation rules for the splinter ideal τ_R under Henselizations and completions.

Proposition 1.0.4. (see Proposition 4.1.7 and Corollary 4.3.4) *Let $(R, \mathfrak{m})$ be a noetherian normal domain of arbitrary characteristic with geometrically regular formal fibers. Then we have the following:*

(1) If R^h is the Henselization of R with respect to $\mathfrak{m}$, then $\tau_{R^h} \cap R = \tau_R$.
(2) If $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, then $\tau_{\widehat{R}} \cap R = \tau_R$.

If R is additionally F-pure, then in (1) we have $\tau_R R^h = \tau_{R^h}$ and in (2) we have $\tau_R \widehat{R} = \tau_{\widehat{R}}$.

In fact, in part (1) of Proposition 1.0.4 one does not need any assumptions on the formal fibers of R. One should compare Proposition 4.1.7 with the transformation rules for the big tight closure test ideal under Henselizations and completions. We expect the equalities $\tau_R R^h = \tau_{R^h}$ and $\tau_R \widehat{R} = \tau_{\widehat{R}}$ to hold without restrictions on the characteristic or singularities of normal noetherian local domains, although we are unable to show this at present.

As an application of the openness of the splinter locus and the ascent of the splinter property under étale maps [19, Thm. A], we show that the splinter condition for noetherian $\mathbf{N}$-graded rings over fields is detected by the homogenous maximal ideal. This is an analogue of Smith and Lyubeznik's result that weak F-regularity is detected by the homogeneous maximal ideal [61, Cor. 4.6].

Corollary 1.0.5. (See Corollary 4.3.10) *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a noetherian graded ring such that $R_0 = k$ is a field. Let $\mathfrak{m} := \bigoplus_{n>0} R_n$ be the homogeneous maximal ideal of R. Then R is a splinter if and only if $R_{\mathfrak{m}}$ is a splinter.*

We expect the splinter loci to be open for arbitrary excellent schemes in *any* characteristic. At the same time, one can use a meta construction of Hochster [33] to give examples of locally excellent (but not excellent) noetherian domains whose splinter loci

are not open. The surprising aspect of Hochster's construction is that the local rings of these locally excellent domains are essentially of finite type over appropriate fields. We refer the reader to [19, Ex. 4.0.3] for more details.

Structure of the paper: In Section 2 we discuss the splinter property, the notion of approximately Gorenstein rings and the Frobenius map. In Section 3 we discuss uniformly F-compatible ideals and their finiteness, traces of algebras and a closely related notion which we call the ideal trace, and test ideals associated to closure operations arising from algebras. In Section 4 we first identify and prove properties of a candidate ideal that detects the non-splinter locus of noetherian domains in arbitrary characteristic under certain stability assumptions. We also discuss properties of a separable version of this ideal using Singh's work on the separable plus closure [59]. We finally specialize to prime characteristic and prove our main results.

Conventions: All rings are commutative with identity. For the most part rings in this paper will be noetherian. However, we will use the absolute integral closure of a domain, which is a highly non-noetherian ring. We say an R-algebra S is *solid* if there exists a nonzero R-linear map $S \to R$.

2. Preliminaries

2.1. Splinters

Let us introduce the main objects of investigation in this paper.

Definition 2.1.1. A noetherian ring A is a *splinter* if every finite ring map $A \to B$ which is surjective on Spec admits an A-linear left-inverse.

Hochster's famous direct summand conjecture, now a theorem [32,25,6,9], is the assertion that noetherian regular rings are splinters. Splinters are always normal, and conversely, a noetherian normal $\mathbf{Q}$-algebra is always splinter. Thus, splinters are an interesting notion of singularity mainly in prime and mixed characteristics. Many naturally arising classes of rings are splinters. For example, since a direct summand of a regular ring is a splinter, it follows that coordinate rings of normal affine toric varieties, Veronese subrings of polynomial rings over fields and rings of invariants of finite groups acting on regular rings where the order of the group is prime to the characteristic of the ring are splinters. Moreover, generic determinantal rings over fields are splinters since they are normal in characteristic 0 and F-regular in characteristic p [31] and (weakly) F-regular rings are splinters [31].

A simple spreading out argument shows that the splinter condition can be checked locally. Furthermore, because a finite direct product of noetherian rings is a splinter precisely when the individual factors are splinters, questions about splinters often immediately reduce to the domain case. We refer the reader to [32,35,46,8,47,19] for various

properties of these singularities. The definition of a splinter and its derived variant can also be made in a non-Noetherian setting, and interesting non-Noetherian rings, such as valuation rings, are derived splinters [3].

2.2. Approximately Gorenstein rings and purity

Let A be a ring. An A-linear map $M \to N$ is *pure* (also called *universally injective*) if for all A-modules P, the induced map

$$M \otimes_A P \to N \otimes_A P$$

is injective. We say $M \to N$ is *cyclically pure* if for all cyclic A-modules P, $M \otimes_A P \to N \otimes_A P$ is injective, or equivalently, if for all ideals I of A, $M/IM \to N/IN$ is injective.

Split maps are pure and pure maps are cyclically pure. Filtered colimits of (cyclically) pure maps are also (cyclically) pure, and faithfully flat ring maps are pure. Pure ring maps induce the affine covers for the canonical Grothendieck topology, which is the finest topology for which all representable presheaves are sheaves [4].

Hochster characterized noetherian rings A with the property that every A-linear map $A \to M$ that is cyclically pure is also pure [34]. His characterization utilizes the following notion.

Definition 2.2.1. A noetherian local ring $(A, \mathfrak{m})$ is approximately Gorenstein if for all $n \in \mathbf{Z}_{>0}$, there exists an ideal I such that $I \subseteq \mathfrak{m}^n$ and A/I is Gorenstein. A noetherian ring is *approximately Gorenstein* if all its localizations at maximal ideals are approximately Gorenstein local rings.

The ideals I in Definition 2.2.1 can be chosen to be $\mathfrak{m}$-primary [34, Prop. 2.1]. The class of approximately Gorenstein rings include noetherian normal rings, noetherian local rings of depth at least 2, reduced locally excellent rings, and more generally, noetherian rings whose local rings at maximal ideals are formally reduced. All of this is directly proved, or implied by the results in [34]. The point of introducing the approximately Gorenstein property is the following result.

Proposition 2.2.2. *[34, Thm. (2.6)] Let A be a noetherian ring. The following are equivalent:*

(1) A is approximately Gorenstein.
(2) Any ring map $A \to B$ that is cyclically pure is pure.
(3) If M is an A-module, then any A-linear map $A \to M$ that is cyclically pure is pure.

The approximately Gorenstein property implies that a noetherian domain A is a splinter precisely when $A \to A^+$ is cyclically pure [19, Lem. 2.3.1]. This observation

leads to various equivalent interpretations of an ideal that defines the non-splinter locus of a noetherian domain in Section 4, making this ideal easier to work with. One such interpretation involves big test ideals of algebra closures (Proposition 3.3.1(4)), for which we need the following result.

Lemma 2.2.3. *Let A be an approximately Gorenstein ring. Suppose that there exists a maximal ideal $\mathfrak{m}$ of A and an essential extension $A/\mathfrak{m} \hookrightarrow M$, where M is a finitely generated A-module. Then there exists an $\mathfrak{m}$-primary ideal I of A such that A/I is Gorenstein and M embeds in A/I.*

Proof. The annihilator J of M is $\mathfrak{m}$-primary. Since $A_{\mathfrak{m}}$ is approximately Gorenstein, there exists an $\mathfrak{m}$-primary ideal I of A such that $I \subseteq J$ and $A/I = A_{\mathfrak{m}}/IA_{\mathfrak{m}}$ is Gorenstein. Then $A/\mathfrak{m} \hookrightarrow M$ is an essential extension of A/I-modules, and since A/I is a zero-dimensional Gorenstein ring, it is an injective module over itself. As $\mathfrak{m}$ is the associated prime of A/I, we have an inclusion $A/\mathfrak{m} \hookrightarrow A/I$, which extends to a map $f\colon M \to A/I$. Then f is injective because $A/\mathfrak{m} \hookrightarrow M$ is essential. □

2.3. Frobenius

Let R be a ring of prime characteristic $p > 0$. Then we have the (absolute) Frobenius map

$$F\colon R \to R$$

that maps $r \mapsto r^p$. For $e \in \mathbf{Z}_{>0}$ we also have the e-th interate F^e of the Frobenius. The target copy of R regarded as an R-module by restriction of scalars along F^e will be denoted $F^e_* R$, and for $x \in R$ the same element viewed in $F^e_* R$ will be denoted $F^e_* x$. Thus, if $r \in R$ and $F^e_* x \in F^e_* R$, $r \cdot F^e_* x = F^e_* r^{p^e} x$.

We say R is *F-finite* if the Frobenius map is finite (equivalently, of finite type). We say R is *Frobenius split* if F admits an R-linear left-inverse $F_* R \to R$, and R is F-pure if F is a pure map. Frobenius splitting and F-purity do not coincide in general even for nice rings. For example, the first author and Murayama recently constructed examples of excellent Henselian regular local rings of Krull dimension 1 which are not Frobenius split [16]. Regular rings of prime characteristic are always F-pure because the Frobenius map of a regular ring is faithfully flat [43]. When R is noetherian, purity of Frobenius is the same as cyclic purity [19, Rem. 2.2.2], so we will not introduce a new definition for when the Frobenius is cyclically pure even though such a definition would make sense in a non-noetherian context.

If R is a splinter of prime characteristic $p > 0$, then R is F-pure because the Frobenius map can be expressed as a filtered colimit of finite purely inseparable maps which are all split by the definition of the splinter property.

Given a noetherian R of prime characteristic $p > 0$, an R-module M and a submodule N of M, the *tight closure of N in M*, denoted N^*_M is the set of elements $m \in M$ for which there exists $c \in R$ not in any minimal prime such that for all $e \gg 0$,

$$F^e_* c \otimes m \in \operatorname{im}(F^e_* R \otimes_R N \to F^e_* R \otimes_R M).$$

For $M = R$ and $N = I$ an ideal, the tight closure of I in R is usually denoted I^* and consists of elements $r \in R$ for which there exists a c not in any minimal prime such that $cr^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. Here $I^{[p^e]}$ is the ideal generated by the p^e-th powers of elements of I. Alternatively, $F^e_* I^{[p^e]} = IF^e_* R$, the expansion of I to $F^e_* R$.

3. Uniformly F-compatible and trace ideals

Throughout this section, R will denote a ring of prime characteristic $p > 0$.

3.1. Uniformly F-compatible ideals

Following Schwede [58, Def. 3.1], we make the following definition in a non-F-finite setting.

Definition 3.1.1. An ideal I of R is *uniformly F-compatible* if for all $e \in \mathbf{Z}_{>0}$ and all R-linear maps $\varphi\colon F^e_* R \to R$, we have $\varphi(F^e_* I) \subseteq I$.

Said differently, I is uniformly F-compatible if for any φ as above, we have an induced R/I-linear map $\overline{\varphi}\colon F^e_*(R/I) \to R/I$ such that the diagram

$$\begin{array}{ccc} F^e_* R & \xrightarrow{\varphi} & R \\ \downarrow{\scriptstyle F^e_*\pi} & & \downarrow{\scriptstyle \pi} \\ F^e_*(R/I) & \xrightarrow{\overline{\varphi}} & R/I \end{array}$$

commutes, where $\pi\colon R \to R/I$ is the canonical projection.

Uniformly F-compatible ideals are related to Mehta and Ramanathan's notion of compatibly split subschemes of a Frobenius split scheme [52], and indeed were named so by Schwede because of this connection. In the noetherian local case, uniformly F-compatible ideals are dual to Smith and Lyubeznik's $\mathcal{F}(E)$-submodules of E [45], where E is the injective hull of the residue field of the noetherian local ring $(R, \mathfrak{m})$.

We now collect some well-known properties of uniformly F-compatible ideals.

Lemma 3.1.2. *Let R be a ring of prime characteristic $p > 0$ and let Σ denote the collection of uniformly F-compatible ideals of R. Then we have the following:*

(1) Σ is closed under arbitrary sums and arbitrary intersections.

(2) If $I \in \Sigma$ and $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$, then $\mathfrak{p} \in \Sigma$.
(3) If R is noetherian and $I \in \Sigma$, then $\sqrt{I} \in \Sigma$.
(4) If R is noetherian, then $\mathrm{nil}(R) \in \Sigma$.
(5) If R is Frobenius split, then every element of Σ is a radical ideal.

Proof. (1) follows readily from the definition of uniform F-compatibility. For (2), assume $\mathfrak{p}$ is an associated prime of I. Then there exists $a \notin I$ such that $\mathfrak{p} = (I : a)$. Suppose $\varphi\colon F^e_* R \to R$ is an R-linear map. To show that $\varphi(F^e_*\mathfrak{p}) \subseteq \mathfrak{p}$ it suffices to show that for any $r \in \mathfrak{p}$, $\varphi(F^e_* r)a \in I$. But

$$\varphi(F^e_* r)a = \varphi(F^e_* a^{p^e} r) \in I$$

because $a^{p^e} r = a^{p^e-1}(ar) \in I$ and I is uniformly F-compatible by assumption. Part (3) follows from (1) and (2) because $\sqrt{I}$ is the intersection of the minimal primes of I, which are associated primes of R/I when R is noetherian. Part (4) follows from (3) because the zero ideal is uniformly F-compatible. Finally, (5) follows from the fact that if $\varphi\colon F_* R \to R$ is a Frobenius splitting of R, then the induced map $\overline{\varphi}\colon F_*(R/I) \to R/I$ is a Frobenius splitting of R/I, and any Frobenius split ring is reduced. □

Remark 3.1.3. Fixing an R-linear map $\varphi\colon F^e_* R \to R$, one can also consider the set of ideals I of R that are *φ-compatible* in the sense that $\varphi(F^e_* I) \subseteq I$. Thus, a uniformly F-compatible ideal is one that is compatible with any map $F^e_* R \to R$, for all $e > 0$. Properties (1)-(4) of Lemma 3.1.2 are also satisfied by the set of ideals compatible with a fixed φ, while property (5) is satisfied if φ is a Frobenius splitting.

Remark 3.1.4. For a fixed R-linear map $\varphi\colon F^e_* R \to R$, one should make a point to contrast the φ-compatible ideals as in Remark 3.1.3 with those ideals I satisfying the stronger condition that $\varphi(F^e_* I) = I$. There is no distinction when φ is surjective; when φ is not surjective, such ideals are known to satisfy finiteness properties akin to those for the ideals compatible with Frobenius splittings. Similar remarks apply to the definition of uniformly compatible ideals in general. See [7,11] for further details.

3.2. Trace and ideal trace

The most important examples of uniformly F-compatible ideals in this paper are traces of ring maps.

Definition 3.2.1. Let A be a ring (not necessarily of prime characteristic) and B be an A-algebra. Then the *trace of B over A*, denoted $\tau_{B/A}$, is

$$\tau_{B/A} \coloneqq \mathrm{im}(\mathrm{Hom}_A(B, A) \xrightarrow{\mathrm{eval}\,@1} A).$$

Thus, $\tau_{B/A} \neq 0$ precisely when B admits a nonzero A-linear map $B \to A$, that is, if B is a *solid* A-algebra in the terminology of Hochster [36]. Similarly, A is a direct summand of B precisely when $\tau_{B/A} = A$.

The next example shows that traces are related to the field trace from linear algebra.

Example 3.2.2. Suppose $A \hookrightarrow B$ is a finite extension of noetherian normal domains which is étale in codimension 1 (that is, for all height 1 prime ideals $\mathfrak{q}$ of B, $A_{\mathfrak{q}\cap A} \to B_{\mathfrak{q}}$ is essentially (aka local) étale). If K (resp. L) is the fraction field of A (resp. B), then the field trace

$$\mathrm{Tr}\colon L \to K$$

is a nonzero map because L/K is separable [69, Tag 0BIL]. Since A is normal, $\mathrm{Tr}(B) \subseteq A$ because the minimal polynomial of any element of B over K has coefficients in A [69, Tag 0BIH]. Thus, restricting Tr to B induces a nonzero A-linear map $B \to A$, which, abusing notation, we also denote by Tr. Then it is well-known that $\mathrm{Hom}_A(B, A)$ is generated as a B-module by Tr (one can apply [72, Prop. 4.8] using the fact that the ramification divisor of a map étale in codimension 1 is trivial). That is, any A-linear map $B \to A$ is of the form $\mathrm{Tr}(b \cdot \_)$, for some $b \in B$. Consequently, $\tau_{B/A}$ coincides with the image of the trace map $\mathrm{Tr}\colon B \to A$.

Lemma 3.2.3. *Let A be a ring and B be an A-algebra.*

(1) $\tau_{B/A} = \sum \mathrm{im}(\varphi)$, *where φ ranges over all elements of* $\mathrm{Hom}_A(B, A)$.
(2) If A has prime characteristic $p > 0$, then $\tau_{B/A}$ is uniformly F-compatible.
(3) If B is finitely presented as an A-module, then for all prime ideals $\mathfrak{p}$ of A, $\tau_{B_{\mathfrak{p}}/A_{\mathfrak{p}}} = (\tau_{B/A})_{\mathfrak{p}}$.
(4) If $A \to C$ is a flat ring map and B is a finitely presented A-module, then $\tau_{B/A}C = \tau_{B\otimes_A C/C}$.

Proof. (1) The inclusion $\tau_{B/A} \subseteq \sum \mathrm{im}(\varphi)$ follows by the definition of trace. Suppose $a \in \mathrm{im}(\varphi)$. Choose $b \in B$ such that $\varphi(b) = a$. Then a is the image of $1 \in B$ under the composition

$$B \xrightarrow{b\cdot} B \xrightarrow{\varphi} A,$$

and so, $a \in \tau_{B/A}$. Thus, $\mathrm{im}(\varphi) \subseteq \tau_{B/A}$ for all $\varphi \in \mathrm{Hom}_A(B, A)$, so we get the other containment.

In the generality stated (i.e. without assuming A is noetherian or B is module finite over A), (2) is proved in [17, Prop. 8.5(i)]. We reproduce the proof for the reader's convenience since the argument is straightforward and the result is crucial for this paper. We want that if $\phi : F^e_* A \to A$ is an A-linear map, then $\phi(F^e_* \tau_{B/A}) \subseteq \tau_{B/A}$. So let $F^e_* b \in F^e_* \tau_{B/A}$. Choose an A-linear map

$$\eta_b : B \to A$$

such that $\eta_b(1) = b$. Then the composition

$$B \xrightarrow{F^e} F^e_* B \xrightarrow{F^e_* \eta_b} F^e_* A \xrightarrow{\phi} A$$

is an A-linear map that sends $1 \mapsto \phi(F^e_* b)$. Thus, $\phi(F^e_* b) \in \tau_{B/A}$, and so, $\phi(F^e_* \tau_{B/A}) \subseteq \tau_{B/A}$.

For (3), the hypothesis that B is finitely presented as an A module implies that

$$A_{\mathfrak{p}} \otimes_A \operatorname{Hom}_A(B, A) = \operatorname{Hom}_{A_{\mathfrak{p}}}(B_{\mathfrak{p}}, A_{\mathfrak{p}}).$$

Applying $A_{\mathfrak{p}} \otimes_A -$ to $\operatorname{Hom}_A(B, A) \xrightarrow{\text{eval } @1} A$ gives

$$\operatorname{Hom}_{A_{\mathfrak{p}}}(B_{\mathfrak{p}}, A_{\mathfrak{p}}) \xrightarrow{\text{eval } @1} A_{\mathfrak{p}}.$$

Since localization commutes with taking images of linear maps, we get the desired result.

(4) is a generalization of (3). We have an exact sequence

$$\operatorname{Hom}_A(B, A) \xrightarrow{\text{eval } @1} A \to A/\tau_{B/A} \to 0.$$

Applying $C \otimes_A -$ to the above sequence and using the fact that Hom commutes with flat base change when the first argument is finitely presented, we get the exact sequence

$$\operatorname{Hom}_C(B \otimes_A C, C) \xrightarrow{\text{eval } @1} C \to C/\tau_{B/A}C \to 0.$$

Then $\tau_{B \otimes_A C/C} = \operatorname{im}(\operatorname{Hom}_C(B \otimes_A C, C) \xrightarrow{\text{eval } @1} C) = \ker(C \to C/\tau_{B/A}C) = \tau_{B/A}C$. □

We now deduce some non-obvious consequences of the previous results.

Corollary 3.2.4. *Let R be a ring of prime characteristic $p > 0$ and S be an R-algebra.*

(1) If R is Frobenius split, then $\tau_{S/R}$ is a radical ideal.
(2) If $(R, \mathfrak{m})$ is a complete local noetherian domain and S is a big Cohen–Macaulay R-algebra (for example, $S = R^+$), then $\tau_{S/R}$ is a nonzero uniformly F-compatible ideal.

Proof. (1) follows by Lemma 3.2.3(2) and Lemma 3.1.2(3).

For (2), by the defining property of a big Cohen–Macaulay R-algebra, it follows that if $d = \dim(R)$, then

$$H^d_{\mathfrak{m}}(S) \neq 0.$$

Consequently, $\operatorname{Hom}_R(S, R) \neq 0$ by [36, Cor. 2.4], or equivalently, $\tau_{S/R}$ is nonzero. □

Example 3.2.5. There are other important examples of uniformly F-compatible ideals.

(1) Suppose R is a reduced F-finite noetherian ring. The *big* or *non-finitistic* test ideal $\tau_b(R)$ of R is a uniformly F-compatible ideal, which is also the smallest (with respect to inclusion) uniformly F-compatible ideal of R that is not contained in any minimal prime [75,40,58]. Blickle, Schwede and Tucker have shown that when R is additionally a normal, F-finite and $\mathbf{Q}$-Gorenstein domain, then $\tau_b(R)$ can be realized as the trace of some finite and generically étale extension S of R [13, Thm. 4.6]. In a similar vein, Polstra and Schwede have proved that in the $\mathbf{Q}$-Gorenstein setting, one can often realize any uniformly F-compatible ideal of R as a trace of some finite extension of R [56]. However, it is not known if $\tau_b(R)$ can be realized as trace of a solid R-algebra S when R is not $\mathbf{Q}$-Gorenstein. We will show in Corollary 3.3.5 that the finitistic test ideal is recoverable as a trace of a big Cohen-Macaulay algebra for an arbitrary complete local domain of prime characteristic.

(2) Suppose $\mathcal{C}$ is a collection of ideals of a noetherian ring R of prime characteristic $p > 0$ that is closed under Frobenius powers, that is, $I \in \mathcal{C} \Rightarrow I^{[p^e]} \in \mathcal{C}$ for all $e \in \mathbf{Z}_{>0}$. We claim that

$$\mathcal{I} \coloneqq \bigcap_{I \in \mathcal{C}} (I : I^*)$$

is uniformly F-compatible. Here I^* denotes the tight closure of I. Suppose $c \in \mathcal{I}$ and $\varphi \colon F^e_* R \to R$ is an R-linear map. We have to show that $\varphi(F^e_* c) \in \mathcal{I}$, that is, for $I \in \mathcal{C}$ and $z \in I^*$, we want $\varphi(F^e_* c)z \in I$. Now $z \in I^*$ implies $z^{p^e} \in (I^{[p^e]})^*$. This follows by choosing $d \in R^\circ$ such that $d(z^{p^e})^{p^f} = dz^{p^{e+f}} \in I^{[p^{e+f}]} = (I^{[p^e]})^{[p^f]}$ for $f \gg 0$. Now $I \in \mathcal{C} \Rightarrow I^{[p^e]} \in \mathcal{C}$. So by the choice of c, we have $cz^{p^e} \in c(I^{[p^e]})^* \subseteq I^{[p^e]}$. Then $\varphi(F^e_* c)z = \varphi(F^e_* cz^{p^e}) \in \varphi(F^e_* I^{[p^e]}) \subseteq I$, as desired. If R is an approximately Gorenstein ring (for example, an excellent reduced ring or a normal ring), then taking $\mathcal{C}$ to be the collection of all ideals of R, the ideal $\mathcal{I}$ is the *finitistic test ideal* of R [29, Prop. (8.15)]. Similarly, taking $\mathcal{C}$ to be the collection of parameter ideals of R, the ideal $\mathcal{I}$ is the *parameter test ideal* of R [63, Def. 4.3]. Thus, both types of test ideals are uniformly F-compatible for nice rings.

The trace $\tau_{B/A}$ detects whether $A \to B$ splits. However, splitting is not a good notion for maps $A \to B$ without finiteness assumptions on B. The better notion then is that of purity, and we now introduce a uniformly F-compatible that detects purity of $A \to B$ in most cases of interest.

Definition 3.2.6. Let A be a ring and B be an A-algebra. Then the *ideal trace* of B/A, denoted $T_{B/A}$, is

$$T_{B/A} \coloneqq \bigcap_I (I : IB \cap A),$$

where the intersection ranges over all ideal I of A.

Ideal traces satisfy the following elementary properties.

Lemma 3.2.7. *Let A be a ring and B be an A-algebra. Suppose $\mathcal{C}$ is the set of ideals of A.*

(1) $T_{B/A} = A$ if and only if $A \to B$ is cyclically pure.
(2) If A is an approximately Gorenstein ring, then $T_{B/A} = A$ if and only if $A \to B$ is pure.
(3) We have $\tau_{B/A} \subseteq T_{B/A}$.
(4) Suppose A has prime characteristic $p > 0$. Then for a fixed ideal I of A,

$$T_I \coloneqq \bigcap_{e \geq 0} (I^{[p^e]} : I^{[p^e]}B \cap A)$$

is uniformly F-compatible.
(5) Suppose A has prime characteristic $p > 0$. Then $T_{B/A}$ is uniformly F-compatible.

Proof. (1) follows from the fact that for an ideal I of A, $A/I \to B/IB$ is injective if and only if $I = IB \cap A$, or equivalently, that $(I : IB \cap A) = A$.

(2) follows from (1) and Proposition 2.2.2 due to Hochster.

(3) Let $c \in \tau_{B/A}$, and choose $\varphi \in \mathrm{Hom}_A(B, A)$ such that $\varphi(1) = c$. Then for any ideal I of A,

$$c(IB \cap A) = \varphi(1)(IB \cap A) = \varphi(IB \cap A) \subseteq \varphi(IB) \subseteq I,$$

where the second equality and the last containment of sets follow by A-linearity of φ.

(4) Suppose $c \in T_I$ and $\varphi : F^e_* A \to A$ is an A-linear map. For $f \geq 0$, let $z \in I^{[p^f]}B \cap A$. Then $z^{p^e} \in I^{[p^{e+f}]}B \cap A$, and so,

$$\varphi(F^e_* c)z = \varphi(F^e_* c z^{p^e}) \in \varphi(F^e_* I^{[p^{e+f}]}) \subseteq I^{[p^f]}.$$

Thus, $\varphi(F^e_* c)(I^{[p^f]}B \cap A) \subseteq I^{[p^f]}$ for all $f \geq 0$, which shows $\varphi(F^e_* c) \in T_I$, and hence, the uniform F-compatibility of T_I.

(5) It is clear that

$$T_{B/A} = \bigcap_{I \in \mathcal{C}} T_I,$$

and since an arbitrary intersection of uniformly F-compatible ideals is uniformly F-compatible (Lemma 3.1.2), by (4) we conclude that $T_{B/A}$ is uniformly F-compatible. □

Remark 3.2.8. In general, the containment $\tau_{B/A} \subseteq T_{B/A}$ is strict even for nice rings A. For example, the first author and Murayama have recently constructed examples of

excellent Henselian regular local rings A of Krull dimension 1 and prime characteristic $p > 0$ that admit no nonzero A-linear maps $F_*A \to A$ [16]. For such a ring, $\tau_{F_*A/A} = 0$. However, the Frobenius $F : A \to F_*A$ is faithfully flat [43], hence is pure, and hence is also cyclically pure. Therefore $T_{F_*A/A} = A$. This example is extreme in the sense that F_*A is not a solid A-algebra. Thus one can ask the following question: suppose B is a solid A-algebra and $T_{B/A} = A$. Then does it follow that $\tau_{B/A} = A$ when A is approximately Gorenstein? The question has an affirmative answer if $A \to B$ is finite or if A is complete, because then $A \to B$ is pure, and hence split, by [39, Cor. 5.2] in the finite case and by a lemma due to Auslander in the complete case (see proof of Corollary 3.4.4).

3.3. Algebra closures, traces and ideal traces

Let A be a noetherian ring of arbitrary characteristic. Then for any A-algebra B, Pérez and R.G. define an associated closure operation cl_B that satisfies many of the properties of tight closure [55, Def. 2.4]. Namely, for an arbitrary A-module M and a submodule N of M, an element $m \in M$ is in $N_M^{\mathrm{cl}_B}$, the cl_B *closure of* N *in* M, if

$$1 \otimes m \in \mathrm{im}(B \otimes_A N \to B \otimes_A M),$$

where $B \otimes_A N \to B \otimes_A M$ is obtained by tensoring $N \subseteq M$ by id_B.

For example, if $M = A$ and $N = I$ is an ideal, then

$$I_A^{\mathrm{cl}_B} = IB \cap A.$$

If $B = A^+$, then cl_{A^+} is commonly known as the *plus closure*.

Analogous to tight closure, one defines the *big test ideal* of cl_B, denoted $\tau_{\mathrm{cl}_B}(A)$, as

$$\tau_{\mathrm{cl}_B}(A) = \bigcap_{N \subseteq M} (N :_A N_M^{\mathrm{cl}_B}),$$

where the intersection ranges over all A-modules M and submodules N of M. Similarly, the *finitistic test ideal* of cl_B, denoted $\tau_{\mathrm{cl}_B}^{\mathrm{fg}}(A)$, is defined as

$$\tau_{\mathrm{cl}_B}^{\mathrm{fg}}(A) = \bigcap_{M \text{ is fin. gen.}} (N :_A N_M^{\mathrm{cl}_B}).$$

Now the intersection runs over all finitely generated R-modules M and submodules N.

Despite the many parallels between tight closure and cl_B, these latter closure operations are better behaved. In particular, our next result answers [55, Question 3.7] in the affirmative for closure operations that arise from A-algebras (see also Remark 3.3.2(2) for a partial result for closure operations arising from A-modules).

Proposition 3.3.1. *Let A be a noetherian ring and B be an A-algebra. Let M be an A-module and N be a submodule of M. Then we have the following:*

(1) If M' is a submodule of M, then $(N \cap M')^{\mathrm{cl}_B}_{M'} \subseteq N^{\mathrm{cl}_B}_M$.
(2) Let $\{M_i\}_i$ be the collection of finitely generated A-submodules of M. Then $\{(N \cap M_i)^{\mathrm{cl}_B}_{M_i}\}_i$ is a filtered poset of submodules of M under inclusion and

$$N^{\mathrm{cl}_B}_M = \bigcup_i (N \cap M_i)^{\mathrm{cl}_B}_{M_i}.$$

(3) $\tau_{\mathrm{cl}_B}(A) = \tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A)$, that is, the big and finitistic test ideals of cl_B coincide.
(4) If A is approximately Gorenstein, then $\tau_{\mathrm{cl}_B}(A) = T_{B/A}$.
(5) If $\mathcal{C}$ is the collection of ideals of A primary to maximal ideals, then $T_{B/A} = \bigcap_{I \in \mathcal{C}} (I : IB \cap A)$.
(6) $\tau_{B/A} \subseteq \tau_{\mathrm{cl}_B}(A)$.

Proof. For ease of notation, for any A-module M and a submodule N of M, we use $\xi_{N,M}$ to denote the canonical map $B \otimes_A N \to B \otimes_A M$ obtained by tensoring $N \subseteq M$ by id_B. By definition, $m \in N^{\mathrm{cl}_B}_M$ if and only if $1 \otimes m \in \mathrm{im}(\xi_{N,M})$.

(1) We have a commutative diagram

$$\begin{array}{ccc} B \otimes_A (N \cap M') & \xrightarrow{\xi_{N\cap M', M'}} & B \otimes_A M' \\ \downarrow{\scriptstyle \xi_{N\cap M', N}} & & \downarrow{\scriptstyle \xi_{M', M}} \\ B \otimes_A N & \xrightarrow{\xi_{N,M}} & B \otimes_A M \end{array}$$

If $m' \in (N \cap M')^{\mathrm{cl}_B}_{M'}$, then $1 \otimes m' \in \mathrm{im}(\xi_{N\cap M', M'})$. By the commutativity of the above diagram, it follows that $1 \otimes m' \in \mathrm{im}(\xi_{N,M})$. Thus, $(N \cap M')^{\mathrm{cl}_B}_{M'} \subseteq N^{\mathrm{cl}_B}_M$.

(2) $\{M_i\}_i$ is a filtered poset of R-submodules of M under inclusion such that

$$M = \mathrm{colim}_i M_i \quad \text{and} \quad N = \mathrm{colim}_i N \cap M_i.$$

The colimits here are unions. By (1), we get

$$\bigcup_i (N \cap M_i)^{\mathrm{cl}_B}_{M_i} \subseteq N^{\mathrm{cl}_B}_M.$$

Since tensor products commute with filtered colimits, it follows that

$$\xi_{N,M} \colon B \otimes_A N \to B \otimes_A M = \mathrm{colim}_i \left(\xi_{N\cap M_i, M_i} \colon B \otimes_A (N \cap M_i) \to B \otimes_A M_i\right).$$

Consequently, by the exactness of filtered colimits in Mod_A [69, Tag 00DB], it follows that

$$\operatorname{im}(\xi_{N,M}) = \operatorname{colim}_i \operatorname{im}(\xi_{N\cap M_i,M_i}).$$

Thus, for $m \in M$, in order for $1 \otimes m$ to be in $\operatorname{im}(\xi_{N,M})$, there must exist an index i such that $m \in M_i$ and $1 \otimes m \in \operatorname{im}(\xi_{N\cap M_i,M_i})$. Unravelling the definition of cl_B, this gives (2).

(3) The containment $\tau_{\mathrm{cl}_B}(A) \subseteq \tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A)$ follows by the definitions of the big and finitistic test ideals of cl_B. Let $a \in \tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A)$, and $N \subseteq M$ be a pair of A-modules. Given $m \in N_M^{\mathrm{cl}_B}$, by (2) there exists some finitely generated submodule M' of M such that $m \in (N \cap M')_{M'}^{\mathrm{cl}_B}$. Since

$$a\big((N \cap M')_{M'}^{\mathrm{cl}_B}\big) \subseteq N \cap M',$$

it follows that $am \in N \cap M' \subseteq N$. Thus,

$$a(N_M^{\mathrm{cl}_B}) \subseteq N,$$

and since M and N are arbitrary, we get $a \in \tau_{\mathrm{cl}_B}(A)$. This shows that $\tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A) \subseteq \tau_{\mathrm{cl}_B}(A)$.

(4) Recall that

$$T_{B/A} = \bigcap_I (I : IB \cap A) = \bigcap_I (I : I_A^{\mathrm{cl}_B}),$$

where I ranges over all ideals of A. By (3) it suffices to show that $\tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A) = T_{B/A}$. We will follow the proof of [29, Prop. (8.15)], where the analogous fact is shown for the finitistic tight closure test ideal. Let $c \in T_{B/A}$. By [55, Lem. 3.3], it suffices to show that if M is a finitely generated A-module, then

$$c \in (0 :_A 0_M^{\mathrm{cl}_B}),$$

that is, c annihilates $0_M^{\mathrm{cl}_B}$. Assume for contradiction that this is not the case. Then there exists $m \in 0_M^{\mathrm{cl}_B}$ such that $cm \neq 0$. Since M is a noetherian A-module, let N be a submodule of M maximal with respect to the property that $cm \notin N$. Then $m \in N_M^{\mathrm{cl}_B}$ and $cm \notin N$. Replacing M by M/N, N by 0 and m by its image in M/N, we may assume there exists $m \in 0_M^{\mathrm{cl}_B}$ such that $cm \neq 0$ and for all submodules $0 \subsetneq M' \subsetneq M$, $cm \in M'$. Thus, $A(cm) \subset M$ is an essential extension, and moreover, $A(cm)$ has to be a nontrivial simple A-module. This means that there exists a maximal ideal $\mathfrak{m}$ of A such that $A(cm) \simeq A/\mathfrak{m}$. Since A is approximately Gorenstein, M embeds in A/I for an $\mathfrak{m}$-primary ideal I by Lemma 2.2.3. Then

$$0_M^{\mathrm{cl}_B} \subseteq 0_{A/I}^{\mathrm{cl}_B}.$$

Since $c \in T_{B/A} = (I : IB \cap A) = (I : I_A^{\mathrm{cl}_B})$, [55, Lem. 2.15] shows that c annihilates $0_{A/I}^{\mathrm{cl}_B}$, and so, c also annihilates $0_M^{\mathrm{cl}_B}$. This is a contradiction.

(5) Recall $\mathcal{C}$ is the collection of ideals of A primary to maximal ideals. It suffices to show that

$$\bigcap_{I\in\mathcal{C}}(I : IB \cap A) \subseteq T_{B/A}.$$

Let J be an arbitrary ideal of A and let $c \in \bigcap_{I\in\mathcal{C}}(I : IB \cap A)$. Then for any maximal ideal $\mathfrak{m}$ of A and $n \in \mathbf{Z}_{>0}$,

$$c(JB \cap A) \subseteq c((J + \mathfrak{m}^n)B \cap A) \subseteq J + \mathfrak{m}^n.$$

Thus,

$$c(JB \cap A) \subseteq \bigcap_{n\in\mathbf{Z}_{>0}} \bigcap_{\mathfrak{m}} J + \mathfrak{m}^n = J,$$

where the inner intersection runs over all maximal ideals of A. It follows that $c \in T_{B/A}$.

(6) If A is approximately Gorenstein, then (6) follows from (4) and Lemma 3.2.7(3) because $\tau_{B/A} \subseteq T_{B/A}$. We will show that (6) holds for any noetherian ring A. Let $c \in \tau_{B/A}$ and choose an A-linear map $f\colon B \to A$ such that $f(1) = c$. Then for any A-module N, we get an A-linear map

$$\begin{aligned} \eta_N\colon B \otimes_A N &\longrightarrow N. \\ b \otimes n &\longmapsto f(b)n \end{aligned}$$

Note that η_N is functorial in N, that is, for any A-linear map $\varphi\colon N \to M$,

$$\begin{array}{ccc} B \otimes_A N & \xrightarrow{\mathrm{id}_B \otimes \varphi} & B \otimes_A M \\ \downarrow{\scriptstyle \eta_N} & & \downarrow{\scriptstyle \eta_M} \\ N & \xrightarrow{\varphi} & M. \end{array}$$

commutes. Applying this commutative diagram when N is a submodule of an A-module M and φ is the inclusion map, we see that if $m \in N_M^{\mathrm{cl}_B}$, that is, if $1 \otimes m \in \mathrm{im}(B \otimes_A N \to B \otimes_A M)$, then

$$cm = f(1)m = \eta_M(1 \otimes m)$$

is in the image of $N \subseteq M$, that is, $cm \in N$. Thus, $c \in (N :_A N_M^{\mathrm{cl}_B})$, and since N, M are arbitrary, we get $c \in \tau_{\mathrm{cl}_B}(A)$. □

Remark 3.3.2.

(1) Proposition 3.3.1(6) is also observed when A is local in [55, Thm. 3.12] (see [55, Rem. 3.13]), where it is additionally shown that equality holds in part (6) if B is a finite A-algebra or if A is complete and B is an arbitrary A-algebra. We will use [55, Thm. 3.12] to show that the trace $\tau_{B/A}$ and the ideal trace $T_{B/A}$ are equal for module finite extensions in most cases of interest (Corollary 3.3.3(1)). This basic observation will be useful in identifying and proving properties of an ideal that detects the non-splinter locus of a noetherian domain in Section 4.

(2) One can define a closure operation cl_B for an A-module B that is not necessarily an A-algebra [55, Def. 2.4]. The proof of Proposition 3.3.1(3) can be modified to show that if B is a finitely generated A-module, then the big and finitistic cl_B-closure test ideals coincide. Indeed, if B is generated as an A-module by $b_1, \ldots, b_n$, then $m \in N_M^{\mathrm{cl}_B}$ precisely when $b_i \otimes m \in \mathrm{im}(B \otimes_A N \to B \otimes_A M)$ for all i. But one can always find a large finitely generated A-submodule M' of M such that $m \in M'$ and $b_i \otimes m \in \mathrm{im}(B \otimes_A (N \cap M') \to B \otimes_A M')$ for all i. However, the closure operations associated with modules that are not algebras are often not related to tight closure. For example, there exist even finitely generated Cohen–Macaulay modules B over a non-regular but weakly F-regular local ring for which cl_B is non-trivial [55, Rem. 2.23].

(3) The arguments of Proposition 3.3.1 will not work to show that the big and finitistic test ideals from tight closure theory are equal because tight closure involves checking algebra closure type relations for infinitely many B's. However, see Corollary 3.3.5 for a deep characterization of finitistic tight closure in terms of an algebra closure due to Hochster.

We can now exhibit relations between the ideals $\tau_{B/A}$ and $T_{B/A}$ for nice rings. In particular, we will see that the ideal trace $T_{B/A}$ often localizes for finite extensions $A \hookrightarrow B$, a fact that is not obvious from its definition which involves an infinite intersection of ideals.

Corollary 3.3.3. *Let A be an approximately Gorenstein noetherian ring and B be an A-algebra.*

(1) If B is a finite A-algebra, then $\tau_{B/A} = \tau_{\mathrm{cl}_B}(A) = \tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A) = T_{B/A}$.
(2) If B is a finite A-algebra, then for all prime ideals $\mathfrak{p}$ of A, $T_{B_{\mathfrak{p}}/A_{\mathfrak{p}}} = (T_{B/A})_{\mathfrak{p}}$.
(3) If A is complete local, then $\tau_{B/A} = \tau_{\mathrm{cl}_B}(A) = \tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A) = T_{B/A}$.
(4) If A is a complete local domain, then $\tau_{A^+/A} = \tau_{\mathrm{cl}_{A^+}}(A) = \tau^{\mathrm{fg}}_{\mathrm{cl}_{A^+}}(A) = T_{A^+/A}$ and $\tau_{A^+/A} \neq 0$.

All the ideals in (1) and (3) are nonzero precisely when B is a solid A-algebra.

Proof. (1) The equalities $\tau_{\mathrm{cl}_B}(A) = \tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A) = T_{B/A}$ follow by Proposition 3.3.1. Thus, it suffices to show that

$$\tau_{B/A} = T_{B/A}. \tag{3.3.3.1}$$

This equality holds when A is local and B is a finite A-algebra by [55, Thm. 3.12], where it is shown that $\tau_{B/A} = \tau_{\mathrm{cl}_B}(A)$. Now note that

$$(T_{B/A})_{\mathfrak{p}} = \left(\bigcap_I (I : IB \cap A)\right)_{\mathfrak{p}} \subseteq \bigcap_I (I : IB \cap A)_{\mathfrak{p}} = \bigcap_I (IA_{\mathfrak{p}} : IB_{\mathfrak{p}} \cap A_{\mathfrak{p}}) = T_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}. \tag{3.3.3.2}$$

The first equality is by definition of $T_{B/A}$, and the inclusion is a well-known property of localization. The second equality follows by flatness of localization. Indeed, we have $(I : IB \cap A)_{\mathfrak{p}} = (IA_{\mathfrak{p}} : (IB \cap A)A_{\mathfrak{p}})$ by flatness and the fact that $IB \cap A$ is finitely generated, and

$$A/(IB \cap A) \hookrightarrow B/IB$$

stays an inclusion upon localizing at $\mathfrak{p}$, which implies that $(IB \cap A)A_{\mathfrak{p}} = IB_{\mathfrak{p}} \cap A_{\mathfrak{p}}$. The final equality follows by definition of $T_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}$ and the fact that all ideals of $A_{\mathfrak{p}}$ are expanded from A.

By the veracity of (3.3.3.1) in the local case, we have $T_{B_{\mathfrak{p}}/A_{\mathfrak{p}}} = \tau_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}$, and consequently,

$$(\tau_{B/A})_{\mathfrak{p}} \subseteq (T_{B/A})_{\mathfrak{p}} \subseteq T_{B_{\mathfrak{p}}/A_{\mathfrak{p}}} = \tau_{B_{\mathfrak{p}}/A_{\mathfrak{p}}} = (\tau_{B/A})_{\mathfrak{p}}.$$

Here the first inclusion follows because $\tau_{B/A} \subseteq T_{B/A}$ by Lemma 3.2.7(3), the second inclusion follows by (3.3.3.2) and the last equality follows by Lemma 3.2.3(3). Thus, for all prime ideals $\mathfrak{p}$ of A, $(\tau_{B/A})_{\mathfrak{p}} = (T_{B/A})_{\mathfrak{p}}$, which implies that $\tau_{B/A} = T_{B/A}$.

(2) follows from (1) and the fact that $(\tau_{B/A})_{\mathfrak{p}} = \tau_{B_{\mathfrak{p}}/A_{\mathfrak{p}}}$ by Lemma 3.2.3(3).

(3) The equalities $\tau_{\mathrm{cl}_B}(A) = \tau^{\mathrm{fg}}_{\mathrm{cl}_B}(A) = T_{B/A}$ again follow by Proposition 3.3.1, and [55, Thm. 3.12] shows that $\tau_{B/A} = \tau_{\mathrm{cl}_B}(A)$.

The first part of (4) follows by (3). It remains to show that A^+ is a solid A-algebra. Since A is complete local, by Cohen's structure theorem, there exists a complete regular local ring R and a module finite extension $R \hookrightarrow A$. Then $A^+ = R^+$ and the map $R \hookrightarrow A \hookrightarrow A^+$ is pure because R is a splinter by the direct summand theorem [6,32]. Since R is complete, a well-known result due to Auslander (see for example [17, Lem. 2.3.3]) implies $R \to A^+$ splits. Since A is a finite extension domain of R, A^+ must also be a solid A-algebra by [36, Cor. 2.3]. □

Remark 3.3.4. Corollary 3.3.3(4) was announced a number of years ago by Hochster and Zhang. To the best of our knowledge, their result is not publicly available and our proof is independent of theirs.

Granting another unpublished but widely available result of Hochster in his course notes [37], we show that the finitistic test ideal from tight closure theory can always be recovered as a trace ideal for *any* complete local domain. This result is an analog of [13, Thm. 4.6].

Corollary 3.3.5. *Let $(R, \mathfrak{m})$ be a complete local noetherian domain of prime characteristic $p > 0$. Then there exists a big Cohen-Macaulay R-algebra B that satisfies all of the following properties:*

(1) B is an R^+-algebra.
(2) For all finitely generated R-modules M and submodules N of M, $N_M^{\mathrm{cl}_B} = N_M^*$, where N_M^* denotes the tight closure of N in M.
(3) $\tau_{B/R} = \tau_*^{\mathrm{fg}}(R)$, where $\tau_*^{\mathrm{fg}}(R)$ is the finitistic tight closure test ideal of R.

Proof. The existence of a big Cohen–Macaulay R-algebra B that satisfies (1) and (2) follows by [37, Thm. on pg. 250]. This is a significant strengthening of [36, Thm. (11.1)], where Hochster shows that given a finitely generated R-module M and a submodule N of M, there exists a big Cohen–Macaulay R-algebra B depending on M, N such that $N_M^{\mathrm{cl}_B} = N_M^*$.

So choose B satisfying (1) and (2). Then B is a solid R-algebra because R is complete and B is big Cohen–Macaulay. Now by [55, Thm. 3.12],

$$\tau_{B/R} = \tau_{\mathrm{cl}_B}(R).$$

But Proposition 3.3.1(3) tells us that

$$\tau_{\mathrm{cl}_B}(R) = \tau_{\mathrm{cl}_B}^{\mathrm{fg}}(R) \coloneqq \bigcap_{M \text{ is fin. gen.}} (N :_R N_M^{\mathrm{cl}_B}) = \bigcap_{M \text{ is fin. gen.}} (N :_R N_M^*).$$

The last intersection is precisely the finitistic tight closure test ideal of R [29, Def. 8.22]. □

Remark 3.3.6. Let B be Hochster's (very large) big Cohen–Macaulay R-algebra from Corollary 3.3.5. It is not known if cl_B coincides with tight closure for submodules of arbitrary R-modules. Indeed, an affirmative answer to this question would imply that $\tau_{B/R}$ is also the non-finitistic or big test ideal $\tau_b(R)$ of R from tight closure, thereby showing that the big and finitistic tight closure test ideals coincide. This would then settle the outstanding problem of whether weak F-regularity is equivalent to strong F-regularity for complete local domains.

3.4. Finiteness of uniformly F-compatible ideals

Let R be a noetherian ring, and let Σ be a collection of radical ideals of R that is closed under finite intersections and such that for $I \in \Sigma$, any minimal prime of I is also in Σ.

The main example for us is where Σ is the collection of uniformly F-compatible ideals of R when R is Frobenius split; see Lemma 3.1.2. It follows that Σ is finite precisely when $\Sigma \cap \operatorname{Spec}(R)$ is finite because every ideal of Σ (apart from the unit ideal) is a finite intersection of elements of $\Sigma \cap \operatorname{Spec}(R)$. We use this basic observation to recall the following well-known result which is a key technical ingredient of our paper.

Proposition 3.4.1. *Let R be a noetherian Frobenius split ring of prime characteristic $p > 0$. Then R has finitely many uniformly F-compatible ideals in each of the following cases:*

(1) R is F-finite.
(2) $(R, \mathfrak{m})$ is local.

In fact, if $\varphi\colon F_*R \to R$ is a Frobenius splitting, then R has finitely many φ-compatible ideals in both cases.

We will reprove Proposition 3.4.1, in part to highlight the similarity between the proofs of the global and local cases, and also to give a proof of the global F-finite case that does not rely on F-adjunction or the language of divisor pairs. But first, we highlight a bound on the multiplicity of F-pure noetherian local rings that gives a proof of the local case of Proposition 3.4.1.

Proposition 3.4.2. *[41, Thm. 3.1] Let $(R, \mathfrak{m}, \kappa)$ be a noetherian local ring of prime characteristic $p > 0$ that is F-pure. Suppose d is the dimension of R and $v = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$. Then*

$$e(R) \leq \binom{v}{d},$$

where $e(R)$ is the Hilbert–Samuel multiplicity of R.

Proof of Proposition 3.4.1. Since R is Frobenius split, all uniformly F-compatible ideals are radical. Part (1) is precisely [57, Cor. 5.10]. For the reader's convenience, we translate Schwede's proof, avoiding the machinery of F-adjunction and pairs. Fix a Frobenius splitting

$$\varphi : F_*R \to R.$$

We will show the stronger statement that there are finitely many φ-compatible ideals in the sense of Remark 3.1.3. As discussed in the beginning of this subsection, it suffices to show that there are finitely many φ-compatible ideals that are prime. If not, there will be infinitely many distinct φ-compatible prime ideals $\{\mathfrak{p}_\alpha\}_{\alpha \in A}$ such that $\dim(R/\mathfrak{p}_\alpha) = d$, for some fixed $0 < d \leq \dim(R)$ ($\dim(R) < \infty$ because R is F-finite [44, Prop. 1.1]). Let

$$\mathfrak{q} \coloneqq \bigcap_{\alpha \in A} \mathfrak{p}_\alpha.$$

We claim that if $\mathfrak{p}$ is a minimal prime of $\mathfrak{q}$, then $\mathfrak{p}$ is the intersection of the $\mathfrak{p}_\alpha$ such that $\mathfrak{p} \subseteq \mathfrak{p}_\alpha$. Indeed, let

$$\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$$

be the distinct minimal primes of $\mathfrak{q}$. For all $i \in \{1, \dots, n\}$, let I_i be the intersection of all the $\mathfrak{p}_\alpha$ that contain the minimal prime $\mathfrak{p}_i$ (the intersection should be interpreted to be R if no such $\mathfrak{p}_\alpha$ exist, but our claim will show this is impossible). Our claim is that $I_i = \mathfrak{p}_i$, for all $i \in \{1, \dots, n\}$. Without loss of generality, assume $i = 1$. Note that every prime ideal $\mathfrak{p}_\alpha$ must contain some minimal prime $\mathfrak{p}_i$ because $\mathfrak{q} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n \subseteq \mathfrak{p}_\alpha$. Thus,

$$I_1 \cap \cdots \cap I_n = \bigcap_{\alpha \in A} \mathfrak{p}_\alpha = \mathfrak{q} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n.$$

Since $\mathfrak{p}_1$ is prime, there exists $i \in \{1, \dots, n\}$ such that $I_i \subseteq \mathfrak{p}_1$. But $\mathfrak{p}_i \subseteq I_i$ for all i, so by minimality, $I_2, \dots, I_n \nsubseteq \mathfrak{p}_1$. Hence, we must have $I_1 \subseteq \mathfrak{p}_1$, and by definition of I_1, we have $\mathfrak{p}_1 \subseteq I_1$, proving our claim.

Since $\mathfrak{q}$ has finitely many minimal primes, by the pigeonhole principle, there exists a minimal prime of $\mathfrak{q}$ that is the intersection of infinitely many of the $\mathfrak{p}_\alpha$'s. Thus, replacing $\mathfrak{q}$ by this minimal prime, we may assume that $\mathfrak{q}$ is prime. Note that for all α, $\mathfrak{q} \subsetneq \mathfrak{p}_\alpha$ since there are no inclusion relations among the $\mathfrak{p}_\alpha$. Moreover, $\mathfrak{q}$ is uniformly φ-compatible, so we get an induced Frobenius splitting

$$\overline{\varphi} : F_*(R/\mathfrak{q}) \to R/\mathfrak{q}$$

of the domain $R/\mathfrak{q}$. For all α, $\mathfrak{p}_\alpha/\mathfrak{q}$ is a nonzero $\overline{\varphi}$-compatible ideal of $R/\mathfrak{q}$ and

$$\bigcap_\alpha (\mathfrak{p}_\alpha/\mathfrak{q}) = (\bigcap_\alpha \mathfrak{p}_\alpha)/\mathfrak{q} = (0). \tag{3.4.2.1}$$

Since $R/\mathfrak{q}$ is an F-finite domain and $\overline{\varphi}$ is a nonzero map, there is a smallest *nonzero* $\overline{\varphi}$-compatible ideal with respect to inclusion, namely the (big or non-finitistic) test ideal $\tau(R/\mathfrak{q}, \overline{\varphi})$ [71, Thm. 3.8] (this is where F-finiteness is used seriously for the first time). In particular,

$$(0) \neq \tau(R/\mathfrak{q}, \overline{\varphi}) \subseteq \mathfrak{p}_\alpha/\mathfrak{q},$$

for all α. This contradicts (3.4.2.1).

We prove (2) following [41, Rem. 3.4]. As above, for a Frobenius splitting φ of $(R, \mathfrak{m})$, it suffices to show there are only finitely many φ-compatible prime ideals $\mathfrak{p}$ of coheight d, for any fixed $0 \leq d \leq \dim(R)$. So suppose $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec}(R)$ are prime ideals of coheight d. Since $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ is uniformly F-compatible,

$$R/\bigcap_{i=1}^{n}\mathfrak{p}_i$$

is a Frobenius split equidimensional local ring of Krull dimension d. Suppose v (resp. v') is the embedding dimension of R (resp. $R/\bigcap_{i=1}^{n}\mathfrak{p}_i$). Then $v' \leq v$. Since $R/\bigcap_{i=1}^{n}\mathfrak{p}_i$ is reduced and equidimensional, we get

$$n \leq \sum_{i=1}^{n} e(R/\mathfrak{p}_i) = e(R/\bigcap_{i=1}^{n}\mathfrak{p}_i) \leq \binom{v'}{d} \leq \binom{v}{d}.$$

The first inequality follows because the multiplicity of a local domain is a positive integer, the equality follows by [50, Thm. 14.7] because $R/\bigcap_{i=1}^{n}\mathfrak{p}_i$ is reduced and equidimensional, the second inequality follows by Proposition 3.4.2 applied to the F-pure local ring $R/\bigcap_{i=1}^{n}\mathfrak{p}_i$, and the final inequality follows because $v' \leq v$. Thus, for each $0 \leq d \leq \dim(R)$, the number of φ-compatible prime ideals of coheight d is bounded above by $\binom{v}{d}$, which only depends on R and d. □

Remark 3.4.3.

(1) In the proof of Proposition 3.4.1 in the F-finite case, we used the highly non-trivial fact that if R is a noetherian F-finite domain and $\varphi\colon F_*R \to R$ is a nonzero R-linear map, then R has a *smallest nonzero* φ-compatible ideal with respect to inclusion. The existence of this ideal is based on a deep result of Hochster and Huneke in tight closure theory on the existence of completely stable test elements for noetherian F-finite rings. For more details, please see [71, Lem. 3.6] and [30, Thm. 5.10].

(2) The finiteness of the set of uniformly F-compatible ideals of a Frobenius split noetherian local ring $(R, \mathfrak{m})$ in the excellent case also follows by [20, Cor. 3.2], which is a characteristic independent result. The advantage of Proposition 3.4.2 is that it allows one to obtain explicit bounds on the number of uniformly F-compatible prime ideals (and also without any excellence hypothesis). The same explicit bounds were also obtained in [70, Thm. 4.2] in the excellent local case, albeit via more involved considerations. It is a different matter that the authors do not know an example of a non-excellent Frobenius split noetherian local ring. Indeed, the most common method of constructing non-excellent noetherian local rings in prime characteristic is in the dimension 1 regular case via *arc valuations*; see [18]. However, [17, Thm. 7.4.1] shows that any Frobenius split normal noetherian domain of dimension 1 has to be excellent. Thus, the 'usual' method of constructing non-excellent noetherian local rings in prime characteristic do not give examples that are Frobenius split. Consequently, it is unclear if Proposition 3.4.1 gives additional cases of finiteness of F-compatible ideals outside the excellent setting in the local case.

(3) The proofs of the finiteness of the set Σ of uniformly F-compatible ideals of Frobenius split noetherian rings that are F-finite or local only used the facts that Σ is closed under arbitrary intersections, consists of radical ideals, and is also closed under taking

minimal primes of ideals. The property of Σ being closed under sums of ideals is never used, in contrast with [20, Cor. 3.2] and [70, Thm. 4.1].

Corollary 3.4.4. *If $(R, \mathfrak{m})$ is a complete local noetherian ring of prime characteristic that is F-pure, then R has finitely many uniformly F-compatible ideals.*

Proof. This follows by Proposition 3.4.1, because purity of a ring map $R \to S$, when R is complete, is equivalent to splitting by a result due to Auslander; see [21, Lem. 1.2] (or [17, Lem. 2.3.3] for a more general statement). □

4. The splinter locus

Suppose X is a locally noetherian scheme. We define

$$\mathrm{Spl}(X) \coloneqq \{x \in X : \mathcal{O}_{X,x} \text{ is a splinter}\}.$$

Note that for any open subscheme U of X, $\mathrm{Spl}(U) = \mathrm{Spl}(X) \cap U$. If R is a noetherian ring, we define $\mathrm{Spl}(R)$ to be $\mathrm{Spl}(\mathrm{Spec}(R))$. Since splinters are integrally closed domains, it follows that $\mathrm{Spl}(X)$ is contained in the normal locus

$$\mathrm{Nor}(X) \coloneqq \{x \in X : \mathcal{O}_{X,x} \text{ is an integrally closed domain}\}$$

of X. Thus, if $\mathrm{Nor}(X)$ is open (for example, if X is has an open regular locus [24, Cor. (6.13.5)]), then to show that $\mathrm{Spl}(X)$ is open it suffices to assume that X is normal. Since openness of the splinter locus can be checked on a sufficiently fine affine open cover of X, one may further assume that $X = \mathrm{Spec}(A)$, where A is a noetherian integrally closed domain (in particular, A is approximately Gorenstein). Thus, we will analyze when the splinter locus of a noetherian domain is open.

4.1. Traces and splinter loci

Recall that if B is an A-algebra, then the ideal trace of B/A is

$$T_{B/A} \coloneqq \bigcap_I (I : IB \cap A),$$

where the intersection ranges over all ideals I of A. Similarly, the trace of B/A is

$$\tau_{B/A} \coloneqq \mathrm{im}(\mathrm{Hom}_A(B, A) \xrightarrow{\mathrm{eval}\,@1} A).$$

In general, $\tau_{B/A} \subseteq T_{B/A}$, and equality holds when A is an approximately Gorenstein noetherian domain and B is a finite extension of A (Corollary 3.3.3(1)). We begin with the following observation:

Lemma 4.1.1. *Let A be a noetherian domain, $\mathcal{C}$ be the collection of finite A-subalgebras of A^+ and*

$$\Sigma_\tau \coloneqq \{\tau_{B/A} : B \in \mathcal{C}\} \text{ and } \Sigma_T \coloneqq \{T_{B/A} : B \in \mathcal{C}\}.$$

(1) For all $B \in \mathcal{C}$, $\tau_{B/A} \neq 0$ and $T_{B/A} \neq 0$.
(2) If $B, B' \in \mathcal{C}$ such that $B \subseteq B'$, then $\tau_{B'/A} \subseteq \tau_{B/A}$ (resp. $T_{B'/A} \subseteq T_{B/A}$). Thus, Σ_τ (resp. Σ_T) is a cofiltered poset of ideals of A under inclusion.
(3) If a minimal element of Σ_τ (resp. Σ_T) exists, then this is the smallest element of Σ_τ (resp. Σ_T) in the sense that it is contained in every other element of Σ_τ (resp. Σ_T).
(4) If Σ_τ (resp. Σ_T) is finite, then it has a smallest element under inclusion.
(5) A is a splinter $\iff \bigcap_{B\in\mathcal{C}} \tau_{B/A} = A \iff \bigcap_{B\in\mathcal{C}} T_{B/A} = A$.

We say a partially order set $(\Sigma, \leq)$ is *cofiltered* if for all $x, y \in \Sigma$, there exists $z \in \Sigma$ such that $z \leq x, y$.

Proof. (1) For all $B \in \mathcal{C}$, $\operatorname{Hom}_A(B, A) \neq 0$ because

$$\operatorname{Frac}(A) \otimes_A \operatorname{Hom}_A(B, A) = \operatorname{Hom}_{\operatorname{Frac}(A)}(\operatorname{Frac}(B), \operatorname{Frac}(A)) \neq 0.$$

Thus, for all $B \in \mathcal{C}$, $\tau_{B/A} \neq 0$. Since $T_{B/A}$ contains $\tau_{B/A}$ (Lemma 3.2.7(3)), it follows that for all $B \in \mathcal{C}$, $T_{B/A} \neq 0$.

(2) If $B \subseteq B'$, then $\operatorname{Hom}_A(B', A) \xrightarrow{\text{eval @1}} R$ factors as

$$\operatorname{Hom}_A(B', A) \to \operatorname{Hom}_A(B, A) \xrightarrow{\text{eval @1}} A,$$

where $\operatorname{Hom}(B', A) \to \operatorname{Hom}(B, A)$ is given by restriction to B. It follows that $\tau_{B'/A} \subseteq \tau_{B/A}$. The set $\mathcal{C}$ is a filtered poset under inclusion because if $B, B' \in \mathcal{C}$, then $B[B'] \in \mathcal{C}$. Thus, Σ_τ is a cofiltered poset under inclusion.

Analogously, if $B \subseteq B'$, then $IB \cap A \subseteq IB' \cap A$. This means that $(I : IB' \cap A) \subseteq (I : IB \cap A)$, and so, $T_{B'/A} \subseteq T_{B/A}$. Again, because $\mathcal{C}$ is a filtered poset, Σ_T is a cofiltered poset.

(3) holds generally for any cofiltered poset.

(4) follows from (3) because any finite poset has a minimal element.

(5) If A is a splinter, then $\tau_{B/A} = A$ for any finite extension of A by definition of the splinter property. This shows that if A is a splinter, then $\bigcap_{B\in\mathcal{C}} \tau_{B/A} = A$.

If $\bigcap_{B\in\mathcal{C}} \tau_{B/A} = A$, then $\bigcap_{B\in\mathcal{C}} T_{B/A} = A$ because $\tau_{B/A} \subseteq T_{B/A}$.

It remains to show that if

$$\bigcap_{B\in\mathcal{C}} T_{B/A} = A,$$

then A is splinter. If $1 \in T_{B/A}$, then for all ideals I of A we have

$$I = IB \cap A.$$

Thus $A \to B$ is cyclically pure for all $B \in \mathcal{C}$. Since $A^+ = \operatorname{colim}_{B\in\mathcal{C}} B$, it follows that $A \to A^+$ is also cyclically pure. Then A is a splinter by [19, Lem. 2.3.1]. □

Remark 4.1.2.

(1) Using the notation of Lemma 4.1.1, if Σ_τ has a smallest element under inclusion, then there exists a finite A-subalgebra B_0 of A^+ such that

$$\tau_{B_0/A} = \bigcap_{B\in\mathcal{C}} \tau_{B/A}.$$

This implies that for all finite A-subalgebras B' of A^+ containing B_0,

$$\tau_{B_0/A} = \tau_{B'/A},$$

that is, the traces of the finite A-subalgebras of A^+ stabilize. We will see that this stable trace ideal, when it exists, defines the splinter locus of A.

(2) A similar stabilization result also holds if Σ_T has a smallest element under inclusion. If A is additionally approximately Gorenstein, then $\Sigma_\tau = \Sigma_T$ by Corollary 3.3.5(1), that is, we get the same stable ideal as in the previous remark.

Definition 4.1.3. Let A be a noetherian domain and $\mathcal{C}$ be the collection of finite A-subalgebras of A^+. We define the *trace of* A, denoted τ_A, to be

$$\tau_A \coloneqq \bigcap_{B\in\mathcal{C}} \tau_{B/A}.$$

The *ideal trace of* A, denoted T_A, is defined to be

$$T_A \coloneqq \bigcap_{B\in\mathcal{C}} T_{B/A}.$$

Proposition 4.1.4. *Let A be a noetherian domain and $\mathcal{C}$ be the collection of finite A-subalgebras of A^+. Then we have the following:*

(1) A is a splinter $\iff \tau_A = A \iff T_A = A$.
(2) $T_A = T_{A^+/A}$.
(3) If A is approximately Gorenstein, then $\tau_A = T_A$ and τ_A is the big plus closure test ideal.
(4) If $(A, \mathfrak{m})$ is complete local, then $\tau_A = \tau_{A^+/A}$.

Assume that $\Sigma_\tau \coloneqq \{\tau_{B/A} \colon B \in \mathcal{C}\}$ has a smallest element under inclusion. Then:

(5) *There exists* $B_0 \in \mathcal{C}$ *such that* $\tau_A = \tau_{B_0/A}$.
(6) *If* A *is approximately Gorenstein, there exists* $B_0 \in \mathcal{C}$ *such that* $T_A = T_{B_0/A} = \tau_{B_0/A}$.
(7) *If* $(A, \mathfrak{m})$ *is complete local, there exists* $B_0 \in \mathcal{C}$ *such that* $\tau_{A^+/A} = \tau_{B_0/A} = T_{B_0/A} = T_{A^+/A}$.
(8) *If* $\mathfrak{p} \in \operatorname{Spec}(A)$, *then* $\tau_{A_\mathfrak{p}} = (\tau_A)_\mathfrak{p}$.
(9) *For* $\mathfrak{p} \in \operatorname{Spec}(A)$, $A_\mathfrak{p}$ *is a splinter if and only if* $\tau_A \nsubseteq \mathfrak{p}$. *Thus, the splinter locus of* A *is the complement in* $\operatorname{Spec}(A)$ *of* $\mathbf{V}(\tau_A)$.
(10) *There exists a finite* A*-subalgebra* B_0 *of* A^+ *such that if* $A \hookrightarrow B_0$ *splits, then* A *is a splinter.*

Proof. (1) is precisely Lemma 4.1.1(5).

(2) For all $B \in \mathcal{C}$, we have $B \subseteq A^+$. Therefore,

$$T_{A^+/A} = \bigcap_I (I : IA^+ \cap A) \subseteq \bigcap_I (I : IB \cap A) = T_{B/A},$$

where the intersections are indexed by all ideals I of A and the middle containment follows because $IB \cap A \subseteq IA^+ \cap A$. Thus,

$$T_{A^+/A} \subseteq T_A.$$

Now suppose $c \in T_A$ and let I be an ideal of A. To prove $T_A \subseteq T_{A^+/A}$ we have to show that $c(IA^+ \cap A) \subseteq I$. Let $z \in IA^+ \cap A$ and choose a finite A-subalgebra B of A^+ such that $z \in IB \cap A$. Since $c \in T_{B/A}$, it follows that $cz \in I$, and so, $c(IA^+ \cap A) \subseteq I$.

(3) If A is approximately Gorenstein, then for all $B \in \mathcal{C}$,

$$\tau_{B/A} = T_{B/A}$$

by Corollary 3.3.3(1). So $\tau_A = T_A$ by the definition of these ideals. By (2), we then get $\tau_A = T_{A^+/A}$, and taking $B = A^+$ in Proposition 3.3.1(4), it follows that $T_{A^+/A} = \tau_{\mathrm{cl}_{A^+}}(A)$. Since cl_{A^+} is precisely plus closure, $\tau_{\mathrm{cl}_{A^+}}(A)$ is the big plus closure test ideal.

(4) A noetherian complete local domain is approximately Gorenstein because reduced excellent rings are approximately Gorenstein. Then $\tau_A = T_A$ by (3), $T_A = T_{A^+/A}$ by (2), and $T_{A^+/A} = \tau_{A^+/A}$ by Corollary 3.3.3(4).

(5) follows by the hypothesis that Σ_τ has a smallest element.

(6) Since A is approximately Gorenstein, (3) implies

$$T_A = \tau_A.$$

Now choose $B_0 \in \mathcal{C}$ that satisfies the conclusion of (5). Then $T_A = \tau_A = \tau_{B_0/A} = T_{B_0/A}$, where the last equality again follows by the approximately Gorenstein property and Corollary 3.3.3(1).

(7) Since complete local domains are approximately Gorenstein, by (2), (3) and (4),

$$\tau_{A^+/A} = \tau_A = T_A = T_{A^+/A},$$

and by (6), there exists $B_0 \in \mathcal{C}$ such that $T_A = T_{B_0/A} = \tau_{B_0/A}$.

(8) If Σ_τ has a smallest element $\tau_{B_0/A}$, then by exactness of localization, for $\mathfrak{p} \in \operatorname{Spec}(A)$, the collection

$$\Sigma_{\tau,\mathfrak{p}} \coloneqq \{(\tau_{B/A})_\mathfrak{p} : B \in \mathcal{C}\}$$

also has a smallest element under inclusion, namely $(\tau_{B_0/A})_\mathfrak{p} = \tau_{(B_0)_\mathfrak{p}/A_\mathfrak{p}}$. Here the last equality follows by Lemma 3.2.3(3). Thus,

$$(\tau_A)_\mathfrak{p} = (\bigcap_{B\in\mathcal{C}} \tau_{B/A})_\mathfrak{p} = (\tau_{B_0/A})_\mathfrak{p} = \bigcap_{B\in\mathcal{C}} (\tau_{B/A})_\mathfrak{p} = \bigcap_{B\in\mathcal{C}} \tau_{B_\mathfrak{p}/A_\mathfrak{p}}.$$

It suffices to show that

$$\bigcap_{B\in\mathcal{C}} \tau_{B_\mathfrak{p}/A_\mathfrak{p}} = \tau_{A_\mathfrak{p}}. \tag{4.1.4.1}$$

Let T be a finite $A_\mathfrak{p}$-subalgebra of $(A_\mathfrak{p})^+ = (A^+)_\mathfrak{p}$. Suppose $T = A_\mathfrak{p}[t_1, \ldots, t_n]$. Since each t_i is integral over $A_\mathfrak{p}$, there exists $s \in A \setminus \mathfrak{p}$ such that for all i, st_i is integral over A. As s is a unit in T, replacing t_i by st_i does not change T, so we may assume that each t_i is integral over A. Then the A-subalgebra

$$T' \coloneqq A[t_1, \ldots, t_n]$$

of T has the property that T' is a finite extension of A contained in A^+ and $(T')_\mathfrak{p} = T$. Thus,

$$\tau_{T/A_\mathfrak{p}} = \tau_{(T')_\mathfrak{p}/A_\mathfrak{p}} = (\tau_{T'/A})_\mathfrak{p}.$$

Said differently, this argument shows that every trace of a finite $A_\mathfrak{p}$-subalgebra of $(A_\mathfrak{p})^+$ is the localization at $\mathfrak{p}$ of the trace of some finite A-subalgebra of A^+. Unravelling the definition of $\tau_{A_\mathfrak{p}}$, this implies (4.1.4.1).

(9) By (1), $A_\mathfrak{p}$ is a splinter if and only if $\tau_{A_\mathfrak{p}} = A_\mathfrak{p}$. Now by (7), $\tau_{A_\mathfrak{p}} = (\tau_A)_\mathfrak{p}$. Thus, $\tau_{A_\mathfrak{p}} = A_\mathfrak{p}$ if and only if $(\tau_A)_\mathfrak{p} = A_\mathfrak{p}$, and this last equality holds precisely when $\tau_A \not\subseteq \mathfrak{p}$. It follows that the splinter locus of A is the complement of the closed set $\mathbf{V}(\tau_A)$ of $\operatorname{Spec}(A)$.

(10) Choose a finite A-subalgebra B_0 of A^+ that satisfies (5). If $A \hookrightarrow B_0$ splits, then $\tau_A = \tau_{B_0/A} = A$, and so, A is a splinter by (1). □

Remark 4.1.5. The equality $\tau_A = \tau_{A^+/A}$ in part (4) and the equality $\tau_{A^+/A} = \tau_{B_0/A}$ in part (6) of Proposition 4.1.4 fail quite dramatically if we do not assume A is complete,

even for excellent regular local rings. Indeed, by [16] choose an excellent Henselian regular local ring A of Krull dimension 1 and prime characteristic $p > 0$ such that A admits no nonzero A-linear maps $F_*A \to A$. Since F_*A embeds in A^+, this implies that there are no nonzero A-linear maps $A^+ \to A$. Thus, $\tau_{A^+/A} = 0$ while $\tau_A = A$ because regular rings of prime characteristic are splinters by [32]. Note that in this case τ_A equals $\tau_{B/A}$ for any finite A-subalgebra B of A^+.

Example 4.1.6. Suppose R is a noetherian F-finite normal domain that is $\mathbf{Q}$-Gorenstein. Then [13] shows that the set Σ_τ of Proposition 4.1.4 has a smallest element, namely the big test ideal $\tau_b(R)$ of R. Thus, $\tau_R = \tau_b(R)$, and so, Proposition 4.1.4(9) shows that the splinter locus of R coincides with the complement of $\mathbf{V}(\tau_b(R))$, which is the strongly F-regular locus of R. This is not surprising because Singh showed that in the affine $\mathbf{Q}$-Gorenstein setting, the splinter condition is the same as being F-regular [60] and it is known that for F-finite $\mathbf{Q}$-Gorenstein rings, (weak) F-regularity is equivalent to strong F-regularity [48].

We will now use Proposition 4.1.4 to compare the traces under Henselizations and completions.

Proposition 4.1.7. *Let $(A, \mathfrak{m})$ be a noetherian normal domain. We have the following:*

(1) If A^h is the Henselization of A with respect to $\mathfrak{m}$, then $\tau_{A^h} \cap A = \tau_A$.
(2) If A has geometrically regular formal fibers and $\widehat{A}$ is the $\mathfrak{m}$-adic completion, then $\tau_{\widehat{A}} \cap A = \tau_A$.

We need the following lemma, which is interesting in its own right.

Lemma 4.1.8. *Let $A \to B$ be a cyclically pure map of noetherian domains. Then $T_B \cap A \subseteq T_A$.*

Proof of Lemma 4.1.8. By Proposition 4.1.4(2), $T_A = T_{A^+/A}$ and $T_B = T_{B^+/B}$. Cyclic purity implies $A \to B$ is injective. So we may assume that $A \subseteq B$ and $A^+ \subseteq B^+$.

Let $c \in T_B \cap A = T_{B^+/B} \cap A$, and pick any ideal I of A. Then

$$c(IA^+ \cap A) \subseteq c((IB)B^+ \cap B) \cap A \subseteq IB \cap A = I,$$

where the first containment follows because $IA^+ \cap A \subseteq (IB)B^+ \cap B$, the second containment because $c \in T_{B^+/B}$, and the equality because $A \to B$ is cyclically pure. Thus, $c \in \bigcap_I (I : IA^+ \cap A) = T_{A^+/A} = T_A$. □

We now prove Proposition 4.1.7 utilizing some ideal-theoretic results of [19].

Proof of Proposition 4.1.7. Note that a noetherian integrally closed domain is approximately Gorenstein.

(1) A^h is also a noetherian integrally closed domain [69, Tag 06DI]. Thus, by Proposition 4.1.4(3) and (2),

$$\tau_A = T_{A^+/A} \quad \text{and} \quad \tau_{A^h} = T_{(A^h)^+/A^h}.$$

Therefore it suffices to show that $T_{(A^h)^+/A^h} \cap A = T_{A^+/A}$. Since $A \to A^h$ is faithfully flat, Lemma 4.1.8 and Proposition 4.1.4(2) give

$$T_{(A^h)^+/A^h} \cap A \subseteq T_{A^+/A}.$$

Now suppose $c \in T_{A^+/A}$. It remains to show that $c \in T_{(A^h)^+/A^h}$. If $\mathcal{C}$ is the collection of $\mathfrak{m}A^h$-primary ideals of A^h, then Proposition 3.3.1(5) shows that

$$T_{(A^h)^+/A^h} = \bigcap_{J \in \mathcal{C}} (J : J(A^h)^+ \cap A^h).$$

Therefore it suffices to show that $c(J(A^h)^+ \cap A^h) \subseteq J$, for any $\mathfrak{m}A^h$-primary ideal J of A^h. Note that

$$A \to A^h$$

is a local homomorphism such that the induced map on completions is an isomorphism. Therefore any $\mathfrak{m}A^h$-primary ideal of A^h is expanded from an $\mathfrak{m}$-primary ideal of A (for example, see [19, Lem. 3.1.2]). So choose an $\mathfrak{m}$-primary ideal I of A such that

$$J = IA^h.$$

By [19, Prop. 3.1.4(2)],

$$(J(A^h)^+ \cap A^h) \cap A = J(A^h)^+ \cap A = I(A^h)^+ \cap A = IA^+ \cap A.$$

Moreover, $J \subseteq J(A^h)^+ \cap A^h$, which means that $J(A^h)^+ \cap A^h$ is also an $\mathfrak{m}A^h$-primary ideal expanded from some $\mathfrak{m}$-primary ideal of A. Then it must be the case that

$$J(A^h)^+ \cap A^h = (IA^+ \cap A)A^h.$$

Since $c \in T_{A^+/A}$, we have $c(IA^+ \cap A) \subseteq I$. Consequently,

$$c(J(A^h)^+ \cap A^h) = c\big((IA^+ \cap A)A^h\big) = (c(IA^+ \cap A))A^h \subseteq IA^h = J,$$

as desired.

(2) Since A is a normal domain with geometrically regular formal fibers, $\widehat{A}$ is also a normal domain [69, Tag 0BFK] and both $A, \widehat{A}$ are approximately Gorenstein. Now the rest of the proof of (2) follows from the argument given in (1) but with A^h replaced by $\widehat{A}$, $(A^h)^+$ replaced by $\widehat{A}^+$, and [19, Prop. 3.1.4(2)] replaced by [19, Prop. 3.2.2], which says that for an ideal I of A, $I\widehat{A}^+ \cap A = IA^+ \cap A$. □

4.2. Separable traces and splinter loci

Let R be a noetherian domain with fraction field K. Recall that we say R is *N-1* if the integral closure of R in K is a finite R-algebra.

Excellent domains are *N-1*, although the *N-1* assumption is substantially more general. For example, any noetherian normal domain is *N-1*, although noetherian normal domains are far from being excellent in general. Moreover, in the context of singularity theory, especially in prime characteristic, most notions of F-singularities such as F-injective, F-pure, Frobenius split, splinter, F-rational and all avatars of F-regular imply the *N-1* property at the level of local rings [17, Lem. 7.1.4].

If R is a domain, then we will use $(R^+)^{\text{sep}}$ to denote the subring of R^+ consisting of those elements whose minimal polynomials over K are separable. Thus, if K^{sep} is the maximal separable extension of K in $\overline{K} = \text{Frac}(R^+)$, then $(R^+)^{\text{sep}}$ is the integral closure of R in K^{sep}, or equivalently, $(R^+)^{\text{sep}} = R^+ \cap K^{\text{sep}}$.

We will now specialize to the setting where R is a noetherian domain of prime characteristic $p > 0$. Then recall that for an ideal I of R, the *Frobenius closure of* I, denoted $I^{[F]}$, is defined as

$$I^{[F]} \coloneqq \{r \in R \colon r^{p^e} \in I^{[p^e]} \text{ for some } e \in \mathbf{Z}_{\geq 0}\}.$$

One can verify that $I^{[F]}$ is an ideal of R that is contained in the tight closure I^*.

The following Proposition, proved by Singh [59], is the key result that motivates this section (see also [68]). We state the Proposition with a more general hypothesis than in [59], and explain why Singh's arguments only need this weaker hypothesis.

Proposition 4.2.1. *[59] Let R be a noetherian N-1 domain of prime characteristic $p > 0$ and fraction field K. Let I be an ideal of R.*

(1) If $r \in I^{[F]}$, then there exists a finite generically étale R-subalgebra S of R^+ such that $r \in IS$.
(2) For all ideals I of R, $IR^+ \cap R = I(R^+)^{\text{sep}} \cap R$.

Indication of proof. Singh proves (1) in [59, Thm. 3.1] assuming that R is excellent. In the proof, he considers roots $u_1, \ldots, u_n$ of certain Artin-Schreier polynomials over K and then takes S to be the integral closure of R in the fraction field L of $R[u_1, \ldots, u_n]$. Note that L is a finite separable extension of K by construction. The only place excellence appears to be used in the proof of [59] is to conclude that S is a finite R-algebra. We claim this follows as long as R is *N-1*. Indeed, if $\overline{R}$ is the integral closure of R in K, then S is also the integral closure of $\overline{R}$ in L. Since $\overline{R}$ is a normal noetherian domain (it is module finite over R by the *N-1* hypothesis), S is then a finite $\overline{R}$-algebra by [69, Tag 032L]. The point here is that since L/K is a finite separable extension, it admits a nonzero trace $\text{Tr}_{L/K}$ that restricts to give a nonzero $\overline{R}$-linear map $S \to \overline{R}$ because $\overline{R}$ is

normal. However, any generically finite solid algebra extension of noetherian domains is actually finite [18, Prop. 3.7]. Consequently, S is a finite R-algebra, and the rest of the proof of [59, Thm. 3.1] applies without change.

Similarly, (2) is [59, Cor. 3.4], but again where R is assumed to be excellent. As in [59], for $z \in IR^+ \cap R$ choose a finite R-subalgebra R_1 of R^+ such that $z \in IR_1$. If R_2 is the largest separable extension of R is R_1, then $z \in (IR_2)^{[F]}$ because $R_2 \hookrightarrow R_1$ is purely inseparable and module finite. Let $L = \mathrm{Frac}(R_2)$, which is a finite separable extension of K. Since R is *N-1*, the argument in the previous paragraph shows that the integral closure of R in L is a finite R-algebra. But this integral closure is also the integral closure of R_2 in $L = \mathrm{Frac}(R_2)$. Thus, R_2 is *N-1*. Now by (1), one can find a finite generically étale R_2-subalgebra of $(R_2)^+ = R^+$ such that $z \in IS$. Then $R \hookrightarrow S$ is a finite generically étale extension, and we are done. □

As a consequence, we obtain the following characterization of splinters in prime characteristic for *N-1* domains without any excellence or approximately Gorenstein hypotheses.

Corollary 4.2.2. (cf. [59, Cor. 3.9]) *Let R be a noetherian N-1 domain of prime characteristic $p > 0$. Then R is a splinter if and only if $R \hookrightarrow S$ is cyclically pure for every generically étale finite extension domain S.*

Proof. The 'if' implication is the non-trivial one. If $R \hookrightarrow S$ is cyclically pure for every generically étale finite extension domain S, then $R \to (R^+)^{\mathrm{sep}}$ is cyclically pure because $(R^+)^{\mathrm{sep}}$ is a filtered union of generically étale finite R-subalgebras. Now since R is *N-1*, by Proposition 4.2.1(2), $R \to R^+$ is cyclically pure. Then R is a splinter by [19, Lem. 2.3.1].

Again, Singh proves the same result assuming R is excellent [59, Cor. 3.9] and that R is a direct summand of every generically étale finite extension domain S. □

Motivated by Corollary 4.2.2 we introduce the following definition.

Definition 4.2.3. Let R be a noetherian domain and $\mathcal{C}^{\mathrm{sep}}$ be the collection of generically étale finite R-subalgebras of R^+. We define the *separable trace of R*, denoted τ_R^{sep}, to be

$$\tau_R^{\mathrm{sep}} := \bigcap_{S \in \mathcal{C}^{\mathrm{sep}}} \tau_{S/R}.$$

The *separable ideal trace* of R, denoted T_R^{sep}, is

$$T_R^{\mathrm{sep}} := \bigcap_{S \in \mathcal{C}^{\mathrm{sep}}} T_{S/R}.$$

Remark 4.2.4.

(1) If $\mathrm{Frac}(R)$ has characteristic 0, then $\tau_R^{\mathrm{sep}} = \tau_R$ and $T_R^{\mathrm{sep}} = T_R$.

(2) If $\mathrm{Frac}(R)$ has characteristic $p > 0$, we have $\tau_R \subseteq \tau_R^{\mathrm{sep}}$ and $T_R \subseteq T_R^{\mathrm{sep}}$.
(3) In general, $\tau_R^{\mathrm{sep}} \subseteq T_R^{\mathrm{sep}}$.
(4) Since $\mathcal{C}^{\mathrm{sep}}$ is a filtered poset under inclusion, the same argument as in Lemma 4.1.1 demonstrates that

$$\Sigma_\tau^{\mathrm{sep}} \coloneqq \{\tau_{S/R} \colon S \in \mathcal{C}^{\mathrm{sep}}\}$$

and

$$\Sigma_T^{\mathrm{sep}} \coloneqq \{T_{S/R} \colon S \in \mathcal{C}^{\mathrm{sep}}\}$$

are cofiltered collections of ideals of R under inclusion. In particular, if $\Sigma_\tau^{\mathrm{sep}}$ (resp. Σ_T^{sep}) has a minimal element, then it has a smallest element.

We then have the following analogue of Proposition 4.1.4. We state it for noetherian domains of prime characteristic because for mixed characteristic and equal characteristic 0 domains, the separable trace provides no new information over the usual trace.

Proposition 4.2.5. *Let R be a noetherian N-1 domain of prime characteristic $p > 0$. Let $\mathcal{C}^{\mathrm{sep}}$ be the collection of generically étale finite R-subalgebras of R^+. Then we have the following:*

(1) R is a splinter $\iff \tau_R^{\mathrm{sep}} = R \iff T_R^{\mathrm{sep}} = R$.
(2) $T_R^{\mathrm{sep}} = T_{(R^+)^{\mathrm{sep}}/R}$.
(3) If R is approximately Gorenstein, then $\tau_R^{\mathrm{sep}} = T_R^{\mathrm{sep}}$ and τ_R^{sep} is the big separable plus closure test ideal.
(4) If $(R, \mathfrak{m})$ is complete local, then $\tau_R^{\mathrm{sep}} = \tau_{(R^+)^{\mathrm{sep}}/R}$.

Assume $\Sigma_\tau^{\mathrm{sep}} \coloneqq \{\tau_{S/R} \colon S \in \mathcal{C}^{\mathrm{sep}}\}$ has a smallest element under inclusion. Then:

(5) There exists $S_0 \in \mathcal{C}^{\mathrm{sep}}$ such that $\tau_R^{\mathrm{sep}} = \tau_{S_0/R}$.
(6) If R is approximately Gorenstein, there exists $B_0 \in \mathcal{C}^{\mathrm{sep}}$ such that $T_R^{\mathrm{sep}} = T_{S_0/R} = \tau_{S_0/R}$.
(7) If $(R, \mathfrak{m})$ is complete local, there exists $S_0 \in \mathcal{C}^{\mathrm{sep}}$ such that $\tau_{(R^+)^{\mathrm{sep}}/R} = \tau_{S_0/R} = T_{S_0/R} = T_{(R^+)^{\mathrm{sep}}/R}$.
(8) If $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\tau_{R_\mathfrak{p}}^{\mathrm{sep}} = (\tau_R^{\mathrm{sep}})_\mathfrak{p}$.
(9) For $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_\mathfrak{p}$ is a splinter if and only if $\tau_R^{\mathrm{sep}} \not\subseteq \mathfrak{p}$. Thus, the splinter locus of R is the complement in $\mathrm{Spec}(R)$ of $\mathbf{V}(\tau_R^{\mathrm{sep}})$.
(10) There exists a finite generically étale R-subalgebra S_0 of R^+ such that if $R \hookrightarrow S_0$ splits, then R is a splinter.

Sketch of proof. (1) follows by Corollary 4.2.2.

(2) follows using the proof of Proposition 4.1.4(2) verbatim, but with A (resp. A^+) replaced by R (resp. $(R^+)^{\mathrm{sep}}$).

(3) The equality $\tau_R^{\mathrm{sep}} = T_R^{\mathrm{sep}}$ follows by Corollary 3.3.3(1). Then τ_R^{sep} is the big separable plus closure test ideal because it equals $T_{(R^+)^{\mathrm{sep}}/R}$ using (2), and the latter ideal is the big separable plus closure test ideal by Proposition 3.3.1(4) applied to $B = (R^+)^{\mathrm{sep}}$.

(4) A complete local domain is approximately Gorenstein. Therefore $\tau_R^{\mathrm{sep}} = T_R^{\mathrm{sep}} = T_{(R^+)^{\mathrm{sep}}/R}$ by (2) and (3), and $T_{(R^+)^{\mathrm{sep}}/R} = \tau_{(R^+)^{\mathrm{sep}}/R}$ by Corollary 3.3.3(3) applied to $B = (R^+)^{\mathrm{sep}}$, which is a solid R-algebra because R^+ is a solid R-algebra.

(5) follows because $\Sigma_\tau^{\mathrm{sep}}$ has a smallest element.

(6) follows by (2), (3), (5) and Corollary 3.3.3(1) because $\tau_{S_0/R} = T_{S_0/R}$.

(7) follows by (2), (3), (4) and (6).

(8) follows using the same line of reasoning as in Proposition 4.1.4(8). The key point is that if $\Sigma_\tau^{\mathrm{sep}}$ has a smallest element, then the a priori infinite intersection

$$\tau_R^{\mathrm{sep}} = \bigcap_{S \in \mathcal{C}^{\mathrm{sep}}} \tau_{S/R}$$

behaves like a finite intersection, and hence it commutes with localization at $\mathfrak{p}$, giving

$$(\tau_R^{\mathrm{sep}})_{\mathfrak{p}} = \bigcap_{S \in \mathcal{C}^{\mathrm{sep}}} \tau_{S_{\mathfrak{p}}/R_{\mathfrak{p}}}.$$

One can then show $\bigcap_{S \in \mathcal{C}^{\mathrm{sep}}} \tau_{S_{\mathfrak{p}}/R_{\mathfrak{p}}} = \tau_{R_{\mathfrak{p}}}^{\mathrm{sep}}$ by a similar spreading out argument. Indeed, the R-subalgebra T' of T constructed in the proof of Proposition 4.1.4(8) will be generically étale if T is a generically étale finite $R_{\mathfrak{p}}$-subalgebra of $((R_{\mathfrak{p}})^+)^{\mathrm{sep}} = ((R^+)^{\mathrm{sep}})_{\mathfrak{p}}$.

(9) follows from (8) and (1) and because the property of being *N-1* localizes [69, Tag 032G].

Finally for (10), any S_0 satisfying the conclusion of (5) works by (1). □

We obtain the following non-obvious consequence of the previous results.

Corollary 4.2.6. *Let R be a noetherian N-1 domain of prime characteristic $p > 0$ that is approximately Gorenstein. Then*

$$\tau_R = \tau_R^{\mathrm{sep}}.$$

Proof. By Proposition 4.1.4 parts (2) and (3) and Proposition 4.2.5 parts (2) and (3), we have

$$\tau_R = T_{R^+/R} \quad \text{and} \quad \tau_R^{\mathrm{sep}} = T_{(R^+)^{\mathrm{sep}}/R}.$$

Now by Proposition 4.2.1(2),

$$T_{R^+/R} = \bigcap_I (I : IR^+ \cap R) = \bigcap_I (I : I(R^+)^{\text{sep}} \cap R) = T_{(R^+)^{\text{sep}}/R},$$

where the intersections range over all ideals I of R. This completes the proof. □

4.3. Openness of splinter loci in prime characteristic

We will now show that the splinter locus is open for schemes in prime characteristic that are of most interest in arithmetic and geometry. In particular, we will show that the splinter locus of any scheme of finite type over an excellent local ring of prime characteristic is open. In fact, our results will hold more generally for some quasi-excellent schemes and even some schemes that are not quasiexcellent (see Remarks 4.3.2 and 4.3.8). Recall that a noetherian ring R is *quasi-excellent* if the local rings of R have geometrically regular formal fibers and if the regular locus of any finite type R-algebra is open. Thus an excellent ring is a quasi-excellent ring that is universally catenary. Our first result is affine in nature.

Theorem 4.3.1. *Let R be a noetherian F-pure domain of prime characteristic $p > 0$ and assume that R satisfies any of the following conditions:*

(i) R is F-finite.
(ii) R is local (not necessarily excellent).
(iii) $(A, \mathfrak{m})$ is a noetherian local ring of prime characteristic $p > 0$ with geometrically regular formal fibers and R is essentially of finite type over A.

Let $\mathcal{C}$ be the collection of finite R-subalgebras of R^+. Then we have the following:

(1) $\Sigma_\tau := \{\tau_{S/R} : S \in \mathcal{C}\}$ is a finite set of radical ideals of R.
(2) The splinter locus $\operatorname{Spl}(R)$ of $\operatorname{Spec}(R)$ is open and its complement is $\mathbf{V}(\tau_R) = \mathbf{V}(\tau_R^{\text{sep}})$.
(3) τ_R and τ_R^{sep} are radical ideals and $\tau_R = \tau_R^{\text{sep}}$.
(4) There exists a finite generically étale extension domain S of R such that $\tau_R = \tau_R^{\text{sep}} = \tau_{S/R}$.
(5) There exists a finite generically étale extension domain S of R such that if $R \hookrightarrow S$ splits, then R is a splinter.
(6) If R is complete local, there exists a finite generically étale R-subalgebra S of R^+ such that $\tau_{R^+/R} = \tau_R = \tau_R^{\text{sep}} = \tau_{(R^+)^{\text{sep}}/R} = \tau_{S/R}$.

Proof. If R satisfies (i), (ii) or (iii), we first claim that R is approximately Gorenstein and *N-1*.

By definition, R is approximately Gorenstein if $R_{\mathfrak{m}}$ is approximately Gorenstein for all maximal ideals $\mathfrak{m}$ of R. If R is F-pure, so is $R_{\mathfrak{m}}$. Then $R_{\mathfrak{m}}$ is approximately Gorenstein by [15, Cor. 3.6(ii)], where it is shown more generally that F-injective noetherian rings are approximately Gorenstein.

A quasi-excellent domain is Nagata [69, Tag 07QV], hence *N-1*. Thus, the *N-1* property follows when R is F-finite because F-finite rings are excellent [44, Thm. 2.5]. If $(A, \mathfrak{m})$ has geometrically regular formal fibers, then A is quasi-excellent [49, (33.D), Thm. 3.6] (or see [42, Prop. 5.5.1]). Therefore if R is essentially of finite type over A, then R is also quasi-excellent [69, Tag 07QU], hence *N-1*. Finally, if $(R, \mathfrak{m})$ is local and F-pure, then R is *N-1* by [17, Lem. 7.1.4].

(1) An F-finite noetherian F-pure ring is Frobenius split. By Proposition 3.4.1, the collection of uniformly F-compatible ideals is finite in case (i) and also in case (ii) when $(R, \mathfrak{m})$ is additionally Frobenius split, and by Lemma 3.1.2, every uniformly F-compatible is radical when R is Frobenius split. Let $\mathcal{C}$ be the collection of finite R-subalgebras of R^+ and $\mathcal{C}^{\text{sep}}$ be the subset of $\mathcal{C}$ consisting of those R-subalgebras that are also generically étale. Then

$$\Sigma_\tau \coloneqq \{\tau_{S/R} \colon S \in \mathcal{C}\}$$

and

$$\Sigma_\tau^{\text{sep}} \coloneqq \{\tau_{S/R} \colon S \in \mathcal{C}^{\text{sep}}\}$$

are collections of uniformly F-compatible ideals by Lemma 3.2.3. In particular, both Σ and Σ^{sep} are finite sets of radical ideals in case (i) and in case (ii) when $(R, \mathfrak{m})$ is Frobenius split. We will now show that Σ_τ (and hence Σ_τ^{sep}) is also a finite set of radical ideals in case (iii) and also in case (ii) when we drop the hypothesis that $(R, \mathfrak{m})$ is Frobenius split. Note that in the generality of (ii) and (iii), an F-pure noetherian domain R need not be Frobenius split (or admit any nonzero R-linear map $F_*R \to R$) [16], so it is not at all obvious that Σ_τ is finite or that its elements are radical ideals.

If R is essentially of finite type over a local G-ring $(A, \mathfrak{m})$, then it suffices to show that there is a faithfully flat map $R \to R'$ such that R' is F-finite and Frobenius split. Indeed, suppose one can find such a cover of R. Then for any $\tau_{S/R} \in \Sigma_\tau$,

$$\tau_{S/R}R' = \tau_{S \otimes_R R'/R'}$$

by Lemma 3.2.3(4) because S is a finite extension of R. Thus,

$$\{\tau_{S/R}R' : S \in \mathcal{C}\}$$

is a set of uniformly F-compatible ideals of the Frobenius split F-finite ring R' because all the expansion ideals are traces. Therefore this set is finite and each $\tau_{S/R}R'$ is a radical ideal by the argument in the previous paragraph. Since $R \to R'$ is faithfully flat,

$$\tau_{S/R} = \tau_{S/R}R' \cap R.$$

As contractions of radical ideals are radical, Σ_τ must be a finite set of radical ideals as well.

We now show the existence R'. Let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. By our assumption, $A \to \widehat{A}$ is a regular map. Since R is essentially of finite type over A, by [69, Tag 07C1] and the fact that property of being regular is preserved under localization,

$$R \to R \otimes_A \widehat{A}$$

is also a regular map. Therefore the relative Frobenius

$$F_* R \otimes_R (R \otimes_A \widehat{A}) \to F_*(R \otimes_A \widehat{A})$$

is faithfully flat by results of Radu [73, Thm. 4] and André [5, Thm. 1]. Since $R \to F_* R$ is pure, by base change

$$R \otimes_A \widehat{A} \to F_* R \otimes_R (R \otimes_A \widehat{A})$$

is pure, hence so is the composition

$$R \otimes_A \widehat{A} \to F_* R \otimes_R (R \otimes_A \widehat{A}) \to F_*(R \otimes_A \widehat{A}).$$

This last map is the Frobenius of $R \otimes_A \widehat{A}$. Thus, $R \otimes_A \widehat{A}$ is an F-pure ring which is essentially of finite type over a complete local ring of prime characteristic (see also [27, Sec. 2] for a generalization of this argument). Therefore by the gamma construction, there exists a faithfully flat local map

$$\widehat{A} \to \widehat{A}^{\Gamma}$$

such that $\widehat{A}^{\Gamma}$ is F-finite and $R \otimes_A \widehat{A}^{\Gamma} = (R \otimes_A \widehat{A}) \otimes_{\widehat{A}} \widehat{A}^{\Gamma}$ is F-pure [54, Thm. 3.4(ii)]. Consequently, $R \otimes_A \widehat{A}^{\Gamma}$ is Frobenius split since it is F-finite. Then we can take $R' = R \otimes_A \widehat{A}^{\Gamma}$ to be the faithfully flat F-finite cover of R that is Frobenius split.

It remains to show that if $(R, \mathfrak{m})$ is a local F-pure ring that is not Frobenius split, then Σ_τ is a finite set of radical ideals. The strategy is similar to the one above for case (iii). Since R is F-pure, so is its completion $\widehat{R}$ [38, Cor. 6.13] (without any restrictions on the formal fibers of R). Then $\widehat{R}$ is Frobenius split because F-purity and Frobenius splitting coincide for complete local rings. We now have that $\widehat{R}$ has finitely many uniformly F-compatible ideals by Proposition 3.4.1, all of which are radical by Lemma 3.1.2 because we have a splitting. Then, as above, Σ_τ is a finite set of radical ideals of R because the expansions of these ideals in $\widehat{R}$ (which is faithfully flat over R) are again trace ideals, and hence uniformly F-compatible, radical and finite in number.

Since $\Sigma_\tau^{\text{sep}} \subseteq \Sigma_\tau$, we have shown that if R satisfies (i), (ii) or (iii), then Σ_τ (hence also Σ_τ^{sep}) is a finite set of radical ideals of R. This proves (1).

(2) By (1), Lemma 4.1.1 and Remark 4.2.4 we conclude that Σ_τ and Σ_τ^{sep} have smallest elements under inclusion. We can then apply Proposition 4.1.4(9) and Proposition 4.2.5(9) to conclude that

$$\operatorname{Spec}(R) \setminus \mathbf{V}(\tau_R) = \operatorname{Spl}(R) = \operatorname{Spec}(R) \setminus \mathbf{V}(\tau_R^{\mathrm{sep}}).$$

This proves (2).

There are two ways to prove (3). We have already observed that R is *N-1* and approximately Gorenstein if it satisfies (i), (ii) or (iii). Then we can apply Corollary 4.2.6 to get (3).

Alternatively, both τ_R and τ_R^{sep} are radical ideals (they are intersections of ideals in Σ_τ) that define the non-splinter locus of $\operatorname{Spec}(R)$ by (2), so they must be equal.

(4) follows by (3) and Proposition 4.2.5(5).

For (5) choose an S that satisfies the conclusion of (4). If $R \hookrightarrow S$ splits, we have $\tau_R = \tau_R^{\mathrm{sep}} = R$, that is R is a splinter (Proposition 4.2.5(1)).

(6) The equalities

$$\tau_{R^+/R} = \tau_R \quad \text{and} \quad \tau_{(R^+)^{\mathrm{sep}}/R} = \tau_R^{\mathrm{sep}}$$

follow by Proposition 4.1.4(4) and Proposition 4.2.5(4). We are then done by (4). □

Remark 4.3.2. In Theorem 4.3.1(iii), the formal fibers of $(A, \mathfrak{m})$ are assumed to be geometrically regular in order to ensure that if R is an essentially of finite type A-algebra that is F-pure, then the base change $R_{\widehat{A}} \coloneqq R \otimes_A \widehat{A}$ is also F-pure. However, one can get by with weaker assumptions on the formal fibers of A in order to get F-purity of $R_{\widehat{A}}$ from that of R, which is all that is needed to construct a faithfully flat F-finite cover of R that is Frobenius split. One such condition is discussed in the present remark, and another condition will be discussed in Remark 4.3.3. Define a noetherian algebra R over a field k of characteristic $p > 0$ to be *geometrically F-pure* if for all finite purely inseparable extensions ℓ of k, $R \otimes_k \ell$ is F-pure.

We now claim that if $(A, \mathfrak{m})$ is a noetherian local ring whose formal fibers are Gorenstein and geometrically F-pure, then for an essentially of finite type A-algebra R, if R is F-pure then so is $R_{\widehat{A}}$. Moreover, if R is a domain then it is *N-1*. We briefly indicate the steps needed to prove this result, following the strategy of [15] that shows an analogous result for 'Cohen–Macaulay and geometrically F-injective'. The techniques in [15] are inspired by arguments of Vélez [76].

(1) It is known that if $(S, \mathfrak{m}) \to (T, \mathfrak{n})$ is a flat local homomorphism of noetherian local rings whose closed fiber is Gorenstein and F-pure, then F-purity ascends from S to T. This is proved in [2, Prop. 3.3] when R and S are F-finite, and the general case appears in [51, Thm. 7.3].
(2) By (1) it suffices to show that if the formal fibers of A are Gorenstein and geometrically F-pure, then $R \to R_{\widehat{A}}$ has Gorenstein and F-pure fibers.
(3) Let k be a field of characteristic $p > 0$. If R is a noetherian k-algebra that is Gorenstein and geometrically F-pure, then we claim that for all finitely generated field extensions k' of k, $R_{k'} = R \otimes_k k'$ is Gorenstein and F-pure. By [69, Tag 0C03],

the Gorenstein property is preserved by base change along finitely generated field extensions. Thus, $R_{k'}$ is Gorenstein. Since F-purity satisfies faithfully flat descent, by the proof strategy of [15, Prop. 4.10] and [15, Lem. 4.9], it suffices to show $R_{k'}$ is F-pure when k' is a finitely generated separable extension and when k' is a finite purely inseparable extension. If k' is a finitely generated separable extension of k, then $R \to R_{k'}$ is a regular homomorphism, so $R_{k'}$ is F-pure by (1) or the argument in the proof of Theorem 4.3.1(1). If k' is a finite purely inseparable extension of k, then $R_{k'}$ is F-pure by the definition of geometrically F-pure.

(4) We now claim that if $S \to T$ is a homomorphism of noetherian rings whose fibers are Gorenstein and geometrically F-pure, then for every essentially of finite type S-algebra R, the fibers of $R \to R \otimes_S T$ are also Gorenstein and F-pure. The proof of this reduces to (3) by [24, Lem. 7.3.7]. Finally, (2) follows by (4) upon taking $S = A$ and $T = \widehat{A}$. Therefore if R is F-pure, so is $R_{\widehat{A}}$.

(5) The formal fibers of A are geometrically reduced since they are geometrically F-pure. By the Zariski-Nagata theorem [24, Thm. 7.6.4], A is a Nagata ring, that is, for all prime ideals $\mathfrak{p}$ of A, $A/\mathfrak{p}$ is a Japanese ring. Then A is universally Japanese by [69, Tag 0334]. Hence R is universally Japanese by [69, Tag 032S], and so, R is *N-1* if it is a domain.

Remark 4.3.3. The proof of Theorem 4.3.1 shows, more generally, that if R is a noetherian F-pure domain that admits a faithfully flat cover $R \to S$, where S has finitely many trace ideals of finite extensions of S, then the splinter locus of R is open in $\operatorname{Spec}(R)$. To illustrate the utility of this observation, suppose $(A, \mathfrak{m}) \xrightarrow{\varphi} R$ is an essentially of finite type F-pure homomorphism of noetherian rings in the sense of Hashimoto [27, (2.3)], where $(A, \mathfrak{m})$ is a noetherian F-pure local ring and R is a domain. Being an F-pure homomorphism means that the relative Frobenius

$$F_{R/A} \colon F_* A \otimes_R R \to F_* R$$

is a pure ring map. For example, φ is F-pure if $F_{R/A}$ is faithfully flat, or equivalently, if φ is geometrically regular by [73,5]. The hypothesis that φ is F-pure and A is F-pure implies that R is also F-pure [27, Prop. 2.4(4)]. Consider the commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{A}^{\Gamma} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi \otimes \widehat{A}} & & \downarrow{\scriptstyle \varphi \otimes \widehat{A}^{\Gamma}} \\ R & \longrightarrow & R \otimes_A \widehat{A} & \longrightarrow & R \otimes_A \widehat{A}^{\Gamma}, \end{array}$$

where $\widehat{A}$ is F-pure by Remark 4.3.2(1), and $\widehat{A}^{\Gamma}$ is chosen to be noetherian local, F-finite and F-pure (equivalently, Frobenius split) via the Γ-construction. Then $\varphi \otimes \widehat{A}^{\Gamma}$ is also essentially of finite type, and so, $R \otimes_A \widehat{A}^{\Gamma}$ is noetherian and F-finite because $\widehat{A}^{\Gamma}$ is. Since F-pure homomorphisms are stable under arbitrary base change [27, Prop. 2.4(7)],

it follows that $\varphi \otimes \widehat{A^\Gamma}$ is F-pure. Consequently, $R \otimes_A \widehat{A^\Gamma}$ is a faithfully flat cover of R that is noetherian, F-finite and F-pure, and so, it has finitely many trace ideals by Proposition 3.4.1 and Lemma 3.2.3(2). Thus, the splinter locus of $\operatorname{Spec}(R)$ is open.

The formation of trace commutes with Henselizations and completions under certain conditions, giving a refinement of Proposition 4.1.7.

Corollary 4.3.4. *Let $(R, \mathfrak{m})$ be a noetherian local domain which is F-pure and normal.*

(1) If R is Frobenius split, then $\tau_R R^h = \tau_{R^h}$.
(2) If R has geometrically regular formal fibers, then $\tau_R R^h = \tau_{R^h}$.
(3) If R has geometrically regular formal fibers, then $\tau_R \widehat{R} = \tau_{\widehat{R}}$.

Proof. (1) R^h is a normal domain by [69, Tag 06DI]. We claim that R^h is also Frobenius split. Indeed, since R^h is a filtered colimit of étale R-algebras, the relative Frobenius

$$F_* R \otimes_R R^h \to F_* R^h$$

is an isomorphism because $R \to R^h$ is weakly étale [69, Tag 092N] and the relative Frobenius of a weakly étale map is an isomorphism [69, Tag 0F6W]. Then R^h is Frobenius split because splittings are preserved under base change. By Theorem 4.3.1(1), τ_R (resp. τ_{R^h}) defines the non-splinter locus of $\operatorname{Spec}(R)$ (resp. $\operatorname{Spec}(R^h)$). Moreover, τ_R and τ_{R^h} are radical by Theorem 4.3.1(3). Since $R \to R^h$ is a regular homomorphism,

$$R/\tau_R \to R^h/\tau_R R^h$$

is a regular homomorphism as well by finite type base change [69, Tag 07C1]. Then $R^h/\tau_R R^h$ is reduced by [69, Tag 07QK] because R/τ_R is reduced. In other words, $\tau_R R^h$ is a radical ideal of R^h. Therefore to show that $\tau_R R^h = \tau_{R^h}$, it suffices to check that $\tau_R R^h$ defines the non-splinter locus of $\operatorname{Spec}(R^h)$ as well.

By Proposition 4.1.7(1), we have

$$\tau_R R^h \subseteq \tau_{R^h}.$$

Therefore $\mathbf{V}(\tau_{R^h}) \subseteq \mathbf{V}(\tau_R R^h)$. On the other hand, if $\mathfrak{q} \in \mathbf{V}(\tau_R R^h)$, then

$$\tau_R \subseteq \mathfrak{q} \cap R,$$

that is, $R_{\mathfrak{q} \cap R}$ is not a splinter. However

$$R_{\mathfrak{q} \cap R} \to (R^h)_{\mathfrak{q}}$$

is faithfully flat (because $R \to R^h$ is), and the splinter property satisfies faithfully flat descent. Consequently, $(R^h)_{\mathfrak{q}}$ cannot be a splinter, and so, $\mathfrak{q} \in \mathbf{V}(\tau_{R^h})$. This establishes the other inclusion $\mathbf{V}(\tau_R R^h) \subseteq \mathbf{V}(\tau_{R^h})$, completing the proof of (1).

(2) Since the relative Frobenius $F_*R \otimes_R R^h \to F_*R^h$ is an isomorphism, R^h is also F-pure when R is F-pure. Moreover, the formal fibers of R^h are also geometrically regular [23, Thm. 5.3(i)] because $A \to A^h$ is ind-étale and hence absolutely flat. Therefore by Theorem 4.3.1(2) the non-splinter locus of R (resp. R^h) is defined by τ_R (resp. τ_{R^h}). One can now use the same argument as in (1) to get (2).

(3) Note that $\widehat{R}$ is F-pure. This is true for the completion of any F-pure noetherian local ring by Remark 4.3.2(1), but in our setting this also follows by the regularity of $R \to \widehat{R}$ and the Radu-André theorem using the argument given in the proof of Theorem 4.3.1(1). Proposition 4.1.7(2) shows that $\tau_R \widehat{R} \subseteq \tau_{\widehat{R}}$ and Theorem 4.3.1(2) shows that τ_R (resp. $\tau_{\widehat{R}}$) defines the non-splinter locus of R (resp. $\widehat{R}$). One can then mimic the argument of (1) to prove (3). We omit the details. □

Remark 4.3.5. The proof of Corollary 4.3.4 shows more generally that if $(A, \mathfrak{m})$ is a noetherian normal domain of arbitrary characteristic that satisfies the hypotheses of Proposition 4.1.7, and if τ_A, τ_{A^h} and $\tau_{\widehat{A}}$ define the non-splinter locus A, A^h and $\widehat{A}$ respectively, then $\tau_A A^h$ and τ_{A^h} agree up to radical, as do $\tau_A \widehat{A}$ and $\tau_{\widehat{A}}$.

A natural question one can ask is whether τ_R is nonzero. Indeed, if τ_R is the ideal that defines the non-splinter locus in general, then τ_R has to be nonzero because a domain is generically a splinter. The next result implies that showing $\tau_R \neq 0$ is equivalent to a long-standing conjecture in tight closure theory on the existence of test elements.

Proposition 4.3.6. *Let R be a noetherian domain of characteristic $p > 0$ whose local rings at maximal ideals have geometrically regular formal fibers (i.e. R is a G-ring) and whose regular locus is open. Let $\tau_*^{\mathrm{fg}}(R)$ be the finitistic tight closure test ideal of R. Then we have the following:*

(1) $\tau_*^{\mathrm{fg}}(R) \subseteq \tau_R$.
(2) If $T_{F_*R/R} \neq 0$, *then* $\tau_*^{\mathrm{fg}}(R) \neq 0$.
(3) $\tau_R \neq 0$ *if and only if* $\tau_*^{\mathrm{fg}}(R) \neq 0$.
(4) If R is F-pure, then $\tau_R \neq 0$.
(5) If $\tau_{F_*R/R} \neq 0$, *that is, if there is a nonzero R-linear map $F_*R \to R$, then* $\tau_R \neq 0$.

Proof. The completions of the local rings of R at maximal ideals are reduced because the property of being reduced is preserved under completion when the formal fibers are geometrically regular. Thus, R is approximately Gorenstein.

(1) Since R is approximately Gorenstein, $\tau_R = T_{R^+/R} = \bigcap_I (I : IR^+ \cap R)$ by Proposition 4.1.4(3). For all ideals I of R, we have

$$IR^+ \cap R \subseteq I^*$$

by [31, Cor. (5.23)]. Thus, $(I : I^*) \subseteq (I : IR^+ \cap R)$, and so,

$$\tau_*^{\mathrm{fg}}(R) = \bigcap_I (I : I^*) \subseteq \bigcap_I (I : IR^+ \cap R) =: \tau_R.$$

Here the first equality follows by [29, Prop. (8.15)] because R is approximately Gorenstein. This proves (1).

(2) Since R is a domain and the regular locus of R is open, one can find a $c \neq 0$ such that R_c is regular and $c \in T_{F_*R/R}$. Then the result follows by [1, Thm. 1.2].

(3) If $\tau_*^{\mathrm{fg}}(R) \neq 0$, then $\tau_R \neq 0$ by (1). Conversely, suppose $\tau_R = T_{R^+/R} \neq 0$. Since F_*R embeds in R^+, this means that

$$0 \neq T_{R^+/R} \subseteq T_{F_*R/R} = \bigcap_I (I : IF_*R \cap R).$$

Then $\tau_*^{\mathrm{fg}}(R) \neq 0$ by (2).

(4) follows (2) and (3) because if R is F-pure, then $T_{F_*R/R} = R$.

(5) If $\tau_{F_*R/R} \neq 0$, then $T_{F_*R/R} \neq 0$ by Lemma 3.2.7 because $\tau_{F_*R/R} \subseteq T_{F_*R/R}$. We are then done by (2) and (3). □

We now deduce our main global result.

Theorem 4.3.7. *Let X be a scheme of prime characteristic $p > 0$. Suppose that X satisfies any of the following conditions:*

(i) X is locally noetherian and F-finite.
(ii) X is locally essentially of finite type over a noetherian local ring $(A, \mathfrak{m})$ of prime characteristic $p > 0$ with geometrically regular formal fibers.

Then

$$\mathrm{Spl}(X) = \{x \in X : \mathcal{O}_{X,x} \text{ is a splinter}\}$$

is open in X.

Recall that we say that X is *locally essentially of finite type over A* if there exists an affine open cover $\mathrm{Spec}(B_i)$ of X such that for all i, B_i is an essentially of finite type A-algebra.

Proof. Let $\mathrm{Nor}(X)$ denote the normal locus of X and $fp(X)$ denote the locus of points $x \in X$ such that $\mathcal{O}_{X,x}$ is F-pure. We claim that both these loci are open if X satisfies (i) or (ii). Indeed, in either case X is quasi-excellent and has an open regular locus. Then $\mathrm{Nor}(X)$ is open by [24, Cor. (6.13.5)]. If X is locally noetherian and F-finite, then $fp(X)$ is open and coincides with the locus of points at which X is Frobenius split. If X

is locally essentially of finite type over $(A, \mathfrak{m})$ as in (ii), then $fp(X)$ is open by [54, Cor. 3.5] (this result is attributed to Hoshi when A is excellent in [28, Thm. 3.2]). Note that [54, Cor. 3.5] is stated assuming X is quasi-compact, but for openness of loci, one can always work on an affine cover of X.

If $\mathcal{O}_{X,x}$ is a splinter, then it is normal and F-pure (the latter follows because the Frobenius $F : \mathcal{O}_{X,x} \to F_*\mathcal{O}_{X,x}$ is an integral extension). Thus,

$$\mathrm{Spl}(X) \subseteq \mathrm{Nor}(X) \cap fp(X).$$

Therefore, replacing X by $\mathrm{Nor}(X) \cap fp(X)$, we may assume X is locally F-pure and normal. Now there exists an affine open cover $\{\mathrm{Spec}(R_\alpha)\}_\alpha$ of X, where each R_α is an F-pure domain that satisfies condition (i) or (iii) of Theorem 4.3.1 depending on whether X satisfies conditions (i) or (ii) in the statement of this theorem (you can even choose R_α to be normal). Then for all α, $\mathrm{Spl}(X) \cap \mathrm{Spec}(R_\alpha) = \mathrm{Spl}(R_\alpha)$ is open in $\mathrm{Spec}(R_\alpha)$ by Theorem 4.3.1(2), and hence, also in X. Then $\mathrm{Spl}(X) = \bigcup_\alpha \mathrm{Spl}(R_\alpha)$ is open in X as well. □

Remark 4.3.8. In Remark 4.3.2 we showed that Theorem 4.3.1(1)-(6) still holds if in part (iii) of Theorem 4.3.1 we assume that the formal fibers of $(A, \mathfrak{m})$ are Gorenstein and geometrically F-pure. We claim that the same hypotheses on the formal fibers of $(A, \mathfrak{m})$ also work for Theorem 4.3.7. The two things we need to check are:

(1) *If A has Gorenstein and geometrically F-pure formal fibers, then for any essentially of finite type A-algebra R, the F-pure locus of R is open*: We follow the proof strategy of [15, Thm. B] which establishes the analogous fact for the property 'Cohen–Macaulay and geometrically F-injective'. Since the induced map $\mathrm{Spec}(R_{\widehat{A}}) \to \mathrm{Spec}(R)$ is faithfully flat and quasi-compact, by [24, Cor. 2.3.12], it suffices to show that the inverse image of the F-pure locus of $\mathrm{Spec}(R)$ is open in $\mathrm{Spec}(R_{\widehat{A}})$. But this inverse image is the F-pure locus of $\mathrm{Spec}(R_{\widehat{A}})$ by Remark 4.3.2(1), (4) and by faithfully flat descent of F-purity. That the F-pure locus of $R_{\widehat{A}}$ is open now follows by [54, Cor. 3.5] because $\widehat{A}$ is excellent.

(2) *If A has Gorenstein and geometrically F-pure formal fibers, then the normal locus of any essentially of finite type A-algebra R is open*: by Remark 4.3.2(5), A is universally Japanese, and so, for all finite A-algebras S such that S is domain, the normal locus of S is open by [24, Cor. 6.13.3] since S is Japanese. Then the normal locus of R is open by [24, Prop. 6.13.7 (b) $\Longrightarrow$ (a)].

Example 4.3.9. In order to show the openness of splinter loci in prime characteristic, it suffices to restrict ones attention to the intersection of the F-pure and normal loci. This intersection is open as long as the F-pure locus and the normal locus are both open. Furthermore, we have shown that in the local case, a noetherian F-pure domain always has an open splinter locus (Theorem 4.3.1). Thus one may naturally wonder if

the splinter loci of an F-pure and normal noetherian domain is always open. We now use a construction of Hochster [33] to give examples of locally excellent F-pure and normal domains of prime characteristic $p > 0$ whose splinter loci are not open. We begin by choosing an algebraically closed field k of prime characteristic $p > 0$ and a local domain $(R, \mathfrak{m})$ essentially of finite type over k such that $(R, \mathfrak{m})$ is F-pure and normal, R is not a splinter and $R/\mathfrak{m} = k$. The last hypothesis ensures that if K/k is any field extension, then $\mathfrak{m}(K \otimes_k R)$ is a maximal ideal of $K \otimes_k R$. For an explicit example, if k is a field of characteristic not equal to 3, then the local ring at the origin of the Fermat cubic

$$R = k[x, y, z]_{(x,y,z)}/(x^3 + y^3 + z^3)$$

is not F-rational [65, Ex. 6.2.5], hence also not a splinter since excellent splinters are F-rational [62]. By Fedder's criterion, R is F-pure, for instance, when the characteristic of k is 7 and R is normal (it is $R_1 + S_2$) when the characteristic of $k \neq 3$. Coming back to our example, once we have such an R, Hochster then constructs [33, Prop. 1] a noetherian domain S using R such that

(a) S has infinitely many maximal ideals;
(b) for any maximal ideal $\mathfrak{M}$ of S,

$$S_{\mathfrak{M}} \cong (L_{\mathfrak{M}} \otimes_k R)_{\mathfrak{m}(L_{\mathfrak{M}} \otimes_k R)}$$

for a suitable field extension $L_{\mathfrak{M}}/k$ that depends on $\mathfrak{M}$;
(c) every nonzero element of S is contained in only finitely many maximal ideals.

In particular, this implies that S is a locally excellent domain; in fact the local rings of S are essentially of finite type over appropriate field extensions of k. Furthermore, (a) and (c) imply that the intersection of all the maximal ideals of S is (0).

We now claim that since k is algebraically closed, for all field extensions K/k, $K \otimes_k R$ is also F-pure and normal. Indeed, since k does not have any non-trivial finite purely inseparable extensions,

$$k \hookrightarrow K$$

is trivially a regular map (i.e. a flat map with geometrically regular fibers). As R is essentially of finite type over k, the base change map

$$R \to K \otimes_k R$$

is also a regular map [49, (33.D), Lem. 4]. We now observe that both F-purity and normality ascend from the base to the target over regular maps, proving that $K \otimes_k R$ is both F-pure and normal. The ascent of F-purity over regular maps follows by the Radu-André theorem because the relative Frobenius of $F_{K \otimes_k R/R}$ is faithfully flat, hence

pure, and so the Frobenius on $K \otimes_k R$ can be expressed as the composition of the pure maps

$$K \otimes_k R \xrightarrow{\mathrm{id}_K \otimes_k F_R} K \otimes_k F_* R \xrightarrow{F_{K \otimes_k R/R}} F_*(K \otimes_k R),$$

where the first map in the composition is pure because it is the base change of the pure map $F_R \colon R \to F_* R$. The ascent of normality over regular maps follows because the R_n and S_n properties ascend over regular maps; see for instance [69, Tag 0BFK]. The upshot of this discussion is that for the locally excellent noetherian domain S and for any maximal ideal $\mathfrak{M}$ of S, $S_{\mathfrak{M}} \cong (L_{\mathfrak{M}} \otimes_k R)_{\mathfrak{m}(L_{\mathfrak{M}} \otimes_k R)}$ is both F-pure and normal. Since F-purity and normality can be checked locally at the maximal ideals, it follows that S is a locally excellent F-pure and normal domain. However, R was chosen so that it is not a splinter, and the splinter property satisfies faithfully-flat descent. Therefore for all maximal ideals $\mathfrak{M}$ of S, $S_{\mathfrak{M}}$ is not a splinter because the map $R \to (L_{\mathfrak{M}} \otimes_k R)_{\mathfrak{m}(L_{\mathfrak{M}} \otimes_k R)} \cong S_{\mathfrak{M}}$ is faithfully flat. Thus the non-splinter locus of S contains all the maximal ideals, whose intersection is (0). This means that the non-splinter locus of S cannot be closed as otherwise the splinter locus would be empty, which it is not since S, being a domain, is a splinter at its generic point.

For noetherian graded rings over fields of prime characteristic, the splinter property is detected by the homogeneous maximal ideal. This is well-known over fields of characteristic 0 because then splinter is the same as being normal.

Corollary 4.3.10. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a noetherian graded ring such that $R_0 = k$ is a field of characteristic $p > 0$. Let $\mathfrak{m} \coloneqq \bigoplus_{n>0} R_n$ be the homogeneous maximal ideal of R. Then R is a splinter if and only if $R_{\mathfrak{m}}$ is a splinter.*

Proof. The splinter property localizes. So the backward implication is the non-trivial one.

We first assume that k is infinite. By Theorem 4.3.7, the splinter locus of R is open. Let I be the radical ideal of R that defines the non-splinter locus. Note that $k^\times$ acts on R by automorphisms (for $c \in k^\times$ and $x \in R_n$, $c \cdot x = c^n x$), and automorphisms clearly preserve the non-splinter locus of R. This means that I is stable under the action of $k^\times$, and since k is infinite, I must be a homogeneous ideal of R. If $R_{\mathfrak{m}}$ is a splinter, then $I \nsubseteq \mathfrak{m}$, which means that $I = R$, that is, R is a splinter.

Suppose k is finite and that $R_{\mathfrak{m}}$ is a splinter. Finite fields of prime characteristic are perfect. This means that an algebraic closure $\overline{k}$ is the filtered union of finite étale subextensions ℓ/k. Let

$$R_\ell \coloneqq \ell \otimes_k R.$$

Then $R \hookrightarrow R_\ell$ is a finite étale map and $\mathfrak{m}_\ell \coloneqq \mathfrak{m} R_\ell$ is the homogeneous maximal ideal of R_ℓ. It follows that

$$R_{\mathfrak{m}} \hookrightarrow (R_\ell)_{\mathfrak{m}_\ell}$$

is essentially étale, and so, $(R_\ell)_{\mathfrak{m}_\ell}$ is a splinter by [19, Thm. A]. Now

$$R_{\overline{k}} \coloneqq \overline{k} \otimes_k R,$$

is graded noetherian over $\overline{k}$ with homogeneous maximal ideal $\mathfrak{m}_{\overline{k}} \coloneqq \mathfrak{m}R_{\overline{k}}$. Then

$$(R_{\overline{k}})_{\mathfrak{m}_{\overline{k}}} = \operatorname{colim}_\ell (R_\ell)_{\mathfrak{m}_\ell}$$

is a splinter because a filtered colimit of splinters is a splinter [3, Prop. 5.2.5(ii)]. Since $\overline{k}$ is infinite, it follows by the previous paragraph that $R_{\overline{k}}$ is a splinter. By faithfully flat descent along

$$R \hookrightarrow R_{\overline{k}},$$

we then get that R is a splinter. □

Remark 4.3.11. In the situation of Corollary 4.3.10, even when $R_0 = k$ is a finite field, the radical ideal I that defines the closed non-splinter locus of R is still a homogeneous ideal. Indeed, if we look at the faithfully flat graded homomorphism $R \hookrightarrow R_{\overline{k}}$, then we claim that $IR_{\overline{k}}$ is the radical ideal that defines the non-splinter locus of $R_{\overline{k}}$. If our claim holds, then since $IR_{\overline{k}}$ will be graded because $\overline{k}$ is infinite (see proof above), it will follow that $I = IR_{\overline{k}} \cap R$ is also graded (contractions of graded ideals along graded homomorphisms are graded). We will now prove our claim. Expressing $\overline{k}$ as a filtered union of finite (and hence étale) subextensions ℓ/k, by ascent of the splinter property along essentially étale maps [19, Thm. A] and faithfully flat descent of the splinter property, it follows that IR_ℓ is the radical ideal that defines the non-splinter locus of R_ℓ. Note IR_ℓ is radical because $R/I \hookrightarrow R_\ell/IR_\ell$ is étale by base change [69, Tag 00U2] and the property of being reduced ascends under étale maps [69, Tag 033B]. Then

$$IR_{\overline{k}} = \operatorname{colim}_{\ell/k} IR_\ell$$

is a radical ideal of $R_{\overline{k}}$. Let $\mathfrak{p}$ be a prime ideal of $R_{\overline{k}}$. Let $\mathfrak{p}_\ell$ be the contraction of $\mathfrak{p}$ to R_ℓ. Then

$$(R_{\overline{k}})_{\mathfrak{p}} = \operatorname{colim}_{\ell/k} (R_\ell)_{\mathfrak{p}_\ell}$$

If $(R_{\overline{k}})_{\mathfrak{p}}$ is a splinter, then by faithfully flat descent along $(R_\ell)_{\mathfrak{p}_\ell} \hookrightarrow (R_{\overline{k}})_{\mathfrak{p}}$, $(R_\ell)_{\mathfrak{p}_\ell}$ is a splinter as well. Thus, $IR_\ell \nsubseteq \mathfrak{p}_\ell$, and consequently, $IR_{\overline{k}} \nsubseteq \mathfrak{p}$ as well, as otherwise, by contracting the inclusion $IR_{\overline{k}} \subseteq \mathfrak{p}$ to R_ℓ, we would get $IR_\ell \subseteq \mathfrak{p}_\ell$. Thus,

$$\operatorname{Spl}(\operatorname{Spec}(R_{\overline{k}})) \subseteq \operatorname{Spec}(R_{\overline{k}}) \setminus \mathbf{V}(IR_{\overline{k}}).$$

Now suppose $\mathfrak{p}$ is a prime ideal of $R_{\overline{k}}$ such that $IR_{\overline{k}} \not\subseteq \mathfrak{p}$. Then there exists a finite extension ℓ of k contained in $\overline{k}$ such that for all finite subextensions ℓ'/ℓ of $\overline{k}/\ell$,

$$IR_{\ell'} \not\subseteq \mathfrak{p}_{\ell'}.$$

For each such ℓ', $(R_{\ell'})_{\mathfrak{p}_{\ell'}}$ is a splinter since $IR_{\ell'}$ defines the non-splinter locus of $R_{\ell'}$. Thus,

$$(R_{\overline{k}})_{\mathfrak{p}} = \operatorname{colim}_{\ell'/\ell}(R_{\ell'})_{\mathfrak{p}_{\ell'}},$$

is a splinter as well because filtered colimits of splinters are splinters [3, Prop. 5.2.5(ii)]. This shows,

$$\operatorname{Spec}(R_{\overline{k}}) \setminus \mathbf{V}(IR_{\overline{k}}) \subseteq \operatorname{Spl}(\operatorname{Spec}(R_{\overline{k}})),$$

proving that $IR_{\overline{k}}$ is indeed the radical ideal that defines the non-splinter locus of $R_{\overline{k}}$.

Data availability

No data was used for the research described in the article.

Acknowledgments

We thank Linquan Ma, Takumi Murayama, Rebecca R.G. and Karl Schwede for helpful conversations and their interest. We are very grateful to Takumi for allowing us to include the results in Remarks 4.3.2 and 4.3.8 on permanence properties of Gorenstein and geometrically F-pure fibers and their relation to openness of F-pure loci since he was aware of these results from his work in [54]. Additionally, we thank Linquan and Benjamin Antieau for comments on a draft. We thank the referees for their comments on previous drafts. We especially thank the referee who suggested that we include Remark 4.3.11.

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