the treatment of [41], [1], an equal emphasis was given in providing explicit representation formulae for the quantities of interest: $\left\{y, y_t, y_{tt}, \left.\frac{\partial y}{\partial \nu}\right|_\Gamma\right\}$ in terms of the Dirichlet or Neumann boundary datum g. In the canonical case $\gamma = 0$, these turn out to be the same representation formulae expressed via cosine/sine operator theory of the corresponding wave equation. As the parameter γ contributes only lower order terms, the interior/boundary regularity theory is the same, whether $\gamma = 0$ or $0 \neq \gamma \in L^\infty(\Omega)$. The use of cosine/sine operator theory to describe by explicit formulae second order hyperbolic equations with either Dirichlet or Neumann boundary control was introduced in [40] and systematically employed in [18]–[21], [17, Section 3], [22], [23]. Such theory must be combined with boundary elliptic theory. The actual (optimal) interior/boundary regularity results require PDE-proofs, those for the Neumann case requiring pseudo-differential/microlocal analysis tools [22],[23], [37], unlike the differential multiplier-based proofs in the Dirichlet case [17]. In each case, we invoke the corresponding optimal regularity theory from [17] in the Dirichlet case or from [22],[23], [37] in the Neumann case. In short, we proceed from 'lower-regularity' in g (L^2-level) to 'higher-regularity' in g (at the H^1-level, at the H^2-level, etc) in the Dirichlet boundary case. Similarly for the Neumann boundary case. In Section 7, we consider the Neumann boundary control with regularity in space less than $L^2(\Gamma)$, by relying on the corresponding wave problem [44]. In Section 8, we provide optimal interior and boundary regularity results when the third order equation is subject to interior point control.
We refer to the by now abundant literature regarding the nonlinear third order (in time) equation and the numerous applications where it arises. We shall call it here SMGTJ equation [for G. G. Stokes (1851), F. K. Moore & W. E. Gibson (1960), P. A. Thompson (1972) and P. M. Jordan (2004)], see [36], [32], [38], [9],[10]. In this paper we focus on its linearized version.

1.2. **Literature review.**

1. For comparison purposes, the only reference in the literature of the third order problem with a boundary control g "smoother" than $g \in L^2(0, T; L^2(\Gamma))$ is [3], which considers only the case where g is a Dirichlet control. [The complementary case $g = 0$ was studied on several function spaces in [30], see Appendix B, for both the Dirichlet or the Neumann case.] More precisely, in this reference, the main result is [3, Theorem 1], which consists of two cases, part (a) and part (b).

   In a first comparison with [3, Theorem 1(a)], we note the following differences regarding the interior and the boundary regularity. As to the interior regularity, our Theorem 3.2.2 Part A(i) is slightly more precise than [3, Theorem 1(a)] in that the additional assumption $g_t \in C(0, T; H^{\frac{1}{2}}(\Gamma))$ in [3, Eq. (9)] is needed only for $y_t, y_{tt} \in C(0, T; H^1(\Omega) \times L^2(\Omega))$, but not for $y \in C(0, T; H^2(\Omega))$. Our Theorem 3.2.2 for the third order equation gives the same interior regularity properties as for the wave equation [17, Theorem 3.8, p. 180–182]; see our Section 3.2. More importantly, as to the boundary regularity, our Theorem 3.2.2 Part A (ii) [same as for the wave equation case [17, Theorem 2.2, p. 152]] establishes $\left.\frac{\partial y}{\partial \nu}\right|_\Sigma \in H^1(\Sigma)$ under $g \in H^2(\Sigma)$, thus answering in the positive a question raised in [3, Remark 2(iii)].

   In our next comparison with [3, Theorem 1(b)], we note that our Theorem 3.1.3 yields the same conclusions however under weaker assumptions. For

to the case $g \in L^2(\Sigma)$. The procedure was a tour de force. It provided only $L^2(0,T;\cdot)$ time regularity. It had to be supplemented by a "soft" procedure in [19] to obtain $C([0,T];\cdot)$ time regularity. Its abstract version is in [21], see also [24, Vol II, Thm 7.3.1, p. 65]. The approach by duality in [17] has emerged as being far preferable and has influenced the regularity properties of other dynamics (Schrödinger, plates, shells, etc.) subject to boundary controls.

7. Regarding the 'starting' case of Dirichlet $g \in L^2(0,T;L^2(\Gamma))$ - which is not the object of the present paper - an approach different from [41], [1] is proposed in [2]. In this paper the SMGTJ equation is reduced to a Volterra integral equation: the definition of solution is indirect via a solver of the Volterra equation. More details on the different approaches are given in [1].
8. A new approach yielding optimal interior regularity of the SMGTJ equation with control $g \in L^2(0,T;L^2(\Gamma))$ in either Dirichlet or Neumann B.C. is given in [25]. Here a general 3×3 system approach is presented, based on the vector state solution {position, velocity, acceleration}. It yields an explicit representation formula. This is close to, but also distinct from, the abstract variation of parameter formula that arises in more traditional boundary control problems for PDEs [23], [24].

PART A: Dirichlet boundary control

1.3. **Linear third order SMGTJ-equation with non-homogeneous Dirichlet boundary term.** Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\Gamma = \partial\Omega$, as specified below. If the linear third order equation is written in terms of the pressure, then Dirichlet non-homogeneous boundary terms are appropriate [4, 10]. We then consider the following mixed problem in the unknown $y(t,x)$:

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t = f \quad \text{in } Q = (0,T] \times \Omega & \text{(1.1a)} \\ y\big|_{t=0} = y_0; \quad y_t\big|_{t=0} = y_1; \quad y_{tt}\big|_{t=0} = y_2 \quad \text{in } \Omega & \text{(1.1b)} \\ y\big|_{\Sigma} = g \quad \text{in } \Sigma = (0,T] \times \Gamma & \text{(1.1c)} \end{cases}$$

where $b > 0, a^2 > 0$. The following quantity, introduced in [30, 11]

$$\gamma = \alpha - \frac{c^2}{b}, \tag{1.1d}$$

plays a critical role in the stability of the homogeneous case $f \equiv 0, g \equiv 0$. When $\gamma =$ constant, such system is uniformly stable in the appropriate functional setting, and with optimal decay rate, if and only if $\gamma > 0$ [30, Section 5]. Moreover, under Neumann boundary dissipation, uniform stabilization results (with 'minimal' geometric conditions) and strong stabilization results (without geometric conditions) are given in [1] assuming $\gamma \in L^\infty(\Omega), \gamma(x) \geq 0$, a.e. in Ω. In the case of interior/boundary regularity of the present paper, we may take $\gamma \equiv 0$ without loss of generality, as $0 \neq \gamma \in L^\infty(\Omega)$ contributes only lower order terms. See [1, Step 1 of Section 3 and Appendix A] after [41]. On the other hand, it is the case $\gamma = 0$ that generates explicit representation formulae based on cosine/sine operator theory and corresponding Dirichlet or Neumann maps from elliptic theory. This will be done also in the present paper, e.g. Appendix A. By the principle of superposition, one may consider separately two cases: $\{y_0, y_1, y_2\} \neq 0, f \not\equiv 0$ and $g = 0$, and $\{y_0, y_1, y_2\} = 0, f \equiv 0$ and $g \neq 0$. The first case was already considered in [30], [1] and will only be briefly reviewed in Appendix B. In this paper, we shall henceforth consider the second more challenging case

$$f \equiv 0, \qquad \{y_1, y_2, y_3\} = 0, \qquad g \neq 0. \tag{1.2}$$

As noted in the introduction, Section 2 reviews optimal results and representation formulae for the case of Dirichlet boundary term $g \in L^2(0,T; L^2(\Gamma))$ obtained in [1] after [41] and starts from these results to obtain optimal regularity results with g smoother by 'one-unit' in Section 3.1, and by 'two-units' in Section 3.2, etc.

2. **Optimal regularity for the y-problem with $g \in L^2(0,T;L^2(\Gamma))$, $f \equiv 0$, $\{y_1, y_2, y_3\} = 0$ and corresponding explicit representation formulae when $\gamma = 0$ [1].** Regarding problem (1.1a)–(1.1c) with $f = 0, \{y_1, y_2, y_3\} = 0$, the goal is to obtain sharp regularity of the map

$$g \longrightarrow \left\{ y, y_t, y_{tt}, \left. \frac{\partial y}{\partial \nu} \right|_{\Sigma} \right\} \tag{2.1}$$

from the Dirichlet boundary datum g of low regularity such as $L^2(0,T;L^2(\Gamma))$ to the interior solution $\{y, y_t, y_{tt}\}$ and the Neumann boundary trace $\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma}$. Paper [1], after [41], proceeds along the following two steps.

Step 1. We assume at first that

$$\gamma = 0 \text{ or } \alpha = \frac{c^2}{b}, \qquad y_{ttt} + \frac{c^2}{b} y_{tt} - c^2 \Delta y - b \Delta y_t = 0 \tag{2.2}$$

so that in view of the simplified version (2.2), the y–problem (1.1a)–(1.1c) with $f = 0, \{y_0, y_1, y_2\} = 0$ can be rewritten as

$$\begin{cases} \dfrac{d}{dt}\left[y_{tt} - b\Delta y\right] + \dfrac{c^2}{b}\left[y_{tt} - b\Delta y\right] = 0 & \text{in } Q = (0,T] \times \Omega & \text{(2.3a)} \\ \left[y_{tt} - b\Delta y\right]_{t=0} = y_2 - b\Delta y_0 = 0 & \text{in } \Omega & \text{(2.3b)} \\ y\big|_{\Sigma} = g & \text{in } \Sigma = (0,T] \times \Gamma & \text{(2.3c)} \end{cases}$$

Lemma 2.1. *y is a solution of problem (2.3a)–(2.3c) if and only if $y = w$ is a solution of*

$$\begin{cases} w_{tt} = b\Delta w & \textit{in } Q = (0,T] \times \Omega & \text{(2.4a)} \\ w|_{t=0} = w_t|_{t=0} = 0 & \textit{in } \Omega & \text{(2.4b)} \\ w\big|_{\Sigma} = g & \textit{in } \Sigma = (0,T] \times \Gamma\ . & \text{(2.4c)} \end{cases}$$

Thus, in this canonical case $\gamma = 0$, the regularity of the map (2.1) coincides with the regularity of the by now well–known map $g \to \left\{w, w_t, w_{tt}, \frac{\partial w}{\partial \nu}\right\}$ for which we quote [17, p. 172], [19] and [24, Chapter 10, Section 5]. See Theorem 2.2 below.

Step 2. The claim is that $0 \neq \gamma \in L^\infty(\Omega)$ – the case $\alpha \in L^\infty(\Omega)$, and c^2, b positive constants being the most relevant case we wish to cover – produces only lower order terms in the analysis of the regularity of the map in (2.1). [1, Section 3, Appendix A], after [41],

Conclusion. The optimal regularity of the map (2.1) for the SMGTJ–mixed problem (1.1a)–(1.1c) with zero initial data, $f \equiv 0$ and $\gamma \in L^\infty(\Omega)$ is the same as in the canonical case $\gamma = \alpha - \frac{c^2}{b} = 0$; in which case $y = w$ and all the desired quantities are given by the w–problem (2.4a)–(2.4c) as reported in Theorem 2.2 below. In this

as it follows from (2.4a) *and* (2.9) *via* [26, Proposition 12.1, p. 85]. *Additional versions may be obtained by differentiating* (2.10)*:*

$$w_{tt} = \begin{cases} (-A)\left[A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau - Dg(t)\right] \in L^2(0,T;[\mathcal{D}(A)]') & (2.11b) \\ -A^2\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau + ADg(t) & \\ \quad \in C([0,T];[\mathcal{D}(A)]') + L^2(0,T;[\mathcal{D}(A^{3/4+\varepsilon})]') & (2.11c) \end{cases}$$

by (2.6c)

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma} = -\frac{1}{b}D^*AA\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau \in H^{-1}(\Sigma), \tag{2.12}$$

recalling (2.6d) *in* (2.12)*, where* $H^{-1}(\Sigma)$ = *dual of* $\{h \in H_0^1(\Sigma)\}$ *i.e. with* $h(\cdot,0) = 0$ *and* $h(\cdot,T) = 0$ *on* Γ *(but actually,* $h(\cdot,T) = 0$ *is not needed).*
Moreover, from (2.11b)

$$g \in C([0,T];L^2(\Gamma)) \to w_{tt} \in C([0,T];[\mathcal{D}(A)]') \tag{2.13}$$

Remark 2.1. We recover (2.11a) from (2.11c) as follows as $[w - Dg] \in \mathcal{D}(A)$

$$\begin{aligned}(-A)\left[A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau - Dg(t)\right] &= b\Delta\left[w - Dg(t)\right] \\ &= b\Delta w \ \in C([0,T];H^{-2}(\Omega)).\end{aligned}$$

The main result of the present Section 2 is the following

Theorem 2.3. [1] **(i)** *With reference to problem* (1.1a)–(1.1c) *with zero initial conditions* $\{y_0, y_1, y_2\} = 0$, $f = 0$, *and* $\gamma \in L^\infty(\Omega)$ *we have the following optimal interior regularity results:*

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow \begin{cases} y \in C([0,T];L^2(\Omega)) & (2.14) \\ y_t \in C([0,T];[\mathcal{D}(A^{\frac{1}{2}})]' = H^{-1}(\Omega)), & (2.15) \end{cases}$$

$$y_{tt} \begin{cases} \in C([0,T];H^{-2}(\Omega)) & (2.16a) \\ \in L^2(0,T;[\mathcal{D}(A)]'), & (2.16b) \\ \in C([0,T];[\mathcal{D}(A)]') \oplus L^2(0,T;[\mathcal{D}(A^{3/4+\varepsilon})]') & (2.16c) \end{cases}$$

as well as the following boudary trace result:

$$\Longrightarrow \left.\frac{\partial y}{\partial \nu}\right|_{\Sigma} \in H^{-1}(\Sigma). \tag{2.17}$$

Moreover,

$$g \in C([0,T];L^2(\Gamma)) \quad \Longrightarrow y_{tt} \in C([0,T];[\mathcal{D}(A)]'), \tag{2.18}$$

all the maps being continuous.
(ii) *Let now* $\gamma = 0$*. Then by Lemma 2.1,*

$$y = w = \text{ a solution of the problem (2.4a)–(2.4c)} \tag{2.19}$$

so that, in this case, the same representation formulae for $\left\{w, w_t, w_{tt}, \frac{\partial w}{\partial \nu}\right\} = \left\{y, y_t, y_{tt}, \frac{\partial y}{\partial \nu}\right\}$ *of Theorem 2.2 hold for the* y*–problem with zero initial data and* $f \equiv 0$.

Proof. We recall the proof of (3.1.2) from [17, p. 177] as this is needed to show the implication (3.1.5) $\Longrightarrow$ (3.1.6), which is not explicitly noted in [17].
Integrating by parts formula (3.1.2), we obtain via (2.8) and $g(0) = 0$:

$$\begin{aligned} w(t) &= A \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau = \int_0^t \frac{d}{d\tau} \mathcal{C}(t-\tau) D g(\tau) d\tau \\ &= \mathcal{C}(t-\tau) D g(\tau) \Big|_{\tau=0}^{\tau=t} - \int_0^t \mathcal{C}(t-\tau) D \dot{g}(\tau) d\tau \end{aligned} \tag{3.1.8}$$

$$w(t) = D g(t) - \int_0^t \mathcal{C}(t-\tau) D \dot{g}(\tau) d\tau \in C\left([0,T]; H^1(\Omega)\right), \tag{3.1.9}$$

since $Dg(t) \in C\left([0,T]; H^1(\Omega)\right)$ by (2.6b) with $s = \frac{1}{2}$ and applying A^{-1} to (2.10), with $g \in L^2(0,T; L^2(\Gamma))$ replaced by $\dot{g} \in L^2(0,T; L^2(\Gamma))$. Now we prove (3.1.6) as a consequence of (3.1.5). Differentiate (3.1.9) thus obtaining

$$w_t(t) = \cancel{D\dot{g}(t)} - \cancel{D\dot{g}(t)} + A \int_0^t \mathcal{S}(t-\tau) D \dot{g}(\tau) d\tau \in C\left([0,T]; H^1(\Omega)\right) \tag{3.1.10}$$

by invoking (3.1.2) with g there subject to assumption (3.1.1a)–(3.1.1b) replaced now by $\dot{g}$ satisfying the corresponding assumption (3.1.5). Next, differentiating (3.1.10), we obtain

$$w_{tt}(t) = A \int_0^t \mathcal{C}(t-\tau) D \dot{g}(\tau) d\tau \in C\left([0,T]; L^2(\Omega)\right) \tag{3.1.11}$$

recalling formula (3.1.3) with g there satisfying (3.1.1a)–(3.1.1b) with $\dot{g}$ now satisfying the corresponding assumption (3.1.5). ◻

Theorem 3.1.2. *[*[17, Theorem 3.5, p. 178]*] With reference to* (2.4a)–(2.4c)*, let*

$$g \in H^1(\Sigma). \tag{3.1.12}$$

Then, continuously,

$$\frac{\partial w}{\partial \nu}\Big|_{\Gamma} = -\frac{1}{b} D^* A \left[A \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau \right] \in L^2(\Sigma). \tag{3.1.13}$$

Assume further

$$\dot{g} \in H^1(\Sigma). \tag{3.1.14}$$

Then (recalling also (3.1.9)*)*

$$\frac{\partial w_t}{\partial \nu}\Big|_{\Gamma} = -\frac{1}{b} D^* A \left[A \int_0^t \mathcal{S}(t-\tau) D \dot{g}(\tau) d\tau \right] \in L^2(\Sigma). \tag{3.1.15}$$

In view of Remark 2.3 and Lemma 2.1, we then obtain the following results for the original y-problem in (1.1a)–(1.1c).

Theorem 3.1.3. *Part A:* **(i)** *(Interior regularity) With reference to the Dirichlet non-homogeneous mixed problem* (1.1a)–(1.1c)*, with $f \equiv 0$, $\{y_0, y_1, y_2\} = 0$ and $\gamma \in L^\infty(\Omega)$, let*

$$g \in C\left([0,T]; H^{\frac{1}{2}}(\Gamma)\right) \cap H^1\left(0,T; L^2(\Gamma)\right) \tag{3.1.16a}$$

along with the compatibility condition

$$g\big|_{t=0} = y_0\big|_{\Gamma} = 0. \tag{3.1.16b}$$

Then, continuously,

$$w(t) = A \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau \in C\left([0,T]; H^2(\Omega)\right). \tag{3.2.2}$$

In addition, assume

$$g_t \in C\left([0,T]; H^{\frac{1}{2}}(\Gamma)\right). \tag{3.2.3}$$

Then, continuously,

$$w_t(t) = A \int_0^t \mathcal{C}(t-\tau) D g(\tau) d\tau \in C\left([0,T]; H^1(\Omega)\right) \tag{3.2.4}$$

$$\begin{aligned} w_{tt}(t) = b\Delta w &= (-A)\left[A \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau - Dg(t)\right] \\ &= b\Delta[w - Dg] \in C\left([0,T]; L^2(\Omega)\right) \end{aligned} \tag{3.2.5}$$

[refer to (3.1.4) *or Remark 2.1]. Finally, assume the stronger assumption*

$$g \in H^2(\Sigma). \tag{3.2.6}$$

Then, continuously,

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Gamma} = -\frac{1}{b} D^* A \left[A \int_0^t \mathcal{S}(t-\tau) D g(\tau) d\tau\right] \in H^1(\Sigma). \tag{3.2.7}$$

In view of Remark 2.3, we then obtain the following results for the original y-problem in (1.1a)–(1.1c).

Theorem 3.2.2. *Part A:* **(i)** *(Interior regularity) With reference to the Dirichlet non-homogeneous mixed problem* (1.1a)–(1.1c)*, with $f \equiv 0$, $\{y_0, y_1, y_2\} = 0$ and $\gamma \in L^\infty(\Omega)$, let*

$$g \in C\left([0,T]; H^{\frac{3}{2}}(\Gamma)\right) \cap H^2\left(0,T; L^2(\Gamma)\right) \tag{3.2.8a}$$

along with the compatibility conditions

$$g\big|_{t=0} = y_0\big|_{\Gamma} = 0; \qquad g_t\big|_{t=0} = y_1\big|_{\Gamma} = 0. \tag{3.2.8b}$$

Then, continuously

$$y \in C\left([0,T]; H^2(\Omega)\right). \tag{3.2.9}$$

In addition, assume

$$g_t \in C\left([0,T]; H^{\frac{1}{2}}(\Gamma)\right). \tag{3.2.10}$$

Then, continuously

$$y_t \in C\left([0,T]; H^1(\Omega)\right) \tag{3.2.11}$$

$$y_{tt} \in C\left([0,T]; L^2(\Omega)\right). \tag{3.2.12}$$

(ii) *(Boundary regularity) Assume now the stronger hypothesis*

$$g \in H^2(\Sigma). \tag{3.2.13}$$

Then, continuously

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma} \in H^1(\Sigma). \tag{3.2.14}$$

Part B: Now let $\gamma = 0$. Then, by Lemma 2.1, $y = w =$ solution of problem (2.4a) – (2.4c)*, so that, in this case, the same representation formulae* (3.2.2) *of Theorem 3.2.1 hold true for $w = y$ under assumption* (3.2.8a)–(3.2.8b) *on g, as well as the representation formulae* (3.2.4)–(3.2.5) *of Theorem 3.2.1 holds true for*

$\{w_t, w_{tt}\} = \{y_t, y_{tt}\}$ under the additional assumption (3.2.10) on g_t; finally, the representation formula (3.2.7) of Theorem 3.2.1 holds true for $\left.\frac{\partial w}{\partial \nu}\right|_\Gamma = \left.\frac{\partial y}{\partial \nu}\right|_\Gamma$ under the stronger assumption (3.2.6) on g.

3.3. **The case of $g \in H^3(\Sigma)$.** We return to the wave problem (2.4a)–(2.4c), this time with further smoother $g \in H^3(\Sigma)$.

Theorem 3.3.1. [17, Theorem 2.5, p. 165] *With reference to (2.4a)–(2.4c), let*

$$g \in H^3(\Sigma) \tag{3.3.1a}$$

along with the Compatibility Conditions

$$g\big|_{t=0} = y_0\big|_\Gamma = 0; \qquad g_t\big|_{t=0} = y_1\big|_\Gamma = 0; \qquad g_{tt}\big|_{t=0} = 0. \tag{3.3.1b}$$

Then, continuously,

$$w(t) = A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau \in C\left([0,T]; H^3(\Omega)\right) \tag{3.3.2}$$

$$w_t(t) = A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau \in C\left([0,T]; H^2(\Omega)\right) \tag{3.3.3}$$

$$\begin{aligned} w_{tt}(t) = b\Delta w &= (-A)\left[A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau - Dg(t)\right] \\ &= b\Delta[w - Dg] \in C\left([0,T]; H^1(\Omega)\right) \end{aligned} \tag{3.3.4}$$

$$\left.\frac{\partial w}{\partial \nu}\right|_\Gamma = -\frac{1}{b}DA^*\left[A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau\right] \in H^2(\Sigma) \tag{3.3.5}$$

[Regarding (3.3.4), refer to (3.2.5), or (3.1.4) or Remark 2.1.]

In view of Remark 2.3, we then obtain the following results for the original y-problem in (1.1a)–(1.1c).

Theorem 3.3.2. *With reference to the Dirichlet non-homogeneous mixed problem (1.1a)–(1.1c), with $f \equiv 0$, $\{y_0, y_1, y_2\} = 0$ and $\gamma \in L^\infty(\Omega)$, let*

$$g \in H^3(\Sigma) \tag{3.3.6a}$$

along with the compatibility conditions

$$g\big|_{t=0} = y_0\big|_\Gamma = 0; \qquad g_t\big|_{t=0} = y_1\big|_\Gamma = 0; \qquad g_{tt}\big|_{t=0} = \Delta y_0\big|_\Gamma + f(0) = 0. \tag{3.3.6b}$$

Part A: Then, continuously,

$$y \in C\left([0,T]; H^3(\Omega)\right) \tag{3.3.7}$$

$$y_t \in C\left([0,T]; H^2(\Omega)\right) \tag{3.3.8}$$

$$y_{tt} \in C\left([0,T]; H^1(\Omega)\right) \tag{3.3.9}$$

and

$$\frac{\partial y}{\partial \nu} \in H^2(\Sigma). \tag{3.3.10}$$

Part B: Let now $\gamma = 0$. Then, by Lemma 2.1

$$y = w = \text{solution of problem (2.4a) – (2.4c)} \tag{3.3.11}$$

PART B: Neumann boundary control, $\dim \Omega \geq 2$

4. **Linear Third order SMGTJ–equation with non–homogeneous Neumann boundary term in $L^2(0,T;L^2(\Gamma))$, $\dim \Omega \geq 2$.** If the SMGTJ equation (1.1a) is written in terms of the scalar velocity potential, where pressure $= k\partial_t$ (velocity potential), then the Neumann non-homogeneous boundary terms are appropriate [4, 10].

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t = 0 & \text{in } Q = (0,T] \times \Omega & (4.1a) \\ y\big|_{t=0} = y_0 = 0; \quad y_t\big|_{t=0} = y_1 = 0; \quad y_{tt}\big|_{t=0} = y_2 = 0 & \text{in } \Omega & (4.1b) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma} = g & \text{in } \Sigma = (0,T] \times \Gamma & (4.1c) \end{cases}$$

In this case, we seek to obtain optimal regularity of the map

$$g \longrightarrow \{y, y_t, y_{tt}, y|_{\Sigma}.\} \tag{4.2}$$

We proceed along the same approach as for Dirichlet boundary control.

Step 1. When $\gamma = \alpha - \dfrac{c^2}{b} = 0$, the argument in the Section 2, Step 1 below (2.1) yielding (2.3a)–(2.3c), ultimately Lemma 2.1, does not depend on the boundary conditions. Hence we likewise obtain that problem (4.1a)–(4.1c) can be rewritten for $\gamma = 0$ as

$$\begin{cases} \dfrac{d}{dt}\left[y_{tt} - b\Delta y\right] + \dfrac{c^2}{b}\left[y_{tt} - b\Delta y\right] = 0 & \text{in } Q = (0,T] \times \Omega & (4.3a) \\ \left[y_{tt} - b\Delta y\right]_{t=0} = y_2 - b\Delta y_0 = 0 & \text{in } \Omega & (4.3b) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma} = g & \text{in } \Sigma = (0,T] \times \Gamma & (4.3c) \end{cases}$$

Lemma 4.1. *We have that y is a solution of problem (4.3a)–(4.3c) if and only if $y = \eta$ is a solution of*

$$\begin{cases} \eta_{tt} = b\Delta\eta & \textit{in } Q = (0,T] \times \Omega & (4.4a) \\ \eta|_{t=0} = \eta_t|_{t=0} = 0 & \textit{in } \Omega & (4.4b) \\ \left.\dfrac{\partial \eta}{\partial \nu}\right|_{\Sigma} = g & \textit{in } \Sigma = (0,T] \times \Gamma_0 & (4.4c) \end{cases}$$

Thus, in this canonical case $\gamma = 0$, the regularity of the map (4.2) coincides with the regularity of the well–known map $g \to \{\eta, \eta_t, \eta_{tt}, \eta|_{\Sigma}\}$ for which we quote [24, Vol II, Sect 8], [22]–[23], [37]. To this end we introduce the parameter $\widehat{\alpha}$:

$$\widehat{\alpha} = \frac{2}{3} \text{ for a general sufficiently smooth domain } \Omega \subset \mathbb{R}^d, d \geqslant 2. \tag{4.5a}$$

$$\widehat{\alpha} = \frac{3}{4} \text{ for a parallelepiped in } \mathbb{R}^d, d \geqslant 2. \tag{4.5b}$$

Moreover, throughout Part B, we introduce the operators (not to be confused with the Dirichlet-Laplacian in (2.5) of Part A)

Theorem 4.3. **(i)** *With reference to problem* (4.1a)–(4.1c) *and* $\gamma \in L^\infty(\Omega)$ *we have the following optimal interior and boundary regularity results:*

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow \begin{cases} y \in C([0,T];H^{\widehat{\alpha}}(\Omega) \equiv \mathcal{D}(A^{\widehat{\alpha}/2})) & (4.16) \\ y_t \in C([0,T];H^{\widehat{\alpha}-1}(\Omega) \equiv [\mathcal{D}(\mathcal{A}^{(1-\hat{\alpha})/2})]'), & (4.17) \end{cases}$$

$$y_{tt} \in \begin{cases} C([0,T];H^{\widehat{\alpha}-2}(\Omega)) & (4.18a) \\ L^2(0,T;[\mathcal{D}(A^{1-\widehat{\alpha}/2})]'), & (4.18b) \\ C([0,T];[\mathcal{D}(A^{1-\widehat{\alpha}/2})]') \oplus L^2(0,T;[\mathcal{D}(A^{1/4+\varepsilon})]') & (4.18c) \end{cases}$$

as well as the following boudary trace result:

$$\Longrightarrow y|_\Sigma \in H^{2\widehat{\alpha}-1}(\Sigma). \tag{4.19}$$

Moreover, still continuously

$$g \in C([0,T];L^2(\Gamma)) \quad \Longrightarrow y_{tt} \in C([0,T];[\mathcal{D}(A^{1-\widehat{\alpha}/2})]'), \tag{4.20}$$

all the maps being continuous.
(ii) *Let now* $\gamma = 0$. *Then, by Lemma 4.1,* $y = \eta$ *is the solution of the problem* (4.4a)–(4.4c) *with corresponding representation formulae as in Theorem 4.2.*
Counterparts of Remarks 2.2 and 2.3 in the Dirichlet case apply now.

Remark 4.2. For $\dim \Omega = 1$, higher regularity results are available for problem (4.4a)–(4.4c) [24, Theorem 9.8.4.1]

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow \{\eta, \eta_t, \eta_{tt}\} \in C([0,T];H^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)). \tag{4.21}$$

Correspondingly higher regularity results are available for the SMGTJ problem (4.1a)–(4.1c) for $\dim \Omega = 1$. We shall not discuss them explicitly, however.

5. **Optimal regularity theory and corresponding explicit representation formulae for the Neumann y-problem (4.1a)–(4.1c) with $g \in C([0,T];H^{\widehat{\alpha}-\frac{1}{2}}(\Omega) \cap H^1(0,T;L^2(\Gamma))$.** We return to the wave problem (4.4a)–(4.4c), this time with g further smoother, as in the title of the present section. We then appeal to the corresponding optimal interior and boundary regularity [23, Theorem A], [24, Theorem 8A.2, p. 756].

Theorem 5.1. *With reference to non-homogeneous Neumann problem* (4.4a)–(4.4c), *assume*
(i) *(Interior regularity)*

$$g \in C([0,T];H^{\widehat{\alpha}-\frac{1}{2}}(\Omega) \cap H^1(0,T;L^2(\Gamma)) \tag{5.1}$$

[a result that is a-fortiori true, if we assume

$$g \in C([0,T];H^{2\widehat{\alpha}-1,1}(\Sigma) \equiv L^2(0,T;H^{2\widehat{\alpha}-1}(\Gamma)) \cap H^1(0,T;L^2(\Gamma)), \quad g(0) = 0 \tag{5.2}$$

by virtue of [26, Theorem 3.1, $m = 1, j = 0$, p. 19]. *Then the unique solution of* (4.4a)–(4.4c) *satisfies continuously*

$$\eta(t) = A \int_0^t \mathcal{S}(t-\tau) N g(\tau) d\tau \in C([0,T];H^{\widehat{\alpha}+1}(\Omega)) \tag{5.3}$$

$$\eta_t(t) = A \int_0^t \mathcal{C}(t-\tau) N g(\tau) d\tau \in C([0,T];H^{\widehat{\alpha}}(\Omega)) \tag{5.4}$$

$$\eta_{tt}(t) = b\Delta\eta \in C([0,T];H^{\widehat{\alpha}-1}(\Omega)) \tag{5.5}$$

Then, continuously,

$$y \in C([0,T]; H^1(\Omega)) \tag{7.2}$$

$$y_t \in C([0,T]; L^2(\Omega)) \tag{7.3}$$

$$y_{tt} \in C([0,T]; H^{-1}(\Omega)). \tag{7.4}$$

(ii) *Let now $\gamma = 0$, so that by Lemma 4.1, y is a solution of the problem (4.1a)–(4.1c) if and only if $\eta = y$ is a solution of problem (4.4a)–(4.4c), whereby then the following representation formulae hold true*

$$y(t) = \eta(t) = A \int_0^t \mathcal{S}(t-\tau) N g(\tau) d\tau \in C([0,T]; H^1(\Omega)) \tag{7.5}$$

$$y_t(t) = \eta_t(t) = A \int_0^t \mathcal{C}(t-\tau) N g(\tau) d\tau \in C([0,T]; L^2(\Omega)) \tag{7.6}$$

$$y_{tt}(t) = \eta_{tt}(t) = b\Delta\eta \in C([0,T]; H^{-1}(\Omega)). \tag{7.7}$$

Proof. (ii) The regularity results of $\{\eta, \eta_t\}$ given in (7.5)–(7.7) for problem (4.4a)–(4.4c) with g satisfying assumption (7.1a)–(7.1c) is given by [44, Theorem B, p. 495, same as Theorem 4.5, p. 500]. Then (7.5) implies (7.7) by [26, Proposition 12.1, p. 85]. The representation formulae were noted in Theorem 4.2.
(i) Then the regularity results in (7.2)–(7.4) with $\gamma \in L^\infty(\Omega)$ are the same as in the case $\gamma = 0$, as noted repeatedly. □

Theorem 7.2. **(i)** *Consider problem (4.1a)–(4.1c) with $\gamma \in L^\infty(\Omega)$ and Neumann boundary control g satisfying*

$$g \in H^1(0,T; H^{-\frac{1}{2}}(\Gamma)) \cap C([0,T]; H^{\widehat{\alpha}-1}(\Gamma)), \quad g(0) = 0. \tag{7.8}$$

Then, continuously,

$$y \in C([0,T]; H^{\widehat{\alpha}+\frac{1}{2}}(\Omega)) \tag{7.9}$$

$$y_t \in C([0,T]; H^{\widehat{\alpha}-\frac{1}{2}}(\Omega)) \tag{7.10}$$

$$y_{tt} \in C([0,T]; H^{\widehat{\alpha}-\frac{3}{2}}(\Omega)). \tag{7.11}$$

(ii) *Let now $\gamma = 0$, so that by Lemma 4.1, y is a solution of problem (4.1a)–(4.1c) if and only if $\eta = y$ is a solution of problem (4.4a)–(4.4c), whereby then the following representation formulae hold true*

$$y(t) = \eta(t) = A \int_0^t \mathcal{S}(t-\tau) N g(\tau) d\tau \in C([0,T]; H^{\widehat{\alpha}+\frac{1}{2}}(\Omega)) \tag{7.12}$$

$$y_t(t) = \eta_t(t) = A \int_0^t \mathcal{C}(t-\tau) N g(\tau) d\tau \in C([0,T]; H^{\widehat{\alpha}-\frac{1}{2}}(\Omega)) \tag{7.13}$$

$$y_{tt}(t) = \eta_{tt}(t) = b\Delta\eta \in C([0,T]; H^{\widehat{\alpha}-\frac{3}{2}}(\Omega)). \tag{7.14}$$

Proof. (ii) The regularity results of $\{\eta, \eta_t\}$ given in (7.12)–(7.13) for problem (4.4a)–(4.4c) with g satisfying assumption (7.8) is given by [44, Theorem A, p. 495, same as Theorem 4.3, p. 498]. The representation formulae were noted in Theorem 4.2.
(i) Then the regularity results in (7.9)–(7.11) with $\gamma \in L^\infty(\Omega)$ are the same as in the case $\gamma = 0$, as noted repeatedly. □

$$\mathcal{D}(A_d^{1/2}) = H_0^1(\Omega), \qquad \left[\mathcal{D}(A_d^{1/2})\right]' = H^{-1}(\Omega), \qquad \mathcal{D}(A_d^{1/4}) = H_{00}^{1/2}(\Omega). \tag{8.1.7}$$

Below we shall use that

$$\delta \in \left[H^{\theta}(\Omega)\right]', \qquad \theta = \begin{cases} \dfrac{3}{2} + \epsilon, & n = 3 \\ 1 + \epsilon, & n = 2 \\ \dfrac{1}{2} + \epsilon, & n = 1. \end{cases} \tag{8.1.8}$$

Theorem 8.1.1. **(a)** *Let $\gamma = 0$ as in (8.1.2), so that y is a solution of (8.1.1a)–(8.1.1c) if and only if y is a solution of (8.1.4a)-(8.1.4d). In this case, by [24,* Theorem 9.8.1.1, p. 844*], [42], the following results hold true.*
(a1) *Interior regularity, with the following representation formulae (for $b = 1$)*

$$y(t) = \int_0^t \mathcal{S}(t-\tau)\delta(\cdot)v(\tau)d\tau \in C([0,T]; Y_1) \tag{8.1.9}$$

$$y_t(t) = \int_0^t \mathcal{C}(t-\tau)\delta(\cdot)v(\tau)d\tau \in C([0,T]; Y_2) \tag{8.1.10}$$

$$y_{tt}(t) = -A_d y(t) + \delta(\cdot)v(t) = -A_d \int_0^t \mathcal{S}_d(t-\tau)\delta(\cdot)v(\tau)d\tau + \delta(\cdot)v(t)$$

$$\in \begin{cases} C\left([0,T]; [\mathcal{D}(A_d)]' + L^2(0,T;[H^{3/2+\epsilon}(\Omega)]')\right), & n = 3 \quad (8.1.11a) \\ C\left([0,T]; [\mathcal{D}(A_d^{3/4})]' + L^2(0,T;[H^{1+\epsilon}(\Omega)]')\right), & n = 2 \quad (8.1.11b) \\ C\left([0,T]; [\mathcal{D}(A_d^{1/2})]' + L^2(0,T;[H^{1/2+\epsilon}(\Omega)]')\right), & n = 1 \quad (8.1.11c) \end{cases}$$

a-fortiori

$$y_{tt} \in L^2(0,T;Y_3), \qquad Y_3 \equiv \begin{cases} H^{-2}(\Omega), & n = 3 \quad (8.1.12a) \\ (\mathcal{D}(A_d^{3/4}))' \subset [H_{00}^{3/2}(\Omega)]', & n = 2 \quad (8.1.12b) \\ H^{-1}(\Omega) = (\mathcal{D}(A_d^{1/2}))', & n = 1. \quad (8.1.12c) \end{cases}$$

(a2) *Boundary regularity, ((2.6d) with $b = 1$)*

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Gamma} = -D^* A_d \int_0^t \mathcal{S}_d(t-\tau)\delta(\cdot)v(\tau)d\tau \in \begin{cases} H^{-1}(\Sigma), & n = 3 \quad (8.1.13a) \\ H^{-1/2}(\Sigma), & n = 2 \quad (8.1.13b) \\ L^2(\Sigma), & n = 1. \quad (8.1.13c) \end{cases}$$

(b) *For $\gamma \in L^\infty(\Omega)$, the same regularity results hold true.*

An alternative way using the new variable z

$$z = \frac{c^2}{b}y + y_t = (\alpha y + y_t) - \gamma y, \qquad \gamma = \alpha - \frac{c^2}{b}$$

introduced in [30] and used critically in [41], [1] is given in Appendix A.

CASE 2. (homogeneous Neumann B.C.) We now consider the following problem

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t = \delta(x)u(t) & \text{in } Q = (0,T]\times\Omega & (8.1.14a)\\ y\big|_{t=0} = 0; \quad y_t\big|_{t=0} = 0; \quad y_{tt}\big|_{t=0} = 0 & \text{in } \Omega & (8.1.14b)\\ \qquad y\big|_{\Sigma_0} = 0; \quad \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = 0 & \text{in } \Sigma = (0,T]\times\Gamma & (8.1.14c) \end{cases}$$

with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_1 open in Γ, Γ_0 possibly empty.
The approach now is a perfect counterpart of Case 1. Here we recall the Neumann Laplacian A_n in (4.6) for $\Gamma_0 = \emptyset$ or otherwise

$$A_n f = -b\Delta f, \qquad \mathcal{D}(A_n) = \left\{ f \in H^2(\Omega),\ f\big|_{\Gamma_0} = 0,\ \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = 0 \right\}, \tag{8.1.15}$$

as well as the corresponding Neumann map (same as in (4.7) if $\Gamma_0 = 0$)

$$Ng = \phi \Longleftrightarrow \left\{ \Delta\phi = 0 \text{ in } \Omega,\ f\big|_{\Gamma_0} = 0,\ \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = g \right\} \tag{8.1.16a}$$

$$N^* A_n f = \begin{cases} 0 \text{ in } \Gamma_0 & (8.1.16b)\\ -bf\big|_{\Gamma_1} \text{ in } \Gamma_1, \quad f \in \mathcal{D}(A_n). & (8.1.16c) \end{cases}$$

Step 1. Again we assume $\gamma = 0$ as in (8.1.2) at first, so that the problem corresponding to (8.1.3a)–(8.1.3c) starting from (8.1.14a)–(8.1.14c) is now

$$\begin{cases} \dfrac{d}{dt}\left[y_{tt} - b\Delta y\right] + \dfrac{c^2}{b}\left[y_{tt} - b\Delta y\right] = \delta(x)u(x) & \text{in } Q & (8.1.17a)\\ \left[y_{tt} - b\Delta y\right]_{t=0} = y_2 - b\Delta y_0 = 0 & \text{in } \Omega & (8.1.17b)\\ y\big|_{\Sigma_0} = 0, \quad \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = 0 & \text{in } \Sigma & (8.1.17c) \end{cases}$$

or solving problem (8.1.17a)–(8.1.17c)

$$\begin{cases} y_{tt} = b\Delta y + \delta(x)v(t) & \text{in } Q & (8.1.18a)\\ y\big|_{t=0} = 0; \quad y_t\big|_{t=0} = 0 & \text{in } \Omega & (8.1.18b)\\ y\big|_{\Sigma_0} = 0, \quad \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = 0 & \text{in } \Sigma & (8.1.18c)\\ v(t) = \displaystyle\int_0^t e^{-\frac{c^2}{b}(t-\tau)}u(\tau)d\tau \in L^2(0,T) & & (8.1.18d) \end{cases}$$

where y satisfies problem (8.1.17a)-(8.1.17c) – that is, problem (8.1.14a)-(8.1.14c) – if and only if y satisfies problem (8.1.18a)-(8.1.18d). Let, in the present Case 2, $\mathcal{C}_n(t)/\mathcal{S}_n(t)$ be the cosine/sine operators generated by the strictly negative self-adjoint operator $(-A_n)$ in (8.1.15) on $H = L^2(\Omega)$ (or $H = L^2(\Omega)/\mathbb{R}$ if $\Gamma_0 = \emptyset$).

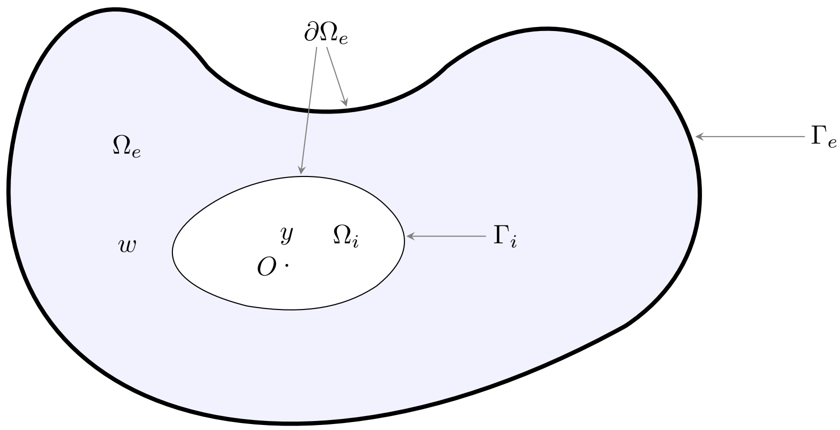

FIGURE 1. The y-problem feeds the w-problem

Theorem 8.2.1. *(a) Case $d = 3$. Consider the SMGTJ equation in the unknown $y(t, x)$ with point control defined in the 3-dimensional interior domain Ω_i and I.C. $\{y_0, y_1, y_2\} = 0$:*

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t = \delta(x)u(t) & \text{in } Q_i = (0,T] \times \Omega_i & (8.2.1a) \\ y\big|_{t=0} = 0; \quad y_t\big|_{t=0} = 0; \quad y_{tt}\big|_{t=0} = 0 & \text{in } \Omega_i & (8.2.1b) \\ y\big|_{\Sigma_i} = 0 & \text{in } \Sigma_i = (0,T] \times \Gamma_i & (8.2.1c) \end{cases}$$

feeding through time integration of its Neumann trace on $\Gamma_i = \partial\Omega_i$ the following w-wave mixed problem defined on the external domain Ω_e with Neumann homogeneous B.C. on the external boundary Γ_e:

$$\begin{cases} w_{tt} = \Delta w & \text{in } Q_e = (0,T] \times \Omega_e & (8.2.2a) \\ w(0,\cdot) = 0, \quad w_t(0,\cdot) = 0 & \text{in } \Omega_e & (8.2.2b) \\ \dfrac{\partial w}{\partial \nu}\bigg|_{\Sigma_i} = \displaystyle\int_0^t \dfrac{\partial y}{\partial \nu}\bigg|_{\Sigma_i} d\tau & \text{in } \Sigma_i = (0,T] \times \Gamma_i & (8.2.2c) \\ \dfrac{\partial w}{\partial \nu}\bigg|_{\Sigma_e} \equiv 0 & \text{in } \Sigma_e = (0,T] \times \Gamma_e. & (8.2.2d) \end{cases}$$

Assume:

$$u \in L^2(0,T). \tag{8.2.3}$$

Then, continuously ($\widehat{\alpha}$ in (4.5))

$$w = L\left(\frac{\partial w}{\partial \nu}\bigg|_{\Sigma_i}\right) \in C([0,T]; H^{\widehat{\alpha}-1}(\Omega_e)) \tag{8.2.4}$$

$$w_t = \frac{d}{dt} L\left(\frac{\partial w}{\partial \nu}\bigg|_{\Sigma_i}\right) \in C([0,T]; H^{\widehat{\alpha}-2}(\Omega_e)). \tag{8.2.5}$$

(b) Case $d = 2$. When Ω_i and Ω_e are 2-dimensional, consider the same y-problem (8.2.1a)–(8.2.1c) and w-problem (8.2.2a)–(8.2.2c), except that now

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma_i} = \left.\frac{\partial y}{\partial \nu}\right|_{\Sigma_i} \quad \text{in } \Sigma_i = (0,T] \times \Gamma_i \tag{8.2.6}$$

replaces (8.2.2c). Assume again (8.2.3) for the scalar control u. Then, continuously,

$$w = L\left(\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma_i}\right) \in C([0,T]; H^{\widehat{\alpha}-\frac{1}{2}}(\Omega)) \tag{8.2.7}$$

$$w_t = \frac{d}{dt} L\left(\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma_i}\right) \in C([0,T]; H^{\widehat{\alpha}-\frac{3}{2}}(\Omega)) \tag{8.2.8}$$

Proof. (a) *Case* $d = 3$: Under assumption (8.2.3) of the point control, we have, continuously, the following regularity of the y-problem on Ω_i by Theorem 8.1.1, Eq. (8.1.9), (8.1.6a) as well as (8.1.13a):

$$\{y, y_t\} \in C([0,T]; L^2(\Omega_i) \times H^{-1}(\Omega_i)) \tag{8.2.9}$$

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma_i} \in H^{-1}(\Sigma_i), \text{ hence } \int_0^t \left.\frac{\partial y}{\partial \nu}\right|_{\Sigma_i} d\tau \in L^2(0,T; H^{-1}(\Gamma_i))) \tag{8.2.10}$$

Then with $g(t) = \displaystyle\int_0^t \left.\frac{\partial y}{\partial \nu}\right|_{\Sigma_i} d\tau$, $g(0) = 0$, for the Neumann control acting in (8.2.2c), we invoke [44, Theorem 5.4, same as Theorem D, along with (4.15b), (5.3.2)] to the w-problem and obtain (8.2.4), (8.2.5).

(b) *Case* $d = 2$: Here, under assumption (8.2.3) of the point control we have, continuously, the following regularity of the y-problem on Ω_i by Theorem 8.1.1, Eq. (8.1.9), (8.1.10), (8.1.6b) as well as (8.1.11b):

$$\{y, y_t\} \in C([0,T]; H^{1/2}_{00}(\Omega_i) \times [H^{1/2}_{00}(\Omega_i)]') \tag{8.2.11}$$

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma_i} \in H^{-1/2}(\Sigma_i) = [H^{1/2}(\Sigma_i)]'. \tag{8.2.12}$$

Then, with $g(t) = \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_i}$, $g(0) = 0$ by (8.2.1b), the Neumann control (8.2.6), we invoke [44, Eq. (4.13), (4.15b)] and obtain (8.2.7), (8.2.8). □

Illustration #2: Let Ω_i and Ω_e be as in Illustration #1.

Theorem 8.2.2. *(a) Case $d = 3$. Consider the same SMGTJ equation (8.2.1a)–(8.2.1c) of Illustration #1 in the unknown $y(t,x)$ with point control defined in the 3-dimensional interior domain Ω_i and I.C. $\{y_0, y_1, y_2\} = 0$, this time feeding through its Neumann trace on $\Gamma_i = \partial\Omega_i$ the following w-wave mixed problem defined in the external domain Ω_e, this time with Dirichlet homogeneous B.C. on the external boundary Γ_e:*

$$\begin{cases} w_{tt} = \Delta w & \text{in } Q_e = (0,T] \times \Omega_e & (8.2.13a) \\ w(0,\cdot) = w_0, \quad w_t(0,\cdot) = w_1 & \text{in } \Omega_e & (8.2.13b) \\ w\big|_{\Sigma_i} = \displaystyle\int_0^t \left.\frac{\partial y}{\partial \nu}\right|_{\Sigma_i} d\tau & \text{in } \Sigma_i = (0,T] \times \Gamma_i & (8.2.13c) \\ w\big|_{\Sigma_e} \equiv 0 & \text{in } \Sigma_e = (0,T] \times \Gamma_e. & (8.2.13d) \end{cases}$$

Integrating by parts ($\mathcal{S}(0)=0$) on $z^{(2)}(t)$ in (A.4a), we obtain

$$
\begin{aligned}
z^{(2)}(t) &= A\int_0^t \mathcal{S}(t-\tau)D\left(y_t(\tau)\big|_\Gamma\right)d\tau \\
&= \Big[A\mathcal{S}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)\Big]_{\tau=0}^{\tau=t} + A\int_0^t \mathcal{C}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau \qquad \text{(A.5a)}\\
&= A\mathcal{S}(0)D\left(y(t)\big|_\Gamma\right) - A\mathcal{S}(t)D\left(y(0)\big|_\Gamma\right) + A\int_0^t \mathcal{C}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau. \qquad \text{(A.5b)}
\end{aligned}
$$

At this point, we notice that since the component y of the solution of (1.1a)–(1.1c) with $f=0, \{y_0,y_1,y_2\}=0$ was taken to be smooth, then compatibility conditions apply and yield

$$
y(0)\big|_\Gamma = y_0\big|_\Gamma = 0 \tag{A.6}
$$

as $y_0=0$ throughout. Then, by (A.6) used in (A.5b) we see that the second term in (A.5b) also vanishes and thus we obtain

$$
z^{(2)}(t) = A\int_0^t \mathcal{C}(t-\tau)D(y(\tau)|_\Gamma)d\tau. \tag{A.7}
$$

Thus, combining (A.7) in (A.4a), we obtain

$$
z(t) = \frac{c^2}{b}A\int_0^t \mathcal{S}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau + A\int_0^t \mathcal{C}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau \tag{A.8}
$$

originally for smooth trace $y(\cdot)|_\Gamma$. Extending the integral term, by closedness and density, we finally obtain

$$
g\in L^2(0,T;L^2(\Gamma)) \Longrightarrow z(t) = \frac{c^2}{b}A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau + A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)d\tau \tag{A.9a}
$$

$$
= \frac{c^2}{b}w(t) + w_t(t) \in C([0,T];H^{-1}(\Omega)\equiv[\mathcal{D}(A^{1/2})]') \tag{A.9b}
$$

recalling w in (2.9) and w_t in (2.10) of the w–problem (2.4a)–(2.4b). Next recall that $z=\frac{c^2}{b}y+y_t$ from (A.1) and compare with (A.9b). By subtraction we find

$$
(y-w)_t = -\frac{c^2}{b}(y-w), \qquad (y-w)(0)=0, \tag{A.10}
$$

and since $y(0)=w(0)=0$, (A.10) implies

$$
y(t)=w(t)=A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)d\tau \in C([0,T];L^2(\Omega)) \tag{A.11}
$$

and (2.19) is established. Then $y_t=w_t, y_{tt}=w_{tt}$ in (2.10), (2.11) follow at once. Lemma 2.1 is proved. □

Appendix B. **Case** $g\equiv 0$ **in** (1.1c)**.** In this case, problem (1.1a)–(1.1c) can be rewritten abstractly as

$$
u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t = 0 \quad \text{on } H = L^2(\Omega), \tag{B.1}
$$

along with I.C. u_0, u_1, u_2. We re-write it as a first order problem as

$$\frac{d}{dt}\begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix} = G \begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix} + f; \qquad G = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2 A & -bA & -\alpha I \end{bmatrix}. \tag{B.2}$$

It is established in [30] that the operator G (with appropriate domain) is the generator of a s.c. group e^{Gt} in several function spaces:

$$U_0 = H \times H \times H \tag{B.3a}$$

$$U_1 \equiv \mathcal{D}(A^{\frac{1}{2}}) \times \mathcal{D}(A^{\frac{1}{2}}) \times H; \qquad U_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \tag{B.3b}$$

$$U_3 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H; \qquad U_4 \equiv \mathcal{D}(A^{\frac{3}{2}}) \times \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \tag{B.3c}$$

$$U_5 \equiv V_3 \equiv H \times [\mathcal{D}(A^{1/2})]' \times [\mathcal{D}(A)]' \tag{B.3d}$$

whereby the solution of problem (1.1a)–(1.1c) (with $g \equiv 0$) is given by

$$\begin{bmatrix} u(t) \\ u_t(t) \\ u_{tt}(t) \end{bmatrix} = e^{Gt} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} + \int_0^t e^{G(t-\tau)} f(\tau) d\tau \in C([0,T]; U_i) \tag{B.4}$$

continuously for $[u_0, u_1, u_2] \in U_i$ and $f \in L^1(0,T;U_i), i = 1,2,3,4,5$. The case that couples with Theorem 3.2.2 is $U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H$, yielding the required interior regularity by (B.4), while the boundary regularity is given by [41, Theorem 6.2], [1, Theorem 5.2].

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