AN EXOTIC II_1 FACTOR WITHOUT PROPERTY GAMMA

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Abstract. We introduce a new iterative amalgamated free product construction of II_1 factors, and use it to construct a separable II_1 factor which does not have property Gamma and is not elementarily equivalent to the free group factor $\mathrm{L}(\mathbb{F}_n)$, for any $2 \leq n \leq \infty$. This provides the first explicit example of two non-elementarily equivalent II_1 factors without property Gamma. Moreover, our construction also provides the first explicit example of a II_1 factor without property Gamma that is also not elementarily equivalent to any ultraproduct of matrix algebras. Our proofs use a blend of techniques from Voiculescu's free entropy theory and Popa's deformation/rigidity theory.

1. Introduction

The study of the continuous model theory of II_1 factors was initiated by Farah, Hart and Sherman in [FHS14], who adapted the notion of elementary equivalence (requiring that the objects considered satisfy the same first-order sentences) to the context of II_1 factors. By the continuous version of the Keisler-Shelah theorem, two II_1 factors M, N are *elementarily equivalent* if and only if they admit isomorphic ultrapowers, $M^{\mathcal{U}} \cong N^{\mathcal{V}}$, for some ultrafilters $\mathcal{U}, \mathcal{V}$ on arbitrary sets [FHS14, HI02]. Ultrapowers of II_1 factors have been a major tool in operator algebras since the works of McDuff [McD70] and Connes [Con76] in the 1970s. In spite of this, proving that two given II_1 factors have no isomorphic ultrapowers, and so are not elementarily equivalent, remains a challenging task.

As shown in [FHS14] (see also [FGL06]), for separable II_1 factors, Murray and von Neumann's property Gamma [MvN43] and McDuff's property [McD70] are axiomatizable and thus are remembered by ultrapowers. This implies that the hyperfinite II_1 factor R, the free group factor $\mathrm{L}(\mathbb{F}_2)$ and any separable non-McDuff II_1 factor with property Gamma (see [DL69]) are not elementarily equivalent. It was then realized by Goldbring and Hart that a II_1 factor introduced in [ZM69] provides a fourth elementary equivalence class (see [GH17]). However, besides these examples, it was unclear how to find any additional elementary equivalence classes of II_1 factors. This problem was solved by Boutonnet and two of the authors in [BCI17] who proved that the continuum of non-isomorphic separable II_1 factors $(M_\alpha)_{\alpha\in\{0,1\}^{\mathbb{N}}}$ constructed by McDuff in [McD69] are pairwise non elementarily equivalent. More precisely, the main result of [BCI17] shows that M_α and M_β do not admit isomorphic ultrapowers, whenever $\alpha \neq \beta$. Subsequently, explicit sentences witnessing that M_α and M_β are not elementarily equivalent were given in [GH17, GHT18].

The proofs of the main result of [BCI17] and in fact of all of the existing results providing non-elementarily equivalent II_1 factors are based on analyzing central sequences. As a result, it remained a fundamental open question to find any non-elementarily equivalent II_1 factors that do not have property Gamma and thus admit no non-trivial central sequences.

We settle this question in the present work. A main novelty of our approach, that allows us to bypass the above difficulty, is the use of 1-bounded entropy from Voiculescu's free probability theory. For a finite tuple X of self-adjoint operators in a tracial von Neumann algebra (N, τ), one has the

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1-*bounded entropy* $h(X)$, implicit in Jung's work [Jun07] and defined explicitly by Hayes [Hay18], see Subsection 2.2. This quantity, unlike Voiculescu' free entropy dimension $\delta_0(X)$ [Voi94], is known to be an invariant of the von Neumann algebra generated by X as shown in [Hay18, Theorem A.9]. Hence, we have a well-defined notion of 1-bounded entropy $h(N)$ for a finitely generated tracial von Neumann algebra (N,τ). Moreover, $h(N)$ extends to arbitrary, possibly non-separable, tracial von Neumann algebras (N,τ) by [Hay18, Definition A.2].

The main result of this paper is the following:

Theorem A. *There exists a separable II_1 factor M which does not have property Gamma and is not elementarily equivalent to any tracial von Neumann algebra (N,τ) satisfying $h(N)>0$. For instance, M is not elementarily equivalent to* $\mathrm{L}(\mathbb{F}_2)$.

Moreover, for any ultrafilters $\mathcal{U},\mathcal{V}$ on sets I,J, respectively, there does not exist an embedding of $M^{\mathcal{U}}$ into $N^{\mathcal{V}}$ that contains the diagonal inclusion of N.

Examples of tracial von Neumann algebras (N,τ) with $h(N)>0$ include the interpolated free group factors $\mathrm{L}(\mathbb{F}_t)$, for all $1<t\leq\infty$, and, more generally, any free product $N_1 * N_2$ of two Connes-embeddable diffuse tracial von Neumann algebras (N_1,τ_1) and (N_2,τ_2). (Moreover, $h(N)=\infty$ for such N; for this and additional examples, see Fact 2.7). By Theorem A, M is not elementarily equivalent to any such N, including $\mathrm{L}(\mathbb{F}_2)$. This gives the first explicit example of two non-elementarily equivalent non-Gamma II_1 factors, thus settling a problem posed at a 2018 workshop at the American Institute of Mathematics [AIM, Problem 1.3], see also [IP] and [Pet, Problem U.2].

It has been speculated for some time that free probability theory is likely to shed light on the model-theoretic study of II_1 factors, see for instance Farah's ICM survey [Far14, Section 5] and [FGSW]. Offering positive evidence in this direction, Theorem A represents the first application of free probability to the model theory of II_1 factors.

Now we describe the key facets of our construction that allows us to prove Theorem A. The II_1 factor from Theorem A is built via a new iterative construction involving amalgamated free products (see Section 4). By using techniques from Popa's deformation/rigidity theory, notably [IPP08], and the notion of property (T), we are able to guarantee that M is indeed non-Gamma. The main property of our construction is presented in our second main theorem below.

Theorem B. *There exists a separable II_1 factor M without property Gamma which satisfies the following. For every countably cofinal ultrafilter $\mathcal{U}$ on a set I and $u_1,u_2\in\mathcal{U}(M^{\mathcal{U}})$ with $u_1^2=u_2^3=1$ and $\{u_1\}''\perp\{u_2\}''$, there exist Haar unitaries $v_1,v_2\in M^{\mathcal{U}}$ such that $[u_1,v_1]=[u_2,v_2]=[v_1,v_2]=0$.*

Two von Neumann subalgebras P,Q of a tracial von Neumann algebra (M,τ) are called orthogonal (written $P\perp Q$) if $\tau(xy)=\tau(x)\tau(y)$, for every $x\in P, y\in Q$. For the notion of a countably cofinal ultrafilter, see Definition 5.2. Here we only note that every free ultrafilter on $\mathbb{N}$ is countably cofinal.

The construction in Theorem B is designed to imply the following estimate for the 1-bounded entropy, which we present as our next main theorem.

Theorem C. *Let M be any II_1 factor satisfying the properties of Theorem B. Then $h(M^{\mathcal{U}})\leq 0$, for every ultrafilter $\mathcal{U}$ on a set I.*

The above estimate allows us to prove the desired non-isomorphism of ultrapowers. Indeed, let M be as in Theorem B. If (N,τ) is a tracial von Neumann algebra which is elementarily equivalent to M, then $M^{\mathcal{U}}\cong N^{\mathcal{V}}$, for some ultrafilters $\mathcal{U},\mathcal{V}$. Properties of the 1-bounded entropy give that $h(N)\leq h(N^{\mathcal{V}})$ (see Facts 2.3 and 2.4). The conclusion of Theorem A then follows immediately. We refer the reader to Remark 5.9, pointed out to us by I. Goldbring and D. Jekel, for an explicit sequence which differentiates the elementary classes of M and N.

Note that if M is a II_1 factor with property Gamma, then $h(M^{\mathcal{U}}) \leq 0$, for every ultrafilter $\mathcal{U}$ on a set I. Prior to the writing of this paper no examples of non-Gamma II_1 factors which satisfy this inequality were known. Hence, Theorem C is also of independent interest.

A II_1 factor is called *pseudocompact* if it is elementarily equivalent to a matrix ultraproduct (see [FHS14, Section 5]). Pseudocompact factors cannot have property Gamma by [FH11, Section 4] and [FHS14, Theorem 5.1]. By combining Theorem C with recent work of Jekel [Jek22] on matrix ultraproducts we obtain the first example of a non-Gamma II_1 factor which is not pseudocompact.

Corollary D. *There exists a separable II_1 factor M without property Gamma which is not elementarily equivalent to $\prod_{\mathcal{U}} \mathbb{M}_{k_n}(\mathbb{C})$, for any sequence $(k_n) \subset \mathbb{N}$ and any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

Remark 1.1. The Connes Embedding Problem (CEP) asks if every separable II_1 factor embeds into $R^{\mathcal{U}}$, where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$ [Con76]. A negative answer to the CEP has been announced in the preprint [JNV$^+$]. Assuming M_0 is a non-Connes-embeddable separable II_1 factor, then $M = M_0 * \mathrm{L}(\mathbb{Z})$ is a non-Gamma separable II_1 factor which is still not embeddable. Any such M is neither elementarily equivalent to any embeddable non-Gamma II_1 factor (e.g., $\mathrm{L}(\mathbb{F}_2)$) nor pseudocompact. Moreover, assuming a negative answer to the CEP, [GH, Corollary 5.5] implies the existence of infinitely many elementary equivalence classes of non-Gamma II_1 factors. In contrast, our construction of a non-Gamma II_1 factor which is not elementarily equivalent to $\mathrm{L}(\mathbb{F}_2)$ and not pseudocompact is explicit and does not depend on the answer to the CEP, nor does it use techniques from [JNV$^+$]. We note that it is open whether the II_1 factor we construct is Connes-embeddable. As such, it remains an open question to find examples of Connes-embeddable non-Gamma II_1 factors which are not elementarily equivalent.

Comments on the proofs of Theorems B and C. The proof of Theorem B relies on a new construction of II_1 factors which is of independent interest and is presented in Section 4. This associates, via a 2-step amalgamated free product procedure, to every II_1 factor M_1 and unitaries $u_1, u_2 \in M_1$, a tracial von Neumann algebra M_2 generated by M_1 and Haar unitaries $v_1, v_2 \in M_2$ satisfying $[u_1, v_1] = [u_2, v_2] = [v_1, v_2] = 0$. When $\{u_1\}'' \perp \{u_2\}''$, we use deformation/rigidity results from [IPP08] to deduce that M_2 is a II_1 factor. Moreover, under this assumption, we show that any irreducible subfactor $Q \subset M_1$ is still irreducible in M_2, see Theorem 4.2.

In Section 5, assuming that M_1 has property (T) and iterating the above construction, we get an increasing sequence of II_1 factors $(M_n)_{n \geq 1}$ whose inductive limit $M := (\cup_{n \geq 1} M_n)''$ is non-Gamma and has the following property. For a countable dense set of unitaries $u_1, u_2 \in M$ with $u_1^2 = u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$ there are Haar unitaries $v_1, v_2 \in M$ such that $[u_1, v_1] = [u_2, v_2] = [v_1, v_2] = 0$. Using a result which allows us to lift unitaries $u_1, u_2 \in M^{\mathcal{U}}$ with $u_1^2 = u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$ (see Lemma 3.1) we conclude that $M^{\mathcal{U}}$ satisfies the conclusion of Theorem B. The restriction to unitaries u_1 and u_2 of orders 2 and 3 is due to the fact that Lemma 3.1 only applies in this case.

The statement of Theorem B is partially inspired by [Hay18, Corollary 4.8]. This shows that if a diffuse tracial von Neumann algebra (M, τ) has property (C$'$) introduced in [GP17, Definition 3.6], then $h(M) \leq 0$. In particular, [Hay18, Corollary 4.8] implies that $h(M) \leq 0$, for any diffuse von Neumann algebra (M, τ) that is generated by $u_1, \cdots, u_k \in \mathcal{U}(M)$ so that there exist pairwise commuting Haar unitaries $v_1, \cdots, v_k \in \mathcal{U}(M^{\mathcal{U}})$ with $[u_i, v_i] = 0$, for any $1 \leq i \leq k$. Property (C$'$) is an asymptotic commutativity property which weakens Popa's property (C) [Pop84]. The latter, itself a weakening of property Gamma, was shown to fail for $\mathrm{L}(\mathbb{F}_n)$, $2 \leq n \leq \infty$, in [Dyk97].

To outline the proof of Theorem C, let M be as in Theorem B and $\mathcal{U}$ be a countably cofinal ultrafilter on a set I. Using an observation made in the proof of [Hay18, Corollary 4.8] (see Fact 2.9) we derive that $h(\{u_1, u_2\}'' : M^{\mathcal{U}}) \leq 0$, for any $u_1, u_2 \in \mathcal{U}(M^{\mathcal{U}})$ with $u_1^2 = u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$. Here, $h(N : M)$ denotes the 1-bounded entropy of N in the presence of M, see

Subsection 2.2. On the other hand, $M^{\mathcal{U}}$ can be generated by a family of subalgebras of the form $\{u_1, u_2\}''$, where $u_1, u_2 \in \mathcal{U}(M^{\mathcal{U}})$ satisfy $u_1^2 = u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$, all containing a fixed diffuse subalgebra. Using the behavior of the 1-bounded entropy with respect to joins (see Facts 2.6 and 2.5), we conclude that $h(M^{\mathcal{U}}) \leq 0$, for any countably cofinal ultrafilter $\mathcal{U}$. Since $h(M) \leq 0$ and $M^{\mathcal{U}} \cong M$ for any ultrafilter $\mathcal{U}$ that is not countably cofinal, Theorem C follows.

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2. Preliminaries

2.1. **Tracial von Neumann algebras.** Let (M, τ) be a tracial von Neumann algebra, i.e., a pair consisting of a von Neumann algebra M and a faithful normal tracial state $\tau : M \to \mathbb{C}$. We denote by $\mathcal{U}(M)$ the group of unitaries of M and by M_{sa} the set of self-adjoint elements of M. Given a self-adjoint set $S \subset M$, von Neumann's bicommutant theorem implies that S'' is the smallest unital von Neumann subalgebra of M containing S. For von Neuman subalgebras (M_α) of M, we will use the notation $\vee_\alpha M_\alpha$ for $(\cup_\alpha M_\alpha)''$.

For an ultrafilter $\mathcal{U}$ on a set I, we denote by $M^{\mathcal{U}}$ the tracial ultraproduct: the quotient $\ell^\infty(I, M)/\mathcal{J}$ by the closed ideal $\mathcal{J} \subset \ell^\infty(I, M)$ consisting of $x = (x_n)$ with $\lim\limits_{n \to \mathcal{U}} \|x_n\|_2 = 0$. We have a natural diagonal inclusion $M \subset M^{\mathcal{U}}$ given by $x \mapsto (x_n)$, where $x_n = x$, for all $n \in I$. A separable II_1 factor M has *property Gamma* if $M' \cap M^{\mathcal{U}} \neq \mathbb{C}1$, for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For more details on tracial ultraproducts, we refer the reader to [BO08, Appendix E] and [ADP, Section 5].

Two tracial von Neumann algebras (M_1, τ_1) and (M_2, τ_2) are said to be *elementarily equivalent* if there exist ultrafilters $\mathcal{U}, \mathcal{V}$ on arbitrary sets I, J such that $M_1^{\mathcal{U}} \cong M_2^{\mathcal{V}}$. This is the *semantic* definition of elementary equivalence. The model theoretic (sometimes called syntactic) definition for elementary equivalence will not be stated in this paper, as it is equivalent to the semantic definition by deep results of Keisler-Shelah adapted to the continuous setting, see [FHS14, Section 2] and [HI02, Theorem 10.7].

A key tool in our work is the amalgamated free product construction for tracial von Neumann algebras. Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras with a common von Neumann subalgebra B such that $\tau_{1|B} = \tau_{2|B}$. We denote by $M = M_1 *_B M_2$ the amalgamated free product with its canonical trace τ. See [Pop93] and [VDN92] for more details on the construction.

To prove that the II_1 factors we construct do not have property Gamma, we will use property (T) and Popa's intertwining techniques.

A II_1 factor has *property (T)* [CJ85] (see also [Pop06a]) if for every $\varepsilon > 0$, there are $F \subset M$ finite and $\delta > 0$ such that for any Hilbert M-M-bimodule $\mathcal{H}$ and unit vector $\xi \in \mathcal{H}$ with $\max_{x \in F} \|x\xi - \xi x\| \leq \delta$, there exists $\eta \in \mathcal{H}$ satisfying $\|\eta - \xi\| \leq \varepsilon$ and $x\eta = \eta x$, for every $x \in M$. Let Γ be an icc countable group with property (T); for instance, take $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ by [Kaž67]. Then $M = \mathrm{L}(\Gamma)$ is a II_1 factor with property (T), see [CJ85, Theorem 2] and [Pop86, Theorem 4.1.7].

In this paper, we will use the well-known fact that II_1 factors with property (T) have weak spectral gap (in the sense of [Pop12]) in any inclusion:

Proposition 2.1. *Let M be a II_1 factor and $M_1 \subset M$ be a subfactor with property (T). Then $M_1' \cap M^{\mathcal{U}} = (M_1' \cap M)^{\mathcal{U}}$, for any ultrafilter $\mathcal{U}$ on a set I.*

Conversely, if the equality $M_1' \cap M^{\mathcal{U}} = (M_1' \cap M)^{\mathcal{U}}$ holds for every II_1 factor M containing M_1 and every ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then M_1 must have property (T), as shown recently in [Tan].

Theorem 2.2 (see [Pop06b]). *Let (M,τ) be a separable tracial von Neumann algebra and let $P \subset pMp, Q \subset M$ be von Neumann subalgebras. Then the following conditions are equivalent:*

(1) There exist projections $p_0 \in P, q_0 \in Q$, a $*$-homomorphism $\theta : p_0Pp_0 \to q_0Qq_0$ and a non-zero partial isometry $v \in q_0Mp_0$ such that $\theta(x)v = vx$, for all $x \in p_0Pp_0$.
(2) There is no sequence $u_n \in \mathcal{U}(P)$ satisfying $\|E_Q(x^*u_ny)\|_2 \to 0$, for all $x,y \in pM$.

If one of these equivalent conditions holds, we write $P \prec_M Q$, and say that *a corner of P embeds into Q inside M.*

2.2. **1-bounded entropy.** We recall some background for 1-bounded entropy theory (see [Hay18], [Jun07]) and direct the reader to [HJNS21, Section 2.3] and [HJKE21, Sections 2.2 and 2.3] for a more detailed exposition. For a tracial von Neumann algebra (M,τ) and $X \in M_{\text{sa}}^d$, the *law* of X is the linear functional $\ell_X : \mathbb{C}\langle t_1,\ldots,t_d\rangle \to \mathbb{C}$ given by $\ell_X(f) = \tau(f(X))$. Let $\Sigma_{d,R}$ be the set of all linear maps $\ell : \mathbb{C}\langle t_1,\ldots,t_d\rangle \to \mathbb{C}$ satisfying that there exists a finite von Neumann algebra (M,τ) and $X \in M_{\text{sa}}^d$ such that $\ell = \ell_X$ and $\|x\| \le R$ for all $x \in X$. We equip $\Sigma_{d,R}$ with the weak* topology.

We describe the orbital version of 1-bounded entropy (see Definition A.2 in [Hay18]). Let (M,τ) be a diffuse tracial von Neumann algebra, and $X, Y \subset M_{\text{sa}}$ finite such that $\|x\| \le R$ for all $x \in X \cup Y$. Following [Voi94], for each weak* neighborhood $\mathcal{O}$ of $\ell_{X\sqcup Y}$ in $\Sigma_{d,R}$ and $n \in \mathbb{N}$, we define

$$\Gamma_R^{(n)}(X : Y;\mathcal{O}) = \{A \in \mathbb{M}_n(\mathbb{C})_{\text{sa}}^X : \exists B \in \mathbb{M}_n(\mathbb{C})_{\text{sa}}^Y \text{ such that } \ell_{A\sqcup B} \in \mathcal{O}, \|A_x\|, \|B_y\| \le R, \forall x \in X, y \in Y\}.$$

Given $d,n \in \mathbb{N}$, $\varepsilon > 0$ and $\Omega, \Xi \subseteq \mathbb{M}_n(\mathbb{C})^d$, then Ξ is said to $(\varepsilon, \|\cdot\|_2)$-*cover* Ω if for every $A \in \Omega$, there is $B \in \Xi$ with $\|A - B\|_2 < \varepsilon$. Define the *covering number* $K_\varepsilon(\Omega, \|\cdot\|_2)$ of $\Omega \subseteq \mathbb{M}_n(\mathbb{C})^d$ as the minimal cardinality of a set that $(\varepsilon, \|\cdot\|_2)$-covers Ω. We say that Ξ *orbitally* $(\varepsilon, \|\cdot\|_2)$-*covers* Ω if for every $A \in \Omega$, there is a $B \in \Xi$ and an $n \times n$ unitary matrix V so that $\|A - VBV^*\|_2 < \varepsilon$. Define the *orbital covering number* $K_\varepsilon^{\text{orb}}(\Omega, \|\cdot\|_2)$ as the minimal cardinality of a set that orbitally $(\varepsilon, \|\cdot\|_2)$-covers Ω.

Let $X_0, Y_0 \subset M_{\text{sa}}$ not necessarily finite, satisfying $X_0'' \subset Y_0''$ and $\|x\| \le R$ for all $x \in X_0 \cup Y_0$. Let X, Y be finite subsets of X_0, Y_0 respectively. For a weak*-neighborhood $\mathcal{O}$ of $\ell_{X\sqcup Y}$, we define

$$\begin{aligned}
h_\varepsilon(X : Y;\mathcal{O}) &:= \limsup_{n\to\infty} \frac{1}{n^2} \log K_\varepsilon^{\text{orb}}(\Gamma_R^{(n)}(X : Y;\mathcal{O})),\\
h_\varepsilon(X : Y) &:= \inf_{\mathcal{O}\ni\ell_{X\sqcup Y}} h_\varepsilon(\mathcal{O}),\\
h_\varepsilon(X_0 : Y_0) &:= \sup_{X\subset_{\text{finite}}X_0} \inf_{Y\subset_{\text{finite}}Y_0} h_\varepsilon(X : Y)\\
h(X_0 : Y_0) &:= \sup_{\epsilon>0} h_\varepsilon(X_0 : Y_0)
\end{aligned}$$

Note that $h(X_1 : Y_1) = h(X_2 : Y_2)$ if $X_1'' = X_2''$ and $Y_1'' = Y_2''$ by [Hay18, Theorem A.9]. Hence, given a von Neumann subalgebra $N \subset M$, we unambiguously write $h(N : M)$ (and call it the 1-*bounded entropy of N in the presence of M*) to be $h(X : Y)$ for some generating sets X of N and Y of M. We write $h(M) = h(M : M)$ and call it the 1-*bounded entropy of M*.

For the purposes of this article we recall the following facts about h:

Fact 2.3. (see [HJKE21, 2.3.3]) $h(N_1 : M_1) \le h(N_2 : M_2)$ if $N_1 \subset N_2 \subset M_2 \subset M_1$ and N_1 is diffuse.

Fact 2.4. (see [Hay18, Proposition 4.5]) $h(N : M) = h(N : M^{\mathcal{U}})$ if $N \subset M$ is diffuse, and $\mathcal{U}$ is an ultrafilter on a set I. (Note that [Hay18, Proposition 4.5] asserts this fact for free ultrafilters $\mathcal{U}$. The fact is trivially true also for non-free (i.e., principal) ultrafilters.)

Fact 2.5. (see [Hay18, Lemma A.12]) $h(N_1 \vee N_2 : M) \leq h(N_1 : M) + h(N_2 : M)$ if $N_1, N_2 \subset M$ and $N_1 \cap N_2$ is diffuse. In particular, $h(N_1 \vee N_2) \leq h(N_1) + h(N_2)$.

Fact 2.6. (see [Hay18, Lemma A.10]) Assume that $(N_\alpha)_\alpha$ is an increasing chain of diffuse von Neumann subalgebras of M. Then $h(\bigvee_\alpha N_\alpha : M) = \sup_\alpha h(N_\alpha : M)$.

By [Jun07, Corollary 3.5] and [Hay18, Proosition A.16], $h(N) = \infty$ whenever (N, τ) is a tracial von Neumann algebra admitting a finite generating set $X \subset N_{\rm sa}$ with $\delta_0(X) > 1$, where δ_0 is Voiculescu's modified free entropy dimension (see Section 6 of [Voi96]).

Fact 2.7. The following tracial von Neumann algebras (N, τ) satisfy $h(N) > 0$. The first five examples all arise from identifying generating sets X satisfying $\delta_0(X) > 1$, and thus $h(N) = \infty$.

(1) (see [Jun07, Lemma 3.7])) $N_1 * N_2$ where (N_1, τ_1) and (N_2, τ_2) are Connes-embeddable diffuse tracial von Neumann algebras.
(2) The free perturbation algebras of Voiculescu (see Theorem 4.1 in [Bro05]).
(3) Many examples of amalgamated free products $N_1 *_B N_2$ where B is amenable (see Section 4 of [BDJ08] for precise examples).
(4) (see [CdSH+22]) Graph products of finite dimensional tracial von Neumann algebras over trees where the cardinality of the vertex set is greater than or equal to 4.
(5) (see [Shl09], Theorem 3) Von Neumann algebras of Connes-embeddable nonamenable groups Γ admitting non inner cocycles $c : \Gamma \to \mathbb{C}\Gamma$.
(6) (see [Hay20], [BC], [HJKE22]) Nonamenable von Neumann subalgebras of $\mathrm{L}(\mathbb{F}_t)$ for $t > 0$.

The following recent result of Jekel provides another family of examples:

Fact 2.8. (see [Jek22, Theorem 1.1]) Suppose that $h(N) > 0$. Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of natural numbers and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let $\mathcal{M} = \prod_{\mathcal{U}} \mathbb{M}_{n_k}(\mathbb{C})$. Then there exists an embedding $N \hookrightarrow \mathcal{M}$ such that $h(N : \mathcal{M}) > 0$. In particular $h(\mathcal{M}) > 0$.

The following fact follows easily from Fact 2.5. This observation appears in the proof of Corollary 4.8 in [Hay18]. For completeness, we include a proof here.

Fact 2.9. Assume that $u_1, u_2 \in \mathcal{U}(M)$ such that there are Haar unitaries $v_1, v_2 \in M$ satisfying $[v_1, u_1] = [v_2, u_2] = [v_1, v_2] = 0$. Then $h(\{u_1, u_2\}'' : M) \leq 0$.

Proof. Since $\{u_1, v_1\}'', \{v_1, v_2\}'', \{v_2, u_2\}''$ are abelian, we get

$$h(\{u_1, v_1\}'') = h(\{v_1, v_2\}'') = h(\{v_2, u_2\}'') = 0.$$

Since $\{v_1\}''$ and $\{v_2\}''$ are diffuse, Fact 2.5 implies that

$$h(\{u_1, u_2, v_1, v_2\}'') = h(\{u_1, v_1\}'' \bigvee \{v_1, v_2\}'' \bigvee \{v_2, u_2\}'') \leq 0.$$

Hence, using Fact 2.3 we see that

$$h(\{u_1, u_2\}'' : M) \leq h(\{u_1, u_2\}'' : \{u_1, u_2, v_1, v_2\}'') \leq h(\{u_1, u_2, v_1, v_2\}'') \leq 0,$$

which proves the fact. □

3. A lifting lemma

The goal of this section is to establish the following lifting lemma:

Lemma 3.1. *Let I be a set, $\mathcal{U}$ an ultrafilter on I and $(M_n)_{n\in I}$ be a family of II_1 factors. Consider projections $p,q_1,q_2,q_3\in\prod_{\mathcal{U}}M_n$ such that $q_1+q_2+q_3=1$ and $\{p\}''\perp\{q_1,q_2,q_3\}''$.*

Then we can represent $p=(p_n)$ and $q_i=(q_{i,n})$, where $p_n,q_{i,n}\in M_n$ are projections such that $q_{1,n}+q_{2,n}+q_{3,n}=1$ and $\{p_n\}''\perp\{q_{1,n},q_{2,n},q_{3,n}\}''$, for every $n\in I$.

Lemma 3.1 is an immediate consequence of the following perturbation lemma.

Lemma 3.2. *For every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that the following holds.*

Let M be a II_1 factor and $e,f_1,f_2,f_3\in M$ be projections such that $f_1+f_2+f_3=1$ and we have $|\tau(ef_i)-\tau(e)\tau(f_i)|\leq\delta$, for every $1\leq i\leq 3$. Then there exist projections $p,q_1,q_2,q_3\in M$ such that $q_1+q_2+q_3=1$, $\|p-e\|_1\leq\varepsilon$, $\|q_i-f_i\|_1\leq\varepsilon$ and $\tau(pq_i)=\tau(p)\tau(q_i)$, for every $1\leq i\leq 3$.

Note that if p,q are projections in a II_1 factor M, then $\|p-q\|_2=\|p(p-q)+(p-q)q\|_2\leq 2\|p-q\|_1$. This implies that the statement of Lemma 3.2 still holds if we replace $\|\cdot\|_1$ by $\|\cdot\|_2$. Using this observation, it is standard to derive Lemma 3.1 from Lemma 3.2.

The proof of Lemma 3.2 is based on the next two lemmas.

Lemma 3.3. *Let (M,τ) be a diffuse tracial von Neumann algebra, $\delta>\varepsilon>0$ and $x=x^*\in M$ with $|\tau(x)|\leq\varepsilon$ and $\|x\|_1>\delta$. Then there is a projection $p\in M$ such that $\tau(xp)=0$ and $\tau(p)>\frac{\delta-\varepsilon}{\delta+\varepsilon}$.*

Proof. Let $x=y-z$ be the decomposition of x into its positive and negative parts and $q\in M$ be the support projection of y. Then $y\in qMq$ and $z\in(1-q)M(1-q)$. If $\tau(x)=0$, there is nothing to prove. We may assume that $\tau(x)>0$, since the case $\tau(x)<0$ is analogous.

Since $\tau(y)-\tau(z)=\tau(x)\leq\varepsilon$ and $\tau(y)+\tau(z)=\|x\|_1>\delta$, letting $s=\frac{\delta-\varepsilon}{\delta+\varepsilon}\in(0,1)$, it follows that $\tau(y)s<\tau(z)$. Let $y'\in qMq$ be a self-adjoint operator with finite spectrum such that

$$2\|y'-y\|_1<\tau(z)-\tau(y)s. \tag{3.1}$$

Since M is diffuse and y' has finite spectrum, we can find an increasing net of projections $(e_t)_{t\in[0,1]}$ in qMq such that $e_0=0,e_1=q,\tau(e_t)=\tau(q)t$ and $\tau(y'e_t)=\tau(y')t$, for every $t\in[0,1]$. Then for every $t\in[0,1]$, we have that

$$|\tau(ye_t)-\tau(y)t|\leq|\tau(ye_t)-\tau(y'e_t)|+|(\tau(y)-\tau(y'))t|\leq 2\|y'-y\|_1,$$

and thus $\tau(ye_t)\leq\tau(y)t+2\|y'-y\|_1$.

Combining this inequality for $t=s$ with (3.1) gives that $\tau(ye_s)<\tau(z)$. As $\tau(ye_1)=\tau(y)>\tau(z)$ and the map $t\mapsto\tau(ye_t)$ is continuous, we can find $t\in(s,1)$ such that $\tau(ye_t)=\tau(z)$.

Finally, let $p=e_t+(1-q)$. Then we have $\tau(xp)=\tau(yp)-\tau(zp)=\tau(ye_t)-\tau(z)=0$ and $\tau(p)=\tau(e_t)+\tau(1-q)=t\tau(q)+\tau(1-q)\geq t>s$, which finishes the proof. ☐

Lemma 3.4. *Let $\varepsilon,\delta\geq 0$ such that $\varepsilon<\delta^2$ and (M,τ) be a diffuse tracial von Neumann algebra. Let $p,f_1,f_2,f_3\in M$ be projections such that $f_1+f_2+f_3=1$, $|\tau(pf_i)-\tau(p)\tau(f_i)|\leq\varepsilon$ and $\|f_i(p-\tau(p))f_i\|_1>\delta$, for every $1\leq i\leq 2$.*

Then there exist projections $q_1,q_2,q_3\in M$ such that $q_1+q_2+q_3=1,\tau(pq_i)=\tau(p)\tau(q_i)$ and $\|q_i-f_i\|_1<\frac{4\varepsilon}{\delta^2}$, for every $1\leq i\leq 3$.

Proof. Let $1\leq i\leq 2$ and define $x_i=f_i(p-\tau(p))f_i$. Then we have $x_i=x_i^*\in f_iMf_i$ and $|\tau(x_i)|=|\tau(pf_i)-\tau(p)\tau(f_i)|\leq\varepsilon$. Since $\|x_i\|_1>\delta$ and $\|x_i\|_1\leq\tau(f_i)\|p-\tau(p)\|\leq\tau(f_i)$, we get that $\tau(f_i) > \delta$. Thus, $|\frac{\tau(x_i)}{\tau(f_i)}| \leq \frac{\varepsilon}{\delta}$ and $\frac{\|x_i\|_1}{\tau(f_i)} \geq \|x_i\|_1 > \delta$. Altogether, by applying Lemma 3.3 to $x_i \in f_iMf_i$, we find a projection $q_i \in f_iMf_i$ such that

$$\tau(x_iq_i) = 0 \quad \text{and} \quad \frac{\tau(q_i)}{\tau(f_i)} > \frac{\delta - \frac{\varepsilon}{\delta}}{\delta + \frac{\varepsilon}{\delta}} = \frac{1 - \frac{\varepsilon}{\delta^2}}{1 + \frac{\varepsilon}{\delta^2}} > 1 - \frac{2\varepsilon}{\delta^2}. \tag{3.2}$$

Using (3.2) we get that $\tau((p - \tau(p))q_i) = \tau(x_iq_i) = 0$ and thus $\tau(pq_i) = \tau(p)\tau(q_i)$. Moreover,

$$\|q_i - f_i\|_1 = \tau(f_i) - \tau(q_i) < \frac{2\varepsilon}{\delta^2}\tau(f_i) \leq \frac{2\varepsilon}{\delta^2}.$$

Let $q_3 = 1 - q_1 - q_2$ and $f_3 = 1 - f_1 - f_2$. Then $\tau(pq_3) = \tau(p) - \tau(pq_1) - \tau(pq_2) = \tau(p)(1 - \tau(q_1) - \tau(q_2)) = \tau(p)\tau(q_3)$. Moreover, $\|q_3 - f_3\|_1 = \|(q_1 + q_2) - (f_1 + f_2)\|_1 \leq \|q_1 - f_1\|_1 + \|q_2 - f_2\|_1 < \frac{4\varepsilon}{\delta^2}$. This finishes the proof of the lemma. $\square$

Proof of Lemma 3.2. Assume that the conclusion of Lemma 3.2 fails. Then there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we can find a II_1 factor (M_n, τ_n) and projections $e_n, f_{1,n}, f_{2,n}, f_{3,n} \in M_n$ satisfying the following: $f_{1,n} + f_{2,n} + f_{3,n} = 1$, $|\tau_n(e_nf_{i,n}) - \tau_n(e_n)\tau_n(f_{i,n})| \leq \frac{1}{n}$, for every $1 \leq i \leq 3$, and $\|p_n - e_n\|_1 + \|q_{1,n} - f_{1,n}\|_1 + \|q_{2,n} - f_{2,n}\|_1 + \|q_{3,n} - f_{3,n}\|_1 > \varepsilon$, for all projections $p_n, q_{1,n}, q_{2,n}, q_{3,n} \in M_n$ such that $q_{1,n} + q_{2,n} + q_{3,n} = 1$ and $\tau_n(p_nq_{i,n}) = \tau_n(p_n)\tau_n(q_{i,n})$, for every $1 \leq i \leq 3$.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let τ be the canonical trace of $\prod_{\mathcal{U}} M_n$ given by $\tau(x) = \lim_{n\to\mathcal{U}} \tau_n(x_n)$, for every $x = (x_n) \in \prod_{\mathcal{U}} M_n$. Then $p = (e_n), q_1 = (f_{1,n}), q_2 = (f_{2,n}), q_3 = (f_{3,n}) \in \prod_{\mathcal{U}} M_n$ are projections satisfying that $q_1 + q_2 + q_3 = 1$ and $\{p\}'' \perp \{q_1, q_2, q_3\}''$.

We will get a contradiction by analyzing two cases:

Case 1. The set $\{1 \leq i \leq 3 \mid q_i(p - \tau(p))q_i = 0\}$ has at most one element.

Without loss of generality, assume that $q_i(p - \tau(p))q_i \neq 0$, for all $1 \leq i \leq 2$.

For $n \in \mathbb{N}$ and $1 \leq i \leq 2$, define $\delta_i = \|q_i(p - \tau(p))q_i\|_1, \delta_{i,n} = \|f_{i,n}(e_n - \tau_n(e_n))f_{i,n}\|_1$ and $\kappa_{i,n} = |\tau_n(e_nf_{i,n}) - \tau_n(e_n)\tau_n(f_{i,n})|$. Then $\delta_i > 0$, $\lim_{n\to\mathcal{U}} \delta_{i,n} = \delta_i$ and $0 \leq \kappa_{i,n} \leq \frac{1}{n}$, for every $n \in \mathbb{N}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then the set J of $n \in \mathbb{N}$ such that $\delta_{i,n} > \frac{\delta}{2}$ and $\kappa_{i,n} < \delta_{i,n}^2$, for every $1 \leq i \leq 2$, belongs to $\mathcal{U}$.

By Lemma 3.4, for every $n \in J$, we find projections $q_{i,n} \in M_n$ such that $q_{1,n} + q_{2,n} + q_{3,n} = 1$, $\tau_n(e_nq_{i,n}) = \tau_n(e_n)\tau_n(q_{i,n})$ and $\|q_{i,n} - f_{i,n}\|_1 < \frac{4\kappa_{i,n}}{\delta_{i,n}^2} < \frac{16}{\delta^2n^2}$, for every $1 \leq i \leq 2$. As J is infinite, we can find $n \in J$ such that $\frac{16}{\delta^2n^2} < \frac{\varepsilon}{3}$, for every $1 \leq i \leq 2$. Put $p_n = e_n$. Then $\|p_n - e_n\|_1 + \|q_{1,n} - f_{1,n}\|_1 + \|q_{2,n} - f_{2,n}\|_1 + \|q_{3,n} - f_{3,n}\|_1 < \varepsilon$, contradicting the first paragraph of the proof.

Case 2. The set $\{1 \leq i \leq 3 \mid q_i(p - \tau(p))q_i = 0\}$ has at least two elements.

Without loss of generality, assume that $q_i(p - \tau(p))q_i = 0$, for every $1 \leq i \leq 2$.

We claim that $Q := \{p, q_1, q_2, q_3\}''$ is a type I von Neumann algebra. Let $1 \leq i \leq 2$. Since $q_ipq_i = \tau(p)q_i$, we get that $v_i := \tau(p)^{-\frac{1}{2}}q_ip$ is a partial isometry. Thus, $p_i := v_i^*v_i = \tau(p)^{-1}pq_ip$ is a projection. Recall that any von Neumann algebra generated by two projections is of type I, being a direct sum of type I_1 and I_2 algebras. Since $pQp = \{pq_1p, pq_2p, pq_3p\}'' = \{p_1, p_2, p\}''$ and $p_1, p_2 \in p(\prod_{\mathcal{U}} M_n)p$ are projections, pQp is of type I. Since $q_i((1-p)-\tau(1-p))q_i = q_i(\tau(p)-p)q_i = 0$, for every $1 \leq i \leq 2$, we also get that $(1-p)Q(1-p)$ is of type I. The last two facts imply the claim.

Next, endow $Q \subset \prod_{\mathcal{U}} M_n$ with the restriction of τ to Q. Since Q is of type I, it is hyperfinite. If $n \in \mathbb{N}$, then using that M_n is a II_1 factor we can find a normal $*$-homomorphism $\pi_n : Q \to M_n$ such that $\tau_n(\pi_n(x)) = \tau(x)$, for every $x \in Q$. Then the normal $*$-homomorphism $\pi : Q \to \prod_{\mathcal{U}} M_n$

given by $\pi(x) = (\pi_n(x))$ satisfies that $\tau(\pi(x)) = \lim_{n\to\mathcal{U}} \tau_n(\pi_n(x)) = \tau(x)$, for every $x \in Q$. As is well-known (see, e.g, [HS18, Theorem 1.1]), since Q is hyperfinite, any two trace-preserving $*$-homomorphism from Q to $\prod_{\mathcal{U}} M_n$ are unitarily conjugate. Thus, we can find $u_n \in \mathcal{U}(M_n)$, for every $n \in \mathbb{N}$, such that $x = (u_n\pi_n(x)u_n^*)$, for every $x \in Q$. In particular, $p = (p_n)$ and $q_i = (q_{i,n})$, where $p_n = u_n\pi_n(p)u_n^*$ and $q_{i,n} = u_n\pi_n(q_i)u_n^*$, for every $n \in \mathbb{N}$ and $1 \leq i \leq 3$. Then $q_{1,n} + q_{2,n} + q_{3,n} = 1$, for every $n \in \mathbb{N}$, and $\lim_{n\to\mathcal{U}}(\|p_n - e_n\|_1 + \|q_{1,n} - f_{1,n}\|_1 + \|q_{2,n} - f_{2,n}\|_1 + \|q_{3,n} - f_{3,n}\|_1) = 0$. Moreover,

$$\tau_n(p_nq_{i,n}) = \tau_n(\pi_n(pq_i)) = \tau(pq_i) = \tau(p)\tau(q_i) = \tau_n(\pi_n(p))\tau_n(\pi_n(q_i)) = \tau_n(p_n)\tau_n(q_{i,n}),$$

for every $n \in \mathbb{N}$ and $1 \leq i \leq 3$. Altogether, this also contradicts the first paragraph of the proof. $\square$

4. A construction of II_1 factors

In this section, we introduce a new construction of II_1 factors which we will use iteratively to build the II_1 factor in Theorem B.

Definition 4.1. Let (M, τ) be a tracial von Neumann algebra and $A_1, A_2 \subset M$ be von Neumann subalgebras. We define a tracial von Neumann algebra $\Phi(M, A_1, A_2)$ as follows. Put $B_1 = B_2 = \mathrm{L}(\mathbb{Z})$ and define

$$\Phi(M, A_1) := M *_{A_1} (A_1 \overline{\otimes} B_1) \quad \text{and}$$

$$\Phi(M, A_1, A_2) := \Phi(M, A_1) *_{(A_2 \bigvee B_1)} ((A_2 \vee B_1)\overline{\otimes} B_2).$$

Given $u_1, u_2 \in \mathcal{U}(M)$, we will use the notation $\Phi(M, u_1, u_2) := \Phi(M, \{u_1\}'', \{u_2\}'')$.

More generally, given von Neumann subalgebras $A_1, \cdots A_k \subset M$ one can define $\Phi(M, A_1, \cdots, A_k)$ inductively by letting $B_1 = \cdots = B_k = \mathrm{L}(\mathbb{Z})$ and for every $1 \leq i \leq k$

$$\Phi(M, A_1, \cdots, A_i) := \Phi(M, A_1, \cdots, A_{i-1}) *_{(A_i \bigvee B_1 \bigvee \cdots \bigvee B_{i-1})} ((A_i \vee B_1 \vee \cdots \vee B_{i-1})\overline{\otimes} B_i).$$

Here, we focus on the case $k = 2$ which suffices for the purpose of proving Theorem B. The main result of this section gives necessary conditions which guarantee that $\Phi(M, A_1, A_2)$ is a II_1 factor. Furthermore, we prove:

Theorem 4.2. *Let (M, τ) be a tracial von Neumann algebra and $A_1, A_2 \subset M$ be von Neumann subalgebras such that $A_1 \perp A_2$ and $M \nprec_M A_i$, for every $i = 1, 2$. Put $P = \Phi(M, A_1, A_2)$.*

Then P is a II_1 factor containing Haar unitaries $v_1, v_2 \in P$ so that $v_1 \in A_1' \cap P, v_2 \in A_2' \cap P$ and $[v_1, v_2] = 0$. Moreover, if $Q \subset M$ is a von Neumann subalgebra such that $Q \nprec_M A_i$, for every $1 \leq i \leq 2$, then $Q' \cap P \subset M$.

In the proof of Theorem B, we will use the following immediate corollary of Theorem 4.2

Corollary 4.3. *Let (M, τ) be a tracial von Neumann algebra having no type I direct summand. Let $u_1, u_2 \in \mathcal{U}(M)$ such that $\{u_1\}'' \perp \{u_2\}''$ and put $P = \Phi(M, u_1, u_2)$.*

Then P is a II_1 factor containing Haar unitaries $v_1, v_2 \in P$ so that $[u_1, v_1] = [u_2, v_2] = [v_1, v_2] = 0$. Moreover, if $Q \subset M$ is a von Neumann subalgebra such that $Q \nprec_M \{u_i\}''$, for every $1 \leq i \leq 2$, then $Q' \cap P \subset M$.

Since M has no type I direct summand, $M \nprec_M \{u_i\}''$, for every $1 \leq i \leq 2$, and thus Corollary 4.3 follows from Theorem 4.2.

Remark 4.4. Let us argue that the condition that $\{u_1\}'' \perp \{u_2\}''$ in Corollary 4.3 is necessary in order to get that M is a II_1 factor. Thus, the condition that $A_1 \perp A_2$ in Theorem 4.2 is also necessary. In the context of Corollary 4.3, assume that M is generated by u_2 and $\mathrm{E}_{\{u_1\}''}(u_2)$. Denote $N := \Phi(M, \{u_1\}'') = M *_{\{u_1\}''} (\{u_1\}''\overline{\otimes}\mathrm{L}(\mathbb{Z}))$ and let $v_1 \in \mathrm{L}(\mathbb{Z})$ be a generating Haar unitary. By [IPP08, Theorem 1.1] we get that $\mathrm{L}(\mathbb{Z})' \cap N = \{u_1\}''\overline{\otimes}\mathrm{L}(\mathbb{Z})$. This gives that

$$\mathrm{E}_{\mathrm{L}(\mathbb{Z})'\cap N}(u_2) = \mathrm{E}_{\{u_1\}''\overline{\otimes}\mathrm{L}(\mathbb{Z})}(u_2) = \mathrm{E}_{\{u_1\}''}(u_2).$$

On the other hand, $\mathrm{E}_{\mathrm{L}(\mathbb{Z})'\cap N}(u_2)$ is the $\|\cdot\|_2$-limit of the sequence $(\frac{1}{n}\sum_{k=1}^n v_1^k u_2 v_1^{*k})_n$ and thus belongs to $\{u_2, v_1\}''$. The last two facts together imply that $\mathrm{E}_{\{u_1\}''}(u_2) \in \{u_2, v_1\}''$. Since M is generated by u_2 and $\mathrm{E}_{\{u_1\}''}(u_2)$, we get that $M \subset \{u_2, v_1\}''$. Since N is generated by M and v_1, we get that $\{u_2, v_1\}'' = N$. Thus, $\Phi(M, u_1, u_2) = N\overline{\otimes}\mathrm{L}(\mathbb{Z})$ is not a factor, so the conclusion of Corollary 4.3 does not hold.

Now, the existence of $u_1, u_2 \in \mathcal{U}(M)$ such that $\{u_2, \mathrm{E}_{\{u_1\}''}(u_2)\}'' = M$, can be checked whenever M is generated by two unitaries $u_1, \widetilde{u_2}$ such that $\{u_1\}'' \perp \{\widetilde{u_2}\}''$ (e.g., if $M = \mathrm{L}(\Gamma)$, for any 2-generated group Γ). To see this, write $\widetilde{u_2} = \exp(ih)$, where $h \in \{\widetilde{u_2}\}''$ is a self-adjoint element, let $n \in \mathbb{N}$ such that $\tau(\exp(\frac{ih}{n})) \neq 0$ and define $u_2 = u_1 \exp(\frac{ih}{n})$. Then $\mathrm{E}_{\{u_1\}''}(u_2) = \tau(\exp(\frac{ih}{n}))u_1$ and thus $\{u_2, \mathrm{E}_{\{u_1\}''}(u_2)\}'' = \{u_1, \exp(\frac{ih}{n})\}'' = \{u_1, \widetilde{u_2}\}'' = M$.

The proof of Theorem 4.2 relies on the main technical result of [IPP08]. To recall the latter result, let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras with a common von Neumann subalgebra B such that $\tau_{1|B} = \tau_{2|B}$. Let $M = M_1 *_B M_2$ be the amalgamated free product with its canonical trace τ. By [PV10, Section 5.1], for $0 < \rho < 1$ we have a unital tracial completely positive map $\mathrm{m}_\rho : M \to M$ such that $\mathrm{m}_\rho(b) = b$, for every $b \in B$, and $\mathrm{m}_\rho(x_1x_2\cdots x_n) = \rho^n x_1x_2\cdots x_n$, for every $x_i \in M_{i_j} \ominus B$, where $i_j \in \{1,2\}$, for every $1 \leq j \leq n$, and $i_j \neq i_{j+1}$, for every $1 \leq j \leq n-1$. Then

$$\lim_{\rho\to 1} \|\mathrm{m}_\rho(x) - x\|_2 = 0, \text{ for every } x \in M. \tag{4.1}$$

The following is the main technical result of [IPP08], formulated here as in [PV10, Theorem 5.4], see also [Hou09, Section 5].

Theorem 4.5. *Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras with a common von Neumann subalgebra B such that $\tau_{1|B} = \tau_{2|B}$. Let $M = M_1 *_B M_2$ be the amalgamated free product with its canonical trace τ. Let $Q \subset pMp$ be a von Neumann subalgebra. Assume that there are $0 < \rho < 1$ and $c > 0$ such that $\|\mathrm{m}_\rho(u)\|_2 \geq c$, for every $u \in \mathcal{U}(Q)$.*

Then $Q \prec_M M_1$ or $Q \prec_M M_2$.

As $\tau(\mathrm{m}_{\rho^2}(u)u^*) = \|\mathrm{m}_\rho(u)\|_2^2 \geq c^2$, for every $u \in \mathcal{U}(Q)$, [PV10, Theorem 5.4] implies Theorem 4.5.

Lemma 4.6. *Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras with a common von Neumann subalgebra B such that $\tau_{1|B} = \tau_{2|B}$. Let $M = M_1 *_B M_2$ be the amalgamated free product with its canonical trace τ. For $i \in \{1,2\}$, let $A_i \subset M_i$ be a von Neumann subalgebra with $A_i \perp B$. Let $Q \subset M_1$ be a von Neumann subalgebra such that $Q \prec_M A_1 \vee A_2$ and $Q \nprec_{M_1} B$.*

Then $Q \prec_{M_1} A_1$.

Proof. Denote $A = A_1 \vee A_2$. We first claim that A_1 and A_2 are freely independent inside M and thus $A = A_1 * A_2$. Let $a_j \in A_{i_j} \ominus \mathbb{C}1$ for $i_j \in \{1,2\}$, for every $1 \leq j \leq n$, where $i_j \neq i_{j+1}$, for every $1 \leq j \leq n-1$. Since $A_i \perp B$, for every $i \in \{1,2\}$, we get that $E_B(a_j) = 0$, for every $1 \leq j \leq n$. This implies that $\tau(a_1a_2\cdots a_n) = 0$, proving the claim.

Since $Q \prec_M A$, we can find projections $q \in Q, p \in A$, a nonzero partial isometry $v \in pMq$ and $*$-homomorphism $\varphi : qQq \to pAp$ such that $\varphi(x)v = vx$, for every $x \in qQq$. Moreover, we may assume that the support projection of $E_A(vv^*)$ is equal to p.

Claim 4.7. *$\varphi(qQq) \prec_A A_1$ or $\varphi(qQq) \prec_A A_2$.*

Proof of Claim 4.7. Since m_ρ is a unital tracial completely positive map, using (4.1) and [Pop06a, Corollary, Section 1.1.2] we deduce that

$$\sup_{x\in(M)_1} \|\mathrm{m}_\rho(xv) - \mathrm{m}_\rho(x)v\|_2 \to 0 \quad \text{and} \quad \sup_{x\in(M)_1} \|\mathrm{m}_\rho(vx) - v\mathrm{m}_\rho(x)\|_2 \to 0, \quad \text{as } \rho \to 1. \tag{4.2}$$

Now, if $x \in M_1$, then the definition of m_ρ implies that $\mathrm{m}_\rho(x) = \mathrm{E}_B(x) + \rho(x - \mathrm{E}_B(x))$ and thus $\|\mathrm{m}_\rho(x) - x\|_2 = (1-\rho)\|x - \mathrm{E}_B(x)\|_2 \leq (1-\rho)\|x\|_2$. In particular, since $Q \subset M_1$, we derive that

$$\sup_{x\in(qQq)_1} \|\mathrm{m}_\rho(x) - x\|_2 \to 0, \quad \text{as } \rho \to 1. \tag{4.3}$$

By combining (4.2) and (4.4) and using that $\varphi(x)v = vx$, for every $x \in qQq$, it follows that $\sup_{x\in(qQq)_1} \|\mathrm{m}_\rho(\varphi(x))v - vx\|_2 \to 0$, as $\rho \to 1$. Therefore, we can find $0 < \rho < 1$ such that $\|\mathrm{m}_\rho(\varphi(x))v - vx\|_2 < \|v\|_2/2$, for every $x \in (qQq)_1$. This implies that

$$\|\mathrm{m}_\rho(\varphi(u))\|_2 \geq \|\mathrm{m}_\rho(\varphi(u))v\|_2 > \|v\|_2/2, \quad \text{for every } u \in \mathcal{U}(qQq).$$

In other words, $\|\mathrm{m}_\rho(y)\| > \|v\|_2$, for every $y \in \mathcal{U}(\varphi(qQq))$. Note that the restriction of m_ρ to A is equal to the map m_ρ on A associated with the free product decomposition $A = A_1 * A_2$. Since $\varphi(qQq) \subset pAp$, we can apply Theorem 4.5 to get the claim. $\square$

By Claim 4.7, we have that $\varphi(qQq) \prec_A A_i$, for some $i \in \{1, 2\}$. Since the support projection of $E_A(vv^*)$ is equal to p, [Vae08, Remark 3.8] implies that $Q \prec_M A_i$. Finally, since $qQq \subset M_1$, $A_i \subset M_i$ and $Q \nprec_{M_1} B$, applying [IPP08, Theorem 1.1] gives that $i = 1$ and $Q \prec_{M_1} A_1$. $\square$

Proof of Theorem 4.2. Let $P = \Phi(M, A_1, A_2) = N *_{(A_2\vee B_1)} ((A_2 \vee B_1)\overline{\otimes} B_2)$, where $B_1 = B_2 = \mathrm{L}(\mathbb{Z})$ and $N = \Phi(M, A_1) = M *_{A_1} (A_1\overline{\otimes} B_1)$. Let $v_1 \in B_1$ and $v_2 \in B_2$ be generating Haar unitaries. Since $[A_1, B_1] = [A_2, B_2] = [B_1, B_2] = \{0\}$, we get that $v_1 \in A_1' \cap P, v_2 \in A_2' \cap P$ and $[v_1, v_2] = 0$.

Next, we prove the moreover assertion. Let $Q \subset M$ be a von Neumann subalgebra such that $Q \nprec_M A_i$, for every $1 \leq i \leq 2$. Since $N = M *_{A_1} (A_1\overline{\otimes} B_1)$, $A_2 \perp A_1$, $B_1 \perp A_1$ and $Q \nprec_M A_1$, by Lemma 4.6 we conclude that

$$Q \nprec_N A_2 \vee B_1. \tag{4.4}$$

Since $P = N *_{(A_2\vee B_1)} ((A_2 \vee B_1)\overline{\otimes} B_2)$, using (4.4) and applying [IPP08, Theorem 1.1] we get that $Q' \cap P \subset N$. Since $N = M *_{A_1} (A_1\overline{\otimes} B_1)$ and $Q \nprec_M A_1$, applying [IPP08, Theorem 1.1] again gives that $Q' \cap N \subset M$. Altogether, we get that $Q' \cap P \subset M$, which proves the moreover assertion.

Since $M \nprec_M A_i$, for every $1 \leq i \leq 2$. By applying the moreover assertion to $Q = M$, we get that $M' \cap P \subset M$, hence $\mathcal{Z}(P) = P' \cap M \subset \mathcal{Z}(M)$. Thus, if M is a II_1 factor, then P is a II_1 factor.

In the general case, we first note that [IPP08, Theorem 1.1] gives that $B_1' \cap M = A_1$ and $B_2' \cap M = (B_2' \cap N) \cap M = (A_2 \vee B_1) \cap M$. Thus, $\mathcal{Z}(P) = P' \cap M \subset A_1 \cap (A_2 \vee B_1)$. We claim that $A_1 \perp (A_2 \vee B_1)$. Assuming the claim, it follows that $A_1 \cap (A_2 \vee B_1) = \mathbb{C}1$ and so P is a II_1 factor. To justify the claim and finish the proof, denote $M_1 = M, M_2 = A_1\overline{\otimes} B_1$, $C_1 = A_2$, $C_2 = B_1$ and $B = A_1$. Thus, $N = M_1 *_B M_2$ and the claim is equivalent to $B \perp (C_1 \vee C_2)$. Let $x \in B$ and $y \in C_1 \vee C_2$ of the form $y = y_1 y_2 \cdots y_n$, where $y_j \in C_{i_j} \ominus \mathbb{C}1$ for some $i_j \in \{1, 2\}$, for every $1 \leq j \leq n$, such that $i_j \neq i_{j+1}$, for every $1 \leq j \leq n-1$. Since $C_i \perp B$, for every $1 \leq i \leq 2$, we get that $\mathrm{E}_B(y_j) = 0$ and thus $y_j \in M_{i_j} \ominus B$, for every $1 \leq j \leq n$. Moreover, $\mathrm{E}_B(xy_1) = x\mathrm{E}_B(y_1) = 0$ and thus $xy_1 \in M_{i_1} \ominus B$. This implies that $\tau(xy) = \tau((xy_1)y_2 \cdots y_n) = 0$. Since C_1 and C_2 are freely independent, as shown in the proof of Lemma 4.6, the linear span of elements $y \in C_1 \vee C_2$ of the above form is dense in $(C_1 \vee C_2) \ominus \mathbb{C}1$. Thus, we get that $B \perp (C_1 \vee C_2)$, proving the claim. $\square$

5. Proofs of main results

This section is devoted to the proofs of our main results.

5.1. **Proof of Theorem B.** We start by constructing the II_1 factor from Theorem B by iterating the construction from Section 4.

For a II_1 factor M, we denote by $\mathcal{V}(M)$ the set of pairs $(u_1, u_2) \in \mathcal{U}(M) \times \mathcal{U}(M)$ such that $u_1^2 = u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$. We endow $\mathcal{U}(M) \times \mathcal{U}(M)$ with the product $\|\cdot\|_2$-topology.

Definition 5.1. Let M_1 be a II_1 factor. We construct a new II_1 factor M which contains M_1 and arises as the inductive limit of a sequence $(M_n)_{n\in\mathbb{N}}$ of II_1 factors satisfying $M_n \subset M_{n+1}$, for every $n \in \mathbb{N}$. Let $\sigma = (\sigma_1, \sigma_2) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that $\sigma_1(n) \leq n$, for every $n \in \mathbb{N}$. Assume that $M_1, \dots, M_n$ have been constructed, for some $n \in \mathbb{N}$. Let $\{(u_1^{n,k}, u_2^{n,k})\}_{k\in\mathbb{N}} \subset \mathcal{V}(M_n)$ be a $\|\cdot\|_2$-dense sequence. We define

$$M_{n+1} := \Phi(M_n, u_1^{\sigma(n)}, u_2^{\sigma(n)}).$$

Note that M_{n+1} is well-defined since $\sigma_1(n) \leq n$ and thus $(u_1^{\sigma(n)}, u_2^{\sigma(n)}) \in \mathcal{V}(M_n)$. Then $M_n \subset M_{n+1}$ and Corollary 4.3 implies that M_{n+1} is a II_1 factor. Thus, M defined as follows is a II_1 factor:

$$M := (\cup_{n\in\mathbb{N}} M_n)''.$$

Convention. For the rest of this section, $(M_n)_{n\in\mathbb{N}}$ and M denote the II_1 factors introduced in Definition 5.1.

Definition 5.2. An ultrafilter $\mathcal{U}$ on a set I is called *countably cofinal* if there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ with $\cap_n A_n = \emptyset$.

Proposition 5.3. *Let $u_1, u_2 \in \mathcal{U}(M^{\mathcal{U}})$ such that $u_1^2 = u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$, where $\mathcal{U}$ is a countably cofinal ultrafilter on a set I.*

Then there exist Haar unitaries $v_1, v_2 \in M^{\mathcal{U}}$ such that $[u_1, v_1] = [u_2, v_2] = [v_1, v_2] = 0$.

Proof. Let $p, q_1, q_2, q_3 \in M^{\mathcal{U}}$ be projections such that $u_1 = 2p - 1$ and $u_2 = q_1 + \zeta q_2 + \zeta^2 q_3$, where $\zeta = \exp(\frac{2\pi i}{3})$. We may clearly assume that $u_1 \neq \pm 1$, so that $p \neq 0, 1$.

Since $M = (\cup_{n\in\mathbb{N}} M_n)''$ and $\mathcal{U}$ is cofinal, [BCI17, Lemma 2.2] gives that $p, q_1, q_2, q_3 \in \prod_{n\in\mathcal{U}} M_{k_n}$, for some $(k_n)_{n\in I} \subset \mathbb{N}$. Moreover, the proof of [BCI17, Lemma 2.2] provides a function $f : I \to \mathbb{N}$ such that $\lim_{n\to\mathcal{U}} f(n) = +\infty$.

Since $\{p\}'' \perp \{q_1, q_2, q_3\}''$, by Lemma 3.1, we can represent $p = (p_n)$ and $q_i = (q_{i,n})$, where $p_n, q_{n,i} \in M_{k_n}$ are projections such that $q_{1,n} + q_{2,n} + q_{3,n} = 1$ and $\{p_n\}'' \perp \{q_{1,n}, q_{2,n}, q_{3,n}\}''$, for every $n \in I$. Let $u_{1,n} = 2p_n - 1$ and $u_{2,n} = q_{1,n} + \zeta q_{2,n} + \zeta^2 q_{3,n}$. Then $u_1 = (u_{1,n})$ and $u_2 = (u_{2,n})$. Since $\{u_{1,n}\}'' = \{p_n\}'' \perp \{q_{1,n}, q_{2,n}, q_{3,n}\}'' = \{u_{2,n}\}''$, we get $(u_{1,n}, u_{2,n}) \in \mathcal{V}(M_{k_n})$, for every $n \in I$.

Since $\{(u_1^{k_n,j}, u_2^{k_n,j})\}_{j\in\mathbb{N}}$ is dense in $\mathcal{V}(M_{k_n})$, we can find $j_n \in \mathbb{N}$ such that

$$\|u_{1,n} - u_1^{k_n,j_n}\|_2 + \|u_{2,n} - u_2^{k_n,j_n}\|_2 \leq \frac{1}{f(n)}, \text{ for every } n \in I. \tag{5.1}$$

For $n \in I$, let $l_n \in \mathbb{N}$ such that $\sigma(l_n) = (k_n, j_n)$. Then $M_{\sigma(l_n)+1} = \Phi(M_{\sigma(l_n)}, u_1^{k_n,j_n}, u_2^{k_n,j_n})$. Thus, by Corollary 4.3, we can find Haar unitaries $v_{1,n}, v_{2,n} \in \mathcal{U}(M_{\sigma(l_n)+1}) \subset \mathcal{U}(M)$ such that

$$[u_1^{k_n,l_n}, v_{1,n}] = [u_2^{k_n,l_n}, v_{2,n}] = [v_{1,n}, v_{2,n}] = 0, \text{ for every } n \in I. \tag{5.2}$$

Finally, let $v_1 = (v_{1,n}), v_2 = (v_{2,n}) \in \mathcal{U}(M^{\mathcal{U}})$. Then v_1, v_2 are Haar unitaries and as $\lim_{n\to\mathcal{U}} f(n) = +\infty$, (5.1) and (5.2) together imply that $[u_1, v_1] = [u_2, v_2] = [v_1, v_2] = 0$. $\square$

In order to prove Theorem B, we also need to find instances which guarantee that M is full. This happens if M_1 has property (T):

Proposition 5.4. *Assume that M_1 has property (T). Then M does not have property Gamma.*

Proof. Let $n \in \mathbb{N}$. Recall that $M_{n+1} = \varphi(M_n, u_1^{\sigma(n)}, u_2^{\sigma(n)})$ and $M_1 \subset M_n$. Since M_1 is a II_1 factor, we have that $M_1 \not\prec_{M_n} \{u_1^{\sigma(n)}\}''$ and $M_1 \not\prec_{M_n} \{u_2^{\sigma(n)}\}''$. By applying Corollary 4.3 we derive that $M_1' \cap M_{n+1} = M_1' \cap M_n$. Thus, we get that $M_1' \cap M_n = \mathcal{Z}(M_1) = \mathbb{C}1$. Since this holds for every $n \in \mathbb{N}$, we deduce that $M_1' \cap M = \mathbb{C}1$. Finally, since M_1 has property (T), by Proposition 2.1, we have that $M_1' \cap M^{\mathcal{U}} = (M_1' \cap M)^{\mathcal{U}} = \mathbb{C}1$, where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. Hence, $M' \cap M^{\mathcal{U}} = \mathbb{C}1$ and so M does not have property Gamma. $\square$

Proof of Theorem B. Let M_1 be a II_1 factor with property (T), e.g., take $M_1 = \mathrm{L}(\mathrm{SL}_3(\mathbb{Z}))$. Let M be constructed as in Definition 5.1. The conclusion follows from Propositions 5.3 and 5.4. $\square$

5.2. **Proof of Theorem C and its corollaries.** In this subsection, we prove that the II_1 factor M from Theorem B also satisfies the conclusion of Theorems C and A and Corollary D. To this end, we first show the following:

Corollary 5.5. *Let $p, q_1, q_2, q_3 \in M^{\mathcal{U}}$ be projections such that $q_1 + q_2 + q_3 = 1$, where $\mathcal{U}$ is a countably cofinal ultrafilter on a set I. Assume that $\{p\}'' \perp \{q_1, q_2, q_3\}''$.*

Then $h(\{p, q_1, q_2, q_3\}'', M^{\mathcal{U}}) \leq 0$.

Proof. Define $u_1, u_2 \in \mathcal{U}(M^{\mathcal{U}})$ by $u_1 = 2p - 1$ and $u_2 = q_1 + \zeta q_2 + \zeta^2 q_3$, where $\zeta = \exp(\frac{2\pi i}{3})$. Then $u_1^2 = u_2^3 = 1$ and $\{u_1, u_2\}'' = \{p, q_1, q_2, q_3\}''$. Thus, by combining Lemma 2.9 and Proposition 5.3 we get that $h(\{p, q_1, q_2, q_3\}'' : M) = h(\{u_1, u_2\}'' : M) \leq 0$. $\square$

To prove that $h(M^{\mathcal{U}}) \leq 0$, we will need an additional lemma:

Lemma 5.6. *Let (A, τ) be a diffuse tracial von Neumann algebra and $x \in A$ such that $x = x^*$ and $\tau(x) = 0$. Let $\mathcal{F}$ be the set of projections $p \in A$ such that $\tau(xp) = 0$. Then $\mathcal{F}'' = A$.*

Proof. We first prove the conclusion under the assumption that A is abelian. Let $x = y - z$ be the decomposition of x into positive and negative parts. Let q and r be the support projections of y and z, respectively. Since $0 = \tau(x) = \tau(y) - \tau(z)$, we get that $\tau(y) = \tau(z)$.

Let $e \in Aq$ be a projection. Since A is diffuse and $\tau(ye) \leq \tau(y) = \tau(z) = \tau(zr)$, we can find a projection $f \in Ar$ such that $\tau(zf) = \tau(ye)$. Then we have that $e - f \in \mathcal{F}$. Since $ef = 0$, we get that $e + f = (e - f)^2 \in \mathcal{F}''$ and thus $e \in \mathcal{F}''$, for every projection $e \in Aq$. Thus, $Aq \subset \mathcal{F}''$. Similarly, we conclude that $Ar \subset \mathcal{F}''$. Since $x(1 - q - r) = 0$, we also have that $A(1 - q - r) \subset \mathcal{F}''$. Since A is abelian, it follows that $A \subset \mathcal{F}''$ and thus $\mathcal{F}'' = A$.

For general A, let $B \subset A$ be a diffuse abelian von Neumann subalgebra. Note that $\tau(E_B(x)) = 0$ and that if $p \in B$ is a projection with $\tau(E_B(x)p) = 0$, then $\tau(xp) = \tau(E_B(x)p) = 0$ and so $p \in \mathcal{F}$. By applying the above proof to B and $E_B(x) \in B$, we conclude that $B \subset \mathcal{F}''$. Since this holds for every diffuse abelian von Neumann subalgebra $B \subset A$, we conclude that $\mathcal{F}'' = A$. $\square$

Theorem 5.7. *$h(M^{\mathcal{U}}) \leq 0$, for any ultrafilter $\mathcal{U}$ on any set I.*

Proof. If $\mathcal{U}$ is not countably cofinal, then $M^{\mathcal{U}} = M$ by [BCI17, Lemma 2.3]. Thus, if $\mathcal{V}$ is a free ultrafilter on $\mathbb{N}$, then Facts 2.3 and 2.4 give that $h(M^{\mathcal{U}}) = h(M) = h(M : M^{\mathcal{V}}) \leq h(M^{\mathcal{V}})$. This implies that in order to prove the conclusion, we may assume that $\mathcal{U}$ is countably cofinal.

Assume that $\mathcal{U}$ is a countable cofinal ultrafilter and denote $P = M^{\mathcal{U}}$. Since P is a II_1 factor, we can find a unital, trace-preserving embedding of $S := \mathrm{L}(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$ into P. Let $p, q \in S$ be two projections with $\tau(p) = \tau(q) = \frac{1}{2}$ which generate the two canonical copies of $\mathrm{L}(\mathbb{Z}/2\mathbb{Z})$ inside S.

Then $\|q(2p-1)q\|_2 = \sqrt{\tau(q(2p-1)q(2p-1))} = \frac{1}{2}$ and similarly $\|(1-q)(2p-1)(1-q)\|_2 = \frac{1}{2}$. Let $x = (1-q)(2p-1)(1-q) \in (1-q)P(1-q)$. Then $x = x^*, \tau(x) = 0$ and $x \neq 0$. We define $\mathcal{F}$ to be the set of projections $r \in (1-q)P(1-q)$ such that $\tau(xr) = 0$.

For $r \in \mathcal{F}$ we define $S_r := \{p, q, r, 1-q-r\}''$. Then $\tau((2p-1)q) = 0$, $\tau((2p-1)r) = \tau(xr) = 0$ and $\tau((2p-1)(1-q-r)) = \tau(2p-1) - \tau((2p-1)q) - \tau((2p-1)r) = 0$. Thus, $\{p\}'' \perp \{q, r, 1-q-r\}''$. Altogether, we can apply Corollary 5.5 to deduce that

$$h(S_r : P) \leq 0, \text{ for every } r \in \mathcal{F}. \tag{5.3}$$

Since $S \subset S_r$, for all $r \in \mathcal{F}$, and S is diffuse, combining Facts (2.5) and (2.6) with (5.3) we get that

$$h(\bigvee_{r \in \mathcal{F}} S_r : P) \leq 0. \tag{5.4}$$

On the other hand, by Lemma 5.6 we have that $\mathcal{F}'' = (1-q)P(1-q)$. This implies that

$$\bigvee_{r \in \mathcal{F}} S_r = S \bigvee (1-q)P(1-q). \tag{5.5}$$

Combining (5.4) and (5.5) we get $h(S \bigvee (1-q)P(1-q) : P) \leq 0$. Similarly, $h(S \bigvee qPq : P) \leq 0$. Using again that S is diffuse, Fact 2.5 implies that $h(S \bigvee qPq \bigvee (1-q)P(1-q) : P) \leq 0$. Since the projections q and $1-q$ are equivalent in S, we get that $S \bigvee qPq \bigvee (1-q)P(1-q) = P$, which implies the desired conclusion that $h(P) = h(P : P) \leq 0$. □

Although this is not needed to derive our main results, we mention an easy consequence of the previous proof which seems of independent interest:

Corollary 5.8. *Let M be a II_1 factor such that $h(M) > 0$. Let $\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Then there exists a homomorphism $\pi : \Gamma \to \mathcal{U}(M)$ such that $h(\pi(\Gamma)'' : M) > 0$.*

Proof. As M is a II_1 factor, we can find a unital, trace-preserving embedding of $S := \mathrm{L}(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})$ into M. Let $p, q \in S$ be two projections with $\tau(p) = \tau(q) = \frac{1}{2}$ which generate the two canonical copies of $\mathrm{L}(\mathbb{Z}/2\mathbb{Z})$ inside S. Since $h(M) > 0$ and $S \bigvee qMq \bigvee (1-q)M(1-q) = M$, Fact 2.5 gives that $h(S \bigvee qMq) > 0$ or $h(S \bigvee (1-q)M(1-q)) > 0$. Assume, without loss of generality, that $h(S \bigvee qMq) > 0$. Given a projection $r \in qMq$, let $S_r = \{p, r, q-r, 1-q\}''$. Since $S \bigvee qMq$ is generated by $\{S_r \mid r \in qMq \text{ projection}\}$, Fact 2.5 implies that $h(S_r : M) > 0$, for some projection $r \in qMq$. Since clearly $S_r = \pi(\Gamma)''$, for a homomorphism $\pi : \Gamma \to \mathcal{U}(M)$, the conclusion follows. □

Proof of Theorem C. Let M_1 be a II_1 factor with property (T), e.g., take $M = \mathrm{L}(\mathrm{SL}_3(\mathbb{Z}))$. Let M be constructed as in Definition 5.1. By Theorem 5.7 and Proposition 5.4 we get that $h(M^{\mathcal{U}}) \leq 0$, for every ultrafilter $\mathcal{U}$, and M does not have property Gamma. □

Proof of Theorem A. Let M be as in Theorem C. Suppose that for some ultrafilters $\mathcal{U}, \mathcal{V}$ on sets I, J, there exists an embedding of $M^{\mathcal{U}}$ into $N^{\mathcal{V}}$ that contains the diagonal inclusion of N. By combining Theorem C and Facts 2.4 and 2.3 we get the following chain of inequalities:

$$0 < h(N) = h(N : N^{\mathcal{V}}) \leq h(N : M^{\mathcal{U}}) \leq h(M^{\mathcal{U}} : M^{\mathcal{U}}) = h(M^{\mathcal{U}}) \leq 0,$$

which is a contradiction. □

Proof of Corollary D. Let M be as in Theorem C. For a sequence $(k_n) \subset \mathbb{N}$ and free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ with $\lim_{n \to \mathcal{U}} k_n = +\infty$, let $\mathcal{M} = \prod_{\mathcal{U}} \mathbb{M}_{k_n}(\mathbb{C})$. Then Fact 2.8 implies that $h(\mathcal{M}) > 0$. By Theorem A, we deduce that M is not elementarily equivalent to $\mathcal{M}$. □

The following remark was communicated to us separately by I. Goldbring and D. Jekel.

Remark 5.9. We give an explicit sentence distinguishing up to elementary equivalence any II_1 factor M satisfying the properties of Theorem B and any tracial von Neumann algebra (N, τ) with $h(N) > 0$, in particular $\mathrm{L}(\mathbb{F}_2)$. This follows readily from Lemma 3.2. For unitaries $u_1, u_2, v_1, v_2 \in M$, we define the formulae

$$\phi(u_1, u_2) = \|u_1^2 - 1\|_2 + \|u_2^3 - 1\|_2 + |\tau(u_1 u_2) - \tau(u_1)\tau(u_2)| + |\tau(u_1 u_2^2) - \tau(u_1)\tau(u_2^2)|$$

$$\psi(u_1, u_2, v_1, v_2) = \|u_1 v_1 - v_1 u_1\|_2 + \|u_2 v_2 - v_2 u_2\|_2 + \|v_1 v_2 - v_2 v_1\|_2 + \sum_{k \in \mathbb{Z} \setminus \{0\}} 2^{-k}(|\tau(v_1^k)| + |\tau(v_2^k)|).$$

Note that $\phi(u_1, u_2) = 0$ means that $u_1^2 = u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$. We also note that D. Jekel observed that Lemma 3.2 implies that the set $\{u_1, u_2 \in \mathcal{U}(M) \mid \phi(u_1, u_2) = 0\}$ is a definable set over the theory of II_1 factors.

Theorem B shows that M satisfies $\sup_{u_1, u_2 \in \mathcal{U}(M), \phi(u_1, u_2) = 0} \left(\inf_{v_1, v_2 \in \mathcal{U}(M)} \psi(u_1, u_2, v_1, v_2) \right) = 0$. In combination with Lemma 3.2, we derive the existence of a function $\delta : [0, \infty) \to [0, \infty)$ such that $\delta(0) = 0$, $\delta((0, \infty)) \subset (0, \infty)$ and for all $\varepsilon > 0$, the following implication holds for $u_1, u_2 \in \mathcal{U}(M)$: if $\phi(u_1, u_2) < \delta(\varepsilon)$, then $\phi'(u_1, u_2) := \inf_{v_1, v_2 \in \mathcal{U}(M)} \psi(u_1, u_2, v_1, v_2) < \varepsilon$. Moreover, δ is independent of the II_1 factor M, and can be taken to be continuous and strictly increasing. Then we have that $\delta(\phi'(u_1, u_2)) \leq \phi(u_1, u_2)$, for every $u_1, u_2 \in \mathcal{U}(M)$, and we can thus write the distinguishing sentence as follows:

$$\sup_{u_1, u_2 \in \mathcal{U}(M)} \max(0, \delta(\phi'(u_1, u_2)) - \phi(u_1, u_2)).$$

In fact, it is easy to see that a II_1 factor M satisfies this sentence if and only if it satisfies the conclusion of Theorem B.

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