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MANUSCRIPT

The unconditional uniqueness for the energy-supercritical NLS

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Abstract
We consider the cubic and quintic nonlinear Schrödinger equations (NLS) under the $\mathbb{R}^d$ and $\mathbb{T}^d$ energy-supercritical setting. Via a newly developed unified scheme, we prove the unconditional uniqueness for solutions to NLS at critical regularity for all dimensions. Thus, together with [19, 20], the unconditional uniqueness problems for H^1-critical and H^1-supercritical cubic and quintic NLS are completely and uniformly resolved at critical regularity for these domains. One application of our theorem is to prove that defocusing blowup solutions of the type in [59] are the only possible $C([0, T); \dot{H}^{s_c})$ solutions if exist in these domains.

Keywords Energy-supercritical NLS · Gross-Pitaevskii Hierarchy · Klainerman-Machedon Board Game · Multilinear Estimates

Mathematics Subject Classification Primary 35Q55 · 35A02 · 81V70

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1 Introduction

We consider the nonlinear Schrödinger equation (NLS)

$$\begin{cases} i\partial_t u = -\Delta u \pm |u|^{p-1}u, & (t,x) \in [0,T] \times \Lambda^d \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

where $\Lambda^d = \mathbb{R}^d$ or $\mathbb{T}^d$ and $\pm$ denotes defocusing/focusing. In Euclidean spaces, the NLS (1.1) enjoys the scaling invariance

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{1.2}$$

which preserves the homogeneous Sobolev norm $\|u_0\|_{\dot{H}^{s_c}}$ where the critical scaling exponent is given by

$$s_c := \frac{d}{2} - \frac{2}{p-1}. \tag{1.3}$$

Accordingly, the initial value problem (1.1) for $u_0 \in \dot{H}^{s_c}$ can be classified as energy subcritical, critical or supercritical depending on whether the critical Sobolev exponent s_c lies below, equal to or above the energy exponent $s = 1$.

In this paper, we focus on the cubic and quintic cases under the energy-supercritical setting ($s_c > 1$) where

$$s_c = \begin{cases} \frac{d-2}{2} & for\ d \geq 5,\ p = 3, \\ \frac{d-1}{2} & for\ d \geq 4,\ p = 5. \end{cases} \tag{1.4}$$

In the energy-supercritical setting, the global well-posedness of (1.1) is fully open, away from the classical local well-posedness and L^2-supercritical blowup results [5, 30]. But it has been, for a long time, believed that, even under the energy-supercritical setting, the defocusing version of (1.1) is globally well-posed and the solution scatters when $\Lambda = \mathbb{R}$, just like the energy-critical and subcritical cases, especially after the breakthrough [3, 22, 32, 45, 62][1] on the $\mathbb{R}^d$ energy-critical cubic and quintic cases. (See, for example [47].) Surprisingly, the recent work [59] unexpectedly constructed the first instance of finite time blowup solution for the defocusing energy-supercritical NLS. Thus it is of interest to know if there could exist a scattering global solution in $\dot{H}^{s_c}$ but may not be in $C([0,T);\dot{H}^{s_c})\bigcap L_{t,x}^{p(d+2)/2}$ when blowups of this type exist.

There are certainly multiple routes for such a problem. But one way is the classical unconditional uniqueness theorem in $\dot{H}^{s_c}$ which itself has remained open at least for $\mathbb{T}^d$. With an unconditional uniqueness result, we know that there could be at most one solution in $C([0,T);\dot{H}^{s_c})$ regardless of auxiliary spaces. One application is to prove that blowup solutions of the type in [59] is the only possible $C([0,T);\dot{H}^{s_c})$ solution if exist in these domains. In this paper, we prove the $\dot{H}^{s_c}$ unconditional uniqueness for (1.1) as follows and address this issue.

Theorem 1.1 [2] *Let $s_c > 1$ and $p = 3$ or 5. There is at most one $C([0,T_0];\dot{H}^{s_c}(\Lambda^d))$[3] solution to (1.1).*

The fundamental concept of unconditional uniqueness was first raised by Kato in [43, 44] when proving well-posedness in Strichartz type spaces had made vast progress. In $\mathbb{R}^d$, these unconditional uniqueness problems at critical regularity are usually proved by showing any solution must agree with the Strichartz solution, if exists, using the inhomogeneous (retarded) Strichartz estimate. Such a method has been proven to be successful even in the $\mathbb{R}^3$ quintic energy-critical case, see for example [22]. (This is a very active field, see for example [1, 29, 34, 50, 54, 60, 61, 69] and the reference within for work on other dispersive equations along this line.)

However, such arguments for the Euclidean setting are no longer effective if (1.1) is posed on $\mathbb{T}^d$, as the Strichartz estimate is rather weak in the periodic case. The L_x^2 Strichartz estimate does not hold in the periodic case and hence the dual Strichartz estimate also fails. On the other hand, the well-posedness on $\mathbb{T}^d$ is more intricate, such as using the $X_{s,b}$ space [2] and the atomic U^p and V^p spaces [37, 42]. Thus the unconditional uniqueness problems on $\mathbb{T}^d$ under the critical setting are much more difficult to handle. Nevertheless, a unified method has recently unexpectedly arisen from the study of the derivation of (1.1) on the $\mathbb{T}^d$ case in [36] and under the energy-critical setting in [19, 20].[4]

We find that one could use the scheme of [20] to perfectly solve the unconditional uniqueness problem under the energy-supercritical setting for both $\mathbb{R}^d$ and $\mathbb{T}^d$. The proof comes from the Gross-Pitaevskii(GP) hierarchy, which seems to be weaker than the NLS analysis, as it originates from the derivation of NLS. However, we will see that such an argument is also powerful and worthy for further study. Here, we focus on

[1] See [23] for a more detailed survey.

[2] One could extend the domain Λ^d to more general manifolds, as long as the multilinear estimates which relies on Fourier analysis and Strichartz estimates in Section 5 hold.

[3] We consider H^{s_c} for the $\mathbb{T}^d$ case and $\dot{H}^{s_c}$ for the $\mathbb{R}^d$ case as $\dot{H}^{s_c}$ does not generate much differences for the $\mathbb{T}^d$ case.

[4] We mention [36] 1st here and in the related places in the rest of the paper. Even though [19] was posted on arXiv one month before [36], X. Chen and Holmer were not aware of the unconditional uniqueness implication of [19] until [36].

the quintic GP hierarchy, also see [20] for the cubic case. The quintic GP hierarchy is a sequence $\{\gamma^{(k)}(t)\}_{k=1}^{\infty}$ which satisfies the infinitely coupled hierarchy of equations:

$$i\partial_t\gamma^{(k)} = \sum_{j=1}^{k}[-\Delta_{x_j}, \gamma^{(k)}] \pm b_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1,k+2}[\delta(x_j - x_{k+1})\delta(x_j - x_{k+2}), \gamma^{(k+2)}] \tag{1.5}$$

where b_0 is some coupling constant, $\pm$ denotes defocusing/focusing. Given any solution u of (1.1), it generates a solution to (1.5) by letting

$$\gamma^{(k)} = |u\rangle\langle u|^{\otimes k} \tag{1.6}$$

in operator form or

$$\gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^{k} u(t, x_j)\overline{u}(t, x'_j)$$

in kernel form where $\mathbf{x}_k = (x_1, ..., x_k)$.

The hierarchy approach was first suggested by Spohn [67] for the derivation of NLS from quantum many-body dynamic. Around 2005, it was Erdös, Schlein, and Yau who first rigorously derived the 3D cubic defocusing NLS from a 3D quantum many-body dynamic in their fundamental papers [24–28]. The proof for the uniqueness of the GP hierarchy was the principal part and also surprisingly dedicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. With a sophisticated Feynman graph analysis in [25], they proved the H^1-type unconditional uniqueness of the $\mathbb{R}^3$ cubic GP hierarchy. The first series of ground breaking papers have motivated a large amount of work.

Subsequently in 2007, with imposing an additional a-prior condition on space-time norm, Klainerman and Machedon [52], inspired by [25, 51], gave an another proof of the uniqueness of the GP hierarchy in a different space of density matrices defined by Strichartz type norms. They provided a different combinatorial argument, the now so-called Klainerman-Machedon (KM) board game argument, to combine the inhomogeneous terms effectively reducing their numbers and then derived a space-time estimate to control these terms. At that time, it was open to prove that the limits coming from the N-body dynamics satisfy the space-time bound. Nonetheless, [52] has made the delicate analysis of the GP hierarchy approachable from the perspective of PDE. Later, Kirkpatrick, Schlein, and Staffilani [48] obtained the KM space-time bound via a simple trace theorem in both $\mathbb{R}^2$ and $\mathbb{T}^2$ and derived the 2D cubic defocusing NLS from the 2D quantum many-body dynamic. Such a scheme also motivated many works [8, 13, 16, 18, 31, 63, 65, 68] for the uniqueness of GP hierarchies.

Later in 2008, T. Chen and Pavlović [8] initiated the study of the quintic GP hierarchy and provided a proof for the quintic KM board game argument, which laid the foundation for the further study of the quintic GP hierarchy. They also showed that the 2D quintic case, which is usually considered the same as the 3D cubic case since they share the same scaling criticality, satisfied the KM space-time bound while it was still open for the 3D cubic case at that time. To attack the problem, they also considered the well-posedness theory with more general data in [7, 9, 11]. (See also [12, 56–58, 65, 66]). Then in 2011, they proved that the 3D cubic KM space-time bound holds for the defocusing $\beta < 1/4$ case in [10]. The result was quickly improved to $\beta < 2/7$ by X. Chen in [14] and then extended to the almost optimal case, $\beta < 1$, by X. Chen and Holmer in [15, 17]. Around the same period of time, Gressman, Sohinger, and Staffilani [31] studied the uniqueness of the GP hierarchy on $\mathbb{T}^3$ and proved that the sharp space-time estimate on $\mathbb{T}^3$ needed an additional ε derivatives than the $\mathbb{R}^3$ setting in which one derivative is needed. Later, Herr and Sohinger generalized this fact to more general cases in [35].

In 2013, by introducing quantum de Finetti theorem from [55], T. Chen, Hainzl, Pavlović and Seiringer [6] provided a simplified proof of the $L_T^\infty H_x^1$-type 3D cubic uniqueness theorem in [25]. With the quantum de Finetti theorem, one can replace the space-time estimates by Sobolev multilinear estimates. The scheme in [6], which consists of the KM board game argument, the quantum de Finetti theorem and the Sobolev multilinear estimates, is robust to deal with such uniqueness problems. Following the scheme in [6], Sohinger [64] solved the aforementioned ε-loss problem for the defocusing $\mathbb{T}^3$ cubic case. In [40], Hong, Taliaferro, and Xie used the scheme to obtain unconditional uniqueness theorems in $\mathbb{R}^d$, $d = 1, 2, 3$, with regularities matching the NLS analysis. Then in [41], they proved H^1 small solution uniqueness for the $\mathbb{R}^3$ quintic case. For other refined uniqueness theorems, see also [21].

The uniqueness analysis of GP hierarchy started to unexpectedly yield new NLS results with regularity lower than the NLS analysis all of a sudden since [36] and [19, 20]. In [36], with the scheme in [6], Herr and Sohinger discovered new unconditional uniqueness results for the cubic NLS on $\mathbb{T}^d$, which covered the full scaling-subcritical regime for $d \geq 4$. (See also the later work [49] using NLS analysis.)

On the other hand, the $\mathbb{T}^3$ quintic energy-critical case at H^1 regularity was not known until recently [19]. By discovering the new hierarchical uniform frequency localization (HUFL) property for the GP hierarchy, X. Chen and Holmer established a new H^1-type uniqueness theorem for the $\mathbb{T}^3$ quintic energy-critical GP hierarchy. The new uniqueness theorem, though neither conditional nor unconditional for the GP hierarchy implies the H^1 unconditional uniqueness result for the $\mathbb{T}^3$ quintic energy-critical NLS. Then in [20], they proved the unconditional uniqueness for the $\mathbb{T}^4$ cubic energy-critical case by working out new combinatorics and extending the KM board game argument. As the previously used Sobolev multilinear estimates fail on $\mathbb{T}^4$, they develop the new combinatorics which enable the application of U-V multilinear estimates, which is indeed weaker than Sobolev multilinear estimates. The scheme in [20], which effectively combines the quantum de Finetti theorem, the U-V space techniques, the multilinear estimates proved by using the scale invariant Strichartz estimates / l^2-decoupling theorem and the HUFL properties, provides a unified proof of the large solution uniqueness.

2 Proof of the Main Theorem

2.1 Outline of the Proof

Our proof will focus on the $\mathbb{T}^d$ case, as it works the same for $\mathbb{R}^d$.[5] Our argument follows the scheme of [20] where an extended version of KM board game argument which is compatible with U-V estimates was discovered. We summarize our proof below, especially for the quintic case.

To conclude the uniqueness for NLS (1.1), one usually proves that

$$w(t) = u_1(t,x) - u_2(t,x) \equiv 0$$

where u_1 and u_2 are two solutions to (1.1) with the same initial datum. Instead, we turn to prove that

$$\gamma^{(k)}(t) := \prod_{j=1}^{k} u_1(t,x_j)\overline{u}_1(t,x_j') - \prod_{j=1}^{k} u_2(t,x_j)\overline{u}_2(t,x_j'), \tag{2.1}$$

which is a solution to (1.5) with zero initial datum, vanishes identically on $[0, T_0]$. The formulation (2.1) endows the NLS (1.1) with an extra linear structure via the GP hierarchy so that one could iteratively use multilinear estimates to yield smallness, instead of constructing a closed inequality in some Strichartz space.

Hence, we first prove Theorem 2.2, which is a uniqueness theorem for the GP hierarchy, and then Theorem 1.1 comes as a corollary of Theorem 2.2 and Lemma 2.7. As Theorem 2.2 requires the uniform in time frequency localization (UTFL) condition, we prove that every $C([0,T_0]; H^{s_c})$ solution to (1.1) satisfies UTFL condition by Lemma 2.7. Thus we would have established Theorem 1.1 once we have proved Theorem 2.2.

The GP hierarchy argument does not require the dual Strichartz estimate or the existence of a Strichartz solution. However, we have to carefully combine and estimate the $(2k-1)!!2^k$ summands in iterated Duhamel expansions. More precisely,

$$\gamma^{(1)}(t_1) = \sum_{(\mu,sgn)} \int_{t_1 \geq t_3 \geq \cdots \geq t_{2k+1}} J^{(2k+1)}_{\mu,sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \tag{2.2}$$

with $\underline{t}_{2k+1} = (t_3, t_5, ..., t_{2k+1})$ and

$$\begin{aligned} J^{(2k+1)}_{\mu,sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) = {} & U^{(1)}(t_1 - t_3) B^{sgn(2)}_{\mu(2);2,3} U^{(3)}(t_3 - t_5) B^{sgn(4)}_{\mu(4);4,5} \\ & \cdots U^{(2k-1)}(t_{2k-1} - t_{2k+1}) B^{sgn(2k)}_{\mu(2k);2k,2k+1} \gamma^{(2k+1)}(t_{2k+1}) \end{aligned} \tag{2.3}$$

where $U^{(2j+1)}(t)$ is the propagator and $B^{\pm}_{i;2j,2j+1}$ is the collapsing operator, and thus there are $(2k-1)!!2^k$ terms in $\gamma^{(1)}(t_1)$. Hence handling the $(2k-1)!!2^k$ terms in the

[5] By using the classical methods, we also give a more usual proof for the $\mathbb{R}^d$ case at the appendix.

critical setting is the main difficulty. Now, we divide the proof of Theorem 2.2 into two main parts.

Part 1: The Estimate Part. Usually, one employs the original KM board game argument to sort the $(2k-1)!!2^k$ summands of $\gamma^{(1)}$ into a sum of KM upper echelon forms with a time integration domain, which is a union of a very large number of high dimensional simplexes. As Sobolev type multilinear estimates work regardless of the time integration domain, one can iteratively use them to yield smallness. Nevertheless, if we have some combinatorics which is compatible with space-time type multilinear estimates, we could exploit the multilinear estimates in U-V spaces. Indeed, based on the combinatorics part (Part 2), we can write

$$\gamma^{(1)}(t_1) = \sum_{reference\ (\hat{\mu}, s\hat{g}n)} \int_{T_C(\hat{\mu}, s\hat{g}n)} J^{(2k+1)}_{\hat{\mu}, s\hat{g}n}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \tag{2.4}$$

where the number of reference pairs in Definition 4.22 can be controlled by 16^k, which is substantially smaller than the original $(2k-1)!!2^k$. More importantly, the time integration domain $T_C(\hat{\mu}, s\hat{g}n)$ is compatible with space-time type multilinear estimates. Hence it comes down to how to estimate

$$\int_{T_C(\mu, sgn)} J^{(2k+1)}_{\mu, sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}. \tag{2.5}$$

In Section 3.1, we start with a Duhamel tree to represent the Duhamel expansion. Then in Section 3.2, we introduce the time integration domain $T_C(\mu, sgn)$ which is compatible with space-time type multilinear estimates. Subsequently in Section 3.3, after giving a short introduction to U-V spaces, we show how to apply the U-V multilinear estimates to estimate (2.5). We will use the following U-V multilinear estimates.

$$\left\| \int_a^t e^{i(t-s)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5) ds \right\|_{X^s} \leq C\|u_1\|_{X^s}\|u_2\|_{X^{s_c}}\|u_3\|_{X^{s_c}}\|u_4\|_{X^{s_c}}\|u_5\|_{X^{s_c}}, \tag{2.6}$$

$$\begin{aligned} &\left\| \int_a^t e^{i(t-\tau)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5) d\tau \right\|_{X^s} \\ &\leq C\|u_1\|_{X^s}\left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{3(d+3)}} \|P_{\leq M_0}u_2\|_{X^{s_c}} + \|P_{>M_0}u_2\|_{X^{s_c}} \right) \|u_3\|_{X^{s_c}}\|u_4\|_{X^{s_c}}\|u_5\|_{X^{s_c}}, \end{aligned} \tag{2.7}$$

where $\widetilde{u} \in \{u, \overline{u}\}$ and $s \in \{s_c, s_c - 2\}$. The proof highly relies on the scale invariant Strichartz estimates / l^2-decoupling theorem [4, 46] and hence is postponed to Section 5. Compared with Sobolev multilinear estimates, the proof of U-V multilinear estimates is simpler and less technical. (Although the representation (2.4) is also compatible with $X_{s,b}$ multilinear estimates, they need an additional ε derivatives in time and hence cannot be used to deal with the critical problem.) On the one hand, to prove Sobolev multilinear estimates, the $L^\infty_T H^{-s}$ space, which is usually used in the duality argument, is an endpoint case in Littlewood-Paley theory and does not carry any Strichartz regularity. On the other hand, U-V techniques have been proven to be successful and adaptive for NLS in many different general domains.

Together with assumptions in Theorem 2.2, we are able to prove the following key estimate.

$$\left\| \langle \nabla_{x_1} \rangle^{s_c-2} \langle \nabla_{x_1'} \rangle^{s_c-2} \int_{T_C(\hat{\mu}, s\hat{g}n)} J^{(2k+1)}_{\hat{\mu}, s\hat{g}n}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \right\|_{L^\infty_T L^2_{x_1} L^2_{x_1'}} \leq \delta^k \tag{2.8}$$

where $\delta(T, \varepsilon, C_0, M_0)$ can be sufficiently small by properly choosing these parameters. More specifically, the smallness comes from the UTFL property, that is,

$$\| \langle \nabla \rangle^{s_c} P_{>M(\varepsilon)} u \|_{L^\infty_{[0,T_0]} L^2_x} \leq \varepsilon.$$

By iteratively using (2.7) at least $\frac{4}{5}k$ times, we obtain the factor of smallness as follows

$$\left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} C_0 + \varepsilon \right)^{\frac{4}{5}k}.$$

Thus, we are left to prove the representation (2.4), especially, the compatibility part.

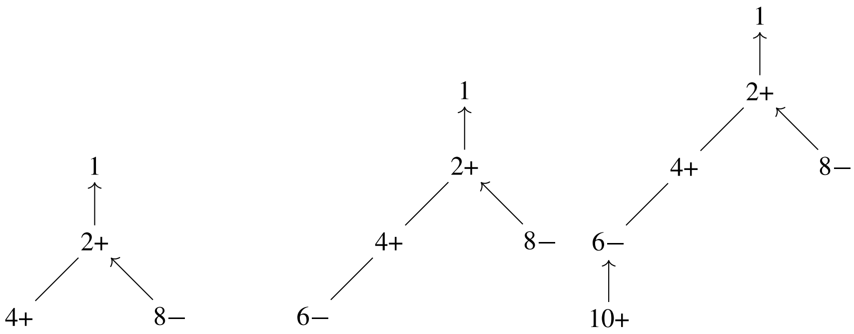
Part 2: The Combinatorics Part. In Section 4, by working out new combinatorics to reconstruct the quintic KM board game argument from the ground up, we could represent $\gamma^{(1)}$ in the form of (2.4), which is compatible with the U-V multilinear estimates. The combinatorics analysis is independent of the multilinear estimates or the regularity settings, so it could be applied for more general cases. Such a representation (2.4), which enables the application of U-V multilinear estimates, would also be helpful for further study of GP hierarchy.

In Section 4.1, we first give a brief review of the quintic KM board game argument as in [8, 52]. Then, we give an introduction to an admissible tree diagram representation used to represent collapsing map pairs. For example[6], given a collapsing map pair (μ_1, sgn_1) as follows,

$2j$	2	4	6	8	10
μ_1	1	1	1	3	6
sgn_1	+	+	−	−	+

we generate the following trees in turn by Algorithm 6.

6 We will not use this example again in the paper, as we can generate as many as we want.



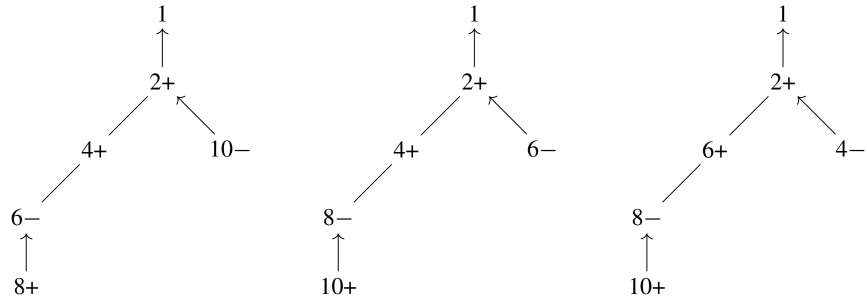
Such a tree diagram reprentation could provide a more elaborated proof of the original quintic KM board game argument as well.

In Section 4.2, we give the signed Klainerman-Machedon acceptable moves in Chen-Pavlović format (signed acceptable moves), which sorts $(2k-1)!!2^k$ collapsing map pairs (μ, sgn) into various equivalence classes, the number of which can be controlled by 16^k. Moreover, the signed acceptable moves preserve the signed tree structures. Here are all the collapsing map pairs and the corresponding trees equivalent to (μ_1, sgn_1).

$2j$	2	4	6	8	10
μ_2	1	1	1	6	3
sgn_2	+	+	−	+	−

$2j$	2	4	6	8	10
μ_3	1	1	3	1	8
sgn_3	+	+	−	−	+

$2j$	2	4	6	8	10
μ_4	1	3	1	1	8
sgn_4	+	−	+	−	+



(Notice that the above trees have the same skeleton.)

However, extending to signed move is not sufficient for our proposes. To be compatible with the U-V multilinear estimates, we have to further combine the integrals and enlarge the time integration domain. For this purpose, we need the wild moves defined by (4.18). The wild moves, unlike the signed acceptable moves, do change the tree structure. However, KM upper echelon forms are not invariant under the wild moves.

Thus in Section 4.3, we prove that there exists a unique special form, for which we call the tamed form, in each equivalent class and hence arrive at

$$\gamma^{(1)}(t_1) = \sum_{(\mu_*, sgn_*) \text{ tamed}} \int_{T_D(\mu_*)} J^{(2k+1)}_{\mu_*, sgn_*}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \tag{2.9}$$

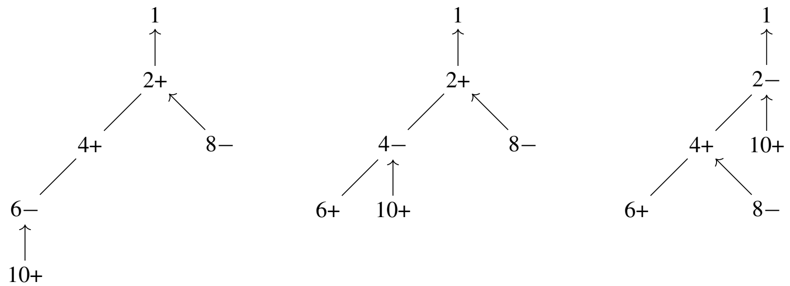
where $T_D(\mu_*)$ can be read out from the corresponding tree. Here, (μ_1, sgn_1) is the unique tamed form in the equivalent class $\{(\mu_i, sgn_i)\}_{i=1}^4$.

Subsequently in Section 4.4,we exploit wild moves for a further reduction of (2.9), which keeps the tamed form invariant, to sort the tamed forms into tamed classes. All the tamed forms and trees wildly relatable to (μ_1, sgn_1) are shown as follows.

$2j$	2	4	6	8	10
μ_1	1	1	1	3	6
sgn_1	+	+	−	−	+

$2j$	2	4	6	8	10
μ_5	1	1	1	3	4
sgn_5	+	−	+	−	+

$2j$	2	4	6	8	10
μ_6	1	1	1	5	2
sgn_6	−	+	+	−	+



(Notice the changes on the tree structures under the wild move.)

Finally, in Section 4.5, we prove that, given a tamed class, there exists a unique reference form representing the tamed class, and the time integration domain for the whole tamed class can be directly read out from the reference form. Moreover, the time integration domain is compatible. For instance, as (μ_1, sgn_1) is also the unique reference form, we could directly read $T_C(\mu_1, sgn_1)$ out as follows

$$T_C(\mu_1, sgn_1) = \{t_1 \geq t_3, t_3 \geq t_5, t_7 \geq t_{11}, t_3 \geq t_9\},$$

which is indeed compatible with U-V multilinear estimates. (See Example 3.7 on how to estimate.) Hence, we arrive at the representation (2.4).

At the end, we have justified the representation (2.4) and the key estimate (2.8) is now valid. Hence, we have

$$\left\| \langle \nabla_{x_1} \rangle^{s_c - 2} \langle \nabla_{x_1'} \rangle^{s_c - 2} \gamma^{(1)}(t_1) \right\|_{L_T^\infty L_{x_1}^2 L_{x_1'}^2} \leq (16\delta)^k \to 0 \text{ as } k \to \infty,$$

which implies that $\gamma^{(1)} \equiv 0$ which finishes the proof of Theorem 2.2.

To sum up, we prove unconditional uniqueness for solutions to the $\mathbb{R}^d$ and $\mathbb{T}^d$ energy-supercritical cubic and quintic NLS at critical regularity for all dimensions via a newly developed unified scheme. Thus, together with [19, 20], the unconditional uniqueness problems for H^1-critical and H^1-supercritical cubic and quintic NLS are completely and uniformly resolved at critical regularity for these domains. The novelty of this paper is that our procedure works uniformly in all dimensions regardless of the domain.

2.2 The Uniqueness for GP Hierarchy

In this section, we first prove Theorem 2.2, which is a uniqueness theorem for the GP hierarchy. Theorem 1.1 then comes as a corollary of Theorem 2.2 and Lemma 2.7. Here, we consider the $\mathbb{T}^d$ case with the inhomogeneous norm H^{s_c}, as the homogeneous norm $\dot{H}^{s_c}$ is special for the $\mathbb{R}^d$ case.

Definition 2.1 ([6]) A nonnegative trace class symmetric operator sequence $\Gamma = \left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$, is called admissible if for all k, one has

$$\operatorname{Tr} \gamma^{(k)} = 1, \ \gamma^{(k)} = \operatorname{Tr}_{k+1} \gamma^{(k+1)}. \tag{2.10}$$

Here, a trace class operator is called symmetric, if, written in kernel form

$$\gamma^{(k)}(x_1, ..., x_k; x_1', ..., x_k') = \overline{\gamma^{(k)}(x_1', ..., x_k'; x_1, ..., x_k)},$$
$$\gamma^{(k)}(x_{\pi(1)}, ..., x_{\pi(k)}; x_{\pi(1)}', ..., x_{\pi(k)}') = \gamma^{(k)}(x_1, ..., x_k; x_1', ..., x_k'),$$

for all permutations π on $\{1, 2, ..., k\}$. Let $\mathcal{H}^s \equiv \mathcal{H}^s(\Lambda^d)$ denote the set of all symmetric operator sequences $\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ of density matrices such that, for each $k \in \mathbb{N}$

$$\left(\prod_{j=1}^{k} \langle \nabla_{x_j} \rangle^{s_c}\right) \gamma^{(k)} \left(\prod_{j=1}^{k} \langle \nabla_{x_j} \rangle^{s_c}\right) \in \mathcal{L}_k^1,$$

where $\mathcal{L}_k^1$ denotes the space of trace class on $L^2(\Lambda^{dk} \times \Lambda^{dk})$.

Theorem 2.2 *Let* $\Gamma = \left\{\gamma^{(k)}\right\} \in \bigoplus_{k \geq 1} C([0, T_0]; \mathcal{H}_k^{s_c})$ *be a solution to (1.5) in* $[0, T_0]$ *in the sense that*

(1) Γ *is admissible in the sense of Definition 2.1.*
(2) Γ *satisfies the kinetic energy condition that* $\exists\, C_0 > 0$ *such that*

$$\sup_{t \in [0, T_0]} \operatorname{Tr} \left(\prod_{j=1}^{k} \langle \nabla_{x_j} \rangle^{s_c}\right) \gamma^{(k)}(t) \left(\prod_{j=1}^{k} \langle \nabla_{x_j} \rangle^{s_c}\right) \leq C_0^{2k}. \tag{2.11}$$

Then there is a threshold $\eta(C_0) > 0$ *such that the solution is unique in* $[0, T_0]$ *provided that*

$$\sup_{t\in[0,T_0]} \operatorname{Tr}\left(\prod_{j=1}^{k} P_{>M}^{j}\langle\nabla_{x_j}\rangle^{s_c}\right)\gamma^{(k)}(t)\left(\prod_{j=1}^{k} P_{>M}^{j}\langle\nabla_{x_j}\rangle^{s_c}\right) \leq \eta^{2k} \tag{2.12}$$

for some frequency M*, which is allowed to depend on* $\gamma^{(k)}$ *but must apply uniformly on* $[0, T_0]$*. Here,* $P_{>M}^{j}$ *is the projection onto frequencies* $> M$ *acting on functions of* x_j*.*

By letting $\gamma^{(k)} = |u\rangle\langle u|^{\otimes k}$, we could obtain Corollary 2.3, which is a special case of Theorem 2.2 as follows.

Corollary 2.3 *Given an initial datum* $u_0 \in H^{s_c}$*, there is at most one* $C([0, T_0]; H^{s_c})$ *solution to (1.1) satisfying the following conditions*

(1) There is a $C_0 > 0$ *such that*

$$\sup_{t\in[0,T_0]} \|u(t)\|_{H^{s_c}} \leq C_0$$

(2) There is some frequency M *such that*

$$\sup_{t\in[0,T_0]} \|P_{>M}u(t)\|_{H^{s_c}} \leq \eta(C_0)$$

for the threshold $\eta(C_0) > 0$ *concluded in Theorem 2.2.*

Before the proof, we set up some notations. We rewrite $\gamma^{(k)}$ in Duhamel form

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) \mp i\int_0^t U^{(k)}(t-s)B^{(k+2)}\gamma^{(k+2)}(s)ds \tag{2.13}$$

where $U^{(k)}(t) = \prod_{j=1}^{k} e^{it(\Delta_{x_j}-\Delta_{x'_j})}$ and

$$\begin{aligned}
&B^{(k+2)}\gamma^{(k+2)}\\
&\quad= \sum_{j=1}^{k} B_{j;k+1,k+2}\gamma^{(k+2)}\\
&\quad= \sum_{j=1}^{k}\left(B^{+}_{j;k+1,k+2} - B^{-}_{j;k+1,k+2}\right)\gamma^{(k+2)}\\
&\quad= \sum_{j=1}^{k} \operatorname{Tr}_{k+1,k+2}\Big(\delta(x_j - x_{k+1})\delta(x_j - x_{k+2})\gamma^{(k+2)}\\
&\qquad -\gamma^{(k+2)}\delta(x_j - x_{k+1})\delta(x_j - x_{k+2})\Big).
\end{aligned}$$

In the above, products are interpreted as the compositions of operators. For example, in kernels,

$$\begin{aligned}&\mathrm{Tr}_{k+1,k+2}\left(\gamma^{(k+2)}\delta(x_j-x_{k+1})\delta(x_j-x_{k+2})\right)(\mathbf{x}_k;\mathbf{x}_k')\\&=\int\gamma^{(k+2)}(\mathbf{x}_k,x_{k+1}',x_{k+2}';\mathbf{x}_k',x_{k+1}',x_{k+2}')\delta(x_j'-x_{k+1}')\delta(x_j'-x_{k+2}')dx_{k+1}'dx_{k+2}'\end{aligned}$$

where $\mathbf{x}_k=(x_1,...,x_k)$.

We will prove that if $\Gamma_1=\left\{\gamma_1^{(k)}\right\}_{k=1}^{\infty}$ and $\Gamma_2=\left\{\gamma_2^{(k)}\right\}_{k=1}^{\infty}$ are two solutions to (2.13), with the same initial datum and assumptions in Theorem 2.2, then $\Gamma=\left\{\gamma^{(k)}=\gamma_1^{(k)}-\gamma_2^{(k)}\right\}$ is identically zero. We start using a representation of Γ given by the quantum de Finetti theorem.

Lemma 2.4 (quantum de Finetti Theorem [6, 55])[7] *Let $\{\gamma^{(k)}\}_{k=1}^{\infty}$ be admissible. Then there exists a probability measure $d\mu_t(\phi)$ supported on the unit sphere of $L^2(\Lambda^d)$ such that*

$$\gamma^{(k)}(t)=\int|\phi\rangle\langle\phi|^{\otimes k}d\mu_t(\phi).$$

By Lemma 2.4, there exist $d\mu_{1,t}$ and $d\mu_{2,t}$ representing the two solutions Γ_1 and Γ_2. Then the Chebyshev argument as in [6,Lemma 4.5] turns the assumptions in Theorem 2.2 to the support property that $d\mu_{j,t}$ is supported in the set

$$S=\left\{\phi\in\mathbb{S}(L^2(\Lambda^d)):\|P_{>M}\langle\nabla\rangle^{s_c}\phi\|_{L^2}\leq\varepsilon\right\}\bigcap\left\{\phi\in\mathbb{S}(L^2(\Lambda^d)):\|\phi\|_{H^{s_c}}\leq C_0\right\}. \tag{2.14}$$

Let the signed measure $d\mu_t=d\mu_{1,t}-d\mu_{2,t}$, we have

$$\gamma^{(k)}(t)=\left(\gamma_1^{(k)}-\gamma_2^{(k)}\right)(t)=\int|\phi\rangle\langle\phi|^{\otimes k}d\mu_t(\phi) \tag{2.15}$$

and $d\mu_t$ is supported in the set S defined in (2.14).

It suffices to prove $\gamma^{(1)}(t)=0$ as the proof is the same for the general k case. For notational convenience, we set the $\pm i$ in (2.13) to be 1. Since (2.13) is linear, Γ is a solution to (2.13) with zero initial datum. Thus after iterating (2.13) k times, we can write

$$\gamma^{(1)}(t_1)=\int_0^{t_1}\int_0^{t_3}\cdots\int_0^{t_{2k-1}}J^{(2k+1)}(\gamma^{(2k+1)})(t_1,\underline{t}_{2k+1})d\underline{t}_{2k+1} \tag{2.16}$$

where

[7] We in fact do not need Lemma 2.4 to prove Theorem 1.1, but it is the origin of the ideas of this proof.

$$
\begin{aligned}
&J^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) \\
&\quad = U^{(1)}(t_1 - t_3) B^{(3)} U^{(3)}(t_3 - t_5) B^{(5)} \\
&\qquad \cdots U^{(2k-1)}(t_{2k-1} - t_{2k+1}) B^{(2k+1)} \gamma^{(2k+1)}(t_{2k+1})
\end{aligned} \tag{2.17}
$$

with $\underline{t}_{2k+1} = (t_3, t_5, ..., t_{2k+1})$.

We notice that there are $(2k-1)!!2^k$ summands inside $\gamma^{(1)}(t_1)$. Exactly,

$$
\gamma^{(1)}(t_1) = \sum_{(\mu, sgn)} \int_{t_1 \geq t_3 \geq \cdots \geq t_{2k+1}} J^{(2k+1)}_{\mu, sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}, \tag{2.18}
$$

where

$$
\begin{aligned}
J^{(2k+1)}_{\mu, sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) =& U^{(1)}(t_1 - t_3) B^{sgn(2)}_{\mu(2);2,3} U^{(3)}(t_3 - t_5) B^{sgn(4)}_{\mu(4);4,5} \\
& \cdots U^{(2k-1)}(t_{2k-1} - t_{2k+1}) B^{sgn(2k)}_{\mu(2k);2k,2k+1} \gamma^{(2k+1)}(t_{2k+1})
\end{aligned} \tag{2.19}
$$

with sgn meaning the signature array $(sgn(2), sgn(4), ..., sgn(2k))$ and $B^{sgn(2j)}_{\mu(2j);2j,2j+1}$ stands for $B^{+}_{\mu(2j);2j,2j+1}$ or $B^{-}_{\mu(2j);2j,2j+1}$ depending on the sign of the $2j$-th signature element. Here, $\{\mu\}$ is a set of maps from $\{2, 4, ..., 2k\}$ to $\{1, 2, ..., 2k-1\}$ satisfying that $\mu(2) = 1$ and $\mu(2j) < 2j$ for all j.

Now, we get into the proof, which spans Section 3-Section 5, so we first state two propositions which are the main results in Section 3 and Section 4.

In Section 3, the main result is the following proposition.

Proposition 2.5 *Let $\gamma^{(k)}(t) = \int |\phi\rangle\langle\phi|^{\otimes k} d\mu_t(\phi)$. Then we have*

$$
\begin{aligned}
&\left\| \langle \nabla_{x_1} \rangle^{s_c - 2} \langle \nabla_{x_1'} \rangle^{s_c - 2} \int_{T_C(\mu, sgn)} J^{(2k+1)}_{\mu, sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \right\|_{L^\infty_T L^2_{x_1} L^2_{x_1'}} \\
&\quad \leq C^k \int_0^T \int \|\phi\|^{\frac{16}{5}k+2}_{H^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}} \right)^{\frac{4k}{5}} \\
&\quad d|\mu_{t_{2k+1}}|(\phi) dt_{2k+1}
\end{aligned}
$$

where $T(\mu, sgn)$ is the compatible time integration domain defined by (3.2).

In Section 4, our goal is to represent $\gamma^{(1)}$ as follows.

Proposition 2.6 *We can write*

$$
\gamma^{(1)}(t_1) = \sum_{reference\ (\hat{\mu}, \hat{sgn})} \int_{T_C(\hat{\mu}, \hat{sgn})} J^{(2k+1)}_{\hat{\mu}, \hat{sgn}}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \tag{2.20}
$$

where the number of reference pairs in Definition 4.22 can be controlled by 16^k, which is substantially smaller than the original $(2k-1)!!2^k$. More importantly, the summands *are endowed with the time integration domain $T_C(\hat{\mu}, s\hat{g}n)$, which is compatible with the estimate part in Section 3.*

With the above two propositions, we could complete the proofs of Theorem 2.2 and Corollary 2.3.

Proof of Theorem 2.2 By Proposition 2.6, we can write

$$\gamma^{(1)}(t_1) = \sum_{reference\ (\hat{\mu}, s\hat{g}n)} \int_{T_C(\hat{\mu}, s\hat{g}n)} J^{(2k+1)}_{\hat{\mu}, s\hat{g}n}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}$$

where the number of reference pairs can be controlled by 16^k. Then we have

$$\begin{aligned}
&\left\| \langle \nabla_{x_1} \rangle^{s_c-2} \langle \nabla_{x_1'} \rangle^{s_c-2} \gamma^{(1)}(t_1) \right\|_{L_T^\infty L_{x_1}^2 L_{x_1'}^2} \\
&\leq \sum_{reference\ (\hat{\mu}, s\hat{g}n)} \Big\| \langle \nabla_{x_1} \rangle^{s_c-2} \langle \nabla_{x_1'} \rangle^{s_c-2} \int_{T_C(\hat{\mu}, s\hat{g}n)} J^{(2k+1)}_{\hat{\mu}, s\hat{g}n}(\gamma^{(2k+1)}) \\
&\quad (t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \Big\|_{L_T^\infty L_{x_1}^2 L_{x_1'}^2}
\end{aligned}$$

By Proposition 2.5,

$$\begin{aligned}
&\leq (16C)^k \int_0^T \int \|\phi\|_{H^{s_c}}^{\frac{16}{5}k+2} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{s_c}} + \|P_{>M_0}\phi\|_{H^{s_c}} \right)^{\frac{4k}{5}} \\
&\quad d|\mu_{t_{2k+1}}|(\phi) dt_{2k+1}
\end{aligned}$$

Put in the support property (2.14),

$$\begin{aligned}
&\leq 2T(16C)^k C_0^{\frac{16}{5}k+2} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} C_0 + \varepsilon \right)^{\frac{4}{5}k} \\
&\leq 2TC_0^2 \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} C^2 C_0^5 + C^2 C_0^4 \varepsilon \right)^{\frac{4}{5}k}
\end{aligned}$$

for all k. Select ε small enough such that $C^2 C_0^4 \varepsilon < \frac{1}{4}$ and then select T small enough such that $T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} C^2 C_0^5 < \frac{1}{4}$, we thus have for T small enough,

$$\left\| \langle \nabla_{x_1} \rangle^{s_c-2} \langle \nabla_{x_1'} \rangle^{s_c-2} \gamma^{(1)}(t_1) \right\|_{L_T^\infty L_{x_1}^2 L_{x_1'}^2} \leq 2TC_0^2 \left(\frac{1}{2}\right)^k \to 0 \text{ as } k \to \infty.$$

We can then bootstrap to fill the whole $[0, T_0]$ interval. □

Proof of Corollary 2.3 Given the solution μ of (1.1), it generates a solution to the GP hierarchy (1.5) taking the following form

$$\left\{\prod_{j=1}^{k} u(t,x_j)\overline{u}(t,x_j')\right\}_{k=1}^{\infty}. \tag{2.21}$$

Thus, one could just apply the proof of Theorem 2.2 to the special case

$$\gamma^{(k)}(t)=\int |\phi\rangle\langle\phi|^{\otimes k}d\mu_t(\phi)=\prod_{j=1}^{k} u_1(t,x_j)\overline{u}_1(t,x_j')-\prod_{j=1}^{k} u_2(t,x_j)\overline{u}_2(t,x_j'),$$

where u_1 and u_2 are two solutions to (1.1) and μ_t is the signed measure $\delta_{u_{1,t}}-\delta_{u_{2,t}}$. Especially, when $k=1$, we have proved that

$$u_1(t,x)\overline{u}_1(t,x')=u_2(t,x)\overline{u}_2(t,x'), \tag{2.22}$$

which directly implies the uniqueness for the trivial solution $u\equiv 0$. Then by Lemma A.1 which concludes that the phase difference is zero, we complete the proof. □

As Corollary 2.3 requires condition (2), uniform in time frequency (UTFL) condition, we prove that every $C([0,T_0];H^{s_c})$ solution to (1.1) satisfies UTFL condition by Lemma 2.7. Immediately, Theorem 1.1 follows from Corollary 2.3 and Lemma 2.7.

Lemma 2.7 *Let u be a $C([0,T_0];H^{s_c})$ solution, then u satisfies uniform in time frequency localization (UTFL), that is, for each $\varepsilon>0$ there exists $M(\varepsilon)$ such that*

$$\|\langle\nabla\rangle^{s_c}P_{>M(\varepsilon)}u\|_{L^\infty_{[0,T_0]}L^2_x}\leq\varepsilon. \tag{2.23}$$

Proof We compute

$$\begin{aligned}\left|\partial_t\|\langle\nabla\rangle^{s_c}P_{\leq M}u\|_{L^2_x}^2\right| =&2\left|\operatorname{Im}\int P_{\leq M}\langle\nabla\rangle^{s_c}u\cdot P_{\leq M}\langle\nabla\rangle^{s_c}(|u|^{p-1}u)dx\right|\\ \leq&2\|P_{\leq M}\langle\nabla\rangle^{s_c}u\|_{L^2}\|P_{\leq M}\langle\nabla\rangle^{s_c}(|u|^{p-1}u)\|_{L^2}.\end{aligned}$$

Noting that $\|P_{\leq M}\langle\nabla\rangle^{s}f\|_{L^2}\lesssim M^s\|P_{\leq M}f\|_{L^2}$, then by Sobolev embedding A.3 and (A.11), we have

$$\begin{aligned}\left|\partial_t\|\langle\nabla\rangle^{s_c}P_{\leq M}u\|_{L^2_x}^2\right| \lesssim&2M^2\|P_{\leq M}\langle\nabla\rangle^{s_c}u\|_{L^2}\|P_{\leq M}\langle\nabla\rangle^{s_c-2}(|u|^{p-1}u)\|_{L^2}\\ \lesssim&2M^2\|u\|_{H^{s_c}}^{p+1}.\end{aligned}$$

Hence there exists $\delta'>0$ such that for any $t_0\in[0,T_0]$, it holds that for $t\in(t_0-\delta',t_0+\delta')\cap[0,T]$,

$$\left|\|\langle\nabla\rangle^{s_c}P_{\leq M}u(t)\|_{L^2_x}^2-\|\langle\nabla\rangle^{s_c}P_{\leq M}u(t_0)\|_{L^2_x}^2\right|\leq\frac{1}{16}\varepsilon^2. \tag{2.24}$$

On the other hand, since $u \in C([0, T_0]; H^{s_c})$, for each t_0, there exists $\delta'' > 0$ such that for any $t \in (t_0 - \delta'', t_0 + \delta'') \cap [0, T_0]$,

$$\left| \| \langle \nabla \rangle^{s_c} u(t) \|_{L_x^2}^2 - \| \langle \nabla \rangle^{s_c} u(t_0) \|_{L_x^2}^2 \right| \leq \frac{1}{16} \varepsilon^2. \tag{2.25}$$

Let $\delta = \min \left(\delta', \delta'' \right)$. Then we have that for any $t \in (t_0 - \delta, t_0 + \delta) \cap [0, T_0]$,

$$\left| \| \langle \nabla \rangle^{s_c} P_{>M} u(t) \|_{L_x^2}^2 - \| \langle \nabla \rangle^{s_c} P_{>M} u(t_0) \|_{L_x^2}^2 \right| \leq \frac{1}{4} \varepsilon^2.$$

For each $t \in [0, T_0]$, there exists M_t such that

$$\| \langle \nabla \rangle^{s_c} P_{>M_t} u(t) \|_{L_x^2} \leq \frac{1}{2} \varepsilon.$$

By the above, there exists $\delta_t > 0$ such that on $(t - \delta_t, t + \delta_t) \cap [0, T_0]$, we have

$$\| \langle \nabla \rangle^{s_c} P_{>M_t} u \|_{L^\infty_{(t-\delta_t, t+\delta_t) \cap [0, T_0]} L_x^2} \leq \varepsilon.$$

Since the collection of interval $(t - \delta_t, t + \delta_t) \cap [0, T_0]$, as t ranges over $[0, T_0]$, is an open cover of $[0, T_0]$. By compactness, we might as well assume that

$$(t_1 - \delta_{t_1}, t_1 + \delta_{t_1}) \cap [0, T_0], ..., (t_J - \delta_{t_J}, t_J + \delta_{t_J}) \cap [0, T_0]$$

be a finite open cover of $[0, T_0]$. Letting

$$M = \left(M_{t_1}, ..., M_{t_J} \right),$$

we have established (2.23). □

3 Estimates for the Compatible Time Integration Domain

3.1 Duhamel Expansion and Duhamel Tree

We start the analysis of the Duhamel expansions. We will create a Duhamel tree (we write D-tree for short) and show how to obtain the Duhamel expansion $J^{(2k+1)}_{\mu, sgn}$ from the D-tree. At first, we present an algorithm to generate a Duhamel tree from a collapsing map pair (μ, sgn) and then show this by an example. Subsequently, we are able to calculate the Duhamel expansion by a general algorithm. Finally, we exhibit an example by employing the above algorithms.

Algorithm 1 (Duhamel Tree)

(1) Let $D^{(0)}$ be a starting node in the D-tree. Find the pair of indices l and r so that $l \geq 1, r \geq 1$ and

$$\mu(2l) = 1,\ sgn(2l) = +,$$
$$\mu(2r) = 1,\ sgn(2r) = -,$$

and moreover l and r are the minimal indices for which the above equalities hold. Then place $D^{(2l)}$ or $D^{(2r)}$ as the left or right child of $D^{(0)}$ in the D-tree. If there is no such l or r, place $F_{1,+}$ or $F_{1,-}$ as the left or right child of $D^{(0)}$ in the D-tree.

(2) Set counter $j = 1$.

(3) Given j, find the indices $\{k_i\}_{i=1}^5$ so that $k_i > j$ and

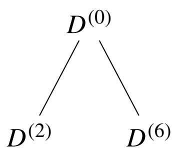
$$\begin{cases} \mu(2k_1) = \mu(2j),\ sgn(2k_1) = sgn(2j), \\ \mu(2k_2) = 2j,\ sgn(2k_2) = +, \\ \mu(2k_3) = 2j,\ sgn(2k_3) = -, \\ \mu(2k_4) = 2j+1,\ sgn(2k_4) = +, \\ \mu(2k_5) = 2j+1,\ sgn(2k_5) = -, \end{cases} \tag{3.1}$$

and k_i is the minimal index for which the corresponding equalities hold. Then place $D^{(2k_i)}$ as the i-th child of $D^{(2j)}$ in the D-tree. If there is no such $k_1/k_2/k_3/k_4/k_5$, then place $F_{\mu(2j),sgn(2j)}/F_{2j,+}/F_{2j,-}/F_{2j+1,+}/F_{2j+1,-}$ as the i-th child of $D^{(2j)}$ in the D-tree.

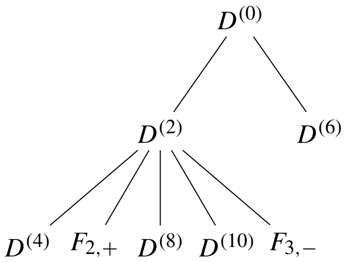
(4) If $j = k$, then stop, otherwise set $j = j + 1$ and go to step (3).

Example 3.1 Let us work with the following example

$2j$	2	4	6	8	10	12	14
$\mu(2j)$	1	1	1	2	3	6	6
$sgn(2j)$	+	+	−	−	+	+	−



Let $D^{(0)}$ be a starting node in the D-tree, so we need to find the minimal $l \geq 1, r \geq 1$ such that $\mu(2l) = 1$, $sgn(2l) = +$ and $\mu(2r) = 1$, $sgn(2r) = -$. In the case, it is $2l = 2$ and $2r = 6$ so we put $D^{(2)}$ and $D^{(6)}$ as left and right children of $D^{(0)}$, respectively, in the D-tree as shown in the left.

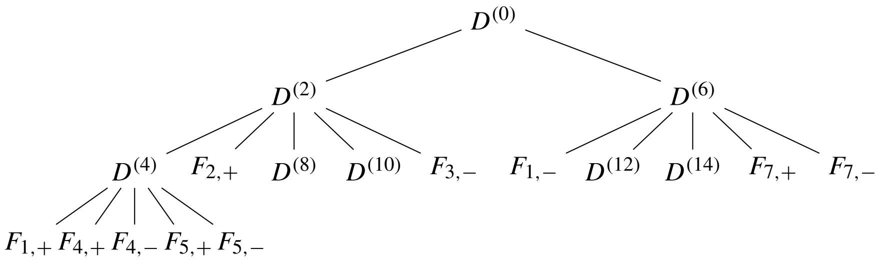


Now we start with counter $j = 1$ so we need to find the minimal $k_i > j$ such that

$$\mu(2k_1) = \mu(2),\ sgn(2k_1) = sgn(2),$$
$$\mu(2k_2) = 2,\ sgn(2k_2) = +,$$
$$\mu(2k_3) = 2,\ sgn(2k_3) = -,$$
$$\mu(2k_4) = 3,\ sgn(2k_4) = +,$$
$$\mu(2k_5) = 3,\ sgn(2k_5) = -.$$

We find $2k_1 = 4$, $2k_3 = 8$, $2k_4 = 10$ and there is no such k_2 and k_5. Thus, we put $D^{(4)}/F_{2,+}/D^{(8)}/D^{(10)}/F_{3,-}$ as the i-th child of $D^{(2)}$ (shown at left).

Next, the counter turns to $j = 2$ and we find that there is no k_i satisfying (3.1), so we put $F_{1,+}/F_{4,+}/F_{4,-}/F_{5,+}/F_{5,-}$ as the i-th child of $D^{(4)}$. Then we move to $j = 3$ and find that $2k_2 = 12$ and $2k_3 = 14$ satisfy (3.1) so we put $F_{1,-}/D^{(12)}/D^{(14)}/F_{7,+}/F_{7,-}$ as the i-th child of $D^{(6)}$ shown as follows.



Finally, by repeating the above step, we jump to the full D-tree shown as follows where we use F to replace $F_{i,\pm}$ for short.

Next, we present the following algorithm to obtain the Duhamel expansion from the D-tree. For convenience, let $U_{\pm j} := e^{\pm i t_j \Delta}$.

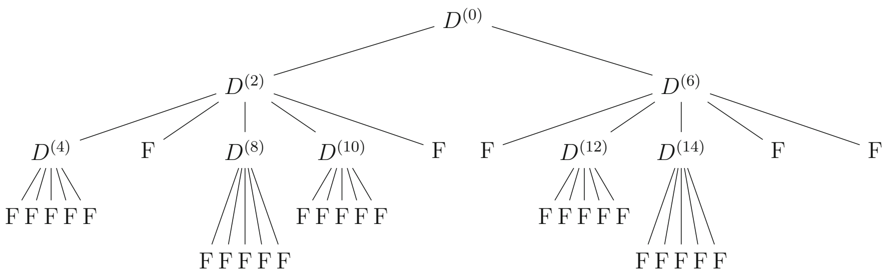


Fig. 1 Duhamel Tree

Algorithm 2 (From D-tree to Duhamel expansion)

(1) Set $F_{i,+} = U_{-2k-1}\phi$, $F_{i,-} = \overline{U_{-2k-1}\phi}$ for $i = 1, 2, ..., 2k$. If $sgn(2k) = +$, set

$$D^{(2k)}(t_{2k+1}) = U_{-2k-1}(|\phi|^4\phi),$$

if $sgn(2k) = -$, set

$$D^{(2k)}(t_{2k+1}) = \overline{U_{-2k-1}(|\phi|^4\phi)}.$$

(2) Set counter $l = k - 1$.

(3) Given l, if $sgn(2l) = +$, set

$$D^{(2l)}(t_{2l+1}) = U_{-2l-1}\left[(U_{2l+1}C_1)(U_{2l+1}C_2)\left(\overline{U_{2l+1}\overline{C_3}}\right)(U_{2l+1}C_4)\left(\overline{U_{2l+1}\overline{C_5}}\right)\right]$$

if $sgn(2l) = -$, set

$$D^{(2l)}(t_{2l+1}) = \overline{U_{-2l-1}\left[\left(U_{2l+1}\overline{C_1}\right)\left(\overline{U_{2l+1}C_2}\right)\left(U_{2l+1}\overline{C_3}\right)\left(\overline{U_{2l+1}C_4}\right)\left(U_{2l+1}\overline{C_5}\right)\right]}$$

where C_i is the i-th child of $D^{(2l)}$ in the D-tree.

(4) Set $l = l - 1$. If $l = 0$, set

$$D^{(0)}(t_1, \underline{t}_{2k+1}) = (U_1C_l)(x_1)(\overline{U_1\overline{C_r}})(x_1'),$$

where C_l/C_r is the left/right child of $D^{(0)}$ in the D-tree, and stop, otherwise go to step (3).

Proposition 3.2 *By Algorithm 2 (From D-tree to Duhamel expansion), we have*

$$J^{(2k+1)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{\otimes(2k+1)})(t_1, \underline{t}_{2k+1}) = D^{(0)}(t_1, \underline{t}_{2k+1}).$$

Proof It follows from Algorithms 1 and 2. □

Example 3.3 We calculate the Duhamel expansion in Example 3.1. By Algorithm 1 and 2, we obtain

$$\begin{cases} D^{(14)}(t_{15}) = \overline{U_{-15}(|\phi|^4\phi)}, \\ D^{(12)}(t_{13}, t_{15}) = U_{-13}(|U_{13,15}\phi|^4 U_{13,15}\phi), \\ D^{(10)}(t_{11}, t_{15}) = U_{-11}(|U_{11,15}\phi|^4 U_{11,15}\phi), \\ D^{(8)}(t_9, t_{15}) = \overline{U_{-9}(|U_{9,15}\phi|^4 U_{9,15}\phi)}, \\ D^{(6)}(t_7, t_{13}, t_{15}) = \overline{U_{-7}\left[(U_{7,15}\phi)(\overline{U_7 D^{(12)}})(U_7\overline{D^{(14)}})(\overline{U_{7,15}\phi})(U_{7,15}\phi)\right]}, \\ D^{(4)}(t_5, t_{15}) = U_{-5}(|U_{5,15}\phi|^4 U_{5,15}\phi), \\ D^{(2)}(t_3, t_5, t_9, t_{11}, t_{15}) = U_{-3}\left[(U_3 D^{(4)})(U_{3,15}\phi)(\overline{U_3\overline{D^{(8)}}})(U_3 D^{(10)})(\overline{U_{3,15}\phi})\right], \\ D^{(0)}(t_1, \underline{t}_{15}) = (U_1 D^{(2)})(\overline{U_1\overline{D^{(6)}}}). \end{cases}$$

where $U_{i,j} := U_i U_{-j}$.

On the one hand, expanding $D^{(0)}(t_1, \underline{t}_{15})$ gives the Duhamel expansion

$$
\begin{aligned}
&D^{(0)}(t_1, \underline{t}_{15})\\
&= (U_1 D^{(2)})(\overline{U_1 \overline{D^{(6)}}})\\
&= \left(U_{1,3}\left[(U_3 D^{(4)})(U_{3,15}\phi)(\overline{U_3 \overline{D^{(8)}}})(U_3 D^{(10)})(\overline{U_{3,15}\phi})\right]\right)\\
&\quad \cdot \left(\overline{U_{1,7}\left[(U_{7,15}\phi)(\overline{U_7 D^{(12)}})(U_7 \overline{D^{(14)}})(\overline{U_{7,15}\phi})(U_{7,15}\phi)\right]}\right)\\
&= U_{1,3}\left[(U_{3,5}(|U_{5,15}\phi|^4 U_{5,15}\phi))(|U_{3,15}\phi|^2)(\overline{U_{3,9}(|U_{9,15}\phi|^4 U_{9,15}\phi)})(U_{3,11}(|U_{11,15}\phi|^4 U_{11,15}\phi))\right]\\
&\quad \cdot \overline{U_{1,7}\left[(|U_{7,15}\phi|^2 U_{7,15}\phi)(\overline{U_{7,13}(|U_{13,15}\phi|^4 U_{13,15}\phi)})(U_{7,15}(|\phi|^4\phi))\right]}
\end{aligned}
$$

On the other hand, we calculate $J^{(15)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{\otimes 15})(t_1, \underline{t}_{2k+1})$ by step. Note that

$$
\begin{aligned}
&J^{(15)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{\otimes 15})(t_1, \underline{t}_{15})\\
&\quad = U^{(1)}_{1,3} B^{+}_{1;2,3} U^{(3)}_{3,5} B^{+}_{1;4,5} U^{(5)}_{5,7} B^{-}_{1;6,7} U^{(7)}_{7,9} B^{-}_{2;8,9} U^{(9)}_{9,11}\\
&\quad\quad B^{+}_{3;10,11} U^{(11)}_{11,13} B^{+}_{6;12,13} U^{(13)}_{13,15} B^{-}_{6;14,15}(|\phi\rangle\langle\phi|^{\otimes 15}).
\end{aligned}
$$

At first,

$$
U^{(13)}_{13,15} B^{-}_{6;14,15}(|\phi\rangle\langle\phi|^{\otimes 15}) = (U_{13,15}\phi)(x_6)(\overline{U_{13,15}(|\phi|^4\phi)})(x_6')|U_{13,15}\phi\rangle\langle U_{13,15}\phi|^{\otimes 12}.
$$

Adding $U^{(11)}_{11,13} B^{+}_{6;12,13}$ gives

$$
(U_{11,13}(|U_{13,15}\phi|^4 U_{13,15}\phi))(x_6)(\overline{U_{11,15}(|\phi|^4\phi)})(x_6')|U_{11,15}\phi\rangle\langle U_{11,15}\phi|^{\otimes 10}.
$$

Then adding $U^{(7)}_{7,9} B^{-}_{2;8,9} U^{(9)}_{9,11} B^{+}_{3;10,11}$ gives

$$
\begin{aligned}
&(U_{7,15}\phi)(x_2)(\overline{U_{7,9}(|U_{9,15}\phi|^4 U_{9,15}\phi)})(x_2')\\
&(U_{7,11}(|U_{11,15}\phi|^4 U_{11,15}\phi))(x_3)(\overline{U_{7,15}\phi})(x_3')\\
&(U_{7,13}(|U_{13,15}\phi|^4 U_{13,15}\phi))(x_6)(\overline{U_{7,15}(|\phi|^4\phi)})(x_6')|U_{7,15}\phi\rangle\langle U_{7,15}\phi|^{\otimes 4}
\end{aligned}
$$

Finally adding $U^{(1)}_{1,3} B^{+}_{1;2,3} U^{(3)}_{3,5} B^{+}_{1;4,5} U^{(5)}_{5,7} B^{-}_{1;6,7}$ gives

$$
\begin{aligned}
&U_{1,3}\left[(U_{3,5}(|U_{5,15}\phi|^4 U_{5,15}\phi))(|U_{3,15}\phi|^2)(\overline{U_{3,9}(|U_{9,15}\phi|^4 U_{9,15}\phi)})(U_{3,11}(|U_{11,15}\phi|^4 U_{11,15}\phi))\right](x_1)\\
&\quad \cdot \left(\overline{U_{1,7}\left[(|U_{7,15}\phi|^2 U_{7,15}\phi)(\overline{U_{7,13}(|U_{13,15}\phi|^4 U_{13,15}\phi)})(U_{7,15}(|\phi|^4\phi))\right]}\right)(x_1')
\end{aligned}
$$

which shows that $J^{(15)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{\otimes 15})(t_1, \underline{t}_{15}) = D^{(0)}(t_1, \underline{t}_{15})$.

3.2 Compatible Time Integration Domain

To enable the application of U-V multilinear estimates, we have to take into account the compatible time integration domain. Combining with the Duhamel tree, we present a general algorithm to compute the Duhamel expansion with the compatible time integration domain.

Definition 3.4 Define the compatible time integration domain as follows

$$T_C(\mu, sgn) = \left\{ t_{2j+1} \geq t_{2l+1} : D^{(2l)} \to D^{(2j)} \right\}. \tag{3.2}$$

where $D^{(2l)} \to D^{(2j)}$ denotes that $D^{(2l)}$ is the child of $D^{(2j)}$. Moreover, we say that $D^{(2l)}$ is the offspring of $D^{(2j)}$ if there exist $2l_1$,...,$2l_r$ such that $D^{(2l)} \to D^{(2l_1)} \to \cdots \to D^{(2l_r)} \to D^{(2j)}$.

Example 3.5 Continuing the Example 3.3, we will expand

$$\int_{T_C} J^{(15)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{\otimes 15})(t_1, \underline{t}_{15}) d\underline{t}_{15}.$$

From the D-tree (Fig. 1), the compatible time integration domain is as follows

$$\int_{t_3=0}^{t_1} \int_{t_7=0}^{t_1} \int_{t_5=0}^{t_3} \int_{t_9=0}^{t_3} \int_{t_{11}=0}^{t_3} \int_{t_{13}=0}^{t_7} \int_{t_{15}=0}^{t_7}.$$

Write $\int_{t_{15}=0}^{t_1}$ on the outside and hence $\int_{t_7=0}^{t_1}$ changes into $\int_{t_7=t_{15}}^{t_1}$. Then it turns to

$$\int_{t_{15}=0}^{t_1} \int_{t_3=0}^{t_1} \int_{t_7=t_{15}}^{t_1} \int_{t_5=0}^{t_3} \int_{t_9=0}^{t_3} \int_{t_{11}=0}^{t_3} \int_{t_{13}=0}^{t_7}.$$

So we can rewrite

$$\begin{aligned}
&\int_{T_C} J^{(15)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{\otimes 15})(t_1, \underline{t}_{15}) d\underline{t}_{15} \\
&= \int_{t_{15}=0}^{t_1} \left[\int_{t_3=0}^{t_1} \int_{t_5=0}^{t_3} \int_{t_9=0}^{t_3} \int_{t_{11}=0}^{t_3} U_1 D^{(2)} \right] \left[\int_{t_7=t_{15}}^{t_1} \int_{t_{13}=0}^{t_7} \overline{U_1 \overline{D^{(6)}}} \right] \\
&= \int_{t_{15}=0}^{t_1} \int_{t_3=0}^{t_1} U_{1,3} \left[\left(U_3 \int_{t_5=0}^{t_3} D^{(4)} \right) U_{3,15}\phi \left(\overline{U_3 \int_{t_9=0}^{t_3} \overline{D^{(8)}}} \right) \left(U_3 \int_{t_{11}=0}^{t_3} D^{(10)} \right) \overline{U_{3,15}\phi} \right] \\
&\quad \cdot \overline{\int_{t_7=t_{15}}^{t_1} U_{1,7} \left[(U_{7,15}\phi) \left(U_7 \int_{t_{13}=0}^{t_7} D^{(12)} \right) (U_7 \overline{D^{(14)}}) (\overline{U_{7,15}\phi}) (U_{7,15}\phi) \right]}
\end{aligned}$$

where $D^{(2j)}$ is shown in Example 3.3. We can see that all the Duhamel structures are fully compatible with U-V multilinear estimates, which we will show in Section 3.3.

Next, we give a general form of the algorithm.

Algorithm 3 (From D-tree to Duhamel integration)

(1) Let

$$Q^{(2k)}(t_{2k+1}) = D^{(2k)}(t_{2k+1}).$$

and replace $D^{(2k)}$ by $Q^{(2k)}(t_{2k+1})$ in the D-tree.

(2) Set counter $l = k - 1$.

(3) Given l, there exists only one j such that $D^{(2l)} \to D^{(2j)}$. Then there will be four cases as follows.

Case 1. $D^{(2k)}$ is the offspring of $D^{(2l)}$ and $sgn(2l) = +$. Then set

$$\begin{aligned}&Q^{(2l)}(t_{2j+1}, t_{2k+1})\\&= \int_{t_{2l+1}=t_{2k+1}}^{t_{2j+1}} U_{-2l-1}\left[\left(U_{2l+1}C_1\right)\left(U_{2l+1}C_2\right)\left(\overline{U_{2l+1}\overline{C_3}}\right)\left(U_{2l+1}C_4\right)\left(\overline{U_{2l+1}\overline{C_5}}\right)\right]dt_{2l+1}.\end{aligned}$$

Case 2. $D^{(2k)}$ is the offspring of $D^{(2l)}$ and $sgn(2l) = -$. Then set

$$\begin{aligned}&Q^{(2l)}(t_{2j+1}, t_{2k+1})\\&= \int_{t_{2l+1}=t_{2k+1}}^{t_{2j+1}} \overline{U_{-2l-1}\left[\left(U_{2l+1}\overline{C_1}\right)\left(\overline{U_{2l+1}C_2}\right)\left(U_{2l+1}\overline{C_3}\right)\left(\overline{U_{2l+1}C_4}\right)\left(U_{2l+1}\overline{C_5}\right)\right]}dt_{2l+1}.\end{aligned}$$

Case 3. $D^{(2k)}$ is not the offspring of $D^{(2l)}$ and $sgn(2l) = +$. Then set

$$\begin{aligned}&Q^{(2l)}(t_{2j+1}, t_{2k+1})\\&= \int_{t_{2l+1}=0}^{t_{2j+1}} U_{-2l-1}\left[\left(U_{2l+1}C_1\right)\left(U_{2l+1}C_2\right)\left(\overline{U_{2l+1}\overline{C_3}}\right)\left(U_{2l+1}C_4\right)\left(\overline{U_{2l+1}\overline{C_5}}\right)\right]dt_{2l+1}.\end{aligned}$$

Case 4. $D^{(2k)}$ is not the offspring of $D^{(2l)}$ and $sgn(2l) = -$. Then set

$$\begin{aligned}&Q^{(2l)}(t_{2j+1}, t_{2k+1})\\&= \int_{t_{2l+1}=0}^{t_{2j+1}} \overline{U_{-2l-1}\left[\left(U_{2l+1}\overline{C_1}\right)\left(\overline{U_{2l+1}C_2}\right)\left(U_{2l+1}\overline{C_3}\right)\left(\overline{U_{2l+1}C_4}\right)\left(U_{2l+1}\overline{C_5}\right)\right]}dt_{2l+1}.\end{aligned}$$

In the above, C_i is the i-th child of $D^{(2l)}$ in the D-tree.

(4) Update the D-tree by using $Q^{(2l)}(t_{2j+1}, t_{2k+1})$ to replace $D^{(2l)}$.

(5) Set $l = l - 1$. If $l = 0$, set

$$Q^{(0)}(t_1) = \int_{t_{2k+1}=0}^{t_1} (U_1 C_l)(\overline{U_1 \overline{C_r}})dt_{2k+1}$$

where C_l or C_r is the left or right child of $D^{(0)}$ in the updated D-tree, and stop, otherwise go to step (3).

Hence, we arrive at a representation as follows.

Proposition 3.6

$$\int_{T_C} J^{(2k+1)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{(2k+1)})(t_1,\underline{t}_{2k+1})d\underline{t}_{2k+1} = Q^{(0)}(t_1). \tag{3.3}$$

Proof It follows from Algorithm 3. □

3.3 Estimates using the *U*-*V* multilinear estimates

Referring to the standard text [53] for the definition of U^p_t and V^p_t, we define $X^s([0,T))$ and $Y^s([0,T))$ to be the spaces of all functions $u : [0,T) \mapsto H^s(\mathbb{T}^d)$ such that for every $\xi \in \mathbb{Z}^d$ the map $t \mapsto \widehat{e^{-it\Delta}u(t)}(\xi)$ is in $U^2([0,T);\mathbb{C})$ and $V^2_{rc}([0,T);\mathbb{C})$, respectively, with norms given by

$$\|u\|_{X^s([0,T))} := \left(\sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{2s}\|\widehat{e^{-it\Delta}u(t)}(\xi)\|^2_{U^2}\right)^{1/2},$$

$$\|u\|_{Y^s([0,T))} := \left(\sum_{\xi\in\mathbb{Z}^d}\langle\xi\rangle^{2s}\|\widehat{e^{-it\Delta}u(t)}(\xi)\|^2_{V^2}\right)^{1/2}$$

as in [37, 38, 42, 46]. In particular, we have the usual properties,

$$\|u\|_{L^\infty_t H^s_x} \lesssim \|u\|_{Y^s} \lesssim \|u\|_{X^s}, \tag{3.4}$$

$$\|e^{it\Delta}f\|_{Y^s} \le \|e^{it\Delta}f\|_{X^s} \le \|f\|_{H^s}, \tag{3.5}$$

which were proved in [37,Propositions 2.8-2.10].

By quintilinear estimates in Lemma 5.6 and the trivial estimate $\|u\|_{Y^s} \lesssim \|u\|_{X^s}$, we have that

$$\left\|\int_a^t e^{i(t-s)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5)ds\right\|_{X^{\frac{d-5}{2}}} \le C\|u_1\|_{X^{\frac{d-5}{2}}}\|u_2\|_{X^{\frac{d-1}{2}}}\|u_3\|_{X^{\frac{d-1}{2}}}\|u_4\|_{X^{\frac{d-1}{2}}}\|u_5\|_{X^{\frac{d-1}{2}}} \tag{3.6}$$

$$\left\|\int_a^t e^{i(t-s)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5)ds\right\|_{X^{\frac{d-1}{2}}} \le C\|u_1\|_{X^{\frac{d-1}{2}}}\|u_2\|_{X^{\frac{d-1}{2}}}\|u_3\|_{X^{\frac{d-1}{2}}}\|u_4\|_{X^{\frac{d-1}{2}}}\|u_5\|_{X^{\frac{d-1}{2}}} \tag{3.7}$$

$$\left\|\int_a^t e^{i(t-s)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5)ds\right\|_{X^{\frac{d-5}{2}}} \le C\|u_1\|_{X^{\frac{d-5}{2}}}\left(T^{\frac{1}{2(d+3)}}M_0^{\frac{2d+3}{2(d+3)}}\|P_{\le M_0}u_2\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0}u_2\|_{X^{\frac{d-1}{2}}}\right)\|u_3\|_{X^{\frac{d-1}{2}}} \|u_4\|_{X^{\frac{d-1}{2}}}\|u_5\|_{X^{\frac{d-1}{2}}} \tag{3.8}$$

$$
\begin{aligned}
&\left\| \int_a^t e^{i(t-s)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5)ds \right\|_{X^{\frac{d-1}{2}}} \\
&\quad \leq C\|u_1\|_{X^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}u_2\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0}u_2\|_{X^{\frac{d-1}{2}}} \right) \|u_3\|_{X^{\frac{d-1}{2}}} \\
&\quad \|u_4\|_{X^{\frac{d-1}{2}}} \|u_5\|_{X^{\frac{d-1}{2}}} \qquad (3.9)
\end{aligned}
$$

where $\widetilde{u} \in \{u, \overline{u}\}$.

Before moving into the estimate part, we first mark the D-tree as a preparation, as we will use the above U-V multilinear estimates according to the marked D-tree. Here, we give a general algorithm to mark a D-Tree.

Algorithm 4 (Marked D-Tree)

(1) We put a subscript R at $D^{(2k)}$, that is, $D_R^{(2k)}$. Here, we use the subscript R to denote the roughest term.
(2) Set counter $l = k - 1$. If $D^{(2k)}$ is the offspring (see Definition 3.4) of $D^{(2l)}$, put a subscript R at $D^{(2l)}$, that is, $D_R^{(2l)}$. Moreover, if one of the children of $D^{(2l)}$ is F, then put a subscript ϕ at $D^{(2l)}$, that is, $D_\phi^{(2l)}$ or $D_{\phi,R}^{(2l)}$.
(3) Set $l = l - 1$. If $l = 0$, then stop, otherwise go to step (2).

Example 3.7 We estimate the Duhamel expansion in Example 3.3 with the corresponding time integration domain to show how to apply the U-V multilinear estimates. First, Applying Algorithm 4 to the D-tree in Fig. 1, we obtain a marked D-tree as in Fig. 2.

Next, we get into the estimate part. By Proposition 3.6, it suffices to estimate $Q^{(0)}(t_1)$. Combining the D-tree (Fig. 1) and Algorithm 3, we obtain

$$
Q^{(0)}(t_1) = \int_{t_{15}=0}^{t_1} (U_1 Q^{(2)}(t_1, t_{15}))\overline{(U_1 \overline{Q^{(6)}(t_1, t_{15})})}dt_{15},
$$

$$
Q^{(2)}(t_1, t_{15}) = \int_{t_3=0}^{t_1} U_{-3}\left[\left(U_3 Q^{(4)}\right)\left(U_{3,15}\phi\right)\left(\overline{U_3\overline{Q^{(8)}}}\right)\left(U_3 Q^{(10)}\right)\left(\overline{U_{3,15}\phi}\right)\right]dt_3,
$$

$$
Q^{(6)}(t_1, t_{15}) = \int_{t_7=t_{15}}^{t_1} \overline{U_{-7}\left[\left(U_{7,15}\phi\right)\left(\overline{U_7 Q^{(12)}}\right)\left(U_7\overline{Q^{(14)}}\right)\left(\overline{U_{7,15}\phi}\right)\left(U_{7,15}\phi\right)\right]}dt_7.
$$

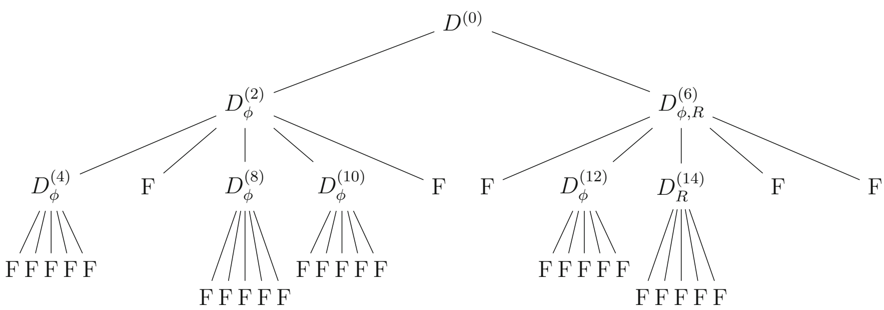


Fig. 2 Marked Duhamel Tree

At first, we use Minkowski to obtain

$$
\begin{aligned}
&\|\langle\nabla_{x_1}\rangle^{\frac{d-5}{2}}\langle\nabla_{x_1'}\rangle^{\frac{d-5}{2}} Q^{(0)}(t_1)\|_{L^\infty_T L^2_{x_1} L^2_{x_1'}}\\
&\quad\le \int_0^T \|U_1 Q^{(2)}(t_1,t_{15})\|_{L^\infty_{t_1} H^{\frac{d-5}{2}}_{x_1}} \|\overline{U_1\overline{Q^{(6)}(t_1,t_{15})}}\|_{L^\infty_{t_1} H^{\frac{d-5}{2}}_{x_1'}} dt_{15}
\end{aligned}
$$

Note that $D^{(2)}_\phi$ carries no R subscript, so we can bump it to $H^{\frac{d-1}{2}}$ and then use estimate (3.4) to get

$$
\le \int_0^T \|U_1 Q^{(2)}(t_1,t_{15})\|_{X^{\frac{d-1}{2}}} \|U_1\overline{Q^{(6)}(t_1,t_{15})}\|_{X^{\frac{d-5}{2}}} dt_{15}
$$

By multilinear estimate (3.9),

$$
\begin{aligned}
\|U_1 Q^{(2)}\|_{X^{\frac{d-1}{2}}} \le & C\|U_3 Q^{(4)}\|_{X^{\frac{d-1}{2}}}\\
&\left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\le M_0} U_{3,15}\phi\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0} U_{3,15}\phi\|_{X^{\frac{d-1}{2}}}\right)\\
&\|U_3\overline{Q^{(8)}}\|_{X^{\frac{d-1}{2}}} \|U_3 Q^{(10)}\|_{X^{\frac{d-1}{2}}} \|U_{3,15}\phi\|_{X^{\frac{d-1}{2}}}.
\end{aligned}
$$

As $D^{(6)}_{\phi,R}$ carries subscript ϕ and R, we use multilinear estimate (3.8) to get

$$
\begin{aligned}
\|U_1\overline{Q^{(6)}}\|_{X^{\frac{d-5}{2}}} \le & C\|U_7\overline{Q^{(14)}}\|_{X^{\frac{d-5}{2}}}\\
&\left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\le M_0} U_{7,15}\phi\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0} U_{7,15}\phi\|_{X^{\frac{d-1}{2}}}\right)\\
&\|U_7 Q^{(12)}\|_{X^{\frac{d-1}{2}}} \|U_{7,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{7,15}\phi\|_{X^{\frac{d-1}{2}}}.
\end{aligned}
$$

From Algorithm 3, we have

$$
\begin{aligned}
Q^{(4)} &= \int_{t_5=0}^{t_3} U_{-5}\left[\left(U_{5,15}\phi\right)\left(U_{5,15}\phi\right)\left(\overline{U_{5,15}\phi}\right)\left(U_{5,15}\phi\right)\left(\overline{U_{5,15}\phi}\right)\right] dt_5,\\
Q^{(8)} &= \int_{t_9=0}^{t_3} \overline{U_{-9}\left[\left(U_{9,15}\phi\right)\left(\overline{U_{9,15}\phi}\right)\left(U_{9,15}\phi\right)\left(\overline{U_{9,15}\phi}\right)\left(U_{9,15}\phi\right)\right]} dt_9,\\
Q^{(10)} &= \int_{t_{11}=0}^{t_3} U_{-11}\left[\left(U_{11,15}\phi\right)\left(U_{11,15}\phi\right)\left(\overline{U_{11,15}\phi}\right)\left(U_{11,15}\phi\right)\left(\overline{U_{11,15}\phi}\right)\right] dt_{11},\\
Q^{(12)} &= \int_{t_{13}=0}^{t_7} U_{-13}\left[(U_{13,15}\phi)(U_{13,15}\phi)(\overline{U_{13,15}\phi})(U_{13,15}\phi)(\overline{U_{13,15}\phi})\right] dt_{13},\\
Q^{(14)} &= \overline{U_{-15}(|\phi|^4\phi)}.
\end{aligned}
$$

Notice that $D_\phi^{(4)}$, $D_\phi^{(6)}$, $D_\phi^{(8)}$ and $D_\phi^{(10)}$ only carry subscript ϕ, so we use multilinear estimate (3.9) to obtain

$$\begin{aligned}
\|U_3 Q^{(4)}\|_{X^{\frac{d-1}{2}}} \leq & C\|U_{5,15}\phi\|_{X^{\frac{d-1}{2}}} \\
& \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0} U_{5,15}\phi\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0} U_{5,15}\phi\|_{X^{\frac{d-1}{2}}}\right) \\
& \|U_{5,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{5,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{5,15}\phi\|_{X^{\frac{d-1}{2}}} \\
\leq & C\|\phi\|^4_{H^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}}\right) \\
\|U_3 \overline{Q^{(8)}}\|_{X^{\frac{d-1}{2}}} \leq & C\|U_{9,15}\phi\|_{X^{\frac{d-1}{2}}} \\
& \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0} U_{9,15}\phi\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0} U_{9,15}\phi\|_{X^{\frac{d-1}{2}}}\right) \\
& \|U_{9,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{9,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{9,15}\phi\|_{X^{\frac{d-1}{2}}} \\
\leq & C\|\phi\|^4_{H^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}}\right) \\
\|U_3 Q^{(10)}\|_{X^{\frac{d-1}{2}}} \leq & C\|U_{11,15}\phi\|_{X^{\frac{d-1}{2}}} \\
& \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0} U_{11,15}\phi\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0} U_{11,15}\phi\|_{X^{\frac{d-1}{2}}}\right) \\
& \|U_{11,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{11,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{11,15}\phi\|_{X^{\frac{d-1}{2}}} \\
\leq & C\|\phi\|^4_{H^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}}\right) \\
\|U_7 Q^{(12)}\|_{X^{\frac{d-1}{2}}} \leq & C\|U_{13,15}\phi\|_{X^{\frac{d-1}{2}}} \\
& \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0} U_{13,15}\phi\|_{X^{\frac{d-1}{2}}} + \|P_{>M_0} U_{13,15}\phi\|_{X^{\frac{d-1}{2}}}\right) \\
& \|U_{13,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{13,15}\phi\|_{X^{\frac{d-1}{2}}} \|U_{13,15}\phi\|_{X^{\frac{d-1}{2}}} \\
\leq & C\|\phi\|^4_{H^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}}\right)
\end{aligned}$$

Finally, to deal with the roughest term $Q^{(14)}$, we use Sobolev inequality (A.11),

$$\|U_7 \overline{Q^{(14)}}\|_{X^{\frac{d-5}{2}}} = \|U_{7,15}(|\phi|^4\phi)\|_{X^{\frac{d-5}{2}}} \leq \||\phi|^4\phi\|_{H^{\frac{d-5}{2}}} \leq C\|\phi\|^5_{H^{\frac{d-1}{2}}}.$$

Together with the above estimates, we arrive at

$$\begin{aligned}
& \|\langle\nabla_{x_1}\rangle^{\frac{d-5}{2}} \langle\nabla_{x_1'}\rangle^{\frac{d-5}{2}} Q^{(0)}(t_1)\|_{L_T^\infty L_{x_1}^2 L_{x_1'}^2} \\
& \leq C^7 \int_0^T \|\phi\|^{24}_{H^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}}\right)^6 dt_{15}.
\end{aligned}$$

From Example 3.7, one can immediately tell that a decay power comes from estimates (3.8) and (3.9). Actually, such a decay power is at least proportional to k.

Definition 3.8 ([19, 20]) For $l < k$, we say the l-th coupling is an unclogged coupling, if one of the children of $D^{(2l)}$ is F. If the l-th coupling is not unclogged, we will call it a congested coupling.

Lemma 3.9 *[19,Lemma 5.14] For large k, there are at least $\frac{4}{5}k$ unclogged couplings in k couplings.*

The main result of this section is the following proposition.

Proposition 3.10 *Let $\gamma^{(k)}(t) = \int |\phi\rangle\langle\phi|^{\otimes k} d\mu_t(\phi)$. Then we have*

$$\begin{aligned}&\left\| \langle\nabla_{x_1}\rangle^{\frac{d-5}{2}} \langle\nabla_{x_1'}\rangle^{\frac{d-5}{2}} \int_{T_C} J^{(2k+1)}_{\mu,sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \right\|_{L^\infty_T L^2_{x_1} L^2_{x_1'}} \\ &\leq C^k \int_0^T \int \|\phi\|^{\frac{16}{5}k+2}_{H^{\frac{d-1}{2}}} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} \right. \\ &\quad \left. + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}} \right)^{\frac{4k}{5}} d|\mu_{t_{2k+1}}|(\phi) dt_{2k+1}.\end{aligned}$$

Proof We rewrite

$$\begin{aligned}&\|\langle\nabla_{x_1}\rangle^{\frac{d-5}{2}} \langle\nabla_{x_1'}\rangle^{\frac{d-5}{2}} \int_{T_C} J^{(2k+1)}_{\mu,sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}\|_{L^\infty_T L^2_{x_1} L^2_{x_1'}} \\ &= \|\langle\nabla_{x_1}\rangle^{\frac{d-5}{2}} \langle\nabla_{x_1'}\rangle^{\frac{d-5}{2}} \int_{T_C} \int J^{(2k+1)}_{\mu,sgn}(|\phi\rangle\langle\phi|^{\otimes(2k+1)})(t_1, \underline{t}_{2k+1}) d\mu_{t_{2k+1}}(\phi) d\underline{t}_{2k+1}\|_{L^\infty_T L^2_{x_1} L^2_{x_1'}}\end{aligned}$$

By Proposition 3.6, we then use Minkowski and estimate (3.4)

$$\begin{aligned}&\leq \int_0^T \int \|U_1 C_l(t_1, t_{2k+1})\|_{L^\infty_{t_1} H^{\frac{d-5}{2}}_{x_1}} \|\overline{U_1 \overline{C_r(t_1, t_{2k+1})}}\|_{L^\infty_{t_1} H^{\frac{d-5}{2}}_{x_1'}} d|\mu_{t_{2k+1}}|(\phi) dt_{2k+1} \\ &\leq \int_0^T \int \|U_1 C_l(t_1, t_{2k+1})\|_{X^{\frac{d-5}{2}}} \|U_1 \overline{C_r(t_1, t_{2k+1})}\|_{X^{\frac{d-5}{2}}} d|\mu_{t_{2k+1}}|(\phi) dt_{2k+1}.\end{aligned}$$

where C_l or C_r is the left or right child of $Q^{(0)}$ by Algorithm 3. Only one of C_l and C_r carries the subscript R, so bump the other one into $X^{\frac{d-1}{2}}$.

We can now present the algorithm which proves the general case.

Algorithm 5 (Estimate)

(1) Set counter $l = 1$.
(2) Given l, there exists only one j such that $D^{(2l)} \to D^{(2j)}$. There will be four cases as follows.
Case 1. $D^{(2l)} = D^{(2l)}_{\phi,R}$. Then apply estimate (3.8), put the factor carrying the subscript R in $X^{\frac{d-5}{2}}$ and replace all the $X^{\frac{d-1}{2}}$ norm of $U\phi$ by $H^{\frac{d-1}{2}}$ norm of ϕ.

Case 2. $D^{(2l)} = D_{\phi}^{(2l)}$. Then apply estimate (3.9) and replace all the $X^{\frac{d-1}{2}}$ norm of $U\phi$ by $H^{\frac{d-1}{2}}$ norm of ϕ.
Case 3. $D^{(2l)} = D_{R}^{(2l)}$. Then apply estimate (3.6), put the factor carrying the subscript R in $X^{\frac{d-5}{2}}$ and replace all the $X^{\frac{d-1}{2}}$ norm of $U\phi$ by $H^{\frac{d-1}{2}}$ norm of ϕ.
Case 4. $D^{(2l)} = D^{(2l)}$. Then apply estimate (3.7) and replace all the $X^{\frac{d-1}{2}}$ norm of $U\phi$ by $H^{\frac{d-1}{2}}$ norm of ϕ.

(3) Set counter $l = l + 1$. If $l < k$, go to step (2), otherwise go to step (4).
(4) We are now at the k-th coupling and would have applied (3.8) and (3.9) at least $\frac{4}{5}k$ times, so we arrive at

$$\left\| \langle \nabla_{x_1} \rangle^{\frac{d-5}{2}} \langle \nabla_{x_1'} \rangle^{\frac{d-5}{2}} \int_{T_C} J_{\mu,sgn}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \right\|_{L_T^\infty L_{x_1}^2 L_{x_1'}^2}$$
$$\leq C^{k-1} \int_0^T \int \|\phi\|_{H^{\frac{d-1}{2}}}^{\frac{16}{5}k-3} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}} \right)^{\frac{4k}{5}}$$
$$\||\phi|^4\phi\|_{H^{\frac{d-5}{2}}} d|\mu_{t_{2k+1}}|(\phi) dt_{2k+1}$$

Apply Sobolev inequality (A.11) to $\||\phi|^4\phi\|_{H^{\frac{d-5}{2}}}$,

$$\leq C^k \int_0^T \int \|\phi\|_{H^{\frac{d-1}{2}}}^{\frac{16}{5}k+2} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{2(d+3)}} \|P_{\leq M_0}\phi\|_{H^{\frac{d-1}{2}}} + \|P_{>M_0}\phi\|_{H^{\frac{d-1}{2}}} \right)^{\frac{4k}{5}}$$
$$d|\mu_{t_{2k+1}}|(\phi) dt_{2k+1}.$$

□

4 Existence of Compatible Time Integration Domain

In this section, our main goal is to prove that

$$\gamma^{(1)}(t_1) = \sum_{reference\ (\hat{\mu}, s\hat{g}n)} \int_{T_C(\hat{\mu}, s\hat{g}n)} J_{\hat{\mu}, s\hat{g}n}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}, \quad (4.1)$$

where the number of reference pairs can be controlled by 16^k, and the summands are endowed with the compatible time integration domain $T_C(\hat{\mu}, s\hat{g}n)$ that we introduce in Section 3.2. We divide this section into two main parts. In Section 4.1, we first recall the quintic KM board game argument and then give an introduction to an admissible tree diagram representation as a preparation for the subsequent sections. Such type of tree also gives an elaborated proof of the quintic KM board game argument. Then in Section 4.2- 4.5, we prove the extended quintic KM board game argument, which allows to sort the summands in the initial Duhamel-Born expansion $\gamma^{(1)}$ into a sum of reference forms with the compatible time integration domain.

4.1 Admissible Tree

We first give a brief review of the quintic KM board game argument as in [8, 52]. In short, one could sort $(2k-1)!!2^k$ summands into a sum of upper echelon forms with the time integration domain, denoted by D_m, which is a union of a very large number of high dimensional simplexes. The number of upper echelon forms can be controlled by 8^k. Then, we give an introduction to an admissible tree diagram representation which could provide an elaborated proof of the quintic KM board game argument. Besides, one could use it to calculate D_m explicitly, which was unknown.

Recall that $\{\mu\}$ is a set of maps from $\{2, 4, ..., 2k\}$ to $\{1, 2, 3, ..., 2k-1\}$ satisfying $\mu(2) = 1$ and $\mu(2l) < 2l$ for all $2l$. For convenience, we extend the domain to $\{2, 3, 4, ..., 2k\}$ and define

$$\mu(2l+1) := \mu(2l) \quad l \in \{1, 2, ..., k-1\}. \tag{4.2}$$

Moreover, if μ satisfies $\mu(2j) \le \mu(2j+2)$ for $1 \le j \le k-1$, then it is in upper echelon form as they are called in [8, 52].

Let $P = \{\rho\}$ be a set of permutations of $\{2, 4, ..., 2k\}$. To be compatible with the definition (4.2), we also extend the domain to $\{2, 3, 4, ..., 2k+1\}$ and define

$$\rho(2l+1) := \rho(2l) + 1, \quad l \in \{1, 2, ..., k\}. \tag{4.3}$$

We note that P is closed under the composition and inverse operations.

Associated to each μ and $\sigma \in P$, we define the Duhamel integrals

$$I(\mu, \sigma, f^{(2k+1)}) = \int_{t_1 \ge t_{\sigma(3)} \ge \cdots \ge t_{\sigma(2k+1)}} J_{\mu}^{(2k+1)}(f^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}. \tag{4.4}$$

where

$$\begin{aligned} J_{\mu}^{(2k+1)}(f^{(2k+1)})(t_1, \underline{t}_{2k+1}) =& U^{(1)}(t_1 - t_3) B_{\mu(2);2,3} U^{(3)}(t_3 - t_5) B_{\mu(4);4,5} \\ & \cdots U^{(2k-1)}(t_{2k-1} - t_{2k+1}) B_{\mu(2k);2k,2k+1} f^{(2k+1)}(t_{2k+1}) \end{aligned}$$

and $f^{(2k+1)}$ is a symmetric density.

Definition 4.1 For fixed $j \in \{2, 3, .., k-1\}$ and a permutation $\rho = (2j, 2j+2) \circ (2j+1, 2j+3) \in P$, if $\mu(2j) \ne \mu(2j+2)$ and $\mu(2j+2) < 2j$, we define the action as follows:

$$\begin{aligned} \mu' &= (2j, 2j+2) \circ (2j+1, 2j+3) \circ \mu \circ (2j, 2j+2) \circ (2j+1, 2j+3), \\ \sigma' &= (2j, 2j+2) \circ (2j+1, 2j+3) \circ \sigma. \end{aligned}$$

We call the action induced by ρ, which we simply denote $KM(\rho)$, a Klainerman-Machedon acceptable move in Chen-Pavlović format of μ, or an acceptable move of μ for simplicity.

For general case, we also call a permutation ρ a Klainerman-Machedon acceptable move in Chen-Pavlović format of μ, if $\rho = \rho_r \circ \rho_{r-1} \circ \cdots \circ \rho_1$ where ρ_1 is an acceptable move of μ and $\rho_i = (2j_i, 2j_i + 2) \circ (2j_i + 1, 2j_i + 3)$ is an acceptable move of $\mu_i = KM(\rho_{i-1}) \circ \cdots \circ KM(\rho_1)(\mu)$ for $2 \leq i \leq r$. Moreover, we define the action $(\mu', \sigma') = KM(\rho)(\mu, \sigma)$:

$$\mu' = \rho \circ \mu \circ \rho^{-1},$$
$$\sigma' = \rho \circ \sigma.$$

If μ and μ' are such that there exists ρ as above for which $(\mu', \sigma') = KM(\rho)(\mu, \sigma)$ then we say that μ' and μ are KM-relatable. This is an equivalence relation that partitions the set of collapsing maps into equivalence classes.

Now, we could describe the quintic KM board game argument in [8, 52]. Namely, for every μ, there is exactly one μ_m in upper echelon form, which is KM-related to μ and the number of upper echelon forms can be controlled by 8^k. Moreover, it follows from [8,Lemma 7.1] that

$$I(\mu, \sigma, f^{(2k+1)}) = I(\mu', \sigma', f^{(2k+1)}). \tag{4.5}$$

With the equality (4.5), one has

$$\sum_{\mu \sim \mu_m} I(u, id, \gamma^{(2k+1)}) = \int_{D_m} J_{\mu}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \tag{4.6}$$

where the time integration domain D_m is a union of the simplexes $\{t_1 \geq t_{\sigma(3)} \geq \cdots \geq t_{\sigma(2k+1)}\}$.

The time integration domain D_m is obviously very complicated for large k, as it is a union of a very large number of simplexes in high dimension. To calculate D_m, we construct a ternary tree with the following algorithm.

Algorithm 6 (1) Set counter $j = 1$.

(2) Given j, find the indices l, m, r so that $l > j, m > r, r > j$ and

$$\mu(2l) = \mu(2j),$$
$$\mu(2m) = 2j,$$
$$\mu(2r) = 2j + 1,$$

and l, m and r are the minimal indices for which the above equalities hold. Then place $2l/2m/2r$ as the left/middle/right child of node $2j$ in the tree. If there is no such $l/m/r$, the node $2j$ will be missing a left/middle/right child.

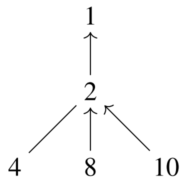
(3) If $j = k$, then stop, otherwise set $j = j + 1$ and go to step (2).

Since it requires $\mu(2j) < 2j$, one can check that every node $2j$ has a parent by induction argument. Hence, the generated tree by μ, which we denote by $T(\mu)$, is a connected ternary tree with child node's label strictly larger than its parent node's label.

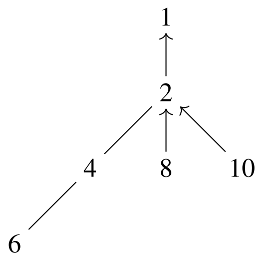
Example 4.2 Let us work with the following example

$2j$	2	4	6	8	10
$\mu(2j)$	1	1	1	2	3

By Algorithm 2, we start with $j = 1$ and note that $\mu(2) = 1$. For the left, middle and right child of node 2, we need to respectively find the minimal $a > 1$, $b > 1$ and $c > 1$ such that $\mu(2a) = 1$, $\mu(2b) = 2$ and $\mu(2c) = 3$. In the case, it is $a = 2$, $b = 4$ and $c = 5$, so we put 2, 8, and 10 as left, middle and right children of node 2, respectively, in the tree[8].



Next we turn to $j = 2$. Since $\mu(4) = 1$, we find the minimal $a > 2$, $b > 2$ and $c > 2$ such that $\mu(2a) = \mu(4) = 1$, $\mu(2b) = 4$ and $\mu(2c) = 5$. We find $a = 3$ and there is no such b or c satisfying the above condition, so we only put 6 as the left child of node 4 in the tree.

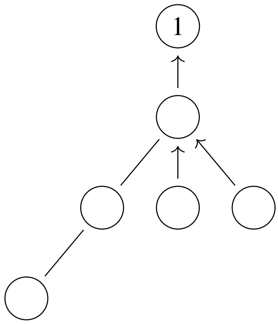


Since all indices appear in the tree, it is complete.

Definition 4.3 A ternary tree is called an admissible tree if every child node's label is strictly larger than its parent node's label. For an admissible tree, we call, the graph of the tree without any labels in its nodes, the skeleton of the tree.

For example, the skeleton of the tree in Example 4.2 is shown as follows.

[8] We use a line to link the left child and an arrow to link the middle/right child, as we would like to emphasize the differences between the left child and the middle/right child. Besides, by this way, it is convenient to calculate the tier value which we introduce in Section 4.3.



Given an admissible ternary tree, we can uniquely reconstruct a collapsing map μ that generates it. For notational convenience, we take the following notations.

$$2l \xrightarrow{L} 2j : \text{node } 2l \text{ is the left child of node } 2j,$$
$$2l \xrightarrow{M} 2j : \text{node } 2l \text{ is the middle child of node } 2j,$$
$$2l \xrightarrow{R} 2j : \text{node } 2l \text{ is the right child of node } 2j,$$
$$2l \rightarrow 2j : \text{node } 2l \text{ is a child of node } 2j.$$

Algorithm 7 (From admissible tree to collapsing map)

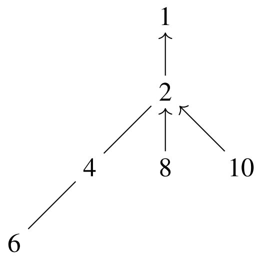
(1) Set counter $j = 1$ and $\mu(2) = 1$.
(2) Given j, in the admissible tree α,

$$\text{if there exists } 2k_1 \text{ such that } 2k_1 \xrightarrow{L} 2j, \text{ then } \mu(2k_1) := \mu(2j);$$
$$\text{if there exists } 2k_2 \text{ such that } 2k_2 \xrightarrow{M} 2j, \text{ then } \mu(2k_2) := 2j;$$
$$\text{if there exists } 2k_3 \text{ such that } 2k_3 \xrightarrow{R} 2j, \text{ then } \mu(2k_3) := 2j + 1.$$

Otherwise, go to step (3).
(3) Set $j = j + 1$. If $j = k$, then stop, otherwise go to step (2).

Since it is an admissible tree α, one can see that, if $j = l$, we have defined $\mu(2i)$ for $1 \leq i \leq l$ and $\mu(2i) < 2i$ by the step (2). Especially, when $j = k$, we generate a collapsing map μ. Moreover, one has that $T(\mu)$ equals to tree α.

Example 4.4 Suppose we are given the tree as follows.



At first, let $\mu(2) = 1$. As there are left, middle and right children of node 2, we define $\mu(4) = 1$, $\mu(8) = 2$, and $\mu(10) = 3$. Next, turn to node 4. There only exists the left child and hence we define $\mu(6) = \mu(4) = 1$. Finally we arrive at

$2j$	2	4	6	8	10
$\mu(2j)$	1	1	1	2	3

Note that the upper echelon form μ_m is unique in every equivalent class. Given a skeleton tree, there also exists a unique upper echelon tree. We give an algorithm to uniquely produce an upper echelon tree.

Algorithm 8 (Generate an upper echelon tree)

(1) Given a skeleton tree with $k+1$ nodes, label the top node with 1 and set counter $j=1$.
(2) If the node labeled $2j$ has a left child, then label that left child node with $2(j+1)$, set counter $j=j+1$ and go to step (4). If not, go to step (3).
(3) In the already labeled nodes which has an unlabeled middle or right child, search for the node with the smallest label. If such a node has an unlabeled middle child, label the middle child with $2(j+1)$, set counter $j=j+1$, and go to step (4). If such a node has no unlabeled middle child but an unlabeled right child, label the right child with $2(j+1)$, set counter $j=j+1$, and go to step (4). If none of the labeled nodes has an unlabeled middle or right child, then stop.
(4) If $j=k$, then stop, otherwise go to step (2).

Next, we are able to show that acceptable moves preserve the tree structures but permute the labeling under the admissibility requirement.

Proposition 4.5 *Two collapsing maps μ and μ' are KM-relatable if and only if the trees corresponding to μ and μ' have the same skeleton. Moreover, if $\mu' = KM(\rho)(\mu)$, then $T(\mu')$ has the same skeleton to $T(\mu)$ with node $2j$ replaced by $\rho(2j)$.*

Proof Without loss, we might as well assume that $\rho=(2j_0,2j_0+2)\circ(2j_0+1,2j_0+3)$ and $\mu'=KM(\rho)(\mu)$. With node $2j$ in the tree $T(\mu)$ replaced by $\rho(2j)$, it generates a tree α'' with the same skeleton as $T(\mu)$. Since $\rho\in P$ is an acceptable move with respect to μ, we have $\mu(2j_0+2)<2j_0$ and $\mu(2j_0)\neq\mu(2j_0+2)$, which implies that α'' is also an admissible tree. By Algorithm 7, it generates a collapsing map μ''. Thus it suffices to prove $\mu'=\mu''$, or equivalently, $\mu''=KM(\rho)(\mu)$. Note that

$$2l \overset{L/M/R}{\longrightarrow} 2j \text{ in the tree } T(\mu) \Longleftrightarrow \rho(2l) \overset{L/M/R}{\longrightarrow} \rho(2j) \text{ in the tree } \alpha''.$$

By Algorithm 7, it implies that

$$\begin{cases} \mu(2l)=\mu(2j) & \Longleftrightarrow \mu''(\rho(2l))=\mu''(\rho(2j)),\\ \mu(2l)=2j & \Longleftrightarrow \mu''(\rho(2l))=\rho(2j),\\ \mu(2l)=2j+1 & \Longleftrightarrow \mu''(\rho(2l))=\rho(2j)+1. \end{cases} \tag{4.7}$$

With $\mu(2)=\mu''(2)=1$, by induction argument we obtain

$$\rho\circ\mu(2l)=\mu''(\rho(2l)),$$

that is, $\mu''=KM(\rho)(\mu)$.

Conversely, we suppose that $T(\mu)$ has the same skeleton as $T(\mu')$. By Algorithm 8 and Algorithm 7, it generates a unique collapsing map μ_s which is in upper echelon form for the skeleton of $T(\mu)$. On the other hand, there exist an acceptable move σ with respect to μ as well as μ_m, which is in an upper echelon form, such that $\mu_m = KM(\sigma)(\mu)$. Since $T(\mu_m)$ also has the same skeleton as $T(\mu)$, it gives that $\mu_m = \mu_s$. In the same way, we also have $\mu'_m = \mu_s$, which implies that μ and μ' are KM-relatable. □

Given k, we would like to have the number of different ternary tree structures of k nodes, which equals to the number of equivalent classes. This number is exactly defined as the generalized Catalan number (see [39]), that is,

$$\frac{1}{k}\binom{3k}{k-1} \tag{4.8}$$

which can be controlled by 8^k by Stirling's approximation to $k!$. Hence, we just provide a proof of the quintic KM board game argument.

Now, let us get to the main part, namely, how to compute D_m for a given upper echelon class. We define a map T_D which maps an admissible tree $T(\mu)$ to a time integration domain

$$T_D(\mu) = \left\{t_{2j+1} \geq t_{2l+1} : 2l \to 2j \text{ in the tree } T(\mu)\right\} \bigcap \{t_1 \geq t_3\}. \tag{4.9}$$

where $2l \to 2j$ denotes that node $2l$ is a child of node $2j$.

Proposition 4.6 *Given a μ_m in upper echelon form, we have*

$$\sum_{\mu\sim\mu_m} \int_{t_1\geq t_3\geq \ldots \geq t_{2k+1}} J_{\mu}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}$$
$$= \int_{T_D(\mu_m)} J_{\mu_m}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}.$$

and hence

$$\gamma^{(1)}(t_1) = \sum_{\mu_m: upper\ echelon\ form} \int_{T_D(\mu_m)} J_{\mu_m}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}. \tag{4.10}$$

Proof Let $\Sigma(\mu_m)$ be the set of all acceptable moves with respect to μ_m. Then by the equality (4.5), we have

$$\sum_{\mu\sim\mu_m} I(\mu, id, \gamma^{(2k+1)}) = \sum_{\rho\in\Sigma(\mu_m)} I(\mu_m, \rho^{-1}, \gamma^{(2k+1)}).$$

By Proposition 4.5, we see that

$$\Sigma(\mu_m) = \{\rho \in P : \rho(2j) < \rho(2l), \text{ if } 2l \to 2j \text{ in the tree } T(\mu_m)\} \tag{4.11}$$

Table 1 Acceptable moves and Time integration domain

$2j$	2	4	6	8	10	$2j$	2	4	6	8	10	Time integration domain
$\rho_1(2j)$	2	4	6	8	10	$\rho_1^{-1}(2j)$	2	4	6	8	10	$\{t_1 \geq t_3 \geq t_5 \geq t_7 \geq t_9 \geq t_{11}\}$
$\rho_2(2j)$	2	4	6	10	8	$\rho_2^{-1}(2j)$	2	4	6	10	8	$\{t_1 \geq t_3 \geq t_5 \geq t_7 \geq t_{11} \geq t_9\}$
$\rho_3(2j)$	2	4	8	6	10	$\rho_3^{-1}(2j)$	2	4	8	6	10	$\{t_1 \geq t_3 \geq t_5 \geq t_9 \geq t_7 \geq t_{11}\}$
$\rho_4(2j)$	2	4	8	10	6	$\rho_4^{-1}(2j)$	2	4	10	6	8	$\{t_1 \geq t_3 \geq t_5 \geq t_{11} \geq t_7 \geq t_9\}$
$\rho_5(2j)$	2	4	10	6	8	$\rho_5^{-1}(2j)$	2	4	8	10	6	$\{t_1 \geq t_3 \geq t_5 \geq t_9 \geq t_{11} \geq t_7\}$
$\rho_6(2j)$	2	4	10	8	6	$\rho_6^{-1}(2j)$	2	4	10	8	6	$\{t_1 \geq t_3 \geq t_5 \geq t_{11} \geq t_9 \geq t_7\}$
$\rho_7(2j)$	2	6	8	4	10	$\rho_7^{-1}(2j)$	2	8	4	6	10	$\{t_1 \geq t_3 \geq t_9 \geq t_5 \geq t_7 \geq t_{11}\}$
$\rho_8(2j)$	2	6	8	10	4	$\rho_8^{-1}(2j)$	2	10	4	6	8	$\{t_1 \geq t_3 \geq t_{11} \geq t_5 \geq t_7 \geq t_9\}$
$\rho_9(2j)$	2	6	10	4	8	$\rho_9^{-1}(2j)$	2	8	4	10	6	$\{t_1 \geq t_3 \geq t_9 \geq t_5 \geq t_{11} \geq t_7\}$
$\rho_{10}(2j)$	2	6	10	8	4	$\rho_{10}^{-1}(2j)$	2	10	4	8	6	$\{t_1 \geq t_3 \geq t_{11} \geq t_5 \geq t_9 \geq t_7\}$
$\rho_{11}(2j)$	2	8	10	4	6	$\rho_{11}^{-1}(2j)$	2	8	10	4	6	$\{t_1 \geq t_3 \geq t_9 \geq t_{11} \geq t_5 \geq t_7\}$
$\rho_{12}(2j)$	2	8	10	6	4	$\rho_{12}^{-1}(2j)$	2	10	8	4	6	$\{t_1 \geq t_3 \geq t_{11} \geq t_9 \geq t_5 \geq t_7\}$

and hence

$$\bigcup_{\rho \in \Sigma(\mu_m)} \left\{t_1 \geq t_{\rho^{-1}(3)} \geq \cdots \geq t_{\rho^{-1}(2k+1)}\right\} = T_D(\mu_m). \tag{4.12}$$

□

Example 4.7 Let us demonstrate Proposition 4.6 by an example. Recall the upper echelon tree $T(\mu)$ in Example 4.2.

$2j$	2	4	6	8	10
$\mu_1(2j)$	1	1	1	2	3

There are 12 acceptable moves with respect to μ_1 such that

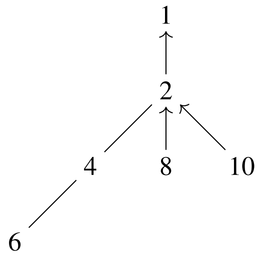
$$u_i = KM(\rho_i)(\mu_1).$$

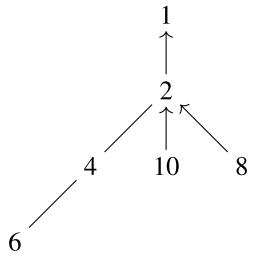
Here are all the admissible trees equivalent to $T(\mu_1)$.

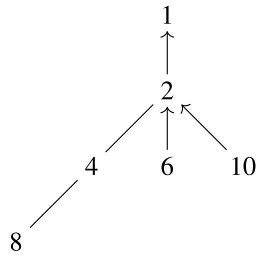
$2j$	2	4	6	8	10
$\mu_1(2j)$	1	1	1	2	3

$2j$	2	4	6	8	10
$\mu_2(2j)$	1	1	1	3	2

$2j$	2	4	6	8	10
$\mu_3(2j)$	1	1	3	1	2



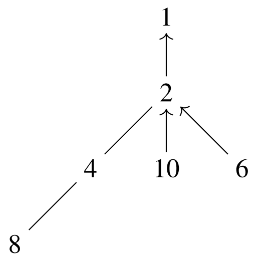


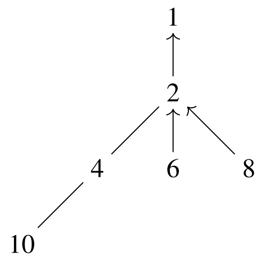


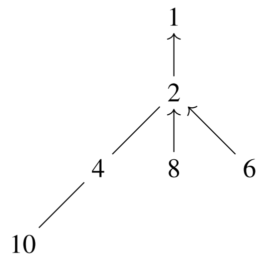
$2j$	2	4	6	8	10
$\mu_4(2j)$	1	1	3	1	2

$2j$	2	4	6	8	10
$\mu_5(2j)$	1	1	2	3	1

$2j$	2	4	6	8	10
$\mu_6(2j)$	1	1	3	2	1



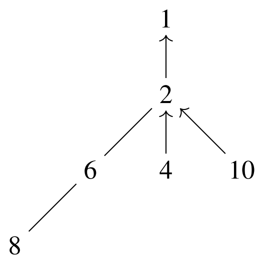


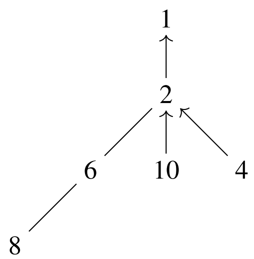


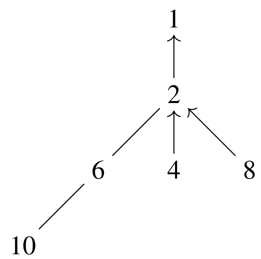
$2j$	2	4	6	8	10
$\mu_7(2j)$	1	2	1	1	3

$2j$	2	4	6	8	10
$\mu_8(2j)$	1	3	1	1	2

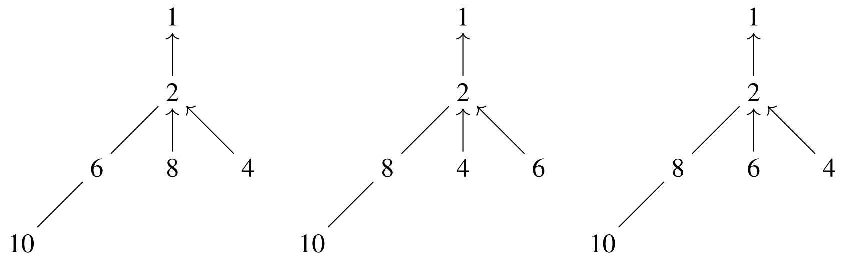
$2j$	2	4	6	8	10
$\mu_9(2j)$	1	2	1	3	1







$2j$	2	4	6	8	10	$2j$	2	4	6	8	10	$2j$	2	4	6	8	10
$\mu_{10}(2j)$	1	3	1	2	1	$\mu_{11}(2j)$	1	2	3	1	1	$\mu_{12}(2j)$	1	3	2	1	1



From the above time integration domains in Table 1, we have

$$\bigcup_{\rho_i \in \Sigma(\mu_1)} \left\{ t_1 \geq t_{\rho_i^{-1}(3)} \geq t_{\rho_i^{-1}(5)} \geq t_{\rho_i^{-1}(7)} \geq t_{\rho_i^{-1}(9)} \geq t_{\rho_i^{-1}(11)} \right\}$$
$$= \{t_1 \geq t_3, t_3 \geq t_5 \geq t_7, t_3 \geq t_9, t_3 \geq t_{11}\}.$$

By the definition,

$$T_D(\mu_1) = \{t_1 \geq t_3, t_3 \geq t_5 \geq t_7, t_3 \geq t_9, t_3 \geq t_{11}\}.$$

Hence,

$$\bigcup_{\rho_i \in \Sigma(\mu_1)} \left\{ t_1 \geq t_{\rho_i^{-1}(3)} \geq t_{\rho_i^{-1}(5)} \geq t_{\rho_i^{-1}(7)} \geq t_{\rho_i^{-1}(9)} \geq t_{\rho_i^{-1}(11)} \right\} = T_D(\mu_1).$$

4.2 Signed KM Acceptable Moves

In Section 4.1, we provide a method to compute $T_D(\mu)$. A nontrivial application is that, the original KM board game argument is not compatible with space-time multilinear estimates. Indeed, depending on the sign combination in the Duhamel expansion $J_{\mu_m}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1})$, one could run into the problem that one needs to estimate the x part and the x' part using the same time integral. (See [20,Example 4].) To be compatible with the estimate part in Section 3, we restart with the signed collapsing pair (μ, sgn).

First, we rewrite

$$\gamma^{(1)} = \sum_{(\mu,sgn)} I(\mu, id, sgn, \gamma^{(2k+1)}), \tag{4.13}$$

where

$$I(\mu, \sigma, sgn, \gamma^{(2k+1)}) = \int_{t_1 \geq t_{\sigma(3)} \geq \cdots \geq t_{\sigma(2k+1)}} J_{\mu,sgn}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \tag{4.14}$$

and

$$J_{\mu,sgn}^{(2k+1)}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) = U^{(1)}(t_1 - t_3) B_{\mu(2);2,3}^{sgn(2)} U^{(3)}(t_3 - t_5) B_{\mu(4);4,5}^{sgn(4)} \cdots U^{(2k-1)}(t_{2k-1} - t_{2k+1}) B_{\mu(2k);2k,2k+1}^{sgn(2k)} \gamma^{(2k+1)}(t_{2k+1}) \quad (4.15)$$

where the notations have been introduced in (2.19).

For convenience, we define

$$sgn(2l+1) := sgn(2l)$$

for $l \in \{1, 2, ..., k\}$.

Definition 4.8 Let ρ be an acceptable move of μ. We define a signed version of the KM acceptable move in Chen-Pavlović format, still denoted by $KM(\rho)$, as follows:

$$(\mu', \sigma', sgn') = KM(\rho)(\mu, \sigma, sgn)$$

where

$$\begin{aligned} \mu' &= \rho \circ \mu \circ \rho^{-1}, \\ \sigma' &= \rho \circ \sigma, \\ sgn' &= sgn \circ \rho^{-1}. \end{aligned}$$

If (μ, σ, sgn) and (μ', σ', sgn') are such that there exists ρ as above for which

$$(\mu', \sigma', sgn') = KM(\rho)(\mu, \sigma, sgn)$$

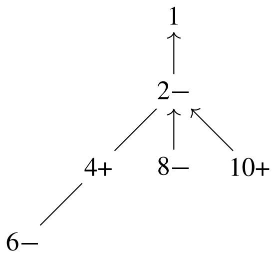
then we say that (μ, σ, sgn) and (μ', σ', sgn') are KM-relatable. With a slight modification of the argument in [8, 52], we also have

$$I(\mu', \sigma', sgn', \gamma^{(2k+1)}) = I(\mu, \sigma, sgn, \gamma^{(2k+1)}). \quad (4.16)$$

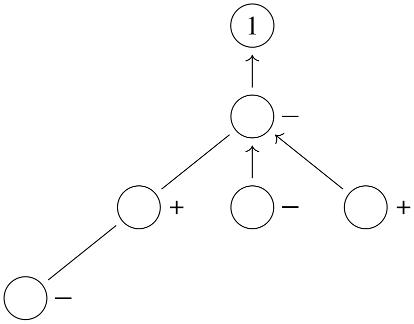
Example 4.9 We consider the following pair (μ, sgn)

$2j$	2	4	6	8	10
$\mu(2j)$	1	1	1	2	3
$sgn(2j)$	$-$	$+$	$-$	$-$	$+$

By Algorithm 6, with adding the sign, it generates a signed admissible tree as follows



Definition 4.10 For a skeleton tree, we call it the signed skeleton tree if we add the sign. For example, the signed skeleton of the tree in Example 4.9 is shown as follows.



Similarly, the signed acceptable moves also preserve the signed tree structures.

Proposition 4.11 *Two collapsing map pairs are KM-relatable if and only if they have the same signed skeleton tree.*

Proof By Proposition 4.5, it suffices to prove that ρ keeps the sign invariant. Indeed, we note that node $2j$ in the tree $T(\mu)$ is corresponding to node $\rho(2j)$ in the tree $T(\mu')$ and hence $sgn'(\rho(2j)) = sgn(2j)$. □

4.3 Tamed Form

We will prove that there exists a unique special form, which we call the tamed form, in every equivalent class. First, through an example, we present an algorithm for producing the tamed enumeration of a signed skeleton. Then we exhibit how to reduce a signed tree with same skeleton but different enumeration into the tamed form using signed KM acceptable moves. In the end, we arrive at

$$\gamma^{(1)}(t_1) = \sum_{(\mu_*, sgn_*) \text{ tamed}} \int_{T_D(\mu_*)} J^{(2k+1)}_{\mu_*, sgn_*}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}. \tag{4.17}$$

which is an adaptation of representation (4.10).

Definition 4.12 We call $2j \geq 4$ is tier of q if

$$\mu^q(2j) = 1 \quad \text{but} \quad \mu^{q-1}(2j) > 1$$

where $\mu^q = \mu \circ \cdots \circ u$, the composition taken q times. We write $t(2j)$ for the tier value of $2j$[9].

Definition 4.13 A pair (μ, sgn) is tamed if it meets the following four requirements:

(1) If $t(2l) < t(2r)$, then $2l < 2r$.
(2) If $t(2l) = t(2r)$, $\mu^2(2l) = \mu^2(2r)$, $sgn(\mu(2l)) = sgn(\mu(2r))$ and $\mu(2l) < \mu(2r)$, then $2l < 2r$.
(3) If $t(2l) = t(2r)$, $\mu^2(2l) = \mu^2(2r)$, $sgn(\mu(2l)) = +$, $sgn(\mu(2r)) = -$, then $2l < 2r$.
(4) If $t(2l) = t(2r)$, $\mu^2(2l) \neq \mu^2(2r)$, $\mu(2l) < \mu(2r)$, then $2l < 2r$.

Conditions (2), (3), and (4) specify the ordering for $2l$ and $2r$ belonging to the same tier. More precisely, rule (2) says that the ordering of middle child is prior to the one of right child and the ordering follows the parental ordering if two different parents belong to the same left branch with the same sign. Rule (3) says that if the parents belong to the same left branch, a positive parent dominates over a negative parent. Finally, if the parents do not belong to the same left branch, rule (4) says that the ordering follows the parental ordering regardless of the signs of the parents.

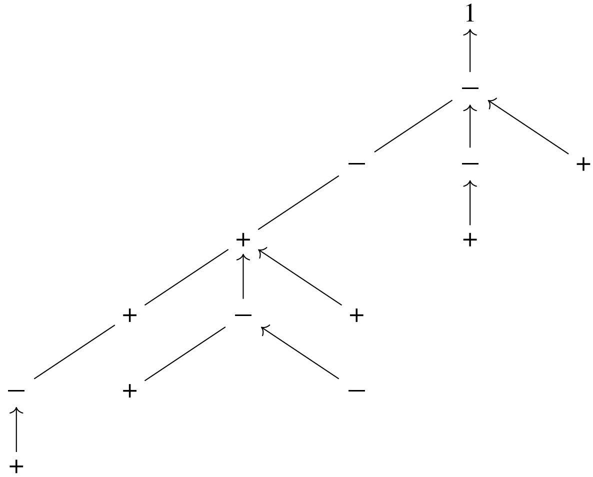
Example 4.14 The pair (μ_*, sgn_*) in the following chart

$2j$	2	4	6	8	10	12	14	16	18	20	22	24	26
$\mu_*(2j)$	1	1	1	1	1	6	6	7	2	3	10	13	18
$sgn_*(2j)$	−	−	+	+	−	−	+	+	−	+	+	−	+
$t(2j)$	1	1	1	1	1	2	2	2	2	2	2	3	3

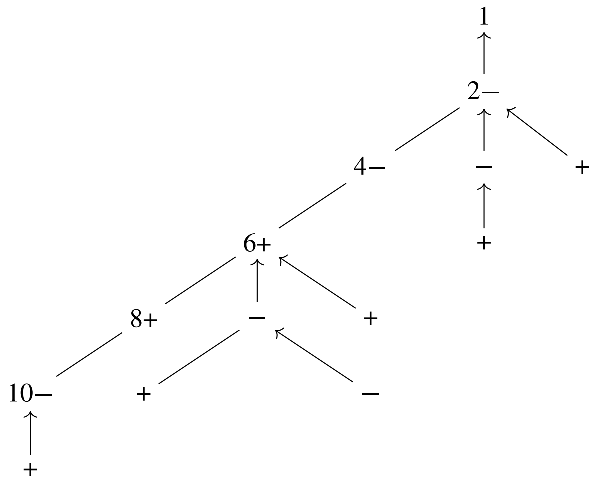
is tamed. We illustrate an algorithm for determining the unique tamed enumeration of a signed skeleton tree.

[9] The tier value of $2j$ equals to the number of the arrows from node $2j$ to node 1. That is why we use arrows to link the middle/right child.

We start with the skeleton (on the right) of the tree generated by (μ_*, sgn_*) with only the signs indicated.



We consider the left branch attached to node 1 where there are five nodes. Then we label the left branch in order with 2, 4, 6, 8, 10.



Now, we set a queue where we list the nodes + first and then the − nodes

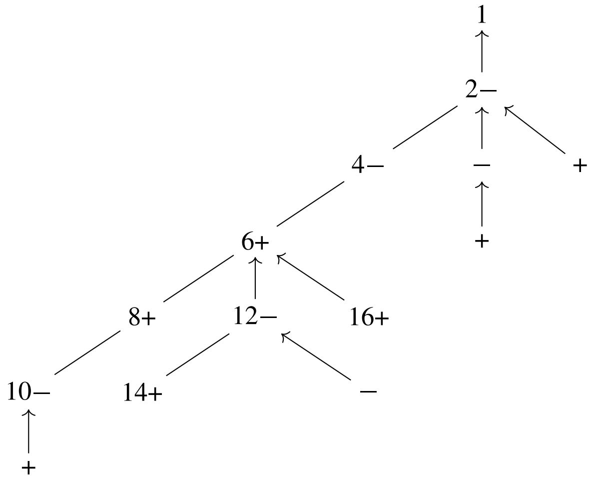
$$Queue : 6+, 8+, 2-, 4-, 10-$$

Then we work along the queue from left to right. Since 6+ has both a middle and right child, we first label the middle child and its left branch with the next available number 12 and 14 and add these numbers to the queue putting the + nodes before the − nodes

$$Queue : 6+, 8+, 2-, 4-, 10-, 14+, 12-$$

Next we label the right child with number 16 and add it to the queue

$$Queue : 6+, 8+, 2-, 4-, 10-, 14+, 12-, 16+$$



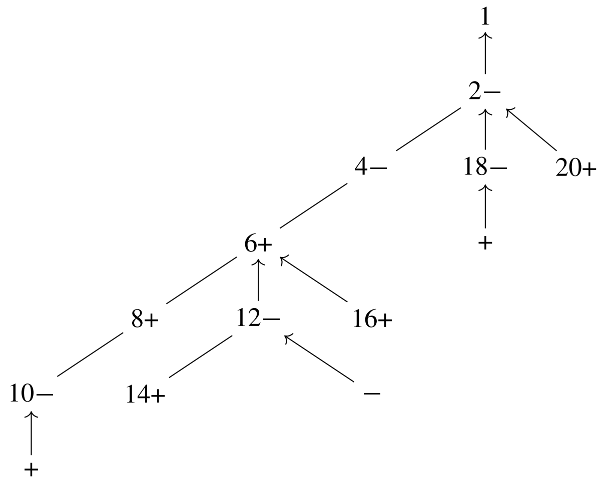
Since we have already dealt with 6+, we can pop it from the queue

$$Queue : 8+, 2-, 4-, 10-, 14+, 12-, 16+$$

Subquently, we come to the next node in the queue which is 8+. Since the node 8+ has no child, we skip and pop it from the queue

$$Queue : 2-, 4-, 10-, 14+, 12-, 16+$$

Then, we come to the node 2−, which has both a middle and right child. We first label the middle child with 18 and then the right child with 20.



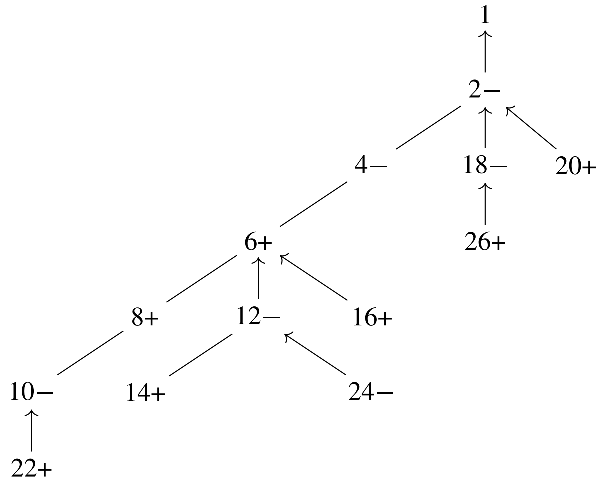
From the queue, we pop 2− and add 18−, 20+:

$$Queue : 4-, 10-, 14+, 12-, 16+, 18-, 20+$$

Since 4− has no child, we pop it and proceed to 10−, which has a middle child. We label it with 22. The queue is updated:

$$Queue : 14+, 12-, 16+, 18-, 20+, 22+$$

By turn, we arrive at the fully enumberated tree.



Here is the general algorithm to generate a tamed tree from a given signed skeleton tree.

Algorithm 9 (Generate a Tamed tree)

(1) Start with a queue that first contains only 1.

(2) If the queue is empty, then stop. If not, dequeue the leftmost entry l of the queue and go to step (3).
(3) If there is a middle child of l, pass to the middle child of l, and label its left branch with the next available label $2j, 2(j+1), ..., 2(j+q)$. If not, go to step (5).
(4) Take the left branch enumerated in step (3) and first list all $+$ nodes in order from $2j, 2(j+1), ..., 2(j+q)$ and add them to the right side of the queue, and then list in order all $-$ nodes from $2j, 2(j+1), ..., 2(j+q)$ and add them to the right side of the queue. Set the next available label to be $2(j+q+1)$, and go to step (5).
(5) If there is a right child of l, pass to the right child of l, and label its left branch with the next available label $2j, 2(j+1), ..., 2(j+q)$. If not, go to step (2).
(6) Take the left branch enumerated in step (5) and first list all $+$ nodes in order from $2j, 2(j+1), ..., 2(j+q)$ and add them to the right side of the queue, and then list in order all $-$ nodes from $2j, 2(j+1), ..., 2(j+q)$ and add them to the right side of the queue. Set the next available label to be $2(j+q+1)$, and go to step (2).

Next, we will explain how to execute a sequence of signed KM acceptable moves to bring a collapsing map pair (μ, sgn) into the tamed form. After presenting an example, we will give the general form of the algorithm.

Example 4.15 We consider the following collapsing map

$2j$	2	4	6	8	10	12	14	16	18	20	22	24	26
$\mu(2j)$	1	1	1	6	1	6	7	1	2	16	9	18	3
$sgn(2j)$	$-$	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$

By Algorithm 6, it generates $T(\mu, sgn)$ as shown in Fig. 3, which has the same signed skeleton tree with the collapsing map (μ_*, sgn_*) in Example 4.14.

Comparing Fig. 3 $T(u, sgn)$ with Fig. 4 $T(\mu_*, sgn_*)$, we note that the node 8 on the $T(\mu_*, sgn_*)$ is the first one that differs from the one on the $T(\mu, sgn)$, which is labeled 10. To change node 10 into node 8, we do KM(8,10) on (μ, sgn).

The KM(8,10) move is

$$\mu_1 = (8, 10) \circ (9, 11) \circ \mu \circ (8, 10) \circ (9, 11),$$
$$sgn_1 = sgn \circ (8, 10) \circ (9, 11).$$

It gives that

$2j$	2	4	6	8	10	12	14	16	18	20	22	24	26
$\mu_1(2j)$	1	1	1	1	6	6	7	1	2	16	11	18	3
$sgn_1(2j)$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$

Next, we compare Fig. 5 $T(u_1, sgn_1)$ with Fig. 6 $T(\mu_*, sgn_*)$ and find that the next different node is 10 in the tree $T(u_*, sgn_*)$, which is corresponding to node

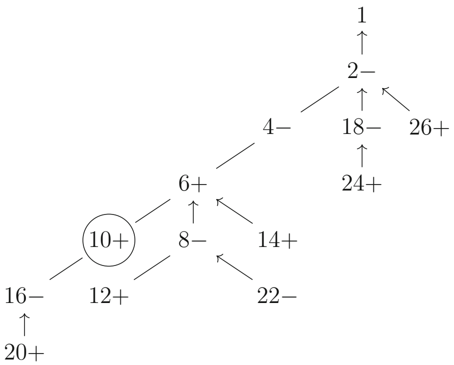


Fig. 3 $T(\mu, sgn)$

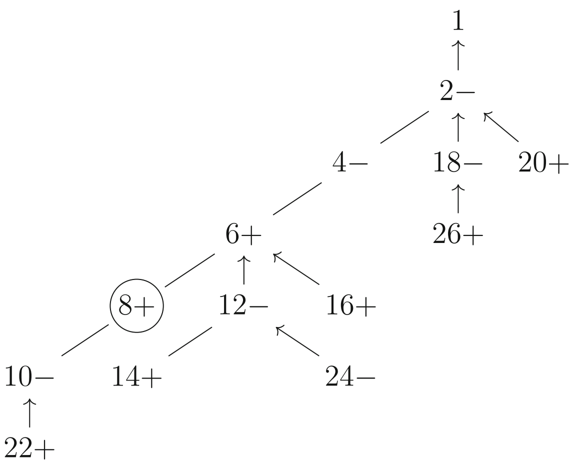


Fig. 4 $T(\mu_*, sgn_*)$

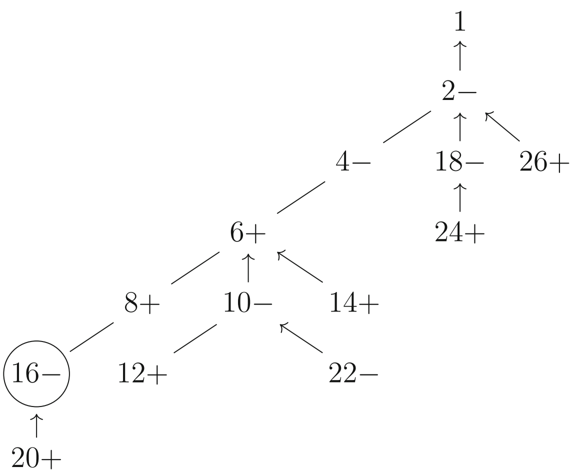


Fig. 5 $T(\mu_1, sgn_1)$

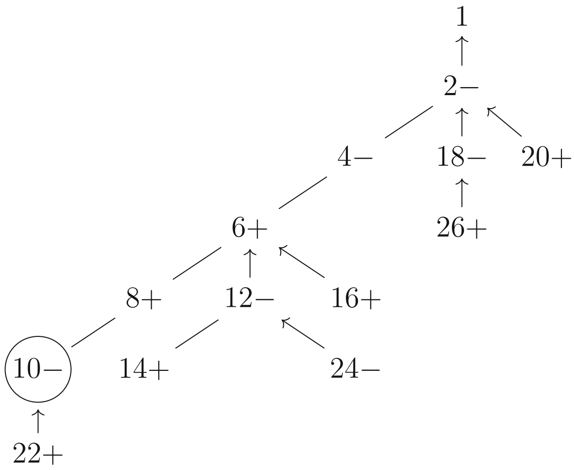


Fig. 6 $T(\mu_*, sgn_*)$

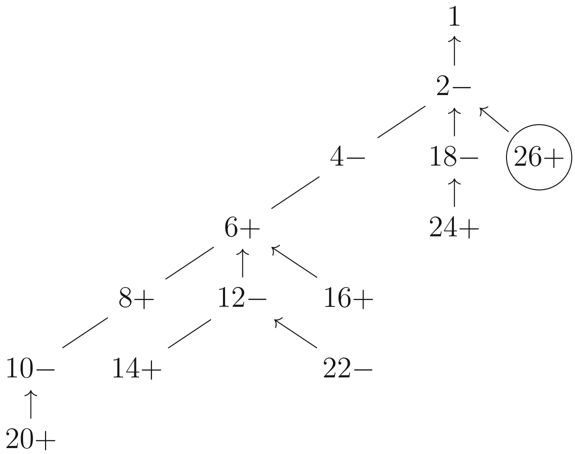


Fig. 7 $T(\mu_2, sgn_2)$

16 in the tree $T(u_1, sgn_1)$. Hence we do KM(14,16), KM(12,14) and KM(10,12) on (μ_1, sgn_1). Then we have

$$(\mu_2, sgn_2) = KM(10, 12) \circ KM(12, 14) \circ KM(14, 16)(\mu_1, sgn_1)$$

and

$2j$	2	4	6	8	10	12	14	16	18	20	22	24	26
$\mu_2(2j)$	1	1	1	1	1	6	6	7	2	10	13	18	3
$sgn_2(2j)$	−	−	+	+	−	−	+	+	−	+	−	+	+

Comparing Fig. 7 $T(u_2, sgn_2)$ and Fig. 8 $T(\mu_*, sgn_*)$, we find that the next different node is 20 in the tree $T(\mu_*, sgn_*)$, which is corresponding to node 26 in the tree

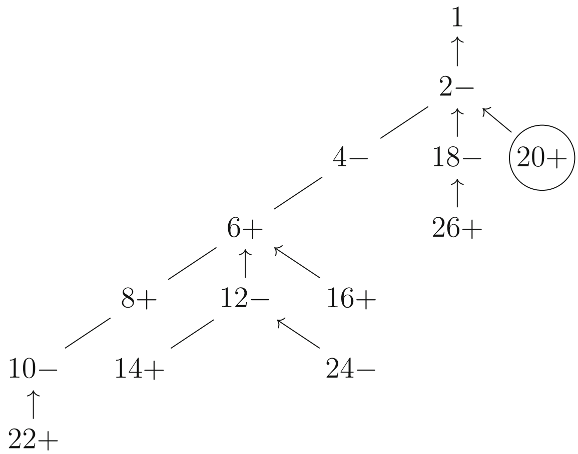


Fig. 8 $T(\mu_*, sgn_*)$

$T(\mu_3, sgn_3)$. Hence, we do KM(24,26), KM(22,24) and KM(20,22) on (u_2, sgn_2) to obtain

$$(\mu_3, sgn_3) = KM(20, 22) \circ KM(22, 24) \circ KM(24, 26)(\mu_2, sgn_2)$$

and

$2j$	2	4	6	8	10	12	14	16	18	20	22	24	26
$\mu_3(2j)$	1	1	1	1	1	6	6	7	2	3	10	13	18
$sgn_3(2j)$	−	−	+	+	−	−	+	+	−	+	+	−	+

We see that (μ_3, sgn_3) is just the tamed pair (μ_*, sgn_*) as shown in Example 4.14.

Here is a general algorithm to bring a collapsing map (μ, sgn) into the tamed form.

Algorithm 10 (Tamed form)

(1) Given a collapsing map pair (μ, sgn), by Algorithm 6, we obtain a signed admissible tree $T(\mu, sgn)$. From the signed skeleton tree, by Algorithm 9, it generates a tamed tree α.
(2) Set counter $j = 1$.
(3) If the node $2j$ in the tame tree α is corresponding to $2l$ in the tree $T(\mu, sgn)$, then set

$$(u', sgn') = KM(2j, 2j+2) \circ \cdots \circ KM(2l-4, 2l-2) \circ KM(2l-2, 2l)(\mu, sgn).$$

(4) Set $(\mu, sgn) = (\mu', sgn')$. If $j = k$, then stop, otherwise set $j = j + 1$ and go to step (3).

Next, we arrive at the main part, that is, the following adaptation of Proposition 4.6.

Proposition 4.16 *Within a signed KM-relatable equivalence class of collapsing map pairs* (μ, sgn)*, there is a unique tamed* (μ_*, sgn_*)*. Moreover,*

$$\sum_{(\mu,sgn)\sim(\mu_*,sgn_*)} I(\mu, id, sgn, \gamma^{(2k+1)}) = \int_{T_D(\mu_*)} J^{(2k+1)}_{\mu_*,sgn_*}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1})d\underline{t}_{2k+1} \tag{4.18}$$

where $T_D(\mu_*)$ *is defined by (4.9). Consequently,*

$$\gamma^{(1)}(t_1) = \sum_{(\mu_*,sgn_*)\ tamed} \int_{T_D(\mu_*)} J^{(2k+1)}_{\mu_*,sgn_*}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1})d\underline{t}_{2k+1} \tag{4.19}$$

where the number of tamed forms can be controlled by 16^k.

Proof The existence and uniqueness follow from Algorithm 9. For (4.18), the proof is the same as Proposition 4.6. As shown in (4.8), the number of different ternary tree structures of k nodes can be controlled by 8^k. By paying an extra factor of 2^k, which comes from the signs, there are at most 16^k tamed forms. □

The next step will be to rearrange the tamed pairs (μ_*, sgn_*) via wild moves, as defined and discussed in the next section. This will produce a further reduction of (4.19).

4.4 Wild Moves

We then introduce wild moves which keeps the tamed form invariant so that we can partition the class of tamed pairs (μ, sgn) into equivalence classes of wildly relatable forms.

Definition 4.17 Given a collapsing map pair (μ, sgn), define $G_i = \{2j : \mu(2j) = i\}$ for $i = 1, 2, ..., 2k-1$. We call $\rho \in P$ allowable with respect to (μ, sgn) if it satisfies the following two conditions:

(1) $\rho(G_i) = G_i$ for $i = 1, 2, ..., 2k-1$.
(2) If $2q < 2s$, $\mu(2q) = \mu(2s)$ and $sgn(2q) = sgn(2s)$, then $\rho(2q) < \rho(2s)$.
We denote the set of all allowable permutations ρ with respect to (μ, sgn) by $P(\mu, sgn)$. Note that condition (1) is equivalent to $\mu \circ \rho = \mu \circ \rho^{-1} = \mu$, which leaves all left branchs invariant. Moreover, if (μ, sgn) is in tamed form, G_i will be the form $\{2l, 2l+2, ..., 2r\}$.

Definition 4.18 (Wild move) Given a signed collapsing map (μ, sgn) and $\rho \in P(\mu, sgn)$, then the wild move $W(\rho)$ is defined as an action on a ternary (μ, σ, sgn), where

$$(\mu', \sigma', sgn') = W(\rho)(\mu, \sigma, sgn)$$

with

$$\mu' = \rho \circ \mu \circ \rho^{-1} = \rho \circ \mu,$$

$$\sigma' = \rho \circ \sigma,$$
$$sgn' = sgn \circ \rho^{-1}.$$

It is fairly straightforward to show that wild moves preserve the tamed class by using the definition of tamed form. It is noteworthy that the analogous statement for upper echelon forms does not hold and it is the purpose of introducing the tamed class.

Proposition 4.19 *Suppose (μ, sgn) is in tamed form, and $W(\rho)$ is a wild move defined as above. Then (μ', sgn') is also tamed.*

Proof It follows from the definition of tamed form. □

Thus we can say that two tamed forms (μ, sgn) and (μ', sgn') are wildly relatable if there exists an allowable permutation ρ such that

$$(\mu', \sigma', sgn') = W(\rho)(\mu, \sigma, sgn).$$

This is an equivalence relation that partitions the set of tamed forms into equivalence classes of wildly relatable forms.

The main result of this section is

Proposition 4.20 *Given a signed collapsing map (u, sgn) in tamed form and $\rho \in P(\mu, sgn)$ as in Definition 4.17, let*

$$(\mu', \sigma', sgn') = W(\rho)(\mu, \sigma, sgn).$$

Then for any symmetric density $\gamma^{(2k+1)}$, we have

$$J^{(2k+1)}_{\mu',sgn'}(\gamma^{(2k+1)})(t_1, \sigma'^{-1}(\underline{t}_{2k+1})) = J^{(2k+1)}_{\mu,sgn}(\gamma^{(2k+1)})(t_1, \sigma^{-1}(\underline{t}_{2k+1})). \quad (4.20)$$

Consequently,

$$\int_{\sigma[T_D(\mu)]} J^{(2k+1)}_{\mu,sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1})d\underline{t}_{2k+1}$$
$$= \int_{\sigma'[T_D(\mu)]} J^{(2k+1)}_{\mu',sgn'}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1})d\underline{t}_{2k+1} \quad (4.21)$$

where $\sigma[T_D(\mu)]$ is defined as follows

$$\sigma[T_D(\mu)] =: \left\{t_{\sigma(2j)+1} \geq t_{\sigma(2l)+1} : 2l \to 2j \text{ in the tree } T(\mu)\right\} \bigcap \left\{t_1 \geq t_{\sigma(2)+1}\right\}.$$

Proof Since (μ, sgn) is a tamed pair, G_i will be the form $\{2p, 2p+2, ..., 2q\}$. Then, $\rho \in P(\mu, sgn)$ can be written as a composition of permutations

$$\rho = \tau_1 \circ \cdots \circ \tau_s$$

with the property that each $\tau = (2l, 2l+2)\circ(2l+1, 2l+3)$ and $sgn(2l) \neq sgn(2l+2)$. Thus, it suffices to prove

$$U^{(2j-1)}(-t_{2j+1})B^{-}_{\mu(2j);2j,2j+1}U^{(2j+1)}(t_{2j+1}-t_{2j+3})B^{+}_{\mu(2j+2);2j+2,2j+3}U^{(2j+3)}(t_{2j+3})$$
$$= U^{(2j-1)}(-t_{2j+3})B^{+}_{\mu(2j);2j+2,2j+3}\widetilde{U}^{(2j+1)}(t_{2j+3}-t_{2j+1})B^{-}_{\mu(2j+2);2j,2j+1}U^{(2j+3)}(t_{2j+1})$$

where $\widetilde{U}^{(2j+1)}(t) = U^{(2j-1)}(t)e^{it(\Delta_{x_{2j+2}}-\Delta_{x'_{2j+2}})}e^{it(\Delta_{x_{2j+3}}-\Delta_{x'_{2j+3}})}$.

Without loss, we might as well take $j = 1$ and $\mu(2) = \mu(4) = 1$ so that this becomes

$$U^{(1)}(-t_3)B^{-}_{1;2,3}U^{(3)}(t_3-t_5)B^{+}_{1;4,5}U^{(5)}(t_5) = U^{(1)}(-t_5)B^{+}_{1;4,5}\widetilde{U}^{(3)}(t_5-t_3)B^{-}_{1;2,3}U^{(5)}(t_3). \quad (4.22)$$

For simplicity, we take the following notations

$$U^{(1)}(-t_3) = U^{1}_{-3}U^{1'}_{-3},$$
$$U^{(3)}(t_3-t_5) = U^{1}_{3,5}U^{1'}_{3,5}U^{2}_{3,5}U^{2'}_{3,5}U^{3}_{3,5}U^{3'}_{3,5},$$
$$U^{(5)}(t_5) = U^{1}_{5}U^{1'}_{5}U^{2}_{5}U^{2'}_{5}U^{3}_{5}U^{3'}_{5}U^{4}_{5}U^{4'}_{5}U^{5}_{5}U^{5'}_{5}.$$

where $U^{l}_{\pm j} = e^{\pm it_j\Delta_{x_l}}$, $U^{l'}_{\pm j} = e^{\mp it_j\Delta_{x'_l}}$, $U^{l}_{j,k} = U^{l}_{j}U^{l}_{-k}$ and $U^{l'}_{j,k} = U^{l'}_{j}U^{l'}_{-k}$.

Expanding $U^{(1)}(-t_3)$, $U^{(3)}(t_3-t_5)$ and $U^{(5)}(t_5)$ gives

$$U^{(1)}(-t_3)B^{-}_{1;2,3}U^{(3)}(t_3-t_5)B^{+}_{1;4,5}U^{(5)}(t_5)$$
$$= U^{1}_{-3}U^{1'}_{-3}B^{-}_{1;2,3}U^{1}_{3,5}U^{1'}_{3,5}U^{2}_{3,5}U^{2'}_{3,5}U^{3}_{3,5}U^{3'}_{3,5}B^{+}_{1;4,5}U^{1}_{5}U^{1'}_{5}U^{2}_{5}U^{2'}_{5}U^{3}_{5}U^{3'}_{5}U^{4}_{5}U^{4'}_{5}U^{5}_{5}U^{5'}_{5}$$

Since $B^{-}_{1;2,3}$ acts only on the 2, $2'$, 3, $3'$ and $1'$ coordinates, we exchange $B^{-}_{1;2,3}$ with $U^{1}_{3,5}$. In the same way, $B^{+}_{1;4,5}$ acts only on 4, $4'$, 5, $5'$ and 1 coordinates, so we exchange $B^{+}_{1;4,5}$ with $U^{1'}_{3,5}U^{2}_{3,5}U^{2'}_{3,5}U^{3}_{3,5}U^{3'}_{3,5}$. Thus, we have

$$U^{(1)}(-t_3)B^{-}_{1;2,3}U^{(3)}(t_3-t_5)B^{+}_{1;4,5}U^{(5)}(t_5)$$
$$= U^{1}_{-3}U^{1'}_{-3}U^{1}_{3,5}B^{-}_{1;2,3}B^{+}_{1;4,5}U^{1'}_{3,5}U^{2}_{3,5}U^{2'}_{3,5}U^{3}_{3,5}U^{3'}_{3,5}U^{1}_{5}U^{1'}_{5}U^{2}_{5}U^{2'}_{5}U^{3}_{5}U^{3'}_{5}U^{4}_{5}U^{4'}_{5}U^{5}_{5}U^{5'}_{5}$$

Exchanging $B^{-}_{1;2,3}$ with $B^{+}_{1;4,5}$ gives

$$= U^{1}_{-3}U^{1'}_{-3}U^{1}_{3,5}B^{+}_{1;4,5}B^{-}_{1;2,3}U^{1'}_{3,5}U^{2}_{3,5}U^{2'}_{3,5}U^{3}_{3,5}U^{3'}_{3,5}U^{1}_{5}U^{1'}_{5}U^{2}_{5}U^{2'}_{5}U^{3}_{5}U^{3'}_{5}U^{4}_{5}U^{4'}_{5}U^{5}_{5}U^{5'}_{5}$$

With $U^{1}_{-3}U^{1'}_{-3}U^{1}_{3,5} = U^{1}_{-5}U^{1'}_{-5}U^{1'}_{5,3}$ and $U^{1'}_{3,5}U^{1}_{5}U^{1'}_{5} = U^{1}_{5,3}U^{1}_{3}U^{1'}_{3}$, we obtain

$$= U^{1}_{-5}U^{1'}_{-5}U^{1'}_{5,3}B^{+}_{1;4,5}B^{-}_{1;2,3}U^{1}_{5,3}U^{1}_{3}U^{1'}_{3}U^{2}_{3}U^{2'}_{3}U^{3}_{3}U^{3'}_{3}U^{4}_{5}U^{4'}_{5}U^{5}_{5}U^{5'}_{5}$$

Exchanging $U^{1'}_{5,3}$ with $B^{+}_{1;4,5}$ and $B^{-}_{1;2,3}$ with $U^{1}_{5,3}U^{4}_{3}U^{4'}_{3}U^{5}_{3}U^{5'}_{3}$, we have

$$
\begin{aligned}
&= U^{1}_{-5}U^{1'}_{-5}B^{+}_{1;4,5}U^{1'}_{5,3}U^{1}_{5,3}U^{4}_{5,3}U^{4'}_{5,3}U^{5}_{5,3}U^{5'}_{5,3}B^{-}_{1;2,3}U^{1}_{3}U^{1'}_{3}U^{2}_{3}U^{2'}_{3}U^{3}_{3}U^{3'}_{3}U^{4}_{3}U^{4'}_{3}U^{5}_{3}U^{5'}_{3} \\
&= U^{(1)}(-t_5)B^{+}_{1;4,5}\widetilde{U}^{(3)}(t_5-t_3)B^{-}_{1;2,3}U^{(5)}(t_3).
\end{aligned}
$$

Since $\gamma^{(2k+1)}$ is a symmetric density, one can permute

$$
(x_2, x_3, x_4, x_5; x_2', x_3', x_4', x_5') \leftrightarrow (x_4, x_5, x_2, x_3; x_4', x_5', x_2', x_3').
$$

Then it gives that

$$
\begin{aligned}
&U^{(1)}(-t_5)B^{+}_{1;4,5}\widetilde{U}^{(3)}(t_5-t_3)B^{-}_{1;2,3}U^{(5)}(t_3) \longmapsto U^{(1)}(-t_5)B^{+}_{1;2,3}U^{(3)}(t_5-t_3)B^{-}_{1;4,5}U^{(5)}(t_3),\\
&B^{\pm}_{\mu(2l);2l,2l+1} \longmapsto B^{\pm}_{(2,4)\circ(3,5)\circ\mu(2l);2l,2l+1}, \quad l \geq 3,
\end{aligned}
$$

which proves equality (4.20). □

Example 4.21 Let us work with the following pair (μ_1, sgn_1)

$2j$	2	4	6	8	10	12	14
$\mu_1(2j)$	1	1	1	2	3	7	7
$sgn_1(2j)$	+	+	−	−	+	+	−

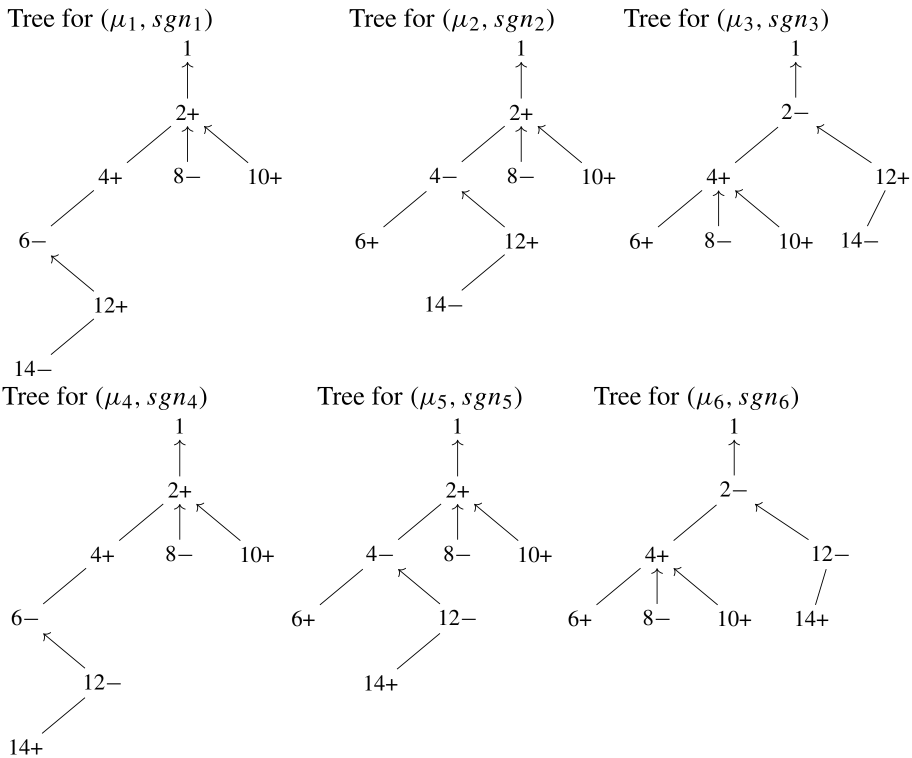
There are six wild moves as follows:

$$
(\mu_j, \sigma_j, sgn_j) = W(\rho_j)(\mu_1, id, sgn_1),
$$

	2	4	6	12	14		2	4	6	12	14	$\rho_j^{-1}[T_D(\mu_j)]$
ρ_1	2	4	6	12	14	ρ_1^{-1}	2	4	6	12	14	$\{t_3 \geq t_5 \geq t_7, t_7 \geq t_{13} \geq t_{15}, t_3 \geq t_9, t_3 \geq t_{11}\}$
ρ_2	2	6	4	12	14	ρ_2^{-1}	2	6	4	12	14	$\{t_3 \geq t_7 \geq t_5, t_7 \geq t_{13} \geq t_{15}, t_3 \geq t_9, t_3 \geq t_{11}\}$
ρ_3	4	6	2	12	14	ρ_3^{-1}	6	2	4	12	14	$\{t_7 \geq t_3 \geq t_5, t_7 \geq t_{13} \geq t_{15}, t_3 \geq t_9, t_3 \geq t_{11}\}$
ρ_4	2	4	6	14	12	ρ_4^{-1}	2	4	6	14	12	$\{t_3 \geq t_5 \geq t_7, t_7 \geq t_{15} \geq t_{13}, t_3 \geq t_9, t_3 \geq t_{11}\}$
ρ_5	2	6	4	14	12	ρ_5^{-1}	2	6	4	14	12	$\{t_3 \geq t_7 \geq t_5, t_7 \geq t_{15} \geq t_{13}, t_3 \geq t_9, t_3 \geq t_{11}\}$
ρ_6	4	6	2	14	12	ρ_6^{-1}	6	2	4	14	12	$\{t_7 \geq t_3 \geq t_5, t_7 \geq t_{15} \geq t_{13}, t_3 \geq t_9, t_3 \geq t_{11}\}$

The collapsing mappings (μ_j, sgn_j) and corresponding trees are indicated below. We notice that all (μ_j, sgn_j) are tamed and also that wild moves, unlike the KM moves, do change the skeleton.

$2j$	2	4	6	8	10	12	14		2	4	6	8	10	12	14
$\mu_1(2j)$	1	1	1	2	3	7	7	$sgn_1(2j)$	+	+	−	−	+	+	−
$\mu_2(2j)$	1	1	1	2	3	5	5	$sgn_2(2j)$	+	−	+	−	+	+	−
$\mu_3(2j)$	1	1	1	4	5	3	3	$sgn_3(2j)$	−	+	+	−	+	+	−
$\mu_4(2j)$	1	1	1	2	3	7	7	$sgn_4(2j)$	+	+	−	−	+	−	+
$\mu_5(2j)$	1	1	1	2	3	5	5	$sgn_5(2j)$	+	−	+	−	+	−	+
$\mu_6(2j)$	1	1	1	4	5	3	3	$sgn_6(2j)$	−	+	+	−	+	−	+



4.5 Reference Form and Proof of Compatibility

We will prove that, given a tamed class, there is a reference form representing the tamed class. Moreover, the tamed time integration domain for the whole tamed class, which can be directly read out from the reference form, is just the compatible time integration domain introduced in Section 3.2.

Definition 4.22 A tamed pair $(\hat{\mu}, \hat{sgn})$ will be called a reference pair provided that in every left branch, all the $+$ nodes come before all the $-$ nodes.

Example 4.23 The collapsing pair (μ_1, sgn_1) in Example 4.21 is a reference pair.

$2j$	2	4	6	8	10	12	14
$sgn_1(2j)$	$+$	$+$	$-$	$-$	$+$	$+$	$-$

From the table, we can see that the $+$ nodes come before all the $-$ nodes in left branches (2, 4, 6) and (12, 14).

As we infer from the examples, each class can be represented by a unique reference pair (μ, sgn). Exactly, we have the following proposition.

Proposition 4.24 *An equivalence class of wildly relatable tamed pairs*

$$Q = \{(\mu, sgn)\}$$

contains a unique reference pair $(\hat{\mu}, \hat{sgn})$. *For every* $(u, sgn) \in Q$, *there is a unique allowable permutation* $\rho \in P(\hat{\mu}, \hat{sgn})$ *such that*

$$(\mu, sgn) = W(\rho)(\hat{\mu}, \hat{sgn}).$$

Proof Note that wild moves will not destroy the left branch but permute the signs. Thus, there exists an allowable permutation such that the + nodes come before all the − nodes in every left branch. The uniqueness follows from the conditions (1) and (2) in Definition 4.17. □

Next, we get into the analysis of the main result.

Proposition 4.25 *The Duhamel expansion to coupling order k can be grouped into at most* 16^k *terms:*

$$\gamma^{(1)}(t_1) = \sum_{reference\ (\hat{\mu}, \hat{sgn})} \int_{T_R(\hat{\mu}, \hat{sgn})} J^{(2k+1)}_{\hat{\mu}, \hat{sgn}}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \quad (4.23)$$

where

$$T_R(\hat{\mu}, \hat{sgn}) = \bigcup_{\rho \in P(\hat{\mu}, \hat{sgn})} \rho^{-1}[T_D(\rho \circ \hat{\mu})]. \quad (4.24)$$

and $T_D(\mu)$ *is defined by (4.9).*

Proof Recall

$$\gamma^{(1)}(t_1) = \sum_{(\mu_*, sgn_*)\ \text{tamed}} \int_{T_D(\mu_*)} J^{(2k+1)}_{\mu_*, sgn_*}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}$$

where the number of tamed forms can be controlled by 16^k. In this sum, group together equivalence classes Q of wildly relatable (μ, sgn).

$$\gamma^{(1)}(t_1) = \sum_{\text{class } Q} \sum_{(\mu, sgn) \in Q} \int_{T_D(\mu)} J^{(2k+1)}_{\mu, sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}.$$

There exists exact one reference $(\hat{\mu}, \hat{sgn})$ in each equivalence class Q. By Proposition 4.24, for each $(\mu, sgn) \in Q$, there is a unique allowable $\rho \in P(\hat{\mu}, \hat{sgn})$ such that

$$(\mu, sgn) = W(\rho)(\hat{\mu}, \hat{sgn}).$$

Since W is an action, we can write

$$(\hat{\mu}, \rho^{-1}, s\hat{g}n) = W(\rho^{-1})(\mu, id, sgn).$$

Then by Proposition 4.20,

$$\int_{T_D(\mu)} J^{(2k+1)}_{\mu,sgn}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} = \int_{\rho^{-1}[T_D(\mu)]} J^{(2k+1)}_{\hat{\mu},s\hat{g}n}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}.$$

Consequently, we obtain

$$\gamma^{(1)}(t_1) = \sum_{reference\ (\hat{\mu},s\hat{g}n)} \sum_{\rho \in P(\hat{\mu},s\hat{g}n)} \int_{\rho^{-1}[T_D(\rho\circ\hat{\mu})]} J^{(2k+1)}_{\hat{\mu},s\hat{g}n}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1}.$$

Since $\left\{\rho^{-1}[T_D(\rho \circ \hat{\mu})]\right\}$ is a collection of disjoint sets, we obtain the equality (4.23). □

We are left to calculate the time integration domain $T_D(\mu)$ and $T_R(\hat{\mu}, s\hat{g}n)$.

Proposition 4.26 *Let $\rho \in P(\hat{\mu}, s\hat{g}n)$ and $(\mu, sgn) = W(\rho)(\hat{\mu}, s\hat{g}n)$, then*

$$\begin{aligned} T_D(\mu) =& \left\{t_{2j+1} \geq t_{2l+1} : \hat{\mu}(2j) = \hat{\mu}(2l), 2j < 2l\right\} \\ & \bigcap \left\{t_{\rho(2j)+1} \geq t_{\rho(2l)+1} : \hat{\mu}(2l) = 2j \text{ or } \hat{\mu}(2l) = 2j+1\right\}, \quad (4.25) \\ T_R(\hat{\mu}, s\hat{g}n) =& \left\{t_{2j+1} \geq t_{2l+1} : 2j < 2l, \hat{\mu}(2l) = \hat{\mu}(2j),\ s\hat{g}n(2j) = s\hat{g}n(2l)\right\} \\ & \bigcap \left\{t_{2j+1} \geq t_{2l+1} : \hat{\mu}(2l) = 2j \text{ or } \hat{\mu}(2l) = 2j+1\right\}. \quad (4.26) \end{aligned}$$

Proof Since $\mu = \rho \circ \hat{\mu}$, we can write

$$\begin{aligned} & T_D(\mu) \\ & = \left\{t_{2j+1} \geq t_{2l+1} : \hat{\mu}(2j) = \hat{\mu}(2l), 2j < 2l\right\} \bigcap \\ & \quad \left\{t_{2j+1} \geq t_{2l+1} : \mu(2l) = 2j \text{ or } \mu(2l) = 2j+1\right\}. \end{aligned}$$

It remains to prove

$$\begin{aligned} & \left\{t_{2j+1} \geq t_{2l+1} : \mu(2l) = 2j \text{ or } \mu(2l) = 2j+1\right\} \\ & = \left\{t_{\rho(2j)+1} \geq t_{\rho(2l)+1} : \hat{\mu}(2l) = 2j \text{ or } \hat{\mu}(2l) = 2j+1\right\}. \quad (4.27) \end{aligned}$$

Actually, with $\mu = \rho \circ \hat{\mu}$ and $\hat{\mu} \circ \rho^{-1} = \hat{\mu}$, we have

$$\begin{aligned} & \mu(2l) = 2j \Longleftrightarrow \hat{\mu}(\rho^{-1}(2l)) = \rho^{-1}(2j), \\ & \mu(2l) = 2j+1 \Longleftrightarrow \hat{\mu}(\rho^{-1}(2l)) = \rho^{-1}(2j)+1, \end{aligned}$$

which implies (4.27).

Then by (4.25), we can rewrite

$$\rho^{-1}[T_D(\rho\circ\hat{\mu})] = \left\{t_{\rho^{-1}(2j)+1} \geq t_{\rho^{-1}(2l)+1} : \hat{\mu}(2j) = \hat{\mu}(2l), 2j < 2l\right\} \bigcap \left\{t_{2j+1} \geq t_{2l+1} : \hat{\mu}(2l) = 2j \text{ or } \hat{\mu}(2l) = 2j+1\right\}.$$

It suffices to prove

$$\bigcup_{\rho\in P(\hat{\mu},\hat{sgn})} \left\{t_{\rho^{-1}(2j)+1} \geq t_{\rho^{-1}(2l)+1} : \hat{\mu}(2j) = \hat{\mu}(2l), 2j < 2l\right\} = \left\{t_{2j+1} \geq t_{2l+1} : 2j < 2l, \hat{\mu}(2l) = \hat{\mu}(2j),\ \hat{sgn}(2j) = \hat{sgn}(2l)\right\}. \tag{4.28}$$

For simplicity, we take the notations

$$A_{j,l}(\rho) = \left\{t_{\rho^{-1}(2j)+1} \geq t_{\rho^{-1}(2l)+1} : 2j < 2l, \hat{\mu}(2l) = \hat{\mu}(2j)\right\},$$
$$B_{j,l} = \left\{t_{2j+1} \geq t_{2l+1} : 2j < 2l, \hat{\mu}(2l) = \hat{\mu}(2j),\ \hat{sgn}(2j) = \hat{sgn}(2l)\right\},$$

where $A_{j,l}$ and $B_{j,l}$ will be the full space if (j,l) does not satisfy the corresponding requirement. We are left to prove that

$$\bigcup_{\rho\in P(\hat{\mu},\hat{sgn})} \bigcap_{j,l} A_{j,l}(\rho) = \bigcap_{j,l} B_{j,l}.$$

Given $\rho \in P(\hat{\mu}, \hat{sgn})$, we will prove $\bigcap_{j,l} A_{j,l}(\rho) \subset B_{j_0,l_0}$ for every pair (j_0, l_0) which satisfies $2j_0 < 2l_0$, $\hat{\mu}(2l_0) = \hat{\mu}(2j_0)$ and $\hat{sgn}(2j_0) = \hat{sgn}(2l_0)$. Let $2j_1 = \rho(2j_0)$ and $2l_1 = \rho(2j_0)$. Since $\rho \in P(\hat{\mu}, \hat{sgn})$, we obtain $\hat{\mu}(2l_1) = \hat{\mu}(2j_1)$ and $2j_1 < 2l_1$. Hence,

$$\bigcap_{j,l} A_{j,l}(\rho) \subset A_{j_1,l_1}(\rho) = B_{j_0,l_0}.$$

Conversely, suppose that $(t_1, t_3, ..., t_{2k+1}) \in \bigcap_{j,l} B_{j,l}$. Note that $\left\{G_i = \left\{2r : \hat{\mu}(2r) = i\right\}\right\}_{i=1}^{2k-1}$ is a partition of $\{2, 4, ..., 2k\}$. Thus there exists a unique $\sigma \in P$ such that

$$\begin{cases} \sigma(G_i) = G_i, \\ t_{\sigma^{-1}(2j)+1} \geq t_{\sigma^{-1}(2l)+1}. \end{cases} \tag{4.29}$$

where $2j < 2l$ and $\hat{\mu}(2j) = \hat{\mu}(2l)$. It implies that $(t_1, t_3, ..., t_{2k+1}) \in \bigcap_{j,l} A_{j,l}(\sigma)$.

We are left to prove that $\sigma \in P(\hat{\mu}, \hat{sgn})$. For any pair (j_0, l_0) which satisfies $2l_0 < 2j_0$, $\hat{\mu}(2l_0) = \hat{\mu}(2j_0)$ and $\hat{sgn}(2j_0) = \hat{sgn}(2l_0)$, we have $(t_1, ..., t_{2k+1}) \in B_{j_0,l_0}$, which implies that $t_{2j_0+1} \geq t_{2l_0+1}$. Combining with (4.29), we obtain $\sigma(2l_0) < \sigma(2j_0)$, which shows that $\sigma \in P(\hat{\mu}, \hat{sgn})$. □

With Propositions 4.25 and 4.26, we arrive at the main result as follows.

Proposition 4.27 *The time integration domain obtained in (4.24) is compatible in the sense that*

$$T_R(\hat{\mu}, s\hat{g}n) = T_C(\hat{\mu}, s\hat{g}n) \tag{4.30}$$

and hence

$$\gamma^{(1)}(t_1) = \sum_{reference\ (\hat{\mu}, s\hat{g}n)} \int_{T_C(\hat{\mu}, s\hat{g}n)} J^{(2k+1)}_{\hat{\mu}, s\hat{g}n}(\gamma^{(2k+1)})(t_1, \underline{t}_{2k+1}) d\underline{t}_{2k+1} \tag{4.31}$$

where $T_C(\hat{\mu}, s\hat{g}n) = \left\{t_{2j+1} \geq t_{2l+1} : D^{(2l)} \to D^{(2j)}\right\}$ *is the compatible time integration domain defined by (3.4).*

Proof From the definition of $T_C(\hat{\mu}, s\hat{g}n)$, we have that $t_{2j+1} \geq t_{2l+1}$ if and only if one of the following cases holds

$$\begin{cases} \hat{\mu}(2j) = \hat{\mu}(2l), \quad s\hat{g}n(2j) = s\hat{g}n(2l), \\ \hat{\mu}(2l) = 2j, s\hat{g}n(2l) = +, \\ \hat{\mu}(2l) = 2j, s\hat{g}n(2l) = -, \\ \hat{\mu}(2l) = 2j+1, s\hat{g}n(2l) = +, \\ \hat{\mu}(2l) = 2j+1, s\hat{g}n(2l) = -, \end{cases}$$

where $2l > 2j$ is the the minimal index for which the corresponding equalities hold. The requirement that $2l$ is the minimal index can be removed by induction argument. Thus, these cases are respectively corresponding to

$$\begin{cases} \hat{\mu}(2l) = \hat{\mu}(2j), \ s\hat{g}n(2j) = s\hat{g}n(2l), \\ \hat{\mu}(2l) = 2j, \\ \hat{\mu}(2l) = 2j+1, \end{cases}$$

which implies that $T_R(\hat{\mu}, s\hat{g}n) = T_C(\hat{\mu}, s\hat{g}n)$. □

5 *U*-*V* Multilinear Estimates

Our proof of U-V multilinear estimates will focus on the $\mathbb{T}^d$ case, as it works the same for $\mathbb{R}^d$ with the homogeneous norm. We recall the definition of U-V spaces in Section 3.3 and use the following tools to prove U-V multilinear estimates.

Lemma 5.1 *[37,Propositions 2.11] For* $f \in L^1(0, T; H^s(\mathbb{T}^d))$*, we have*

$$\left\| \int_a^t e^{i(t-\tau)\Delta} f(\tau, \cdot) d\tau \right\|_{X^s([0,T))} \leq \sup_{g \in Y^{-s}([0,T)): \|g\|_{Y^{-s}}=1} \left| \int_0^T \int_{\mathbb{T}^d} f(t,x) \overline{g(t,x)} dt dx \right|, \tag{5.1}$$

for all $a \in [0, T)$.

Lemma 5.2 (Strichartz estimate on $\mathbb{T}^d$ [4, 46]) *For* $p > \frac{2(d+2)}{d}$,

$$\|P_{\le M} u\|_{L^p_{t,x}} \lesssim M^{\frac{d}{2}-\frac{d+2}{p}} \|P_{\le M} u\|_{Y^0([0,T))} \tag{5.2}$$

Lemma 5.3 *Let M be a dyadic value and let Q be a(possibly) noncentered M-cube in Fourier space*

$$Q = \{\xi_0 + \eta : |\eta| < M\}.$$

Let P_Q be the corresponding Littlewood-Paley projection, then by the Galilean invariance, we have

$$\|P_Q u\|_{L^p_{t,x}} \lesssim M^{\frac{d}{2}-\frac{d+2}{p}} \|P_Q u\|_{Y^0([0,T))} \tag{5.3}$$

for $p > \frac{2(d+2)}{d}$.

Lemma 5.4 (Bernstein with noncentered frequency projection) *Let M and Q be as in Lemma 5.3, then for* $1 \le p \le q \le \infty$

$$\|P_Q f\|_{L^q_x} \lesssim M^{\frac{d}{p}-\frac{d}{q}} \|P_Q f\|_{L^p_x}. \tag{5.4}$$

Lemmas 5.3 and 5.4 are very well-known, and are available in many references, for example, see [20].

5.1 Trilinear Estimates

To deal with the cubic energy-supercritical NLS, we prove the following U-V trilinear estimates at critical regularity. Let $\widetilde{u} \in \{u, \overline{u}\}$.

Lemma 5.5 *On $\mathbb{T}^d$ with $d \ge 4$ and $s \in \left\{\frac{d-6}{2}, \frac{d-2}{2}\right\}$, we have the high frequency estimate*

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{g}(t,x)\,dxdt \lesssim \|u_1\|_{Y^s} \|u_2\|_{Y^{\frac{d-2}{2}}} \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}}, \tag{5.5}$$

and the low frequency estimate

$$\begin{aligned}&\iint_{x,t} \widetilde{u}_1(t,x)(P_{\le M_0}\widetilde{u}_2)(t,x)\widetilde{u}_3(t,x)\widetilde{g}(t,x)\,dxdt \\ &\lesssim T^{\frac{1}{d+3}} M_0^{\frac{2(d+2)}{3(d+3)}} \|u_1\|_{Y^s} \|P_{\le M_0} u_2\|_{Y^{\frac{d-2}{2}}} \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}},\end{aligned} \tag{5.6}$$

for all $T \leq 1$ *and all frequencies* $M_0 \geq 1$. *Then by Lemma 5.1, (5.5) and (5.6), we have*

$$\Big\| \int_a^t e^{i(t-\tau)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3)d\tau \Big\|_{X^s} \lesssim \|u_1\|_{Y^s}\|u_2\|_{Y^{\frac{d-2}{2}}}\|u_3\|_{Y^{\frac{d-2}{2}}} \tag{5.7}$$

and

$$\Big\| \int_a^t e^{i(t-\tau)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3)d\tau \Big\|_{X^s} \lesssim \|u_1\|_{Y^s}\left(T^{\frac{1}{d+3}}M_0^{\frac{2(d+2)}{3(d+3)}}\|P_{\leq M_0}u_2\|_{Y^{\frac{d-2}{2}}} + \|P_{>M_0}u_2\|_{Y^{\frac{d-2}{2}}}\right)\|u_3\|_{Y^{\frac{d-2}{2}}}. \tag{5.8}$$

Proof It suffices to prove high and low frequency estimates (5.5) and (5.6). For simplicity, we take $\widetilde{u} = u$ and $\widetilde{g} = g$.

For the high frequency estimate (5.5), decompose the 4 factors into Littlewood-Paley pieces so that

$$I = \sum_{M_1,M_2,M_3,M_4} I_{M_1,M_2,M_3,M_4}$$

where

$$I_{M_1,M_2,M_3,M_4} = \iint_{x,t} u_{1,M_1}u_{2,M_2}u_{3,M_3}g_{M_4}dxdt$$

with $u_{j,M_j} = P_{M_j}u_j$ and $g_{M_4} = P_{M_4}g$. By orthogonality, we know that these cases are as follows

$$M_{\sigma(1)} \sim M_{\sigma(2)} \geq M_{\sigma(3)} \geq M_{\sigma(4)}$$

where σ is a permutation on $\{1, 2, 3, 4\}$. By symmetry, we might as well assume without loss that $M_2 \geq M_3$.

First, we consider the most difficulty case, namely, Case A. $M_1 \sim M_4 \geq M_2 \geq M_3$. Then, we need only to deal with one such as Case B. $M_1 \sim M_2 \geq M_4 \geq M_3$, since other cases can be treated in the same way.

Let I_A denote the integral restricted to the Case A. Decompose the M_1 and M_4 dyadic spaces into M_2 size cubes. Due to the frequency constraint $\xi_2 \sim -(\xi_1+\xi_3+\xi_4)$, for each choice Q of an M_2 size cube within the ξ_1 space, the variable ξ_2 is constrained to at most 3^d of M_2 size cubes. For convenience, we denote these cubes by a single cube Q_c that corresponds to Q. Then

$$\begin{aligned} I_A &\lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1\sim M_4\geq M_2\geq M_3}} \sum_Q \|P_Q u_{1,M_1}u_{2,M_2}u_{3,M_3}P_{Q_c}g_{M_4}\|_{L^1_{t,x}} \\ &\lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1\sim M_4\geq M_2\geq M_3}} \sum_Q \|P_Q u_{1,M_1}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}}\|u_{2,M_2}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}}\|u_{3,M_3}\|_{L^{d+3}_{t,x}}\|P_{Q_c}g_{M_4}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} \end{aligned}$$

where the factor corresponding to the smallest size cubes (here M_3 size cubes) is put in $L_{t,x}^{d+3}$ and the others are put in $L_{t,x}^{\frac{3(d+3)}{d+2}}$. By (5.2) and (5.3),

$$\lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1\sim M_4\geq M_2\geq M_3}} \sum_{Q} M_2^{\frac{d-2}{2}-\frac{1}{d+3}} \|P_Q u_{1,M_1}\|_{Y^0} \|u_{2,M_2}\|_{Y^0} M_3^{\frac{d-2}{2}+\frac{1}{d+3}} \|u_{3,M_3}\|_{Y^0} \|P_{Q_c} g_{M_4}\|_{Y^0}$$

Applying Cauchy-Schwarz to sum in Q,

$$\lesssim \sum_{\substack{M_1,M_4 \\ M_1\sim M_4}} M_1^{-s} M_4^{s} \|u_{1,M_1}\|_{Y^s} \|g_{M_4}\|_{Y^{-s}} \sum_{\substack{M_2,M_3 \\ M_2\geq M_3}} M_2^{-\frac{1}{d+3}} M_3^{\frac{1}{d+3}} \|u_{2,M_2}\|_{Y^{\frac{d-2}{2}}} \|u_{3,M_3}\|_{Y^{\frac{d-2}{2}}} \tag{5.9}$$

Applying Cauchy-Schwarz,

$$\begin{aligned}\lesssim & \left(\sum_{M_1} \|u_{1,M_1}\|_{Y^s}^2\right)^{\frac{1}{2}} \left(\sum_{M_4} \|g_{4,M_4}\|_{Y^{-s}}^2\right)^{\frac{1}{2}} \\ & \left(\sum_{\substack{M_2,M_3 \\ M_2\geq M_3}} \left(\frac{M_3}{M_2}\right)^{\frac{1}{d+3}} \|u_{2,M_2}\|_{Y^{\frac{d-2}{2}}}^2\right)^{\frac{1}{2}} \left(\sum_{\substack{M_2,M_3 \\ M_2\geq M_3}} \left(\frac{M_3}{M_2}\right)^{\frac{1}{d+3}} \|u_{3,M_3}\|_{Y^{\frac{d-2}{2}}}^2\right)^{\frac{1}{2}} \\ \lesssim & \|u_1\|_{Y^s} \|u_2\|_{Y^{\frac{d-2}{2}}} \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}}.\end{aligned}$$

Case B. $M_1 \sim M_2 \geq M_4 \geq M_3$. Decompose the M_1 and M_2 dyadic spaces into M_4 size cubes and we have

$$\begin{aligned} I_B \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1\sim M_2\geq M_4\geq M_3}} \sum_{Q} \|P_Q u_{1,M_1} P_{Q_c} u_{2,M_2} u_{3,M_3} g_{M_4}\|_{L_{t,x}^1} \\ \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1\sim M_2\geq M_4\geq M_3}} \sum_{Q} \|P_Q u_{1,M_1}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} \|P_{Q_c} u_{2,M_2}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} \|u_{3,M_3}\|_{L_{t,x}^{d+3}} \|g_{M_4}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}}\end{aligned}$$

By (5.2) and (5.3),

$$\lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1\sim M_2\geq M_4\geq M_3}} \sum_{Q} M_4^{\frac{d-2}{2}-\frac{1}{d+3}} \|P_Q u_{1,M_1}\|_{Y^0} \|P_{Q_c} u_{2,M_2}\|_{Y^0} M_3^{\frac{d-2}{2}+\frac{1}{d+3}} \|u_{3,M_3}\|_{Y^0} \|g_{M_4}\|_{Y^0}$$

Applying Cauchy-Schwarz to sum in Q,

$$\lesssim \sum_{\substack{M_1,M_2\\M_1\sim M_2}} \Bigg(M_1^{-s} M_2^{-\frac{d-2}{2}} \|u_{1,M_1}\|_{Y^s} \|u_{2,M_2}\|_{Y^{\frac{d-2}{2}}}$$

$$\sum_{\substack{M_3,M_4\\M_1\sim M_2\geq M_4\geq M_3}} M_4^{\frac{d-2}{2}-\frac{1}{d+3}+s} M_3^{\frac{1}{d+3}} \|u_{3,M_3}\|_{Y^{\frac{d-2}{2}}} \|g_{4,M_4}\|_{Y^{-s}} \Bigg)$$

If $s+\frac{d-2}{2}=0$, it can be estimated in the same way as (5.9). Thus we need only to treat the case $s+\frac{d-2}{2}>0$. Supping out $\|u_{3,M_3}\|_{Y^{\frac{d-2}{2}}}$ and $\|g_{M_4}\|_{Y^{-s}}$ in M_3 and M_4, we have

$$\lesssim \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}} \sum_{\substack{M_1,M_2\\M_1\sim M_2}} M_1^{-s} M_2^{-\frac{d-2}{2}} \|u_{1,M_1}\|_{Y^s} \|u_{2,M_2}\|_{Y^{\frac{d-2}{2}}}$$

$$\sum_{\substack{M_3,M_4\\M_1\sim M_2\geq M_4\geq M_3}} M_4^{\frac{d-2}{2}-\frac{1}{d+3}+s} M_3^{\frac{1}{d+3}}$$

By the fact that $\frac{d-2}{2}+s>0$,

$$\begin{aligned}
&\lesssim \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}} \sum_{\substack{M_1,M_2\\M_1\sim M_2}} M_1^{-s} M_2^{-\frac{d-2}{2}} \|u_{1,M_1}\|_{Y^s} \|u_{2,M_2}\|_{Y^{\frac{d-2}{2}}} M_2^{\frac{d-2}{2}+s}\\
&= \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}} \sum_{\substack{M_1,M_2\\M_1\sim M_2}} M_1^{-s} M_2^{s} \|u_{1,M_1}\|_{Y^{-s}} \|u_{2,M_2}\|_{Y^{\frac{d-2}{2}}}
\end{aligned}$$

Applying Cauchy-Schwarz,

$$\lesssim \|u_1\|_{Y^s} \|u_2\|_{Y^{\frac{d-2}{2}}} \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}}.$$

Case B requires that $\frac{d-2}{2}+s\geq 0$. If we exchange M_1 and M_4 in Case B, we find another requirement that $\frac{d-2}{2}-s\geq 0$ is also needed. In this case, it becomes Case A again if $s=\frac{d-2}{2}$ and it becomes similar but a bit different if $s=\frac{d-6}{2}$ as M_1 and M_2 are not symmetric.

Proof of the low frequency estimate (5.6). We first deal with the most difficult Case A. $M_1\sim M_4\geq M_3\geq M_2$. Decompose the M_1 and M_4 dyadic spaces into M_3 size cubes, we have

$$I_{M_1,M_2,M_3,M_4} \lesssim \sum_Q \|P_Q u_{1,M_1}(P_{\leq M_0} u_{2,M_2}) u_{3,M_3} P_{Q^c} g_{M_4}\|_{L^1_{t,x}}$$

$$\leq \sum_{Q} \| P_Q u_{1,M_1} \|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} \| P_{\leq M_0} u_{2,M_2} \|_{L_{t,x}^{d+3}} \| u_{3,M_3} \|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} \| P_{Q_c} g_{M_4} \|_{L_{t,x}^{\frac{3(d+3)}{d+2}}}$$

where the factor corresponding to the smallest size cubes (here M_2 size cubes) is put in $L_{t,x}^{d+3}$ and the others are put in $L_{t,x}^{\frac{3(d+3)}{d+2}}$.

By Hölder, Bernstein inequalities and (3.4),

$$\begin{aligned} \| P_{\leq M_0} u_{2,M_2} \|_{L_{t,x}^{d+3}} &\lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} M_2^{\frac{d-2}{2}+\frac{1}{d+3}} \| P_{\leq M_0} u_{2,M_2} \|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} M_2^{\frac{d-2}{2}+\frac{1}{d+3}} \| P_{\leq M_0} u_{2,M_2} \|_{Y^0} \end{aligned}$$

By (5.2) and (5.3),

$$\begin{aligned} &I_{M_1,M_2,M_3,M_4} \\ &\quad \lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} \sum_{Q} M_2^{\frac{d-2}{2}+\frac{1}{d+3}} M_3^{\frac{d-2}{2}-\frac{1}{d+3}} \| P_Q u_{1,M_1} \|_{Y^0} \\ &\quad \| P_{\leq M_0} u_{2,M_2} \|_{Y^0} \| u_{3,M_3} \|_{Y^0} \| P_{Q_c} g_{M_4} \|_{Y^0} \end{aligned}$$

Applying Cauchy-Schwarz to sum in Q, we arrive at

$$\begin{aligned} &I_{M_1,M_2,M_3,M_4} \\ &\quad \lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} M_3^{\frac{d-2}{2}-\frac{1}{d+3}} M_2^{\frac{d-2}{2}+\frac{1}{d+3}} \| u_{1,M_1} \|_{Y^0} \| P_{\leq M_0} u_{2,M_2} \|_{Y^0} \| u_{3,M_3} \|_{Y^0} \| g_{M_4} \|_{Y^0}. \end{aligned}$$

Then by $M_1 \sim M_4$, we obtain

$$\begin{aligned} I_{M_1,M_2,M_3,M_4} &\lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} M_3^{\frac{d-2}{2}-\frac{1}{d+3}} M_2^{\frac{d-2}{2}+\frac{1}{d+3}} M_1^{-s} \| u_{1,M_1} \|_{Y^s} M_2^{-\frac{d-2}{2}} \| P_{\leq M_0} u_{2,M_2} \|_{Y^{\frac{d-2}{2}}} \\ &\quad M_3^{-\frac{d-2}{2}} \| u_{3,M_3} \|_{Y^{\frac{d-2}{2}}} M_4^s \| g_{M_4} \|_{Y^{-s}} \\ &\lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} M_3^{-\frac{1}{d+3}} M_2^{\frac{1}{d+3}} \| u_{1,M_1} \|_{Y^s} \| P_{\leq M_0} u_{2,M_2} \|_{Y^{\frac{d-2}{2}}} \| u_{3,M_3} \|_{Y^{\frac{d-2}{2}}} \| g_{M_4} \|_{Y^{-s}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} I_A &\lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1 \sim M_4 \geq M_3 \geq M_2}} \| u_{1,M_1} \|_{Y^s} \| g_{M_4} \|_{Y^{-s}} M_3^{-\frac{1}{d+3}} \\ &\quad M_2^{\frac{1}{d+3}} \| P_{\leq M_0} u_2 \|_{Y^{\frac{d-2}{2}}} \| u_{3,M_3} \|_{Y^{\frac{d-2}{2}}} \\ &\lesssim T^{\frac{1}{d+3}} M_0^{\frac{2}{d+3}} \| u_1 \|_{Y^s} \| P_{\leq M_0} u_2 \|_{Y^{\frac{d-2}{2}}} \| u_3 \|_{Y^{\frac{d-2}{2}}} \| g \|_{Y^{-s}} \end{aligned}$$

Next, we deal with one such as Case B. $M_1 \sim M_2 \geq M_4 \geq M_3$, since other cases can be treated in the same way. Decompose the M_1 and M_2 dyadic spaces into M_4 size cubes and we have

$$I_{M_1,M_2,M_3,M_4} \leq \sum_Q \|P_Q u_{1,M_1}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} \|P_{Q_c} P_{\leq M_0} u_{2,M_2}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} \|u_{3,M_3}\|_{L_{t,x}^{d+3}} \|g_{M_4}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}}$$

By Hölder, Bernstein inequalities and (3.4),

$$\begin{aligned} \|P_{Q_c} P_{\leq M_0} u_{2,M_2}\|_{L_{t,x}^{\frac{3(d+3)}{d+2}}} &\lesssim T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} M_4^{\frac{d-2}{6} - \frac{1}{3(d+3)}} \|P_{Q_c} P_{\leq M_0} u_{2,M_2}\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} M_4^{\frac{d-2}{6} - \frac{1}{3(d+3)}} \|P_{Q_c} P_{\leq M_0} u_{2,M_2}\|_{Y^0} \end{aligned}$$

By (5.2) and (5.3),

$$\begin{aligned} I_{M_1,M_2,M_3,M_4} \lesssim & T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} M_4^{\frac{d-2}{2} - \frac{1}{d+3}} M_3^{\frac{d-2}{2} + \frac{1}{d+3}} \\ & \sum_Q \|P_Q u_{1,M_1}\|_{Y^0} \|P_{Q_c} P_{\leq M_0} u_{2,M_2}\|_{Y^0} \|u_{3,M_3}\|_{Y^0} \|g_{M_4}\|_{Y^0} \end{aligned}$$

Applying Cauchy-Schwarz to sum in Q, we arrive at

$$\begin{aligned} & I_{M_1,M_2,M_3,M_4} \\ & \lesssim T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} M_4^{\frac{d-2}{2} - \frac{1}{d+3}} M_3^{\frac{d-2}{2} + \frac{1}{d+3}} \|u_{1,M_1}\|_{Y^0} \|P_{\leq M_0} u_{2,M_2}\|_{Y^0} \|u_{3,M_3}\|_{Y^0} \|g_{M_4}\|_{Y^0} \\ & \lesssim T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} M_1^{-s} \|u_{1,M_1}\|_{Y^s} M_2^{-\frac{d-2}{2}} \|P_{\leq M_0} u_{2,M_2}\|_{Y^{\frac{d-2}{2}}} \\ & \quad M_3^{\frac{1}{d+3}} \|u_{3,M_3}\|_{Y^{\frac{d-2}{2}}} M_4^{\frac{d-2}{2} - \frac{1}{d+3} + s} \|g_{M_4}\|_{Y^{-s}} \end{aligned}$$

If $s + \frac{d-2}{2} = 0$, it can be estimated in the same way as (5.9). Thus we need only to treat the case $s + \frac{d-2}{2} > 0$. Supping out $\|u_{3,M_3}\|_{Y^{\frac{d-2}{2}}}$ and $\|g_{M_4}\|_{Y^{-s}}$ in M_3 and M_4, we have

$$\begin{aligned} I_B \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1 \sim M_2 \geq M_4 \geq M_3}} I_{M_1,M_2,M_3,M_4} \\ \lesssim & T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}} \\ & \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1 \sim M_2 \geq M_4 \geq M_3}} \Bigg(M_1^{-s} \|u_{1,M_1}\|_{Y^s} M_2^{-\frac{d-2}{2}} \|P_{\leq M_0} u_{2,M_2}\|_{Y^{\frac{d-2}{2}}} \\ & M_3^{\frac{1}{d+3}} M_4^{\frac{d-2}{2} - \frac{1}{d+3} + s} \Bigg) \end{aligned}$$

$$\lesssim T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}} \sum_{\substack{M_1, M_2 \\ M_1 \sim M_2}} M_1^{-s} \|u_{1,M_1}\|_{Y^s} M_2^s \|P_{\leq M_0} u_{2,M_2}\|_{Y^{\frac{d-2}{2}}}$$

Applying Cauchy-Schwarz,

$$\lesssim T^{\frac{d+2}{3(d+3)}} M_0^{\frac{2(d+2)}{3(d+3)}} \|u_1\|_{Y^s} \|P_{\leq M_0} u_2\|_{Y^{\frac{d-2}{2}}} \|u_3\|_{Y^{\frac{d-2}{2}}} \|g\|_{Y^{-s}}.$$

□

5.2 Quintilinear Estimates

Lemma 5.6 *On $\mathbb{T}^d$ with $d \geq 3$ and $s \in \left\{\frac{d-5}{2}, \frac{d-1}{2}\right\}$, we have the high frequency estimate*

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{u}_4(t,x)\widetilde{u}_5(t,x)\widetilde{g}(t,x)dxdt \\ \lesssim \|u_1\|_{Y^s} \|u_2\|_{Y^{\frac{d-1}{2}}} \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \|g\|_{Y^{-s}}, \tag{5.10}$$

and the low frequency estimate

$$\iint_{x,t} \widetilde{u}_1(t,x)(P_{\leq M_0}\widetilde{u}_2)(t,x)\widetilde{u}_3(t,x)\widetilde{u}_4(t,x)\widetilde{u}_5(t,x)\widetilde{g}(t,x)dxdt \\ \lesssim T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{3(d+3)}} \|u_1\|_{Y^s} \|P_{\leq M_0} u_2\|_{Y^{\frac{d-1}{2}}} \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \|g\|_{Y^{-s}}, \tag{5.11}$$

for all $T \leq 1$ and all frequencies $M_0 \geq 1$. Then by Lemma 5.1, (5.10) and (5.11), we have

$$\left\| \int_a^t e^{i(t-\tau)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5)d\tau \right\|_{X^s} \lesssim \|u_1\|_{Y^s} \|u_2\|_{Y^{\frac{d-1}{2}}} \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \tag{5.12}$$

and

$$\left\| \int_a^t e^{i(t-\tau)\Delta}(\widetilde{u}_1\widetilde{u}_2\widetilde{u}_3\widetilde{u}_4\widetilde{u}_5)d\tau \right\|_{X^s} \\ \lesssim \|u_1\|_{Y^s} \left(T^{\frac{1}{2(d+3)}} M_0^{\frac{2d+3}{3(d+3)}} \|P_{\leq M_0} u_2\|_{Y^{\frac{d-1}{2}}} + \|P_{>M_0} u_2\|_{Y^{\frac{d-1}{2}}} \right) \|u_3\|_{Y^{\frac{d-1}{2}}} \\ \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}}. \tag{5.13}$$

Remark 5.7 Notice that (5.10) for $d = 3$ is implied by [19,Lemma 5.15]. One can compare the proof of the stronger $L_t^1 H^s$ estimate with the proof here and see that the proof of the weaker U-V estimates are indeed much less technical, and the current method to incorporate these weaker estimates is indeed stronger.

Proof For the high frequency estimate (5.10), decompose the 6 factors into Littlewood-Paley pieces so that

$$I=\sum_{M_1,M_2,M_3,M_4,M_5,M_6} I_{M_1,M_2,M_3,M_4,M_5,M_6}$$

where

$$I_{M_1,M_2,M_3,M_4,M_5,M_6}=\iint_{x,t} u_{1,M_1}u_{2,M_2}u_{3,M_3}u_{4,M_4}u_{5,M_5}g_{M_6}dxdt$$

with $u_{j,M_j}=P_{M_j}u_j$ and $g_{M_6}=P_{M_6}g$.

We first take care of the most difficult Case A. $M_1\sim M_6\geq M_2\geq M_3\geq M_4\geq M_5$. Then, we need only to deal with one such as Case B. $M_1\sim M_2\geq M_6\geq M_3\geq M_4\geq M_5$, since other cases can be treated in the same way. Decompose the M_1 and M_6 dyadic spaces into M_2 size cubes, then

$$\begin{aligned}
I_A &\lesssim \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6\\ M_1\sim M_6\geq M_2\geq M_3\geq M_4\geq M_5}}\sum_Q \|P_Q u_{1,M_1}P_{Q^c}u_{2,M_2}u_{3,M_3}u_{4,M_4}u_{5,M_5}g_{M_6}\|_{L^1_{t,x}}\\
&\lesssim \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6\\ M_1\sim M_6\geq M_2\geq M_3\geq M_4\geq M_5}}\sum_Q \Bigg(\|P_Q u_{1,M_1}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}}\|u_{2,M_2}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}}\|u_{3,M_3}\|_{L^{2(d+3)}_{t,x}}\\
&\qquad \|u_{4,M_4}\|_{L^{2(d+3)}_{t,x}}\|u_{5,M_5}\|_{L^{2(d+3)}_{t,x}}\|P_{Q^c}g_{M_6}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}}\Bigg)
\end{aligned}$$

where three factors corresponding to small size cubes (here M_3, M_4, M_5 size cubes) are put in $L^{2(d+3)}_{t,x}$ and the others are put in $L_{t,x}^{\frac{6(d+3)}{2d+3}}$. By (5.2) and (5.3),

$$\begin{aligned}
&\lesssim \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6\\ M_1\sim M_6\geq M_2\geq M_3\geq M_4\geq M_5}}\sum_Q \Bigg(M_2^{\frac{d-1}{2}-\frac{3}{2(d+3)}}\|P_Q u_{1,M_1}\|_{Y^0}\|u_{2,M_2}\|_{Y^0}M_3^{\frac{d-1}{2}+\frac{1}{2(d+3)}}\|u_{3,M_3}\|_{Y^0}\\
&\qquad M_4^{\frac{d-1}{2}+\frac{1}{2(d+3)}}\|u_{4,M_4}\|_{Y^0}M_5^{\frac{d-1}{2}+\frac{1}{2(d+3)}}\|u_{5,M_5}\|_{Y^0}\|P_{Q^c}g_{M_6}\|_{Y^0}\Bigg).
\end{aligned}$$

Applying Cauchy-Schwarz to sum in Q,

$$\begin{aligned}
&\lesssim \sum_{\substack{M_1,M_6\\ M_1\sim M_6}} M_1^{-s}M_6^{s}\|u_{1,M_1}\|_{Y^s}\|g_{M_6}\|_{Y^{-s}}\sum_{\substack{M_2,M_3,M_4,M_5\\ M_2\geq M_3\geq M_4\geq M_5}}\Bigg(M_2^{-\frac{3}{2(d+3)}}M_3^{\frac{1}{2(d+3)}}M_4^{\frac{1}{2(d+3)}}M_5^{\frac{1}{2(d+3)}}\\
&\qquad \|u_{2,M_2}\|_{Y^{\frac{d-1}{2}}}\|u_{3,M_3}\|_{Y^{\frac{d-1}{2}}}\|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}}\|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}}\Bigg) \qquad (5.14)
\end{aligned}$$

Supping out $\|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}}$ and $\|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}}$ in M_4 and M_5, and then applying Cauchy-Schwarz as shown in (5.9),

$$\lesssim \|u_1\|_{Y^s}\|u_2\|_{Y^{\frac{d-1}{2}}}\|u_3\|_{Y^{\frac{d-1}{2}}}\|u_4\|_{Y^{\frac{d-1}{2}}}\|u_5\|_{Y^{\frac{d-1}{2}}}\|g\|_{Y^{-s}}.$$

Case B. $M_1 \sim M_2 \geq M_6 \geq M_3 \geq M_4 \geq M_5$. Decompose the M_1 and M_2 dyadic spaces into M_6 size cubes and we have

$$\begin{aligned} I_B \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1\sim M_2\geq M_6\geq M_3\geq M_4\geq M_5}} \sum_Q \|P_Q u_{1,M_1} P_{Q^c} u_{2,M_2} u_{3,M_3} u_{4,M_4} u_{5,M_5} g_{M_6}\|_{L^1_{t,x}} \\ \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1\sim M_2\geq M_6\geq M_3\geq M_4\geq M_5}} \sum_Q \Bigg(\|P_Q u_{1,M_1}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \|P_{Q^c} u_{2,M_2}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \|u_{3,M_3}\|_{L_{t,x}^{2(d+3)}} \\ & \|u_{4,M_4}\|_{L_{t,x}^{2(d+3)}} \|u_{5,M_5}\|_{L_{t,x}^{2(d+3)}} \|g_{M_6}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \Bigg) \end{aligned}$$

By (5.2) and (5.3),

$$\begin{aligned} \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1\sim M_2\geq M_6\geq M_3\geq M_4\geq M_5}} \sum_Q \Bigg(M_6^{\frac{d-1}{2}-\frac{3}{2(d+3)}} \|P_Q u_{1,M_1}\|_{Y^0} \|P_{Q^c} u_{2,M_2}\|_{Y^0} M_3^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{3,M_3}\|_{Y^0} \\ & M_4^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{4,M_4}\|_{Y^0} M_5^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{5,M_5}\|_{Y^0} \|g_{M_6}\|_{Y^0} \Bigg). \end{aligned}$$

Applying Cauchy-Schwarz to sum in Q,

$$\begin{aligned} \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1\sim M_2\geq M_6\geq M_3\geq M_4\geq M_5}} \Bigg(M_6^{\frac{d-1}{2}-\frac{3}{2(d+3)}} \|u_{1,M_1}\|_{Y^0} \|u_{2,M_2}\|_{Y^0} M_3^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{3,M_3}\|_{Y^0} \\ & M_4^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{4,M_4}\|_{Y^0} M_5^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{5,M_5}\|_{Y^0} \|g_{M_6}\|_{Y^0} \Bigg) \\ \lesssim & \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1\sim M_2\geq M_6\geq M_3\geq M_4\geq M_5}} \Bigg(M_1^{-s} \|u_{1,M_1}\|_{Y^s} M_2^{-\frac{d-1}{2}} \|u_{2,M_2}\|_{Y^{\frac{d-1}{2}}} M_3^{\frac{1}{2(d+3)}} \|u_{3,M_3}\|_{Y^{\frac{d-1}{2}}} \\ & M_4^{\frac{1}{2(d+3)}} \|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}} M_5^{\frac{1}{2(d+3)}} \|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}} M_6^{\frac{d-1}{2}-\frac{3}{2(d+3)}+s} \|g_{M_6}\|_{Y^{-s}} \Bigg) \end{aligned}$$

If $s = \frac{d-1}{2}$, it can be estimated in the same way as (5.14). Thus, we need only to treat the case $\frac{d-1}{2} + s > 0$. Supping out $\|u_{3,M_3}\|_{Y^{\frac{d-1}{2}}}$, $\|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}}$, $\|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}}$ and

$\|g_{M_6}\|_{Y^{-s}}$, in M_3, M_4, M_5 and M_6, we have

$$\lesssim \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \|g\|_{Y^{-s}} \sum_{\substack{M_1, M_2, M_3, M_4, M_5, M_6 \\ M_1 \sim M_2 \geq M_6 \geq M_3 \geq M_4 \geq M_5}} \Big(M_1^{-s} \|u_{1,M_1}\|_{Y^s} M_2^{-\frac{d-1}{2}} \|u_{2,M_2}\|_{Y^{\frac{d-1}{2}}} M_6^{\frac{d-1}{2} - \frac{3}{2(d+3)} + s} M_3^{\frac{1}{2(d+3)}} M_4^{\frac{1}{2(d+3)}} M_5^{\frac{1}{2(d+3)}} \Big)$$

By the fact that $\frac{d-1}{2} + s > 0$,

$$\begin{aligned} &\lesssim \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \|g\|_{Y^{-s}} \sum_{\substack{M_1, M_2 \\ M_1 \sim M_2}} M_1^{-s} \|u_{1,M_1}\|_{Y^s} M_2^{-\frac{d-1}{2}} \|u_{2,M_2}\|_{Y^{\frac{d-1}{2}}} M_2^{\frac{d-1}{2}+s} \\ &= \|u_1\|_{Y^s} \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \sum_{\substack{M_1, M_2 \\ M_1 \sim M_2}} M_1^{-s} \|u_{1,M_1}\|_{Y^s} M_2^{s} \|u_{2,M_2}\|_{Y^{\frac{d-1}{2}}} \end{aligned}$$

Applying Cauchy-Schwarz,

$$\lesssim \|u_1\|_{Y^s} \|u_2\|_{Y^{\frac{d-1}{2}}} \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \|g\|_{Y^{-s}}.$$

Case B requires that $\frac{d-1}{2} + s \geq 0$. If we exchange M_1 and M_6 in Case B, we find another requirement that $\frac{d-1}{2} - s \geq 0$ is also needed. In this case, it becomes Case A again if $s = \frac{d-1}{2}$ and it becomes similar but a bit different if $s = \frac{d-5}{2}$ as M_1 and M_2 are not symmetric.

Proof of the low frequency estimate (5.11). At first, we deal with the most difficult Case A. $M_1 \sim M_6 \geq M_5 \geq M_4 \geq M_3 \geq M_2$. Decompose the M_1 and M_6 dyadic spaces into M_5 size cubes,

$$\begin{aligned} I_{M_1,M_2,M_3,M_4,M_5,M_6} &\lesssim \sum_Q \|P_Q u_{1,M_1} (P_{\leq M_0} u_{2,M_2}) u_{3,M_3} u_{4,M_4} u_{5,M_5} P_{Q_c} g_{M_6}\|_{L^1_{t,x}} \\ &\leq \sum_Q \Big(\|P_Q u_{1,M_1}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \|P_{\leq M_0} u_{2,M_2}\|_{L_{t,x}^{2(d+3)}} \|u_{3,M_3}\|_{L_{t,x}^{2(d+3)}} \\ &\qquad \|u_{4,M_4}\|_{L_{t,x}^{2(d+3)}} \|u_{5,M_5}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \|P_{Q_c} g_{M_6}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \Big) \end{aligned}$$

where three factors corresponding to small size cubes (here M_4, M_3, M_2 size cubes) are put in $L_{t,x}^{2(d+3)}$ and the others are put in $L_{t,x}^{\frac{6(d+3)}{2d+3}}$.

By Hölder, Bernstein inequalities and (3.4),

$$\begin{aligned} \|P_{\leq M_0} u_{2,M_2}\|_{L_{t,x}^{2(d+3)}} &\lesssim T^{\frac{1}{2(d+3)}} M_0^{\frac{1}{d+3}} M_2^{\frac{d-1}{2} + \frac{1}{2(d+3)}} \|P_{\leq M_0} u_{2,M_2}\|_{L_t^\infty L_x^2} \\ &\lesssim T^{\frac{1}{2(d+3)}} M_0^{\frac{1}{d+3}} M_2^{\frac{d-1}{2} + \frac{1}{2(d+3)}} \|P_{\leq M_0} u_{2,M_2}\|_{Y^0} \end{aligned}$$

By (5.2) and (5.3),

$$
\begin{aligned}
&I_{M_1,M_2,M_3,M_4,M_5,M_6}\\
&\lesssim T^{\frac{1}{2(d+3)}} M_0^{\frac{1}{d+3}} \sum_{Q} \Big(\|P_Q u_{1,M_1}\|_{Y^0} M_2^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|P_{\le M_0} u_{2,M_2}\|_{Y^0} M_3^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{3,M_3}\|_{Y^0}\\
&\quad M_4^{\frac{d-1}{2}+\frac{1}{2(d+3)}} \|u_{4,M_4}\|_{Y^0} M_5^{\frac{d-1}{2}-\frac{3}{2(d+3)}} \|u_{5,M_5}\|_{Y^0} \|P_{Q^c} g_{M_6}\|_{Y^0} \Big)
\end{aligned}
$$

Applying Cauchy-Schwarz to sum in Q,

$$
\begin{aligned}
I_{M_1,M_2,M_3,M_4,M_5,M_6} \lesssim & T^{\frac{1}{2(d+3)}} M_0^{\frac{1}{d+3}} M_5^{\frac{d-1}{2}-\frac{3}{2(d+3)}} M_4^{\frac{d-1}{2}+\frac{1}{2(d+3)}} M_3^{\frac{d-1}{2}+\frac{1}{2(d+3)}} M_2^{\frac{d-1}{2}+\frac{1}{2(d+3)}}\\
& \|u_{1,M_1}\|_{Y^0} \|P_{\le M_0} u_{2,M_2}\|_{Y^0} \|u_{3,M_3}\|_{Y^0} \|u_{4,M_4}\|_{Y^0} \|u_{5,M_5}\|_{Y^0} \|g_{M_6}\|_{Y^0}.
\end{aligned}
$$

Then by $M_1 \sim M_6$, we obtain

$$
\begin{aligned}
&I_{M_1,M_2,M_3,M_4,M_5,M_6}\\
&\lesssim T^{\frac{1}{2(d+3)}} M_0^{\frac{1}{d+3}} M_5^{-\frac{3}{2(d+3)}} M_4^{\frac{1}{2(d+3)}} M_3^{\frac{1}{2(d+3)}} M_2^{\frac{1}{2(d+3)}}\\
&\quad \|u_{1,M_1}\|_{Y^s} \|P_{\le M_0} u_{2,M_2}\|_{Y^{\frac{d-1}{2}}} \|u_{3,M_3}\|_{Y^{\frac{d-1}{2}}} \|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}} \|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}} \|g_{M_6}\|_{Y^{-s}}.
\end{aligned}
$$

and hence

$$
\begin{aligned}
I_A \lesssim & T^{\frac{1}{2(d+3)}} M_0^{\frac{1}{d+3}} \sum_{\substack{M_1,M_6\\ M_1\sim M_6}} \|u_{1,M_1}\|_{Y^s} \|g_{M_6}\|_{Y^{-s}}\\
& \sum_{\substack{M_2,M_3,M_4,M_5\\ M_5\ge M_4\ge M_3\ge M_2}} \Big(M_5^{-\frac{3}{2(d+3)}} M_4^{\frac{1}{2(d+3)}} M_3^{\frac{1}{2(d+3)}} M_2^{\frac{1}{2(d+3)}}\\
& \|P_{\le M_0} u_{2,M_2}\|_{Y^{\frac{d-1}{2}}} \|u_{3,M_3}\|_{Y^{\frac{d-1}{2}}} \|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}} \|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}} \Big)
\end{aligned}
$$

Estimating it in the same way as (5.14), we have

$$
I_A \lesssim T^{\frac{1}{2(d+3)}} M_0^{\frac{3}{2(d+3)}} \|u_1\|_{Y^s} \|P_{\le M_0} u_2\|_{Y^{\frac{d-1}{2}}} \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \|g\|_{Y^{-s}}.
$$

Next, we deal with one such as Case B. $M_1 \sim M_2 \ge M_3 \ge M_4 \ge M_5 \ge M_6$, since other cases can be treated in the same way. Decompose the M_1 and M_2 dyadic spaces into M_3 size cubes and we have

$$
\begin{aligned}
I_{M_1,M_2,M_3,M_4,M_5,M_6} \le & \sum_{Q} \|P_Q u_{1,M_1}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \|P_{Q_c} P_{\le M_0} u_{2,M_2}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} \|u_{3,M_3}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}}\\
& \|u_{4,M_5}\|_{L_{t,x}^{2(d+3)}} \|u_{4,M_5}\|_{L_{t,x}^{2(d+3)}} \|g_{M_6}\|_{L_{t,x}^{2(d+3)}},
\end{aligned}
$$

By Hölder, Bernstein inequalities and (3.4),

$$\begin{aligned}\|P_{Q_c}P_{\le M_0}u_{2,M_2}\|_{L_{t,x}^{\frac{6(d+3)}{2d+3}}} &\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}M_3^{\frac{d-1}{6}-\frac{1}{2(d+3)}}\|P_{Q_c}P_{\le M_0}u_{2,M_2}\|_{L_t^\infty L_x^2}\\ &\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}M_3^{\frac{d-1}{6}-\frac{1}{2(d+3)}}\|P_{Q_c}P_{\le M_0}u_{2,M_2}\|_{Y^0}\end{aligned}$$

By (5.2) and (5.3),

$$\begin{aligned}&I_{M_1,M_2,M_3,M_4,M_5,M_6}\\ &\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}M_3^{\frac{d-1}{2}-\frac{3}{2(d+3)}}M_4^{\frac{d-1}{2}+\frac{1}{2(d+3)}}M_5^{\frac{d-1}{2}+\frac{1}{2(d+3)}}M_6^{\frac{d-1}{2}+\frac{1}{2(d+3)}}\\ &\quad\times\sum_Q\|P_Qu_{1,M_1}\|_{Y^0}\|P_{Q_c}P_{\le M_0}u_{2,M_2}\|_{Y^0}\|u_{3,M_3}\|_{Y^0}\|u_{4,M_4}\|_{Y^0}\|u_{5,M_5}\|_{Y^0}\|g_{M_6}\|_{Y^0}\end{aligned}$$

Applying Cauchy-Schwarz to sum in Q,

$$\begin{aligned}&\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}M_3^{\frac{d-1}{2}-\frac{3}{2(d+3)}}M_4^{\frac{d-1}{2}+\frac{1}{2(d+3)}}M_5^{\frac{d-1}{2}+\frac{1}{2(d+3)}}M_6^{\frac{d-1}{2}+\frac{1}{2(d+3)}}\\ &\quad\|u_{1,M_1}\|_{Y^0}\|P_{\le M_0}u_{2,M_2}\|_{Y^0}\|u_{3,M_3}\|_{Y^0}\|u_{4,M_4}\|_{Y^0}\|u_{5,M_5}\|_{Y^0}\|g_{M_6}\|_{Y^0}\\ &\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}M_1^{-s}M_2^{-\frac{d-1}{2}}M_3^{-\frac{3}{2(d+3)}}M_4^{\frac{1}{2(d+3)}}M_5^{\frac{1}{2(d+3)}}M_6^{\frac{d-1}{2}+\frac{1}{2(d+3)}+s}\\ &\quad\|u_{1,M_1}\|_{Y^s}\|P_{\le M_0}u_{2,M_2}\|_{Y^{\frac{d-1}{2}}}\|u_{3,M_3}\|_{Y^{\frac{d-1}{2}}}\|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}}\|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}}\|g_{M_6}\|_{Y^{-s}}\end{aligned}$$

If $s+\frac{d-1}{2}=0$, it can be estimated in the same way as (5.14). Thus we need only to treat the case $s+\frac{d-1}{2}>0$. Supping out $\|u_{3,M_3}\|_{Y^{\frac{d-1}{2}}}$, $\|u_{4,M_4}\|_{Y^{\frac{d-1}{2}}}$, $\|u_{5,M_5}\|_{Y^{\frac{d-1}{2}}}$ and $\|g_{M_6}\|_{Y^{-s}}$ in M_3, M_4, M_5 and M_6, we have

$$\begin{aligned}I_B &\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}\|u_3\|_{Y^{\frac{d-1}{2}}}\|u_4\|_{Y^{\frac{d-1}{2}}}\|u_5\|_{Y^{\frac{d-1}{2}}}\|g\|_{Y^{-s}}\sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6\\ M_1\sim M_2\ge M_3\ge M_4\ge M_5\ge M_6}}\Big(M_1^{-s}\\ &\quad\|u_{1,M_1}\|_{Y^s}M_2^{-\frac{d-1}{2}}\|P_{\le M_0}u_{2,M_2}\|_{Y^{\frac{d-1}{2}}}M_3^{-\frac{3}{2(d+3)}}M_4^{\frac{1}{2(d+3)}}M_5^{\frac{1}{2(d+3)}}M_6^{\frac{d-1}{2}+\frac{1}{2(d+3)}+s}\Big)\end{aligned}$$

By the fact that $s+\frac{d-1}{2}>0$,

$$\begin{aligned}&\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}\|u_3\|_{Y^{\frac{d-1}{2}}}\|u_4\|_{Y^{\frac{d-1}{2}}}\|u_5\|_{Y^{\frac{d-1}{2}}}\|g\|_{Y^{-s}}\sum_{\substack{M_1,M_2\\ M_1\sim M_2}}\Big(M_1^{-s}\|u_{1,M_1}\|_{Y^s}\\ &\quad M_2^{-\frac{d-1}{2}}\|P_{\le M_0}u_{2,M_2}\|_{Y^{\frac{d-1}{2}}}M_2^{\frac{d-1}{2}+s}\Big)\\ &\lesssim T^{\frac{2d+3}{6(d+3)}}M_0^{\frac{2d+3}{3(d+3)}}\|u_3\|_{Y^{\frac{d-1}{2}}}\|u_4\|_{Y^{\frac{d-1}{2}}}\|u_5\|_{Y^{\frac{d-1}{2}}}\|g\|_{Y^{-s}}\sum_{\substack{M_1,M_2\\ M_1\sim M_2}}\|u_{1,M_1}\|_{Y^s}\|P_{\le M_0}u_{2,M_2}\|_{Y^{\frac{d-1}{2}}}\end{aligned}$$

Applying Cauchy-Schwarz,

$$\lesssim T^{\frac{2d+3}{6(d+3)}} M_0^{\frac{2d+3}{3(d+3)}} \|u_1\|_{Y^s} \|P_{\leq M_0} u_2\|_{Y^{\frac{d-1}{2}}} \|u_3\|_{Y^{\frac{d-1}{2}}} \|u_4\|_{Y^{\frac{d-1}{2}}} \|u_5\|_{Y^{\frac{d-1}{2}}} \|g\|_{Y^{-s}}.$$

□

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Appendix A. Miscellaneous Lemmas

We provide the following lemmas under the $\mathbb{T}^d$ setting, as they work the same for the $\mathbb{R}^d$ case with the homogeneous norm.

Lemma A.1 *Let u_1 and u_2 be the $C([0, T_0]; H^{s_c})$ solutions to (1.1) with the same initial datum such that*

$$u_1(t,x)\bar{u}_1(t,x') = u_2(t,x)\bar{u}_2(t,x'). \tag{A.1}$$

Then $u_1(t,x) = u_2(t,x)$.

Proof From the proof of Corollary 2.3, we have obtained the uniqueness for the trivial solution $u \equiv 0$, so we might as well assume that $u_1(t) \neq 0$ for all $t \in [0, T_0]$. On the other hand, we note that

$$\langle\nabla\rangle^{s_c} u_1(t,x) \|\langle\nabla\rangle^{s_c} u_1(t)\|_{L^2}^2 = \langle\nabla\rangle^{s_c} u_2(t,x) \langle\langle\nabla\rangle^{s_c} u_2(t), \langle\nabla\rangle^{s_c} u_1(t)\rangle \tag{A.2}$$

which implies that

$$\langle\nabla\rangle^{s_c} u_1(t) = a(t) \langle\nabla\rangle^{s_c} u_2(t), \tag{A.3}$$

where

$$a(t) = \frac{\langle\langle\nabla\rangle^{s_c} u_2(t), \langle\nabla\rangle^{s_c} u_1(t)\rangle}{\|\langle\nabla\rangle^{s_c} u_1(t)\|_{L^2}^2}.$$

Since $u_1 \in C([0, T_0]; H^{s_c})$, we have that

$$c_0 := \inf_{t\in[0,T_0]} \|\langle\nabla\rangle^{s_c} u_1\|_{L^2} > 0, \tag{A.4}$$

which implies that $a(t)$ is well-defined. We are left to prove $a(t) = 1$ for every $t \in [0, T_0]$. Taking differences gives that

$$P_{\leq M}(u_2 - u_1) = -i \int_0^t e^{i(t-\tau)\Delta} P_{\leq M}(|u_1|^{p-1} u_1)(\tau, x)(a(\tau) - 1) d\tau, \tag{A.5}$$

where we used (A.3) for u_2.

On the one hand, by (A.3) and the UTFL property in Lemma 2.7, we obtain

$$\|P_{\leq M}(u_2 - u_1)\|_{H^{s_c}} = \|P_{\leq M}(a(t)-1)u_1\|_{H^{s_c}} \geq \frac{c_0|a(t)-1|}{2}. \tag{A.6}$$

On the other hand, by $\|P_{\leq M}\langle\nabla\rangle^s f\|_{L^2} \lesssim M^s\|P_{\leq M} f\|_{L^2}$ and Sobolev embedding A.3 and (A.11), we get

$$\begin{aligned}
&\left\|\int_0^t e^{i(t-\tau)\Delta} P_{\leq M}(|u_1|^{p-1}u_1)(\tau,x)(a(\tau)-1)d\tau\right\|_{H^{s_c}}\\
&\lesssim \int_0^t |a(\tau)-1|\|P_{\leq M}(|u_1|^{p-1}u_1)\|_{H^{s_c}}d\tau\\
&\lesssim M^2\int_0^t |a(\tau)-1|\|P_{\leq M}(|u_1|^{p-1}u_1)\|_{H^{s_c-2}}d\tau\\
&\lesssim M^2\int_0^t |a(\tau)-1|\|u_1\|_{H^{s_c}}^p d\tau\\
&\leq M^2 C_0^p\int_0^t |a(\tau)-1|d\tau.
\end{aligned} \tag{A.7}$$

Combining estimates (A.6) and (A.7), we have

$$|a(t)-1| \lesssim \int_0^t |a(\tau)-1|d\tau \tag{A.8}$$

which implies that $a(t) \equiv 1$ by Gronwall's inequality. □

Lemma A.2

$$\|fg\|_{H^s(\mathbb{T}^d)} \lesssim \|f\|_{H^{s+s_1}(\mathbb{T}^d)}\|g\|_{H^{s_2}(\mathbb{T}^d)} + \|f\|_{H^{\widetilde{s}_1}(\mathbb{T}^d)}\|g\|_{H^{s+\widetilde{s}_2}(\mathbb{T}^d)} \tag{A.9}$$

where $s \geq 0$, $s_i > 0$, $\widetilde{s}_i > 0$, $s_1 + s_2 = \frac{d}{2}$ *and* $\widetilde{s}_1 + \widetilde{s}_2 = \frac{d}{2}$.

Proof This has certainly been studied by many authors. For completeness, we include a proof. We note that

$$P_N(P_{<N-1}f\,P_{<N-1}g) = 0$$

for $N \geq 2$ and hence

$$P_N(fg) = P_N\left[(P_{\geq N-1}f)g + (P_{<N-1}f)(P_{\geq N-1}g)\right].$$

We expand

$$\|fg\|_{H^s(\mathbb{T}^d)}^2 \simeq \sum_{N=0}\langle N\rangle^{2s}\|P_N(fg)\|_{L^2}^2$$

$$
\begin{aligned}
&\lesssim \| fg\|_{L^2}^2 + \sum_{N=2} \langle N\rangle^{2s} \| P_N \left[(P_{\geq N-1} f) g + (P_{<N-1} f)(P_{\geq N-1} g) \right] \|_{L^2}^2 \\
&\lesssim \| fg\|_{L^2}^2 + I^2 + II^2
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \| \langle N\rangle^s P_N \left[(P_{\geq N-1} f) g \right] \|_{l^2 L^2}, \\
II &= \| \langle N\rangle^s P_N \left[(P_{<N-1} f)(P_{\geq N-1} g) \right] \|_{l^2 L^2}.
\end{aligned}
$$

For II, by Hölder inequality, we have

$$
\begin{aligned}
II =& \| \langle N\rangle^s P_N \left[(P_{<N-1} f)(P_{\geq N-1} g) \right] \|_{l^2 L^2} \\
\leq & \| \langle N\rangle^s (P_{<N-1} f)(P_{\geq N-1} g) \|_{l^2 L^2} \\
\leq & \| P_{<N-1} f \|_{l^\infty L^{p_1}} \| \langle N\rangle^s P_{\geq N-1} g \|_{l^2 L^{p_2}},
\end{aligned}
$$

where $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$. Then by Sobolev inequality,

$$
\begin{aligned}
II \lesssim & \| f \|_{H^{s_1}} \| \langle N\rangle^s \langle \nabla\rangle^{s_2} P_{\geq N-1} g \|_{l^2 L^2} \\
=& \| f \|_{H^{s_1}} \left\| \sum_{M\geq N} \langle N\rangle^s \langle M\rangle^{-s} \langle \nabla\rangle^{s_2} \langle M\rangle^s P_M g \right\|_{L^2 l^2}
\end{aligned}
$$

where $s_i \in (0, \frac{d}{2})$ for $i = 1, 2$. By Young's inequality,

$$
II \lesssim \| f \|_{H^{s_1}} \| \langle N\rangle^s \langle \nabla\rangle^{s_2} P_N g \|_{L^2 l^2} \lesssim \| f \|_{H^{s_1}} \| g \|_{H^{s+s_2}}
$$

I can be estimated in the same way as II. □

Lemma A.3 (Sobolev embedding)

$$
\| f_1 f_2 f_3 \|_{H^s(\mathbb{T}^d)} \lesssim \prod_{j=1}^{3} \| f_j \|_{H^{\frac{s+d}{3}}(\mathbb{T}^d)} \tag{A.10}
$$

$$
\| f_1 f_2 f_3 f_4 f_5 \|_{H^s(\mathbb{T}^d)} \lesssim \prod_{j=1}^{5} \| f_j \|_{H^{\frac{s+2d}{5}}(\mathbb{T}^d)} \tag{A.11}
$$

for $s \in (-\frac{d}{2}, \frac{d}{2})$.

Proof For $s = 0$, it follows from Hölder and Sobolev inequalities.

For $s \in (-\frac{d}{2}, 0)$, by duality, we have

$$
\begin{aligned}
&\| f_1 f_2 f_3 \|_{H^s} \lesssim \| f_1 f_2 f_3 \|_{L^{\frac{2d}{d-2s}}}, \\
&\| f_1 f_2 f_3 f_4 f_5 \|_{H^s} \lesssim \| f_1 f_2 f_3 f_4 f_5 \|_{L^{\frac{2d}{d-2s}}}.
\end{aligned}
$$

Then by Hölder inequality and the Sobolev embedding,

$$\|f_1 f_2 f_3\|_{H^s} \lesssim \prod_{j=1}^{3} \|f_j\|_{L^{\frac{6d}{d-2s}}} \lesssim \prod_{j=1}^{3} \|f_j\|_{H^{\frac{s+d}{3}}(\mathbb{T}^d)},$$

$$\|f_1 f_2 f_3 f_4 f_5\|_{H^s} \lesssim \prod_{j=1}^{5} \|f_j\|_{L^{\frac{10d}{d-2s}}} \lesssim \prod_{j=1}^{5} \|f_j\|_{H^{\frac{s+2d}{5}}(\mathbb{T}^d)}.$$

For $s \in (0, \frac{d}{2})$, we use Sobolev inequality (A.9). Taking $f = f_1 f_2$ and $g = f_3$ with $s_1 = \frac{d-2s}{6}$, $s_2 = \frac{s+d}{3}$, $\widetilde{s}_1 = \frac{d+4s}{6}$ and $\widetilde{s}_2 = \frac{d-2s}{3}$, we have

$$\|f_1 f_2 f_3\|_{H^s} \lesssim \|f_1 f_2\|_{H^{\frac{d+4s}{6}}} \|f_3\|_{H^{\frac{s+d}{3}}}.$$

For $\|f_1 f_2\|_{H^{\frac{d+4s}{6}}}$, using it again with $s_1 = \frac{d-2s}{6}$, $s_2 = \frac{s+d}{3}$, $\widetilde{s}_1 = \frac{s+d}{3}$ and $\widetilde{s}_2 = \frac{d-2s}{6}$, we obtain

$$\|f_1 f_2\|_{H^{\frac{d+4s}{6}}} \lesssim \|f_1\|_{H^{\frac{s+d}{3}}} \|f_2\|_{H^{\frac{s+d}{3}}}.$$

Taking $f = f_1 f_2$ and $g = f_3 f_4 f_5$ with $s_1 = \frac{3(d-2s)}{10}$, $s_2 = \frac{3s+d}{5}$, $\widetilde{s}_1 = \frac{4s+3d}{10}$ and $\widetilde{s}_2 = \frac{d-2s}{5}$, we obtain

$$\begin{aligned}\|f_1 f_2 f_3 f_4 f_5\|_{H^s} &\lesssim \|f_1 f_2\|_{H^{s+s_1}} \|f_3 f_4 f_5\|_{H^{s_2}} + \|f_1 f_2\|_{H^{\widetilde{s}_1}} \|f_3 f_4 f_5\|_{H^{s+\widetilde{s}_2}} \\ &\lesssim \prod_{j=1}^{2} \|f_j\|_{H^{\frac{2(s+s_1)+d}{4}}} \prod_{j=3}^{5} \|f_j\|_{H^{\frac{s_2+d}{3}}} + \prod_{j=1}^{2} \|f_j\|_{H^{\frac{2\widetilde{s}_1+d}{4}}} \prod_{j=3}^{5} \|f_j\|_{\frac{s+\widetilde{s}_2+d}{3}} \\ &\lesssim \prod_{j=1}^{5} \|f_j\|_{H^{\frac{s+2d}{5}}}.\end{aligned}$$

□

Appendix B. Results for Some H^1-subcritical Cases

Note that the proof of Theorem 1.1 works uniformly in all dimensions, $d \geq 4$ for quintic case and $d \geq 5$ for cubic case. For completeness, we present some results for low dimensions using our method. As we are limited by the Sobolev embedding in Lemma A.3, the regularity requirements are higher than the critical scaling exponent s_c. Certainly, it is still an open problem to push s down to s_c for H^1-subcritical problems in both $\mathbb{R}^d$ and $\mathbb{T}^d$.

Theorem B.1 *(a). There is at most one $C([0, T_0]; \dot{H}^{\frac{d}{4}}(\Lambda^d))$ solution to (1.1) where $p = 3$ and $d = 2, 3$.*

(b). There is at most one $C([0, T_0]; \dot{H}^{\frac{2}{3}}(\Lambda^2))$ solution to (1.1) where $p = 5$.

Lemma B.2 *On* $\mathbb{T}^2$,

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{g}(t,x)dtdx \lesssim T^{\frac{1}{2}} \|u_1\|_{Y^{-\frac{1}{2}}} \|u_2\|_{Y^{\frac{1}{2}}} \|u_3\|_{Y^{\frac{1}{2}}} \|g\|_{Y^{\frac{1}{2}}}, \tag{B.1}$$

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{g}(t,x)dtdx \lesssim T^{\frac{1}{2}} \|u_1\|_{Y^{\frac{1}{2}}} \|u_2\|_{Y^{\frac{1}{2}}} \|u_3\|_{Y^{\frac{1}{2}}} \|g\|_{Y^{-\frac{1}{2}}}. \tag{B.2}$$

On $\mathbb{T}^3$,

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{g}(t,x)dtdx \lesssim T^{\frac{1}{4}} \|u_1\|_{Y^{-\frac{3}{4}}} \|u_2\|_{Y^{\frac{3}{4}}} \|u_3\|_{Y^{\frac{3}{4}}} \|g\|_{Y^{\frac{3}{4}}}, \tag{B.3}$$

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{g}(t,x)dtdx \lesssim T^{\frac{1}{4}} \|u_1\|_{Y^{\frac{3}{4}}} \|u_2\|_{Y^{\frac{3}{4}}} \|u_3\|_{Y^{\frac{3}{4}}} \|g\|_{Y^{-\frac{3}{4}}} \tag{B.4}$$

Proof By the symmetry of u_1 and g, it suffices to prove (B.1) and (B.3). For simplicity, we take $\widetilde{u} = u$ and $\widetilde{g} = g$. Decompose the 4 factors into Littlewood-Paley pieces so that

$$I = \sum_{M_1,M_2,M_3,M_4} I_{M_1,M_2,M_3,M_4}$$

where

$$I_{M_1,M_2,M_3,M_4} = \iint_{x,t} u_{1,M_1} u_{2,M_2} u_{3,M_3} g_{M_4} dxdt$$

with $u_{j,M_j} = P_{M_j} u_j$ and $g_{M_4} = P_{M_4} g$.

It suffices to consider the most difficult case A. $M_1 \sim M_2 \geq M_3 \geq M_4$ while other cases can be dealt with in a similar way. Decompose the M_1 and M_2 dyadic into M_3 size cubes.

$$\begin{aligned} I_A &\lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}} \sum_Q \|P_Q u_{1,M_1} P_{Q_c} u_{2,M_2} u_{3,M_3} g_{M_4}\|_{L^1_{t,x}} \\ &\lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}} \sum_Q T^{\frac{1}{2}} \|P_Q u_{1,M_1}\|_{L^\infty_t L^2_x} \|P_{Q_c} u_{2,M_2}\|_{L^5_{t,x}} \|u_{3,M_3}\|_{L^5_{t,x}} \|g_{M_4}\|_{L^{10}_{t,x}} \end{aligned}$$

By (5.2)and (5.3),

$$\lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}}$$

$$\sum_{Q} T^{\frac{1}{2}} \|P_Q u_{1,M_1}\|_{Y^0} M_3^{\frac{1}{5}} \|P_{Q_c} u_{2,M_2}\|_{Y^0} M_3^{\frac{1}{5}} \|u_{3,M_3}\|_{Y^0} M_4^{\frac{3}{5}} \|g_{M_4}\|_{Y^0}$$

Applying Cauchy-Schwarz to sum in Q,

$$\lesssim T^{\frac{1}{2}} \sum_{\substack{M_1, M_2 \\ M_1 \sim M_2}} M_1^{\frac{1}{2}} M_2^{-\frac{1}{2}} \|u_{1,M_1}\|_{Y^{-\frac{1}{2}}} \|u_{2,M_2}\|_{Y^{\frac{1}{2}}} \sum_{\substack{M_3, M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}} M_3^{-\frac{1}{10}} M_4^{\frac{1}{10}} \|u_{3,M_3}\|_{Y^{\frac{1}{2}}} \|g_{M_4}\|_{Y^{\frac{1}{2}}}$$

Applying Cauchy-Schwarz,

$$\begin{aligned} &\lesssim T^{\frac{1}{2}} \left(\sum_{M_1} \|u_{1,M_1}\|^2_{Y^{-\frac{1}{2}}} \right)^{\frac{1}{2}} \left(\sum_{M_2} \|u_{2,M_2}\|^2_{Y^{\frac{1}{2}}} \right)^{\frac{1}{2}} \\ &\quad \left(\sum_{\substack{M_3, M_4 \\ M_3 \geq M_4}} \left(\frac{M_4}{M_3} \right)^{\frac{1}{10}} \|u_{3,M_3}\|^2_{Y^{\frac{1}{2}}} \right)^{\frac{1}{2}} \left(\sum_{\substack{M_3, M_4 \\ M_3 \geq M_4}} \left(\frac{M_4}{M_3} \right)^{\frac{1}{10}} \|g_{M_4}\|^2_{Y^{\frac{1}{2}}} \right)^{\frac{1}{2}} \\ &\lesssim T^{\frac{1}{2}} \|u_1\|_{Y^{-\frac{1}{2}}} \|u_2\|_{Y^{\frac{1}{2}}} \|u_3\|_{Y^{\frac{1}{2}}} \|g\|_{Y^{\frac{1}{2}}}. \end{aligned}$$

For $d = 3$, we have that

$$\begin{aligned} I_A &\lesssim \sum_{\substack{M_1, M_2, M_3, M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}} \sum_{Q} \|P_Q u_{1,M_1} P_{Q_c} u_{2,M_2} u_{3,M_3} g_{M_4}\|_{L^1_{t,x}} \\ &\lesssim \sum_{\substack{M_1, M_2, M_3, M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}} \sum_{Q} \|P_Q u_{1,M_1}\|_{L^4_{t,x}} \|P_{Q_c} u_{2,M_2}\|_{L^4_{t,x}} \|u_{3,M_3}\|_{L^\infty_t L^2_x} \|g_{M_4}\|_{L^2_t L^\infty_x} \end{aligned}$$

By (5.3) and Bernstein,

$$\lesssim \sum_{\substack{M_1, M_2, M_3, M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}} \sum_{Q} M_3^{\frac{1}{4}} \|P_Q u_{1,M_1}\|_{Y^0} M_3^{\frac{1}{4}} \|P_{Q_c} u_{2,M_2}\|_{Y^0} \|u_{3,M_3}\|_{Y^0} T^{\frac{1}{4}} M_4 \|g_{M_4}\|_{Y^0}$$

Applying Cauchy-Schwarz to sum in Q,

$$\lesssim T^{\frac{1}{4}} \sum_{\substack{M_1, M_2 \\ M_1 \sim M_2}} M_1^{\frac{3}{4}} M_2^{-\frac{3}{4}} \|u_{1,M_1}\|_{Y^{-\frac{3}{4}}} \|u_{2,M_2}\|_{Y^{\frac{3}{4}}} \sum_{\substack{M_3, M_4 \\ M_1 \sim M_2 \geq M_3 \geq M_4}} M_3^{-\frac{1}{4}} M_4^{\frac{1}{4}} \|u_{3,M_3}\|_{Y^{\frac{3}{4}}} \|g_{M_4}\|_{Y^{\frac{3}{4}}}$$

$$\lesssim T^{\frac{1}{4}} \|u_1\|_{Y^{-\frac{3}{4}}} \|u_2\|_{Y^{\frac{3}{4}}} \|u_3\|_{Y^{\frac{3}{4}}} \|g\|_{Y^{\frac{3}{4}}}.$$

□

Lemma B.3 *On* $\mathbb{T}^2$,

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{u}_4(t,x)\widetilde{u}_5(t,x)\widetilde{g}(t,x)dtdx \lesssim T^{\frac{1}{3}} \|u_1\|_{Y^{-\frac{2}{3}}} \|u_2\|_{Y^{\frac{2}{3}}} \|u_3\|_{Y^{\frac{2}{3}}} \|u_4\|_{Y^{\frac{2}{3}}} \|u_5\|_{Y^{\frac{2}{3}}} \|g\|_{Y^{\frac{2}{3}}}. \tag{B.5}$$

$$\iint_{x,t} \widetilde{u}_1(t,x)\widetilde{u}_2(t,x)\widetilde{u}_3(t,x)\widetilde{u}_4(t,x)\widetilde{u}_5(t,x)\widetilde{g}(t,x)dtdx \lesssim T^{\frac{1}{3}} \|u_1\|_{Y^{\frac{2}{3}}} \|u_2\|_{Y^{\frac{2}{3}}} \|u_3\|_{Y^{\frac{2}{3}}} \|u_4\|_{Y^{\frac{2}{3}}} \|u_5\|_{Y^{\frac{2}{3}}} \|g\|_{Y^{-\frac{2}{3}}}. \tag{B.6}$$

Proof Decompose the 6 factors into Littlewood-Paley pieces so that

$$I = \sum_{M_1,M_2,M_3,M_4,M_5,M_6} I_{M_1,M_2,M_3,M_4,M_5,M_6}$$

where

$$I_{M_1,M_2,M_3,M_4,M_5,M_6} = \iint_{x,t} u_{1,M_1} u_{2,M_2} u_{3,M_3} u_{4,M_4} u_{5,M_5} g_{M_6} dxdt$$

with $u_{j,M_j} = P_{M_j} u_j$ and $g_{M_6} = P_{M_6} g$.

In the same way as trilinear estimates in Lemma B.2, it suffices to take care of the most difficult case A. $M_1 \sim M_6 \geq M_2 \geq M_3 \geq M_4 \geq M_5$. Decompose the M_1 and M_6 dyadic spaces into M_2 size cubes, then

$$I_A \lesssim \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1 \sim M_6 \geq M_2 \geq M_3 \geq M_4 \geq M_5}} \sum_Q \|P_Q u_{1,M_1} P_{Q_c} u_{2,M_2} u_{3,M_3} u_{4,M_4} u_{5,M_5} g_{M_6}\|_{L^1_{t,x}}$$

$$\lesssim \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1 \sim M_6 \geq M_2 \geq M_3 \geq M_4 \geq M_5}} \sum_Q \Big(\|P_Q u_{1,M_1}\|_{L^{\frac{9}{2}}_{t,x}} \|u_{2,M_2}\|_{L^{\frac{9}{2}}_{t,x}} \|u_{3,M_3}\|_{L^9_{t,x}} \|u_{4,M_4}\|_{L^9_{t,x}} \|u_{5,M_5}\|_{L^9_{t,x}} \|P_{Q_c} g_{M_6}\|_{L^{\frac{9}{2}}_{t,x}} \Big)$$

By (5.2)and (5.3),

$$\lesssim \sum_{\substack{M_1,M_2,M_3,M_4,M_5,M_6 \\ M_1 \sim M_6 \geq M_2 \geq M_3 \geq M_4 \geq M_5}} \sum_Q \Big(M_2^{\frac{1}{9}} \|P_Q u_{1,M_1}\|_{Y^0} M_2^{\frac{1}{9}} \|P_{Q_c} u_{2,M_2}\|_{Y^0} T^{\frac{1}{9}} M_3^{\frac{7}{9}} \|u_{3,M_3}\|_{Y^0} T^{\frac{1}{9}} M_4^{\frac{7}{9}} \|u_{4,M_4}\|_{Y^0} T^{\frac{1}{9}} M_5^{\frac{7}{9}} \|u_{5,M_5}\|_{Y^0} M_2^{\frac{1}{9}} \|g_{M_6}\|_{Y^0} \Big)$$

Applying Cauchy-Schwarz to sum in Q,

$$\lesssim T^{\frac{1}{3}} \sum_{\substack{M_1, M_6\\ M_1 \sim M_6}} M_1^{\frac{2}{3}} M_6^{-\frac{2}{3}} \|u_{1,M_1}\|_{Y^{-\frac{2}{3}}} \|g_{M_6}\|_{Y^{\frac{2}{3}}}$$

$$\sum_{\substack{M_2, M_3, M_4, M_5\\ M_2 \geq M_3 \geq M_4 \geq M_5}} M_2^{-\frac{1}{3}} M_3^{\frac{1}{9}} M_4^{\frac{1}{9}} M_5^{\frac{1}{9}} \|u_{2,M_2}\|_{Y^{\frac{2}{3}}} \|u_{3,M_3}\|_{Y^{\frac{2}{3}}} \|u_{4,M_4}\|_{Y^{\frac{2}{3}}} \|u_{5,M_5}\|_{Y^{\frac{2}{3}}}$$

$$\lesssim T^{\frac{1}{3}} \|u_1\|_{Y^{-\frac{2}{3}}} \|u_2\|_{Y^{\frac{2}{3}}} \|u_3\|_{Y^{\frac{2}{3}}} \|u_4\|_{Y^{\frac{2}{3}}} \|u_5\|_{Y^{\frac{2}{3}}} \|g\|_{Y^{\frac{2}{3}}}.$$

□

Appendix C. A More Usual Proof for the $\mathbb{R}^d$ Case

With the dual Strichartz estimate and the existence of a better solution, we could give a more usual proof of the unconditional uniqueness under the energy-supercritical setting for $\mathbb{R}^d$ case. Such an argument has been used by many authors and we summarize it below, but it does not work for the $\mathbb{T}^d$ case. For simplicity, we prove it for the cubic case, as it works the same for the quintic case. At first, we need the following lemmas.

Lemma C.1 (Strichartz Estimate) *Let I be a compact time interval, and let $u : I \times \mathbb{R}^3 \mapsto \mathbb{C}$ be a Schwartz solution to the forced Schrödinger equation*

$$i\partial_t u + \Delta u = \sum_{m=1}^{M} F_m$$

for some Schwartz functions F_1,..., F_m. Then

$$\||\nabla|^s u\|_{L_t^q L_x^r} \lesssim \||\nabla|^s u_0\|_{L_x^2} + \sum_{m=1}^{M} \||\nabla|^s F_m\|_{L_t^{q_m'} L_x^{r_m'}}$$

for $s \geq 0$ and any admissible exponents (q_i, r_i) for $i = 1, 2, ..., m$, where p' denotes the dual exponent to p.

Lemma C.2 (Leibniz Rule [33]) *Let $s \geq 0$ and $1 < r, r_1, r_2, q_1, q_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_i} + \frac{1}{q_i}$ for $i = 1, 2$. Then,*

$$\||\nabla|^s (fg)\|_{L^r} \lesssim \|f\|_{L^{r_1}} \||\nabla|^s g\|_{L^{q_1}} + \||\nabla|^s f\|_{L^{r_2}} \|g\|_{L^{q_2}}.$$

Let u be a maximal-lifespan solution constructed in [47] and v be a $C([0, T); \dot{H}^{s_c})$ solution to NLS with the same initial datum. We write $w = v - u$ and observe that w

obeys a difference equation, which we write in integral form as

$$
\begin{aligned}
w(t) =& -i \int_0^t e^{i(t-s)\Delta}(|u+w|^2(u+w)(s) - |u|^2 u(s))ds \\
=& \int_0^t e^{i(t-s)\Delta} \sum_{j=0}^{2} \varnothing(u^j(s) w^{3-j}(s))ds,
\end{aligned}
\tag{C.1}
$$

where $\varnothing(u^j w^{3-j})$ is a finite linear combination of expressions which could be possibly replaced by their complex conjugates.

By Sobolev inequality, we have

$$\||\nabla|^{s_c-1} w\|_{L_x^{\frac{2d}{d-2}}} \lesssim \||\nabla|^{s_c} w\|_{L^2},$$

and hence $\||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}}$ is finite. Then by Strichartz estimate in Lemma C.1, we have that

$$
\begin{aligned}
\||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \lesssim& \sum_{j=0}^{1} \||\nabla|^{s_c-1}(u^j w^{3-j})\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + \||\nabla|^{s_c-1}(u^2 w)\|_{L_t^1 L_x^2} \\
:=& I_0 + I_1 + I_2.
\end{aligned}
$$

For I_0, by Lemma C.2, we have that

$$I_0 = \||\nabla|^{s_c-1}(|w|^2 w)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|w\|_{L_t^\infty L_x^d}^2$$

Then by Sobolev inequality,

$$\lesssim \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \||\nabla|^{s_c} w\|_{L_t^\infty L_x^2}^2.$$

For I_1, by Lemma C.2, we have that

$$
\begin{aligned}
I_1 =& \||\nabla|^{s_c-1}(uw^2)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
\lesssim& \||\nabla|^{s_c-1} u\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \|w^2\|_{L_t^2 L_x^{\frac{d}{2}}} + \|u\|_{L_t^\infty L_x^d} \||\nabla|^{s_c-1}(w^2)\|_{L_t^2 L_x^2} \\
\lesssim& \||\nabla|^{s_c-1} u\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \|w^2\|_{L_t^2 L_x^{\frac{d}{2}}} + \|u\|_{L_t^\infty L_x^d} \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|w\|_{L_t^\infty L_x^d}
\end{aligned}
$$

By Hölder and Sobolev inequalities,

$$\lesssim \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \||\nabla|^{s_c} w\|_{L_t^\infty L_x^2}.$$

For I_2, by Lemma C.2, we obtain

$$\begin{aligned} I_2 =& \||\nabla|^{s_c-1}(u^2 w)\|_{L_t^1 L_x^2} \\ \lesssim& \||\nabla|^{s_c-1}(u^2)\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|w\|_{L_t^2 L_x^d} + \|u\|^2_{L_t^4 L_x^{2d}} \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\ \lesssim& \||\nabla|^{s_c-1} u\|_{L_t^4 L_x^{\frac{2d}{d-3}}} \|u\|_{L_t^4 L_x^{2d}} \|w\|_{L_t^2 L_x^d} + \|u\|^2_{L_t^4 L_x^{2d}} \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \end{aligned}$$

By Sobolev inequality,

$$\lesssim \||\nabla|^{s_c} u\|^2_{L_t^4 L_x^{\frac{2d}{d-1}}} \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}}.$$

where $(4, \frac{2d}{d-1})$ is a Strichartz pair.

Together with the above estimates, we get

$$\begin{aligned} &\||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\ &\quad\lesssim \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\ &\qquad \left(\||\nabla|^{s_c} w\|^2_{L_t^\infty L_x^2} + \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \||\nabla|^{s_c} w\|_{L_t^\infty L_x^2} + \||\nabla|^{s_c} u\|^2_{L_t^4 L_x^{\frac{2d}{d-1}}} \right) \end{aligned}$$

Note that $w \in C_t^0 \dot{H}^{s_c}$ and $w(0) = 0$, so we can ensure $\||\nabla|^{s_c} w\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)} \le \varepsilon$ by choosing I sufficiently small. Also, from the Strichartz analysis in [47,Theorem 3.1 and Remarks. 1.] , $|\nabla|^{s_c} u$ has finite S^0 norm, and in particular it has finite $L_t^4 L_x^{\frac{2d}{d-1}}$. Thus we can also ensure that $\||\nabla|^{s_c} u\|_{L_t^4 L_x^{\frac{2d}{d-1}}} (I \times \mathbb{R}^d) \le \varepsilon$ by choosing I sufficiently small. From our choice of I, we have

$$\||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \le C\varepsilon \||\nabla|^{s_c-1} w\|_{L_t^2 L_x^{\frac{2d}{d-2}}},$$

which implies that w vanishes identically on $I \times \mathbb{R}^d$ provided that ε is sufficiently small.

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