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Coercivity of the Dirichlet-to-Neumann Operator and Applications to the Muskat Problem

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To Professor Duong Minh Duc, with gratitude, respect and admiration

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Abstract
We consider the Dirichlet-to-Neumann operator in strip-like and half-space domains with Lipschitz boundary. It is shown that the quadratic form generated by the Dirichlet-to-Neumann operator controls some sharp homogeneous fractional Sobolev norms. As an application, we prove that the global Lipschitz solutions constructed in Dong et al. (2021) for the one-phase Muskat problem decays exponentially in time in any Hölder norm C^α, $\alpha \in (0, 1)$.

Keywords Dirichlet-to-Neumann operator · Coercivity · Muskat problem · Time decay of solutions

Mathematics Subject Classification (2010) 35J25 · 35Q35 · 35B40

1 Introduction

The Dirichlet-to-Neumann operator arises naturally in free boundary problems in fluid mechanics as a result of dimension reduction. To name a few, the water wave, the Muskat and the Hele-Shaw problem [1–3, 8].

Let M be either the real line $\mathbb{R}$ or the circle $\mathbb{T}$. We consider either the strip-like domain

$$\Omega = \{(x, y) \in M^d \times \mathbb{R} : b(x) < y < f(x)\} \tag{1.1}$$

or the half-space

$$\Omega = \{(x, y) \in M^d \times \mathbb{R} : y < f(x)\}. \tag{1.2}$$

The boundary functions f and b are Lipschitz continuous and satisfy

$$\inf_{x\in M^d} f(x) - b(x) \geq h > 0. \tag{1.3}$$

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We also refer to (1.1) as the finite depth case and to (1.2) as the infinite depth case.

Given a function $g : M^d \to \mathbb{R}$, $d \geq 1$, we consider the boundary value problem

$$\begin{cases} \Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \\ \phi(x, f(x)) = g(x), \\ \partial_\nu \phi(x, b(x)) = 0, \end{cases} \tag{1.4}$$

where $\nu = \frac{1}{\sqrt{|\nabla_x b|^2+1}}(\nabla_x b, -1)$ is the outward unit normal to the bottom boundary $\{y = b(x)\}$. In the infinite depth case, the Neumann condition in (1.4) is replaced by the decay condition

$$\lim_{(x,y)\to\infty} \nabla_{x,y}\phi = 0.$$

The Dirichlet-Neumann operator G associated to Ω is defined by

$$G(g) = \left(\partial_y \phi - \nabla_x f \cdot \nabla_x \phi\right)|_{y=f(x)} = (-\nabla_x f, 1) \cdot \nabla_{x,y}\phi|_{y=f(x)}.$$

In other words, $G(g)$ is the normal derivative of the harmonic function ϕ on the top boundary $\{y = f(x)\}$.

For the perfect half-space, i.e., $f = 0$, we have $G(g) = |D|g$, where $|D|$ is the Fourier multiplier $|\xi|$. In other words, $|D|$ is the square root of the Laplacian $-\Delta_x$. The quadratic form generated by $|D|$ is coercive

$$\int_{M^d} g|D|g = \||D|^{\frac{1}{2}} g\|^2_{L^2(M^d)} = \|g\|^2_{\dot{H}^{\frac{1}{2}}(M^d)}. \tag{1.5}$$

On the other hand, for straight strip domains, i.e., $f = 0$ and $b(x) \equiv -a$ with $a > 0$, we have $G(g) = |D| \tanh(a|D|)$, whence

$$\int_{M^d} g|D| \tanh(a|D|)g = \|[|D| \tanh(a|D|)]^{\frac{1}{2}} g\|^2_{M^d}, \tag{1.6}$$

where the right hand-side is equivalent to the seminorm $\dot{H}^{\frac{1}{2}}(\mathbb{T}^d)$ when $M^d = \mathbb{T}^d$ and to the seminorm

$$\|g\|^2_{\widetilde{H}^{\frac{1}{2}}(\mathbb{R}^d)} := \int_{M^d} \min\{|\xi|, |\xi|^2\}|\widehat{g}(\xi)|^2 d\xi \tag{1.7}$$

when $M^d = \mathbb{R}^d$. The Sobolev type space $\widetilde{H}^s$ is studied in detail in [7]. We also refer to [2, 4, 5] for pointwise lower bounds for $g|D|g$ and $gG(g)$.

With applications to free boundary problems in mind, we are interested in generalizing (1.5) and (1.6) to non flat boundary, i.e., to domains of the form (1.1) and (1.2) with nontrivial boundary functions f and g.

It is known that when g belongs to the fractional Sobolev space $H^{\frac{1}{2}}(M^d)$, $G(g)$ is well-defined in $H^{-\frac{1}{2}}(M^d)$. See Proposition 2.1 below. We shall prove the following coercive inequalities that generalize (1.5) and (1.6) to the domain (1.2) and (1.1) respectively:

$$\langle G(g), g\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)} \geq M\|g\|^2_X, \tag{1.8}$$

where X is either $\widetilde{H}^{\frac{1}{2}}$ or $\dot{H}^{\frac{1}{2}}$ depending on M and the depth of Ω; the constant M depends explicitly on the boundary of Ω. See Propositions 2.2 and 2.3 below.

In Proposition 2.4 we establish the coercive inequality

$$\langle G(g), \Phi'(g)\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)} \geq M\|\Psi(g)\|^2_X,$$

where Φ is any C^2 convex function such that $\Phi'(z)/z$ is continuous, and $\Psi(z) = \int_0^z \sqrt{\Phi''(z')}dz'$. As a consequence, when $M = \mathbb{T}$ and g has zero mean, $\langle G(g), \Phi'(g)\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)}$ controls the L^p norm of g.

In Section 3, we apply (1.8) to obtain time decay of the global Lipschitz solutions constructed in [6] for the one-phase Muskat problem. It is shown that for any data $f_0 \in W^{1,\infty}(\mathbb{T})$, the global solution f satisfies

$$f \in L^2((0,\infty); \dot{H}^{\frac{1}{2}}(\mathbb{T})), \quad \partial_t f \in L^2((0,\infty); \dot{H}^{-\frac{1}{2}}(\mathbb{T})).$$

If f_0 has zero mean, we prove that all the Hölder norms $C^\alpha(\mathbb{T})$, $\alpha \in (0,1)$ of f decay exponentially.

2 Coercive Inequalities for the Dirichlet-to-Neumann Operator

We denote

$$\mathrm{Lip}(M^d) = \left\{u : M^d \to \mathbb{R} : \exists C > 0,\ \forall x, x' \in M^d,\ |u(x) - u(x')| \le C|x - x'|\right\}.$$

We first recall the following proposition on the boundedness of the Dirichlet-to-Neumann operator.

Proposition 2.1 ([1, 8]) *Let $d \ge 1$.*
(1) *(The finite depth case) Assume that $b, f \in \mathrm{Lip}(M^d)$ such that $f - b \in L^\infty(M^d)$ and (1.3) holds. Let $\widetilde{H}^{\frac{1}{2}}(\mathbb{R}^d)$ be the space of $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ functions whose Fourier transform are locally L^2 in the complement of the origin such that the seminorm (1.7) is finite. For notational convenience, we set $\widetilde{H}^{\frac{1}{2}}(\mathbb{T}^d) = \dot{H}^{\frac{1}{2}}(\mathbb{T}^d)$.*

For any $g \in \widetilde{H}^{\frac{1}{2}}(M^d)$, there exists a unique solution $\phi \in \dot{H}^1(\Omega)$ to (1.4) and we have $G(g) \in H^{-\frac{1}{2}}(M^d)$ together with the bound

$$\|G(g)\|_{H^{-\frac{1}{2}}(M^d)} \le C\left(\|\nabla f\|_{L^\infty(M^d)} + \|\nabla b\|_{L^\infty(M^d)}\right)\|g\|_{\widetilde{H}^{\frac{1}{2}}(M^d)},$$

where $C = C(h, d)$.
(2) *(The infinite depth cases) Let $f \in \mathrm{Lip}(M^d)$. For any $g \in \dot{H}^{\frac{1}{2}}(M^d)$, there exists a unique solution $\phi \in \dot{H}^1(\Omega)$ to (1.4) and we have $G(g) \in H^{-\frac{1}{2}}(M^d)$ together with the bound*

$$\|G(g)\|_{H^{-\frac{1}{2}}(M^d)} \le C\left(\|\nabla f\|_{L^\infty(M^d)}\right)\|g\|_{\dot{H}^{\frac{1}{2}}(M^d)}, \tag{2.1}$$

where $C = C(d)$.

Coercive inequalities for $\langle G(g), g\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)}$ are established in Propositions 2.2 and 2.3 for the finite and infinite depth cases respectively.

Proposition 2.2 *Let Ω be the strip-like domain (1.1), where $b, f \in \mathrm{Lip}(M^d)$ such that $f - b \in L^\infty(M^d)$ and (1.3) holds. There exists a constant $C = C(d) > 0$ such that for any $g \in H^{\frac{1}{2}}(M^d)$, we have*

$$\langle G(g), g\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)} \ge \frac{Ch}{1 + \|\nabla f\|^2_{L^\infty} + \|f - b\|^2_{W^{1,\infty}}}\|g\|^2_{\widetilde{H}^{\frac{1}{2}}(M^d)}, \tag{2.2}$$

where h is given by (1.3).

Proof We flatten Ω using the Lipschitz diffeomorphism

$$M^d \times (-1,0) \ni (x,z) \mapsto \mathcal{S}(x,z) = (x, \varrho(x,z)) \in \Omega,$$

where

$$\varrho(x,z) = (z+1) f(x) - z b(x)$$

satisfies $\partial_z \varrho(x,z) = f(x) - b(x) \geq h$ and $\nabla_{x,z}\varrho \in L^\infty(M^d \times (-1,0))$. By the chain rule, the function $v = \phi \circ \mathcal{S}$ satisfies

$$\operatorname{div}_{x,z}(\mathcal{A}\nabla_{x,z} v)(x,z) = \partial_z \varrho (\Delta_{x,y}\phi)(\mathcal{S}(x,z)) = 0, \tag{2.3}$$

where

$$\mathcal{A} = \begin{bmatrix} \partial_z \varrho \mathbb{I}_{d\times d} & -\nabla_x \varrho \\ -(\nabla_x \varrho)^T & \frac{1+|\nabla_x \varrho|^2}{\partial_z \varrho} \end{bmatrix}.$$

Here we regard the gradient as a column matrix. In terms of v we have

$$G(g)(x) = -\nabla_x \varrho(x,0) \cdot \nabla_x v(x,0) + \frac{1+|\nabla_x \varrho(x,0)|^2}{\partial_z \varrho(x,0)} \partial_z v(x,0) = e_{d+1} \cdot (\mathcal{A}\nabla_{x,z} v)(x,0). \tag{2.4}$$

We recall the following Stokes formula

$$\begin{aligned} &\langle e_{d+1} \cdot u(\cdot,0), w(\cdot,0)\rangle_{H^{-\frac12}(M^d), H^{\frac12}(M^d)} \\ &- \langle e_{d+1} \cdot u(\cdot,-a), w(\cdot,-a)\rangle_{H^{-\frac12}(M^d), H^{\frac12}(M^d)} \\ =&(u, \nabla_{x,z} w)_{L^2(M^d\times(-a,0))} + (\operatorname{div}_{x,z} u, w)_{L^2(M^d\times(-a,0))}, \quad a>0, \end{aligned} \tag{2.5}$$

provided that $u \in L^2(M^d \times (-1,0))^{d+1}$, $\operatorname{div}_{x,z} u \in L^2(M^d \times (-1,0))$ and $w \in H^1(M^d \times (-1,0))$.

We check that (2.5) is applicable with $u = \mathcal{A}\nabla_{x,z} v$ and $w = v$. Indeed, since $\nabla_{x,y}\phi \in L^2(\Omega)$ (by Proposition 2.1) and $\nabla_{x,z}\varrho \in L^\infty(M^d \times (-1,0))$, we have $\nabla_{x,z} v \in L^2(M^d \times (-1,0))$, and thus $\mathcal{A}\nabla_{x,z} v \in L^2(M^d \times (-1,0))$. In addition, since $v(\cdot,0) = g(\cdot) \in L^2(M^d)$ and Ω has finite depth, it follows that $v \in L^2(M^d \times (-1,0))$. By the chain rule, we have

$$\begin{aligned} e_{d+1} \cdot (\mathcal{A}\nabla_{x,z} v)|_{z=-1} &= -\nabla_x \varrho \cdot \nabla_x v + \frac{1+|\nabla_x \varrho|^2}{\partial_z \varrho} \partial_z v|_{z=-1} \\ &= -\nabla_x \varrho \cdot \nabla_x \phi + \partial_y \phi|_{z=-1} \\ &= -\nabla_x b \cdot \nabla_x \phi + \partial_y \phi|_{z=-1} \\ &= -\sqrt{1+|\nabla_x b|^2} \partial_\nu \phi(x, b(x)) = 0. \end{aligned} \tag{2.6}$$

Then applying (2.5) and invoking (2.3), (2.4) and (2.6), we deduce

$$\begin{aligned} &\langle G(g), g\rangle_{H^{-\frac12}, H^{\frac12}} \\ =& \int_{-1}^0 \int_{M^d} \mathcal{A}\nabla_{x,z} v \cdot \nabla_{x,z} v \, dx dz \\ =& \int_{-1}^0 \int_{M^d} \partial_z \varrho \left\{ |\nabla_x v|^2 - 2\frac{\nabla_x \varrho}{\partial_z \varrho} \cdot \nabla_x v \partial_z v + \frac{1+|\nabla_x \varrho|^2}{|\partial_z \varrho|^2} |\partial_z v|^2 \right\} dx dz \\ =& \int_{-1}^0 \int_{M^d} \partial_z \varrho \left\{ \left| \nabla_x v - \frac{\nabla_x \varrho}{\partial_z \varrho} \partial_z v \right|^2 + \frac{|\partial_z v|^2}{|\partial_z \varrho|^2} \right\} dx dz. \end{aligned} \tag{2.7}$$

In the remainder of this proof, we only treat the more difficult case $M^d = \mathbb{R}^d$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function that is identically 1 on $(-1/3, \infty)$ and vanishes on

$(-\infty, -2/3)$. Then $w(x,z) := \chi(z)v(x,z)$ satisfies $w(x,0) = g(x)$ and w vanishes near $z = -1$. Consequently,

$$|\widehat{g}(\xi)|^2 = |\widehat{w}(\xi,0)|^2 = \Re \int_{-1}^{0} \partial_z \widehat{w}(\xi,z)\overline{\widehat{w}(\xi,z)}dz$$
$$= \Re \int_{-1}^{0} \left[\chi'(z)\widehat{v}(\xi,z) + \chi(z)\partial_z \widehat{v}(\xi,z)\right]\chi(z)\overline{\widehat{v}(\xi,z)}dz,$$

where $\widehat{w}$ is the Fourier transform of w with respect to $x \in \mathbb{R}^d$. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \min\{|\xi|, |\xi|^2\}|\widehat{g}(\xi)|^2 \\
&\le C \int_{-1}^{0}\int_{\mathbb{R}^d} |\xi|^2|\widehat{v}(\xi,z)|^2 + |\partial_z\widehat{v}(\xi,z)||\xi||\widehat{v}(\xi,z)|d\xi dz \\
&\le C \int_{-1}^{0}\int_{\mathbb{R}^d} |\widehat{\nabla_x v}(\xi,z)|^2 + |\partial_z\widehat{v}(\xi,z)||\widehat{\nabla_x v}(\xi,z)|d\xi dz \\
&\le C\|\nabla_x v\|^2_{L^2(\mathbb{R}^d\times(-1,0))} + C\|\nabla_x v\|_{L^2(\mathbb{R}^d\times(-1,0))}\|\partial_z v\|_{L^2(\mathbb{R}^d\times(-1,0))} \\
&\le C \int_{-1}^{0}\int_{\mathbb{R}^d} |\nabla_x v|^2 + |\partial_z v|^2 dxdz.
\end{aligned}$$

It follows from this and the triangle inequality

$$|\nabla_x v| \le \left|\nabla_x v - \frac{\nabla_x \varrho}{\partial_z \varrho}\partial_z v\right| + \frac{|\nabla_x\varrho|}{\partial_z\varrho}|\partial_z v|$$

that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \min\{|\xi|, |\xi|^2\}||\widehat{g}(\xi)|^2 \\
&\le C\int_{-1}^{0}\int_{\mathbb{R}^d} \left|\nabla_x v - \frac{\nabla_x\varrho}{\partial_z\varrho}\partial_z v\right|^2 + \left(|\partial_z\varrho|^2 + |\nabla_x\varrho|^2\right)\frac{|\partial_z v|^2}{|\partial_z\varrho|^2}dxdz \\
&\le C\int_{-1}^{0}\int_{\mathbb{R}^d} \partial_z\varrho\left\{\left|\nabla_x v - \frac{\nabla_x\varrho}{\partial_z\varrho}\partial_z v\right|^2 + \frac{|\partial_z v|^2}{|\partial_z\varrho|^2}\right\}\frac{1+|\partial_z\varrho|^2+|\nabla_x\varrho|^2}{\partial_z\varrho}dxdz.
\end{aligned}$$

Using

$$h \le \partial_z\varrho = f(x) - b(x) \le \|f-b\|_{L^\infty} \quad \text{and} \quad \|\nabla_x\varrho\|_{L^\infty} \le \|\nabla f\|_{L^\infty} + \|\nabla(f-b)\|_{L^\infty},$$

we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^d} \min\{|\xi|, |\xi|^2\}|\widehat{g}(\xi)|^2 \\
&\le C\frac{1+\|\nabla f\|^2_{L^\infty}+\|f-b\|^2_{W^{1,\infty}}}{h}\int_{-1}^{0}\int_{\mathbb{R}^d}\partial_z\varrho\left\{\left|\nabla_x v - \frac{\nabla_x\varrho}{\partial_z\varrho}\partial_z v\right|^2 + \frac{|\partial_z v|^2}{|\partial_z\varrho|^2}\right\}dxdz.
\end{aligned} \tag{2.8}$$

In view of (2.7) and (2.8) we conclude the proof of (2.2). □

Proposition 2.3 *Let Ω be the half-space domain (1.2) with $f \in \mathrm{Lip}(M^d)$. There exists a constant $C = C(d) > 0$ such that for any $g \in H^{\frac{1}{2}}(M^d)$, we have*

$$\langle G(g), g\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)} \ge \frac{C}{1+\|\nabla f\|_{L^\infty(M^d)}}\|g\|^2_{\dot{H}^{\frac{1}{2}}(M^d)}. \tag{2.9}$$

Proof We flatten $\Omega = \{(x, y) \in M^d \times \mathbb{R} : y < f(x)\}$ using the Lipschitz diffeomorphism

$$M^d \times (-\infty, 0) \ni (x, z) \mapsto \mathcal{S}(x, z) = (x, \varrho(x, z)) \in \Omega,$$

where $\varrho(x, z) = z + f(x)$. The formula (2.4) holds with $v = \phi \circ \mathcal{S}$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $\chi(z) = 1$ on $(-1/3, \infty)$ and $\chi(z) = 0$ on $(-\infty, -2/3)$. We apply the Stokes formula (2.5) with $u = \mathcal{A}\nabla_{x,z}v$, $w = v(x, z)\chi(\frac{z}{-n})$ and $a = n$ to have

$$\langle G(g), g\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} = \int_{M^d}\int_{-n}^{0} \mathcal{A}\nabla_{x,z}v\cdot\nabla_{x,z}v-\frac{1}{n}\big(e_{d+1}\cdot\mathcal{A}\nabla_{x,z}v\big)\chi'\left(\frac{z}{-n}\right) v dz dx. \quad (2.10)$$

We shall prove that

$$I := \lim_{n\to\infty}\frac{1}{n}\int_{M^d}\int_{-n}^{0}\big(e_{d+1}\cdot\mathcal{A}\nabla_{x,z}v\big)\chi'\left(\frac{z}{-n}\right) v dz dx = 0. \quad (2.11)$$

Since $v(x, 0) = g(x)$, we have

$$\begin{aligned} |v(x,z)| &\le |g(x)| + |z|^{\frac{1}{2}}\left|\int_z^0 |\partial_z v(x,z')|^2 dz'\right|^{\frac{1}{2}} \\ &\le |g(x)| + n^{\frac{1}{2}}\left|\int_{-n}^0 |\partial_z v(x,z')|^2 dz'\right|^{\frac{1}{2}}, \quad z \in [-n, 0], \end{aligned}$$

whence

$$\begin{aligned} I &\le \frac{1}{n}\int_{M^d}\int_{-n}^{0}\left|e_{d+1}\cdot\mathcal{A}\nabla_{x,z}v\right|\left|\chi'\left(\frac{z}{-n}\right)\right||g(x)| dz dx \\ &\quad + \frac{1}{\sqrt{n}}\int_{M^d}\int_{-n}^{0}\left|e_{d+1}\cdot\mathcal{A}\nabla_{x,z}v\right|\left|\chi'\left(\frac{z}{-n}\right)\right|\left|\int_{-n}^0 |\partial_z v(x,z')|^2 dz'\right|^{\frac{1}{2}} dz dx \\ &:= I_1 + I_2. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} I_1 &\le \frac{1}{n}\|\mathcal{A}\nabla_{x,z}v\|_{L^2(M^d\times(-n,0))}\left\|g\chi'\left(\frac{\cdot}{-n}\right)\right\|_{L^2(M^d\times(-n,0))} \\ &\le \frac{C}{n^{\frac{1}{2}}}\|\mathcal{A}\nabla_{x,z}v\|_{L^2(M^d\times(-n,0))}\|g\|_{L^2(M^d)} \to 0 \quad \text{as } n\to\infty \end{aligned}$$

and

$$\begin{aligned} I_2 &\le \frac{1}{\sqrt{n}}\left\|\mathcal{A}\nabla_{x,z}v\chi'\left(\frac{\cdot}{-n}\right)\right\|_{L^2(M^d\times(-n,0))}\left\|\int_{-n}^0\left|\partial_z v(\cdot,z')|^2 dz'\right|^{\frac{1}{2}}\right\|_{L^2(M^d\times(-n,0))} \\ &\le \left\|\mathcal{A}\nabla_{x,z}v\chi'\left(\frac{\cdot}{-n}\right)\right\|_{L^2(M^d\times(-n,0))}\|\partial_z v\|_{L^2(M^d\times(-n,0))}. \end{aligned}$$

Since $\chi'(\frac{z}{-n}) \to 0$ as $n \to \infty$ and $\mathcal{A}\nabla_{x,z}v \in L^2(M^d \times (-\infty, 0))$, the dominated convergence theorem implies that $\lim_{n\to\infty} I_2 = 0$. Therefore, passing $n \to \infty$ in (2.10) we obtain

$$\begin{aligned} \langle G(g), g\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} &= \int_{M^d}\int_{-\infty}^{0}\mathcal{A}\nabla_{x,z}v\cdot\nabla_{x,z}v dz dx \\ &= \int_{M^d}\int_{-\infty}^{0}|\nabla_x v - \nabla f\partial_z v|^2 + |\partial_z v|^2 dz dx, \end{aligned} \quad (2.12)$$

where we have used that $\nabla_x \varrho = \nabla_x f$ and $\partial_z \varrho = 1$.

We only consider the more difficult case $M^d = \mathbb{R}^d$ in the remainder of this proof. For $w(x,z) = \chi(\frac{z}{-n})v(x,z)$, we have $w(x,0) = v(x,0) = g(x)$ and w vanishes near $z = -n$. Consequently,

$$\begin{aligned}
\int_{\mathbb{R}^d} |\xi||\widehat{g}(\xi)|^2 d\xi &= \int_{\mathbb{R}^d} |\xi| \int_{-n}^0 \partial_z |\widehat{w}(\xi,z)|^2 = \Re \int_{\mathbb{R}^d}\int_{-n}^0 \partial_z \widehat{w}(\xi,z)|\xi|\overline{\widehat{w}(\xi,z)}dzd\xi \\
&= \Re \int_{\mathbb{R}^d}\int_{-n}^0 \chi^2\left(\frac{z}{-n}\right)\partial_z \widehat{v}(\xi,z)\overline{\widehat{|D|v}(\xi,z)}dzd\xi \\
&\quad -\frac{1}{n}\Re \int_{\mathbb{R}^d}\int_{-n}^0 \chi\left(\frac{z}{-n}\right)\chi'\left(\frac{z}{-n}\right)\widehat{v}(\xi,z)\overline{\widehat{|D|v}(\xi,z)}dzd\xi \\
&= 2\int_{\mathbb{R}^d}\int_{-n}^0 \chi^2\left(\frac{z}{-n}\right)\partial_z v(x,z)|D|v(x,z)dzdx \\
&\quad -\frac{2}{n}\int_{\mathbb{R}^d}\int_{-n}^0 \chi\left(\frac{z}{-n}\right)\chi'\left(\frac{z}{-n}\right)v(x,z)|D|v(x,z)dzdx.
\end{aligned}$$

Since $\partial_z v$ and $|D|v$ belong to $L^2(\mathbb{R}^d \times \mathbb{R}_-)$, arguing as in (2.11), we can pass to the limit $n \to \infty$ and obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |\xi||\widehat{g}(\xi)|^2 d\xi &= 2\int_{\mathbb{R}^d}\int_{-\infty}^0 \partial_z v(x,z)|D|v(x,z)dzdx \\
&= 2\int_{\mathbb{R}^d}\int_{-\infty}^0 \partial_z v(x,z)\mathcal{R}\cdot\nabla_x v(x,z)dzdx \\
&= \int_{\mathbb{R}^d}\int_{-\infty}^0 2\partial_z v\mathcal{R}\cdot(\nabla_x v - \nabla f\partial_z v) + 2\partial_z v\mathcal{R}\cdot(\nabla f\partial_z v)dzdx \\
&= \int_{\mathbb{R}^d}\int_{-\infty}^0 |\partial_z v|^2 + |\mathcal{R}\cdot(\nabla_x v - \nabla f\partial_z v)|^2 + 2\partial_z v\mathcal{R}\cdot(\nabla f\partial_z v) \\
&\qquad - \left[\partial_z v - \mathcal{R}\cdot(\nabla_x v - \nabla f\partial_z v)\right]^2 dzdx,
\end{aligned}$$

where $\mathcal{R}$ denotes the Riesz transform, $\widehat{\mathcal{R}u}(\xi) = \frac{-i\xi}{|\xi|}\widehat{u}(\xi)$. Using Hölder's inequality and the boundedness of $\mathcal{R}$ in L^2, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |\xi||\widehat{g}(\xi)|^2 d\xi &\le C\|\partial_z v\|_{L^2}\|\nabla_x v - \nabla f\partial_z v\|_{L^2} + C\|\nabla f\|_{L^\infty}\|\partial_z v\|_{L^2}^2 \\
&\le C(1 + \|\nabla f\|_{L^\infty})\int_{\mathbb{R}^d}\int_{-\infty}^0 |\nabla_x v - \nabla f\partial_z v|^2 + |\partial_z v|^2 dzdx.
\end{aligned} \tag{2.13}$$

Finally, (2.9) follows from (2.12) and (2.13). □

Next, we generalize (2.2) and (2.9) to the pairing $\langle G(g), \Phi'(g)\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)}$ for convex functions Φ.

Proposition 2.4 *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a C^2 convex function such that $\Phi'(z)/z$ is continuous on $\mathbb{R}$. Set*

$$\Psi(z) = \int_0^z \sqrt{\Phi''(z')}dz'.$$

Let $g \in H^{\frac{1}{2}}(M^d) \cap L^\infty(M^d)$.

(1) *(The finite depth case) If* $b,\ f \in \mathrm{Lip}(M^d)$ *such that* $f - b \in L^\infty(M^d)$ *and* (1.3) *holds, then there exists a constant* $C = C(d) > 0$ *such that*

$$\langle G(g), \Phi'(g)\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)} \geq \frac{Ch}{1 + \|\nabla f\|^2_{L^\infty} + \|f - b\|^2_{W^{1,\infty}}} \|\Psi(g)\|^2_{\widetilde{H}^{\frac{1}{2}}(M^d)}.$$

(2) *(The infinite depth case) If* $f \in \mathrm{Lip}(M^d)$*, then there exists a constant* $C = C(d) > 0$ *such that*

$$\langle G(g), \Phi'(g)\rangle_{H^{-\frac{1}{2}}(M^d), H^{\frac{1}{2}}(M^d)} \geq \frac{C}{1 + \|\nabla f\|_{L^\infty}} \|\Psi(g)\|^2_{\dot{H}^{\frac{1}{2}}(M^d)}.$$

Proof We shall only consider the more difficult case $M^d = \mathbb{R}^d$. Since $\Phi'(z)/z$ is continuous and $g \in H^{\frac{1}{2}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, it can be shown that $\Phi'(g) \in H^{\frac{1}{2}}(\mathbb{R}^d) \subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^d) \subset \widetilde{H}^{\frac{1}{2}}(\mathbb{R}^d)$. Let $v = \phi \circ \mathcal{S}$ as given in the proof of Propositions 2.2 and 2.3. By the maximum principle for the harmonic function ϕ, we have

$$\|v\|_{L^\infty(M^d \times J)} = \|\phi\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(M^d)}, \tag{2.14}$$

where $J = (-1, 0)$ in the finite depth case and $J = (-\infty, 0)$ in the infinite depth case. From (2.14) and the assumption that $\Phi'(z)/z$ is continuous, we deduce that $\Phi'(v) \in L^2(M^d \times J)$.

(1) The finite depth case. Lemma 2.6 below implies that $\nabla_{x,z}\Phi'(v) = \Phi''(v)\nabla_{x,z} v \in L^2(M^d \times (-1, 0))$. Thus we can apply the Stokes formula (2.5) with $u = \mathcal{A}\nabla_{x,z} v$ and $w = \Phi'(v) \in H^1(M^d \times (-1, 0))$ to have

$$\begin{aligned}
\langle G(g), \Phi'(g)\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} &= \int_{-1}^0 \int_{M^d} \mathcal{A}\nabla_{x,z} v \cdot \nabla_{x,z} v \Phi''(v) dx dz \\
&= \int_{-1}^0 \int_{M^d} \mathcal{A}\nabla_{x,z}\Psi(v) \cdot \nabla_{x,z}\Psi(v) dx dz \\
&= \int_{-1}^0 \int_{M^d} \partial_z \varrho \left\{ \left| \nabla_x \Psi(v) - \frac{\nabla_x \varrho}{\partial_z \varrho} \partial_z \Psi(v) \right|^2 + \frac{|\partial_z \Psi(v)|^2}{|\partial_z \varrho|^2} \right\} dx dz.
\end{aligned}$$

We then conclude by following the proof of (2.8) with $\Psi(g)$ in place of g and $\Psi(v)$ in place of v.

(2) The infinite depth case. The proof proceeds similarly to that of Proposition 2.3 and the finite depth case (1) above. We only remark that in place of (2.11), we need to prove

$$\lim_{n \to \infty} \frac{1}{n} \int_{M^d} \int_{-n}^0 \left(e_{d+1} \cdot \mathcal{A}\nabla_{x,z} v\right) \chi'\left(\frac{z}{-n}\right) \Phi'(v) dx dz = 0.$$

Since $\Phi'(z)/z$ is continuous and v is bounded, we can replace $\Phi'(v)$ by v in the preceding limit and argue as in the proof of (2.11). □

Corollary 2.5 *For any* $p \geq 2$*, there exist positive constants* $C = C(d)$ *and* $C' = C'(p, d)$ *such that for any* $g \in H^{\frac{1}{2}}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$ *satisfying* $\int_{\mathbb{T}^d} g = 0$*, we have*

$$\langle G(g), p|g|^{p-2}g\rangle_{H^{-\frac{1}{2}}(\mathbb{T}^d), H^{\frac{1}{2}}(\mathbb{T}^d)} \geq M\left(\||g|^{p/2-1}g\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^d)} + C'\|g\|^p_{L^p(\mathbb{T}^d)} \right),$$

where

$$M = \begin{cases} \frac{Ch}{1+\|\nabla f\|^2_{L^\infty}+\|f-b\|^2_{W^{1,\infty}}} & \text{in the finite depth case,} \\ \frac{C}{1+\|\nabla f\|_{L^\infty(M^d)}} & \text{in the infinite depth case.} \end{cases} \tag{2.15}$$

Proof For $p \geq 2$, Proposition 2.4 is applicable with $\Phi(z) = |z|^p$ and $\Psi(z) = 2\sqrt{\frac{p-1}{p}}|z|^{p/2-1}z$. We obtain

$$\begin{aligned} \langle G(g), p|g|^{p-2}g\rangle_{H^{-\frac{1}{2}}(\mathbb{T}^d), H^{\frac{1}{2}}(\mathbb{T}^d)} &\geq M\frac{p-1}{p}\||g|^{p/2-1}g\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^d)} \\ &\geq M\frac{1}{2}\||g|^{p/2-1}g\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^d)}, \end{aligned}$$

where M is given by (2.15). It then suffices to prove that for some $C' = C'(p, d) > 0$,

$$\|g\|_{L^p(\mathbb{T}^d)} \leq C'\||g|^{p/2-1}g\|^{\frac{2}{p}}_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^d)} + C'\int_{\mathbb{T}^d} g.$$

For the sake of contradiction, assume that for all $n \in \mathbb{N}$, there exists $g_n \neq 0$ such that

$$\frac{1}{n}\|g_n\|_{L^p(\mathbb{T}^d)} \geq \||g_n|^{p/2-1}g_n\|^{\frac{2}{p}}_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^d)} + \int_{\mathbb{T}^d} g_n. \tag{2.16}$$

By the homogeneity of (2.16) in g_n, we can assume that $\|g_n\|_{L^p(\mathbb{T}^d)} = 1$ for all n. Set $q_n = |g_n|^{p/2-1}g_n$. We have $\|q_n\|_{L^2} = \|g_n\|^{p/2}_{L^p} = 1$ and thus the sequence (q_n) is bounded in $H^{\frac{1}{2}}(\mathbb{T}^d)$. By the compact embedding $H^{\frac{1}{2}}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$, there exists a subsequence, which we renumber (q_n), that converges weakly to q in $H^{\frac{1}{2}}(\mathbb{T}^d)$ and converges strongly to q in $L^2(\mathbb{T}^d)$. In particular, we have $\|q\|_{L^2} = 1$. On the other hand, (2.16) implies that $\|q_n\|_{\dot{H}^{\frac{1}{2}}} \leq 1/n$, whence $\|q\|_{\dot{H}^{\frac{1}{2}}} = 0$ and hence $q = c$ is a constant. Since $\|q\|_{L^2} = 1$, c must be nonzero. Assume without loss of generality that $c > 0$. From (2.16) we deduce

$$0 = \lim_{n\to\infty}\int_{\mathbb{T}^d} g_n(x)dx = \lim_{n\to\infty}\int_{\mathbb{T}^d} |q_n(x)|^{2/p}\mathrm{sign}(q_n(x))dx.$$

Since $q_n \to q = c$ in L^2, there exists a subsequence, which we renumber q_n, such that $q_n(x) \to c$ a.e. $\mathbb{T}^d$ and there exists $Q \in L^2(\mathbb{T}^d)$ such that for all n, $|q_n(x)| \leq Q(x)$ a.e. $\mathbb{T}^d$. Then $|q_n|^{2/p}\mathrm{sign}(q_n) \to c^{2/p}$ and $||q_n|^{2/p}\mathrm{sign}(q_n)| \leq |Q|^{2/p}$ a.e. $\mathbb{T}^d$. Since $Q \in L^2(\mathbb{T}^d)$, we have $|Q|^{2/p} \in L^p(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$ for all $p \geq 1$. Therefore, the dominated convergence theorem yields

$$0 = \int_{\mathbb{T}^d} c^{2/p} = c^{2/p}|\mathbb{T}^d|.$$

This contradicts the fact that $c > 0$. □

Lemma 2.6 *Let $U \subset \mathbb{R}^N$ be an open set and let $\Gamma : \mathbb{R} \to \mathbb{R}$ be a C^1 function. If $u \in L^\infty_{\mathrm{loc}}(U)$ and $\nabla u \in L^1_{\mathrm{loc}}(U)$, then $\nabla\Gamma(u) = \Gamma'(u)\nabla u$.*

Proof Let $V \Subset W \Subset U$. Let ρ_n be the standard mollifier at scale $1/n$ and set $u_n = (u1_W) * \rho_n$, where 1_V is the indicator function of V. Since $u \in L^\infty_{\mathrm{loc}}(U)$ and $\nabla u \in L^1_{\mathrm{loc}}(U)$, we have that

$$\begin{gathered} \nabla u_n \to \nabla u \quad \text{in } W^{1,1}(V), \\ \exists M > 0,\ \forall n,\ \|u_n\|_{L^\infty(\mathbb{R}^N)} + \|u\|_{L^\infty(U)} \leq M. \end{gathered}$$

It follows that

$$\int_V |\Gamma(u_n) - \Gamma(u)| \le \max_{[-M,M]} |\Gamma'| \int_V |u_n - u| \to 0 \quad \text{as } n \to \infty$$

and

$$\begin{aligned} &\int_V |\Gamma'(u_n)\nabla u_n - \Gamma'(u)\nabla u| \\ &\le \int_V |\Gamma'(u_n)||\nabla u_n - \nabla u| + \int_V |\Gamma'(u_n) - \Gamma'(u)||\nabla u| \\ &\le \max_{[-M,M]} |\Gamma'| \int_V |\nabla u_n - \nabla u| + \int_V |\Gamma'(u_n) - \Gamma'(u)||\nabla u|. \end{aligned}$$

A subsequence of (u_n), which we renumber (u_n), must converge a.e. to u in V. Hence the last integral converges to 0 by the dominated convergence theorem. Consequently the sequences $(\Gamma(u_m))$, $(\Gamma'(u_n)\nabla u_n)$ converge to $\Gamma(u)$, $\Gamma'(u)\nabla u$ respectively in $L^1(V)$. Since $\nabla\Gamma(u_n) = \Gamma'(u_n)\nabla u_n$ and V is arbitrary, we conclude that $\nabla\Gamma(u) = \Gamma'(u)\nabla u$. □

3 Time Decay for the One-Phase Muskat Problem

The one-phase Muskat problem concerns the dynamics of the free boundary of a fluid occupying a region in a porous medium. The fluid motion is modeled by Darcy's law with gravity. When the fluid domain has the form (1.1) or (1.2), the free boundary f obeys the equation

$$\partial_t f = -G_f(f), \tag{3.1}$$

where we write G_f to emphasize the dependence of G on the free boundary f. Some physical constants have been normalized in (3.1). We refer to [8] for a derivation of (3.1).

We recall the following global well-posedness result.

Theorem 3.1 ([6, Theorem 1.2]) *Let Ω be the domain (1.2) with $M = \mathbb{T}$. For any initial data $f_0 \in W^{1,\infty}(\mathbb{T})$, (3.1) has a unique viscosity solution*

$$f \in C(\mathbb{T} \times [0,\infty)) \cap L^\infty([0,\infty); W^{1,\infty}(\mathbb{T})), \quad \partial_t f \in L^\infty([0,\infty); L^2(\mathbb{T})). \tag{3.2}$$

In particular, (3.1) is satisfied in the $L^\infty_t L^2_x$ sense. Moreover, we have

$$\|f(t)\|_{L^\infty(\mathbb{T})} \le \|f(0)\|_{L^\infty(\mathbb{T})}, \quad \int_{\mathbb{T}} f(x,t)dx = \int_{\mathbb{T}} f(x,0)dx, \quad \forall t > 0 \tag{3.3}$$

and

$$\|\partial_x f(t)\|_{L^\infty(\mathbb{T})} \le \|\partial_x f(0)\|_{L^\infty(\mathbb{T})} \quad a.e.\ t > 0. \tag{3.4}$$

The precise definition of viscosity solutions of (3.1) is given in [6, Definition 6.1]. In what follows, we will only need the fact that (3.1) is satisfied in $L^\infty_t L^2_x$.

We now apply the coercive estimates in the preceding section to prove the following result on time decay of the solutions.

Proposition 3.2 *For any $f_0 \in W^{1,\infty}(\mathbb{T})$, we have*

$$f \in L^2([0,\infty); \dot{H}^{\frac{1}{2}}(\mathbb{T})), \quad \partial_t f \in L^2([0,\infty); H^{-\frac{1}{2}}(\mathbb{T})).$$

If in addition $\int_{\mathbb{T}} f_0 = 0$, then $\|f(t)\|_{H^\alpha}$, $\|f(t)\|_{C^\alpha}$ and $\|\partial_t f(t)\|_{H^{-\varepsilon}}$ decay exponentially as $t \to \infty$ for any $\alpha \in (0,1)$ and any $\varepsilon > 0$.

Proof Thanks to the regularity (3.2), the following calculation is justified

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} f^2(x,t)dx = (\partial_t f(t), f(t))_{L^2,L^2} = \langle \partial_t f(t), f(t)\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}$$
$$= -\langle G_f(f), f\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}.$$

Applying Proposition 2.3 and the maximum principle (3.4), we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f(t)\|^2_{L^2(\mathbb{T})} &\le -\frac{C}{1+\|\partial_x f(t)\|_{L^\infty(\mathbb{T})}}\|f(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{T})}\\ &\le -\frac{C}{1+\|\partial_x f(0)\|_{L^\infty(\mathbb{T})}}\|f(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{T})}\end{aligned} \tag{3.5}$$

for a.e. $t > 0$. It follows that

$$f \in L^2([0,\infty); \dot{H}^{\frac{1}{2}}(\mathbb{T})). \tag{3.6}$$

Combining (3.6) and (2.1) yields

$$\partial_t f = -G_f(f) \in L^2([0,\infty); H^{-\frac{1}{2}}(\mathbb{T})).$$

Assume now that f_0 has zero mean, then (3.3) implies that $f(t)$ has zero mean for all $t > 0$. Consequently, $\|f(t)\|_{\dot{H}^{\frac{1}{2}}} \ge \|f(t)\|_{L^2}$ and thus (3.5) yields

$$\frac{d}{dt}\|f(t)\|^2_{L^2(\mathbb{T})} \le -\frac{C}{1+\|\partial_x f(0)\|_{L^\infty(\mathbb{T})}}\|f(t)\|^2_{L^2(\mathbb{T})}, \quad \forall t > 0.$$

Therefore, the L^2 norm of f decays exponentially,

$$\|f(t)\|_{L^2} \le \|f_0\|_{L^2} e^{-\frac{Ct}{1+\|\partial_x f(0)\|_{L^\infty(\mathbb{T})}}}, \quad \forall t > 0. \tag{3.7}$$

Combining (3.7) with the uniform bounds (3.3) and (3.4), we deduce that f decays exponentially in any norms that interpolate between $L^2(\mathbb{T})$ and $W^{1,\infty}(\mathbb{T})$. In particular, all the $H^\alpha(\mathbb{T})$ and $C^\alpha(\mathbb{T})$ norms, $\alpha \in [0,1)$, of f decay exponentially. Next, we recall from [6] that

$$\|G_f(f)\|_{H^{\sigma-1}} \le C(1+\|\partial_x f\|_{L^\infty})^2\|f\|_{\dot{H}^\sigma}, \quad \sigma \in \left[\frac{1}{2}, 1\right].$$

Therefore, $\partial_t f = -G_f(f)$ decays exponentially in $H^{-\varepsilon}(\mathbb{T})$ for any $\varepsilon > 0$. □

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