

Quasi-local mass on unit spheres at spatial infinity

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In this note, we compute the limit of the Wang-Yau quasi-local mass on unit spheres at spatial infinity of an asymptotically flat initial data set. Similar to the small sphere limit of the Wang-Yau quasi-local mass, we prove that the leading order term of the quasi-local mass recovers the stress-energy tensor. For a vacuum spacetime, the quasi-local mass decays faster and the leading order term is related to the Bel-Robinson tensor. Several new techniques of evaluating quasilocal mass are developed in this note.

1	Introduction	746
2	Review of the Wang-Yau quasi-local mass and asymptotical flatness	749
3	The Brown-York mass	752
4	Optimal embedding equation and the Jang equation	755
5	Limit of the Wang-Yau mass	757
6	Small sphere limit	770
	References	774

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1. Introduction

In general relativity, a spacetime is a 4-manifold N with a Lorentzian metric $g_{\alpha\beta}$ satisfying the *Einstein equation*

$$R_{\alpha\beta} - \frac{R}{2}g_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where $R_{\alpha\beta}$ and R are the Ricci curvature and the scalar curvature of the metric $g_{\alpha\beta}$, respectively. On the right hand side of the Einstein equation, $T_{\alpha\beta}$ is the stress-energy tensor of the matter field, a divergence free and symmetric 2-tensor. For most matter fields, $T_{\alpha\beta}$ satisfies the dominant energy condition. For a vacuum spacetime where $T_{\alpha\beta} = 0$ (which implies $R_{\alpha\beta} = 0$), one way of measuring the gravitational energy is to consider the *Bel-Robinson tensor* [1]

$$(1.1) \quad Q_{\mu\nu\alpha\beta} = W_{\mu}^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta} W_{\rho\nu\sigma\beta} + W_{\mu}^{\rho}{}_{\beta}{}^{\sigma}{}_{\alpha} W_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}W_{\alpha}^{\rho\sigma\tau}W_{\beta\rho\sigma\tau},$$

where $W_{\alpha\beta\gamma\delta}$ is the Weyl curvature tensor of the spacetime N . For a vacuum spacetime, the Bel-Robinson tensor is a divergence free and totally symmetric 4-tensor which also satisfies a certain positivity condition [11, Lemma 7.1.1].

We recall that given a spacelike 2-surface Σ in a spacetime N , the Wang-Yau quasi-local energy $E(\Sigma, \mathcal{X}, T_0)$ (see (2.1)) is defined in [27, 28] with respect to each pair $(\mathcal{X}, T_0)$ of an isometric embedding $\mathcal{X}$ of Σ into the Minkowski space $\mathbb{R}^{3,1}$ and a constant future timelike unit vector $T_0 \in \mathbb{R}^{3,1}$. If the spacetime satisfies the dominant energy condition and the pair $(\mathcal{X}, T_0)$ is admissible (see [28, Definition 5.1]), it is proved that $E(\Sigma, \mathcal{X}, T_0) \geq 0$. The Wang-Yau quasi-local mass is defined to be the infimum of the quasi-local energy among all admissible pairs $(\mathcal{X}, T_0)$. The Euler-Lagrange equation for the critical points of the quasi-local energy is derived in [28]. The Euler-Lagrange equation, coupled with the isometric embedding equation, is referred to as the optimal embedding equation, see (2.2). A solution to the equation is referred to as an optimal embedding.

When studying different notions of quasi-local energy, it is natural to evaluate the large sphere and the small sphere limits of the quasi-local energy and compare with the known measures of the gravitational energy in these situations. One expects the following [12, 23]:

1) For a family of surfaces approaching the spatial/null infinity of an isolated system (the large sphere limit), the limit of the quasi-local energy recovers the total energy-momentum of the isolated system.

2) For a family of surfaces approaching a point p (the small sphere limit), the limit of the quasi-local energy recovers the stress-energy tensor for spacetimes with matter fields and the Bel-Robinson tensor for vacuum spacetimes.

There are many works on evaluating the large sphere and the small sphere limits of different notions of quasi-local energy. See for example [3–6, 14, 15, 18–20, 22, 29–31]. The list here is by no means exhaustive. For a more comprehensive review of different notions of quasi-local energy and their limiting behaviors, see [25] and the references therein.

In a series of papers [5, 6, 29], the above expectations for the Wang-Yau quasi-local energy were verified. One of the key observations in [29] (see [29, Theorem 2.1]) is that for a family of surfaces Σ_r and isometric embeddings $\mathcal{X}_r$, the limit of $E(\Sigma_r, \mathcal{X}_r, T_0)$ is a linear function of T_0 under the compatibility condition

$$(1.2) \quad \lim_{r \rightarrow r_0} \frac{|H_0|}{|H|} = 1,$$

where H and H_0 are the mean curvature vectors of Σ_r in N and the image of the isometric embedding $\mathcal{X}_r$ in $\mathbb{R}^{3,1}$, respectively. The compatibility condition (1.2) holds naturally in the large sphere limit ($r_0 = \infty$) at both spatial and null infinity and the small sphere limit ($r_0 = 0$) around a point. In particular, [29, Theorem 2.1] is used throughout the sequence of papers [5, 6, 29].

In addition to the large sphere limit and the small sphere limit, there is another interesting situation where the compatibility condition holds naturally, namely, the limit of the quasi-local mass on unit spheres at infinity of an asymptotically flat spacetime. In a series of papers [9, 10], we evaluated the limit at null infinity to capture the information of gravitational radiation. In particular, this is carried out in [10] for the Vaidya spacetime. In this note, we evaluate the limit for unit spheres at spatial infinity of an asymptotically flat spacetime, namely, at infinity of an asymptotically flat initial data set.

Theorem 1.1. *Let (M, g, k) be an asymptotically flat initial data set as in (2.1). Let γ be a geodesic on M which is parametrized by arc-length and extends to infinity. Let $p = \gamma(d)$ be a point on γ and Σ be the unit geodesic*

sphere in M that is centered at $p = \gamma(d)$. The quasi-local mass $E(\Sigma, \mathcal{X}, T_0)$ for $T_0 = (a_0, -a_1, -a_2, -a_3)$ has the following asymptotic behavior as $d \rightarrow \infty$ for each of the following isometric embeddings $\mathcal{X}$ of Σ .

- 1) For the isometric embedding $\mathcal{X} : \Sigma \rightarrow \mathbb{R}^3$, we have

$$E(\Sigma, \mathcal{X}, T_0) = \frac{1}{6} (a_0 \mu(p) - a_i J^i(p)) + O(d^{-3-2\alpha}),$$

where μ and J^i are defined in (2.5) and (2.6).

- 2) Suppose the initial data set (M, g, k) satisfies the vacuum constraint equation (2.7). Let N be the future development of (M, g, k) with Weyl curvature $\bar{W}$. Let e_0, e_1, e_2, e_3 be an orthonormal basis at p with e_0 the unit timelike normal of M in N . For $(\mathcal{X}, T_0)$ solving the leading order of the optimal embedding equation (2.2), we have

$$E(\Sigma, \mathcal{X}, T_0) = \frac{1}{90} \left(Q(e_0, e_0, e_0, T_0) + \frac{1}{2a_0} \bar{W}_{0i0j} \bar{W}_0{}^i{}_0{}^j \right) + O(d^{-4-3\alpha}).$$

Here $\bar{W}_{0i0j} = \bar{W}(e_0, e_i, e_0, e_j)(p)$, Q is the Bel-Robinson tensor of N at p , and T_0 is identified with the timelike vector $a_0 e_0 + \sum_{i=1}^3 a_i e_i$ at p .

Our investigation begins with the Brown-York mass. We compute the derivative of the Brown-York mass and use it to rewrite the Brown-York mass as a bulk integral. The integrand consists of the scalar curvature and quadratic terms of the difference of the physical and the reference data, see Lemma 3.1. We use Lemma 3.1 to evaluate the limit in Theorem 3.3. The scalar curvature corresponds to the stress-energy tensor whereas the quadratic terms, which decay faster than the scalar curvature, correspond to the Bel-Robinson tensor.

In the remaining part of this article, we consider the Wang-Yau quasi-local mass for initial data sets which are not necessarily time-symmetric. We start by solving the optimal embedding equation. The structure of the equation is similar to that of [6] for the small sphere limit and of [10] for unit spheres at null infinity of the Vaidya spacetime. After obtaining the optimal embedding, we use it as the Dirichlet boundary value to solve Jang's equation in the bulk. Using the Schoen-Yau identity from [24] and the canonical gauge for the quasi-local mass from [27], we obtain Theorem 4.1 which generalizes Theorem 3.3 for the Wang-Yau quasi-local mass. While the formula is more complicated, it still consists of the integral of the stress-energy tensor and some quadratic terms. In Section 5, we compute the terms appearing in

Theorem 4.1 explicitly and evaluate the limit. Theorem 1.1 is obtained after assembling these results. We observe that the answer is very similar to the small sphere limit obtained in [6]. In Section 6, we demonstrate how the new approach of this article can be applied to recover the result of [6] for the small sphere limit.

2. Review of the Wang-Yau quasi-local mass and asymptotical flatness

Let Σ be a closed spacelike 2-surface in a spacetime N with spacelike mean curvature vector H . Denote the induced metric and connection one-form of Σ by σ and

$$\alpha_H(\cdot) = \left\langle \nabla_{(\cdot)}^N \frac{J}{|H|}, \frac{H}{|H|} \right\rangle$$

where J is the reflection of H through the incoming light cone in the normal bundle. Given an isometric embedding $\mathcal{X} : \Sigma \rightarrow \mathbb{R}^{3,1}$ and future timelike unit Killing field T_0 in $\mathbb{R}^{3,1}$, we consider the projected embedding $\hat{\mathcal{X}}$ into the orthogonal complement of T_0 , and denote the induced metric and the mean curvature of the image surface $\hat{\Sigma}$ by $\hat{\sigma}$ and $\hat{H}$.

The quasi-local energy with respect to $(\mathcal{X}, T_0)$ is

$$(2.1) \quad E(\Sigma, \mathcal{X}, T_0) = \frac{1}{8\pi} \int_{\hat{\Sigma}} \hat{H} d\hat{\Sigma} - \frac{1}{8\pi} \int_{\Sigma} \left(\sqrt{1 + |\nabla\tau|^2} \cosh \theta |H| - \nabla\tau \cdot \nabla\theta - \alpha_H(\nabla\tau) \right) d\Sigma,$$

where ∇ and Δ are the gradient and Laplace operator of σ , $\tau = -\langle \mathcal{X}, T_0 \rangle$ is considered as a function on the 2-surface, and

$$\theta = \sinh^{-1} \left(\frac{-\Delta\tau}{|H|\sqrt{1 + |\nabla\tau|^2}} \right).$$

Moreover, we say that τ solves the *optimal embedding equation* if

$$(2.2) \quad \operatorname{div}_{\sigma} \left(\rho \nabla\tau - \nabla \left[\sinh^{-1} \left(\frac{\rho \Delta\tau}{|H_0||H|} \right) \right] - \alpha_{H_0} + \alpha_H \right) = 0,$$

where

$$\rho = \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1 + |\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1 + |\nabla\tau|^2}}}{\sqrt{1 + |\nabla\tau|^2}}.$$

Next we recall the definition of an asymptotically flat initial data set.

Definition 2.1. (M^3, g, k) is an asymptotically flat initial data set if, outside a compact set, M^3 is diffeomorphic to $\mathbb{R}^3 \setminus \{|x| \leq r_0\}$ for some $r_0 > 0$ and under the diffeomorphism, we have

$$(2.3) \quad \begin{aligned} g_{ij} - \delta_{ij} &= O(|x|^{-\alpha}), \quad \partial g_{ij} = O(|x|^{-1-\alpha}), \\ \partial^2 g_{ij} &= O(|x|^{-2-\alpha}), \quad \partial^3 g_{ij} = O(|x|^{-3-\alpha}), \end{aligned}$$

and

$$(2.4) \quad k_{ij} = O(|x|^{-1-\alpha}), \quad \partial k_{ij} = O(|x|^{-2-\alpha}), \quad \partial^2 k_{ij} = O(|x|^{-3-\alpha})$$

for some $\alpha > \frac{1}{2}$. Here ∂ denotes the partial differentiation on $\mathbb{R}^3$. Furthermore, we shall assume that for the constraint equation, we have

$$(2.5) \quad \frac{1}{2} (R(g) + (\text{tr} k)^2 - |k|^2) = \mu, \quad \mu = O(|x|^{-3-\alpha}), \quad \partial \mu = O(|x|^{-4-\alpha})$$

$$(2.6) \quad D^i(k_{ij} - (\text{tr} k)g_{ij}) = J_j, \quad J = O(|x|^{-3-\alpha}), \quad \partial J = O(|x|^{-4-\alpha})$$

Recall that an initial data set satisfies the dominant energy condition if

$$\mu \geq |J|.$$

On the other hand, an initial data set satisfies the vacuum constraint equation if

$$(2.7) \quad \mu = 0 \text{ and } J = 0.$$

In this case, there is a unique spacetime N with initial data (M^3, g, k) which solves the vacuum Einstein equation.

Let γ be a geodesic on M which is parametrized by arc-length and extends to infinity. We consider $p = \gamma(d)$ for $d \rightarrow \infty$. Consider the normal coordinate (X^1, X^2, X^3) centered at p and let $\Sigma(s)$ be the sphere of radius s in the normal coordinate. The goal is to evaluate the quasi-local mass of the surface $\Sigma = \Sigma(1)$. In particular, we are interested in the leading order term in d .

The set-up of our calculation can be described as the following. On a unit ball B of $\mathbb{R}^3$, there is a family of Riemannian metrics $g_{ij}(d)$ and symmetric 2-tensors $k_{ij}(d)$ parametrized by d . The metrics $g_{ij}(d)$ (the symmetric 2-tensors $k_{ij}(d)$, respectively) are the pull back of the metrics (the symmetric 2-tensors, respectively) on the unit geodesic ball centered at $\gamma(d)$, $d_0 \leq d < \infty$, a geodesic on M that extends to spatial infinity. We assume that

- 1) the standard Cartesian coordinate system (X^1, X^2, X^3) is a geodesic coordinates system for each $g_{ij}(d)$ such that the origin of the coordinates system corresponds to $\gamma(d)$;
- 2) with respect to (X^1, X^2, X^3) , the asymptotic flat conditions (2.3) and (2.4) are satisfied with $|x|$ replaced by d ;
- 3) the constraint equations (2.5) and (2.6) are satisfied with $|x|$ replaced by d .

Namely,

$$(2.8) \quad \begin{aligned} g_{ij}(d) - \delta_{ij} &= O(d^{-\alpha}), \quad \partial g_{ij}(d) = O(d^{-1-\alpha}), \\ \partial^2 g_{ij}(d) &= O(d^{-2-\alpha}), \quad \partial^3 g_{ij}(d) = O(d^{-3-\alpha}), \end{aligned}$$

and

$$(2.9) \quad k_{ij}(d) = O(d^{-1-\alpha}), \partial k_{ij}(d) = O(d^{-2-\alpha}), \partial^2 k_{ij}(d) = O(d^{-3-\alpha})$$

for some $\alpha > \frac{1}{2}$, where ∂ now denotes the partial differentiation with respect to X^1, X^2, X^3 . Moreover,

$$(2.10) \quad \frac{1}{2} (R(g) + (trk)^2 - |k|^2) = \mu, \quad \mu = O(d^{-3-\alpha}), \partial \mu = O(d^{-4-\alpha})$$

$$(2.11) \quad D^i(k_{ij} - (trk)g_{ij}) = J_j, \quad J = O(d^{-3-\alpha}), \partial J = O(d^{-4-\alpha}).$$

In particular, let $R_{ij}(d)$ denote the Ricci curvature of $g_{ij}(d)$, by the Taylor expansion at a point in B with respect to the geodesic coordinate system (X^1, X^2, X^3) , we have

$$(2.12) \quad g_{ij}(d)(X^1, X^2, X^3) = \delta_{ij} - \frac{1}{3} R_{ikjl}(d)(0, 0, 0) X^k X^l + O(d^{-3-\alpha}),$$

$$(2.13) \quad R_{ij}(d)(X^1, X^2, X^3) = R_{ij}(d)(0, 0, 0) + O(d^{-3-\alpha}),$$

$$(2.14) \quad k_{ij}(d)(X^1, X^2, X^3) = k_{ij}(d)(0, 0, 0) + \partial_m k_{ij}(d)(0, 0, 0) X^m + O(d^{-3-\alpha}).$$

These expansions will be abbreviated as

$$(2.15) \quad g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}(p) X^k X^l + O(d^{-3-\alpha}),$$

$$(2.16) \quad R_{ij} = R_{ij}(p) + O(d^{-3-\alpha}),$$

$$(2.17) \quad k_{ij} = k_{ij}(p) + \partial_m k_{ij}(p) X^m + O(d^{-3-\alpha}).$$

We will also use the spherical coordinate system (s, u^1, u^2) on B such that the coordinate transformation $(s, u^1, u^2) \mapsto (X^1, X^2, X^3)$ is given by $X^i = s\tilde{x}^i(u^1, u^2)$, $i = 1, 2, 3$, where $\tilde{x}^i$, $i = 1, 2, 3$ are the three standard coordinate functions on the standard unit sphere in $\mathbb{R}^3$.

Notation: Einstein summation notation will be used throughout the paper, where $i, j, \dots$ sum from 1 to 3. Since we are working in normal coordinates, we can freely raise or lower indices for tensors at $p = \gamma(d)$.

3. The Brown-York mass

In this section, we consider a time-symmetric initial data set and compute the limit of the Brown-York quasi-local mass. The starting point is the following lemma for the Brown-York quasi-local mass: Given a surface Σ in a 3-manifold (M, g) , let R be the scalar curvature of g . Let Ω be the region in M bounded by Σ . Suppose Ω is foliated by surfaces $\Sigma(s)$ with positive Gauss curvature where $0 < s \leq 1$, $\Sigma(1) = \Sigma$, and $\Sigma(s)$ shrinks to a point as s tends to 0. Let $\sigma(s)$ be the induced metric on $\Sigma(s)$. The positivity of the Gauss curvature of $\sigma(s)$ guarantees an isometric embedding into $\mathbb{R}^3$. Denote the mean curvature of $\Sigma(s)$ in M by $H(s)$ and the mean curvature of the isometric embedding of $\Sigma(s)$ into $\mathbb{R}^3$ by $H_0(s)$. Let $h(s)$ and $h_0(s)$ be the second fundamental form of $\Sigma(s)$ in M and $\mathbb{R}^3$, respectively.

Lemma 3.1. *The Brown-York quasi-local mass, $m_{BY}(\Sigma)$, of Σ is*

$$m_{BY}(\Sigma) = \frac{1}{16\pi} \int_{\Omega} (|h_0(s) - h(s)|^2 - (H_0(s) - H(s))^2 + R)$$

where R is the scalar curvature of g .

Proof. Assume that $\Sigma(s)$ are given by $F(x, s) : \Sigma \times (0, 1] \rightarrow M$ with $DF_{(x,s)}(\frac{\partial}{\partial s}) = f(x, s)\nu(x, s)$ where $\nu(x, s)$ is the unit normal of $\Sigma(s)$. We first show that the derivative of the Brown York quasi-local mass is given by

$$(3.1) \quad \frac{d}{ds}E(\Sigma(s)) = \int_{\Sigma(s)} \frac{f}{2} (|h_0 - h|^2 - (H_0 - H)^2 + R).$$

The above formula is known, see [21, Theorem 3.1] for example. For completeness, we include the proof here. We have

$$\frac{d}{ds}\sigma(s) = 2fh.$$

By Proposition 6.1 of [28],

$$(3.2) \quad \frac{d}{ds} \int_{\Sigma(s)} H_0(s) = \int_{\Sigma(s)} f(H_0 H - h \cdot h_0).$$

On the other hand, from the second variation formula, we have

$$(3.3) \quad \frac{d}{ds} \int_{\Sigma(s)} H(s) = \int_{\Sigma(s)} f(H^2 - Ric(\nu, \nu) - |h|^2).$$

The Gauss equations of $\Sigma(s)$ in $\mathbb{R}^3$ and M imply:

$$\begin{aligned} K &= \frac{1}{2}(H_0^2 - |h_0|^2), \\ K &= \frac{R}{2} - Ric(\nu, \nu) + \frac{1}{2}(H^2 - |h|^2). \end{aligned}$$

Taking the difference of the two Gauss equations, we obtain

$$(3.4) \quad Ric(\nu, \nu) = \frac{R}{2} + \frac{1}{2}(H^2 - |h|^2 - H_0^2 + |h_0|^2).$$

The claim follows from subtracting (3.3) from (3.2) and using (3.4) to replace the Ricci curvature term in the result. The lemma follows from integrating (3.1) along the foliation. $\square$

In our setup, $\Sigma(s)$ is the sphere of radius s in the normal coordinates centered at $p = \gamma(d) \in M$. The induced metric and second fundamental form of $\Sigma(s)$ are given by

$$\begin{aligned} \sigma_{ab} &= s^2(\tilde{\sigma}_{ab} - \frac{1}{3}R_{ikjl}(p)\tilde{x}_a^i\tilde{x}_b^j\tilde{x}^k\tilde{x}^l) + O(d^{-3-\alpha}) \\ h(s)_{ab} &= s\tilde{\sigma}_{ab} + O(d^{-2-\alpha}) \end{aligned}$$

where R_{ikjl} is the Riemann curvature tensor of the metric g on M and $\tilde{x}_a^i$ is a shorthand for $\partial_a \tilde{x}^i$.

We first compute the difference of the mean curvature and second fundamental form of $\Sigma(s)$ in M and $\mathbb{R}^3$.

Lemma 3.2. *Consider the surface $\Sigma(s)$. We have*

$$(3.5) \quad H_0(s) - H(s) = -sR_{ij}(p)\tilde{x}^i\tilde{x}^j + O(d^{-3-\alpha})$$

and

$$(3.6) \quad h_0(s) - h(s) = s^3R_{ij}(p)\tilde{x}_a^i\tilde{x}_b^j + O(d^{-3-\alpha}).$$

Proof. We will use repeatedly an implication of (2.5) that $R = O(d^{-3-\alpha})$. Let $\widehat{h}$ and $\widehat{h}_0$ denote the traceless second fundamental forms. By the Gauss equations,

$$\begin{aligned} 2K &= R - 2Ric(\nu, \nu) + \frac{1}{2}H^2 - \frac{1}{2}|\widehat{h}|^2 \\ &= \frac{1}{2}H_0^2 - \frac{1}{2}|\widehat{h}_0|^2. \end{aligned}$$

Since the unit normal of $\Sigma(s)$ is $\nu = \tilde{x}^i \frac{\partial}{\partial X^i} + O(d^{-3-\alpha})$ and $Ric(\nu, \nu) = R_{ij}\tilde{x}^i\tilde{x}^j + O(d^{-3-\alpha})$, we get $H_0(s) - H(s) = -sR_{ij}\tilde{x}^i\tilde{x}^j + O(d^{-3-\alpha})$.

Taking the difference of the Codazzi equations for $\Sigma(s)$ in M and $\mathbb{R}^3$ implies

$$\begin{aligned} \tilde{\nabla}^a(\widehat{h}_{ab}(s) - \widehat{h}_{0ab}(s)) &= -\frac{1}{2}\partial_b(H_0(s) - H(s)) - Ric(\nu, \partial_b) + O(d^{-4-2\alpha}) \\ &= \partial_b(sR_{ij}\tilde{x}^i\tilde{x}^j) + O(d^{-3-\alpha}). \end{aligned}$$

One readily checks that

$$\widehat{h}_{ab}(s) - \widehat{h}_{0ab}(s) = s^3 R_{ij}\tilde{x}_a^i\tilde{x}_b^j + \frac{s^3}{2}R_{ij}\tilde{x}^i\tilde{x}^j\tilde{\sigma}_{ab}$$

satisfies the above equation. Indeed, we find the unique solution as there is no divergence-free, traceless symmetric 2-tensor on S^2 . $\square$

We obtain the following result for the limit of the Brown-York mass:

Theorem 3.3. *On $\Sigma = \Sigma(1)$, we have*

$$m_{BY}(\Sigma) = \frac{1}{6}\mu(p) + O(d^{-4-\alpha}).$$

For an initial data set satisfying the vacuum constraint equation, we have

$$m_{BY}(\Sigma) = \frac{1}{60}Q(e_0, e_0, e_0, e_0) + O(d^{-5-2\alpha})$$

where Q is the Bel-Robinson tensor (1.1) at p of the solution to the vacuum Einstein equation with the time-symmetric initial data (M, g) .

Proof. On Ω , $R = R(p) + O(d^{-4-\alpha})$ by (2.5). Applying Lemma 3.1, together with

$$(3.7) \quad \begin{aligned} |h(s) - h_0(s)| &= O(d^{-2-\alpha}) \\ H_0(s) - H(s) &= O(d^{-2-\alpha}), \end{aligned}$$

we obtain the first formula.

For a vacuum initial data set, we compute

$$\begin{aligned} |h_0(s) - h(s)|^2 - (H_0(s) - H(s))^2 \\ = s^2 R_{ij} R_{lm} (\delta^{il} \delta^{jm} - \delta^{il} \tilde{x}^j \tilde{x}^m - \delta^{jm} \tilde{x}^i \tilde{x}^l) + O(d^{-5-2\alpha}). \end{aligned}$$

Since $R = 0$, we have

$$m_{BY}(\Sigma) = \frac{1}{16\pi} R_{ij} R^{ij} \cdot \frac{4\pi}{3} \int_0^1 s^4 ds + O(d^{-5-2\alpha}) = \frac{1}{60} R_{ij} R^{ij} + O(d^{-5-2\alpha}).$$

Finally, if N is the solution to the Einstein equation with time-symmetric initial data (M, g) , its Weyl curvature satisfies

$$\begin{aligned} \bar{W}_{0i0j} &= R_{ij}, \\ \bar{W}_{0ijk} &= 0. \end{aligned}$$

This finishes the proof of the theorem. $\square$

4. Optimal embedding equation and the Jang equation

In this section, we describe our strategy to handle the second fundamental form k_{ij} . We study the optimal embedding equation on Σ and the Jang equation on Ω . In particular, we first solve the optimal embedding equation on the boundary. Then we solve the Jang equation on the bulk Ω using the solution of the optimal embedding equation as the boundary value

Consider the product manifold $\Omega \times \mathbb{R}$ with the product metric $dt^2 + g_{ij} dX^i dX^j$. The data k_{ij}, μ, J are extended parallelly along the $\mathbb{R}$ factor. Jang's equation for $u \in C^2(\Omega)$ reads

$$(4.1) \quad \left(g^{ij} - \frac{D^i u D^j u}{1 + |Du|^2} \right) \left(k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}} \right) = 0.$$

Denote the graph of u in $\Omega \times \mathbb{R}$ by $\tilde{\Omega}$ and $\tilde{\Sigma} = \partial\tilde{\Omega}$. Let $\tilde{e}_4$ be the downward normal of $\tilde{\Omega}$ and $Y_i = \left(k - \frac{D^2 u}{\sqrt{1 + |Du|^2}} \right) \left(\frac{\partial}{\partial X^i}, \tilde{e}_4 \right)$.

Let $\tilde{g}$ be the induced metric of $\tilde{\Omega}$. Let H_0 be the mean curvature of the isometric embedding of $\tilde{\Sigma}$ into $\mathbb{R}^3$. We recall

$$\begin{aligned} E(\Sigma, \mathcal{X}, T_0) &= \frac{1}{8\pi} \left(\int H_0 d\tilde{\Sigma} - \int \left[\sqrt{1 + |\nabla\tau|^2} \cosh |H| + \Delta\tau\theta - \alpha_H(\nabla\tau) \right] d\Sigma \right) \end{aligned}$$

where $\sinh \theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$. By [28, Theorem 4.1],

$$\begin{aligned} E(\Sigma, \mathcal{X}, T_0) &= \frac{1}{8\pi} \left(\int \left[H_0 - \tilde{H} + \langle Y, \tilde{e}_3 \rangle \right] d\tilde{\Sigma} \right. \\ &\quad \left. + \int \left[|H| \sqrt{1 + |\nabla\tau|^2} (\cosh \theta' - \cosh \theta) + \Delta\tau(\theta' - \theta) \right] d\Sigma \right), \end{aligned}$$

where θ' is defined by

$$\cosh \phi e_3 - \sinh \phi e_4 = \cosh \theta' e_3^H - \sinh \theta' e_4^H$$

with $\sinh \phi = -\frac{u_3}{\sqrt{1+|\nabla\tau|^2}}$. By the Schoen-Yau identity [24, (2.29)]

$$2(\mu - J(\tilde{e}_4)) = \tilde{R} - \left| k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}} \right|_{\tilde{g}}^2 - 2|Y|_{\tilde{g}}^2 + 2\tilde{D}^i Y_i.$$

Together with Lemma 3.1, we obtain

Theorem 4.1.

$$\begin{aligned} E(\Sigma, X, T_0) &= \frac{1}{8\pi} \int (\mu - J(\tilde{e}_4)) d\tilde{\Omega} \\ &\quad + \frac{1}{16\pi} \int \left[\left| k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}} \right|_{\tilde{g}}^2 + 2|Y|_{\tilde{g}}^2 \right] d\tilde{\Omega} \\ &\quad + \frac{1}{16\pi} \int [|h_0(s) - h(s)|^2 - (H_0(s) - H(s))^2] d\tilde{\Omega} \\ &\quad + \frac{1}{8\pi} \int \left[|H| \sqrt{1 + |\nabla\tau|^2} (\cosh \theta' - \cosh \theta) + \Delta\tau(\theta' - \theta) \right] d\Sigma \end{aligned}$$

5. Limit of the Wang-Yau mass

In this section, we study the optimal embedding equation and the Jang equation with respect to the observer

$$T_0 = (a_0, -a_1, -a_2, -a_3).$$

Before restating our main result, Theorem 1.1, recall that in the vacuum case we view the initial data set (M, g, k) as a spacelike hypersurface in its future development N and denote the Weyl curvature of N by $\bar{W}$. Let e_0, e_1, e_2, e_3 be an orthonormal basis at p with e_0 the unit timelike normal of M in N and identify T_0 with the timelike vector $a_0 e_0 + \sum_{i=1}^3 a_i e_i$ at p .

Theorem 5.1. *The quasi-local mass $E(\Sigma, \mathcal{X}, T_0)$ for $T_0 = (a_0, -a_1, -a_2, -a_3)$ has the following asymptotic behavior as $d \rightarrow \infty$ for each of the following isometric embeddings $\mathcal{X}$ of Σ .*

- 1) *For the isometric embedding $\mathcal{X} : \Sigma \rightarrow \mathbb{R}^3$, we have*

$$E(\Sigma, \mathcal{X}, T_0) = \frac{1}{6} (a_0 \mu(p) - a_i J^i(p)) + O(d^{-3-2\alpha}),$$

where μ and J^i are defined in (2.5) and (2.6).

- 2) *Suppose the initial data set (M, g, k) satisfies the vacuum constraint equation (2.7). For $(\mathcal{X}, T_0)$ solving the leading order of the optimal embedding equation (2.2), we have*

$$E(\Sigma, \mathcal{X}, T_0) = \frac{1}{90} \left(Q(e_0, e_0, e_0, T_0) + \frac{1}{2a_0} \bar{W}_{0i0j} \bar{W}_0{}^i{}_0{}^j \right) + O(d^{-4-3\alpha}).$$

Here $\bar{W}_{0i0j} = \bar{W}(e_0, e_i, e_0, e_j)(p)$ and Q is the Bel-Robinson tensor of N at p .

Remark 5.2. *We mostly work at the initial data level and the error term has order $O(d^{-5-2\alpha})$. Only when the result is expressed in terms of the space-time curvature using the Gauss equation of N , $\bar{W}_{0i0j} = R_{ij} + O(d^{-2-2\alpha})$, does the error become $O(d^{-4-3\alpha})$.*

The outline of this section is as follows. We solve the optimal embedding equation and the Dirichlet problem of Jang's equation in the first two subsections and then evaluate each integral in Theorem 4.1 in the subsequent three subsections. Finally, we put everything together to prove Theorem 5.1.

5.1. Optimal embedding

Let us begin with the optimal embedding equation.

Lemma 5.3. *The following pair $T_0 = (a_0, -a_1, -a_2, -a_3)$ and*

$$\begin{aligned}\mathcal{X}^0 &= \frac{1}{2}k_{ij}(p)\tilde{x}^i\tilde{x}^j + \frac{1}{6}\partial_i k_{jm}(p)\tilde{x}^i\tilde{x}^j\tilde{x}^m + \frac{a_i R_{mn}(p)\tilde{x}^m\tilde{x}^n\tilde{x}^i}{6a_0} \\ \mathcal{X}^i &= \tilde{x}^i - \frac{1}{6}R_{in}(p)\tilde{x}^n - \frac{1}{6}R_{mn}(p)\tilde{x}^m\tilde{x}^n\tilde{x}^i\end{aligned}$$

solves the first two order of the optimal embedding equation. In particular, the above solution gives a time function $\tau = -\langle \mathcal{X}, T_0 \rangle$ with

$$(5.1) \quad \tau = a_i \tilde{x}^i + a_0 \left[\frac{1}{2}k_{ij}(p)\tilde{x}^i\tilde{x}^j + \frac{1}{6}\partial_i k_{jm}(p)\tilde{x}^i\tilde{x}^j\tilde{x}^m \right] - \frac{1}{6}a_i R_n^i(p)\tilde{x}^n$$

Proof. With $T_0 = (a_0, -a_1, -a_2, -a_3)$, the optimal embedding equation reads

$$\begin{aligned}\frac{1}{2}\Delta(\Delta + 2)\mathcal{X}^0 &= \operatorname{div}\alpha_H + \frac{a_i}{a_0} \left[\operatorname{div}((H_0 - |H|)\nabla\tilde{x}^i) + \frac{1}{2}\Delta((H_0 - |H|)\tilde{x}^i) \right] \\ &\quad + O(d^{-3-\alpha}).\end{aligned}$$

See [7, Section 7]. We first compute

$$\begin{aligned}(\alpha_H)_a &= -k(\partial_a, \nu) + \partial_a \left(\frac{\operatorname{tr}_\Sigma k}{H} \right) + O(d^{-3-3\alpha}) \\ &= -2k_{ij}\tilde{x}_a^i\tilde{x}^j - 2\partial_m k_{ij}\tilde{x}_a^i\tilde{x}^j\tilde{x}^m + \frac{1}{2}\partial_m k_{ii}\tilde{x}_a^m \\ &\quad - \frac{1}{2}\partial_m k_{ij}\tilde{x}^i\tilde{x}^j\tilde{x}_a^m + O(d^{-3-\alpha}).\end{aligned}$$

Using the Codazzi equation, $\partial_i k_{im} = \partial_m k_{ii} + O(d^{-3-2\alpha})$, we obtain

$$\begin{aligned}\operatorname{div}\alpha_H &= -2k_{ij}(\delta^{ij} - 3\tilde{x}^i\tilde{x}^j) \\ &\quad + 10\partial_m k_{ij} \left(\tilde{x}^i\tilde{x}^j\tilde{x}^m - \frac{1}{5}\delta^{ij}\tilde{x}^m - \frac{1}{5}\delta^{im}\tilde{x}^j - \frac{1}{5}\delta^{jm}\tilde{x}^i \right) + O(d^{-3-\alpha})\end{aligned}$$

Note that $\delta^{ij} - 3\tilde{x}^i\tilde{x}^j$ and $\tilde{x}^i\tilde{x}^j\tilde{x}^m - \frac{1}{5}\delta^{ij}\tilde{x}^m - \frac{1}{5}\delta^{im}\tilde{x}^j - \frac{1}{5}\delta^{jm}\tilde{x}^i$ are -6 and -12 eigenfunctions respectively.

On the other hand, by Lemma 3.2 and $|H| = \sqrt{H^2 - (\text{tr}_\Sigma k)^2}$, we have

$$H_0 - |H| = -R_{ij}\tilde{x}^i\tilde{x}^j + O(d^{-2-2\alpha})$$

and

$$\begin{aligned} \text{div}((H_0 - |H|)\nabla\tilde{x}^i) + \frac{1}{2}\Delta((H_0 - |H|)\tilde{x}^i) \\ = 10R_{mn}\tilde{x}^m\tilde{x}^n\tilde{x}^i - 4R_{in}\tilde{x}^n + O(d^{-2-2\alpha}) \end{aligned}$$

It follows that the given $\mathcal{X}^0$ satisfies the equation up to error of the order $O(d^{-2-2\alpha})$.

For the $\mathcal{X}^i$, we use the well-known formula of Riemann curvature tensor in 3-dimension

$$(5.2) \quad R_{ikjl} = g_{ij}R_{kl} - g_{il}R_{kj} + g_{kl}R_{ij} - g_{kj}R_{il} - \frac{R}{2}(g_{ij}g_{kl} - g_{il}g_{kj})$$

and (2.5) to show that the induced metric is

$$\sigma_{ab} = \tilde{\sigma}_{ab} - \frac{1}{3}R_{kl}\tilde{x}_a^k\tilde{x}_b^l - \frac{1}{3}R_{ij}\tilde{x}^i\tilde{x}^j\tilde{\sigma}_{ab} + O(d^{-2-2\alpha}).$$

The lemma follows from the linearized isometric embedding equation into $\mathbb{R}^3$. $\square$

5.2. Jang's equation

We work in local coordinates and Ω is identified with $B_1 \subset \mathbb{R}^3$. We discuss the solution of Dirichlet problem of Jang's equation

$$(5.3) \quad \begin{cases} \left(g^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right) \left(k_{ij} - \frac{D_i D_j u}{\sqrt{1+|Du|^2}}\right) = 0 & \text{in } B_1 \\ u = \tau & \text{on } \partial B_1 \end{cases}$$

Let $b_{ij} = g_{ij} - \delta_{ij} + \frac{1}{3}R_{ikjl}(p)X^kX^l$ and $c_{ij} = k_{ij} - k_{ij}(p) - \partial_m k_{ij}(p)X^m$. By Definition 2.1 and (2.12), we have

$$(5.4) \quad \|b_{ij}\|_{C^1(B_1)}, \|c_{ij}\|_{C^1(B_1)} \leq Cd^{-3-\alpha}.$$

Lemma 5.4. *Let u be the solution of the Dirichlet problem of Jang's equation on Ω with boundary value τ given in (5.1). Then $u = a_i X^i + \frac{1}{6}T_i X^i +$*

$\frac{1}{2}a_0k_{ij}X^iX^j + \frac{1}{6}B_{ijm}X^iX^jX^m - \frac{1}{6}a_iR_{in}X^n + v$, where

$$\begin{aligned} T_m &= -\frac{1}{4 + \frac{1}{a_0^2}} \cdot \frac{2}{a_0} a_i a_j \bar{W}_0^i{}^j(p) + \frac{1}{4 + \frac{1}{a_0^2}} \cdot 2a_l R_m^l(p) \\ &\quad + \frac{1}{(4 + \frac{1}{a_0^2})(2 + \frac{3}{a_0^2})} \cdot \frac{4a_m}{a_0} a_i a_j R^{ij}(p) \\ B_{ijm} &= \frac{a_0}{3} (\partial_i k_{jm}(p) + \partial_j k_{im}(p) + \partial_m k_{ij}(p)) - \frac{1}{3} (\delta_{ij} T_m + \delta_{jm} T_i + \delta_{mi} T_j), \end{aligned}$$

and $\|v\|_{C^{2,\beta}} \leq C'd^{-3-\alpha}$ for some constant C' and $0 < \beta < 1$ depending only on C in (5.4).

Proof. We write

$$u = a_i X^i + b_i X^i + \frac{1}{2}a_0k_{ij}X^iX^j + \frac{1}{6}B_{ijm}X^iX^jX^m - \frac{1}{6}a_iR_{in}X^n + v$$

for constants b_i, B_{ijm} to be determined from the leading order of Jang's equation.

Define a symmetric 3-tensor

$$T_{ijm} = \frac{a_0}{3} (\partial_i k_{jm} + \partial_j k_{mi} + \partial_m k_{ij}) - B_{ijm}.$$

We need to show that $\tau - u$ is perpendicular to all -2 and -12 eigenfunctions on $\partial B_1 = S^2$ which, by the expression (5.1) of τ , is equivalent to

$$\begin{aligned} \int_{S^2} \left(\frac{1}{6} T_{ijm} \tilde{x}^i \tilde{x}^j \tilde{x}^m - b_i \tilde{x}^i \right) \tilde{x}^l dS^2 &= 0, \\ \int_{S^2} T_{ijm} \tilde{x}^i \tilde{x}^j \tilde{x}^m (\tilde{x}^l \tilde{x}^p \tilde{x}^q - \frac{1}{5} \tilde{x}^l \delta^{pq} - \frac{1}{5} \tilde{x}^p \delta^{ql} - \frac{1}{5} \tilde{x}^q \delta^{lp}) dS^2 &= 0. \end{aligned}$$

The second equation contains 7 linear equations with 10 variables. Using [8, Lemma 5.3], we solve T_{ijm} by free variables $T_{111}, T_{222}, T_{333}$:

$$T_{ijm} = \frac{1}{3} (\delta_{ij} T_m + \delta_{jm} T_i + \delta_{mi} T_j), \quad T_m := T_{mmm}.$$

and then solve $b_i = \frac{1}{6} T_i$.

It remains to solve T_m from Jang's equation

$$(g^{ij} - \frac{D^i u D^j u}{1 + |Du|^2})(k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}}) = 0.$$

By the Codazzi equation, $\partial_m k_{ij} - \partial_i k_{mj} = -\bar{W}_{0jmi} + O(d^{-3-2\alpha})$, it follows that

$$k_{ij} - \frac{u_{ij}}{\sqrt{1 + |Du|^2}} = S_{ijm} X^m + O(d^{-3-\alpha})$$

where

$$(5.5) \quad S_{ijm} = \frac{1}{3} \left(\frac{1}{a_0} (\delta_{ij} T_m + \delta_{jm} T_i - \delta_{mi} T_j) - \bar{W}_{0imj} - \bar{W}_{0jmi} + \frac{a_l}{a_0} (R_{iljm} + R_{imjl}) \right).$$

The leading order of Jang's equation thus reduces to 3 linear equations

$$\sum_{i,j} (\delta_{ij} - \frac{a_i a_j}{a_0^2}) S_{ijm} = 0.$$

We have

$$\begin{aligned} 0 &= \sum_{i,j} (\delta_{ij} - \frac{a_i a_j}{a_0^2}) (\delta_{ij} T_m + \delta_{jm} T_i + \delta_{mi} T_j) \\ &\quad + \frac{a_i a_j}{a_0} (\bar{W}_0^i{}^j{}_m + \bar{W}_0^j{}^i{}_m) - 2a_l R_m^l \\ &= \sum_i G_{im} T_i + \frac{a_i a_j}{a_0} (\bar{W}_0^i{}^j{}_m + \bar{W}_0^j{}^i{}_m) - 2a_l R_m^l, \end{aligned}$$

where $G_{im} = (4 + \frac{1}{a_0^2}) \delta_{im} - 2 \frac{a_i a_m}{a_0^2}$. We solve for the inverse matrix of G_{im}

$$(G^{-1})_{ml} = \frac{1}{(4 + \frac{1}{a_0^2})(2 + \frac{3}{a_0^2})} \left[(2 + \frac{3}{a_0^2}) \delta_{ml} + 2 \frac{a_m a_l}{a_0^2} \right]$$

to get

$$\begin{aligned} T_m &= -\frac{1}{4 + \frac{1}{a_0^2}} \cdot \frac{2a_i a_j}{a_0} \bar{W}_0^i{}^j{}_m + \frac{1}{4 + \frac{1}{a_0^2}} \cdot 2a_l R_m^l \\ &\quad + \frac{1}{(4 + \frac{1}{a_0^2})(2 + \frac{3}{a_0^2})} \cdot \frac{4a_m}{a_0^2} R^{ij} a_i a_j. \end{aligned}$$

After obtaining the leading order of u , we treat Jang's equation as a quasilinear partial differential equation $Q(x, Dv)v = 0$ in B_1 and $v = 0$ on ∂B_1 . By (5.4), we can choose a constant C' that depends only on C such that $\pm C'd^{-3-\alpha}(|x|^2 - 1)$ is a sub/super solution to this equation. This provides the C^0 -estimate and the boundary gradient estimate. By [16, Theorem 15.1] and [16, Theorem 13.7], we get the gradient estimate and Hölder estimate for the gradient. The $C^{2,\beta}$ a priori estimate for v and the solvability of u then follows from the Schauder estimate and [16, Theorem 11.4]. $\square$

Proof of Theorem 5.1, (1). We first examine the limit for an initial data set with matter fields. We will show below that all terms except $\mu - J(\tilde{e}_4)$ are of the order $O(d^{-4-2\alpha})$. As a result,

$$E(\Sigma) = \frac{1}{8\pi} \int (\mu - J(\tilde{e}_4)) d\tilde{\Omega} + O(d^{-4-2\alpha}).$$

The assertion follows from $d\tilde{\Omega} = a_0 dx + O(d^{-2-\alpha})$ and $\tilde{e}_4 = \frac{1}{a_0}(-1, a_1, a_2, a_3) + O(d^{-1-\alpha})$. $\square$

For a vacuum initial data set, Theorem 4.1 becomes

$$(5.6) \quad E(\Sigma, X, T_0) = \frac{1}{16\pi} \int \left[\left| k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}} \right|_{\tilde{g}}^2 + 2|Y|_{\tilde{g}}^2 \right] d\tilde{\Omega}$$

$$(5.7) \quad + \frac{1}{16\pi} \int [|h_0(s) - h(s)|^2 - (H_0(s) - H(s))^2] d\tilde{\Omega}$$

$$(5.8) \quad + \frac{1}{8\pi} \int [|H|\sqrt{1 + |\nabla\tau|^2}(\cosh\theta' - \cosh\theta) + \Delta\tau(\theta' - \theta)] d\Sigma$$

5.3. Evaluation of (5.6)

Lemma 5.5.

$$\begin{aligned} & \frac{1}{16\pi} \int_{B_1} \left(\left| k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}} \right|_{\tilde{g}}^2 + 2|Y|_{\tilde{g}}^2 \right) d\tilde{\Omega} \\ &= -\frac{1}{54a_0} \sum_m T_m^2 - \frac{1}{27a_0} T_m a_l R^{lm}(p) + \frac{1}{90} \left(a_0 - \frac{1}{a_0} \right) R_{ij}(p) R^{ij}(p) \\ & \quad - \frac{1}{54a_0} \sum_{i,j,l} a_i a_j R^{il}(p) R_l^j(p) + \frac{a_0}{180} \bar{W}_{0imj}(p) \bar{W}_0^{imj}(p) \\ & \quad - \frac{1}{45} a_m \bar{W}_0^{imj}(p) \bar{W}_{0i0j}(p) + O(d^{-5-2\alpha}). \end{aligned}$$

Proof. Recall that we write $k_{ij} - \frac{D_i D_j u}{\sqrt{1+|Du|^2}} = S_{ijm} X^m + O(d^{-3-\alpha})$ in the proof of Lemma 5.4. Since $\tilde{e}_4 = \frac{(-1, a_1, a_2, a_3)}{a_0} + O(d^{-1-\alpha})$, we have $Y = \sum_{i,m,p} \frac{a_p}{a_0} S_{ipm} X^m \frac{\partial}{\partial X^i} + O(d^{-3-\alpha})$ and

$$\begin{aligned} & \frac{1}{16\pi} \int_{B_1} \left| k_{ij} - \frac{D_i D_j u}{\sqrt{1+|Du|^2}} \right|_{\bar{g}}^2 + 2|Y|_{\bar{g}}^2 dV_{\bar{g}} \\ &= \frac{a_0}{60} \sum_{i,j,m,p,q} \left[\left(\delta_{ij} - \frac{a_i a_j}{a_0^2} \right) S_{ipm} S_{jqm} \left(\delta_{pq} - \frac{a_p a_q}{a_0^2} \right) \right. \\ & \quad \left. + 2 \left(\delta_{ij} - \frac{a_i a_j}{a_0^2} \right) \frac{a_p a_q}{a_0^2} S_{ipm} S_{jqm} \right] + O(d^{-5-2\alpha}) \\ &= \frac{a_0}{60} \left[\sum_{i,j,m} S_{ijm}^2 - \sum_{i,j,m,p,q} \frac{a_i a_j a_p a_q}{a_0^4} S_{ipm} S_{jqm} \right] + O(d^{-5-2\alpha}) \\ &= \frac{a_0}{60} \left[\sum_{i,j,m} S_{ijm}^2 - \sum_m \left(\sum_i S_{iim} \right)^2 \right] + O(d^{-5-2\alpha}), \end{aligned}$$

where Jang's equation is used in the last equality. We compute, by (5.5),

$$\begin{aligned} \sum_m \left(\sum_i S_{iim} \right)^2 &= \frac{1}{9a_0^2} \sum_m (5T_m + 2a_l R_m^l)^2 \\ &= \frac{1}{9a_0^2} \left(25 \sum_m T_m^2 + 10a_l T_m R^{lm} + 4a_l a_n R^{lm} R_m^n \right) \end{aligned}$$

and

$$\begin{aligned} \sum_{i,j,m} S_{ijm}^2 &= \frac{1}{9} \left[\frac{15}{a_0^2} \sum_m T_m^2 + \sum_{i,j,m} (\bar{W}_{0imj} + \bar{W}_{0jmi})^2 \right. \\ & \quad + \frac{1}{a_0^2} \sum_{i,j,m} \left(2a_m R_{ij} + 2a_l R_m^l \delta_{ij} - a_j R_{im} - a_i R_{jm} - a_l R_j^l \delta_{im} - a_l R_i^l R_{jm} \right)^2 \\ & \quad \left. - 2 \sum_{i,j,l,m} \frac{a_l}{a_0} (\bar{W}_{0imj} + \bar{W}_{0jmi}) (R_{iljm} + R_{imjl}) \right] \end{aligned}$$

By the first Bianchi identity, $2 \sum_{i,j,m} \bar{W}_{0imj} \bar{W}_{0jmi} = \sum_{i,j,m} \bar{W}_{0imj} \bar{W}_{0imj}$ and hence $\sum_{i,j,m} (\bar{W}_{0imj} + \bar{W}_{0jmi})^2 = 3 \sum_{i,j,m} \bar{W}_{0imj}^2$. Direct computation shows that the third term in the bracket is equal to $\frac{1}{a_0^2} (6 \sum_m a_m^2 R_{ij}^2 - 6a_i a_j R^{il} R_l^j)$.

Finally, by (5.2), the last term in the bracket is equal to

$$-12 \sum_{i,j,m} \frac{a_m}{a_0} \bar{W}_{0imj} \bar{W}_{0i0j}.$$

□

5.4. Evaluation of (5.7)

Lemma 5.6.

$$\begin{aligned} & \frac{1}{16\pi} \int [|h_0(s) - h(s)|^2 - (H_0(s) - H(s))^2] d\tilde{\Omega} \\ &= \frac{1}{60a_0} R_{ij}(p) R^{ij}(p) + O(d^{-5-2\alpha}). \end{aligned}$$

Proof. By Lemma 5.8 below, we have

$$h_0(s) - h(s) = s^3 \frac{\sqrt{1 + |a_i \tilde{\nabla} \tilde{x}^i|^2}}{a_0} R_{ij} \tilde{x}_a^i \tilde{x}_b^j + O(d^{-3-\alpha}).$$

We compute

$$\begin{aligned} & |h_0(s) - h(s)|_\sigma^2 - (H_0(s) - H(s))^2 \\ &= (\det_{\tilde{\sigma}}(\sigma))^{-1} (|h_0(s) - h(s)|_{\tilde{\sigma}}^2 - (\text{tr}_{\tilde{\sigma}} h_0(s) - \text{tr}_{\tilde{\sigma}} h(s))^2) \\ &= \frac{s^2}{a_0^2} \left(R_{ij} R^{ij} - 2R_{ij} R_l^i \tilde{x}^j \tilde{x}^l \right) + O(d^{-5-2\alpha}). \end{aligned}$$

Finally, we note that the volume form $d\tilde{\Omega} = a_0 s^2 dS^2 ds + O(d^{-2-\alpha})$ and hence

$$\begin{aligned} & \int [|h_0(s) - h(s)|^2 - (H_0(s) - H(s))^2] d\tilde{\Omega} \\ &= \frac{1}{a_0} \int_0^1 \int_{S^2} (R_{ij} R^{ij} - 2R_{ij} R_l^i \tilde{x}^j \tilde{x}^l) dS^2 s^4 ds + O(d^{-5-2\alpha}) \\ &= \frac{4}{15a_0} R_{ij} R^{ij} + O(d^{-5-2\alpha}). \end{aligned}$$

□

The rest of this subsection is devoted to computing the difference of second fundamental forms of Σ_s in $\tilde{\Omega}$ and in $\mathbb{R}^3$, Lemma 5.8. We first solve the isometric embedding of Σ_s into $\mathbb{R}^3$ and then compute the second fundamental

form. Recall the solution of Jang's equation is $u = a_i X^i + \frac{1}{2} a_0 k_{ij} X^i X^j + \frac{1}{6} B_{ijm} X^i X^j X^m - \frac{a_i}{6} R_{in} X^n + O(d^{-4})$. The induced metric on the graph of Jang's equation is then given by

$$\begin{aligned} \bar{g}_{ij} = & \delta_{ij} + a_i a_j - \frac{1}{3} R_{ikjl} X^k X^l + a_0 (a_i k_{jm} + a_j k_{im}) X^m \\ & + \frac{1}{2} (a_i B_{jlm} + a_j B_{ilm}) X^l X^m - \frac{1}{6} a_i a_l R_{lj} - \frac{1}{6} a_j a_l R_{li} + O(d^{-3-\alpha}). \end{aligned}$$

In polar coordinates (s, u^a) , we have $\bar{g} = \bar{g}_{ss} ds^2 + 2\bar{g}_{as} ds du^a + \bar{g}_{ab} du^a du^b$ where

$$\begin{aligned} \bar{g}_{ss} = & 1 + (a_i \tilde{x}^i)^2 \\ & + a_i \tilde{x}^i \left(2s a_0 k_{jm} \tilde{x}^j \tilde{x}^m + s^2 B_{jlm} \tilde{x}^j \tilde{x}^l \tilde{x}^m - \frac{1}{3} a_l R_{lj} \tilde{x}^j \right) + O(d^{-3-\alpha}). \\ \bar{g}_{as} = & s a_i \tilde{x}_a^i a_j \tilde{x}^j + s a_i \tilde{x}_a^i \left(s a_0 k_{jm} \tilde{x}^j \tilde{x}^m + \frac{1}{2} s^2 B_{jlm} \tilde{x}^j \tilde{x}^l \tilde{x}^m - \frac{1}{6} a_l R_{lj} \tilde{x}^j \right) \\ & + s a_j \tilde{x}^j \left(s a_0 k_{im} \tilde{x}_a^i \tilde{x}^m + \frac{1}{2} s^2 B_{ilm} \tilde{x}_a^i \tilde{x}^l \tilde{x}^m - \frac{1}{6} a_l R_{li} \tilde{x}_a^i \right) + O(d^{-3-\alpha}). \\ \bar{g}_{ab} = & s^2 \tilde{\sigma}_{ab} + s^2 \left(a_i a_j - \frac{s^2}{3} R_{ikjl} \tilde{x}^k \tilde{x}^l \right) \tilde{x}_a^i \tilde{x}_b^j + s^3 a_0 (a_i k_{jm} + a_j k_{im}) \tilde{x}_a^i \tilde{x}_b^j \tilde{x}^m \\ & + \frac{1}{2} s^4 (a_i B_{jlm} + a_j B_{ilm}) \tilde{x}_a^i \tilde{x}_b^j \tilde{x}^l \tilde{x}^m - \frac{1}{6} s^2 (a_i a_l R_{lj}^l + a_j a_l R_{li}^l) \tilde{x}_a^i \tilde{x}_b^j \\ & + O(d^{-3-\alpha}). \end{aligned}$$

Let $\sigma(s)$ be the induced metric on Σ_s . We consider the isometric embedding of $(\Sigma_s, \sigma(s)_{ab})$ into the hyperplane $X^0 = a_i X^i$ in $\mathbb{R}^4$ with the form $X^0 = a_i s(\tilde{x}^i + \mathfrak{y}^i)$ and $X^i = s(\tilde{x}^i + \mathfrak{y}^i)$ where $\mathfrak{y}^i = O(d^{-1-\alpha})$ and satisfies the linearized isometric embedding equations

$$\begin{aligned} (\delta_{ij} + a_i a_j) (\tilde{x}_a^i \mathfrak{y}_b^j + \tilde{x}_b^j \mathfrak{y}_a^i) = & -\frac{s^2}{3} R_{ikjl} \tilde{x}^k \tilde{x}^l \tilde{x}_a^i \tilde{x}_b^j \\ & + s a_0 (a_i k_{jm} + a_j k_{im}) \tilde{x}^m \tilde{x}_a^i \tilde{x}_b^j + \frac{s^2}{2} (a_i B_{jlm} + a_j B_{ilm}) \tilde{x}^l \tilde{x}^m \tilde{x}_a^i \tilde{x}_b^j \\ & - \frac{1}{6} (a_i R_{lj}^l + a_j R_{li}^l) a_l \tilde{x}_a^i \tilde{x}_b^j + O(d^{-3-\alpha}). \end{aligned}$$

It's not hard to see that $\mathfrak{y}^i = y^{(0)i} + y^i$ with

$$y^{(0)i} = a_i \left(\frac{s}{2a_0} k_{lm} \tilde{x}^l \tilde{x}^m + \frac{s^2}{6a_0^2} B_{jlm} \tilde{x}^j \tilde{x}^l \tilde{x}^m - \frac{1}{6a_0^2} a_m R_n^m \tilde{x}^n \right)$$

and

$$(5.9) \quad (\delta_{ij} + a_i a_j)(\tilde{x}_a^i y_b^j + \tilde{x}_b^j y_a^i) = -\frac{s^2}{3} R_{ikjl} \tilde{x}^k \tilde{x}^l \tilde{x}_a^i \tilde{x}_b^j.$$

Equation (5.9) is a linearized isometric embedding equation on an ellipsoid. Let $\hat{y}^i = (\delta_{ij} + a_i a_j) y^j$. One readily verifies that

$$\hat{y}^i = -\frac{s^2}{6} \left(R_j^i \tilde{x}^j + R_{jk} \tilde{x}^j \tilde{x}^k \tilde{x}^i \right)$$

solves (5.9).

The family of isometric embedding of Σ_s forms a foliation $F : (0, 1] \times S^2 \rightarrow \{X^0 = a_i X^i\} \subset \mathbb{R}^4$. From $F_*(\partial_s) = (\tilde{x}^i + \mathfrak{y}^i + s \frac{\partial \mathfrak{y}^i}{\partial s})(a_i \frac{\partial}{\partial X^0} + \frac{\partial}{\partial X^i})$ and $F_*(\partial_a) = s(\tilde{x}_a^i + \mathfrak{y}_a^i)(a_i \frac{\partial}{\partial X^0} + \frac{\partial}{\partial X^i})$, we could write the flat metric, denoted by $\check{g}$, in (s, u^a) coordinates. Straightforward computation shows that

$$\begin{aligned} \check{g}_{ss} &= \bar{g}_{ss} - 2s^2 R_{ij} \tilde{x}^i \tilde{x}^j + O(d^{-3-\alpha}) \\ \check{g}_{bs} &= \bar{g}_{bs} - s^3 R_{ij} \tilde{x}_b^i \tilde{x}^j + O(d^{-3-\alpha}). \end{aligned}$$

Remark 5.7. *It shouldn't be surprising that $\check{g} - \bar{g}$ does not depend on $y^{(0)}$; namely, $\bar{g}$ remains flat after the graphical perturbation of u . Indeed, if $\bar{g}_{ij} = g_{ij} + u_i u_j$, then the curvature tensors are related by*

$$\bar{R}_{ij}{}^k{}_l = R_{ij}{}^k{}_l - \frac{1}{1 + |\nabla u|^2} R_{ij}{}^p{}_l \nabla_p u \nabla^k u + X_{ij}{}^k{}_l,$$

where X is quadratic in the Hessian of u .

We are ready to compute the second fundamental forms.

Lemma 5.8.

$$\bar{h}_{ab} - \check{h}_{ab} = -s^3 \frac{\sqrt{1 + |a_i \tilde{\nabla} \tilde{x}^i|^2}}{a_0} R_{ij}(p) \tilde{x}_a^i \tilde{x}_b^j + O(d^{-3-\alpha}).$$

Proof. We denote the leading order of τ by $\bar{\tau} = a_i \tilde{x}^i$. The second fundamental form of Σ_s can be computed from the formula

$$\bar{h} = \frac{\frac{1}{2} \partial_s \bar{g}_{ab} - \nabla_a \bar{g}_{bs}}{\sqrt{\bar{g}_{ss} - \sigma^{ab} \bar{g}_{as} \bar{g}_{bs}}}.$$

Here σ is the induced metric of Σ_s . We note that $\sigma_{ab} = s^2(\tilde{\sigma}_{ab} + \bar{\tau}_a \bar{\tau}_b) + O(d^{-1-\alpha})$.

We compute

$$\begin{aligned}\bar{g}_{ss} - \sigma^{ab}\bar{g}_{as}\bar{g}_{bs} &= \frac{a_0^2}{1 + |\tilde{\nabla}\bar{\tau}|^2} + O(d^{-2-\alpha}) \\ \frac{1}{2}\partial_s\bar{g}_{ab} - \nabla_a\bar{g}_{bs} &= s\frac{a_0^2}{1 + |\tilde{\nabla}\bar{\tau}|^2}\tilde{\sigma}_{ab} + O(d^{-2-\alpha}).\end{aligned}$$

Hence, the difference of second fundamental forms is given by

$$\begin{aligned}\bar{h} - \check{h} &= \left(\frac{1}{2}\partial_s\bar{g}_{ab} - \nabla_a\bar{g}_{bs} \right) \cdot \frac{(\check{g}_{ss} - \sigma^{cd}\check{g}_{cs}\check{g}_{ds}) - (\bar{g}_{ss} - \sigma^{cd}\bar{g}_{cs}\bar{g}_{ds})}{2a_0^3} \\ &\quad \cdot (1 + |\tilde{\nabla}\bar{\tau}|^2)^{\frac{3}{2}} + \frac{\nabla_a(\check{g}_{bs} - \bar{g}_{bs})}{a_0} \sqrt{1 + |\tilde{\nabla}\bar{\tau}|^2} + O(d^{-4-2\alpha}) \\ &= \frac{\sqrt{1 + |\tilde{\nabla}\bar{\tau}|^2}}{a_0} \left(\frac{1}{2}(\check{g}_{ss} - \bar{g}_{ss} + \sigma^{cd}\bar{g}_{cs}\bar{g}_{ds} - \sigma^{cd}\check{g}_{cs}\check{g}_{ds}) \cdot s\tilde{\sigma}_{ab} \right. \\ &\quad \left. + \nabla_a(\check{g}_{bs} - \bar{g}_{as}) \right) + O(d^{-3-\alpha}).\end{aligned}$$

We compute

$$\begin{aligned}\nabla_a(\check{g}_{bs} - \bar{g}_{bs}) &= \tilde{\nabla}_a(\check{g}_{bs} - \bar{g}_{bs}) - \frac{\tilde{\nabla}^c\bar{\tau}}{1 + |\tilde{\nabla}\bar{\tau}|^2}\tilde{\nabla}_a\tilde{\nabla}_b\bar{\tau}(\check{g}_{cs} - \bar{g}_{cs}) + O(d^{-3-2\alpha}) \\ &= s^3R_{ij}\tilde{x}^i\tilde{x}^j\tilde{\sigma}_{ab} - s^3R_{ij}\tilde{x}_a^i\tilde{x}_b^j - \frac{\tilde{\nabla}^c\bar{\tau}}{1 + |\tilde{\nabla}\bar{\tau}|^2}\tilde{\nabla}_a\tilde{\nabla}_b\bar{\tau}(\check{g}_{cs} - \bar{g}_{cs}) \\ &\quad + O(d^{-3-\alpha}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{1}{2} \left(\check{g}_{ss} - \bar{g}_{ss} + \sigma^{ab}(\bar{g}_{as}\bar{g}_{bs} - \check{g}_{as}\check{g}_{bs}) \right) \\ &= -s^2R_{ij}\tilde{x}^i\tilde{x}^j + \frac{1}{2}\sigma^{cd}(\bar{g}_{cs}(\bar{g}_{ds} - \check{g}_{ds}) + \check{g}_{ds}(\bar{g}_{cs} - \check{g}_{cs})) + O(d^{-3-\alpha}) \\ &= -s^2R_{ij}\tilde{x}^i\tilde{x}^j + \frac{\tilde{\nabla}\bar{\tau}}{1 + |\tilde{\nabla}\bar{\tau}|^2} \cdot s^{-1}\bar{\tau}(\bar{g}_{cs} - \check{g}_{cs}) + O(d^{-3-\alpha}).\end{aligned}$$

Putting these together, the assertion follows. $\square$

5.5. Evaluation of (5.8)

In this subsection, we evaluate the integral resulted from the difference between the gauge induced by Jang's equation and the canonical gauge.

Recall the solution of optimal isometric embedding equation is $\tau = a_i \tilde{x}^i + \frac{a_0}{2} k_{ij} \tilde{x}^i \tilde{x}^j + \frac{a_0}{6} \partial_i k_{jm} \tilde{x}^i \tilde{x}^j \tilde{x}^m - \frac{a_i}{6} R_{in} \tilde{x}^n$ and the solution of Jang's equation is $u = a_i X^i + \frac{1}{6} T_i \tilde{x}^i + \frac{1}{2} a_0 k_{ij} X^i X^j + \frac{1}{6} B_{ijm} X^i X^j X^m - \frac{a_i}{6} R_{in} X^n$. We will need the following lemma.

Lemma 5.9. *For $\tau = a_i \tilde{x}^i + v$, $v = O(d^{-1-\alpha})$, we have*

$$\Delta \tau = \tilde{\Delta} \tau + \tilde{\Delta} v - \frac{2}{3} a_i R_{ij} \tilde{x}^j + \frac{1}{3} a_i R_{kl} \tilde{x}^i \tilde{x}^k \tilde{x}^l + O(d^{-3-\alpha})$$

Lemma 5.10.

$$\begin{aligned} & \frac{1}{8\pi} \int \left[|H| \sqrt{1 + |\nabla \tau|^2} (\cosh \theta' - \cosh \theta) + \Delta \tau (\theta' - \theta) \right] d\Sigma \\ &= \frac{1}{54a_0} \left(\sum_m T_m^2 + 2T_m a_i R^{im}(p) + a_i a_j R^{il}(p) R_l^j(p) \right) + O(d^{-5-2\alpha}). \end{aligned}$$

Proof. For the canonical gauge, we have

$$\begin{aligned} \langle H, \bar{e}_4 \rangle &= |H| \sinh \theta = \frac{-\Delta \tau}{\sqrt{1 + |\nabla \tau|^2}} \\ &= \frac{1}{\sqrt{1 + |\nabla \tau|^2}} \left[2a_i \tilde{x}^i + 3a_0 k_{ij} (\tilde{x}^i \tilde{x}^j - \frac{1}{3} \delta^{ij}) \right. \\ &\quad \left. + a_0 \partial_i k_{jm} \left(2\tilde{x}^i \tilde{x}^j \tilde{x}^m - \frac{1}{3} (\tilde{x}^i \delta^{jm} + \tilde{x}^j \delta^{mi} + \tilde{x}^m \delta^{ij}) \right) \right. \\ &\quad \left. + \frac{a_i}{3} R_n^i \tilde{x}^n - \frac{1}{3} a_m R_{ij} \tilde{x}^i \tilde{x}^j \tilde{x}^m \right] + O(d^{-3-\alpha}). \end{aligned}$$

For the gauge $\{e'_3, e'_4\}$ induced by Jang's equation, we have [27, Theorem 4.1] $e'_4 = \sinh \phi e_3 + \cosh \phi e_4$ with

$$\begin{aligned} \sinh \phi &= \frac{-u_3}{\sqrt{1 + |\nabla \tau|^2}} \\ &= \frac{-1}{\sqrt{1 + |\nabla \tau|^2}} \left(a_i \tilde{x}^i + \frac{1}{6} T_i \tilde{x}^i + a_0 k_{ij} \tilde{x}^i \tilde{x}^j + \frac{1}{2} B_{ijm} \tilde{x}^i \tilde{x}^j \tilde{x}^m - \frac{1}{6} a_i R_n^i \tilde{x}^n \right) \\ &\quad + O(d^{-3-\alpha}) \end{aligned}$$

and hence

$$\begin{aligned}
 \langle H, e'_4 \rangle &= \cosh \phi \langle H, e_4 \rangle + \sinh \phi \langle H, e_3 \rangle \\
 &= \frac{a_0}{\sqrt{1 + |\nabla \tau|^2}} (-k_{ii} + k_{ij} \tilde{x}^i \tilde{x}^j - \partial_m k_{ii} \tilde{x}^m + \partial_m k_{ij} \tilde{x}^m \tilde{x}^i \tilde{x}^j) \\
 &\quad + \frac{1}{\sqrt{1 + |\nabla \tau|^2}} \left(a_i \tilde{x}^i + \frac{1}{6} T_i \tilde{x}^i + a_0 k_{ij} \tilde{x}^i \tilde{x}^j + \frac{1}{2} B_{ijm} \tilde{x}^i \tilde{x}^j \tilde{x}^m - \frac{1}{6} a_i R_n^i \tilde{x}^n \right) \\
 &\quad \cdot (2 - \frac{1}{3} R_{ij} \tilde{x}^i \tilde{x}^j) + O(d^{-3-\alpha}).
 \end{aligned}$$

By the constraint equations, we get

$$|H|(\sinh \theta - \sinh \theta') = \frac{1}{\sqrt{1 + |\nabla \tau|^2}} \frac{2}{3} (T_m \tilde{x}^m + a_i R_j^i \tilde{x}^j) + O(d^{-3-\alpha}).$$

Next, using two elementary computations

$$\cosh \theta' - \cosh \theta = \frac{\sinh \theta' + \sinh \theta}{\cosh \theta' + \cosh \theta} (\sinh \theta' - \sinh \theta)$$

and (up to error of order $O(d^{-6-3\alpha})$)

$$\begin{aligned}
 \theta' - \theta &= \sinh(\theta' - \theta) = \sinh \theta' \cosh \theta - \sinh \theta \cosh \theta' \\
 &= \cosh \theta (\sinh \theta' - \sinh \theta) - \sinh \theta (\cosh \theta' - \cosh \theta),
 \end{aligned}$$

we get up to a negligible error,

$$\begin{aligned}
 &|H| \sqrt{1 + |\nabla \tau|^2} (\cosh \theta' - \cosh \theta) + \Delta \tau (\theta' - \theta) \\
 &= \left(\sqrt{1 + |\nabla \tau|^2} |H| - \sinh \theta \Delta \tau \right) (\cosh \theta' - \cosh \theta) \\
 &\quad + \Delta \tau \cosh \theta (\sinh \theta' - \sinh \theta) \\
 &= \left(\left(\sqrt{1 + |\nabla \tau|^2} |H| - \sinh \theta \Delta \tau \right) \frac{\sinh \theta + \sinh \theta'}{\cosh \theta + \cosh \theta'} + \Delta \tau \cosh \theta \right) \\
 &\quad \cdot (\sinh \theta' - \sinh \theta) \\
 &= \sqrt{1 + |\nabla \tau|^2} |H| \left(\cosh^2 \theta \frac{\sinh \theta + \sinh \theta'}{\cosh \theta + \cosh \theta'} - \sinh \theta \cosh \theta \right) \\
 &\quad \cdot (\sinh \theta' - \sinh \theta) \\
 &= \frac{1}{\cosh \theta + \cosh \theta'} \sqrt{1 + |\nabla \tau|^2} |H| (\sinh \theta' - \sinh \theta)^2
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 & \frac{1}{8\pi} \int \left[|H| \sqrt{1 + |\nabla \tau|^2} (\cosh \theta' - \cosh \theta) + \Delta \tau (\theta' - \theta) \right] d\Sigma \\
 &= \frac{1}{8\pi} \int \frac{1}{2 \cosh \theta} \sqrt{1 + |\nabla \tau|^2} |H| (\sinh \theta' - \sinh \theta)^2 dS^2 \\
 &= \frac{1}{8\pi} \frac{1}{4a_0} \frac{4}{9} \int (T_m \tilde{x}^m + a_i R_j^i \tilde{x}^j)^2 dS^2 + O(d^{-5-2\alpha}) \\
 &= \frac{1}{54a_0} \left(\sum_m T_m^2 + 2T_m a_i R^{im} + a_i a_j R^{il} R_l^j \right) + O(d^{-5-2\alpha}).
 \end{aligned}$$

□

Proof of Theorem 5.1, (2). Putting Lemma 5.5, 5.6, 5.10 together, we get

$$\begin{aligned}
 E(\Sigma, X, T_0) &= \frac{a_0}{90} R_{ij}(p) R^{ij}(p) + \frac{1}{180a_0} R_{ij}(p) R^{ij}(p) \\
 &\quad + \frac{a_0}{180} \bar{W}_{0imj}(p) \bar{W}_0^{imj}(p) - \frac{1}{45} a_m \bar{W}_0^{imj}(p) \bar{W}_{0i0j}(p).
 \end{aligned}$$

Recall [8, page 4]

$$\begin{aligned}
 Q(e_0, e_0, e_0, T_0) &= \frac{a_0}{2} \bar{W}_{0imj}(p) \bar{W}_0^{imj}(p) + a_0 \bar{W}_{0m0n}(p) \bar{W}_0^{m \ n}(p) \\
 &\quad - 2a_m \bar{W}_0^{imj}(p) \bar{W}_{0i0j}(p).
 \end{aligned}$$

By the Gauss equation, $\bar{W}_{0i0j} = R_{ij} + O(d^{-2-2\alpha})$, we complete the proof. □

6. Small sphere limit

The careful readers would surely find the similarity between our main result and the small sphere limit [8, Theorem 1.1, 1.2]. In this section, we adapt the previous computations to the small sphere setting. Although the family of small spheres is different from that considered in [8], the limit of quasi-local mass turns out to be the same.

Let p be a point in the spacetime. We recall the setup in [8]. Let e_0, e_1, e_2, e_3 be an orthonormal basis at p , $\langle e_\alpha, e_\beta \rangle = \eta_{\alpha\beta}$. Using e_0 , we normalize each null vector L at p by $\langle L, e_0 \rangle = -1$. We consider the null geodesics with initial velocity being the normalized L . Σ_r is defined as the level sets of the affine parameter r . In short, Chen-Wang-Yau considered small spheres approaching p along the light cone.

Theorem 6.1. [8, Theorem 1.1, 1.2]

- 1) For the isometric embeddings $\mathcal{X}_r$ of Σ_r into $\mathbb{R}^3$, the quasi-local energy satisfies

$$(6.1) \quad E(\Sigma_r, \mathcal{X}_r, T_0) = r^3 \cdot \frac{4\pi}{3} T(e_0, T_0) + O(r^4)$$

as r goes to 0.

- 2) Suppose the stress-energy tensor $T_{\alpha\beta}$ vanish in a neighborhood of p . Then, for the pair $(\mathcal{X}_r(T_0), T_0)$ solving the leading order term of the optimal embedding equation of Σ_r , we have

$$(6.2) \quad E(\Sigma_r, \mathcal{X}_r(T_0), T_0) = r^5 \cdot \frac{1}{90} \left[Q(e_0, e_0, e_0, T_0) + \frac{\sum_{m,n} \bar{W}_{0m0n}^2(p)}{2a_0} \right] + O(r^6)$$

as r goes to 0.

In the right-hand side of both formula, we identify $T_0 = (a_0, -a_1, -a_2, -a_3)$ with the timelike vector $a_0 e_0 - \sum_{i=1}^3 a_i e_i$ at p .

To get the same limit using the method of previous sections, we approach p along a spacelike hypersurface. Let X^0, X^1, X^2, X^3 be a normal coordinate near p . The metric has the expansion

$$\bar{g}_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} \bar{R}_{\alpha\gamma\beta\delta} X^\gamma X^\delta + \dots$$

Let M be the slice $\{X^0 = 0\}$. We consider small spheres $\Sigma_r = \{X^0 = 0, (X^1)^2 + (X^2)^2 + (X^3)^2 = r^2\}$ and balls $B_r = \{X^0 = 0, (X^1)^2 + (X^2)^2 + (X^3)^2 \leq r^2\}$, $0 < r < \epsilon$. The timelike unit normal vector and second fundamental form of M are given by

$$\vec{n} = \frac{\partial}{\partial X^0} + \frac{1}{3} \bar{R}_{0jik} X^j X^k \frac{\partial}{\partial X^i} - \frac{1}{6} \bar{R}_{0j0k} X^j X^k \frac{\partial}{\partial X^0} + O(|X|^3)$$

and

$$\begin{aligned} k_{ij} &= \frac{1}{2} (\langle D_{\partial_i} \vec{n}, \partial_j \rangle + \langle D_{\partial_j} \vec{n}, \partial_i \rangle) \\ &= \frac{1}{2} \partial_0 g_{ij} + \frac{1}{6} (\bar{R}_{0ijq} + \bar{R}_{0jiq}) X^q + O(|X|^3) \\ &= -\frac{1}{3} (\bar{R}_{0iqj} + \bar{R}_{0jq i}) X^q + O(|X|^2). \end{aligned}$$

We compute on Σ_r

$$\begin{aligned}\mathrm{tr}_\Sigma k &= O(r^2), \\ \alpha_H &= -k(\partial_a, \nu) + \partial_a \left(\frac{\mathrm{tr}_\Sigma k}{|H|} \right) + O(r^4) = -\frac{r^2}{6} \bar{R}_{j0iq} \tilde{x}_a^i \tilde{x}^j \tilde{x}^q + O(r^4), \\ \mathrm{div} \alpha_H &= O(r^2).\end{aligned}$$

We are ready to use Theorem 4.1 to recover the small sphere limits, Theorem 6.1. Consider the nonvacuum case first. By definition, we have $\mu = 8\pi T(e_0, e_0)$ and $J_i = 8\pi T(e_0, e_i)$. Moreover, integrating over B_r provides a factor of r^3 . Therefore we recover (6.1).

For the non-vacuum case, we again solve the optimal embedding equation and Jang's equation first and then evaluate the three integrals on either B_r or Σ_r .

Lemma 6.2. *The following pair $T_0 = (a_0, -a_1, -a_2, -a_3)$ and*

$$\begin{aligned}\mathcal{X}^0 &= \frac{a_i}{6a_0} r^3 R_{mn}(p) \tilde{x}^m \tilde{x}^n \tilde{x}^i + O(r^4) \\ \mathcal{X}^i &= r \tilde{x}^i - \frac{r^3}{6} R_n^i(p) \tilde{x}^n - \frac{r^3}{6} R_{mn}(p) \tilde{x}^m \tilde{x}^n \tilde{x}^i + O(r^4)\end{aligned}$$

solves the leading order of the optimal embedding equation on Σ_r . In particular, the above solution gives a time function $\tau = -\mathcal{X} \cdot T_0$ with

$$\tau = r a_i \tilde{x}^i - \frac{r^3}{6} a_i R_n^i(p) \tilde{x}^n + O(r^4).$$

Lemma 6.3. *Let u be the solution of the Dirichlet problem of Jang's equation on B_r with boundary value τ . Then*

$$u = a_i X^i + \frac{r^2}{6} T_i X^i + \frac{r^2}{6} B_{ijm} X^i X^j X^m - \frac{r^2}{6} a_i R_n^i X^n,$$

where

$$B_{ijm} = -\frac{1}{3} (\delta_{ij} T_m + \delta_{jm} T_i + \delta_{mi} T_j).$$

The constants T_i can be solved from Jang's equation. As before, their contribution to each integral would cancel and we do not bother to solve them explicitly here.

Lemma 6.4.

$$\begin{aligned}
& \frac{1}{16\pi} \int_{B_r} \left[\left| k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}} \right|_{\bar{g}}^2 + 2|Y|_{\bar{g}}^2 \right] d\tilde{\Omega} \\
&= r^5 \left[-\frac{1}{54a_0} \sum_m T_m^2 - \frac{1}{27a_0} T_m a_l R^{lm}(p) + \frac{1}{90} \left(a_0 - \frac{1}{a_0} \right) R_{ij}(p) R^{ij}(p) \right. \\
&\quad - \frac{1}{54a_0} a_i a_j R^{il}(p) R_l^j(p) + \frac{a_0}{180} \bar{W}_{0imj}(p) \bar{W}_0{}^{imj}(p) \\
&\quad \left. - \frac{1}{45} a_m \bar{W}_0{}^{imj}(p) \bar{W}_{0i0j}(p) \right] + O(r^6).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
k_{ij} - \frac{D_i D_j u}{\sqrt{1 + |Du|^2}} &= \frac{1}{3} \left(\frac{1}{a_0} (\delta_{ij} T_m + \delta_{jm} T_i + \delta_{mi} T_j) \right. \\
&\quad \left. - \bar{W}_{0imj} - \bar{W}_{0jmi} + \frac{a_l}{a_0} (R_{iljm} + R_{imjl}) \right) X^m + O(r^2)
\end{aligned}$$

Since we are integrating on a ball with radius r instead of 1, we get an additional factor r^5 from

$$\frac{1}{16\pi} \int_{B_r} X^m X^n dx = \frac{r^5}{60} \delta^{mn}.$$

□

Lemma 6.5.

$$\frac{1}{16\pi} \int [|h_0(s) - h(s)|^2 - (H_0(s) - H(s))^2] d\tilde{\Omega} = \frac{r^5}{60a_0} R_{ij}(p) R^{ij}(p) + O(r^6).$$

Proof. The argument is almost identical as in the proof of Lemma 5.6. We get an additional factor r^5 in the last step:

$$\int_0^r s^4 ds = \frac{r^5}{5}.$$

□

Lemma 6.6.

$$\begin{aligned} & \frac{1}{8\pi} \int_{\Sigma_r} \left[|H| \sqrt{1 + |\nabla \tau|^2} (\cosh \theta' - \cosh \theta) + \Delta \tau (\theta' - \theta) \right] d\Sigma \\ &= \frac{r^5}{54a_0} \left(\sum_m T_m^2 + 2T_m a_i R^{im}(p) + a_i a_j R^{il}(p) R_l^j(p) \right) + O(r^6). \end{aligned}$$

Proof. A similar computation as in the proof of Lemma 5.10 leads to

$$|H|(\sinh \theta - \sinh \theta') = \frac{r}{\sqrt{1 + |\nabla \tau|^2}} \frac{2}{3} (T_m \tilde{x}^m + a_i R_j^i \tilde{x}^j) + O(r^2).$$

Recall the main term in the integrand is $|H|(\sinh \theta - \sinh \theta')^2$. We get a factor r^3 because $|H| = \frac{2}{r} + O(r^{-2})$ and another factor r^2 from the area form of Σ_r . $\square$

Putting the above three lemma together with Theorem 4.1, we recover (6.2).

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