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The minimal model program for b-log canonical divisors and applications

Daniel Chan[1] · **Kenneth Chan**[2] · **Louis de Thanhoffer de Völcsey**[3] · **Colin Ingalls**[4] · **Kelly Jabbusch**[5] · **Sándor J. Kovács**[2] · **Rajesh Kulkarni**[6] · **Boris Lerner**[1] · **Basil Nanayakkara**[7] · **Shinnosuke Okawa**[8] · **Michel Van den Bergh**[9]

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Abstract
We discuss the minimal model program for *b-log varieties*, which is a pair of a variety and a *b*-divisor, as a natural generalization of the minimal model program for ordinary log varieties. We show that the main theorems of the log MMP work in the setting of the *b*-log MMP. If we assume that the log MMP terminates, then so does the *b*-log MMP. Furthermore, the *b*-log MMP includes both the log MMP and the equivariant MMP as special cases. There are various interesting *b*-log varieties arising from different objects, including the *Brauer pairs*, or "non-commutative algebraic varieties which are finite over their centres." The case of toric Brauer pairs is discussed in further detail.

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✉ Shinnosuke Okawa
okawa@math.sci.osaka-u.ac.jp

Extended author information available on the last page of the article

1 Introduction

Let $\boldsymbol{k}$ be an algebraically closed field of characteristic zero. Let K be a field, finitely generated over $\boldsymbol{k}$. A b-divisor $\mathbf{D}$ associates a $\mathbb{Q}$-divisor $\mathbf{D}_X$ to every normal model X of K, compatibly with pushforward. We assume throughout that the coefficients of our b-divisors are rational numbers in the interval [0, 1) unless otherwise stated (e.g. Theorem 3.7). The main result of this paper is that replacing the canonical divisor K_X with $K_X + \mathbf{D}_X$ everywhere, provides a generalization of the minimal model program, namely the b-log MMP. The b-log MMP includes the G-equivariant MMP and the log MMP as special cases, by using appropriate b-divisors, as explained in Examples 3.6 and 3.5.

There are several natural sources of b-divisors. The canonical divisor K_X for a variety X gives an example of a b-divisor as in Example 2.5. A divisor on a model X can be extended to all models as the proper transform b-divisor as in Definition 2.9. If we have a cohomology class defined on the generic point of a variety X, there are obstructions to extending it to codimension one points of X. These obstructions are typically measured by ramification. One such example is the ramification that occurs when we extend a Galois extension of $\boldsymbol{k}(X)$ to a ramified cover as in Example 2.10. A similar situation occurs when we extend a central simple algebra over $\boldsymbol{k}(X)$ to a maximal order as in Example 2.11. Both of these examples give us a natural ramification b-divisor that also serves as the difference between the canonical divisors of X and its pull back in terms of the Riemann–Hurwitz Theorem, or its analogue for maximal orders.

In [8], Chan and Ingalls studied the MMP for maximal orders over surfaces, and were led to developing the b-log MMP for surfaces. This paper generalizes these results to arbitrary dimensions. We show that the main theorems of the log MMP work in the setting of the b-log MMP. The contractions and flips of the b-log MMP are simply log MMP contractions and flips for the log variety $(X, \mathbf{D}_X)$ and so many of the results for the b-log MMP are direct consequences of those for the log MMP. The existence result to run the b-log MMP is the following theorem.

Theorem 1.1 (= Theorem 3.7) *Let $\pi\colon (X, \mathbf{D}) \to U$ be a b-lc pair over U. If $K_X + \mathbf{D}_X$ is not nef, then there exists a $(K_X + \mathbf{D}_X)$-negative extremal contraction. If it is a flipping contraction, then the flip exists.*

The termination of the b-log MMP is addressed by the following result.

Theorem 1.2 (= Theorem 3.8) *Let $(X, \mathbf{D})$ be a b-canonical pair with $\lfloor \mathbf{D} \rfloor = 0$. Suppose either the dimension of X is at most 3, or $\mathbf{D}_X$ is big and $K_X + \mathbf{D}_X$ is pseudo-effective. Then the pair $(X, \mathbf{D})$ admits a minimal model and the log-canonical divisor of the minimal model is semi-ample.*

Remark 1.3 [=Remark 3.9] Let $(X, \mathbf{D})$ be a b-canonical pair with $\lfloor \mathbf{D} \rfloor = 0$. If we assume that the log MMP terminates for $(X, \mathbf{D}_X)$ then the pair admits a minimal model. If the pair admits a canonical model then it is unique by the proof of [28, Theorem 3.52]. The result [28, Theorem 3.52] is stated for pairs, but the proof for b-divisors is the same

Uniqueness of the minimal model up to flops is given by this combination of Proposition 3.11 and Theorem 3.15

Theorem 1.4 *Let $(X, \mathbf{D})$ and $(X', \mathbf{D})$ be b-terminal minimal models over U of the same b-terminal pair. Then any birational map $\varphi\colon X \dashrightarrow X'$ over U is an isomorphism in codimension* 1. *Also, the birational map φ can be decomposed into a sequence of flops.*

This theorem generalizes [8, Proposition 3.17] to arbitrary dimension.

The b-log MMP differs from the log MMP in terms of what types of singularities are permitted. By using the b-divisor in the definition of discrepancy we obtain the following formula for a birational proper morphism $f : Y \to X$

$$K_Y + \mathbf{D}_Y = f^*(K_X + \mathbf{D}_X) + \sum_E b'(E; X, \mathbf{D})E,$$

where the sum is over f-exceptional divisors. Thus we obtain a modification b' of the usual discrepancy. Let d_E be the coefficient of E in the b-divisor $\mathbf{D}$ and let $r_E = 1/(1 - d_E)$. We also introduce another modification of the discrepancy $b(E; X, \mathbf{D}) = r_E b'(E; X, \mathbf{D})$ which is more natural from several points of view as seen in Corollary 2.23, Corollary 2.24, Remark 2.26 and Example 3.6; in particular, the coefficient r_E coincides with the ramification index in the equivariant setting, as we observe in Example 2.10. Using this definition of discrepancy we obtain notions of b-terminal, b-canonical, b-log terminal and b-log canonical.

Running the MMP usually starts with resolving singularities. In our case, there is no appropriate notion of smoothness, so we must begin by resolving singularities to a b-terminal model. We show that any b-log variety admits a b-terminal resolution of singularities in Theorem 2.30 and Corollary 4.14.

Theorem 1.5 (= Theorem 2.30) *Let $(X, \mathbf{D})$ be a b-log variety with $\lfloor \mathbf{D} \rfloor = 0$ such that X is a quasi-projective variety over $\boldsymbol{k}$. Then there exists a projective birational morphism $f : Y \to X$ such that the b-log variety $(Y, \mathbf{D})$ is b-terminal and Y is $\mathbb{Q}$-factorial.*

In fact, we provide two proofs of this result. The first one is shorter and relies on [7], and the second proof is longer but is more constructive and uses toroidal geometry and hence produces a model Y which is toroidal and $\mathbb{Q}$-factorial. These results generalize [8, Corollary 3.6].

Once we have an appropriate partial resolution, we can start running the log MMP using the existence result Theorem 3.7. The negativity lemma allows us to conclude that contractions and flips preserve the type of singularities.

Corollary 1.6 (= Corollary 3.4) *The notion of b-terminality (resp. b-canonicity, b-log terminality, b-klt, b-log canonicity) is preserved under b-MMP.*

Theorem 3.7 and the above Corollary 3.4 generalize [8, Theorem 3.10]. This establishes the main results of the b-log MMP. Next, we discuss the history and motivation of our application of b-log MMP to noncommutative algebraic geometry.

It was noted by Artin that given a maximal order Λ over a variety X, a tensor power of the dualizing sheaf $\omega_\Lambda^{\otimes n}$ of Λ could be realized as the pull back of a divisor $n(K_X + \Delta)$ on X in codimension one. This suggested that one can use a $\mathbb{Q}$-divisor on X for what would naturally be considered the canonical divisor of Λ. This idea was used by Chan and Kulkarni in [9] to classify del Pezzo orders. In [8], Chan and Ingalls applied this idea and the log minimal model program for surfaces to birationally classify orders over surfaces. This is also treated in [2]. Since then, there remained the issue of extending the results to higher dimension. In [31], Nanayakkara, showed that Brauer pairs (X, α) with $\alpha \in \operatorname{Br} X$ of order 2, have b-terminal resolutions in all dimensions, allowing one to start the minimal model program for Brauer pairs by applying log MMP contractions and flips for the pair (X, Δ). However it was not clear if the steps of the MMP would preserve the notion of Brauer terminal, or if terminal resolutions existed in other cases. In 2014, a meeting was held at the American Institute of Mathematics, in order to solve this problem. This paper is a joint work of all the participants at that meeting.

In Sect. 4, we begin by discussing b-discrepancies for divisors over snc pairs. Then we consider the case of toric b-log varieties and their b-discrepancy in some detail. We give a constructive proof of the existence of b-terminal resolutions in the toric case in Proposition 4.8. We complete this section by using the toric results combined with toroidal geometry to provide another proof of the existence of b-terminal resolutions in Corollary 4.14.

In Sect. 5, we return to our original motivation for b-divisors coming from ramification of Brauer classes. We restrict to the case of toric Brauer classes. Given a non-degenerate toric Brauer class α with toric variety X, we show that the b-log variety $(X, \mathbf{D}_\alpha)$ is b-terminal, etc. if and only if X is terminal, etc. in Proposition 5.1. We characterize the singularities of the b-log variety $(\mathbb{A}^3, \mathbf{D}_\alpha)$ for a toric Brauer class α.

We give an application of the b-log MMP.

Corollary 1.7 *Let K be a field, finitely generated over $\boldsymbol{k}$ of characteristic 0. Let Σ be a central simple K algebra with Brauer class $\alpha \in \operatorname{Br} K$ and ramification b-divisor $\mathbf{D}_\alpha$. Suppose either that the b-divisor $\mathbf{K}+\mathbf{D}_\alpha$ is big or it is pseudo-effective and $\mathbf{D}$ is big. Then the group of outer $\boldsymbol{k}$-automorphisms of Σ is finite.*

Proof Recall that the group of outer $\boldsymbol{k}$-automorphisms of Σ is defined by

$$1 \to \Sigma^*/K^* \to \operatorname{Aut}_{\boldsymbol{k}} \Sigma \to \operatorname{Out}_{\boldsymbol{k}} \Sigma \to 1.$$

On the other hand, take $X \in {}_{K/}\mathcal{M}_{/\boldsymbol{k}}$ such that $(X, \mathbf{D}_{\alpha,X})$ is b-terminal. The Skolem–Noether theorem shows that we have an injective map $\operatorname{Out}_{\boldsymbol{k}} \Sigma \to \operatorname{Bir}(X, \mathbf{D}_\alpha)$, where $\operatorname{Bir}(X, \mathbf{D}_\alpha)$ is the group of birational automorphisms of a model $\sigma : X \dashrightarrow X$ such that $\sigma^*\mathbf{D}_\alpha = \mathbf{D}_\alpha$ (the assumption implies that σ is an isomorphism in codimension one).

Now suppose that $\mathbf{K} + \mathbf{D}_\alpha$ is big. By Theorem 3.8 and Remark 3.9 there is a unique canonical model Y of X, which is birational to X, such that (Y, Δ_Y), where Δ_Y is the boundary divisor induced from $\mathbf{D}_{\alpha,X}$, is a stable pair. Moreover, the uniqueness of the canonical model implies the isomorphism $\operatorname{Bir}(X, \mathbf{D}_\alpha) \simeq \operatorname{Aut}(Y, \Delta_Y)$. By Iitaka's Theorem [22, Theorem 11.12] (see also [15, Theorem 1.2] and [29, Proposition 6.5]) we have the finiteness of this group.

Finally, consider the case that $\mathbf{K}+\mathbf{D}_\alpha$ is only pseudo-effective but $\mathbf{D}_\alpha$ is supposed to be big. Take a small enough ε such that $(X, (1+\varepsilon)\mathbf{D}_\alpha)$ is still terminal. Note that the log canonical divisor of the new pair is big. Now we can use that $\operatorname{Bir}(X, \mathbf{D}_{\alpha,X}) = \operatorname{Bir}(X, (1+\varepsilon)\mathbf{D}_{\alpha,X})$, where the finiteness of the latter is shown by the same argument as above.

We also note that the ideas in this paper are used in [17], where two related results are established.

Theorem 1.8 [17, Theorem 1.3] *Let K be a finitely generated field with a b-divisor $\mathbf{D}$. If X, Y are models of K with $(X, \mathbf{D})$ and $(Y, \mathbf{D})$ have b-canonical singularities and $\ell(K_X + \mathbf{D}_X)$ and $\ell(K_Y + \mathbf{D}_Y)$ are both Cartier then*

$$\bigoplus_{n\geq 0} H^0(X, n\ell(K_X + \mathbf{D}_X)) = \bigoplus_{n\geq 0} H^0(Y, n\ell(K_Y + \mathbf{D}_Y))$$

are naturally isomorphic rings.

This leads to a birationally invariant notion of Kodaira dimension for b-divisors. In addition, for a finite group G, they show the existence of G-equivariant b-terminal partial resolutions of b-log pairs [17, Theorem 4.15] using Theorem 2.30 of this paper.

We conclude the introduction with a brief discussion on how one could apply the b-MMP to the study of maximal orders, which was established in dimension two in [8]. Let K be a field, finitely generated over $\boldsymbol{k}$. Let Σ be a central simple K-algebra with Brauer class α. Let Λ be a maximal order Σ with ramification data $(X, \mathbf{D}_{\alpha,X})$ as in Example 2.11. We may run the minimal model program for Λ in the following way. We first resolve singularities of $(X, \mathbf{D}_\alpha)$ to a b-terminal model by using Theorem 2.30 obtaining a birational morphism $f: Y \to X$. We choose a maximal order Λ_Y containing $f^*\Lambda$. Next, we run the b-log MMP. For a birational contraction or a flip $g: Y \dashrightarrow Y'$, we take reflexive hull $\Lambda_{Y'} = (g_*\Lambda_Y)^{\vee\vee}$ which will be a maximal order by [4, Theorem 1.5]. If the log MMP terminates in a birational model (not a Mori fibre space) then so does the b-log MMP and we will obtain a maximal order Λ_Z on a b-terminal minimal model $(Z, \mathbf{D}_\alpha)$. The pair $(Z, \mathbf{D}_{\alpha,Z})$ is canonically determined by Σ up to log flops. Note further that in dimension two by [8, Theorem 1.2], the order Λ_Z is unique up to Morita equivalence. This result relies heavily on the possible algebraic structure of the order in dimension two and we do not have a similar result for higher dimensions. So we ask the following question.

Question 1.9 To what extent is the maximal order on a minimal model uniquely determined?

Question 1.10 How do Mori fibre spaces for the b-log variety $(X, \mathbf{D}_\alpha)$ interact with a maximal order Λ on X? For instance, is there a semi-orthogonal decomposition of the derived category of the category of Λ-modules?

We work over an algebraically closed field $\boldsymbol{k}$ throughout the paper. The characteristic of $\boldsymbol{k}$ will be assumed to be 0 unless otherwise stated. For a scheme X, the set of points of codimension c will be denoted by $X^{(c)}$. A *variety* is an integral scheme which is separated and of finite type over $\boldsymbol{k}$.

Remark 1.11 We work with $\mathbb{Q}$-divisors in this paper, but almost all results naturally generalize to $\mathbb{R}$-divisors. Our motivation came from the study of b-log varieties coming from Brauer pairs (Example 2.11), which always have $\mathbb{Q}$-coefficients.

One theme of this work is that the standard theorems for b-log pairs hold true as soon as they are established for the corresponding classes of log pairs. For example, many such results have been established for 3-folds in characteristics ≥ 7 after the first draft of this paper appeared on the arXiv. The corresponding results for 3-fold b-log pairs in characteristics ≥ 7 follow from them, by the arguments in this paper.

2 *b*-divisors and *b*-discrepancy

2.1 Recap on *b*-divisors

The notion of *b-divisors* appear in birational geometry, especially in the definition of the moduli part of generalized pairs (see, say, [6] for details). The motivating example of b-divisors for us, however, are those arising from Brauer pairs as we will see in Example 2.11 below.

We recall the notion of *b-divisors* after [10, Section 2.3.2] ('b' stands for 'birational'). We will change notation slightly, by not fixing a particular model. For standard terminologies related to singularities in the Minimal Model Program, readers may refer to [28, Section 2.3] or [27].

Let K be a field, finitely generated over our base field $\boldsymbol{k}$ and let $\eta = \operatorname{Spec} K$. A *model* of K is an irreducible variety X over $\boldsymbol{k}$ with a fixed map $\eta \to X$ over $\boldsymbol{k}$, mapping η isomorphically to the generic point of X.

The category of schemes over $\boldsymbol{k}$ and under η will be denoted by ${}_{K/}\mathfrak{Sch}_{/\boldsymbol{k}}$. We take ${}_{K/}\mathcal{M}_{/\boldsymbol{k}}$ to be the full subcategory of objects X which are normal and proper models of K, where the maps are given by birational morphisms that commute with the fixed map from η. An object of ${}_{K/}\mathcal{M}_{/\boldsymbol{k}}$ will be called a *(proper) model of* K.

Definition 2.1 Let E be a prime divisor in some normal model of K. The divisor E gives us a discrete valuation ν on K such that $\operatorname{trdeg} \kappa(\nu) = \operatorname{trdeg}(K) - 1$. Recall that a *place* is an equivalence class of valuations with equal valuation rings. We will call such valuations and places *geometric*. Let R be the discrete valuation ring of ν. Let X be a normal proper model of K. We have maps $\operatorname{Spec} R \leftarrow \eta \to X$. Since X is proper, we obtain a unique extension $\operatorname{Spec} R \to X$. The closure of the image of the closed point $\xi \in \operatorname{Spec} R$ in X will be denoted by $C_X E = \overline{\{\xi\}}$ and called the *centre of* E *on* X. There exists a normal model Y of K with a birational morphism $f: Y \to X$, where the centre of the valuation ν is an irreducible divisor E. Since Y is normal, we have $\mathcal{O}_{Y,E} = R$ and the closed subset $f(E) \subset X$ is $C_X E$. A divisor E in some model is *exceptional* over a model X if $C_X E$ has codimension greater than 1 in X.

The group of Weil divisors on a normal variety X will be denoted by $\operatorname{Div} X$. One can define the pushforward of a Weil divisor under a proper morphism of normal varieties ([16, Section 1.4]), thus we obtain a functor

$$\operatorname{Div}\colon {}_{K/}\mathcal{M}_{/\boldsymbol{k}} \to \mathbf{Ab}; \quad X \mapsto \operatorname{Div} X, \quad (f: Y \to X) \mapsto (f_*\colon \operatorname{Div} Y \to \operatorname{Div} X) \tag{2.1}$$

to the category of abelian groups $\mathbf{Ab}$. By restricting to effective divisors, we also obtain the functor $\operatorname{Div}_{\geq 0}$ in the obvious way.

Definition 2.2 (=[10, Definition 2.3.8]) An element $\mathbf{D}$ of the limit object

$$\mathbf{Div}(K) := \varprojlim_{X \in {}_{K/}\mathcal{M}_{/\boldsymbol{k}}} \operatorname{Div}(X) \in \mathbf{Ab} \tag{2.2}$$

will be called a(n integral) *b-divisor* on K. Similarly, an element of the subset

$$\mathbf{Div}_{\geq 0}(K) := \varprojlim_{X \in {}_{K/}\mathcal{M}_{/\boldsymbol{k}}} \operatorname{Div}_{\geq 0}(X) \tag{2.3}$$

will be called an effective (integral) b-divisor on K.

A b-divisor on X may equivalently be described as a formal integral sum

$$\mathbf{D} = \sum_{\Gamma} d_\Gamma \Gamma, \tag{2.4}$$

where Γ runs through all the geometric places of K, such that for each normal model X there are only finitely many Γ whose centre on X is divisorial and $d_\Gamma \neq 0$. A b-divisor $\mathbf{D}$ associates a divisor to every normal model X of K, which is called the *trace* of $\mathbf{D}$ on X defined by the natural projection map $\operatorname{tr}_X : \mathbf{Div}(K) \to \operatorname{Div}(X)$. Write $X^{(1)}$ for the set of irreducible divisors in X, or equivalently the set of codimension one points. We write

$$\mathbf{D}_X = \operatorname{tr}_X \mathbf{D} = \sum_{\Gamma \in X^{(1)}} d_\Gamma \Gamma \tag{2.5}$$

(see [10, Notation and Conventions 2.3.10]). Note that this is a finite sum for any particular model, and given a birational morphism $f\colon Y \to X$ we have $f_*\mathbf{D}_Y = \mathbf{D}_X$. The b-divisor $\mathbf{D}$ is effective if and only if all the coefficients d_Γ are non-negative. Note that we can also interpret a b-Divisor $\mathbf{D}$ as function d which associates a number d_ν to every geometric place ν of the field K such that for any model X, the support of d restricted to the divisors of X is finite. We will refer to the value of this function d_ν, or d_E, on a geometric place ν, or a divisor E, as the *coefficient* of $\mathbf{D}$ along E. In addition, if a b-divisor $\mathbf{D}$ is defined for all models Y over a fixed model X, then it extends naturally to all models. Indeed, given any model Z, we can find a common model Y with birational morphism $Y \to X$ and $f\colon Y \to Z$ and so the trace on Z is given by

$$\mathbf{D}_Z = f_*(\mathbf{D}_Y). \tag{2.6}$$

We will freely extend the coefficients of b-divisors to $\mathbb{Q}$. All the notions defined so far are naturally extended to b-$\mathbb{Q}$-divisors. We will work primarily with b-divisors with rational coefficients so we will refer to them simply as b-divisors and we will write $\mathbf{Div}K$ for the set of b-divisors with rational coefficients. Our goal is to develop the minimal model program for *b-log varieties* $(X, \mathbf{D})$ where X is a normal proper variety and $\mathbf{D}$ is a b-divisor in $\mathbf{Div}(\boldsymbol{k}(X))$ with coefficients in $[0, 1) \cap \mathbb{Q}$. In our setting, all coefficients of the b-divisor $\mathbf{D}$ are contained in the interval $[0, 1) \cap \mathbb{Q}$. We also write as $\lfloor \mathbf{D} \rfloor = 0$ to indicate that the coefficients are in the interval $[0, 1)$.

First we will consider some motivating examples of b-divisors that occur naturally. Recall that a divisor D is $\mathbb{Q}$-*Cartier* if there is a non-zero natural number a such that aD is a Cartier divisor.

Example 2.3 (=[11, Example 1.7.2]) Given a $\mathbb{Q}$-Cartier divisor D on X, its *Cartier closure* $\overline{D}$ is the b-divisor whose trace on a model Y over X given by $f\colon Y \to X$ is $\overline{D}_Y = f^*D$. We extend $\overline{D}$ to all models by pushforward as described above (2.6).

Example 2.4 Take a non-zero rational function $\varphi \in K^\times$. We associate a b-divisor $\mathbf{div}(\varphi)$ in $\mathbf{Div}(K)$ whose trace on the model X is defined by

$$\mathbf{div}(\varphi)_X := \mathrm{div}_X(\varphi). \tag{2.7}$$

This will be called the *principal b-divisor* associated to φ. The equality

$$\overline{\mathrm{div}_X(\varphi)} = \mathbf{div}(\varphi), \tag{2.8}$$

where the left hand side is the Cartier closure of the Cartier divisor $\mathrm{div}_X(\varphi)$, is easily seen.

Among others, canonical b-divisors play quite an important role in this paper.

Example 2.5 Fixing a rational differential $\omega \in \bigwedge^{\mathrm{trdeg}\, K} \Omega_{K/k}$ defines a *canonical b-divisor* $\mathbf{K} = \mathbf{div}_X(\omega)$ on X. On each model X, the trace will be defined as $\mathrm{div}_X(\omega)$ associated to the rational global section ω of the canonical sheaf $\mathcal{O}_X(K_X)$.

Remark 2.6 In the example above and the lemma below, a canonical divisor on X means a specific choice of a Weil divisor on X (not its linear equivalence class in the Weil divisor class group).

Remark 2.7 Given a canonical b-divisor $\mathbf{K} \in \mathbf{Div}(K)$, for any model X, we will write $K_X = \mathbf{K}_X$.

Lemma 2.8 *A canonical b-divisor is uniquely determined by its trace on any fixed model.*

Proof Let X be a fixed model, and fix the trace K_X of a canonical b-divisor. Let $f: Y \to X$ be a model over X, and K_Y, K'_Y be two canonical divisors on Y such that $f_* K_Y = K_X = f_* K'_Y$. Then $K_Y - K'_Y = \mathrm{div}_Y(\varphi)$ for some $\varphi \in \boldsymbol{k}(Y) = \boldsymbol{k}(X)$ and the support of $K_Y - K_{Y'}$ is contained in the exceptional locus of f. Since $\mathrm{div}_Y(\varphi) = f^* \mathrm{div}_X(\varphi)$ and $\mathrm{div}_X(\varphi) = f_* \mathrm{div}_Y(\varphi) = 0$ by the assumption, we see $\mathrm{div}_Y(\varphi) = 0$.

Definition 2.9 Consider a Weil divisor D on X. The *proper transform b-divisor* $\widehat{D}$ is the b-divisor whose trace on a model $f: Y \to X$ is defined by $\widehat{D}_Y = \left(f^{-1}\right)_* D$, and naturally extended to all models via push-forward. Note that the coefficient of $\widehat{D}$ on any exceptional divisor over X is zero. In fact, proper transform b-divisors are characterized by the support of the formal sum $\mathbf{D} = \sum d_\Gamma \Gamma$ over all geometric places Γ being finite.

Example 2.10 Let G be a finite group. Recall that an element $[L] \in H^1(K, G)$ is represented by an isomorphism class of a Galois extension L/K with a homomorphism $\mathrm{Gal}(L/K) \to G$ which we can assume to be injective. Let X be a model of K and let $\pi: \widetilde{X} \to X$ be the normalization of X in the field L, so that the field homomorphism $\pi^*: \boldsymbol{k}(X) \to \boldsymbol{k}(\widetilde{X}) \xrightarrow{\sim} L$ is canonically identified with the extension L/K.

Since $\pi: \widetilde{X} \to X$ is again a Galois extension with Galois group a subgroup of G, the Riemann–Hurwitz Theorem tells us that there exists an effective $\mathbb{Q}$-divisor $\mathbf{D}_X$ on X such that

$$K_{\widetilde{X}} = \pi_X^* (K_X + \mathbf{D}_X) \tag{2.9}$$

as $\mathbb{Q}$-divisors on $\widetilde{X}$. One can easily verify that the divisors $\mathbf{D}_X$ give rise to a b-divisor with $\lfloor \mathbf{D} \rfloor = 0$ and $\mathbf{D} \in \mathbf{Div}_{\geq 0}(K)$, which will be called the *ramification b-divisor*.

Let $P \subset X$ be a prime divisor and let $\widetilde{P} \subset \widetilde{X}$ be $\pi_X^{-1}(P)$ equipped with the reduced structure. Let $m \geq 1$ be the integer defined by $\pi_X^* P = m\widetilde{P}$; i.e., the ramification index of π_X along P. Then the coefficient d_P in $\mathbf{D}_X$ of a prime divisor $P \subset X$ is equal to $\frac{m-1}{m}$. In this case we obtain

$$r_P = \frac{1}{1 - d_P} = m \tag{2.10}$$

so that r_P has a natural geometric meaning.

On the other hand, let $n = \mathrm{trdeg}\, K$ and fix $\omega \in \Omega^n_{K/\boldsymbol{k}}$ and consider $\omega \otimes 1 \in \Omega^n_{K/\boldsymbol{k}} \otimes_K L \simeq \Omega^n_{L/\boldsymbol{k}}$. We associate canonical b-divisors $\mathbf{K} = \mathbf{div}(\omega)$ and $\widetilde{\mathbf{K}} = \mathbf{div}(\omega \otimes 1)$ in $\mathbf{Div}(K)$ and $\mathbf{Div}(L)$ respectively. Then we have the equality of b-divisors $\pi^*(\mathbf{K} + \mathbf{D}) = \widetilde{\mathbf{K}}$. As we will see later in Example 3.6, the MMP for b-log varieties applied to the pair $(X, \mathbf{D})$ is equivalent to the G-equivariant MMP for $\widetilde{X}$.

Example 2.11 This example is the original motivation for the authors to establish the Minimal Model Theory for b-log varieties. Let K be a field, finitely generated over $\boldsymbol{k}$ and let $\alpha \in H^2(K, \mathbb{G}_m) = \mathrm{Br}\, K$ be a Brauer class. A *Brauer pair*, (X, α) is a pair of a normal proper model X of K and an $\alpha \in \mathrm{Br}\, K$. Then we can define the effective divisor

$$\mathbf{D}_{\alpha, X} = \sum_{D \in X^{(1)}} \left(1 - \frac{1}{r_D}\right) D, \tag{2.11}$$

where $r_D \in \mathbb{Z}_{\geq 1}$ is the *ramification index* of the Brauer class $\alpha \in \boldsymbol{k}(X)$ along the prime divisor D, which is defined via the Artin–Mumford map [3]. Given $\alpha \in \mathrm{Br}(K)$ we have

$$H^2(\boldsymbol{k}(X), \mathbb{G}_m) \stackrel{\mathrm{ram}}{\to} \bigoplus_{D \in X^{(1)}} H^1(\boldsymbol{k}(D), \mathbb{Q}/\mathbb{Z})$$

and we define r_D above to be the order of $\mathrm{ram}_D(\alpha)$. The divisors $\mathbf{D}_{\alpha,X}$ give a b-divisor. We note that the divisor $\mathbf{K}_X + \mathbf{D}_{\alpha,X}$ can be viewed as the canonical divisor of a maximal order in a central simple K algebra representing α, as noted in [8], or can be interpreted as the canonical divisor of the associated root stack [1, Appendix B].

We also note that, in Example 2.10, if we have a cyclic Galois cover, we can treat it analogously to a Brauer class, if we use the map

$$H^1(\boldsymbol{k}(X), \mu) \stackrel{\mathrm{ram}}{\to} \bigoplus_{D \in X^{(1)}} H^0(\boldsymbol{k}(D), \mathbb{Q}/\mathbb{Z}) \tag{2.12}$$

to define the coefficients of the ramification b-divisor.

Example 2.12 The above example can be generalized to the setting of Rost modules. This includes algebraic K-theory, Chow cohomology, motivic cohomology, and more. In [32], the notion of Rost (cycle) modules is defined. Given a Rost module M and a normal scheme X, we obtain maps $\partial_D \colon M(\boldsymbol{k}(X)) \to M(\boldsymbol{k}(D))$ for all irreducible divisors D in X. Given an element $\alpha \in M(\boldsymbol{k}(X))$ only finitely many $\partial_D(\alpha)$ are non-zero as in [32, Definition 2.1]. So given such an α, if the $\partial_D(\alpha)$ has finite order r_D for all D, (for example if α has finite order), we can define a ramification b-divisor by

$$\mathbf{D}_{\alpha,X} = \sum_{D \in X^{(1)}} \left(1 - \frac{1}{r_D}\right) D. \tag{2.13}$$

This also includes the case of abelian Galois covers from Example 2.10.

We will freely use the following remark.

Remark 2.13 Let P be a property of (Weil) divisors (or their classes) such that if $f \colon X \to Y$ is a projective birational morphism of normal varieties, then a (Weil) divisor (class) D on X satisfies the property P if and only if f_*D does. We say that a b-divisor $\mathbf{D}$ satisfies the property P if the trace $\mathbf{D}_X$ of $\mathbf{D}$ on some model $X \in {}_{K/}\mathcal{M}_{/\boldsymbol{k}}$ satisfies P, which is equivalent to that the same holds on *all* models X. For example, bigness and pseudo-effectivity can be taken as the property P.

2.2 *b*-discrepancy

In this section we introduce the discrepancy for b-divisors. First we will recall some facts about the usual notion of discrepancy before we introduce our modification for b-divisors. Recall the following definition.

Definition 2.14 Let (X, D) be a $\mathbb{Q}$-Gorenstein log variety and let $f \colon Y \to X$ be a birational morphism. The discrepancy of divisors E in Y that are exceptional over X for the log variety (X, D) are defined by the equation

$$K_Y + f_*^{-1}D = f^*(K_X + D) + \sum_E a(E; X, D)E$$

where the sum is taken over f-exceptional divisors E, and $f_*^{-1}D$ denotes the proper transform of D. The discrepancy only depends on the divisor and not the choice of model Y, as reflected in the notation.

Definition 2.15 A *b-log variety* is a pair $(X, \mathbf{D})$ of a normal variety X and an effective b-$\mathbb{Q}$-divisor $\mathbf{D}$ on X. If $K_X + \mathbf{D}_X$ is $\mathbb{Q}$-Cartier we say that the pair $(X, \mathbf{D})$ is $\mathbb{Q}$-Gorenstein. The b-divisor $\mathbf{K} + \mathbf{D}$ will be called the *log canonical b-divisor* of the pair $(X, \mathbf{D})$.

In the rest of this paper, unless otherwise stated, we assume that all b-divisors have coefficients are in $[0, 1) \cap \mathbb{Q}$. We will also tacitly assume all pairs are $\mathbb{Q}$-Gorenstein, unless otherwise stated.

Definition 2.16 Let $(X, \mathbf{D})$ be a b-log variety with $\lfloor \mathbf{D} \rfloor = 0$. For each divisor E over X, let $d_E \in [0, 1) \cap \mathbb{Q}$ be the coefficient of $\mathbf{D}$ along E. The *ramification index* $r_E \in [1, \infty) \cap \mathbb{Q}$ of $\mathbf{D}$ along E is defined by the equivalent equations:

$$r_E = \frac{1}{1 - d_E} \qquad d_E = 1 - \frac{1}{r_E}. \tag{2.14}$$

Definition 2.17 Let $(X, \mathbf{D})$ be a $\mathbb{Q}$-Gorenstein b-log variety and E an exceptional divisor over X. Take a model $f : Y \to X$ such that the centre $C_Y E \subset Y$ is a divisor. Then there exists a $b'(E; X, \mathbf{D}) \in \mathbb{Q}$ such that the following equality of $\mathbb{Q}$-divisors

$$(\mathbf{K} + \mathbf{D})_Y = f^* (\mathbf{K} + \mathbf{D})_X + b'(E; X, \mathbf{D})E \tag{2.15}$$

holds on an open neighbourhood of the generic point of $E \subset Y$. The rational number $b'(E; X, \mathbf{D})$ will be called the *b'-discrepancy* of the b-log variety $(X, \mathbf{D})$ with respect to the divisor E over X.

Example 2.18 Consider a usual log variety (X, D) and the proper transform b-divisor $\widehat{D}$. Then it follows from the definition that for any exceptional divisor E over X,

$$a(E; X, D) = b'(E; X, \widehat{D}) = b(E; X, \widehat{D}). \tag{2.16}$$

In this sense, for exceptional divisors, the usual discrepancy can be regarded as the b-discrepancy of a proper transform b-divisor.

Moreover, when $D = 0$, the equality (2.16) is valid for *any* divisor over X; recall that a geometric valuation of $\boldsymbol{k}(X)$ which admits a centre on X is called exceptional if and only if its centre on X is not divisorial.

Remark 2.19 For a divisor E over X we have the equality

$$b'(E; X, \mathbf{D}) = a(E; X, \mathbf{D}_X) + d_E, \tag{2.17}$$

where $a(E; X, \mathbf{D}_X)$ is the usual discrepancy of the log variety $(X, \mathbf{D}_X)$ with respect to the divisor E.

In particular, if $\mathbf{D}$ is effective, we always have the inequality

$$b'(E; X, \mathbf{D}) \geq a(E; X, \mathbf{D}_X). \tag{2.18}$$

Equality holds precisely if $\mathbf{D}$ is not supported on E.

It is more natural to consider a slight modification of b'-discrepancy. This modification is motivated by Corollary 2.23, Corollary 2.24, Remark 2.26 and Example 3.6. The *b-discrepancy* of the b-log variety $(X, \mathbf{D})$ with respect to the divisor E over X is defined by either of the equivalent equations

$$b(E; X, \mathbf{D}) = b'(E; X, \mathbf{D}) \cdot r_E \tag{2.19}$$

$$b(E; X, \mathbf{D}) + 1 = r_E(a(E; X, \mathbf{D}) + 1). \tag{2.20}$$

Note that one can interpret the second equation above as saying that the b-log discrepancy is a positive multiple of the usual log discrepancy.

We say that the b-log variety $(X, \mathbf{D})$ is *snc* if the associated pair $(X, \mathbf{D}_X)$ is snc. The following lemma will be frequently used in this paper.

Lemma 2.20 *For any b-log variety $(X, \mathbf{D})$, consider any log resolution $f: Y \to X$ of the log variety $(X, \mathbf{D}_X)$. Then $(Y, \mathbf{D}_Y)$ is snc.*

Proof Let $f: Y \to X$ be a log resolution of the pair $(X, \mathbf{D}_X)$. Then, by definition, $\mathrm{Exc}\,(f) \cup (f^{-1})_* \mathbf{D}_X$ is an snc divisor. Since $\mathrm{Supp}\,\mathbf{D}_Y$ is a subset, it is snc as well.

Definition 2.21 Let $(X, \mathbf{D})$ be a $\mathbb{Q}$-Gorenstein b-log variety. The *minimal b-discrepancy* of the pair $(X, \mathbf{D})$ is defined by

$$\text{b-discrep}(X, \mathbf{D}) := \inf \{b(E; X, \mathbf{D}) \mid E \text{ is an exceptional divisor over } X\}. \tag{2.21}$$

Note that the infimum is among all divisors over X which are exceptional.

We say

$$(X, \mathbf{D}) \text{ is } \begin{cases} b - \text{terminal} \\ b - \text{canonical} \\ b - \text{log terminal } (b - \text{lt}) \\ b - \text{log canonical } (b - \text{lc}) \end{cases} \quad \text{if b-discrep}\,(X, \mathbf{D}) \begin{cases} > 0 \\ \geq 0 \\ > -1 \\ \geq -1. \end{cases} \tag{2.22}$$

We also make corresponding definitions using $b'(E; X, \mathbf{D})$ in place of $b(E; X, \mathbf{D})$ and so will refer to b-log varieties $(X, \mathbf{D})$ as being b'-terminal, b'-canonical, b'-log terminal, or b'-log canonical.

Similarly we say $(X, \mathbf{D})$ is *b-Kawamata log terminal* (*b-klt)* if it is b-lt and $\lfloor \mathbf{D} \rfloor = 0$. Finally we define the notion of *b-dlt* pairs as follows; a b-log variety $(X, \mathbf{D})$ is *b-divisorially log terminal* (*b-dlt)* if there exists a log resolution $f: Y \to X$ of $(X, \mathbf{D})$ such that

$$b(E; X, \mathbf{D}) > -1 \tag{2.23}$$

holds for any f-exceptional divisor E.

Lemma 2.22 *Let $(X, \mathbf{D})$ be a $\mathbb{Q}$-Gorenstein b-log variety with $\lfloor \mathbf{D} \rfloor = 0$, and E be a divisor over X. Then $a(E; X, \mathbf{D}_X) > (resp. \geq) - 1 \iff b(E; X, \mathbf{D}) > (resp. \geq) - 1$.*

Proof This follows immediately from (2.20) in the definition of b-discrepancy.

Lemma 2.22 immediately implies the following corollaries.

Corollary 2.23 *Let $(X, \mathbf{D})$ be a b-log variety with $\lfloor \mathbf{D} \rfloor = 0$. Then $(X, \mathbf{D})$ is b-lt (resp. b-lc) if and only if the log variety $(X, \mathbf{D}_X)$ is lt (resp. lc) in the usual sense.*

Corollary 2.24 *A b-log variety $(X, \mathbf{D})$ is b-dlt if and only if $b(E; X, \mathbf{D}) > -1$ holds for any exceptional divisor E over X whose centre on X is contained in the non-snc locus of $(X, \mathbf{D}_X)$.*

Proof The equivalence of corresponding conditions for usual log discrepancy (= equivalence of two different definitions of the notion of dlt pairs) is well known [28, Proposition 2.44], [34]. On the other hand, one can immediately check that each of them is respectively equivalent to the b-counterpart because of Lemma 2.22. □

Remark 2.25 We also note that b-log variety $(X, \mathbf{D})$ is b'-terminal (resp. b'-canonical) if and only if it is b-terminal (resp. b-canonical). This follows immediately from Definition 2.19.

Remark 2.26 If $(X, \mathbf{D}_X)$ is not log canonical then its discrepancy is equal to $-\infty$. This observation allows us to see that it is also true that $(X, \mathbf{D})$ is b'-lc if and only if $(X, \mathbf{D}_X)$ is lc. Similarly, if $(X, \mathbf{D}_X)$ is klt, then $(X, \mathbf{D})$ is b'-klt. On the other hand, as the following example shows, the converse does not hold.

Let X be a cone over an elliptic curve E and let $f: Y \to X$ its minimal resolution. Note that the exceptional divisor is isomorphic to E. Let $\mathbf{D}$ be a b-divisor on X such that $\mathbf{D}_X = 0$ and the coefficient of $\mathbf{D}$ along E satisfies $d_E > 0$. Then one can check that $(X, \mathbf{D})$ is b'-lt, though X is (strictly) lc. Actually one can find a Brauer class $\alpha \in \mathrm{Br}(\mathbf{k}(X))$ whose ramification along E corresponds to an étale double cover of E, so that the associated b-divisor $\mathbf{D}_\alpha$ has $d_E = \frac{1}{2}$. One can similarly check that Corollary 2.24 is not true for b'-discrepancy.

Example 2.27 Let (X, D) be a log variety with $\lfloor D \rfloor = 0$ and consider the proper transform b-divisor $\widehat{D}$. Then

$$(X, D) \text{ is } \begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{purely log terminal} \\ \text{dlt} \\ \text{log canonical} \end{cases} \iff (X, \widehat{D}) \text{ is } \begin{cases} b-\text{terminal} \\ b-\text{canonical} \\ b-\text{klt} \\ b-\text{log terminal} \\ b-\text{dlt} \\ b-\text{log canonical} \end{cases} \tag{2.24}$$

(see [28, Definition 2.34]).

For those readers who would like particular examples, we discuss the b-discrepancy of toric b-log varieties in detail in Sect. 5; in particular, we compute the invariants for toric b-log 3-folds in Example 5.3.

In order to run the b-log MMP with b-terminal singularities, it is necessary to first resolve singularities to a b-terminal model. The existence of such a resolution is established in Theorem 2.30, and the proof of this theorem is the goal of the rest of this section.

Lemma 2.28 *Let $(X, \mathbf{D})$ be a $\mathbb{Q}$-Gorenstein b-log variety and $f: Y \to X$ a model on which the trace $(\mathbf{K} + \mathbf{D})_Y$ is $\mathbb{Q}$-Cartier. Suppose $b(E; X, \mathbf{D}) \leq 0$ holds for any f-exceptional prime divisor E. Then for any exceptional divisor F over Y, we have the inequality*

$$b(F; Y, \mathbf{D}) \geq b(F; X, \mathbf{D}). \tag{2.25}$$

Proof We show the claim for b'-discrepancies, since it is equivalent. Let $g: Z \to Y$ be a model over Y on which the centre of F is divisorial. Then around the generic point of F we have

$$K_Z + \mathbf{D}_Z = g^*(K_Y + \mathbf{D}_Y) + b'(F; Y, \mathbf{D})F \tag{2.26}$$

$$= g^* \left(f^*(K_X + \mathbf{D}_X) + \sum_E b'(E; X, \mathbf{D})E \right) + b'(F; Y, \mathbf{D})F \tag{2.27}$$

$$= (f \circ g)^*(K_X + \mathbf{D}_X) + \left(\left(\sum_E b'(E; X, \mathbf{D}) m_F E \right) + b'(F; Y, \mathbf{D}) \right) F. \tag{2.28}$$

Since $b'(E; X, \mathbf{D}) \leq 0$ and $m_F E \geq 0$ hold for all E, we see

$$b'(F; X, \mathbf{D}) = \sum_E b'(E; X, \mathbf{D}) m_F E + b'(F; Y, \mathbf{D}) \leq b'(F; Y, \mathbf{D}).$$

□

Remark 2.29 Assume that the pair $(X, \mathbf{D})$ is b-lt with $\lfloor \mathbf{D} \rfloor = 0$, so that the associated pair $(X, \mathbf{D}_X)$ is klt. Then by [28, Proposition 2.36 (2)] and Remark 2.19, there are only finitely many exceptional divisors over X with non-positive b-discrepancies.

We will provide a second proof of the result below in Corollary 4.14. The second proof uses toroidal geometry and is longer but it is also more explicit and more elementary in the sense that it does not use the result [19, Exercise 5.41] which depends on [7].

Theorem 2.30 *Let $(X, \mathbf{D})$ be a b-log variety with $\lfloor \mathbf{D} \rfloor = 0$ such that X is a quasi-projective variety over $\boldsymbol{k}$. Then there exists a projective birational morphism $f : Y \to X$ such that the b-log variety $(Y, \mathbf{D})$ is b-terminal and Y is $\mathbb{Q}$-factorial.*

A key ingredient of the first proof of Theorem 2.30 is the following corollary of [7]. For convenience of the reader, we include its proof here. It is taken from [26, Theorem 17.10].

Proposition 2.31 (=[19, Exercise 5.41]) *Let (X, Δ) be a klt pair, and $\mathcal{E}$ be a finite collection of exceptional divisors E over X with $a(E; X, \Delta) \leq 0$. Then there exists a projective birational morphism $f : X \to Y$ from a $\mathbb{Q}$-factorial variety Y such that the set of f-exceptional divisors is precisely $\mathcal{E}$.*

Proof Let $\mathcal{A}$ be the set of all exceptional divisors E over X with $a(E; X, \Delta) \leq 0$, which is a finite set by [28, Proposition 2.36 (2)], and put $\mathcal{E}^c := \mathcal{A} \setminus \mathcal{E}$.

Let $g : Z \to X$ be a log resolution of the pair (X, Δ) which also extracts all members of $\mathcal{E}$. Let $\mathcal{A}$ be the set of exceptional divisors of g, and put $\mathcal{E}^c := \mathcal{A} \setminus \mathcal{E}$. By the assumption, we can take some $c \in (0, 1) \cap \mathbb{Q}$ such that for all $E \in \mathcal{E}^c$ it holds that $c + a(E; X, \Delta) > 0$. Now let $h : Z \dashrightarrow Y$ be a minimal model of the klt pair

$$\left(Z, \Delta_Z := g_*^{-1} \Delta + c \sum_{E \in \mathcal{E}^c} E + \sum_{E \in \mathcal{E}} -a(E; X, \Delta) E \right) \tag{2.29}$$

over X, with the structure morphism $f : Y \to X$, whose existence is guaranteed by [7, Theorem 1.2 (1)]. We check that this is the desired morphism.

The canonical bundle formula with respect to f, with a slight modification, is as follows.

$$K_Y + f_*^{-1} \Delta + c \sum_{E \in \mathcal{E}^c} h_* E + \sum_{E \in \mathcal{E}} -a(E; X, \Delta) h_* E \tag{2.30}$$

$$= f^* (K_X + \Delta) + \sum_{F : f-\text{exceptional}} a(F; X, \Delta) F + c \sum_{E \in \mathcal{E}^c} h_* E + \sum_{E \in \mathcal{E}} -a(E; X, \Delta) h_* E \tag{2.31}$$

By definition, the left hand side of (2.30) is f-nef. Hence so is the following divisor.

$$\sum_{F: f\text{-exceptional}} a(F; X, \Delta)F + c\sum_{E\in\mathcal{E}^c} h_*E + \sum_{E\in\mathcal{E}} -a(E; X, \Delta)h_*E \tag{2.32}$$

Since this is an f-exceptional divisor, [28, Lemma 3.39] implies that this divisor indeed satisfies ≤ 0 (i.e., its negation is an effective divisor). By the choice of c, we conclude that all members of $\mathcal{E}^c$ are contracted by h and hence the set of f-exceptional divisors is a subset of $\mathcal{E}$. On the other hand, if a member $E \in \mathcal{E}$ is contracted by h, then we obtain the following contradiction.

$$a(E; X, \Delta) = a(E; Z, \Delta_Z) \overset{\text{[KM98, Lemma 3.38 (4)]}}{<} a(E; Y, \Delta_Y) = a(E; X, \Delta), \tag{2.33}$$

where $\Delta_Y := h_*\Delta_Z = f_*^{-1}\Delta - \sum_{E\in\mathcal{E}} a(E; X, \Delta)h_*E$. The last equality follows from the log crepancy $K_Y + \Delta_Y = f^*(K_X + \Delta_X)$. Thus we conclude that all members of $\mathcal{E}$ appear on Y, which concludes the proof.

Proof of Theorem 2.30 By Lemma 2.20, we find a projective log resolution $X_1 \to X$ so that the pair $(X_1, \mathbf{D}_{X_1})$ is snc. Since X_1 is snc and $\lfloor \mathbf{D}_{X_1} \rfloor = 0$, the pair $(X_1, \mathbf{D}_{X_1})$ is klt. As noted in Remark 2.29, there are only finitely many exceptional divisors over X_1 whose b-discrepancies are non-positive. Let $\mathcal{E}$ be the set of such divisors. Now we apply Proposition 2.31 to $\mathcal{E}$ and obtain a birational projective morphism

$$g : Y \to X_1 \tag{2.34}$$

from a normal $\mathbb{Q}$-factorial variety Y such that the set of g-exceptional divisors is exactly the set $\mathcal{E}$. By Lemma 2.28, we see that the b-log variety $(Y, \mathbf{D})$ is b-terminal. □

Given a b-log variety $(X, \mathbf{D})$, we call the pair $(Y, \mathbf{D})$, supplied by the above Theorem 2.30, a b-terminal resolution of X. Note that a b-log terminal resolution need not be snc, and it is not clear if any b-log variety $(X, \mathbf{D})$ admits a resolution which is simultaneously snc and b-terminal.

Question 2.32 Let $(X, \mathbf{D})$ be a b-log variety with $\lfloor \mathbf{D} \rfloor = 0$. Is there always a projective birational morphism $Y \to X$ such that the b-log variety $(Y, \mathbf{D})$ is snc and b-terminal?

3 The minimal model program for *b*-log varieties

We generalize various definitions in the minimal model theory to b-log varieties. The following definitions for b-log varieties are *based on* those for the usual log varieties, which can be found in standard references on (log) MMP; see, say, [28, Section 3.7].

Definition 3.1 Let $(X, \mathbf{D})$ be a b-lc pair with $\lfloor \mathbf{D} \rfloor = 0$, and X quasi-projective and $\mathbb{Q}$-factorial, equipped with a projective morphism $\pi : X \to U$ to a normal quasi-projective variety U.

- The pair $(X, \mathbf{D})$ is π-*minimal* if $K_X + \mathbf{D}_X$ is π-nef. Note that the definition of a minimal model does not depend on the type of singularities of the pair.
- An *extremal contraction* of $(X, \mathbf{D})$ over U is a morphism $f : X \to Y$ over U which is an extremal contraction of the lc pair $(X, \mathbf{D}_X)$. We say f is *divisorial/small/a Mori fibre space* if f is divisorial/small/a Mori fibre space in the usual sense.

- A *flip* of the pair $(X, \mathbf{D})$ over U is a birational map $X \dashrightarrow X'$ over U which is a flip of the pair $(X, \mathbf{D}_X)$ in the usual sense. Note that this is consistent with $\mathbf{D}$ being a b-divisor. Since $X \dashrightarrow X'$ is an isomorphism in codimension 1 on both X and X', it follows that $\mathbf{D}_{X'}$ is necessarily the proper transform of $\mathbf{D}_X$ on X'.
- A *minimal model program of* $(X, \mathbf{D})$ *over (or relative to)* U is a sequence of birational maps over U

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n \tag{3.1}$$

which is a minimal model program of the *usual* lc pair $(X, \mathbf{D}_X)$ over U (see Corollary 2.23).

Remark 3.2 If $\varphi\colon X \dashrightarrow Y$ is either a divisorial contraction or a flip of the b-log variety $(X, \mathbf{D})$ over U, then clearly $\mathbf{D}_Y = \varphi_* \mathbf{D}_X$. Therefore any subsequence

$$X_i \dashrightarrow X_{i+1} \dashrightarrow \cdots \dashrightarrow X_j$$

of (3.1) is a b-log MMP for the pair $(X_i, \mathbf{D})$.

Lemma 3.3 *Let $\varphi\colon X \dashrightarrow Y$ be either a divisorial contraction or a flip of the b-log variety $(X, \mathbf{D})$ over U. Then for any exceptional divisor E over X we get the inequality*

$$b(E; X, \mathbf{D}) \le b(E; Y, \mathbf{D}). \tag{3.2}$$

If $C_X E$ or $C_Y E$ is contained in the exceptional locus of φ or φ^{-1}, then (3.2) becomes a strict inequality.

Proof It follows from (2.17) that

$$\frac{b(E; Y, \mathbf{D}) - b(E; X, \mathbf{D})}{r_E} = b'(E; Y, \mathbf{D}) - b'(E; X, \mathbf{D}) = a(E; Y, \mathbf{D}_Y) - a(E; X, \mathbf{D}_X). \tag{3.3}$$

Therefore the conclusions follow from the negativity lemma for usual discrepancies [28, Lemma 3.38]. □

Corollary 3.4 *The notion of b-terminality (resp. b-canonicity, b-log terminality, b-klt, b-log canonicity) is preserved under b-MMP.*

Proof Since the arguments are essentially the same, we only discuss the case of b-terminality. Let $(X, \mathbf{D})$ be a b-terminal pair and consider a step of b-MMP $\varphi\colon X \dashrightarrow Y$. If E is an exceptional divisor over X, then Lemma 3.3 implies

$$b(E; Y, \mathbf{D}) \ge b(E; X, \mathbf{D}) > 0.$$

If φ is a divisorial contraction which contracts the prime divisor $E \subset X$, then since $b(E; X, \mathbf{D}) = 0$ by the definitions, we can use the second claim of Lemma 3.3 to see

$$b(E; Y, \mathbf{D}) > b(E; X, \mathbf{D}) = 0.$$

Example 3.5 Recall that the b-discrepancy of the proper transform b-divisor $(X, \widehat{D})$ is the same as the usual discrepancy as discussed in Examples 2.18, 2.27. In addition, the contractions and flips of the b-log MMP are simply those of the log MMP, so running the b-log MMP for the b-log variety $(X, \widehat{D})$ is identical to running the log MMP for the pair (X, D). One may resolve singularities before running the MMP and we note that if one first resolves $(X, \widehat{D})$ to a model which is terminal, canonical, lt or lc, the minimal model (if it exists) will have the same class of singularities by Corollary 3.4.

Next, we explain how the G-equivariant MMP is a special case of the b-log MMP.

Example 3.6 Recall the b-log variety associated to the equivariant setting, discussed in Example 2.10. Let $E \subset Y$ be a prime f-exceptional divisor and set $\widetilde{E} := \left(\pi_Y^{-1}(E) \subset \widetilde{Y}\right)_{\text{red}}$. Then it follows that $a(\widetilde{E}; \widetilde{X}) = b(E; X, \mathbf{D})$ by the following computation. Let $f : Y \to X$ be a birational morphism and let $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$ be the corresponding map on the normalizations of Y and X in the Galois cover. Write $\pi_X : \widetilde{X} \to X$ and $\pi_Y : \widetilde{Y} \to Y$. We have that

$$\sum_{\tilde{E}} a(\widetilde{E}; \widetilde{X})\widetilde{E} = K_{\widetilde{Y}} - \widetilde{f}^* K_{\widetilde{X}} \tag{3.4}$$

$$= \pi_Y^*(K_Y + \mathbf{D}_Y) - \widetilde{f}^* \pi_X^*(K_X + \mathbf{D}_X) \tag{3.5}$$

$$= \pi_Y^*(K_Y + \mathbf{D}_Y) - \pi_Y^* f^*(K_X + \mathbf{D}_X) \tag{3.6}$$

$$= \pi_Y^* \left(\sum_E b'(E; X, \mathbf{D}) E \right) \tag{3.7}$$

$$\overset{2.10.2}{=} \sum_{\widetilde{E}} b'(E; X, \mathbf{D}) r_E \widetilde{E} \tag{3.8}$$

$$= \sum_{\widetilde{E}} b(E; X, \mathbf{D}) \widetilde{E} \tag{3.9}$$

Since $K_{\tilde{X}} = \pi^*(K_X + \mathbf{D}_X)$ we obtain that the MMP of the pair $(X, \mathbf{D})$ corresponds precisely to the G-equivariant MMP of $\widetilde{X}$.

3.1 Fundamental theorems for *b*-log varieties

In this section we establish some foundational results about the b-log MMP by transplanting the corresponding results from (the ordinary) log MMP.

Theorem 3.7 *Let $\pi : (X, \mathbf{D}) \to U$ be a b-lc pair over U. If $K_X + \mathbf{D}_X$ is not nef, then there exists an extremal contraction. If it is a flipping contraction, then the flip exists.*

Proof Since $(X, \mathbf{D}_X)$ is log canonical, the assertions immediately follow from [14, Theorem 1.19] and [5, Corollary 1.2] or [18, Corollary 1.8], respectively. □

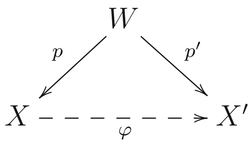
Theorem 3.8 *Let $(X, \mathbf{D})$ be a b-canonical pair with $\lfloor \mathbf{D} \rfloor = 0$. Suppose either the dimension of X is at most 3, or $\mathbf{D}_X$ is big and $K_X + \mathbf{D}_X$ is pseudo-effective, or $K_X + \mathbf{D}_X$ is big. Then the pair admits a minimal model and the log-canonical divisor of the minimal model is semi-ample.*

Proof Since the log variety $(X, \mathbf{D}_X)$ is klt, the assertions immediately follow from the results of [7, 24]. □

Remark 3.9 Let $(X, \mathbf{D})$ be a b-canonical pair with $\lfloor \mathbf{D} \rfloor = 0$. If we assume that the log MMP terminates for $(X, \mathbf{D}_X)$ then the pair admits a minimal model. If the pair admits a canonical model then it is unique by [28, Theorem 3.52].

Definition 3.10 Let $\varphi : X \dashrightarrow X'$ be a birational map between normal varieties. A *common resolution of* φ is a smooth variety W and projective birational morphisms $p, p' : W \rightrightarrows X, X'$ such that $p' = \varphi \circ p$ as rational maps (see Fig. 1). Note that one can be obtained by resolving the singularities of the closure of the graph of ϕ in $X \times X'$.

Fig. 1 Common resolution



Proposition 3.11 *Let* $(X, \mathbf{D})$ *and* $(X', \mathbf{D})$ *be b-terminal minimal models over U of the same b-terminal pair. Then any birational map* $\varphi\colon X \dashrightarrow X'$ *over U is an isomorphism in codimension* 1.

Proof The proof below is a slight modification of the one in [23, p. 420], but we give more details for the convenience of the readers. Take a common resolution of singularities W as in Figure 1. By the symmetry, it is enough to show that any p-exceptional divisor is also p'-exceptional. Consider the canonical bundle formula

$$K_W + \mathbf{D}_W = p^*(K_X + \mathbf{D}_X) + E = (p')^*(K_{X'} + \mathbf{D}_{X'}) + E'. \tag{3.10}$$

Set

$$F = \min(E, E') \tag{3.11}$$

and

$$E = \overline{E} + F, \tag{3.12}$$

$$E' = \overline{E}' + F. \tag{3.13}$$

The assumption is equivalent to $\overline{E} \neq 0$, since $(X, \mathbf{D})$ is b-terminal. By Lemma 3.12 below, one can find an irreducible curve C such that $C \not\subset \operatorname{Supp} \overline{E}'$, $(\overline{E}.C) < 0$, and $p(C) = \text{point}$. This clearly contradicts the equality (3.10), since

$$0 > \left(p^*(K_X + \mathbf{D}_X) + \overline{E}\right).C = \left((p')^*(K_{X'} + \mathbf{D}_{X'}) + \overline{E}'\right).C \geq 0. \tag{3.14}$$

□

Lemma 3.12 *Let* $p\colon W \to X$ *be a birational projective morphism of normal varieties over a field of characteristic zero. Assume that W is smooth, and let* $\overline{E}$ *be a non-trivial effective p-exceptional* $\mathbb{Q}$*-divisor and* $\overline{E}'$ *be an effective divisor on W none of whose component is contracted by p. Then there exists an irreducible projective curve* $C \subset W$ *contracted to a point by p,* $\overline{E}.C < 0$*, and* $C \not\subset \operatorname{Supp} \overline{E}'$.

Proof The proof below is taken from [28, Proof of 3.39]. Consider the decomposition

$$\overline{E} = \sum_{i=2}^{\dim X} \overline{E}_i, \tag{3.15}$$

where $\overline{E}_i$ is the sum of the components $\Gamma \subset \overline{E}$ such that the codimension of $p(\Gamma)$ is i. Let $k \geq 2$ be the minimum integer such that $\overline{E}_k \neq 0$. Take a general complete intersection H^{k-2} of codimension $k-2$ *on* W. Set $\overline{G} := \overline{E}|_{H^{k-2}}$. Then by the genericity we may assume that $\overline{G}_i = \overline{E}_{i+k-2}|_{H^{k-2}}$ for all $i \geq 2$ and that no irreducible component of $\overline{E}' \cap H^{k-2}$ is contracted by the morphism $p|_{H^{k-2}}$. We may also assume that if we let $\overline{H}^{k-2}$ be the normalization of $p\left(H^{k-2}\right)$, then the morphism $p|_{H^{k-2}}\colon H^{k-2} \to \overline{H}^{k-2}$ is projective and birational. Note that if one can find an irreducible projective curve $C \subset H^{k-2}$ which is contracted by $p|_{H^{k-2}}$, $\overline{G}.C < 0$, and $C \not\subset H^{k-2} \cap \overline{E}'$, then as a curve on W it has the required properties as well. Hence we can assume that $k = 2$.

If $k = 2$ take a general complete intersection $S \subset X$ of dimension 2 which is normal [33, Theorem 7], $T = p^{-1}(S) \subset Y$ is smooth, and $T \cap \overline{E}_2 = T \cap \overline{E}$. We may moreover assume that $p|_T : T \to S$ is an isomorphism on an open neighbourhood of $T \cap \operatorname{Supp} \overline{E}'$, since the image of the exceptional locus of $\operatorname{Supp} \overline{E}'$ under the morphism p has codimension at least three. Hence it follows that $N := T \cap \overline{E}_2$ is a non-trivial effective $p|_T$-exceptional divisor none of whose irreducible component is contained in $\operatorname{Supp} \overline{E}'$. By the Hodge index theorem $N^2 < 0$. Since N is an effective divisor, there is at least one component $C \subset N$ such that $N.C < 0$. It is now obvious that the curve C, seen as a curve on W, has all the required properties. □

We next look at some results that hold specifically for surfaces. A b-terminal pair $(S, \mathbf{D})$, where S is a surface will have $(S, \mathbf{D}_S)$ log terminal, so we know that S has quotient singularities.

Corollary 3.13 *Let $(S, \mathbf{D})$ be a b-terminal pair of dimension 2 with non-negative Kodaira dimension. Then it admits a unique minimal model.*

Proof The existence of a minimal model is already settled. The uniqueness follows from the previous proposition and the following well-known lemma. □

Lemma 3.14 *If a birational map $\varphi : S \dashrightarrow S'$ between normal surfaces is an isomorphism in codimension 1 on both S and S′, then it is an isomorphism.*

Proof Consider a common resolution $p, p' : W \to S, S'$ satisfying $p' = \varphi \circ p$. By the assumption, an irreducible curve $C \subset W$ is contracted to a point by p if and only if it is contracted to a point by p'. Therefore, if we consider the image Γ of the morphism $p \times p' : W \to S \times S'$, the natural projections $\Gamma \to S$ and $\Gamma \to S'$ are birational and finite, hence isomorphisms by the Zariski's main theorem [20, Chapter III, Corollary 11.4]. Thus φ extends to an isomorphism whose graph is Γ. □

Theorem 3.15 *Under the assumptions of Proposition 3.11, the birational map φ can be decomposed into a sequence of flops.*

Proof Proposition 3.11 gives the only required modification of the proof of [23, Theorem 1].□

4 Toroidal *b*-log varieties

In this section we will discuss discrepancy and b-terminalizations for toroidal b-log varieties. We will repeat some earlier results, but we include new proofs using toroidal methods, since they are more explicit and constructive. Sncpairs are toroidal and toroidal varieties are naturally stratified, so we begin by proving some results concerning discrepancy for snc pairs.

4.1 Snc stratifications

Let (X, D) be an snc log canonical pair, and D_i be the irreducible components of D. A *stratum* of the pair (X, D) is defined to be an irreducible component of the intersection of some of the divisors D_i. For the blow up $f: Y = \mathbf{Bl}_Z X \to X$ of X along a stratum $Z \subset X$, we define the boundary divisor D_Y by the following equality of $\mathbb{Q}$-divisors.

$$K_Y + D_Y = f^*(K_X + D) \tag{4.1}$$

Since (Y, D_Y) is again an snc log canonical pair, we can repeat the same process as above recursively.

Definition 4.1 We say that an exceptional divisor over X is *extracted by repeatedly performing blowups along the strata* if it appears on a model over X which is obtained by recursing the process above for finitely many steps.

Proposition 4.2 *Let (X, D) be an snc klt pair and E an exceptional divisor over X which can not be extracted by repeatedly performing blowups along the strata. Then*

$$a(E; X, D) > 0. \tag{4.2}$$

The following lemma will be used in the proof of Proposition 4.2.

Lemma 4.3 *Let $f: Y \to X$ and $g: Z \to Y$ be birational morphisms between normal projective varieties. Let D be a $\mathbb{Q}$-divisor on X such that $K_X + D$ and $K_Y + D_Y$, where $D_Y := f_*^{-1} D$, are both $\mathbb{Q}$-Cartier. Let $E \subset Y$ be an f-exceptional divisor, and $F \subset Z$ be a g-exceptional divisor which satisfies $C_Y F \subset E$. Assume that for any f-exceptional divisor $E' \subset Y$ other than E we have $a(E'; X, D) \geq 0$. Then*

$$a(F; X, D) \geq a(F; Y, D_Y) + a(E; X, D). \tag{4.3}$$

Proof Define the divisor D' on Y by the equality

$$K_Y + D' = f^*(K_X + D). \tag{4.4}$$

We see

$$a(F; X, D) - a(F; Y, D_Y) = a(F; Y, D') - a(F; Y, D_Y) \tag{4.5}$$

$$= \sum_{E'} a(E'; X, D) m_F(E') + a(E; X, D) m_F(E) \geq a(E; X, D), \tag{4.6}$$

concluding the proof. □

Proof of Proposition 4.2 Any exceptional divisor E over X can be realized as a codimension 1 regular point on a variety, by repeatedly blowing up its centre for finitely many times (starting with the blow-up of $C_X E$); see [28, 2.45]. Let t be the number of necessary blowups. We prove the statement by induction on t.

Suppose $t = 1$. By replacing X with $X \setminus \mathrm{Sing}(C_X E)$, we may assume $C_X E$ is smooth. Set $c = \mathrm{codim}_X C_X E$, and let $D = \sum a_i D_i$ be the decomposition of D into irreducible divisors. Reorder the D_i so that $C_X E \subset D_i \iff i \leq d$. Note that $\mathrm{mult}_{C_X E} D_i = 1$ for $i \leq d$ and since $C_X E$ is not a stratum, we have the inequality $c > d$. So we obtain the formula

$$a(E; X, D) = c - 1 - \sum_{i=1}^{d} a_i. \tag{4.7}$$

If $d = 0$, we see $a(E; X, D) = c - 1 \geq 1$. If $d > 0$, we see

$$c - 1 - \sum_{i=1}^{d} a_i = c - d - 1 + \sum_{i=1}^{d} (1 - a_i) \geq 0 + (1 - \min\{a_i\}) > 0. \tag{4.8}$$

Now let us consider the induction step. Consider the sequence of blowups which realizes the divisor E:

$$X_t \to X_{t-1} \to \cdots \to X_1 \to X \tag{4.9}$$

Let $E_i \subset X_i$ be the exceptional divisor of the i-th blowup.

Suppose that $C_X E$ is not a stratum. Then by setting $Z = X_t$ and $Y = X_1$, we can apply Lemma 4.3 to obtain the inequality

$$a(E; X, D) \geq a(E; Y, D_Y) + a(E_1; X, D). \tag{4.10}$$

Note that by the case $t = 1$, we know $a(E_1; X, D) > 0$. Moreover, since D_Y, the strict transform of D on Y, is again snc and $C_Y E$ is not a stratum, we can apply the induction hypothesis to see $a(E; Y, D_Y) > 0$. Thus we obtain the conclusion from (4.10).

Finally, suppose that $C_X E$ is a stratum. In this case, define the divisor D' by $K_Y + D' = f^*(K_X + D)$. By applying the induction hypothesis to (Y, D'), we get

$$a(E; X, D) = a(E; Y, D') > 0. \tag{4.11}$$

Thus we conclude the proof. □

Theorem 4.4 *Let $(X, \mathbf{D})$ be an snc b-log variety with $\lfloor \mathbf{D} \rfloor = 0$. Let E be any exceptional divisor over X with $b(E; X, \mathbf{D}) \leq 0$. Then E is extracted by repeatedly performing blowups along strata of the snc pair $(X, \mathbf{D}_X)$.*

Proof Since the log variety $(X, \mathbf{D}_X)$ is snc and $\lfloor \mathbf{D}_X \rfloor = 0$, [28, Corollary 2.31(3)] shows that it is klt. Hence an exceptional divisor E which is *not* extracted by repeatedly performing blowups along strata satisfies the following inequality.

$$b'(E; X, \mathbf{D}) \overset{\text{Remark 2.19}}{\geq} a(E; X, \mathbf{D}_X) \overset{\text{Proposition 4.2}}{>} 0 \tag{4.12}$$

This concludes the proof. □

4.2 Toric b-log varieties

Now we will study b-log varieties where the b-divisor is supported on a toric divisor in a toric variety. In addition to allowing explicit computations, we will provide a second proof of one of the main results of this paper, Theorem 2.30, which shows the existence of b-terminal resolutions, or b-terminalizations. This result is of central importance, since without it, one can not begin the b-log minimal model program with b-terminal singularities.

Let us review some basic facts from toric geometry. We will use results and notation from [12]. Let X be a toric variety with open dense torus $T \simeq \mathbb{G}_m^n \subseteq X$. The variety X is determined by a rational fan Σ in the real vector space spanned by the lattice $N = \mathrm{Hom}(\mathbb{G}_m, T)$. In particular toric geometric valuations of $\boldsymbol{k}(T)$ are given by rational rays in $\mathbb{R}N$. More precisely, there is a correspondence between primitive vectors $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n \simeq N$ (primitive means $\gcd(w_i) = 1$) and toric divisors D_w on some toric model of $\boldsymbol{k}(T)$.

Let $\Sigma(1)$ be the set of rays of the fan Σ. Write u_ρ for the minimal generator in N of a ray ρ in Σ. Write D_ρ or D_{u_ρ} for the divisor associated to ρ. A toric $\mathbb{Q}$-divisor D can be written as $D = \sum d_\rho D_\rho$. If D is $\mathbb{Q}$-Cartier it has an associated *support function* $\phi_D \colon |\Sigma| \to \mathbb{R}$ with the following properties:

(1) ϕ_D is linear on each cone $\sigma \in \Sigma$.
(2) $\phi_D(N) \subseteq \mathbb{Q}$.

(3) $\phi_D(u_\rho) = -d_\rho$.
(4) $D = -\sum_{\rho\in\Sigma(1)} \phi_D(u_\rho) D_\rho$.

Let X be the toric variety associated to a simplicial fan Σ. Recall the following equality:

$$-K_X = \sum_{\rho\in\Sigma(1)} D_\rho \tag{4.13}$$

Note also that support functions are preserved by pullback. More precisely, let $f : \widetilde{X} \to X$ be the map of toric varieties associated to a map of fans $f_\Sigma : \widetilde{\Sigma} \to \Sigma$. Then for a $\mathbb{Q}$-Cartier divisor D on X, we get that

$$\phi_{f^*D} = \phi_D \circ f_\Sigma : |\widetilde{\Sigma}| \to \mathbb{R}.$$

The proof of the following proposition follows the notation and proof of [12, Proposition 11.4.24]. Given a b-divisor $\mathbf{D} \in \mathbf{Div}(K)$ and a normal model X of K, as in Definition 2.16, we will write

$$\mathbf{D}_X = \sum_\Gamma d_\Gamma \Gamma = \sum_\Gamma \left(1 - \frac{1}{r_\Gamma}\right) \Gamma.$$

We say that $(X, \mathbf{D})$ is a toric b-log variety if X is a toric variety and $\mathbf{D}$ is supported on toric divisors for all models.

Proposition 4.5 *Let X be the toric variety associated to the fan Σ and let $(X, \mathbf{D})$ be a toric b-log variety. Let w be an element of the lattice N that is in a (possibly not maximal) simplicial cone σ in Σ, with ramification index r_w. Suppose σ has a set of minimal generators $\sigma = \langle v_1, \ldots, v_m\rangle$ with $v_1, \ldots, v_m$ in the lattice N with ramification indices $r_1, \ldots, r_m$. Write $w = a_1 v_1 + \cdots + a_m v_m$. Then the discrepancy of the divisor D_w associated to w, over $(X, \mathbf{D})$ is given by*

$$b'(D_w; X, \mathbf{D}) = \frac{a_1}{r_1} + \cdots + \frac{a_m}{r_m} - \frac{1}{r_w}. \tag{4.14}$$

$$b(D_w; X, \mathbf{D}) = \frac{a_1 r_w}{r_1} + \cdots + \frac{a_m r_w}{r_m} - 1. \tag{4.15}$$

Proof Let $\widetilde{\Sigma}$ be a simplicial refinement of Σ that contains $\langle w\rangle$ as a ray. Let $E = \widetilde{\Sigma}(1) \setminus \Sigma(1)$ be the set of exceptional divisors. Note that

$$K_X + \mathbf{D}_X = -\sum_{\rho\in\Sigma(1)} D_\rho + \sum_{\rho\in\Sigma(1)} \left(1 - \frac{1}{r_\rho}\right) D_\rho \tag{4.16}$$

$$= -\sum_{\rho\in\Sigma(1)} \left(\frac{1}{r_\rho}\right) D_\rho \tag{4.17}$$

and similarly for $\widetilde{X}$. Now

$$\sum_{\rho\in E} b'(D_\rho, X, \mathbf{D}) D_\rho = K_{\widetilde{X}} + \mathbf{D}_{\widetilde{X}} - f^*(K_X + \mathbf{D}_X) \tag{4.18}$$

$$= -\sum_{\rho\in\widetilde{\Sigma}(1)} \left(\frac{1}{r_\rho}\right) D_\rho + f^*\left(\sum_{\rho\in\Sigma(1)} \left(\frac{1}{r_\rho}\right) D_\rho\right). \tag{4.19}$$

Now let ϕ be the support function associated to the divisor

$$B = \sum_{\rho \in \Sigma(1)} \left(\frac{1}{r_\rho}\right) D_\rho.$$

Since $|\widetilde{\Sigma}| = |\Sigma|$ we will also denote the support function of f^*B by ϕ. Note that

$$f^*B = -\sum_{\rho \in \widetilde{\Sigma}(1)} \phi(u_\rho) D_\rho \tag{4.20}$$

$$= -\sum_{\rho \in \Sigma(1)} \phi(u_\rho) D_\rho - \sum_{\rho \in E} \phi(u_\rho) D_\rho \tag{4.21}$$

$$= \sum_{\rho \in \Sigma(1)} \frac{1}{r_\rho} D_\rho - \sum_{\rho \in E} \phi(u_\rho) D_\rho, \tag{4.22}$$

so that

$$\sum_{\rho \in E} b'(D_\rho, X, \mathbf{D}) D_\rho = -\sum_{\rho \in E} \left(\frac{1}{r_\rho} + \phi(u_\rho)\right) D_\rho. \tag{4.23}$$

To elucidate this sum we consider the coefficient of D_w as above. Since $w = a_1 v_1 + \cdots + a_m v_m$ and $\phi(v_i) = -1/r_i$ we get the desired result. □

The next result follows from Remark 2.29, since toric b-log varieties are b-klt. However, since this also follows directly from toric geometry, we include an alternate proof.

Proposition 4.6 *Let $(X, \mathbf{D})$ be a toric b-log variety. Then there are finitely many toric divisors D_w over X such that $b(D_w, X, \mathbf{D}) \leq 0$.*

Proof Let Σ be the fan associated to X, let $\sigma \in \Sigma$ be a cone, and write $\sigma = \langle v_1, \ldots, v_m \rangle$ for minimal generators v_i in N. Given a primitive vector $w = a_1 v_1 + \cdots + a_m v_m \in \sigma$, the formula (4.15) for discrepancy gives

$$b(D_w; X, \mathbf{D}) = \frac{a_1 r_w}{r_1} + \cdots + \frac{a_m r_w}{r_m} - 1.$$

Since $r_w \geq 1$, and this is positive when the a_i are sufficiently large, there are finitely possible $w \in \sigma$ such that $b'(D_w; X, \mathbf{D}) \leq 0$. The result follows since there are finitely many cones in Σ. □

Proposition 4.7 *Let X be a toric variety associated to the fan Σ. Let $D_{w_1}, \ldots, D_{w_p}$ be a finite set of divisors over X corresponding to the primitive vectors $w_1, \ldots, w_p \in |\Sigma|$ with $\langle w_i \rangle \notin \Sigma$. Then there is a $\mathbb{Q}$-factorial toric variety X' and a birational projective toric morphism $f \colon X' \to X$ such that the exceptional divisors of f are exactly the toric divisors $D_{w_1}, \ldots, D_{w_p}$.*

Proof We first replace Σ by a simplicial refinement by triangulating the non-simplicial cones as described in [12, Proposition 11.1.7]. Given a divisor D_{w_1} to extract, we form the star subdivision $\Sigma_1 := \Sigma^*(w_1)$ as constructed in [12, p. 515, Section 11.1]. This forms a simplicial refinement of Σ with exactly one new ray $\langle w_1 \rangle$. We then repeat for all w_i to obtain Σ' simplicial with new rays corresponding exactly to those primitive vectors with non-positive discrepancy. □

The next result follows from the more general Theorem 2.30, but we will use the statement below in the toroidal setting to provide a more constructive proof of this theorem.

Proposition 4.8 *Any toric b-log variety has a toric $\mathbb{Q}$-factorial b-terminalization.*

Proof By Proposition 4.6, there are only finitely many exceptional divisors over X with non-positive b-discrepancies. Then apply Proposition 4.7 to extract these divisors by a morphism $f: X' \to X$ corresponding to simplicial fan Σ' refining Σ. Now by Lemma 2.28, the discrepancy for X' is larger than the discrepancy for X. Thus we conclude that f is the desired morphism. □

Remark 4.9 The $\mathbb{Q}$-factorial b-terminalization of a toric b-log variety $(X, \mathbf{D})$ we constructed in Proposition 4.8 is dominated by a $\mathbb{Q}$-factorial terminalization (in the usual sense) of the log variety $(X, \mathbf{D}_X)$. Indeed, let $g: Y \to X$ be a projective birational morphism from a $\mathbb{Q}$-factorial variety which extracts exactly those exceptional divisors over X whose discrepancy with respect to the boundary divisor $\mathbf{D}_X$ is non-positive (i.e., a $\mathbb{Q}$-factorization); the existence of such a morphism follows from Proposition 2.31 or Proposition 4.7. Note that, by (2.18), all exceptional divisors of the morphism $f: X' \to X$ constructed in Proposition 4.8 are extracted by g as well. This implies that g is the composition of a contracting birational map $h: Y \to X'$ with f. Since Y is a toric variety, we can replace it with a small $\mathbb{Q}$-factorial modification such that h becomes a genuine morphism (see, say, the last paragraph of the proof of [21, 1.11 Proposition]).

4.3 Toroidal *b*-log varieties

Now we consider toroidal b-log varieties. We will use [25] for definitions, notation and basic results, but we will provide some heuristic explanations. We say a log variety (X, D) is toroidal if $U = X \setminus \operatorname{Supp} D \subset X$ is a toroidal embedding as in [25, p.54]. As explained in [25, p.71], we can associate a *conical polyhedral complex with integral structure* $\Delta = (|\Delta|, \sigma^Y, M^Y)$ to a toroidal embedding. A conical polyhedral complex consists of a finite collection of cones $\{\sigma^Y\}_Y$ with an integral structure $\sigma^Y \subset \mathbb{R}N^Y = \mathbb{R}\operatorname{Hom}(M^Y, \mathbb{Z})$ for each cone, indexed by the natural stratification $\{Y\}$ associated to the toroidal embedding $U \subset X$. The affine toric variety associated to the cone σ_Y corresponding to stratum Y describes the étale local structure of $U \subseteq X$ at the generic point y in Y. A face of a cone in Δ is again a cone in Δ and the cones (with their integral structures) are glued along faces. Unlike the case of a fan used in toric geometry, there is no ambient lattice N so that $|\Delta| \subset \mathbb{R}N$ and we can have more than two faces glued along a face of codimension one. The conical polyhedral complex does not uniquely determine the toroidal variety. However, akin to refinements of fans in toric geometry, there is a correspondence between *finite rational partial polyhedral (f.r.p.p.) decompositions* (see [25, Definition 2, p.86] for the precise definition) Δ' of Δ with $|\Delta'| = |\Delta|$ and proper birational morphisms $X' \to X$ that are *allowable (or toroidal) modifications* by [25, Theorem 6*, p. 90]. We say a b-log variety $(X, \mathbf{D})$ is *toroidal* if $(X, \mathbf{D}_X)$ is toroidal. Note that this implies that for all allowable modifications, $X' \to X$, we have that $(X, \mathbf{D}_{X'})$ is also toroidal. We call the exceptional divisors that are divisors in allowable modifications the *toroidal divisors* over X.

Since a toroidal variety is characterized by being étale locally an affine toric variety with toric boundary, we see that an snc pair (X, D) is toroidal. Suppose we are given a rational ray ρ in a cone $\sigma_Y \subseteq \Delta$ corresponding to the stratum Y. We can form the star subdivision $\Delta^*(\rho)$ where we add one ray $\rho = \langle u_\rho \rangle$ and subdivide every cone $\tau = \langle u_1, \ldots, u_m \rangle$ containing ρ

by forming cones $\langle u_\rho, u_1, \ldots, \hat{u}_i, \ldots, u_m\rangle$ exactly as in [12, p. 515, Section 11.1]. We note the following facts about the star subdivisions:

- $\Delta^*(\rho)$ is a f.r.p.p. decomposition of Δ.
- $|\Delta^*(\rho)| = |\Delta|$.
- There is the corresponding projective allowable modification $X^*(\rho) \to X$.
- If Δ is simplicial, then so is $\Delta^*(\rho)$ and $X^*(\rho)$ is $\mathbb{Q}$-factorial.
- The cones of dimension one (rays) in $\Delta^*(\rho)$ are rays in Δ with the addition of the one new ray ρ.
- The divisor D_ρ is Cartier.

The next lemma follows easily from [25, Theorem 10*, p.90].

Lemma 4.10 *Let (X, D) be toroidal and let $\Delta = (|\Delta|, \sigma^Y, M^Y)$ be the associated conical polyhedral complex with integral structure. Let $\sigma_Y = \langle u_1, \ldots, u_m\rangle$ be minimal primitive generators of σ_Y and $\rho = \langle \sum u_i\rangle$ then $X^*(\rho) \to X$ is the normalization of the blow up of the stratum Y.*

Proof Since the ideal sheaf of any stratum of the log variety (X, D) is a canonical coherent sheaf of fractional ideals in the sense of [25, p.90], any blowup of X along a stratum corresponds to a f.r.p.p. decomposition of Δ by [25, Theorem 10*, p. 93]. This is clearly given by the star subdivision described above. □

Corollary 4.11 *If (X, D) is snc, then the divisors over X obtained by blowing up strata are exactly the toroidal divisors over X.*

Proof It is clear that the divisors obtained by blowing up strata will be toroidal, so we must prove the converse. We first consider the case of affine space as a toric variety $\mathbb{G}_m^n \subset \mathbb{A}^n$. Let $e_1, \ldots, e_n$ be the minimal generators of the cone in the lattice N. In this case, a toric divisor corresponds to primitive vector w with all coordinates non-negative. We will construct a sequence of blow ups at smooth toric subvarieties to obtain $\langle w\rangle$ as a ray. We write $w = \sum a_i e_i$ with $a_i \geq 0$. If all $a_i \leq 1$ we are done. Otherwise we form the star subdivision at $v = \sum \operatorname{sgn}(a_i)e_i$ where $\operatorname{sgn}(a_i)$ is the sign function. Now w will be in a new smooth simplicial cone which includes v as a vertex and we have

$$w = v + \sum_{a_i \neq 0} (a_i - 1)e_i.$$

Now the coefficients of w in terms of generators of the new cone are smaller and the coefficient of v is one. So by repeating this process we will eventually obtain w.

Now given a general snc toroidal pair (X, D), any toroidal divisor corresponds to a ray $\rho = \langle w\rangle$ in some cone σ_Y associated to some stratum Y. Since the cone σ_Y is smooth and simplicial, we can carry out the sequence of star subdivisions described above. This will yield a sequence of blow ups at strata eventually realizing the toroidal divisor corresponding to w. □

Proposition 4.12 *Let $(X, \mathbf{D})$ be an snc b-log variety. Let E be a divisor over X. If E is not toroidal then $b(E; X, D) > 0$. If E is toroidal then the centre of E on X is in a strata Y and E corresponds to a ray $\langle w\rangle$ in the cone $\sigma_Y = \langle v_1, \ldots, v_m\rangle$ with minimal generators v_i. Let r_w be the ramification index of E, let r_i be the ramification index of v_i, and write $w = a_1v_1 + \cdots + a_mv_m$. Then*

$$b(E; X, \mathbf{D}) = \frac{a_1 r_w}{r_1} + \cdots + \frac{a_m r_w}{r_m} - 1.$$

Proof This first statement follows by combining Corollary 4.11 and Theorem 4.4. For the second statement, we know by Corollary 4.11 that E can be obtained by blowing up strata and so will appear in a toroidal morphism that is étale locally toric along Y. So we can apply Proposition 4.5. □

Proposition 4.13 *Let $(X, \mathbf{D})$ be a toroidal b-log variety. Then there is a birational projective toroidal modification $f: X' \to X$ such that X' is $\mathbb{Q}$-factorial and the exceptional divisors of f are exactly the toroidal divisors D_w over X with $b(D_w; X, \mathbf{D}) \leq 0$.*

Proof As noted in Remark 2.29, there are only finitely many exceptional divisors over X whose b-discrepancies are non-positive. Alternatively, this can be seen by combining Theorem 4.4 and Proposition 4.6.

Let $\Delta = (|\Delta|, \sigma^Y, M^Y)$ be the conical polyhedral complex with integral structure associated to X and let S be the set of such divisors with non-positive b-discrepancies. Now take any divisor E in S. By Theorem 4.4, E is obtained by repeatedly blowing up strata. Hence there is a f.r.p.p. decomposition of Δ in which there exists a one-dimensional cone ρ_E corresponding to E.

So there is a vector $w \in M^Y$ in a cone σ^Y in Δ. We take the star subdivision $\Delta(w)$ of Δ, and repeat inductively for all $w \in S$, until we obtain Δ' a f.r.p.p decomposition of Δ whose set of one-dimensional cones is $\{\rho_E \mid E \in S\}$ together with those in Δ.

There exists a corresponding projective allowable modification $Y \to X$ by [25, Theorem 6*, p.90], which extracts only those divisors which are contained in S. □

Corollary 4.14 (Proof of Theorem 2.30 via toroidal modification) *Let $(X, \mathbf{D})$ be a b-log variety. Then there is a projective birational map $Y \to X$ such that $(Y, \mathbf{D})$ is $\mathbb{Q}$-factorial and b-terminal.*

Proof Let $(X, \mathbf{D})$ be a b-log variety. By Lemma 2.20, we find a log resolution $X_1 \to X$ of the log variety $(X, \mathbf{D}_X)$ so that the pair $(X_1, \mathbf{D}_{X_1})$ is snc. Note that $(X_1, \mathbf{D}_{X_1})$ is toroidal. Now by Proposition 4.13 we can find a projective allowable modification $Y \to X_1$ that extracts exactly the toroidal divisors D_w over X_1 with $b(D_w; X_1, \mathbf{D}) \leq 0$. By Theorem 4.4 these are all the divisors over X with $b(D_w; X_1, \mathbf{D}) \leq 0$. So by Lemma 2.28 we see that $(Y, \mathbf{D})$ is b-terminal, and as in the first proof of Theorem 2.30, the composition $Y \to X_1 \to X$ is a desired b-terminalization. □

5 Toric Brauer classes

Let X be a toric variety with open dense torus T of dimension n. We define a toric Brauer pair to be a pair (X, α) where $\alpha \in \operatorname{Br} T \simeq \wedge^2(\operatorname{Hom}(\mu, \mathbb{Q}/\mathbb{Z})^n)$ as noted in [13]. Following [13], we fix a primitive p-th root of unity so we have an isomorphism $\mathbb{Z}/p \simeq \mu_p$. At this point, p is an arbitrary non-zero integer, but we will often restrict to p being prime and note when this occurs. Then we associate a skew symmetric matrix to α $M_\alpha \in (\mathbb{Z}/p)^{n\times n}$, where p is the order of α. Let $\rho = \operatorname{Cone}(w)$ be a ray in $\mathbb{R}N$ generated by the primitive vector $w \in N$, and let $\overline{w} \in (\mathbb{Z}/p)^n$ be the reduction of w modulo p. They also show in [13, Lemma 1.7(b)] that the Brauer class α ramifies on the divisor $D_{\langle w\rangle}$ if $M_\alpha \overline{w}$ in $(\mathbb{Z}/p)^n$ is non-zero, and that the ramification index of α on $D_{\langle w\rangle}$ is the order of $M_\alpha \overline{w}$ in $(\mathbb{Z}/p)^n$.

Proposition 5.1 *Let (X, α) be a toric Brauer pair such that α has odd prime order p. If M_α has full rank then (X, α) is b-terminal (b-canonical, b-lt, b-lc) if and only if X is terminal (canonical, lt, lc).*

Proof Let $w = (a_1, \ldots, a_n)$ be a primitive vector in the lattice N. Since M_α has full rank $M_\alpha \overline{w} = 0$ if and only if $w \in pN$, but then w is not primitive. So the order of $M_\alpha w$ is p for all primitive w. So every toric divisor D has ramification index p, and so $r_D = p$. Then when we compute the discrepancy using the formula of (4.15), all $r_i = r_w = p$, and

$$b(D_w; X, \mathbf{D}) = (a_1 + \cdots + a_n - 1) = a(D_w, X).$$

So all log discrepancies equal the corresponding b-log discrepancies. □

Note that in dimension two, if (X, α) is b-terminal then X is terminal (equivalently smooth) as shown in [8]. This yields the following question.

Question 5.2 Is there a natural condition on a b-divisor $\mathbf{D}$, or a Brauer class α so that $(X, \mathbf{D})$ or (X, α) b-terminal implies X is terminal?

However, when p is odd, M_α must have even rank and so we cannot expect the above results to hold when both n and p are odd. Below we present an example which shows that Proposition 5.1 cannot be generalized to hold dimension 3.

Example 5.3 We present an example of a toric Brauer pair (X, α) in dimension 3 such that X is log terminal with the minimal discrepancy arbitrarily close to -1, whereas the b-log variety $(X, \mathbf{D}_\alpha)$ is b-terminal.

We will let X be the singularity $\frac{1}{r}(1, 1, 0)$, so that the minimal discrepancy is $-1 + 2/r$ and note that

$$-1 + \frac{2}{r} \to -1 \text{ as } r \to \infty.$$

The singularity X can be globally presented as a toric variety using the standard lattice $\mathbb{Z}^3$ and the cone generated by the columns of the following matrix

$$(v_1, v_2, v_3) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We fix a prime p and we let the skew symmetric matrix corresponding to the toric Brauer class α be

$$M_\alpha = \begin{pmatrix} 0 & 0 & -r \\ 0 & 0 & -1 \\ r & 1 & 0 \end{pmatrix} \in (\mathbb{Z}/p)^{3\times 3}. \tag{5.1}$$

In order to check that the pair $(X, \mathbf{D}_\alpha)$ is b-terminal, it is enough to check $b(D_{\langle w \rangle}; X, \mathbf{D}_\alpha) > 0$ for any primitive vector w in the cone. By using the formula (4.15), we can directly check this by elementary arguments.

When the dimension n is odd and we have rank $n - 1$, one can compute the discrepancy for an snc Brauer pair as follows, where we do the case $n = 3$ in detail.

Assume p is a prime so the ramification indices are p or 1. We also identify $\mathbb{Z}/p$ with μ_p^{-1} so that α is represented by a skew-symmetric matrix

$$M = \begin{pmatrix} 0 & c_2 & -c_1 \\ -c_2 & 0 & c_0 \\ c_1 & -c_0 & 0 \end{pmatrix}$$

where $c_j \in \mathbb{Z}/p$. We assume that $M \neq 0$ so that it has rank 2 and $\ker M$ is the $(\mathbb{Z}/p)$ span of the vector (c_0, c_1, c_2). We further assume that at least two of the c_j are non-zero so that α ramifies on all three planes. For $i = 0, \ldots, p-1$, $j = 0, 1, 2$ we let $c_{ij} \in \{0, \ldots, p-1\}$ be the smallest non-negative integer whose residue modulo p is ic_j.

Toric exceptional divisors $E_{(a_0,a_1,a_2)}$ above X correspond via the toric dictionary to primitive triples $(a_0, a_1, a_2) \in \mathbb{N}^3$. From [13, Lemma 1.7b], α is unramified along $E_{(a_0,a_1,a_2)}$ if and only if $(a_0, a_1, a_2) \in \ker M$ modulo p.

Given an integer x, define

$$r_p(x) = \min\{x + yp \mid x + yp \geq 0, y \in \mathbb{Z}\}$$

to be the least non-negative residue of x modulo p.

Proposition 5.4 *We use the above notation and let*

$$c = \min\{r_p(ic_0) + r_p(ic_1) + r_p(ic_2) \mid i \in (\mathbb{Z}/p)^*\}.$$

Then (X, α) is always b-lt, but will be

(1) *b-terminal if $c > p$, in which case $(c_0, c_1, c_2) \equiv (k, a, p-a) \pmod p$ for some $k, a \not\equiv 0$ up to permutation.*
(2) *b-canonical if $c = p$, in which case $\sum c_i \in p\mathbb{Z}$.*
(3) *b-lt and not b-canonical if $c < p$.*

Proof It suffices to compute discrepancy for toric exceptional divisors E. This is $b'(X, \alpha; E) = \frac{1}{p}(a_0 + a_1 + a_2) - \frac{1}{r_E}$ where r_E is the ramification index along $E = E_{(a_0,a_1,a_2)}$. Now $a_0 + a_1 + a_2 > 1$ so this is positive unless $e_E = 1$. In this case, $(a_0, a_1, a_2) \equiv i(c_0, c_1, c_2)$ modulo p for some i. Then b' is minimized when $(a_0, a_1, a_2) = (c_{i0}, c_{i1}, c_{i2})$ for some i whence we obtain $b'(X, \alpha; E) = \frac{1}{p}(c - p)$. Note that the minimum occurs when $r_E = 1$ and so $b'(X, \alpha; E) = b(X, \alpha; E)$, and we obtain the first part of each statement. To obtain the classifications in the first two cases, we use the well known characterization of toric terminal 3-fold singularities described in [30, Example-Claim 14-2-5]. For the last statement, it is clear that $b(X, \alpha; E) > -1$ for all E. □

To compute the discrepancies for b-log varieties that come from ramification information, it is necessary to compute the ramification indices globally before carrying out an étale localization. Since the ramification indices change after étale localization, we note that the discrepancy of a Brauer pair cannot be based on local information in the sense of the following example.

Example 5.5 Suppose that $p = 3$. We let α correspond to $(c_0, c_1, c_2) = (1, 2, 0)$ and α' correspond to $(c_0, c_1, c_2) = (1, 1, 0)$. Note that (X, α) is Brauer canonical but (X, α') is not. Furthermore, if $f : X' \to X$ is the blowup along the coordinate line $C : x_0 = x_1 = 0$, then the discrepancy of (X, α') along the exceptional divisor is negative. However, if $P \in C$ is a general point, then (X, α) and (X, α') are isomorphic in an étale neighbourhood of P.

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Authors and Affiliations

Daniel Chan[1] · Kenneth Chan[2] · Louis de Thanhoffer de Völcsey[3] · Colin Ingalls[4] · Kelly Jabbusch[5] · Sándor J. Kovács[2] · Rajesh Kulkarni[6] · Boris Lerner[1] · Basil Nanayakkara[7] · Shinnosuke Okawa[8] · Michel Van den Bergh[9]

Daniel Chan
danielc@unsw.edu.au

Kenneth Chan
kenhchan@math.washington.edu

Louis de Thanhoffer de Völcsey
louis.dethanhofferdevolcsey@utoronto.ca

Colin Ingalls
cingalls@math.carleton.ca

Kelly Jabbusch
jabbusch@umich.edu

Sándor J. Kovács
skovacs@uw.edu

Rajesh Kulkarni
kulkarni@math.msu.edu

Boris Lerner
blerner@gmail.com

Basil Nanayakkara
bnanayakkara@brocku.ca

Michel Van den Bergh
michel.vandenbergh@uhasselt.be

1 Department of Pure Mathematics, School of Mathematics and Statistics UNSW, Kensington, Australia

2 Department of Mathematics, University of Washington, Seattle, WA, USA

3 Department of Mathematics, University of Toronto, Toronto, ON, Canada

4 School of Mathematics and Statistics, Carleton University, Ottawa, ON, Canada

5 Department of Mathematics and Statistics, University of Michigan-Dearborn, Dearborn, MI, USA

6 Department of Mathematics, Michigan State University, East Lansing, MI, USA

7 Brock University, St. Catharines, ON, Canada

8 Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Japan

9 Universiteit Hasselt, Diepenbeek, Belgium