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Brane brick models for the Sasaki-Einstein 7-manifolds $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$

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Abstract: The 2d $(0,2)$ supersymmetric gauge theories corresponding to the classes of $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ manifolds are identified. The complex cones over these Sasaki-Einstein 7-manifolds are non-compact toric Calabi-Yau 4-folds. These infinite families of geometries are the largest ones for Sasaki-Einstein 7-manifolds whose metrics, toric diagrams, and volume functions are known explicitly. This work therefore presents the largest list of 2d $(0,2)$ supersymmetric gauge theories corresponding to Calabi-Yau 4-folds with known metrics.

Keywords: D-Branes, Differential and Algebraic Geometry, Supersymmetric Gauge Theory

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1 Introduction

The realization of gauge theories in various dimensions in terms of D-branes probing Calabi-Yau (CY) singularities is a powerful approach with multiple applications, including the gauge gravity correspondence, a geometric perspective on gauge theory dynamics, and model building, to name a few (see e.g. [1–5]). When the CYs are toric, the gauge theories are additionally endowed with beautiful combinatorial structures. For example, the $4d$ gauge theories on D3-branes probing toric CY 3-folds are captured by brane tilings [6, 7], also known as dimer models. Brane tilings and their generalizations led to trailblazing developments in connection with the AdS/CFT correspondence [8–11], cluster algebras [12, 13], integrable systems [14–17], scattering amplitudes [18, 19] and many other research areas in mathematics and theoretical physics.

Brane tilings provide a general solution to the correspondence between $4d$ gauge theories and toric CY_3's. They are also powerful enough to concretely deal with infinite families of CY_3's, among which those with explicitly known metrics are particularly interesting. A prominent example is given by the $Y^{p,q}$ family of Sasaki-Einstein manifolds, where $p \geq 0$ and $0 \leq q \leq p$, whose metrics were determined in [20].[1] Before the discovery of the metrics for the $Y^{p,q}$ manifolds, the only Sasaki-Einstein 5-manifolds with known metrics were S^5 and $T^{1,1}$ [21] (and their orbifolds), whose corresponding cones are $\mathbb{C}^3$ and the conifold, respectively. Explicit metrics were later introduced for the $L^{a,b,c}$ family, which contain and generalize the $Y^{p,q}$'s, and the corresponding brane tilings were found in [11, 22]. The matching in these theories between non-trivial gauge theory data, such as central charges and R-charges, and the corresponding geometric computations of volumes represented striking confirmations of the AdS/CFT correspondence with $\mathcal{N}=1$ SUSY. Remarkably, subsequent developments made it possible to determine many of these geometric quantities directly from the toric data, without requiring knowledge of the metric [23–26]. Brane tilings also allow for an elegant treatment of infinite families of orbifolds of various geometries [27, 28].

More recently, a similar program has focused on understanding the $2d$ $\mathcal{N}=(0,2)$ gauge theories on D1-branes probing toric CY 4-folds [29]. This program culminated with the introduction of *brane brick models*, a new class of type IIA brane configurations that are connected to the D1-branes at the singular CY 4-folds by T-duality [30]. Brane brick models fully encode the corresponding $2d$ (0,2) theories. Namely, they specify both the quivers and the J- and E-terms. For this reason, throughout this paper, we will consider determining the brane brick model and finding the $2d$ gauge theory for a given CY_4 as synonyms.[2] Brane brick models significantly simplify the map between geometry and the corresponding gauge theories. We refer the interested reader to [32–34] for further developments. Interestingly, in some cases, the brane brick models for certain CY 4-folds can be derived from brane tilings associated to appropriate CY 3-folds. Algorithms for achieving this include dimensional reduction [29], orbifold reduction [35] and $3d$ printing [36].

Brane brick models have been explicitly constructed for several toric CY 4-folds (see e.g. [29, 30, 32, 33] for various examples). More recently, brane brick models were found for all the 18 smooth Fano 3-folds, whose toric diagrams are given by regular reflexive lattice polytopes in 3 dimensions [37, 38]. In this work, we concentrate on a particular class of toric CY 4-folds that consists of cones over two families of Sasaki-Einstein 7-manifolds known as $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ [39], which can be regarded as natural generalizations of the $Y^{p,q}$ 5-manifolds.

In [40], it was shown that for any positive curvature Kähler-Einstein manifold B_{2n}, there is an infinite family of corresponding Sasaki-Einstein manifolds of the form $Y_{2n+3}(B_{2n})$. When B_4 is either $\mathbb{CP}^1\times\mathbb{CP}^1$ or $\mathbb{CP}^2$, we have the two families of 7-dimensional Sasaki-Einstein manifolds known as $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$, respectively. Our primary focus in this paper will be on these two families of Sasaki-Einstein 7-manifolds and the

[1]For brevity, throughout the paper, we will often use the same name to refer to the Sasaki-Einstein base and the corresponding cone. The distinction should be clear from the context.

[2]More precisely, the correspondence between toric CY_4's and brane brick models is not one-to-one but one-to-many, due to the existence of a low energy equivalence between $2d$ (0,2) gauge theories denoted triality [31, 32].

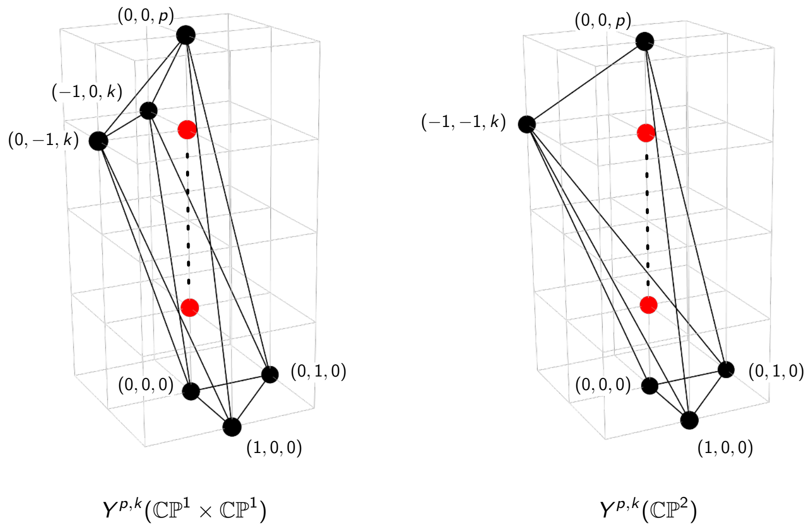

Figure 1. General toric diagrams corresponding to the $Y^{p,k}(\mathbb{CP}^2)$ and $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ families of toric CY 4-folds. The ranges for the parameters p and k are summarized in (2.7).

corresponding toric CY 4-folds. As previously mentioned, from here onwards we will use the names $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ interchangeably for both the Sasaki-Einstein 7-manifolds and for the corresponding CY 4-folds. The two families $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ stand out because their Sasaki-Einstein metrics are known explicitly [39, 40]. The general toric diagrams for the corresponding CY 4-folds are show in figure 1.

In this work, we take these two infinite families of toric CY 4-folds and construct the $2d$ $(0,2)$ gauge theories on D1-branes probing them.[3] The mesonic moduli space for each of these theories agrees with the desired CY_4, confirming that the proposed gauge theories are indeed correct.

The following work is structured as follows. In section 2, a brief overview of brane brick models and the families of Sasaki-Einstein 7-manifolds $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ is given. Then, in sections 3 and 4, we present the general form of the field content and J- and E-terms for the $2d$ $(0,2)$ supersymmetric gauge theories for $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$, respectively. We illustrate the constructions with explicit examples for cases $p = 2, 1$ where $p \leq k$. By calculating the mesonic moduli space, we verify that the proposed $2d$ $(0,2)$ gauge theories and the associated brane brick models correspond to the desired geometries. Moreover, we identify the global symmetries of these theories, which match the isometry groups for the corresponding Sasaki-Einstein 7-manifolds identified in [39, 40].

2 Background

The following sections give a brief introductory overview of the two infinite families of Sasaki-Einstein 7-manifolds $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ and the realization of $2d$ $(0,2)$ supersymmetric gauge theories through brane brick models.

[3] More precisely, we construct one gauge theory for each CY_4 in the family. Generically, we expect multiple dual theories related by triality for each geometry. We focus on phases described by brane brick models, which are commonly referred to as toric phases [32].

2.1 $Y^{p,k}$ Sasaki-Einstein 7-manifolds

In [40], it was found that for every $2n$-dimensional Kähler-Einstein manifold B_{2n} there is an infinite family of compact Sasaki-Einstein manifolds Y_{2n+3} of dimension $2n{+}3$ [39]. For $n = 2$, we have 4-dimensional Kähler-Einstein bases B_4, either $\mathbb{CP}^1\times\mathbb{CP}^1$ or $\mathbb{CP}^2$, and two associated infinite families of Sasaki-Einstein 7-manifolds of the form $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$, respectively. These infinite families of Sasaki-Einstein manifolds can be thought of as Lens space bundles $S^3/\mathbb{Z}_p$ over the base Kähler-Einstein manifolds $\mathbb{CP}^1\times\mathbb{CP}^1$ and $\mathbb{CP}^2$ [39].

The isometry of the Sasaki-Einstein 7-manifolds takes the general form $H \times \mathrm{U}(1)^2$, where H is the isometry of the base B_4. For $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$, the isometries are $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2$ and $\mathrm{SU}(3)\times\mathrm{U}(1)^2$, respectively. These isometries translate into the global symmetries of the corresponding brane brick models.

The toric CY 4-folds X_4 have the following metric

$$\mathrm{d}s^2(X_4) = \mathrm{d}r^2 + r^2\mathrm{d}s^2(Y_7)\,, \tag{2.1}$$

where the local metric $\mathrm{d}s^2(Y_7)$ of the Sasaki-Einstein 7-manifold Y_7 is presented explicitly in [39]. Here, rather than looking at the precise form of the metric, we note that a coordinate α of the metric can be periodically identified where the periods are related to $p, k \in \mathbb{Z}$. The periodic identification of α is associated to a principal $\mathrm{U}(1)$ bundle over a space M_6, which is the total space of an S^2 bundle over the base B_4. The resulting range of k is fixed to be of the following form

$$\frac{hp}{2} \le k \le hp\,, \tag{2.2}$$

where $h \in \mathbb{Z}$ is related to Chern numbers of the principal $\mathrm{U}(1)$ bundles according to [39]. For our two infinite families, the range of k takes therefore the following forms

$$\begin{aligned} Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1):&\quad p \le k \le 2p\,,\\ Y^{p,k}(\mathbb{CP}^2):&\quad \frac{3p}{2} \le k \le 3p\,. \end{aligned} \tag{2.3}$$

Under these ranges for k, the toric diagrams for $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ were identified in [39] as the convex hull of the following extremal points

$$\begin{aligned} Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1):&\quad \left(\begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6\\ \hline 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1\\ 0 & p & 0 & k & 0 & k \end{array}\right),\\ Y^{p,k}(\mathbb{CP}^2):&\quad \left(\begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5\\ \hline 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\\ 0 & p & 0 & 0 & k \end{array}\right), \end{aligned} \tag{2.4}$$

where all points lie on a 4-dimensional plane at distance 1 from the origin.

The toric diagrams of toric CY 4-folds are equivalent up to $\mathrm{GL}(3,\mathbb{Z})$ transformations. As such, we can find a suitable transformation to redefine the coordinates of the toric diagrams for $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ in (2.4) as follows

$$
\begin{aligned}
Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1):\qquad & w_a\cdot\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ p & p & -1\end{pmatrix}\to\left(\begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6\\ \hline 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1\\ 0 & -p & -p & p-k & -p & p-k\end{array}\right),\\
Y^{p,k}(\mathbb{CP}^2):\quad & w_a\cdot\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -p & -p & -1\end{pmatrix}\to\left(\begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5\\ \hline 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\\ 0 & -p & -p & -p & 2p-k\end{array}\right),
\end{aligned}\tag{2.5}
$$

where for each point the $x^{(3)}$-coordinate can be shifted in the positive direction by p to give respectively,

$$
\begin{aligned}
Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1):\quad & \left(\begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6\\ \hline 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1\\ p & 0 & 0 & 2p-k & 0 & 2p-k\end{array}\right),\\
Y^{p,k}(\mathbb{CP}^2):\quad & \left(\begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5\\ \hline 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1\\ p & 0 & 0 & 0 & 3p-k\end{array}\right).
\end{aligned}\tag{2.6}
$$

By swapping the roles of w_1 and w_2, and by redefining the parameter k as $k'=2p-k$ and $k'=3p-k$ for $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$, respectively, we can redefine the bounds on k in (2.3) into a more convenient form as follows

$$
\begin{aligned}
Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1):\quad & 0\le k\le p\,,\\
Y^{p,k}(\mathbb{CP}^2):\quad & 0\le k\le \frac{3}{2}p\,.
\end{aligned}\tag{2.7}
$$

In this work, for both $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ and $Y^{p,k}(\mathbb{CP}^2)$ families, we will include the upper and lower limits for k, since they correspond to additional distinct toric CY 4-folds. Using these new bounds on the redefined parameter k, we note that for $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ when $k=0$, the toric CY 4-fold is the abelian orbifold of the form $\mathcal{C}\times\mathbb{C}/\mathbb{Z}_p$. In the other limit when $k=p$, $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ becomes the abelian orbifold of the form $Q^{1,1,1}/\mathbb{Z}_p$ as illustrated in figure 2. We further note that when $k=p=2m$ with $m\in\mathbb{Z}^+$, $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$ corresponds to another abelian orbifold of the form $Q^{1,1,1}/\mathbb{Z}_2\times\mathbb{Z}_m$ as illustrated in figure 2. Similarly, we note that for $Y^{p,k}(\mathbb{CP}^2)$, when $k=0$, we have the abelian orbifold of the form $\mathbb{C}^4/\mathbb{Z}_3\times\mathbb{Z}_p$, and in the limit when $p=\frac{2}{3}k=2m$ with $m\in\mathbb{Z}^+$, the toric CY 4-fold is the abelian orbifold of the form $M^{3,2}/\mathbb{Z}_m$ as illustrated in figure 3.

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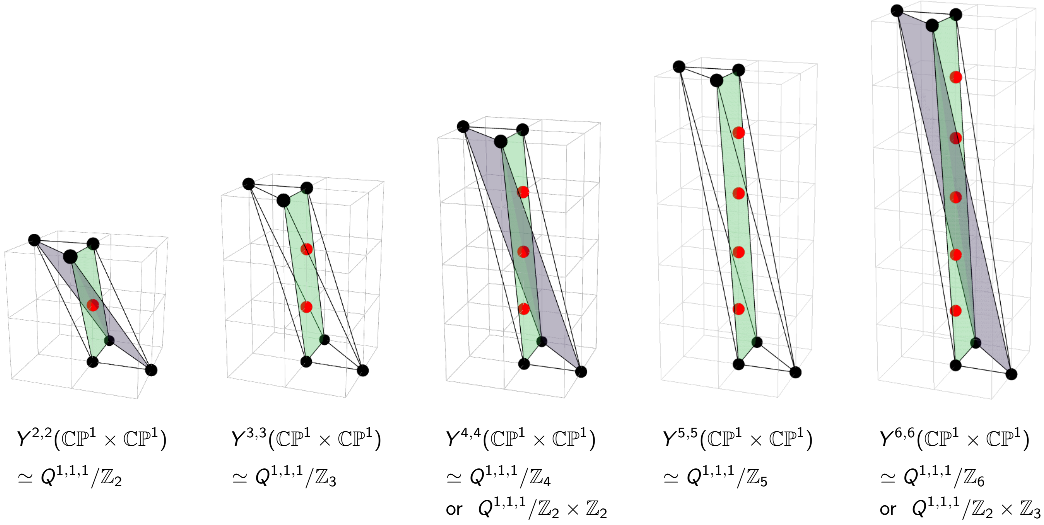

Figure 2. $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ with $p = k$ is equivalent to the abelian orbifolds of the form $Q^{1,1,1}/\mathbb{Z}_p$. If additionally $p = k$ is even, $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is also equivalent to another abelian orbifold of the form $Q^{1,1,1}/\mathbb{Z}_2 \times \mathbb{Z}_p$. The corresponding $2d$ $(0,2)$ theories can be obtained from orbifold reduction [35] of the $4d$ $\mathcal{N} = 1$ theory corresponding to F_0 or an abelian orbifold of F_0. The planes shown in gray correspond to the toric diagram for F_0 and the planes shown in green correspond to the toric diagrams for abelian orbifolds of the form $\mathcal{C}/\mathbb{Z}_m$, where $\mathcal{C}$ is the conifold.

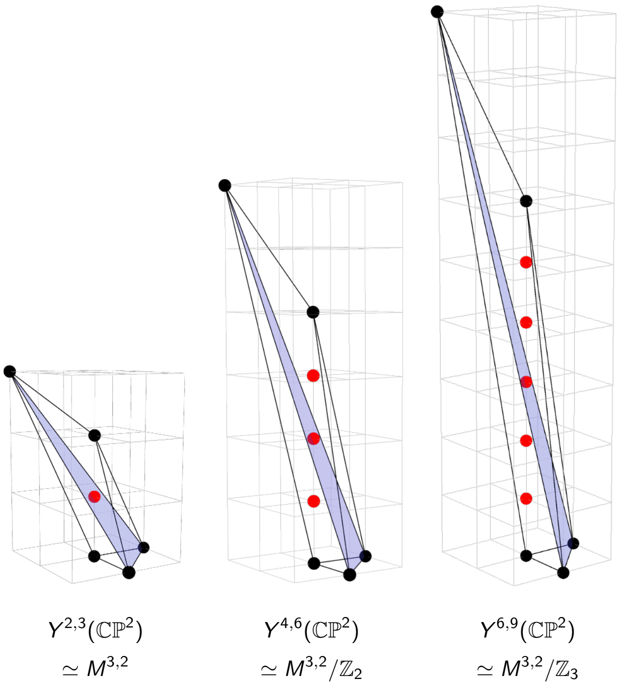

Figure 3. $Y^{p,k}(\mathbb{CP}^2)$ with $p = \frac{2}{3}k = 2m$ and $m \in \mathbb{Z}^+$ is equivalent to the abelian orbifolds of the form $M^{3,2}/\mathbb{Z}_m$. The corresponding $2d$ $(0,2)$ theories can be obtained via orbifold reduction [35] of the $4d$ $\mathcal{N} = 1$ theory corresponding to dP_0. The highlighted plane indicates the toric diagram for dP_0.

	0	1	2	3	4	5	6	7	8	9
D4	×	×	×	·	×	·	×	·	·	·
NS5	×	×	——	——	——	Σ	——	——	·	·

Table 1. The Type IIA brane configuration for brane brick models.

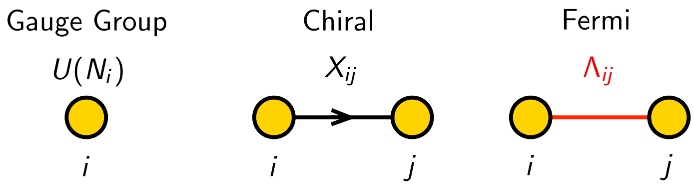

Figure 4. Basic building blocks of $2d$ $(0,2)$ quiver diagrams.

2.2 $2d$ $(0,2)$ theories and brane brick models

Brane brick models are Type IIA brane configurations connected by T-duality to D1-branes probing toric CY 4-folds. They fully encode the $2d$ $(0,2)$ gauge theories on the worldvolume of the D1-branes and significantly simplify the connection between the probed geometry and the corresponding field theory [30]. Brane brick models consist of D4-branes wrapping a 3-torus T^3, suspended from an NS5-brane that wraps a holomorphic surface Σ. Here, the holomorphic surface Σ is the zero locus of the Newton polynomial of the toric diagram of the corresponding toric CY 4-fold. Table 1 summarizes the main ingredients in these configurations.

The gauge symmetry and matter content of the type of $2d$ $(0,2)$ gauge theories considered in this paper can be encoded in terms of generalized quiver diagrams. $\mathrm{U}(N_i)$ gauge groups are represented by nodes, whereas chiral X_{ij} and Fermi Λ_{ij} fields are represented by oriented black and unoriented red lines, respectively, as illustrated in figure 4.

Every Fermi field Λ_{ij} is associated to a pair of holomorphic functions of chiral fields $J[\Lambda_{ij}]$ and $E[\Lambda_{ij}]$ [29, 31, 41]. In order to form gauge invariant terms in the Lagrangian, $E[\Lambda_{ij}]$ has the same gauge quantum numbers as Λ_{ij} while $J[\Lambda_{ij}]$ has the conjugate gauge quantum numbers. The structure of J- and E-terms in theories associated to brane brick models are further constrained to have the following binomial form

$$J[\Lambda_{ij}] = J_+[\Lambda_{ij}] - J_-[\Lambda_{ij}]\,, \;\; E[\Lambda_{ij}] = E_+[\Lambda_{ij}] - E_-[\Lambda_{ij}]\,, \tag{2.8}$$

with $J_\pm[\Lambda_{ij}]$ and $E_\pm[\Lambda_{ij}]$ holomorphic monomials in chiral fields. This constraint is commonly known as the *toric condition* [29] and ensures that the associated CY 4-folds are toric.

The $2d$ $(0,2)$ gauge theories for toric CY 4-folds can be equivalently encoded by periodic quivers living on T^3, which are connected to brane brick models by graph dualization. In this periodic quiver, J- and E-terms are mapped to minimal gauge invariant closed loops, which we denote as plaquettes, which consist of an oriented path of chiral fields and a single Fermi field, the corresponding Λ_{ij}. The Fermi field in a plaquette appears without conjugation for J-terms and conjugated for E-terms. Therefore, as a result of the toric condition, a Fermi field Λ_{ij} is associated to four plaquettes in the periodic quiver, where pairs of plaquettes are identified with the binomials in (2.8). This structure is illustrated in figure 5.

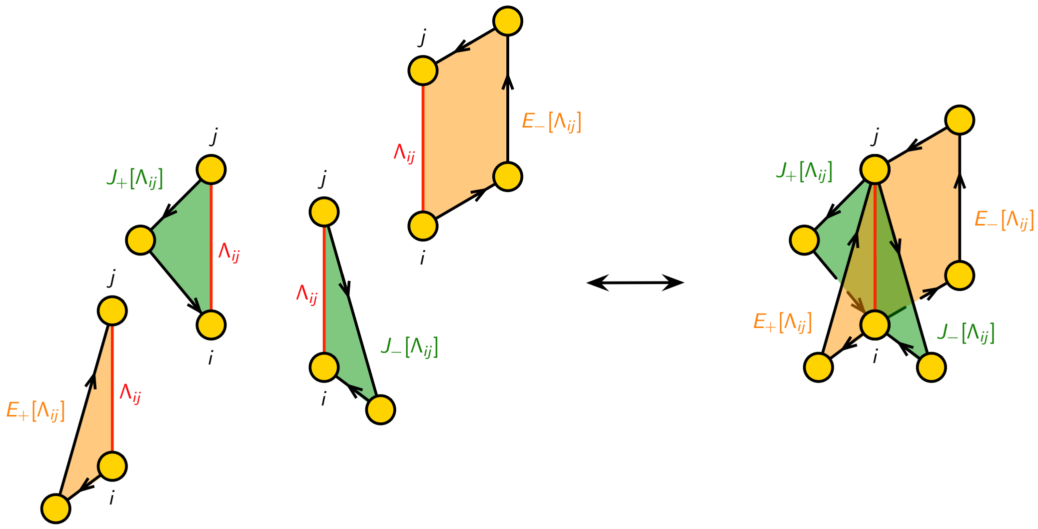

Figure 5. The four plaquettes associated to $J_\pm[\Lambda_{ij}]$ and $E_\pm[\Lambda_{ij}]$ for a Fermi field Λ_{ij} in the periodic quiver dual to a brane brick model.

The J- and E-terms of any $2d$ $(0,2)$ gauge theory must satisfy the *trace condition* [29]

$$\sum_a \mathrm{Tr}(E[\Lambda_a]J[\Lambda_a]) = 0\,, \tag{2.9}$$

where the sum is over the Fermi fields Λ_a. $2d$ $(0,2)$ theories are invariant under the individual exchanges $\Lambda_a \leftrightarrow \overline{\Lambda}_a$, which simultaneously exchanges the roles of $J_a \leftrightarrow E_a$. A discussion on how brane brick models implement the trace condition can be found in [42].

We focus on the case when the ranks of all gauge groups are equal, such that $N_i = N$. In this case, the non-abelian $\mathrm{SU}(N_i)^2$ anomaly cancellation condition at each node i [29] reduces to

$$n_i^\chi - n_i^F = 2\,, \tag{2.10}$$

where n_i^χ is the total number of incoming plus outgoing chiral bifundamental fields at node i, and n_i^F is the total number of bifundamental Fermi fields connected to node i. Since adjoint fields correspond to lines in the quiver that start and end on the same node, each of them contributes 2 to these numbers.

3 $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ models

3.1 Construction of the $2d$ $(0,2)$ theories

The $2d$ $(0,2)$ gauge theories associated to the $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ family are parameterized in terms of $p, k \in \mathbb{Z}$, where we can set the range for k to be $0 \le k \le p$ following the discussion in section 2.1.

The total number of gauge groups is $4p$, which corresponds to the volume of the toric diagram of $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$. For the gauge group labels on the bifundamental chiral and Fermi fields we use the following notation,

$$[m] \equiv ((m-1) \bmod 4p) + 1\,. \tag{3.1}$$

The chiral and Fermi fields form a periodic layered structure with a total number of p layers. Each layer contains 4 gauge groups, which is the same number of gauge groups as in the $4d$ $\mathcal{N}=1$ theories corresponding to F_0 [43], and the periodicity is manifested by the modularity in the notation in (3.1). This periodic layered structure in the quiver originates from the fact that the $2d$ $(0,2)$ theories were obtained using orbifold reduction [35], $3d$ printing [36] and partial resolution via higgsing [29], starting with the $4d$ $\mathcal{N}=1$ gauge theory corresponding to the first phase of F_0 [43, 44].

The gauge group indices are shifted by additional indices between each layer of the quiver,

$$a=0,\dots,p-1\,,\ b=k,\dots,p-1\,,\ c=0,\dots,k-1\,. \tag{3.2}$$

Finally, we also have indices for the two non-abelian factors of the global symmetry $\mathrm{SU}(2)^2\times\mathrm{U}(1)^2$,

$$i,j,k,l=1,2\,. \tag{3.3}$$

Under these labels, the Fermi fields take the form

$$\begin{aligned}&\Lambda^{i}_{[4a+1][4a+2]}\,,\ \Lambda^{2i}_{[4b+6][4b+3]}\,,\ \Lambda^{2i}_{[4b+7][4b+4]}\,,\ \Lambda^{2i}_{[4b+8][4b+1]}\,,\\&\Lambda^{1i}_{[4c+4][4c+7]}\,,\ \Lambda^{2i}_{[4c+8][4c+3]}\,,\end{aligned} \tag{3.4}$$

and the chiral fields take the form

$$\begin{aligned}&X^{i}_{[4a+2][4a+3]}\,,\ X^{i}_{[4a+3][4a+4]}\,,\ X^{i}_{[4a+4][4a+1]}\,,\ X^{i}_{[4b+1][4b+6]}\,,\\&Q_{[4c+5][4c+1]}\,,\ Q_{[4c+6][4c+2]}\,,\ Q_{[4c+7][4c+3]}\,,\ Q_{[4c+8][4c+4]}\,,\\&Q_{[4c+5][4c+3]}\,,\ Q_{[4c+8][4c+2]}\,,\ P_{[4c+1][4c+7]}\,,\ P_{[4c+4][4c+6]}\,.\end{aligned} \tag{3.5}$$

Above, the chiral fields of the form X^i_{rs} originate from the $4d$ $\mathcal{N}=1$ theory corresponding to F_0 with $4d$ chiral fields $\mathcal{X}^i_{rs}$ and superpotential of the form

$$W=\epsilon_{ij}\epsilon_{kl}\mathcal{X}^i_{12}\mathcal{X}^k_{23}\mathcal{X}^j_{34}\mathcal{X}^l_{41}\,. \tag{3.6}$$

The chiral fields of the form X^i_{rs} form layers in the periodic quiver on T^3. Spanned between these layers are chiral fields of the form Q_{rs} and P_{rs} in (3.5). Combined with the Fermi fields in (3.4), two adjacent layers form what is known as the *quiver block* [36] in the periodic quiver of the $2d$ $(0,2)$ theory. The indices a,b,c in (3.4) and (3.5) ensure that p copies of quiver blocks are stacked together in a consistent way such that equivalent fields and gauge groups are identified to each other. The p copies of quiver blocks that are stacked together form the periodic quiver on T^3 for the $2d$ $(0,2)$ theories corresponding to $Y^{p,k}(\mathbb{CP}^1\times\mathbb{CP}^1)$.

From the periodic quiver, we obtain the J-terms, which take the following general form

$$\begin{aligned}J[\Lambda^{i}_{[4a+1][4a+2]}]&=\epsilon_{ij}\epsilon_{kl}X^k_{[4a+2][4a+3]}X^j_{[4a+3][4a+4]}X^l_{[4a+4][4a+1]}\,,\\J[\Lambda^{2i}_{[4b+6][4b+3]}]&=\epsilon_{ij}\epsilon_{kl}X^k_{[4b+3][4b+4]}X^j_{[4b+4][4b+1]}X^l_{[4b+1][4b+6]}\,,\\J[\Lambda^{2i}_{[4b+7][4b+4]}]&=\epsilon_{ij}\epsilon_{kl}X^k_{[4b+4][4b+1]}X^j_{[4b+1][4b+6]}X^l_{[4b+6][4b+7]}\,,\\J[\Lambda^{2i}_{[4b+8][4b+1]}]&=\epsilon_{ij}\epsilon_{kl}X^k_{[4b+1][4b+6]}X^j_{[4b+6][4b+7]}X^l_{[4b+7][4b+8]}\,,\\J[\Lambda^{1i}_{[4c+4][4c+7]}]&=\epsilon_{ij}\epsilon_{kl}X^k_{[4c+7][4c+8]}X^j_{[4c+8][4c+5]}Q_{[4c+5][4c+3]}X^l_{[4c+3][4c+4]}\,,\\J[\Lambda^{2i}_{[4c+8][4c+3]}]&=\epsilon_{ij}\epsilon_{kl}X^k_{[4c+3][4c+4]}X^j_{[4c+4][4c+1]}P_{[4c+1][4c+7]}X^l_{[4c+7][4c+8]}\,,\end{aligned} \tag{3.7}$$

with the corresponding E-terms taking the form

$$
\begin{aligned}
E[\Lambda^{2i}_{[4b+6][4b+3]}] &= X^i_{[4b+6][4b+7]}Q_{[4b+7][4b+3]} - Q_{[4b+6][4b+2]}X^i_{[4b+2][4b+3]}\,,\\
E[\Lambda^{2i}_{[4b+7][4b+4]}] &= X^i_{[4b+7][4b+8]}Q_{[4b+8][4b+4]} - Q_{[4b+7][4b+3]}X^i_{[4b+3][4b+4]}\,,\\
E[\Lambda^{2i}_{[4b+8][4b+1]}] &= X^i_{[4b+8][4b+5]}Q_{[4b+5][4b+1]} - Q_{[4b+8][4b+4]}X^i_{[4b+4][4b+1]}\,,\\
E[\Lambda^{1i}_{[4c+4][4c+7]}] &= X^i_{[4c+4][4c+1]}P_{[4c+1][4c+7]} - P_{[4c+4][4c+6]}X^i_{[4c+6][4c+7]}\,,\\
E[\Lambda^{2i}_{[4c+8][4c+3]}] &= X^i_{[4c+8][4c+5]}Q_{[4c+5][4c+3]} - Q_{[4c+8][4c+2]}X^i_{[4c+2][4c+3]}\,.
\end{aligned}
\tag{3.8}
$$

Additional E-terms plaquettes remain as follows

$$
\begin{aligned}
E_+[\Lambda^i_{[4b+1][4b+2]}] &= X^i_{[4b+1][4b+6]}Q_{[b+5][b+1]}\,,\\
E_-[\Lambda^i_{[4b+5][4b+6]}] &= -Q_{[4b+5][4b+1]}X^i_{[4b+1][4b+6]}\,,\\
E_+[\Lambda^i_{[4c+1][4c+2]}] &= P_{[4c+1][4c+7]}X^i_{[4c+7][4c+8]}Q_{[4c+8][4c+2]}\,,\\
E_-[\Lambda^i_{[4c+5][4c+6]}] &= -Q_{[4c+5][4c+3]}X^i_{[4c+3][4c+4]}P_{[4c+4][4c+6]}\,,
\end{aligned}
\tag{3.9}
$$

which combine to proper E-terms of the form $E = E_+ - E_-$. The E-terms plaquettes are shown here separately because quadratic and cubic plaquettes are combined depending on the value of k in order to form a single E-term.

3.2 Examples

In order to illustrate the construction of the $2d$ $(0,2)$ theories corresponding to $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$, in this section we go through all the cases where $p = 2$. We note that geometries with $p = 2$ and $k > 0$ correspond to smooth Fano 3-folds that have been studied extensively in [38]. For completeness, appendix A.1 summarizes also the J- and E-terms for the $2d$ $(0,2)$ theories when $p = 1$.

3.2.1 $Y^{2,0}(\mathbb{CP}^1 \times \mathbb{CP}^1)$

We begin with the $2d$ $(0,2)$ theory for $Y^{2,0}(\mathbb{CP}^1 \times \mathbb{CP}^1)$, whose quiver diagram is shown in figure 6. The J- and E-terms for this theory take the following form

$$
\begin{array}{rclc}
 & J & & E \\
\Lambda^i_{12} & : & \epsilon_{ij}\epsilon_{kl}X^k_{23}X^j_{34}X^l_{41} & X^i_{16}Q_{62} - Q_{15}X^i_{52} \\
\Lambda^{2i}_{27} & : & \epsilon_{ij}\epsilon_{kl}X^k_{78}X^j_{85}X^l_{52} & X^i_{23}Q_{37} - Q_{26}X^i_{67} \\
\Lambda^{2i}_{38} & : & \epsilon_{ij}\epsilon_{kl}X^k_{85}X^j_{52}X^l_{23} & X^i_{34}Q_{48} - Q_{37}X^i_{78} \\
\Lambda^{2i}_{45} & : & \epsilon_{ij}\epsilon_{kl}X^k_{52}X^j_{23}X^l_{34} & X^i_{41}Q_{15} - Q_{48}X^i_{85} \\
\Lambda^i_{56} & : & \epsilon_{ij}\epsilon_{kl}X^k_{67}X^j_{78}X^l_{85} & X^i_{52}Q_{26} - Q_{51}X^i_{16} \\
\Lambda^{2i}_{63} & : & \epsilon_{ij}\epsilon_{kl}X^k_{34}X^j_{41}X^l_{16} & X^i_{67}Q_{73} - Q_{62}X^i_{23} \\
\Lambda^{2i}_{74} & : & \epsilon_{ij}\epsilon_{kl}X^k_{41}X^j_{16}X^l_{67} & X^i_{78}Q_{84} - Q_{73}X^i_{34} \\
\Lambda^{2i}_{81} & : & \epsilon_{ij}\epsilon_{kl}X^k_{16}X^j_{67}X^l_{78} & X^i_{85}Q_{51} - Q_{84}X^i_{41}
\end{array}\,,
\tag{3.10}
$$

where the global symmetry indices are $i, j, k, l = 1, 2$.

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The P-matrix takes the form

$$P = \left(\begin{array}{c|ccccc|cccc|cccccccc|cccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & q_1 & q_2 & q_3 & q_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\
\hline
X^1_{16} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
X^1_{23} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
X^1_{34} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
X^1_{41} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
X^1_{52} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
X^1_{67} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
X^1_{78} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
X^1_{85} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
X^2_{16} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
X^2_{23} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
X^2_{34} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
X^2_{41} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
X^2_{52} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
X^2_{67} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
X^2_{78} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
X^2_{85} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
Q_{15} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
Q_{26} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
Q_{37} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
Q_{48} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
Q_{51} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
Q_{62} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
Q_{73} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
Q_{84} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1
\end{array}\right). \tag{3.11}$$

The associated J- and E-term U(1)-charge matrix takes the following form

$$Q_{JE} = \left(\begin{array}{ccccc|cccc|cccccccc|cccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & q_1 & q_2 & q_3 & q_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\
\hline
1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 3 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0
\end{array}\right), \tag{3.12}$$

and the corresponding D-term U(1)-charge matrix takes the form

$$Q_D = \left(\begin{array}{ccccc|cccc|cccccccc|cccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & q_1 & q_2 & q_3 & q_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right). \tag{3.13}$$

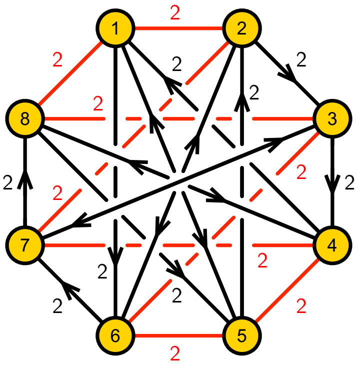

Figure 6. Quiver for $Y^{2,0}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

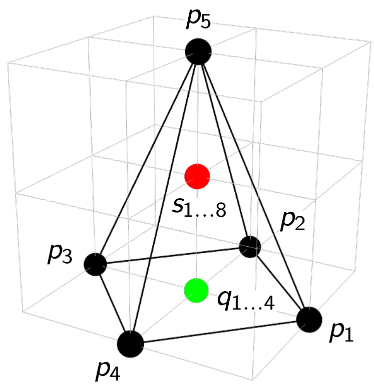

Figure 7. Toric diagram of $Y^{2,0}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

We observe that the charge matrices are invariant under

$$p_1 \leftrightarrow p_3\,, \;\; p_2 \leftrightarrow p_4\,. \tag{3.14}$$

Each of the above Z_2-symmetries correspond to an SU(2) factor in the global symmetry of the $2d$ $(0,2)$ gauge theory.

The corresponding toric data is given by

$$G_t = \left(\begin{array}{ccccc|cccc|cccccccc|cccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & q_1 & q_2 & q_3 & q_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right), \tag{3.15}$$

where the GLSM fields o_k correspond to extra GLSM fields in the nomenclature of [29]. The toric diagram is shown in figure 7 and it agrees with the expectation. We note that the $2d$ $(0,2)$ theory for $Y^{2,0}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ can be constructed using orbifold reduction [35], starting from the $4d$ $\mathcal{N}=1$ theory for the first phase of F_0.

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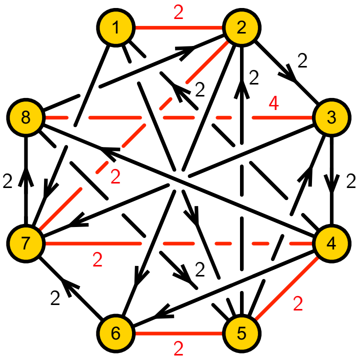

Figure 8. Quiver for $Y^{2,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

3.2.2 $\boldsymbol{Y^{2,1}}(\mathbb{CP}^1 \times \mathbb{CP}^1)$

The J- and E-terms of the 2d $(0,2)$ theory corresponding to $Y^{2,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ are

$$
\begin{array}{rclccc}
 & & J & & E & \\
\Lambda_{12}^{i} & : & \epsilon_{ij}\epsilon_{kl} X_{23}^{k} X_{34}^{j} X_{41}^{l} & P_{17} X_{78}^{i} Q_{82} & - & Q_{15} X_{52}^{i} \\
\Lambda_{27}^{2i} & : & \epsilon_{ij}\epsilon_{kl} X_{78}^{k} X_{85}^{j} X_{52}^{l} & X_{23}^{i} Q_{37} & - & Q_{26} X_{67}^{i} \\
\Lambda_{38}^{2i} & : & \epsilon_{ij}\epsilon_{kl} X_{85}^{k} X_{52}^{j} X_{23}^{l} & X_{34}^{i} Q_{48} & - & Q_{37} X_{78}^{i} \\
\Lambda_{45}^{2i} & : & \epsilon_{ij}\epsilon_{kl} X_{52}^{k} X_{23}^{j} X_{34}^{l} & X_{41}^{i} Q_{15} & - & Q_{48} X_{85}^{i} \\
\Lambda_{47}^{1i} & : & \epsilon_{ij}\epsilon_{kl} X_{78}^{k} X_{85}^{j} Q_{53} X_{34}^{l} & X_{41}^{i} P_{17} & - & P_{46} X_{67}^{i} \\
\Lambda_{56}^{i} & : & \epsilon_{ij}\epsilon_{kl} X_{67}^{k} X_{78}^{j} X_{85}^{l} & X_{52}^{i} Q_{26} & - & Q_{53} X_{34}^{i} P_{46} \\
\Lambda_{83}^{2i} & : & \epsilon_{ij}\epsilon_{kl} X_{34}^{k} X_{41}^{j} P_{17} X_{78}^{l} & X_{85}^{i} Q_{53} & - & Q_{82} X_{23}^{i}
\end{array}
\quad , \tag{3.16}
$$

where the corresponding quiver diagram is shown in figure 8. The global symmetry indices are $i, j, k, l = 1, 2$.

The corresponding P-matrix takes the form

$$
P = \left(
\begin{array}{c|cccccc|ccccccccc|ccccccc|ccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\
\hline
X_{23}^{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
X_{34}^{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
X_{41}^{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
X_{52}^{1} & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
X_{67}^{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
X_{78}^{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
X_{85}^{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
X_{23}^{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
X_{34}^{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
X_{41}^{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
X_{52}^{2} & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
X_{67}^{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
X_{78}^{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
X_{85}^{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
Q_{15} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
Q_{26} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
Q_{37} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
Q_{48} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
Q_{53} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
Q_{82} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
P_{17} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
P_{46} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}
\right) , \tag{3.17}
$$

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and the corresponding U(1)-charge matrix under the J- and E-terms takes the form

$$Q_{JE}=\left(\begin{array}{cccccc|ccccccccc|ccccccc|ccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\
\hline
3 & 1 & 3 & 1 & 0 & 0 & 3 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\
2 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
2 & 1 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 \\
2 & 1 & 2 & 1 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right). \tag{3.18}$$

The charge matrix for the D-terms takes the form

$$Q_{D}=\left(\begin{array}{cccccc|ccccccccc|ccccccc|ccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right). \tag{3.19}$$

We note that the charge matrices are invariant under the following swap of GLSM fields,

$$p_1 \leftrightarrow p_3\,,\ \ p_2 \leftrightarrow p_4\,. \tag{3.20}$$

This further illustrates that the global symmetry is $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2$, which matches with the isometry group on $Y^{2,1}(\mathbb{CP}^1\times\mathbb{CP}^1)$.

The toric data for the brane brick model is given by

$$G_t=\left(\begin{array}{cccccc|ccccccccc|ccccccc|ccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2
\end{array}\right), \tag{3.21}$$

where we identify the o_k as extra GLSM fields [29]. The corresponding toric diagram is shown in figure 9 and it agrees with the expectation. We note that $Y^{2,1}(\mathbb{CP}^1\times\mathbb{CP}^1)$ is one of the Fano 3-folds studied in [38]. In fact, to be precise, it corresponds to Model 9 in [38]. Given the fact that the $2d$ $(0,2)$ theory identified here is different from the one given for the same geometry in [38], we expect both theories to be related by a (sequence of) triality transformation(s) [32].

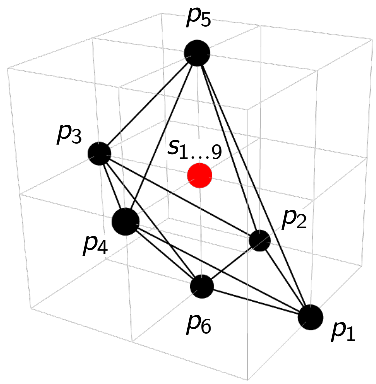

Figure 9. Toric diagram of $Y^{2,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

3.2.3 $\boldsymbol{Y^{2,2}}(\mathbb{CP}^1 \times \mathbb{CP}^1)$

The quiver diagram for the $2d$ $(0,2)$ theory corresponding to $Y^{2,2}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is shown in figure 10. The J- and E-terms for the theory take the following form

$$\begin{array}{rccc} & J & & E \\ \Lambda^{i}_{12}: & \epsilon_{ij}\epsilon_{kl}X^{k}_{23}X^{j}_{34}X^{l}_{41} & & P_{17}X^{i}_{78}Q_{82} - Q_{17}X^{i}_{78}P_{82} \\ \Lambda^{1i}_{47}: & \epsilon_{ij}\epsilon_{kl}X^{k}_{78}X^{j}_{85}Q_{53}X^{l}_{34} & & X^{i}_{41}P_{17} - P_{46}X^{i}_{67} \\ \Lambda^{2i}_{47}: & \epsilon_{ij}\epsilon_{kl}X^{k}_{78}X^{j}_{85}P_{53}X^{l}_{34} & & X^{i}_{41}Q_{17} - Q_{46}X^{i}_{67} \\ \Lambda^{i}_{56}: & \epsilon_{ij}\epsilon_{kl}X^{k}_{67}X^{j}_{78}X^{l}_{85} & & P_{53}X^{i}_{34}Q_{46} - Q_{53}X^{i}_{34}P_{46} \\ \Lambda^{1i}_{83}: & \epsilon_{ij}\epsilon_{kl}X^{k}_{34}X^{j}_{41}Q_{17}X^{l}_{78} & & X^{i}_{85}P_{53} - P_{82}X^{i}_{23} \\ \Lambda^{2i}_{83}: & \epsilon_{ij}\epsilon_{kl}X^{k}_{34}X^{j}_{41}P_{17}X^{l}_{78} & & X^{i}_{85}Q_{53} - Q_{82}X^{i}_{23} \end{array} \quad , \tag{3.22}$$

where the global symmetry indices are $i,j,k,l = 1,2$.

From the J- and E-terms, we can construct the P-matrix which takes the following form

$$P = \left(\begin{array}{c|cccccc|cccccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \\ \hline X^1_{23} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ X^1_{34} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X^1_{41} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ X^1_{67} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ X^1_{78} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X^1_{85} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline X^2_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ X^2_{34} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X^2_{41} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ X^2_{67} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ X^2_{78} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X^2_{85} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline Q_{17} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ Q_{46} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ Q_{53} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ Q_{82} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline P_{17} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ P_{46} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ P_{53} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ P_{82} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right) . \tag{3.23}$$

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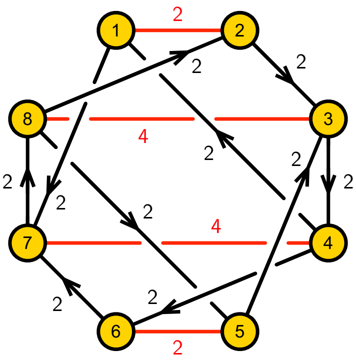

Figure 10. Quiver for $Y^{2,2}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

The corresponding U(1)-charge matrix for the J- and E-terms is given by

$$Q_{JE} = \begin{array}{c} \begin{array}{cccccc|cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \end{array} \\ \left(\begin{array}{cccccc|cccccccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{array}\right) \end{array}, \tag{3.24}$$

and the charge matrix for the D-terms is given by

$$Q_D = \begin{array}{c} \begin{array}{cccccc|cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \end{array} \\ \left(\begin{array}{cccccc|cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}. \tag{3.25}$$

We note that the charge matrices are invariant under

$$p_1 \leftrightarrow p_3\,, \;\; p_2 \leftrightarrow p_4\,. \tag{3.26}$$

This symmetry further indicates that the global symmetry takes the form $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^2$.

The toric data of the $2d$ $(0,2)$ theory is given by

$$G_t = \begin{array}{c} \begin{array}{cccccc|cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \end{array} \\ \left(\begin{array}{cccccc|cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array}. \tag{3.27}$$

The resulting toric diagram is shown in figure 11 and it agrees with the expectation. We note that $Y^{2,2}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ corresponds to $Q^{1,1,1}/\mathbb{Z}_2$. A full classification of the toric phases for this geometry was presented in [36]. The model constructed above indeed corresponds to phase A in this classification. This theory also appeared in [32].

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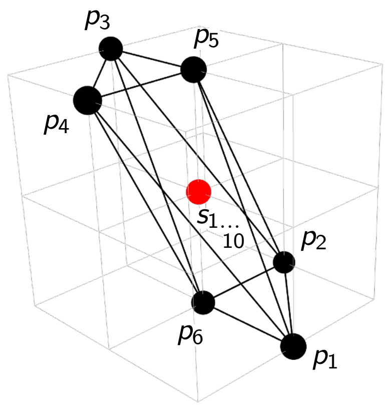

Figure 11. Toric diagram of $Y^{2,2}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

4 $Y^{p,k}(\mathbb{CP}^2)$ models

4.1 Construction of the 2d (0, 2) theories

The 2d (0, 2) gauge theories corresponding to the family of $Y^{p,k}(\mathbb{CP}^2)$ manifolds are parameterized in terms of $p, k \in \mathbb{Z}$, where $0 \leq k \leq \frac{3}{2}p$.

The total number of gauge groups in the 2d (0, 2) theories is 3p, which corresponds to the volume of the toric diagram of $Y^{p,k}(\mathbb{CP}^2)$. The notation used on the gauge group labels for the chiral and Fermi fields is defined as follows

$$[m] \equiv ((m-1) \text{ mod } 3p) + 1\,. \tag{4.1}$$

The chiral and Fermi fields form a periodic layered structure with a total number of p layers. Each layer in the structure contains 3 gauge groups, which is the same number of gauge groups as in the 4d $\mathcal{N} = 1$ theory corresponding to dP_0 [43]. In fact, this structure follows from the particular way in which we constructed the gauge theoories. We made use of orbifold reduction [35], 3d printing [36] and partial resolution via higgsing [29], starting with the 4d $\mathcal{N} = 1$ quiver gauge theory corresponding to dP_0.

Between layers, the gauge group labels are shifted by additional indices given by a, b, c and d. When the parameter k for the $Y^{p,k}(\mathbb{CP}^2)$ models is in the range $0 \leq k \leq p$, only Fermi and chiral fields depending on a, b and c contribute to the 2d (0, 2) theory, with the values of the indices being

$$a = 0, \ldots, p-1\,,\ b = k, \ldots, p-1\,,\ c = 0, \ldots, k-1\,. \tag{4.2}$$

When the model parameter k is in the range $p < k \leq \frac{3}{2}p$, only the Fermi and chiral fields depending on indices a, c and d contribute to the theory, with the values of the indices being

$$a = 0, \ldots, 2p-k\,,\ c = k-p, \ldots, 2p-k\,,\ d = 0, \ldots, k-p-1\,. \tag{4.3}$$

Finally, we also have indices for the $\mathrm{SU}(3) \times \mathrm{U}(1)^2$ global symmetry of the 2d (0, 2) theories corresponding to $Y^{p,k}(\mathbb{CP}^2)$. The indices for the non-abelian part of the global symmetry are

$$i, j, k = 1, 2, 3\,. \tag{4.4}$$

Under these indices, the Fermi fields of the $2d$ $(0,2)$ theories are as follows

$$\Lambda^{i}_{[3a+1][3a+2]}\,,\ \Lambda^{2i}_{[3b+6][3b+1]}\,,\ \Lambda^{2i}_{[3b+5][3b+3]}\,,\ \Lambda^{2i}_{[3c+6][3c+3]}\,,\ \Lambda^{1i}_{[3d+3][3d+6]}\,. \tag{4.5}$$

The chiral fields of the $2d$ $(0,2)$ theories take the following form

$$\begin{aligned}
&X^{i}_{[3a+2][3a+3]}\,,\ X^{i}_{[3a+3][3a+1]}\,,\ X^{i}_{[3b+1][3b+5]}\,,\\
&Q_{[3b+4][3b+1]}\,,\ Q_{[3b+6][3b+3]}\,,\ Q_{[3b+5][3b+2]}\,,\\
&Q_{[3c+4][3c+3]}\,,\ Q_{[3c+6][3c+2]}\,,\ P_{[3c+1][3c+5]}\,,\\
&Q_{[3d+4][3d+2]}\,,\ P_{[3d+1][3d+6]}\,,\ P_{[3d+3][3d+5]}\,.
\end{aligned} \tag{4.6}$$

Here, we note that the chiral fields of the form X^{i}_{rs} originate from the $4d$ $\mathcal{N}=1$ theory corresponding to dP_0 with $4d$ chiral fields $\mathcal{X}^{i}_{rs}$ and superpotential of the form

$$W=\epsilon_{ijk}\mathcal{X}^{i}_{12}\mathcal{X}^{j}_{23}\mathcal{X}^{k}_{31}\,. \tag{4.7}$$

These chiral fields X^{i}_{rs} form layers in the periodic quiver on T^3 of the $2d$ $(0,2)$ theory corresponding to $Y^{p,k}(\mathbb{CP}^2)$. Between these layers in the periodic quiver, we have chiral fields of the form Q_{rs} and P_{rs} in (4.6). Combined with the Fermi fields in (4.5), two adjacent layers form what is known as the *quiver block* [36] in the periodic quiver of the $2d$ $(0,2)$ theory. The indices a,b,c,d in (4.5) and (4.6) ensure that p copies of quiver blocks are stacked together in a consistent way such that equivalent fields and gauge groups are identified to each other. The p copies of quiver blocks that are stacked together form the periodic quiver on T^3 for the $2d$ $(0,2)$ theories corresponding to $Y^{p,k}(\mathbb{CP}^2)$.

From the periodic quiver, we obtain the corresponding J-terms

$$\begin{aligned}
J[\Lambda^{i}_{[3a+1][3a+2]}]&=\epsilon_{ijk}X^{j}_{[3a+2][3a+3]}X^{k}_{[3a+3][3a+1]}\,,\\
J[\Lambda^{2i}_{[3b+6][3b+1]}]&=\epsilon_{ijk}X^{j}_{[3b+1][3b+5]}X^{k}_{[3b+5][3b+6]}\,,\\
J[\Lambda^{2i}_{[3b+5][3b+3]}]&=\epsilon_{ijk}X^{j}_{[3b+3][3b+1]}X^{k}_{[3b+1][3b+5]}\,,\\
J[\Lambda^{2i}_{[3c+6][3c+3]}]&=\epsilon_{ijk}X^{j}_{[3c+3][3c+1]}P_{[3c+1][3c+5]}X^{k}_{[3c+5][3c+6]}\,,\\
J[\Lambda^{1i}_{[3d+3][3d+6]}]&=\epsilon_{ijk}X^{j}_{[3d+6][3d+4]}Q_{[3d+4][3d+2]}X^{k}_{[3d+2][3d+3]}\,,
\end{aligned} \tag{4.8}$$

and the associated E-terms of the following form

$$\begin{aligned}
E[\Lambda^{2i}_{[3b+6][3b+1]}]&=X^{i}_{[3b+6][3b+4]}Q_{[3b+4][3b+1]}-Q_{[3b+6][3b+3]}X^{i}_{[3b+3][3b+1]}\,,\\
E[\Lambda^{2i}_{[3b+5][3b+3]}]&=X^{i}_{[3b+5][3b+6]}Q_{[3b+6][3b+3]}-Q_{[3b+5][3b+2]}X^{i}_{[3b+2][3b+3]}\,,\\
E[\Lambda^{2i}_{[3c+6][3c+3]}]&=X^{i}_{[3c+6][3c+4]}Q_{[3c+4][3c+3]}-Q_{[3c+6][3c+2]}X^{i}_{[3c+2][3c+3]}\,,\\
E[\Lambda^{1i}_{[3d+3][3d+6]}]&=X^{i}_{[3d+3][3d+1]}P_{[3d+1][3d+6]}-P_{[3d+3][3d+5]}X^{i}_{[3d+5][3d+6]}\,.
\end{aligned} \tag{4.9}$$

Additionally, we have E-term plaquettes,

$$\begin{aligned}
E_+[\Lambda^i_{[3b+1][3b+2]}] &= X^i_{[3b+1][3b+5]}Q_{[3b+5][3b+2]}\,,\\
E_-[\Lambda^i_{[3b+4][3b+5]}] &= -Q_{[3b+4][3b+1]}X^i_{[3b+1][3b+5]}\,,\\
E_+[\Lambda^i_{[3c+1],[3c+2]}] &= P_{[3c+1][3c+5]}X^i_{[3c+5][3c+6]}Q_{[3c+6][3c+2]}\,,\\
E_-[\Lambda^i_{[3c+4][3c+5]}] &= -Q_{[3c+4][3c+3]}X^i_{[3c+3][3c+1]}P_{[3c+1][3c+5]}\,,\\
E_+[\Lambda^i_{[3d+1][3d+2]}] &= P_{[3d+1][3d+6]}X^i_{[3d+6][3d+4]}Q_{[3d+4][3d+2]}\,,\\
E_-[\Lambda^i_{[3d+4][3d+5]}] &= -Q_{[3d+4][3d+2]}X^i_{[3d+2][3d+3]}P_{[3d+3][3d+5]}\,,
\end{aligned}\tag{4.10}$$

which combine to proper E-terms of the form $E = E_+ - E_-$. The E-terms plaquettes are shown here separately because quadratic and cubic plaquettes are combined depending on the value of k in order to form a single E-term.

4.2 Examples

In this section, we summarize the construction of $2d$ $(0,2)$ theories corresponding to $Y^{p,k}(\mathbb{CP}^2)$ where $p = 3$. In total there are 5 toric diagrams with $p = 3$ whose toric diagrams have precisely 2 internal points. In appendix A.2 and A.3, the J- and E-terms corresponding to $2d$ $(0,2)$ theories with $p = 1$ and $p = 2$ are summarized for completeness.

4.2.1 $\boldsymbol{Y^{3,0}(\mathbb{CP}^2)}$

The J- and E-terms of the $2d$ $(0,2)$ theory for $Y^{3,0}(\mathbb{CP}^2)$ take the form

$$\begin{array}{rclccc}
 & & J & & E & \\
\Lambda^i_{12} & : & \epsilon_{ijk}X^j_{23}X^k_{31} & X^i_{15}Q_{52} & - & Q_{17}X^i_{72}\\
\Lambda^{2i}_{29} & : & \epsilon_{ijk}X^j_{97}X^k_{72} & X^i_{23}Q_{39} & - & Q_{28}X^i_{89}\\
\Lambda^{2i}_{37} & : & \epsilon_{ijk}X^j_{72}X^k_{23} & X^i_{31}Q_{17} & - & Q_{39}X^i_{97}\\
\Lambda^i_{45} & : & \epsilon_{ijk}X^j_{56}X^k_{64} & X^i_{48}Q_{85} & - & Q_{41}X^i_{15}\\
\Lambda^{2i}_{53} & : & \epsilon_{ijk}X^j_{31}X^k_{15} & X^i_{56}Q_{63} & - & Q_{52}X^i_{23}\\
\Lambda^{2i}_{61} & : & \epsilon_{ijk}X^j_{15}X^k_{56} & X^i_{64}Q_{41} & - & Q_{63}X^i_{31}\\
\Lambda^i_{78} & : & \epsilon_{ijk}X^j_{89}X^k_{97} & X^i_{72}Q_{28} & - & Q_{74}X^i_{48}\\
\Lambda^{2i}_{86} & : & \epsilon_{ijk}X^j_{64}X^k_{48} & X^i_{89}Q_{96} & - & Q_{85}X^i_{56}\\
\Lambda^{2i}_{94} & : & \epsilon_{ijk}X^j_{48}X^k_{89} & X^i_{97}Q_{74} & - & Q_{96}X^i_{64}
\end{array}\,,\tag{4.11}$$

where the global symmetry indices are $i, j, k = 1, 2, 3$. The corresponding quiver diagram is shown in figure 12.

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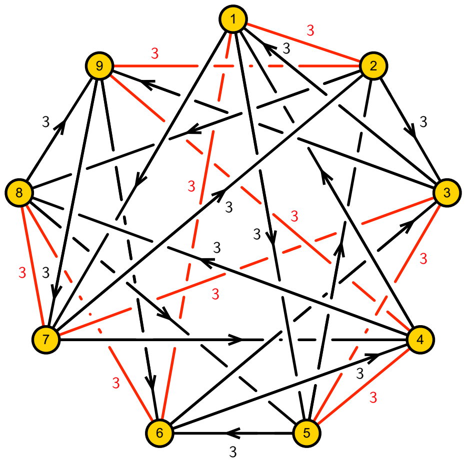

Figure 12. Quiver for $Y^{3,0}(\mathbb{CP}^2)$.

From the J- and E-terms, we can construct the P-matrix, which takes the form

$$
P=\left(\begin{array}{c|cccc|ccc|cccccccccccccccccc|ccccccccc|ccccccccc}
 & p_1 & p_2 & p_3 & p_4 & q_1 & q_2 & q_3 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\
\hline
X^1_{15} & 1&0&0&0 & 0&0&1 & 1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1 & 0&0&0&0&0&1&0&0&0 & 1&0&1&0&0&0&1&1&1 \\
X^1_{23} & 1&0&0&0 & 0&1&0 & 0&0&0&0&1&0&0&1&1&1&0&0&0&0&0&0&0&0 & 0&1&0&0&0&0&0&0&0 & 1&0&0&0&0&1&1&1&1 \\
X^1_{31} & 1&0&0&0 & 1&0&0 & 0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0 & 0&0&1&0&0&0&0&0&0 & 0&1&1&1&1&1&0&0&0 \\
X^1_{48} & 1&0&0&0 & 0&0&1 & 0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&1 & 0&0&0&0&0&0&0&0&1 & 0&1&0&0&1&1&0&1&1 \\
X^1_{56} & 1&0&0&0 & 0&1&0 & 0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0 & 0&0&0&0&0&0&1&0&0 & 0&1&0&1&1&1&0&1&0 \\
X^1_{64} & 1&0&0&0 & 1&0&0 & 0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0 & 0&0&0&0&0&0&0&1&0 & 1&0&1&1&0&0&1&0&1 \\
X^1_{72} & 1&0&0&0 & 0&0&1 & 1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0 & 1&1&1&1&1&0&0&0&0 \\
X^1_{89} & 1&0&0&0 & 0&1&0 & 0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&1&0 & 0&0&0&0&1&0&0&0&0 & 1&0&1&1&1&0&1&0&0 \\
X^1_{97} & 1&0&0&0 & 1&0&0 & 0&0&0&1&0&0&0&0&0&0&0&0&1&1&1&0&0&0 & 0&0&0&1&0&0&0&0&0 & 0&1&0&0&0&1&1&1&1 \\
\hline
X^2_{15} & 0&1&0&0 & 0&0&1 & 1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1 & 0&0&0&0&0&1&0&0&0 & 1&0&1&0&0&0&1&1&1 \\
X^2_{23} & 0&1&0&0 & 0&1&0 & 0&0&0&0&1&0&0&1&1&1&0&0&0&0&0&0&0&0 & 0&1&0&0&0&0&0&0&0 & 1&0&0&0&0&1&1&1&1 \\
X^2_{31} & 0&1&0&0 & 1&0&0 & 0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0 & 0&0&1&0&0&0&0&0&0 & 0&1&1&1&1&1&0&0&0 \\
X^2_{48} & 0&1&0&0 & 0&0&1 & 0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&1 & 0&0&0&0&0&0&0&0&1 & 0&1&0&0&1&1&0&1&1 \\
X^2_{56} & 0&1&0&0 & 0&1&0 & 0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0 & 0&0&0&0&0&0&1&0&0 & 0&1&0&1&1&1&0&1&0 \\
X^2_{64} & 0&1&0&0 & 1&0&0 & 0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0 & 0&0&0&0&0&0&0&1&0 & 1&0&1&1&0&0&1&0&1 \\
X^2_{72} & 0&1&0&0 & 0&0&1 & 1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0 & 1&1&1&1&1&0&0&0&0 \\
X^2_{89} & 0&1&0&0 & 0&1&0 & 0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&1&0 & 0&0&0&0&1&0&0&0&0 & 1&0&1&1&1&0&1&0&0 \\
X^2_{97} & 0&1&0&0 & 1&0&0 & 0&0&0&1&0&0&0&0&0&0&0&0&1&1&1&0&0&0 & 0&0&0&1&0&0&0&0&0 & 0&1&0&0&0&1&1&1&1 \\
\hline
X^3_{15} & 0&0&1&0 & 0&0&1 & 1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1 & 0&0&0&0&0&1&0&0&0 & 1&0&1&0&0&0&1&1&1 \\
X^3_{23} & 0&0&1&0 & 0&1&0 & 0&0&0&0&1&0&0&1&1&1&0&0&0&0&0&0&0&0 & 0&1&0&0&0&0&0&0&0 & 1&0&0&0&0&1&1&1&1 \\
X^3_{31} & 0&0&1&0 & 1&0&0 & 0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0 & 0&0&1&0&0&0&0&0&0 & 0&1&1&1&1&1&0&0&0 \\
X^3_{48} & 0&0&1&0 & 0&0&1 & 0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&1 & 0&0&0&0&0&0&0&0&1 & 0&1&0&0&1&1&0&1&1 \\
X^3_{56} & 0&0&1&0 & 0&1&0 & 0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0 & 0&0&0&0&0&0&1&0&0 & 0&1&0&1&1&1&0&1&0 \\
X^3_{64} & 0&0&1&0 & 1&0&0 & 0&0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0 & 0&0&0&0&0&0&0&1&0 & 1&0&1&1&0&0&1&0&1 \\
X^3_{72} & 0&0&1&0 & 0&0&1 & 1&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 & 1&0&0&0&0&0&0&0&0 & 1&1&1&1&1&0&0&0&0 \\
X^3_{89} & 0&0&1&0 & 0&1&0 & 0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&1&0 & 0&0&0&0&1&0&0&0&0 & 1&0&1&1&1&0&1&0&0 \\
X^3_{97} & 0&0&1&0 & 1&0&0 & 0&0&0&1&0&0&0&0&0&0&0&0&1&1&1&0&0&0 & 0&0&0&1&0&0&0&0&0 & 0&1&0&0&0&1&1&1&1 \\
\hline
Q_{17} & 0&0&0&1 & 0&0&0 & 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1 & 0&0&0&1&1&1&1&1&1 & 0&0&0&0&0&0&1&1&1 \\
Q_{28} & 0&0&0&1 & 0&0&0 & 0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&1 & 0&1&1&0&0&1&1&1&1 & 0&0&0&0&0&1&0&1&1 \\
Q_{39} & 0&0&0&1 & 0&0&0 & 0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&1&1&1 & 0&0&1&0&1&1&1&1&1 & 0&0&1&1&1&0&0&0&0 \\
Q_{41} & 0&0&0&1 & 0&0&0 & 0&0&0&1&1&1&1&0&0&1&0&1&0&0&0&0&0&0 & 1&1&1&1&1&0&0&0&1 & 0&1&0&0&1&1&0&0&0 \\
Q_{52} & 0&0&0&1 & 0&0&0 & 0&1&1&0&0&1&0&0&0&0&0&0&0&1&1&0&1&0 & 1&0&0&1&1&0&1&1&1 & 0&1&0&1&1&0&0&0&0 \\
Q_{63} & 0&0&0&1 & 0&0&0 & 0&0&1&0&1&1&0&0&1&1&0&0&0&0&1&0&0&0 & 1&1&0&1&1&0&0&1&1 & 1&0&0&0&0&0&1&0&1 \\
Q_{74} & 0&0&0&1 & 0&0&0 & 1&1&1&0&0&0&0&1&1&0&1&0&0&0&0&0&0&0 & 1&1&1&0&0&1&1&1&0 & 1&0&1&1&0&0&0&0&0 \\
Q_{85} & 0&0&0&1 & 0&0&0 & 1&0&0&1&1&0&1&0&0&0&0&0&1&0&0&1&0&0 & 1&1&1&1&1&1&0&0&0 & 1&0&1&0&0&0&1&0&0 \\
Q_{96} & 0&0&0&1 & 0&0&0 & 1&1&0&1&0&0&0&1&0&0&0&0&1&1&0&0&0&0 & 1&1&1&1&0&1&1&0&0 & 0&1&0&0&0&1&0&1&0
\end{array}\right). \tag{4.12}
$$

We can also construct the U(1)-charge matrix under the J- and E-terms,

$$
Q_{JE}=\left(\begin{array}{cccc|ccc|cccccccccccccccccc|ccccccccc|ccccccccc}
p_1&p_2&p_3&p_4&q_1&q_2&q_3&u_1&u_2&u_3&u_4&u_5&u_6&u_7&u_8&u_9&u_{10}&u_{11}&u_{12}&u_{13}&u_{14}&u_{15}&u_{16}&u_{17}&u_{18}&v_1&v_2&v_3&v_4&v_5&v_6&v_7&v_8&v_9&o_1&o_2&o_3&o_4&o_5&o_6&o_7&o_8&o_9\\ \hline
1&1&1&0&-1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&2&2&0&3&-1&0&-1&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&-1&0&0&0&0&0&0\\
0&0&0&0&2&2&-1&4&-1&0&-1&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0&-1&0&0&0&0&0&0\\
0&0&0&0&2&1&0&3&-1&0&-1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&2&0&3&-1&0&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&-1&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&1&0&2&-1&0&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&1&0&2&-1&0&-1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&0&0&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&0&0&-1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&0&0&-1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&1&0&1&-1&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&2&0&0&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&2&0&0&-1&0&0&0&-1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&2&0&0&-1&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&2&-1&0&-1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&2&-1&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&0&-1&2&-1&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&0&0&0\\
0&0&0&0&0&1&0&2&-1&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&-1&2&0&0&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&1&0&0&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&-1&1&0&0&-1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&-1&1&-1&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0
\end{array}\right), \tag{4.13}
$$

and also the charge matrix for the D-terms,

$$
Q_{D}=\left(\begin{array}{cccc|ccc|cccccccccccccccccc|ccccccccc|ccccccccc}
p_1&p_2&p_3&p_4&q_1&q_2&q_3&u_1&u_2&u_3&u_4&u_5&u_6&u_7&u_8&u_9&u_{10}&u_{11}&u_{12}&u_{13}&u_{14}&u_{15}&u_{16}&u_{17}&u_{18}&v_1&v_2&v_3&v_4&v_5&v_6&v_7&v_8&v_9&o_1&o_2&o_3&o_4&o_5&o_6&o_7&o_8&o_9\\ \hline
0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&1&-1&1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&1&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right). \tag{4.14}
$$

It can be seen that the charge matrices are invariant under permutations of the extremal GLSM fields (p_1, p_2, p_3). This is the result of the SU(3) non-abelian factor in the global symmetry $\mathrm{SU}(3)\times \mathrm{U}(1)^2$ of the $2d$ $(0,2)$ theory.

The toric data is summarized by

$$
G_t=\left(\begin{array}{cccc|ccc|cccccccccccccccccc|ccccccccc|ccccccccc}
p_1&p_2&p_3&p_4&q_1&q_2&q_3&u_1&u_2&u_3&u_4&u_5&u_6&u_7&u_8&u_9&u_{10}&u_{11}&u_{12}&u_{13}&u_{14}&u_{15}&u_{16}&u_{17}&u_{18}&v_1&v_2&v_3&v_4&v_5&v_6&v_7&v_8&v_9&o_1&o_2&o_3&o_4&o_5&o_6&o_7&o_8&o_9\\ \hline
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&2&2&2&2&2&2&2&2&2\\
1&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&3&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&2&2&2&2&2&2&2&2&2&1&1&1&1&1&1&1&1&1
\end{array}\right), \tag{4.15}
$$

where o_k correspond to extra GLSM fields [29]. The corresponding toric diagram is shown in figure 13. We note that the $2d$ $(0,2)$ theories for $Y^{3,0}(\mathbb{CP}^2)$ can be obtained by orbifold reduction [35] of the $4d$ $\mathcal{N}=1$ theory for dP_0.

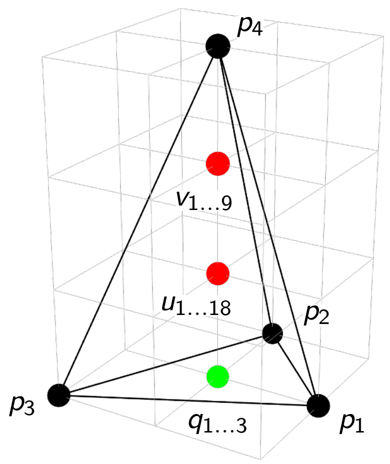

Figure 13. Toric diagram of $Y^{3,0}(\mathbb{CP}^2)$.

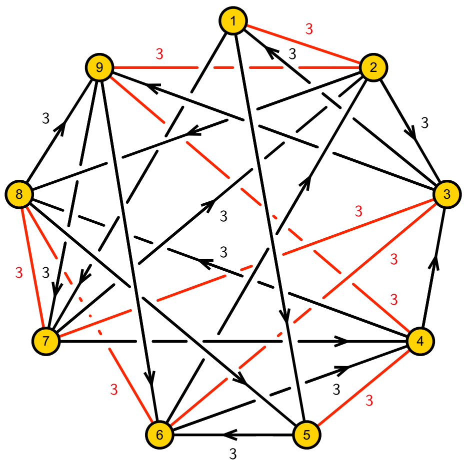

Figure 14. Quiver for $Y^{3,1}(\mathbb{CP}^2)$.

4.2.2 $Y^{3,1}(\mathbb{CP}^2)$

The quiver for the 2d $(0,2)$ theory corresponding to $Y^{3,1}(\mathbb{CP}^2)$ is shown in figure 14. The corresponding J- and E-terms take the following form

$$\begin{array}{rccc} & J & & E \\ \Lambda^{i}_{12}: & \epsilon_{ijk}X^{j}_{23}X^{k}_{31} & & P_{15}X^{i}_{56}Q_{62} - Q_{17}X^{i}_{72} \\ \Lambda^{2i}_{29}: & \epsilon_{ijk}X^{j}_{97}X^{k}_{72} & & X^{i}_{23}Q_{39} - Q_{28}X^{i}_{89} \\ \Lambda^{2i}_{37}: & \epsilon_{ijk}X^{j}_{72}X^{k}_{23} & & X^{i}_{31}Q_{17} - Q_{39}X^{i}_{97} \\ \Lambda^{i}_{45}: & \epsilon_{ijk}X^{j}_{56}X^{k}_{64} & & X^{i}_{48}Q_{85} - Q_{43}X^{i}_{31}P_{15} \\ \Lambda^{2i}_{63}: & \epsilon_{ijk}X^{j}_{31}P_{15}X^{k}_{56} & & X^{i}_{64}Q_{43} - Q_{62}X^{i}_{23} \\ \Lambda^{i}_{78}: & \epsilon_{ijk}X^{j}_{89}X^{k}_{97} & & X^{i}_{72}Q_{28} - Q_{74}X^{i}_{48} \\ \Lambda^{2i}_{86}: & \epsilon_{ijk}X^{j}_{64}X^{k}_{48} & & X^{i}_{89}Q_{96} - Q_{85}X^{i}_{56} \\ \Lambda^{2i}_{94}: & \epsilon_{ijk}X^{j}_{48}X^{k}_{89} & & X^{i}_{97}Q_{74} - Q_{96}X^{i}_{64} \end{array} , \tag{4.16}$$

where the global symmetry indices are $i, j, k = 1, 2, 3$.

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The J- and E-terms can be used to obtain the P-matrix for the theory, which takes the following form

$$
P = \left(\begin{array}{c|ccccc|cccccccccccccccc|ccccccccc|ccccccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} \\
\hline
X^1_{23} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
X^1_{31} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
X^1_{48} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
X^1_{56} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
X^1_{64} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
X^1_{72} & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
X^1_{89} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
X^1_{97} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
X^2_{23} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
X^2_{31} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
X^2_{48} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
X^2_{56} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
X^2_{64} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
X^2_{72} & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
X^2_{89} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
X^2_{97} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
X^3_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
X^3_{31} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
X^3_{48} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
X^3_{56} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
X^3_{64} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
X^3_{72} & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
X^3_{89} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
X^3_{97} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
\hline
Q_{17} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
Q_{28} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
Q_{39} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
Q_{43} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{62} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{74} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
Q_{85} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
Q_{96} & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
P_{15} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right) . \tag{4.17}
$$

The U(1)-charges under the J- and E-terms are summarized in

$$
Q_{JE} = \left(\begin{array}{ccccc|cccccccccccccccc|ccccccccc|ccccccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} \\
\hline
5 & 5 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\
4 & 4 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right) , \tag{4.18}
$$

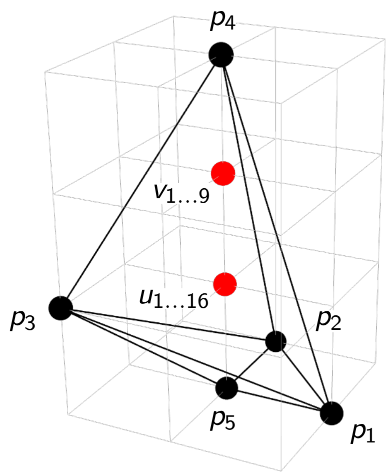

Figure 15. Toric diagram of $Y^{3,1}(\mathbb{CP}^2)$.

and the U(1)-charges for the D-terms are summarized in

$$Q_D = \left(\begin{array}{ccccc|cccccccccccccccc|ccccccccc|ccccccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} \\ \hline
1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right). \tag{4.19}$$

The charge matrices are invariant under permutations of the extremal GLSM fields (p_1, p_2, p_3). This indicates that the global symmetry of the $2d$ $(0, 2)$ theories is enhanced to $\mathrm{SU}(3) \times \mathrm{U}(1)^2$.

The charge matrices are used to obtain the toric data for the $2d$ $(0, 2)$ theories, which is summarized in

$$G_t = \left(\begin{array}{ccccc|cccccccccccccccc|ccccccccc|ccccccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} & o_{11} \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right), \tag{4.20}$$

where o_k correspond to extra GLSM fields [29]. The corresponding toric diagram is shown in figure 15.

4.2.3 $Y^{3,2}(\mathbb{CP}^2)$

The 2d $(0,2)$ theories for $Y^{3,2}(\mathbb{CP}^2)$ have the following J- and E-terms

$$\begin{array}{rcc} & J & E \\ \Lambda^i_{12}: & \epsilon_{ijk}X^j_{23}X^k_{31} & P_{15}X^i_{56}Q_{62} - Q_{17}X^i_{72} \\ \Lambda^{2i}_{29}: & \epsilon_{ijk}X^j_{97}X^k_{72} & X^i_{23}Q_{39} - Q_{28}X^i_{89} \\ \Lambda^{2i}_{37}: & \epsilon_{ijk}X^j_{72}X^k_{23} & X^i_{31}Q_{17} - Q_{39}X^i_{97} \\ \Lambda^i_{45}: & \epsilon_{ijk}X^j_{56}X^k_{64} & P_{48}X^i_{89}Q_{95} - Q_{43}X^i_{31}P_{15} \\ \Lambda^{2i}_{63}: & \epsilon_{ijk}X^j_{31}P_{15}X^k_{56} & X^i_{64}Q_{43} - Q_{62}X^i_{23} \\ \Lambda^i_{78}: & \epsilon_{ijk}X^j_{89}X^k_{97} & X^i_{72}Q_{28} - Q_{76}X^i_{64}P_{48} \\ \Lambda^{2i}_{96}: & \epsilon_{ijk}X^j_{64}P_{48}X^k_{89} & X^i_{97}Q_{76} - Q_{95}X^i_{56} \end{array}, \tag{4.21}$$

with the corresponding quiver diagram shown in figure 16. The global symmetry indices are $i, j, k = 1, 2, 3$.

The P-matrix can be obtained as follows

$$P = \left(\begin{array}{c|ccccc|cccccccccccccc|ccccccccc|cc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\ \hline
X^1_{23} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X^1_{31} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
X^1_{56} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
X^1_{64} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
X^1_{72} & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
X^1_{89} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
X^1_{97} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline
X^2_{23} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X^2_{31} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
X^2_{56} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
X^2_{64} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
X^2_{72} & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
X^2_{89} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
X^2_{97} & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline
X^3_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X^3_{31} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
X^3_{56} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
X^3_{64} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
X^3_{72} & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
X^3_{89} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
X^3_{97} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline
Q_{17} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
Q_{28} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
Q_{39} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
Q_{43} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
Q_{62} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
Q_{76} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
Q_{95} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline
P_{15} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
P_{48} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{array}\right). \tag{4.22}$$

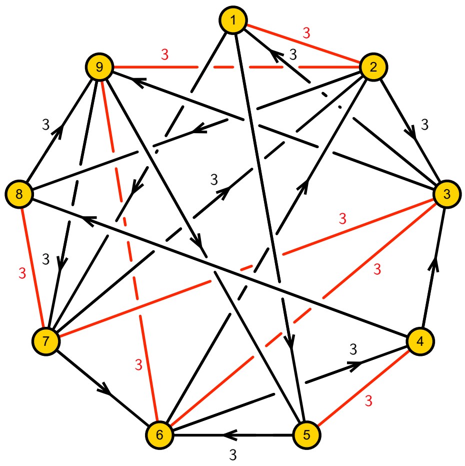

Figure 16. Quiver for $Y^{3,2}(\mathbb{CP}^2)$.

The corresponding U(1)-charges under the J- and E-terms take the form

$$
Q_{JE} = \left(\begin{array}{ccccc|cccccccccccccc|ccccccccc|cc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\
\hline
2 & 2 & 2 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
2 & 2 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) . \tag{4.23}
$$

Furthermore, the U(1)-charges for the D-terms are summarized in

$$
Q_D = \left(\begin{array}{ccccc|cccccccccccccc|ccccccccc|cc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right) . \tag{4.24}
$$

We note that the U(1)-charges under the J- and E-terms and the D-terms remain invariant under any permutation of (p_1, p_2, p_3). This further illustrates that the global symmetry of the $2d$ $(0,2)$ theories is $\mathrm{SU}(3) \times \mathrm{U}(1)^2$.

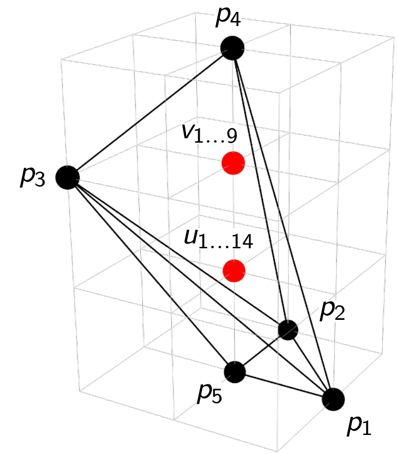

Figure 17. Toric diagram of $Y^{3,2}(\mathbb{CP}^2)$.

The corresponding toric data is given by

$$G_t = \left(\begin{array}{ccccc|cccccccccccccc|ccccccccc|cc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1
\end{array}\right), \tag{4.25}$$

where o_1, o_2 correspond to extra GLSM fields [29]. The toric diagram is shown in figure 17 and it agrees with the expectation.

4.2.4 $Y^{3,3}(\mathbb{CP}^2)$

The J- and E-terms of the $2d$ $(0,2)$ theories for $Y^{3,3}(\mathbb{CP}^2)$ take the following form

$$\begin{array}{rclc}
 & J & & E \\
\Lambda^i_{12} : & \epsilon_{ijk} X^j_{23} X^k_{31} & & P_{15} X^i_{56} Q_{62} - Q_{19} X^i_{97} P_{72} \\
\Lambda^{2i}_{39} : & \epsilon_{ijk} X^j_{97} P_{72} X^k_{23} & & X^i_{31} Q_{19} - Q_{38} X^i_{89} \\
\Lambda^i_{45} : & \epsilon_{ijk} X^j_{56} X^k_{64} & & P_{48} X^i_{89} Q_{95} - Q_{43} X^i_{31} P_{15} \\
\Lambda^{2i}_{63} : & \epsilon_{ijk} X^j_{31} P_{15} X^k_{56} & & X^i_{64} Q_{43} - Q_{62} X^i_{23} \\
\Lambda^i_{78} : & \epsilon_{ijk} X^j_{89} X^k_{97} & & P_{72} X^i_{23} Q_{38} - Q_{76} X^i_{64} P_{48} \\
\Lambda^{2i}_{96} : & \epsilon_{ijk} X^j_{64} P_{48} X^k_{89} & & X^i_{97} Q_{76} - Q_{95} X^i_{56}
\end{array}, \tag{4.26}$$

where the global symmetry indices are $i, j, k = 1, 2, 3$. The corresponding quiver diagram is shown in figure 18.

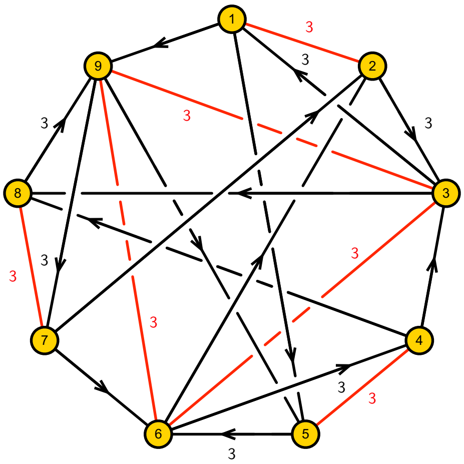

Figure 18. Quiver for $Y^{3,3}(\mathbb{CP}^2)$.

The corresponding P-matrix takes the form

$$P = \left(\begin{array}{c|ccccc|cccccccccccc|ccccccccc|cc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\
\hline
X_{23}^1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{31}^1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
X_{56}^1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{64}^1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
X_{89}^1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{97}^1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
X_{23}^2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{31}^2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
X_{56}^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{64}^2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
X_{89}^2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{97}^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
X_{23}^3 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{31}^3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
X_{56}^3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{64}^3 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
X_{89}^3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
X_{97}^3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
Q_{19} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
Q_{38} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
Q_{43} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
Q_{62} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
Q_{76} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
Q_{95} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\hline
P_{15} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
P_{48} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
P_{72} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right) . \qquad (4.27)$$

The U(1)-charges under J- and E-terms are summarized in

$$
Q_{JE} = \left(\begin{array}{ccccc|cccccccccccc|ccccccccc|cc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\
\hline
1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right), \tag{4.28}
$$

and the U(1)-charges under D-terms are summarized in

$$
Q_D = \left(\begin{array}{ccccc|cccccccccccc|ccccccccc|cc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right). \tag{4.29}
$$

We note that the charge matrices are invariant under permutations of (p_1, p_2, p_3). This illustrates that the global symmetry for the $2d$ $(0,2)$ theory is $\mathrm{SU}(3) \times \mathrm{U}(1)^2$.

The toric data is given by

$$
G_t = \left(\begin{array}{ccccc|cccccccccccc|ccccccccc|cc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & o_1 & o_2 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3
\end{array}\right), \tag{4.30}
$$

where o_1, o_2 correspond to extra GLSM fields [29]. The toric diagram is shown in figure 19 and it agrees with the expectation.

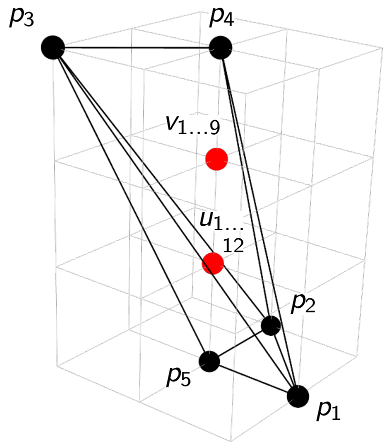

Figure 19. Toric diagram of $Y^{3,3}(\mathbb{CP}^2)$.

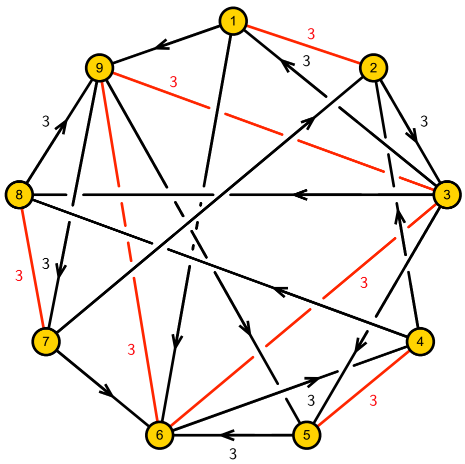

Figure 20. Quiver for $Y^{3,4}(\mathbb{CP}^2)$.

4.2.5 $\mathbf{Y^{3,4}}(\mathbb{CP}^\mathbf{2})$

The model for $Y^{3,4}(\mathbb{CP}^2)$ has a quiver which is shown in figure 20 and the corresponding J- and E-terms are given by

$$\begin{array}{rcc} & J & E \\ \Lambda^i_{12}: & \epsilon_{ijk}X^j_{23}X^k_{31} & P_{16}X^i_{64}Q_{42} - Q_{19}X^i_{97}P_{72} \\ \Lambda^{1i}_{36}: & \epsilon_{ijk}X^j_{64}Q_{42}X^k_{23} & X^i_{31}P_{16} - P_{35}X^i_{56} \\ \Lambda^{2i}_{39}: & \epsilon_{ijk}X^j_{97}P_{72}X^k_{23} & X^i_{31}Q_{19} - Q_{38}X^i_{89} \\ \Lambda^i_{45}: & \epsilon_{ijk}X^j_{56}X^k_{64} & P_{48}X^i_{89}Q_{95} - Q_{42}X^i_{23}P_{35} \\ \Lambda^i_{78}: & \epsilon_{ijk}X^j_{89}X^k_{97} & P_{72}X^i_{23}Q_{38} - Q_{76}X^i_{64}P_{48} \\ \Lambda^{2i}_{96}: & \epsilon_{ijk}X^j_{64}P_{48}X^k_{89} & X^i_{97}Q_{76} - Q_{95}X^i_{56} \end{array} , \tag{4.31}$$

where the global symmetry indices are $i, j, k = 1, 2, 3$.

The P-matrix takes the form

$$P = \left(\begin{array}{c|ccccc|ccccccccccc|cccccccccc|ccccccccc}
 & p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\ \hline
X^1_{23} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
X^1_{31} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
X^1_{56} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
X^1_{64} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
X^1_{89} & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
X^1_{97} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
X^2_{23} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
X^2_{31} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
X^2_{56} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
X^2_{64} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
X^2_{89} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
X^2_{97} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
X^3_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
X^3_{31} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
X^3_{56} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
X^3_{64} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
X^3_{89} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
X^3_{97} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline
Q_{19} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
Q_{38} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
Q_{42} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
Q_{76} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
Q_{95} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline
P_{16} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
P_{35} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
P_{48} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
P_{72} & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right). \tag{4.32}$$

The associated U(1)-charges coming from the J- and E-terms are summarized in

$$Q_{JE} = \left(\begin{array}{ccccc|ccccccccccc|cccccccccc|ccccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\ \hline
3 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{array}\right). \tag{4.33}$$

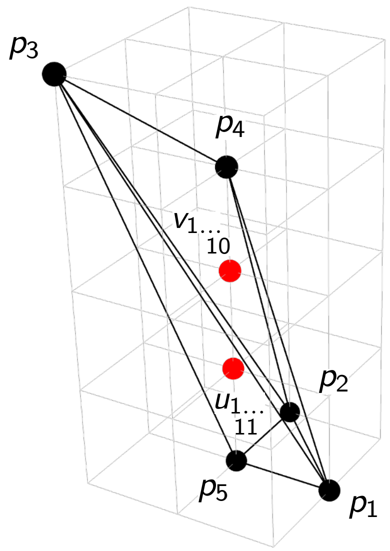

Figure 21. Toric diagram of $Y^{3,4}(\mathbb{CP}^2)$.

The U(1)-charges coming from the D-terms are summarized in

$$Q_D = \left(\begin{array}{ccccc|ccccccccccc|cccccccccc|ccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{4.34}$$

We note that the charge matrices remain invariant under permutations of the extremal GLSM fields (p_1, p_2, p_3) indicating that the global symmetry is enhanced to $SU(3) \times U(1)^2$. This is expected by the construction of the general J- and E-terms in section 4.1 and the isometry group of $Y^{3,4}(\mathbb{CP}^2)$.

The toric data for the $2d$ $(0,2)$ theory is given by

$$G_t = \left(\begin{array}{ccccc|ccccccccccc|cccccccccc|ccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right), \tag{4.35}$$

where o_k correspond to extra GLSM fields [29]. The corresponding toric diagram is shown in figure 21.

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A Additional examples

A.1 J- and E-terms for $Y^{1,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$

$Y^{1,0}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

$$\begin{array}{rclc} & & J & E \\ \Lambda_{12}^{i} & : & \epsilon_{ij}\epsilon_{kl}X_{23}^{k}X_{34}^{j}X_{41}^{l} & X_{12}^{i}Q_{22} - Q_{11}X_{12}^{i} \\ \Lambda_{23}^{2i} & : & \epsilon_{ij}\epsilon_{kl}X_{34}^{k}X_{41}^{j}X_{12}^{l} & X_{23}^{i}Q_{33} - Q_{22}X_{23}^{i} \\ \Lambda_{34}^{2i} & : & \epsilon_{ij}\epsilon_{kl}X_{41}^{k}X_{12}^{j}X_{23}^{l} & X_{34}^{i}Q_{44} - Q_{33}X_{34}^{i} \\ \Lambda_{41}^{2i} & : & \epsilon_{ij}\epsilon_{kl}X_{12}^{k}X_{23}^{j}X_{34}^{l} & X_{41}^{i}Q_{11} - Q_{44}X_{41}^{i} \end{array} \tag{A.1}$$

$Y^{1,1}(\mathbb{CP}^1 \times \mathbb{CP}^1)$.

$$\begin{array}{rclc} & & J & E \\ \Lambda_{12}^{i} & : & \epsilon_{ij}\epsilon_{kl}X_{23}^{k}X_{34}^{j}X_{41}^{l} & P_{13}X_{34}^{i}Q_{42} - Q_{13}X_{34}^{i}P_{42} \\ \Lambda_{43}^{1i} & : & \epsilon_{ij}\epsilon_{kl}X_{34}^{k}X_{41}^{j}Q_{13}X_{34}^{l} & X_{41}^{i}P_{13} - P_{42}X_{23}^{i} \\ \Lambda_{43}^{2i} & : & \epsilon_{ij}\epsilon_{kl}X_{34}^{k}X_{41}^{j}P_{13}X_{34}^{l} & X_{41}^{i}Q_{13} - Q_{42}X_{23}^{i} \end{array} \tag{A.2}$$

A.2 J- and E-terms for $Y^{1,k}(\mathbb{CP}^2)$

$Y^{1,0}(\mathbb{CP}^2)$.

$$\begin{array}{rclc} & & J & E \\ \Lambda_{12}^{i} & : & \epsilon_{ijk}X_{23}^{j}X_{31}^{k} & X_{12}^{i}Q_{22} - Q_{11}X_{12}^{i} \\ \Lambda_{23}^{2i} & : & \epsilon_{ijk}X_{31}^{j}X_{12}^{k} & X_{23}^{i}Q_{33} - Q_{22}X_{23}^{i} \\ \Lambda_{31}^{2i} & : & \epsilon_{ijk}X_{12}^{j}X_{23}^{k} & X_{31}^{i}Q_{11} - Q_{33}X_{31}^{i} \end{array} \tag{A.3}$$

$Y^{1,1}(\mathbb{CP}^2)$.

$$\begin{array}{rclc} & & J & E \\ \Lambda_{12}^{i} & : & \epsilon_{ijk}X_{23}^{j}X_{31}^{k} & P_{12}X_{23}^{i}Q_{32} - Q_{13}X_{31}^{i}P_{12} \\ \Lambda_{33}^{2i} & : & \epsilon_{ijk}X_{31}^{j}P_{12}X_{23}^{k} & X_{31}^{i}Q_{13} - Q_{32}X_{23}^{i} \end{array} \tag{A.4}$$

A.3 J- and E-terms for $Y^{2,k}(\mathbb{CP}^2)$

$\boldsymbol{Y^{2,0}(\mathbb{CP}^2)}$.

$$\begin{array}{lll} & J & E \\ \Lambda^i_{12}: & \epsilon_{ijk}X^j_{23}X^k_{31} & X^i_{15}Q_{52} - Q_{14}X^i_{42} \\ \Lambda^{2i}_{26}: & \epsilon_{ijk}X^j_{64}X^k_{42} & X^i_{23}Q_{36} - Q_{25}X^i_{56} \\ \Lambda^{2i}_{34}: & \epsilon_{ijk}X^j_{42}X^k_{23} & X^i_{31}Q_{14} - Q_{36}X^i_{64} \\ \Lambda^i_{45}: & \epsilon_{ijk}X^j_{56}X^k_{64} & X^i_{42}Q_{25} - Q_{41}X^i_{15} \\ \Lambda^{2i}_{53}: & \epsilon_{ijk}X^j_{31}X^k_{15} & X^i_{56}Q_{63} - Q_{52}X^i_{23} \\ \Lambda^{2i}_{61}: & \epsilon_{ijk}X^j_{15}X^k_{56} & X^i_{64}Q_{41} - Q_{63}X^i_{31} \end{array} \tag{A.5}$$

$\boldsymbol{Y^{2,1}(\mathbb{CP}^2)}$.

$$\begin{array}{lll} & J & E \\ \Lambda^i_{12}: & \epsilon_{ijk}X^j_{23}X^k_{31} & P_{15}X^i_{56}Q_{62} - Q_{14}X^i_{42} \\ \Lambda^{2i}_{26}: & \epsilon_{ijk}X^j_{64}X^k_{42} & X^i_{23}Q_{36} - Q_{25}X^i_{56} \\ \Lambda^{2i}_{34}: & \epsilon_{ijk}X^j_{42}X^k_{23} & X^i_{31}Q_{14} - Q_{36}X^i_{64} \\ \Lambda^i_{45}: & \epsilon_{ijk}X^j_{56}X^k_{64} & X^i_{42}Q_{25} - Q_{43}X^i_{31}P_{15} \\ \Lambda^{2i}_{63}: & \epsilon_{ijk}X^j_{31}P_{15}X^k_{56} & X^i_{64}Q_{43} - Q_{62}X^i_{23} \end{array} \tag{A.6}$$

$\boldsymbol{Y^{2,2}(\mathbb{CP}^2)}$.

$$\begin{array}{lll} & J & E \\ \Lambda^i_{12}: & \epsilon_{ijk}X^j_{23}X^k_{31} & P_{15}X^i_{56}Q_{62} - Q_{16}X^i_{64}P_{42} \\ \Lambda^{2i}_{36}: & \epsilon_{ijk}X^j_{64}P_{42}X^k_{23} & X^i_{31}Q_{16} - Q_{35}X^i_{56} \\ \Lambda^i_{45}: & \epsilon_{ijk}X^j_{56}X^k_{64} & P_{42}X^i_{23}Q_{35} - Q_{43}X^i_{31}P_{15} \\ \Lambda^{2i}_{63}: & \epsilon_{ijk}X^j_{31}P_{15}X^k_{56} & X^i_{64}Q_{43} - Q_{62}X^i_{23} \end{array} \tag{A.7}$$

$\boldsymbol{Y^{2,3}(\mathbb{CP}^2)}$.

$$\begin{array}{lll} & J & E \\ \Lambda^i_{12}: & \epsilon_{ijk}X^j_{23}X^k_{31} & P_{16}X^i_{64}Q_{42} - Q_{16}X^i_{64}P_{42} \\ \Lambda^{1i}_{36}: & \epsilon_{ijk}X^j_{64}Q_{42}X^k_{23} & X^i_{31}P_{16} - P_{35}X^i_{56} \\ \Lambda^{2i}_{36}: & \epsilon_{ijk}X^j_{64}P_{42}X^k_{23} & X^i_{31}Q_{16} - Q_{35}X^i_{56} \\ \Lambda^i_{45}: & \epsilon_{ijk}X^j_{56}X^k_{64} & P_{42}X^i_{23}Q_{35} - Q_{42}X^i_{23}P_{35} \end{array} \tag{A.8}$$

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