On the Global Convergence of Relative Value Iteration for Infinite-Horizon Risk-Sensitive Control of Diffusions

HASSAN HMEDI*, ARI ARAPOSTATHIS*, AND GUODONG PANG†

ABSTRACT. In [8], a multiplicative relative value iteration algorithm (RVI) for infinite-horizon risk-sensitive control of diffusions in $\mathbb{R}^d$ is studied. Assuming that there exists a control for which the diffusion is positive recurrent, the authors have established that the multiplicative value iteration (VI) algorithm converges to the solution of the multiplicative (risk-sensitive) HJB equation starting from an initial condition within the neighborhood of the solution (local convergence). Under a blanket (uniform) exponential ergodicity assumption, the authors have also shown that the RVI algorithm converges to the solution of the multiplicative HJB equation starting from any positive initial condition (global convergence).

In this paper, we revisit this problem without assuming the blanket (uniform) condition. We instead assume a near-monotone running cost, and in addition, a structural assumption relating the running cost function to the solution of the multiplicative HJB equation. We show that this structural assumption implies the existence of a control under which the ground state diffusion is exponentially ergodic. More importantly, a global convergence result of the multiplicative VI/RVI algorithms is established; thus, extending upon the results in [8].

1. INTRODUCTION

Risk-sensitive control has been an active area of research in the past 30 years and has found numerous applications in finance [17], cognitive neuroscience [28], and many more. Unlike the more classical case of the ergodic control problems which lead to the well known Bellman equation, infinite-horizon risk-sensitive (IHRS) control problems lead to a multiplicative dynamic programming equation, since they seek to minimize a cost or maximize a reward given by the exponential growth rate of a multiplicative cost or, resp., reward. This is one of the reasons why the risk-sensitive control penalty is desirable since it captures the higher order moments of the running cost in addition to its expectation.

Risk-sensitive control problems have been extensively examined for discrete time discrete state space controlled Markov chains [13, 14, 18, 19, 25] as well as the continuous time dynamics modeled by controlled diffusions [10, 12, 20, 21, 26, 27]. For the IHRS control problems, the corresponding value iteration (VI) algorithm, which ends up being a multiplicative analog of the classical VI algorithm for ergodic control problems, was also studied for the discrete time discrete state space setting such as in [13, 15, 16]. In [8], the multiplicative VI algorithm was studied for discrete–time Markov chains on a compact Polish space with a compact metric action space.

For controlled diffusions, the corresponding VI algorithms have not thoroughly investigated until the recent work in [8]. In that reference, a first step is taken by proposing multiplicative VI and RVI algorithms for the IHRS control problem of diffusions on the whole Euclidean space. The

* DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING, THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712

† DEPARTMENT OF COMPUTATIONAL APPLIED MATHEMATICS AND OPERATIONS RESEARCH, GEORGE R. BROWN COLLEGE OF ENGINEERING, RICE UNIVERSITY, HOUSTON, TX 77005

E-mail addresses: {ari, hmedi}@utexas.edu, gdpang@rice.edu.

Key words and phrases. Infinite-horizon risk-sensitive control, diffusion, multiplicative (relative) value iteration algorithm, near monotone.

problem was studied under the framework of blanket(uniform) stability of the controlled diffusion. A key component towards that result is the local convergence of the multiplicative VI algorithm assuming that there exists a control for which the ground state diffusion is positive recurrent, that is, the convergence is established only within a neighborhood of the solution to the risk-sensitive HJB equation (see (3.1)). Then the global convergence of the multiplicative RVI algorithm starting from any positive initial condition is established under the blanket (uniform) exponential ergodicity assumption.

Blanket/Uniform exponential ergodicity refers to a process that is positive recurrent and whose transition probabilities converge to the invariant distribution uniformly over all controls at an exponential rate. This condition is usually checked by means of establishing a Foster-Lyapunov inequality. As expected, this is not an easy condition to verify and few dynamical systems are as such, including the large-scale parallel-server queueing networks studied in [9,23]. In this paper, we consider the IHRS control problem of diffusions assuming a near-monotone running cost without the blanket (uniform) ergodicity property, as recently studied in [1, 7]. The near-monotonicity condition essentially penalizes any unstable behavior when the state gets large via the running cost function. In addition, we impose a further structural assumption relating the running cost function to the solution of the multiplicative dynamic programming equation as in Assumption 4.1. Under these assumptions, we improve upon the results in [8] by establishing the following:

- There exists a control under which the ground state diffusion is exponentially ergodic.
- A global convergence result: for any positive initial condition, the multiplicative VI and RVI algorithms converge to the solution of the risk sensitive HJB equation.

As this paper extends the results in [8] towards a better understanding of the RVI algorithm design under an IHRS criterion, we draw upon some of the notations and results from that paper.

1.1. **Organization of the paper.** In the next subsection, we summarize the notation used in the paper. In Section 2, we describe the model and state some of the assumptions used. In Section 3, we review some results concerning the existence and uniqueness of a solution to the risk-sensitive HJB equation under the IHRS criterion. In Section 4, we introduce the multiplicative VI algorithm and establish its global convergence (starting from any positive initial condition) to the solution of the risk-sensitive HJB equation under Assumption 4.1. In Section 5, we establish the global convergence of the multiplicative RVI algorithm. Finally, Section 6 is devoted to summarize the results and propose some future directions.

1.2. **Notation.** We use $\mathbb{R}^d$ (and $\mathbb{R}^d_+$), $d \geq 1$, to denote real-valued d-dimensional (nonnegative) vectors, and write $\mathbb{R}$ for the real line. We use z^{T} to denote the transpose of z. For $x, y \in \mathbb{R}$, $x \wedge y = \min\{x, y\}$.

For a set $A \subseteq \mathbb{R}^d$, $\mathbb{1}_A$ to denote the indicator function of A. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$, and $\langle \cdot\,, \cdot \rangle$ stands for the inner product. A ball of radius $r > 0$ in $\mathbb{R}^d$ around a point x is denoted by $\mathcal{B}_r(x)$, or simply as $\mathcal{B}_r$ if $x = 0$. We also let $\mathcal{B} \equiv \mathcal{B}_1$.

We denote by $\uptau(A)$ the *first exit time* of the process $\{X_t\}_{t\geq 0}$ from the set $A \subset \mathbb{R}^d$, defined by

$$\uptau(A) \;\coloneqq\; \inf\,\{t > 0 \,:\, X_t \notin A\}\,.$$

We let $\uptau_R \coloneqq \uptau(\mathcal{B}_R)$, and $\hat{\uptau}_R \coloneqq \uptau(\mathcal{B}^c_R)$.

Given a strictly positive real function f on $\mathbb{R}^d$, the f-norm of a function $g\colon \mathbb{R}^d \to \mathbb{R}$ is given by

$$\|g\|_f \coloneqq \sup_{x \in \mathbb{R}^d} \frac{|g(x)|}{f(x)}\,.$$

2. Model and Assumptions

All random processes live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Let $X = \{X_t : t \geq 0\}$ be a controlled diffusion on $\mathbb{R}^d$ taking the form

$$\mathrm{d}X_t \,=\, b(X_t, U_t)\,\mathrm{d}t + \upsigma(X_t)\,\mathrm{d}W_t\,, \qquad X_0 = x\,. \tag{2.1}$$

The process $\{W_t\}_{t\geq 0}$ is a d-dimensional standard Wiener process, and the control process $\{U_t\}_{t\geq 0}$ lives in a compact metrizable space $\mathbb{U}$. The sets of admissible controls $\mathfrak{U}$, and stationary Markov controls $\mathfrak{U}_{\mathsf{sm}}$ are defined in the standard manner [4].

We consider the infinite-horizon risk-sensitive (IHRS) control problem: For $U \in \mathfrak{U}$, and a given running cost $c : \mathbb{R}^d \times \mathbb{U} \to [1, \infty)$, the risk-sensitive objective function is defined by

$$\Lambda_U^x \;=\; \Lambda_U^x(c) \;\coloneqq\; \limsup_{T\to\infty}\, \frac{1}{T}\,\log \mathbb{E}_U^x\Bigl[\mathrm{e}^{\int_0^T c(X_t,U_t)\,\mathrm{d}t}\Bigr]\,,$$

and the IHRS optimal values are denoted by

$$\Lambda_*^x \;\coloneqq\; \inf_{U\in\mathfrak{U}}\, \Lambda_U^x\,, \quad \text{ and } \quad \Lambda_* \;\coloneqq\; \inf_{x\in\mathbb{R}^d}\, \Lambda_*^x\,.$$

$$\Lambda_{m,*}^x \;\coloneqq\; \inf_{U\in\mathfrak{U}_{\mathsf{sm}}}\, \Lambda_U^x\,, \quad \text{ and } \quad \Lambda_{m,*} \;\coloneqq\; \inf_{x\in\mathbb{R}^d}\, \Lambda_{m,*}^x\,.$$

We impose the following set of standard assumptions under which (2.1) has a unique solution under any admissible control [4, Theorem 2.2.4].

Assumption 2.1. The following hold.

(i) *Local Lipschitz continuity.* The drift $b\colon \mathbb{R}^d\times\mathbb{U} \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are locally Lipschitz in x: for some positive constants C_R depending on $R > 0$,

$$|b(x,u) - b(y,u)| + \|\upsigma(x) - \upsigma(y)\| \;\leq\; C_R\,|x-y| \quad \forall\, x,y \in \mathcal{B}_R\,,\ \forall\, u\in\mathbb{U}\,,$$

where $\|\upsigma\| \coloneqq \bigl(\mathrm{Tr}\ \upsigma\upsigma^{\mathsf{T}}\bigr)^{1/2}$ denotes the Hilbert–Schmidt norm of the matrix $\upsigma$. In addition, b is continuous.

(ii) *Affine growth condition.* For some $\theta \in [0, 1)$ and a constant κ_0, we have

$$|b(x,u)|^2 + \|\upsigma(x)\|^2 \;\leq\; \kappa_0\bigl(1 + |x|^{2\theta}\bigr)\,, \quad \forall\,(x,u) \in \mathbb{R}^d\times\mathbb{U}\,.$$

(iii) *Nondegeneracy.* Let $a \coloneqq \upsigma\upsigma^{\mathsf{T}}$. For each $R > 0$ and some positive constant C independent from R,

$$\sum_{i,j=1}^{d} a^{ij}(x)\zeta_i\zeta_j \;\geq\; C^{-1}|\zeta|^2 \quad \forall\, x\in B_R\,, \forall\zeta\in\mathbb{R}^d\,.$$

We continue with the following definition of *near-monotonicity*.

Definition 2.1. A continuous map $g : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}$ is said to be *near-monotone* relative to $\lambda\in\mathbb{R}$ if there exists $\epsilon > 0$ such that the set $\mathcal{K}_\epsilon \coloneqq \bigl\{x\in\mathbb{R}^d : \inf_{u\in\mathbb{U}} g(x,u) \leq \lambda+\epsilon\bigr\}$ is compact. We also say that a function is *inf-compact* if it is near-monotone relative to all $\lambda\in\mathbb{R}$.

We impose the following structural assumptions in Assumptions 2.2 and 2.3 on the running cost c, namely *near-monotonicity*, and on the drift b and variance matrix a to ensure the existence of a solution to the risk sensitive Hamilton-Jacobi-Bellman (HJB) equation:

Assumption 2.2. The running cost c is continuous and satisfies

$$\min_{x\in\mathcal{B}_R^c}\,\min_{u\in\mathbb{U}}\, c(x,u) \;\xrightarrow[R\to\infty]{}\; \infty\,, \tag{2.2}$$

Note that (2.2) implies that the running cost c is *inf-compact*.

Assumption 2.3. The following hold for the matrix a, drift b and cost c.

(i) The matrix a is bounded.
(ii) For some $\theta \in [0,1)$ and a constant κ_0, we have

$$|b(x,u)| \,\le\, \kappa_0\bigl(1+|x|^{\theta}\bigr)\,,\quad \text{and}\quad |c(x,u)| \,\le\, \kappa_0\bigl(1+|x|^{2\theta}\bigr)\,,\quad \forall\,(x,u)\in\mathbb{R}^d\times\mathbb{U}\,.$$

(iii) The drift b satisfies the following structural property

$$\max_{x\in\mathcal{B}_R}\,\frac{1}{|x|^{1-\theta}}\,\max_{u\in\mathbb{U}}\,\bigl\langle b(x,u),\,x\bigr\rangle^{+}\,\xrightarrow[R\to\infty]{}\,0\,.$$

Note that Assumptions 2.2 and 2.3 appeared previously in [1] as sufficient conditions for the existence of a solution of the HJB equation of the IHRS control problem and in [7] to establish that the solution of the HJB equation has a polynomial growth.

To describe the risk-sensitive HJB equation below, we introduce the following concepts and notation.

The extended generator of the controlled diffusion X_t in (2.1) is given by

$$\mathcal{L}^u f(x) \,:=\, \frac{1}{2}\,\mathrm{Tr}\,\bigl(a(x)\nabla^2 f(x)\bigr) + \bigl\langle b(x,u),\nabla f(x)\bigr\rangle\,,\qquad u\in\mathbb{U}\,, \tag{2.3}$$

for $f\in C^2(\mathbb{R}^d)$. For ease of notation, we use

$$b_v(x) \,:=\, b\bigl(x,v(x)\bigr)\,,\quad \text{and } c_v(x)\,:=\,c\bigl(x,v(x)\bigr)\qquad \text{for } v\in\mathfrak{U}_{\mathsf{sm}}\,,$$

which we adopt for the rest of this paper and we let $\mathcal{L}_v$ for $v\in\mathfrak{U}_{\mathsf{sm}}$, denote the operator defined as in (2.3), but with $b(x,u)$ replaced by $b_v(x)$. Note that when $v\in\mathfrak{U}_{\mathsf{sm}}$, we use v as a subscript in $\mathcal{L}_v$ to distinguish it from $\mathcal{L}^u$ when $u\in\mathbb{U}$.

Also for $f\in C^2(\mathbb{R}^d)$, let

$$\begin{aligned}\mathcal{G}f(x) \,&:=\, \min_{u\in\mathbb{U}}\bigl[\mathcal{L}^u f(x) + c(x,u)f(x)\bigr] \qquad\qquad (2.4)\\ &=\, \frac{1}{2}\,\mathrm{Tr}\,\bigl(a(x)\nabla^2 f(x)\bigr) + \min_{u\in\mathbb{U}}\bigl[\bigl\langle b(x,u),\nabla f(x)\bigr\rangle + c(x,u)f(x)\bigr]\,.\end{aligned}$$

This is a semilinear operator in $\mathbb{R}^d$. The associated generalized eigenvalue is written as

$$\lambda_* \,=\, \lambda_*(c) \,:=\, \inf\Bigl\{\lambda\in\mathbb{R} : \exists\,\varphi\in\mathscr{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d),\ \varphi>0,\ \mathcal{G}\varphi-\lambda\varphi\le 0 \text{ a.e. in } \mathbb{R}^d\Bigr\}\,. \tag{2.5}$$

We assume that $\lambda_*<\infty$. Note that in specific problems, this is verified via a Foster–Lyapunov equation of the form

$$\mathcal{L}_v\mathcal{V}(x) + c_v(x)\mathcal{V}(x) \,\le\, \kappa_0 - \kappa_1\mathcal{V}(x)$$

for some positive function $\mathcal{V}\in C^2(\mathbb{R}^d)$ which is bounded away from 0, and some $v\in\mathfrak{U}_{\mathsf{sm}}$, and for $\kappa_0,\kappa_1>0$.

3. A brief review of the infinite-horizon risk-sensitive HJB equation

In this section, we review some important results on the HJB equation of the IHRS control problem and the associated ground diffusion. Note that these results have been established in [8, Theorem 3.1]. In [1, Proposition 1.1, Proposition 1.3, Theorem 1.4], the following results are established under the assumption that the running cost c is *near-monotone* relative to Λ_* which implies that $\Lambda_*<\infty$. Let $\mathcal{C}_0$ be the class of nonnegative functions which are not identically equal to 0 and vanish at infinity.

Proposition 3.1. *Under Assumptions 2.1 to 2.3, the following properties hold.*

(i) *The multiplicative (risk-sensitive) HJB equation*

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u\Psi(x) + c(x,u)\,\Psi(x)\bigr] \;=\; \lambda_*\,\Psi(x) \qquad \forall\, x\in\mathbb{R}^d \tag{3.1}$$

has a solution pair (Ψ_*, λ_*) *such that* $\Psi_*\in C^2(\mathbb{R}^d)$*, satisfying* $\inf_{\mathbb{R}^d}\,\Psi_* > 0$ *with* $\Psi_*(0)=1$*, and* $\lambda_* < \infty$*.*

(ii) *Any* $v\in\mathfrak{U}_{\mathsf{sm}}$ *that satisfies the following is stable and optimal:*

$$\mathcal{L}_v\Psi_*(x) + c(x,v(x))\,\Psi_*(x) = \min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u\Psi_*(x) + c(x,u)\,\Psi_*(x)\bigr] \quad a.e.\ x\in\mathbb{R}^d. \tag{3.2}$$

We denote by $\mathfrak{U}^*_{\mathsf{sm}}$ *the set of measurable selectors from the minimizer of* (3.2)*.*

(iii) *We have*

$$\Lambda^x_v = \Lambda^x_* = \Lambda_* = \lambda_* \quad \forall v\in\mathfrak{U}^*_{\mathsf{sm}}\,, \forall x\in\mathbb{R}^d\,.$$

(iv) *It holds that*

$$\Psi_*(x) \;=\; \mathbb{E}^x_{v^*}\Bigl[\mathrm{e}^{\int_0^T [c(X_t,v(X_t))-\lambda_*]\,\mathrm{d}t}\,\Psi_*(X_T)\Bigr] \qquad \forall\,(T,x)\in\mathbb{R}_+\times\mathbb{R}^d\,,$$

for any $v\in\mathfrak{U}^*_{\mathsf{sm}}$*,*

(v) *Let* $\hat{\uptau}_r$ *be as in Subsection 1.2. If it holds that*

$$\Lambda_{m,*}(c+h) > \Lambda_{m,*}(c)\,, \quad \forall h\in\mathcal{C}_0\,, \tag{3.3}$$

the function Ψ_* *has the stochastic representation*

$$\Psi_*(x) \;=\; \mathbb{E}^x_{v^*}\Bigl[\mathrm{e}^{\int_0^{\hat{\uptau}_r} [c(X_t,v(X_t))-\lambda_*]\,\mathrm{d}t}\,\Psi_*(X_{\hat{\uptau}_r})\Bigr] \qquad \forall\, x\in\bar{\mathcal{B}}^c_r\,,$$

for all $r>0$ *and* $v\in\mathfrak{U}^*_{\mathsf{sm}}$*.*

Observe that using the semilinear operator $\mathcal{G}$ in (2.4), the risk-sensitive HJB equation in (3.1) can be written as the following

$$\mathcal{G}\Psi(x) = \lambda\Psi(x)\,.$$

Then finding the risk-sensitive HJB solution (Ψ_*, λ_*) is equivalent to finding the eigenfunction and eigenvalue for the above equation. If there were no control, then the equation is referred to as the multiplicative Poisson equation (MPE) in the literature. In that case, the solution is related to the principal eigenvalue problem studied in the PDE literature [11]. For the controlled diffusion, properties from the associated MPE and principal eigenvalue problem turn out to be useful when one chooses the control u to be Markov controls $v(x)$ in $\mathfrak{U}_{\mathsf{sm}}$.

Proposition 3.1(ii) asserts that any stationary Markov control v which is a measurable selector from the minimizer of (3.2) is stable and optimal, i.e., $\Lambda^x_v = \Lambda_*$ for all $x\in\mathbb{R}^d$ where

$$\Lambda^x_v \;=\; \limsup_{T\to\infty}\,\frac{1}{T}\,\log\mathbb{E}^x_v\Bigl[\mathrm{e}^{\int_0^T c_v(X_t)\,\mathrm{d}t}\Bigr]\,.$$

In addition, Proposition 3.1(iii) asserts that the optimal risk-sensitive value is equal to the generalized eigenvalue defined in (2.5).

Let $v\in\mathfrak{U}^*_{\mathsf{sm}}$. Using Proposition 3.1(iii) in addition to Jensen's inequality, we get the following

$$\limsup_{T\to\infty}\frac{1}{T}\,\mathbb{E}^x_v\biggl[\int_0^T c_v(X_t)\,\mathrm{d}t\biggr] \;\le\; \Lambda^x_v \;\le\; \lambda_*\,.$$

This together with (2.2) implies that the diffusion X_t in (2.1) is positive recurrent under any $v\in\mathfrak{U}^*_{\mathsf{sm}}$. By [1, Lemma 2.1], this implies that Ψ_* is inf-compact, which in turn implies by (3.1) that the diffusion in (2.1) is exponentially ergodic under any $v\in\mathfrak{U}^*_{\mathsf{sm}}$.

So far, we have discussed the assumptions needed to ensure the existence of a solution Ψ_* to (3.1). Before discussing the conditions for uniqueness of the solution, we give the following definition.

Definition 3.1. Let $C_{\mathrm{o}}^{+}(\mathbb{R}^d)$ denote the collection of all non-trivial, nonnegative, continuous functions which vanish at infinity. Recall the definition of λ_* in (2.5). We say that $\lambda_*(c)$ is *strictly monotone at c on the right* if $\lambda_*(c) < \lambda_*(c+h)$ for all $h \in C_{\mathrm{o}}^{+}(\mathbb{R}^d)$, and that $\lambda_*(c)$ is *strictly monotone at c* if $\lambda_*(c-h) < \lambda_*(c)$ for some $h \in C_{\mathrm{o}}^{+}(\mathbb{R}^d)$.

~~Note also that strict monotonicity at c implies strict monotonicity at c on the right by~~ [2, Theorem 2.1].

Uniqueness of the eigenfunction Ψ_*, which we refer to as the *ground state*, is related to the ergodic properties of the *ground state diffusion*.

The ground state diffusion takes the form

$$\mathrm{d}X_t^* \;=\; \bigl(b(X_t^*,U_t) + a(X_t^*)\nabla\psi(X_t^*)\bigr)\,\mathrm{d}t + \upsigma(X_t^*)\,\mathrm{d}W_t^*\,, \tag{3.4}$$

with $\psi := \log\Psi_*$. Recall the definition of $\mathcal{L}^u$ in (2.3). Let

$$\widetilde{\mathcal{L}}^u \;:=\; \mathcal{L}^u + \bigl\langle \nabla\psi(x), a(x)\nabla\bigr\rangle\,, \qquad u\in\mathbb{U}\,. \tag{3.5}$$

This is the extended generator of the ground state diffusion X_t^* under a control $u\in\mathbb{U}$. This definition can be extended to $\widetilde{\mathcal{L}}_v$ for any Markov control v (not necessarily stationary) by replacing $u\in\mathbb{U}$ with v in (3.5). Clearly, $\widetilde{\mathcal{L}}_v$, with $v\in\mathfrak{U}_{\mathsf{sm}}$, is the extended generator of X_t^* under a Markov control v.

Similar to (2.4), we define the following semilinear operator:

$$\mathcal{G}^*\psi(x) \;:=\; \frac{1}{2}\,\mathrm{Tr}\,\bigl(a(x)\nabla^2\psi(x)\bigr) + \min_{u\in\mathbb{U}}\Bigl[\bigl\langle b(x,u) + \tfrac{1}{2}a(x)\nabla\psi(x), \nabla\psi(x)\bigr\rangle + c(x,u)\Bigr]\,. \tag{3.6}$$

Note the coefficient $\frac{1}{2}$ in front of $a(x)\nabla\psi(x)$, so unlike (2.4), $\mathcal{G}^*\psi(x) \neq \min_{u\in\mathbb{U}}\bigl[\widetilde{\mathcal{L}}^u\psi(x) + c(x,u)\bigr]$. However, simple calculations using (3.1) show that

$$\mathcal{G}^*\psi(x) = \lambda_*\,. \tag{3.7}$$

Note that the sets of measurable selectors from the minimizers of (3.1) and (3.6) are naturally equal.

As shown in [2, Theorem 2.3], the ground state diffusion is recurrent if and only if λ_* is *strictly monotone at c on the right*. Moreover, the ground state diffusion is exponentially ergodic if and only if λ_* is *strictly monotone at c* by [2, Theorem 2.2].

In addition, since Proposition 3.1(iii) asserts that $\Lambda_v^x = \lambda_*$, we have that strict monotonicity of λ_* at c on the right implies that the solution pair (Ψ_*,λ_*) is unique by [1, Theorem 1.2] if condition (3.3) holds.

In Section 4, under Assumption 4.1, we will show that the ground state diffusion is exponentially ergodic in Lemma 4.1. Based on the previous discussion, this implies that the solution pair to (3.1) is unique. Thus, studying the convergence of the VI algorithm to the solution of the multiplicative HJB equation is a well-posed problem.

4. The multiplicative VI algorithm and its global convergence

In this section, we prove that the multiplicative VI algorithm converges to the solution Ψ_* of (3.1) starting from any initial positive function under the structural assumption relating the running cost function c to the ground state Ψ_* in Assumption 4.1.

Formally, let

$$C^2_{\Psi,+}(\mathbb{R}^d) \;:=\; \bigl\{g\in C^2(\mathbb{R}^d):\; g\ge 0\,,\;\|g\|_\Psi<\infty\bigr\}\,,$$

where $\|g\|_\Psi$ is defined in Subsection 1.2.

We introduce the multiplicative VI algorithm through the following equation

$$\partial_t\,\overline{\Phi}(t,x) \;=\; \min_{u\in\mathbb{U}}\bigl[\mathcal{L}_u\overline{\Phi}(t,x) + c(x,u)\,\overline{\Phi}(t,x)\bigr] - \lambda_*\,\overline{\Phi}(t,x)\,, \qquad \forall\,(t,x)\in(0,\infty)\times\mathbb{R}^d\,, \tag{4.1}$$

with $\overline{\Phi}(0,x) = \Phi_0(x)$, $\Phi_0 \in C^2_{\Psi_*,+}(\mathbb{R}^d)$.

It is then clear that the VI algorithm is not feasible to implement as it assumes that the value of λ_* is known. This is the reason we resort to the multiplicative RVI algorithm in Section 5 where the unknown λ_* is replaced by the initial value in (5.1).

Before establishing the main result in this paper, we provide the following structural assumption and discuss how it is related to the existing assumptions in the relevant literature. Recall that $\psi = \log \Psi_*$.

Assumption 4.1. There exist positive constants θ_1 and θ_2 such that

$$\min_{u\in\mathbb{U}}\; c(x,u) - \frac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\nabla\psi(x)\bigr|^2 \;\ge\; \theta_1\psi(x) - \theta_2 \qquad \forall x\in\mathbb{R}^d\,.$$

For example, for a bounded non-degenerate matrix a and a quadratic cost c, we have that $\nabla\psi = \frac{\nabla\Psi_*}{\Psi_*}$ has at most a linear growth by [7, Lemma 4.5]. In this case, Assumption 4.1 holds.

Consider the following example in the case of an uncontrolled diffusion:

$$\sigma(x) = 1; \quad b(x) = -3x; \quad c(x) = 4x^2.$$

Then $\Psi_*(x) = e^{x^2}$ is an eigenfunction with an eigenvalue $\lambda_* = 1$. In addition the ground state diffusion in this case is positive recurrent (in fact geometrically ergodic as it will be shown in Lemma 4.1) since $\psi(x) = x^2$ is a Lyapunov function. Note also that

$$c(x) - \frac{1}{2}|\sigma^T\nabla\psi(x)| = 4x^2 - 2x^2 \ge 2x^2 - 1.$$

Here we used $\theta_1 = 2$ and $\theta_2 = 1$.

Note that an analogous assumption to Assumption 4.1 appeared previously in the study of the ergodic control problem as

$$\min_{u\in\mathbb{U}}\; c(x,u) \;\ge\; \theta_1\psi(x) - \theta_2 \qquad \forall x\in\mathbb{R}^d\,, \tag{4.2}$$

where ψ denoted the solution of the ergodic HJB equation. Equation (4.2) was used to establish global convergence of the VI algorithm in continuous time [5, Theorem 3.20] as well as in the discrete setting [3, 24] for ergodic control problems. The global convergence of the multiplicative VI algorithm under Assumption 4.1 was left as an open conjecture in [8]. Observe that Assumption 4.1 is a modification of (4.2) to account for the *relative entropy term* $\frac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\nabla\psi(x)\bigr|^2$ arising from the logarithmic transformation.

Recall that global convergence refers to the convergence of the multiplicative VI algorithm to the solution Ψ_* of (3.1) starting from any positive initial condition, whereas local convergence means converging to this solution starting within a neighborhood of it. Assumption 4.1 will be used to establish the existence of a stable control in Lemma 4.1 and we then establish the global convergence of the multiplicative VI algorithm under Assumption 4.1 in Theorem 4.2. These results extends upon those in [8] where only local convergence of the multiplicative VI algorithm was shown under the assumption that there exits a control for which the ground state diffusion is positive recurrent.

Recall that $\mathfrak{U}^*_{\mathsf{sm}}$ is the set of measurable selectors from the minimizer of (3.2) and let $v^* \in \mathfrak{U}^*_{\mathsf{sm}}$. We start with the following result.

Lemma 4.1. *Under Assumption 4.1 and with $v^* \in \mathfrak{U}^*_{\mathsf{sm}}$, the function $\psi(x)$ satisfies the following Foster-Lyapunov inequality*

$$\widetilde{\mathcal{L}}_{v^*}\psi(x) \;\le\; \lambda_* - \theta_1\psi(x) + \theta_2\,, \quad \forall x\in\mathbb{R}^d\,.$$

This implies that the process $\{X^*_t\}_{t\ge0}$ in (3.4) is geometrically ergodic under any $v^* \in \mathfrak{U}^*_{\mathsf{sm}}$.

Proof. Under Assumption 4.1, we have the following

$$\begin{aligned}
\widetilde{\mathcal{L}}_{v^*}\psi(x) &= \mathcal{L}_{v^*}\psi(x) + \bigl\langle \nabla\psi(x), a(x)\nabla\psi(x)\bigr\rangle \\
&= \mathcal{G}^*\psi(x) - c(x,v^*) + \frac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\nabla\psi(x)\bigr|^2 \\
&= \lambda_* - c(x,v^*) + \frac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\nabla\psi(x)\bigr|^2 \\
&\le \lambda_* - \theta_1\psi(x) + \theta_2\,, \quad \forall x\in\mathbb{R}^d\,,
\end{aligned}$$

where in the first equality we used (3.5), in the second we used (3.6) combined with (2.3), and in the third one we used (3.7). This completes the proof. □

Thus the ground state diffusion X^*_t in (3.4) is positive recurrent under any $v^*\in\mathfrak{U}^*_{\mathsf{sm}}$ which implies that the ground state Ψ_* is unique as discussed previously.

We continue with the following definitions that are needed when establishing the global convergence of the multiplicative VI algorithm defined in (4.1).

Definition 4.1. Let $\{\hat{v}_t\}_{t\ge 0}$ be an a.e. measurable selector from the minimizer of (4.1). We define the corresponding (nonstationary) Markov control

$$\hat{v}^t \coloneqq \bigl\{\hat{v}^t_s \,=\, \hat{v}_{t-s}(x)\,,\ s\in[0,t]\bigr\}\,.$$

Let $\mathcal{S}_t[\Phi_0](x)$, $t\ge 0$, denote the solution of (4.1) starting from an initial condition Φ_0. It is clear that $\mathcal{S}_t[\Psi_*]=\Psi_*$ for all $t\ge 0$ by Proposition 3.1 (i), and that the uniqueness of the ground state Ψ_* implies that any positive initial condition Φ_0 satisfying $\mathcal{S}_t[\Phi_0]=\Phi_0$ for all $t\ge 0$ must equal the ground state Ψ_* up to a positive multiplicative constant.

Let $\mathcal{E}$ denote the set of equilibria of $\mathcal{S}_t$, or equivalently, the set of solutions of the risk-sensitive HJB equation in (3.1), that is,

$$\mathcal{E} \,\coloneqq\, \{r\Psi_*\colon\ r>0\}\,. \tag{4.3}$$

We continue with the following important lemma.

Lemma 4.2. *Let $\overline{\Phi}(t,x)$ be as in (4.1). Under Assumption 4.1, for any $\Phi_0\in C^2_{\Psi_*}(\mathbb{R}^d)$ satisfying $\inf_{\mathbb{R}^d}\Phi_0\ge 1$, we have*

$$\exp\bigl(-\mathrm{e}^{-\theta_1 t}\psi(x) - \tfrac{\lambda_*+\theta_2}{\theta_1}\bigr) \,\le\, \frac{\overline{\Phi}(t,x)}{\Psi_*(x)} \,\le\, \|\Phi_0\|_{\Psi_*} \qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}^d\,.$$

Proof. Let $\overline{\Phi}(t,x)$ denote the canonical solution of (4.1) with initial data Φ_0 and let $\{\hat{v}_t\,,\ t\in\mathbb{R}_+\}$ denote a measurable selector from the corresponding minimizer. Without loss of generality we assume $\Phi_0\ge 1$ since the solution of (4.1) is invariant under scalar multiplication. Let $\overline{\varphi}\coloneqq\log\overline{\Phi}$ and recall that $\psi=\log\Psi_*$.

Note that $\overline{\varphi}$ satisfies

$$\begin{aligned}
\partial_t\,\overline{\varphi}(t,x) &= \min_{u\in\mathbb{U}}\bigl[\mathcal{L}^u\,\overline{\varphi}(t,x) + c(x,u)\bigr] + \tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\,\nabla\overline{\varphi}\bigr|^2 - \lambda_* \\
&= \mathcal{L}_{\hat{v}_t}\,\overline{\varphi}(t,x) + \tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\,\nabla\overline{\varphi}(x)\bigr|^2 + c_{\hat{v}_t}(x) - \lambda_*\,.
\end{aligned} \tag{4.4}$$

In addition,

$$\mathcal{L}_{\hat{v}_t}\,\psi \,\ge\, -\tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}\,\nabla\psi\bigr|^2 - c_{\hat{v}_t} + \lambda_* \tag{4.5}$$

by combining (3.6) and (3.7) since $\hat{v}_t$ is not a measurable selector from the minimizer of (3.6).

Let

$$\chi(t,x) \,\coloneqq\, \overline{\varphi}(t,x) - \bigl(1-\mathrm{e}^{-\theta_1 t}\bigr)\bigl(\psi(x) - \tfrac{\lambda_*+\theta_2}{\theta_1}\bigr)\,. \tag{4.6}$$

Recall (3.5) and let $\widetilde{\mathbb{E}}^t_x$ denote the expectation operator corresponding to the generator $\widetilde{\mathcal{L}}_{\hat{v}_t}$. Combining (4.4)–(4.6), and using Assumption 4.1, we obtain

$$\begin{aligned}
F(t,x) &\coloneqq \partial_t\chi(t,x) - \widetilde{\mathcal{L}}_{\hat{v}_t}\chi(t,x) \\
&\ge c_{\hat{v}_t}(x) - \lambda_* + \tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\,\nabla\overline{\varphi}(x)\bigr|^2 \\
&\qquad - \bigl(1-\mathrm{e}^{-\theta_1 t}\bigr)\bigl(c_{\hat{v}_t}(x)-\lambda_*\bigr) - \bigl\langle\nabla\psi(x),\upsigma(x)\upsigma^{\mathsf{T}}(x)\,\nabla\chi(t,x)\bigr\rangle \\
&\qquad\qquad - \theta_1\mathrm{e}^{-\theta_1 t}\bigl(\psi(x) - \tfrac{\lambda_*+\theta_2}{\theta_1}\bigr) - \tfrac{1}{2}\bigl(1-\mathrm{e}^{-\theta_1 t}\bigr)\,\bigl|\upsigma^{\mathsf{T}}(x)\,\nabla\psi(x)\bigr|^2 \\
&= \tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\,\bigl(\nabla\overline{\varphi}(x)-\nabla\psi(x)\bigr)\bigr|^2 + c_{\hat{v}_t}(x) - \lambda_* - \theta_1\mathrm{e}^{-\theta_1 t}\bigl(\psi(x)-\tfrac{\lambda_*+\theta_2}{\theta_1}\bigr) \\
&\qquad - \bigl(1-\mathrm{e}^{-\theta_1 t}\bigr)\bigl(c_{\hat{v}_t}(x)-\lambda_*\bigr) - \tfrac{1}{2}\mathrm{e}^{-\theta_1 t}\bigl|\upsigma^{\mathsf{T}}(x)\,\nabla\psi(x)\bigr|^2 \\
&= \mathrm{e}^{-\theta_1 t}\biggl(c_{\hat{v}_t}(x) - \tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\,\nabla\psi(x)\bigr|^2 - \theta_1\psi(x) + \theta_2\biggr) + \tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\,\bigl(\nabla\overline{\varphi}(x)-\nabla\psi(x)\bigr)\bigr|^2 \\
&\ge \tfrac{1}{2}\bigl|\upsigma^{\mathsf{T}}(x)\,\bigl(\nabla\overline{\varphi}(x)-\nabla\psi(x)\bigr)\bigr|^2 \ge 0 \qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}^d\,.
\end{aligned} \tag{4.7}$$

Hence, by (4.7) we get

$$\chi(t,x) \,\ge\, \widetilde{\mathbb{E}}^t_x\bigl[\overline{\varphi}(0,x)\bigr] \,=\, \widetilde{\mathbb{E}}^t_x\bigl[\log\Phi_0\bigr] \,\ge\, 0, \qquad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}^d\,.$$

Therefore, we obtain

$$\exp\bigl(-\mathrm{e}^{-\theta_1 t}\psi(x) - \tfrac{\lambda_*+\theta_2}{\theta_1}\bigr) \,\le\, \frac{\overline{\Phi}(t,x)}{\Psi_*(x)}\,.$$

Concerning the upper bound, we note that

$$\begin{aligned}
\mathcal{S}_t[\Phi_0](x) &= \mathcal{S}_t\Bigl[\frac{\Phi_0}{\Psi_*}\Psi_*\Bigr](x) \\
&\le \mathcal{S}_t\bigl[\|\Phi_0\|_{\Psi_*}\Psi_*\bigr](x) \\
&= \|\Phi_0\|_{\Psi_*}\Psi_*(x) \qquad \forall\,\Phi_0>0,
\end{aligned}$$

where the first and the last equalities follow from the definition of $\mathcal{S}_t[.]$ and the fact that $\Psi_*\in\mathcal{E}$ which is the set of equilibria of $\mathcal{S}_t[.]$, and the inequality by the monotonicity of $f\mapsto\mathcal{S}_t[f]$. This completes the proof. □

Before establishing the main result in this paper, we recall the following result from [8, Theorem 3.2] with a slight modification.

For positive constants κ_0 and κ_1, we define the set $G\subset C^2(\mathbb{R}^d)$ by

$$G \coloneqq \bigl\{h\in C^2(\mathbb{R}^d):\, \kappa_0\Psi_* \,\le\, h \,\le\, \kappa_1\,\Psi_*\bigr\}\,. \tag{4.8}$$

We then have the following result.

Theorem 4.1. *Assume that the ground state diffusion in (3.4) is positive recurrent under some $v\in\mathfrak{U}^*_{\mathsf{sm}}$. If $\Phi_0\in G$ as in (4.8) for some positive constants κ_0 and κ_1, then $\mathcal{S}_t[\Phi_0]$ converges to $c_0\Psi_*\in\mathcal{E}$ for some $c_0\in[\kappa_0,\kappa_1]$ as $t\to\infty$ where $\mathcal{E}$ is as in (4.3). Moreover, the only bounded subsets of G, which are invariant under $\mathcal{S}_t[\Phi_0]$, are the points (singletons) of $\mathcal{E}\cap G$.*

The following theorem is the main result in this paper and asserts the global convergence of the multiplicative VI algorithm.

Theorem 4.2. *Suppose Assumptions 2.1 to 2.3 and 4.1 hold. Then, for any $\Phi_0\in C^2_{\Psi_*}(\mathbb{R}^d)$ with $\inf_{\mathbb{R}^d}\Phi_0>0$, then $\mathcal{S}_t[\Phi_0]$ converges, as $t\to\infty$, to a point $c_0\Psi_*\in\mathcal{E}$ satisfying*

$$\mathrm{e}^{-\frac{\lambda_*+\theta_2}{\theta_1}} \,\le\, c_0 \,\le\, \|\Phi_0\|_{\Psi_*}\,.$$

Proof. Here we use the following argument: Recall that under Assumption 4.1 the ground state diffusion was shown to be positive recurrent in Lemma 4.1.

By Lemma 4.2, every ω-limit point h of $\mathcal{S}_t[\Phi_0]$ lies in the set

$$G := \left\{h \in C^2(\mathbb{R}^d) :\ \mathrm{e}^{-\frac{\lambda_*+\theta_2}{\theta_1}}\Psi_* \,\le\, h \,\le\, \|\Phi_0\|_{\Psi_*}\,\Psi_*\right\}$$

Recall that the ω-limit set of Φ_0 is invariant under $\mathcal{S}_t$. Since by Theorem 4.1 the only invariant subsets of G are the points of $\mathcal{E}\cap G$, the result follows. This completes the proof. □

5. The multiplicative RVI algorithm

As mentioned before, the multiplicative VI algorithm is not feasible to implement as it requires λ_* to be known. In this section, we introduce the multiplicative RVI algorithm and show that the convergence of the VI algorithm to the solution Ψ_* of (3.1) implies the convergence of the RVI algorithm to this solution.

Hence we modify (4.1) and introduce the multiplicative RVI algorithm as follows:

$$\partial_t\,\Phi(t,x) \;=\; \min_{u\in\mathbb{U}}\,\bigl[\mathcal{L}_u\Phi(t,x)+c(x,u)\,\Phi(t,x)\bigr]-\Phi(t,0)\,\Phi(t,x)\qquad \forall\,(t,x)\in(0,\infty)\times\mathbb{R}^d\,, \tag{5.1}$$

with $\Phi(0,x)=\Phi_0(x)$.

While the convergence of (5.1) is usually viewed as a stabilization of a nonlinear parabolic PDE in the sense of Has′minskiĭ [22], we prefer to think of it as a continuous-time continuous-space analogue of the RVI algorithm proposed in [29].

We start with the following lemma.

Lemma 5.1. *Suppose that the multiplicative VI algorithm in (4.1) converges to a solution $c_0\Psi_*$ of (3.1) for some constant c_0, then the RVI algorithm in (5.1) also converges to $c_1\Psi_*$ for some constant c_1.*

Proof. The proof is adopted from [8, Section 3.3] and we add it for completeness. Note that if Φ solves (5.1), then

$$\overline{\Phi}(t,x) \;=\; \Phi(t,x)\,\mathrm{e}^{\int_0^t(\Phi(s,0)-\lambda_*)\,\mathrm{d}s} \tag{5.2}$$

solves (4.1).

In addition, from (5.2), we directly obtain

$$\frac{\overline{\Phi}(t,x)}{\Phi(t,x)} \;=\; \frac{\overline{\Phi}(t,0)}{\Phi(t,0)}\qquad \forall\,(t,x)\in(0,\infty)\times\mathbb{R}^d\,, \tag{5.3}$$

which implies that $\frac{\overline{\Phi}(t,x)}{\Phi(t,x)}$ does not depend on x.

By (5.2)–(5.3), we have

$$\begin{aligned}\frac{\partial}{\partial t}\,\frac{\overline{\Phi}(t,x)}{\Phi(t,x)} \;&=\; \bigl(\Phi(t,0)-\lambda_*\bigr)\,\frac{\overline{\Phi}(t,x)}{\Phi(t,x)}\\ &=\; \overline{\Phi}(t,0)-\lambda_*\,\frac{\overline{\Phi}(t,x)}{\Phi(t,x)}\,.\end{aligned} \tag{5.4}$$

Recall that $\Psi_*(0)=1$ by Proposition 3.1(i). Since $\overline{\Phi}(t,0)\to c_0\Psi_*(0)=c_0>0$ as $t\to\infty$ by assumption, it follows by (5.4) that $\frac{\overline{\Phi}(t,x)}{\Phi(t,x)}$ converges to $\frac{c_0}{\lambda^*}$ as $t\to\infty$.

□

We continue with the following theorem on the global convergence of the multiplicative RVI algorithm.

Theorem 5.1. *Let Assumptions 2.1 to 2.3 and 4.1 hold. Then, for any initial condition* $\Phi_0 \in C^2_\Psi(\mathbb{R}^d)$ *with* $\inf_{\mathbb{R}^d} \Phi_0 > 0$*, the solution* Φ *of (5.1) converges as* $t \to \infty$ *to* $\lambda_* \Psi$ *uniformly on compact sets of* $\mathbb{R}^d$*.*

Proof. This follows directly by combining Lemma 5.1 with Theorem 4.2. □

6. Conclusion and Future Directions

In this paper, we considered the IHRS control problem for non-degenerate controlled diffusions with a compact action space, and controlled through the drift. Using only a structural assumption on the running cost function which amounts to a strong form of *near-monotonicity*, we established the existence of a controller for which the ground state diffusion is exponentially ergodic and showed that the multiplicative VI/RVI algorithms converge globally (starting from any positive initial condition) to the solution of the multiplicative HJB equation corresponding to the IHRS control problem.

It would be interesting to check if a similar assumption to Assumption 4.1 can be used to establish these results in the case of a discrete-time Markov chain with a compact Polish state space [3] and for controlled jump diffusions [6].

Acknowledgements

The work of Ari Arapostathis was supported in part by the Army Research Office (ARO) through grant W911NF-17-1-001, and in part by the National Science Foundation (NSF) through grant DMS-1715210. Guodong Pang was partly supported by the ARO grant W911NF-17-1-001 and by the NSF grant DMS-2216765.

References

[1] A. Arapostathis and A. Biswas, *Infinite horizon risk-sensitive control of diffusions without any blanket stability assumptions*, Stochastic Process. Appl. **128** (2018), no. 5, 1485–1524. MR3780687

[2] A. Arapostathis, A. Biswas, and S. Saha, *Strict monotonicity of principal eigenvalues of elliptic operators in* $\mathbb{R}^d$ *and risk-sensitive control*, J. Math. Pures Appl. (9) **124** (2019), 169–219. MR3926044

[3] A. Arapostathis and V. S. Borkar, *Average cost optimal control under weak hypotheses: Relative value iterations*, arXiv e-prints **1902.01048** (2019), available at `https://arxiv.org/abs/1902.01048`.

[4] A. Arapostathis, V. S. Borkar, and M. K. Ghosh, *Ergodic control of diffusion processes*, Encyclopedia of Mathematics and its Applications, vol. 143, Cambridge University Press, Cambridge, 2012. MR2884272

[5] A. Arapostathis, V. S. Borkar, and K. S. Kumar, *Convergence of the relative value iteration for the ergodic control problem of nondegenerate diffusions under near-monotone costs*, SIAM J. Control Optim. **52** (2014), no. 1, 1–31. MR3148068

[6] A. Arapostathis and A. Biswas, *Risk-sensitive control for a class of diffusions with jumps*, arXiv preprint arXiv:1910.05004 (2019).

[7] A. Arapostathis, A. Biswas, V. S Borkar, and K S. Kumar, *A variational characterization of the risk-sensitive average reward for controlled diffusions on* $\mathbb{R}^d$, SIAM Journal on Control and Optimization **58** (2020), no. 6, 3785–3813.

[8] A. Arapostathis and V. S Borkar, *On the relative value iteration with a risk-sensitive criterion*, Banach Center Publications **122** (2020), 9–24.

[9] A. Arapostathis, H. Hmedi, and G. Pang, *On uniform exponential ergodicity of markovian multiclass many-server queues in the halfin–whitt regime*, Mathematics of Operations Research **46** (2021), no. 2, 772–796.

[10] A. Bensoussan and R. J. Elliott, *A finite-dimensional risk-sensitive control problem*, SIAM J. Control Optim. **33** (1995), no. 6, 1834–1846.

[11] H. Berestycki and L. Rossi, *Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains*, Communications on Pure and Applied Mathematics **68** (2015), no. 6, 1014–1065.

[12] A. Biswas, V. S. Borkar, and K. Suresh Kumar, *Risk-sensitive control with near monotone cost*, Appl. Math. Optim. **62** (2010), no. 2, 145–163.

[13] V. S. Borkar and S. P. Meyn, *Risk-sensitive optimal control for Markov decision processes with monotone cost*, Math. Oper. Res. **27** (2002), no. 1, 192–209. MR1886226

[14] R. Cavazos-Cadena and D. Hernández-Hernández, *A characterization of the optimal risk-sensitive average cost in finite controlled Markov chains*, Ann. Appl. Probab. **15** (2005), no. 1A, 175–212.

[15] R. Cavazos-Cadena and R. Montes-de Oca, *The value iteration algorithm in risk-sensitive average Markov decision chains with finite state space*, Math. Oper. Res. **28** (2003), no. 4, 752–776. MR2015911

[16] R. Cavazos-Cadena and R. Montes-de Oca, *Nonstationary value iteration in controlled Markov chains with risk-sensitive average criterion*, J. Appl. Probab. **42** (2005), no. 4, 905–918. MR2203811

[17] M. H. Davis and S. Lleo, *Risk-sensitive investment management*, Vol. 19, World Scientific, 2014.

[18] G. B. Di Masi and L. Stettner, *Risk-sensitive control of discrete-time Markov processes with infinite horizon*, SIAM J. Control Optim. **38** (1999), no. 1, 61–78 (electronic).

[19] G. B. Di Masi and Ł. Stettner, *Infinite horizon risk sensitive control of discrete time Markov processes under minorization property*, SIAM J. Control Optim. **46** (2007), no. 1, 231–252 (electronic).

[20] W. H. Fleming, *Risk sensitive stochastic control and differential games*, Commun. Inf. Syst. **6** (2006), no. 3, 161–177.

[21] W. H. Fleming and W. M. McEneaney, *Risk-sensitive control on an infinite time horizon*, SIAM J. Control Optim. **33** (1995), no. 6, 1881–1915.

[22] R. Z. Has′minskiĭ, *Ergodic properties of recurrent diffusion processes and stabilization of the solution of the Cauchy problem for parabolic equations*, Theory Probab. Appl. **5** (1960), no. 2, 179–196.

[23] H. Hmedi, A. Arapostathis, and G. Pang, *Uniform stability of a class of large-scale parallel server networks*, Accepted in Queueing Systems Theory and Applications, available at https://arxiv.org/abs/1907.04793.

[24] H. Hmedi, J. Carroll, and A. Arapostathis, *Optimal sensor scheduling under intermittent observations subject to network dynamics*, IEEE Transactions on Automatic Control (2022), 1–1.

[25] A. Jaśkiewicz, *Average optimality for risk-sensitive control with general state space*, Ann. Appl. Probab. **17** (2007), no. 2, 654–675.

[26] H. Kaise and S.-J. Sheu, *On the structure of solutions of ergodic type Bellman equation related to risk-sensitive control*, Ann. Probab. **34** (2006), no. 1, 284–320. MR2206349

[27] H. Nagai, *Bellman equations of risk-sensitive control*, SIAM J. Control Optim. **34** (1996), no. 1, 74–101.

[28] A. J Nagengast, D. A Braun, and D. M Wolpert, *Risk-sensitive optimal feedback control accounts for sensorimotor behavior under uncertainty*, PLoS computational biology **6** (2010), no. 7, e1000857.

[29] D. J White, *Dynamic programming, markov chains, and the method of successive approximations*, Journal of Mathematical Analysis and Applications **6** (1963), no. 3, 373–376.