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New families satisfying the dynamical uniform boundedness principle over function fields

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Abstract

We extend a technique, originally due to the first author and Poonen, for proving cases of the Strong Uniform Boundedness Principle (SUBP) in algebraic dynamics over function fields of positive characteristic. The original method applied to unicritical polynomials for which the characteristic does not divide the degree. We show that many new 1-parameter families of polynomials satisfy the SUBP, including the family of all quadratic polynomials in even characteristic. We also give a new family of non-polynomial, non-Lattès rational functions that satisfies the SUBP.

1 Introduction

Our goal is to extend a technique of the first author and Poonen for proving cases of the Strong Uniform Boundedness Conjecture in arithmetic dynamics over function fields. For notation, let K be a field and $f \in K(z)$ a nonconstant rational function. Define the set of K-rational **preperiodic points** of f to be

$$\mathrm{PrePer}(f, K) := \{x \in \mathbb{P}^1(K) \ : \ x \text{ has finite forward orbit under } f\}.$$

Uniform Boundedness Principles. Let K be any field with algebraic closure $\overline{K}$. Let $d > 1$, and let $\mathcal{F} \subset \overline{K}(z)$ be a set of rational functions of degree d.

- We say that $\mathcal{F}$ satisfies the **Uniform Boundedness Principle** (**UBP**) over K if there is a bound $A = A(\mathcal{F}, K)$ such that $\#\mathrm{PrePer}(f, K) \leq A$ for each $f \in \mathcal{F}(K)$.

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- We say that $\mathcal{F}$ satisfies the **Strong Uniform Boundedness Principle** (**SUBP**) over K if for every $D \geq 1$ there is a bound $B = B(\mathcal{F}, D)$ such that $\#\mathrm{PrePer}(f, L) \leq B$ for every extension L/K of degree D and every $f \in \mathcal{F}(L)$.

In order for either of these Uniform Boundedness Principles to hold over a field K, the number of K-rational preperiodic points for maps in the family $\mathcal{F}$ must be finite. Thus, for example, (S)UBP will not be satisfied over an algebraically closed field. On the other hand, when K is a number field, Northcott showed that any rational function of degree at least 2 has only finitely many K-rational preperiodic points [24]. In this setting, Morton and Silverman have conjectured that the set $\mathcal{F} = \mathrm{Rat}_d(\overline{\mathbb{Q}})$ of degree-d rational functions with algebraic number coefficients satisfies the Strong Uniform Boundedness Principle over $\mathbb{Q}$ [21, p. 100]. We give a summary of the current state of knowledge in the next section.

Let k be a field. Throughout this article, a **function field over** k is any finite extension of the rational function field $k(T)$. Equivalently, a function field over k is the field of rational functions of some integral k-curve. We refer to k as the **constant subfield** of the function field.

The primary test case for many ideas in dynamics on the projective line is the family of quadratic polynomials $f_c(z) = z^2 + c$. The first author and Poonen proved the Strong Uniform Boundedness Principle for this family over a function field[1] in characteristic different from 2 [9]. Though global in spirit, the argument for the positive characteristic case in [9] leaned heavily on an analysis of the dynamics of the map $f(z) = z^2 + t$ on the Julia set over the Laurent series field $\mathbb{F}_q((1/t))$. By generalizing this dynamical setup, we can extend this argument to other 1-parameter families.

More precisely, take $f_t \in \mathbb{F}_q(t)(z)$ with $d := \deg_z(f_t) > 1$. We can associate to f_t a countable collection of smooth algebraic $\mathbb{F}_q$-curves, known as *dynatomic curves*, whose L-rational points (for an extension $L/\mathbb{F}_q$) parameterize maps f_s with $s \in L$ together with a marked L-rational preperiodic point of f_s. Each dynatomic curve Z is equipped with a canonical morphism $\varphi_Z \colon Z \to \mathbb{P}^1$, projecting onto the t-coordinate and forgetting the marked preperiodic point. See Sect. 3.2 for more details. For convenience, we will say that a property of dynatomic curves holds "as $Z \to \infty$" if that property holds as $i \to \infty$ for any ordering $Z_1, Z_2, Z_3, \ldots$ of the set of dynatomic curves. Consider the following statements:

(1) The degree of definition and the ramification index of the geometric points in the fiber $\varphi_Z^{-1}(\infty)$ remain uniformly bounded as $Z \to \infty$.
(2) The degree of $\varphi_Z \colon Z \to \mathbb{P}^1$ tends to infinity as $Z \to \infty$.
(3) There is $r \geq 1$ such that $\#Z(\mathbb{F}_{q^r}) \to \infty$ as $Z \to \infty$.
(4) The gonality[2] of Z tends to infinity as $Z \to \infty$.
(5) For any extension $k/\mathbb{F}_q$, and for any function field K over k, the Strong Uniform Boundedness Principle over K holds for the set

$$\mathcal{F} = \{f_s \ : \ s \in \overline{K} \text{ and } f_s \text{ is not } \overline{K}\text{-conjugate to an element of } \overline{k}(z)\}.$$

[1] For the SUBP to hold in this setting, one must generally exclude those parameters c lying in the constant subfield.

[2] The **gonality** of an irreducible k-curve C is the minimum degree of a nonconstant k-morphism $C \to \mathbb{P}^1$.

For brevity, we will say "the Strong Uniform Boundedness Principle over K holds for the family f_t" if (5) is true. The arguments in [9] give the chain of implications

$$(1) + (2) \implies (3) \implies (4) \implies (5).$$

We provide sufficient conditions for (1) and (2) to hold in Sect. 4. Applying them to some special cases, we are able to execute the above plan for new families of polynomials, including the first case where the characteristic of the ground field divides the degree.

Theorem 1.1 *Let $\mathbb{F}_q$ be a finite field, let $d > 1$, and let $\alpha_1, \ldots, \alpha_d \in \mathbb{F}_q[t]$ be distinct polynomials. Set*

$$f_t(z) = (z - \alpha_1) \cdots (z - \alpha_d).$$

Assume that for each $i \neq j$, the following inequality is true:

$$\deg(\alpha_i - \alpha_j) + \sum_{\ell \neq j} \deg(\alpha_\ell - \alpha_j) > \max_\ell \deg(\alpha_\ell).$$

Then, for any extension $k/\mathbb{F}_q$ and any function field K over k, the Strong Uniform Boundedness Principle over K holds for the family f_t.

For example, we could enumerate the elements of $\mathbb{F}_q$ as $\varepsilon_1, \ldots, \varepsilon_q$ and set $\alpha_i = \varepsilon_i t$. The hypothesis of the theorem is satisfied, and we have $f_t(z) = z^q - t^{q-1}z$. Cases where the characteristic of the ground field divides the degree of the family have been a sticking point in much previous work.

The first author and Poonen proved that the Strong Uniform Boundedness Principle holds for the set of all quadratic polynomials over a function field of characteristic $p \neq 2$ [9]. That approach focused on the family z^2+c, which does not capture a general quadratic in even characteristic. We now prove the Strong Uniform Boundedness Principle for this family in all positive characteristics by studying a family of quadratic polynomials which is more appropriate in characteristic 2.

Corollary 1.2 *Let k be any field, and let K be a function field over k. For each $D \geq 1$, there exists a bound $B = B(D)$ such that $\#\mathrm{PrePer}(f, L) \leq B$ for any finite extension L/K of degree at most D and any quadratic polynomial f with coefficients in L that is not conjugate to a polynomial with coefficients in $\overline{k}$.*

Proof We may assume that the characteristic of k is positive, as the characteristic-zero case was handled in [9].

By Theorem 1.1, the Strong Uniform Boundedness Principle holds for the family $f_t(z) = z(z + t)$. Fix $D \geq 1$, and set $D' = 2D$. Then there exists a bound $B = B(D')$ such that for any function field L'/K with $[L' : K] \leq D'$ and any $s \in L' \smallsetminus \overline{k}$, we have

$$\#\mathrm{PrePer}(f_s, L') \leq B.$$

We claim that B is the bound we seek.

Let L/K be an extension of degree D, and let $g(z) = az^2 + bz + c$ be a quadratic polynomial over L. Replacing $g(z)$ with $ag(z/a)$ allows us to assume $a = 1$ without affecting the number of L-rational preperiodic points. Set $L' = L(u)$, where u is a solution to the equation $z^2 + (b-1)z + c = 0$, and define

$$h(z) = g(z+u) - u = z(z + 2u + b).$$

Since $[L' \colon K] \leq 2[L \colon K] \leq D'$, and since h is a member of the quadratic family f_t, we find that

$$\#\mathrm{PrePer}(g, L) \leq \#\mathrm{PrePer}(h, L') \leq B.$$

□

A rational function is of *polynomial* type if it has a totally ramified fixed point. Aside from trivial examples like finite sets, we are aware of three sets of *non-polynomial* type rational functions for which the Strong Uniform Boundedness Principle has been shown to hold over number fields or function fields: Lattès maps, functions with a bounded amount of bad reduction, and twists of a single rational function with nontrivial automorphism group. We give further details and provide references on these examples in Sect. 2. We close this introduction with an example of an algebraic family of non-polynomial type rational functions that avoids all of these special cases.

Theorem 1.3 *Let k be a field of characteristic $p > 0$, and let K be a function field over k. Let $d \geq 2$ and $e \leq d-2$ be nonnegative integers such that p does not divide d. Define*

$$f_t(z) = \frac{z^d - t}{z^e}.$$

Then the Strong Uniform Boundedness Principle over K holds for f_t.

Remark 1.4 By taking $e = 0$ and replacing t with $-t$, Theorem 1.3 applies to the family $z^d + t$ with $p \nmid d$, thus allowing us to recover the main results of [9] in positive characteristic.

2 The dynamical uniform boundedness conjecture

Inspired by the strong uniform boundedness conjecture for torsion points on elliptic curves (later proved by Merel [19]), Morton and Silverman posed the following conjecture, which we state in terms of the Strong Uniform Boundedness Principle. For a field K, write $\mathrm{Rat}_d(K)$ for the set of all degree-d rational functions defined over K.

Conjecture 2.1 (Dynamical Uniform Boundedness Conjecture; [21, p. 100]) *For each integer $d \geq 2$, the family $\mathrm{Rat}_d(\overline{\mathbb{Q}})$ satisfies the Strong Uniform Boundedness Principle over $\mathbb{Q}$.*

Conjecture 2.1 is currently wide open. To illustrate the difficulty in proving this conjecture, we note that just the $d = 4$ case of Conjecture 2.1 is sufficient to prove Merel's theorem, since a point P on an elliptic curve in Weierstrass form is a torsion point if and only if its x-coordinate is preperiodic under the "duplication map" $x(P) \mapsto x(2P)$, which is a degree-4 rational function on $\mathbb{P}^1$.

There has been considerable progress, however, especially when restricting to polynomial maps. We summarize the current state of affairs by describing subsets of $\mathrm{Rat}_d(\overline{\mathbb{Q}})$ for which the (S)UBP has been proven.

(1) Looper has shown that, assuming a strong version of the *abc*-conjecture for number fields, the set $\mathrm{Poly}_d(\overline{\mathbb{Q}})$ of degree-d polynomials satisfies the UBP over every number field [16, 17].

(2) A Lattès map is an endomorphism of $\mathbb{P}^1$ of degree at least 2 that is covered by an endomorphism of an elliptic curve [27, §6.4]. Using deep results of Mazur, Kamienny, and Merel, one can show that the SUBP holds for the set of Lattès maps over a number field [27, §6.7].

(3) For a number field K and a rational function $f \in K(z)$, we say that f has *good reduction* at the prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_K$ if the reduction[3] of f modulo $\mathfrak{p}$ has degree equal to the degree of f. Then for any integer $s \geq 0$, it is known that

$$\mathrm{Rat}_{d,s}(\overline{\mathbb{Q}}) := \bigcup_{K/\mathbb{Q}\text{ finite}} \left\{ f \in \mathrm{Rat}_d(K) \ : \ \begin{array}{l} f \text{ has good reduction away} \\ \text{from a set of } s \text{ primes of } \mathcal{O}_K \end{array} \right\}$$

satisfies the SUBP over $\mathbb{Q}$. There is a substantial literature on this topic; we mention the articles [2, 5, 21, 22], but we recommend [27, Remark 3.16] for a more comprehensive list.

(4) It was conjectured in [11] that a quadratic polynomial in $\mathbb{Q}[z]$ cannot have rational periodic points of period larger than 3, and there is a significant amount of evidence to support this; see [11, 12, 20, 28]. Poonen proved that the set

$$\mathcal{F} := \left\{ f \in \mathbb{Q}[z] \ : \ \begin{array}{l} \deg(f) = 2 \text{ and } f \text{ has no rational} \\ \text{point of period greater than } 3 \end{array} \right\}$$

satisfies the UBP over $\mathbb{Q}$ [25]. In fact, the explicit bound $\#\mathrm{PrePer}(f, \mathbb{Q}) \leq 9$ is given for $f \in \mathcal{F}$.

(5) Recall that the *automorphism group* of a rational function $f \in K(z)$ is the subgroup of elements $g \in \mathrm{PGL}_2(\overline{K})$ such that $g^{-1} \circ f \circ g = f$. Manes has proved that the set

$$\mathcal{F} := \left\{ f \in \mathbb{Q}(z) \ : \ \begin{array}{l} \deg(f) = 2,\ \mathrm{Aut}(f) \neq 1, \text{ and } f \text{ has no} \\ \text{rational point of period greater than } 4 \end{array} \right\}$$

satisfies the UBP over $\mathbb{Q}$ [18]. Moreover, Manes showed that the bound $\#\mathrm{PrePer}(f, \mathbb{Q}) \leq 12$ holds for all $f \in \mathcal{F}$.

[3] To define the reduction modulo $\mathfrak{p}$, one should normalize f so that the minimum $\mathfrak{p}$-adic valuation of the coefficients is 0.

(6) For a fixed number field K and rational function $f \in K(z)$ of degree at least 2, let $[f]$ denote the set of all rational functions in $K(z)$ that are $\mathrm{PGL}_2(\overline{K})$-conjugate to f. Levy, Manes, and Thompson have shown that $[f]$ satisfies the UBP over K [15].[4]

We also recommend [6–8, 13] for additional results toward uniform boundedness over number fields.

A version of Conjecture 2.1 for function fields is also believed to be true, once one removes certain obvious sources of counterexamples. Recall that elements of the group PGL_2 act on rational functions by conjugation.

Conjecture 2.2 (Dynamical Uniform Boundedness Conjecture for function fields) *Let k be a field, and let K be the function field of an integral curve over k. For each integer $d \geq 2$, the family*

$$\mathrm{Rat}_d(\overline{K}) \smallsetminus \mathrm{PGL}_2(\overline{K}).\mathrm{Rat}_d(\overline{k})$$

satisfies the Strong Uniform Boundedness Principle over K.

We must remove the rational functions that are conjugate to an element of $\overline{k}(z)$—the so-called **isotrivial** functions—because they can have infinitely many preperiodic points if the field k is infinite.

Let k be a field, and let K be a function field over k.

(1) Looper showed that if k has characteristic zero and a strong version of the *abc*-conjecture holds for K, then the family

$$\mathrm{Poly}_d(\overline{K}) \smallsetminus \mathrm{Aff}_2(\overline{K}).\mathrm{Poly}_d(\overline{k})$$

satisfies the UBP over K [16, 17]. Here $\mathrm{Aff}_2(\overline{K})$ is the subgroup of $\mathrm{PGL}_2(\overline{K})$ that preserves the point at infinity.

(2) The SUBP holds for the set of Lattès maps over K that do not arise from an elliptic curve whose j-invariant is algebraic over k [23, 26].

(3) Benedetto showed that the space $\mathrm{Poly}_{d,s}(\overline{K})$ of degree-d polynomials with good reduction away from a set of s places satisfies the SUBP over K [1]. Assuming k has characteristic zero, Canci has shown that $\mathrm{Rat}_{d,s}\big(k(T)\big)$ satisfies the UBP over $k(T)$ [4].

(4) The first author and Poonen showed that if the characteristic of k does not divide d, then the family

$$\mathcal{F}(d) := \{z^d + c : c \in \overline{K} \smallsetminus \overline{k}\}$$

satisfies the SUBP over K [9]. (Compare Remark 1.4.)

[4] The content of this statement comes from the fact that a rational function f may admit infinitely many nontrivial **twists**: rational functions conjugate to f over $\overline{K}$, but not over K.

3 Sufficient local conditions for the SUBP

We begin with a technical lemma that is restrictive in its hypotheses, but still general enough to aid with all of our examples. Then we use local considerations to derive consequences for the dynatomic curves associated to a one-parameter family.

3.1 Nonarchimedean preliminaries

Lemma 3.1 *Let k be a nonarchimedean field with nontrivial absolute value $|\cdot|$. Suppose that $D \subset k$ is an open disk and $f\colon D \to D$ is an analytic map such that $f(D) \subsetneq D$, and such that D contains a fixed point of f. Then f has finitely many preperiodic points, and each preperiodic point eventually maps to the fixed point.*

Proof Without loss of generality, we may assume that k is algebraically closed. After a possible change of coordinate, we may assume that $D = D(0, 1)^-$ is the open unit disk and $f(0) = 0$. Write $f(z) = \sum_{n \geq 1} a_n z^n$ with $a_n \in k$. Since $f(D) \subsetneq D$, there is $r < 1$ such that $|a_n| \leq r$ for all $n \geq 0$. For $x \in D$, we have

$$|f(x)| = |x| \cdot \left| \sum_{n \geq 1} a_n x^{n-1} \right| \leq |x| \max_{n \geq 1} |a_n x^{n-1}| \leq r|x|.$$

By induction, we find $|f^j(x)| \leq r^j |x|$ for all $j \geq 1$, and hence, every point in D converges to 0 under iteration. In particular, every preperiodic point must eventually map to 0.

By Weierstrass Preparation, an analytic function on an affinoid domain has only finitely many zeros. Thus, there is $\varepsilon > 0$ such that the only solution to $f(z) = 0$ in the disk $D(0, \varepsilon)^-$ is the origin. Fix a positive integer j such that $r^j < \varepsilon$. Then $|f^j(x)| \leq r^j |x| < \varepsilon$. So if x is preperiodic for f, then we must have $f^j(x) = 0$. Since j depends only on f, we conclude there are finitely many preperiodic points for f, and they all eventually map to the fixed point in D. □

Lemma 3.2 *Let k be a complete nonarchimedean field with nontrivial absolute value $|\cdot|$. Let $f \in k(z)$ be a separable rational function of degree $d > 1$ with an attracting fixed point at ∞. Write D_∞ for the maximal open disk in the immediate basin of ∞, and write Y for the complement of D_∞. Assume that $f^{-1}(Y)$ is a disjoint union of d closed disks. Then the following conclusions hold:*

- *Every preperiodic point either eventually maps to ∞ or else lies in the Julia set for f.*
- *There exists a finite extension k'/k such that every preperiodic point for f lies in $\mathbb{P}^1(k')$.*
- *All of the finite preperiodic points for f lie inside a disk about the origin.*

Proof First, we argue that D_∞ is well defined. Take D_∞ to be the union of all open disks D about ∞ such that $f^n(D)$ converges to ∞ as n grows. There exists at least one such disk since ∞ is an attracting fixed point, so D_∞ is nonempty. Since f has

$d + 1 > 1$ fixed points counted with multiplicity, and since an attracting fixed point must have multiplicity 1, there is some fixed point of f not in D_∞. It follows that $D_\infty \neq \mathbb{P}^1_k$, and hence D_∞ is an open disk.

Now we show that $f(D_\infty) \subset D_\infty$. Since there is a fixed point of f that does not lie in D_∞, we see that $f(D_\infty) \neq \mathbb{P}^1_k$. Thus $f(D_\infty)$ is an open disk that contains ∞, and that is contained in the basin of ∞. By maximality, it follows that $f(D_\infty) \subset D_\infty$.

Set $Y = \mathbb{P}^1_k \smallsetminus D_\infty$. Then Y is a closed disk. Write $f^{-1}(Y) = X_1 \cup \cdots \cup X_d$, where each X_i is a closed disk. We claim that each X_i is properly contained in Y. Take $x \in \bigcup X_i$. If $x \notin Y$, then $x \in D_\infty$. By the preceding paragraph, $f(D_\infty) \subset D_\infty$, so we find that $f(x) \notin Y$, a contradiction. So each $X_i \subset Y$. If $X_i = Y$ for some i, then $X_j \subset X_i$ for $j \neq i$, which contradicts the hypothesis that the X_j are pairwise disjoint. Thus, $X_i \subsetneq Y$.

Next we argue that $f(D_\infty) \neq D_\infty$. Since the X_i are disjoint, we may enlarge each disk slightly to obtain disks $X_i' \subset Y$ that are pairwise disjoint and contain no pole of f, and such that $Y \subsetneq f(X_i')$. Choose any $x \in D_\infty$ that lies in the intersection of the $f(X_i')$. Then $f^{-1}(x)$ consists of d distinct points inside Y. In particular, $x \in D_\infty \smallsetminus f(D_\infty)$.

Write $Y = D(b, r)$, the disk of radius r about some $b \in \overline{k}$. We now argue that $r \in |\overline{k}^\times|$. Let us change coordinates in order to move D_∞ to $D_0 := D(0, 1/r)^-$. More precisely, set

$$g(z) = \frac{1}{f(b + 1/z) - b}.$$

Then the previous paragraph shows g maps D_0 strictly into itself. Now g is given by a series $\sum g_n z^n$ on D_0. If the radius of convergence of this series were larger than $1/r$, then by continuity, there would be a slightly larger disk $D_0' \supset D_0$ such that $g(D_0') \subset D_0$. This would violate maximality of D_∞. Thus, the radius of convergence of the series is $1/r$. But g is a rational function, so the obstruction to extending the domain of convergence of the series is a pole of g. As every pole lies in $\overline{k}^\times$, we see that $r \in |\overline{k}^\times|$, as desired.

We may now define the extension k'/k. Take $Y = D(b, r)$ as before. Let $a_1, \ldots, a_d \in \overline{k}$ be the distinct solutions to $f(z) = b$. Then $X_i = D(a_i, r_i)$. Choose $c_i \in \overline{k}$ such that $|c_i| = r_i$; this is possible since the radius of Y lies in $|\overline{k}^\times|$. Define

$$\mathcal{P} = \{x \in \mathbb{P}^1(\overline{k}) \smallsetminus \{\infty\} \colon f(x) \in D_\infty \text{ and } x \text{ is preperiodic for } f\}.$$

Since $f(D_\infty) \subsetneq D_\infty$, Lemma 3.1 shows that the set $\mathcal{P}$ is finite. (This proves the final conclusion of the lemma.) Define k' to be the extension of k given by adjoining $\mathcal{P} \cup \{a_1, \ldots, a_d, c_1, \ldots, c_d\}$.

In general, if $g \colon D \to D'$ is a separable injective k'-analytic map from a disk D onto a disk D', then the inverse of g exists as an analytic function and is defined over k'. Applying this observation to each of the inverses $h_i \colon Y \to D(a_i, r_i)$, we see that the solutions to $f^n(z) = \infty$ are k'-rational for every $n \geq 0$. Additionally, this setup allows us to use the argument in the proof of Proposition 4.1 of [14] to conclude that the Julia set of f is contained inside $\mathbb{P}^1(k')$, and that the Fatou set of f is the immediate basin of attraction of ∞. In particular, every preperiodic point for f either lies in the Julia set or else eventually maps to ∞, and all such preperiodic points lie in $\mathbb{P}^1(k')$. □

Remark 3.3 The strategy of Proposition 4.1 in [14] shows that the dynamics of f on the Julia set $\mathcal{J}(f)$ is conjugate to the left-shift map on the space of sequences of d symbols. The map is given by sending $x \in \mathcal{J}(f)$ to its itinerary $(i_0, i_1, i_2, \ldots)$, where $f^j(x) \in X_{i_j}$ for each $j \geq 0$.

Remark 3.4 We can rephrase the hypothesis of the lemma in terms of ramification of the morphism of associated analytic spaces. If k is a complete nonarchimedean field, we write $\mathbf{P}_k^1$ for the associated analytic space in the sense of Berkovich. For a rational function $f \in k(z)$, we write $\mathcal{R}_f$ for the ramification locus of f inside $\mathbf{P}_k^1$. If f is separable, then $\mathcal{R}_f \neq \mathbf{P}_k^1$ and f is locally injective on $\mathbf{P}_k^1 \smallsetminus \mathcal{R}_f$. By [3, Thm. 6.3.2], $f^{-1}(Y)$ is a disjoint union of d disks if and only if $f(\mathcal{R}_f) \subset \mathbf{D}_\infty$. Here $\mathbf{D}_\infty$ is the closure of D_∞ inside $\mathbf{P}_k^1$. See [10] for additional mapping properties related to the ramification locus.

3.2 Dynatomic curves

Let k be any field. Given a family of rational functions $f_t(z) \in k(t)(z)$ with $\deg_z(f_t) = d \geq 2$, one can define a dynamical analogue of the classical modular curves, typically referred to as **dynatomic curves**. First, write $f_t(z) = a(z)/b(z)$ with $a, b \in k[t][z]$. Without loss of generality, we may assume that $\gcd_z(a, b) = 1$ and that the coefficients of a and b have no common factor in $k[t]$ of positive degree. This determines a, b up to a common scalar in $k^\times$. Set $A(X, Y) = Y^d a(X/Y)$ and $B(X, Y) = Y^d b(X/Y)$, and let $F = (A, B)$ be the induced family of morphisms on $\mathbb{P}_k^1$. Inductively define polynomials A_m and B_m by

$$A_0 = X, \quad B_0 = Y, \quad A_m = A_{m-1}(A, B), \quad B_m = B_{m-1}(A, B).$$

We set $F^m = (A_m, B_m)$; this corresponds to the m-th iterate of the morphism F. Let H be an irreducible factor of

$$A_{m+n}B_m - A_m B_{m+n} \in k[t][X, Y]$$

for some $m \geq 0$ and $n \geq 1$. Then H is a polynomial in X, Y, t, homogeneous in X and Y, and so its vanishing defines an algebraic curve inside $\mathbb{P}_k^1 \times \mathbb{A}_k^1$. In this paper, a **dynatomic curve** Z is the normalization of the projective closure of the curve $\{H = 0\}$. Two dynatomic curves are considered distinct if they have different associated polynomials. The rational function t gives a morphism $\varphi_Z \colon Z \to \mathbb{P}^1$; by the **degree of** Z we will mean the degree of the morphism φ_Z. For a field extension K/k and $s \in \mathbb{P}^1(K)$, elements of the fiber $\varphi_Z^{-1}(s)$ correspond to points $x \in \mathbb{P}^1(\overline{K})$ which satisfy $f_s(x)^{m+n} = f_s^m(x)$. (They may also satisfy this equation for smaller values of m and n.)

For each place v of $k(t)$, we can form the associated completion $k(t)_v$ with respect to v. These are fields of formal Laurent series. For example, if $v = \mathrm{ord}_\infty$, then $k(t)_v = k((1/t))$, while if $v = \mathrm{ord}_a$ for some $a \in k$, then $k(t)_v = k((t-a))$. Given a family $f_t \in k(t)(z)$ of rational functions parameterized by t, we abuse notation and write $f = f_t$ for the associated rational function defined over the function field $k(t)$ or any of its completions $k(t)_v$.

Proposition 3.5 *Let $f_t \in \mathbb{F}_q(t)(z)$ be a separable family of rational functions satisfying $d := \deg_z(f_t) > 1$. Suppose the following hypotheses hold:*

- *At the place $v = \mathrm{ord}_\infty$, the point at ∞ is an attracting fixed point for f, and the pre-image of the complement of the maximal open disk in the immediate basin of ∞ is a union of d disjoint closed disks.*
- *At each place $v \neq \mathrm{ord}_\infty$, the finite preperiodic points of f are integral.*

Then ∞ is a superattracting fixed point for f, and the following also hold:

(1) There exist integers $r, e \geq 1$ such that for any dynatomic curve Z, the points of the fiber of $\varphi_Z \colon Z \to \mathbb{P}^1$ over ∞ all lie in $Z(\mathbb{F}_{q^r})$ and have ramification index at most e.
(2) The degrees of the dynatomic curves for f_t tend to infinity as $Z \to \infty$.
(3) The gonalities of the dynatomic curves for f_t tend to infinity as $Z \to \infty$.
(4) For each dynatomic curve Z, the irreducible components of the base extension $Z_{\mathbb{F}_{q^r}}$ are geometrically irreducible, where r is the integer in (1).

Remark 3.6 The integers r, e in the statement of the proposition are the residue degree and ramification index of the extension k'/k from Lemma 3.2. The proof of the lemma is constructive, so one can calculate r and e explicitly.

Proof of Proposition 3.5 We begin by arguing that ∞ is superattracting for f. Suppose otherwise, and let $\lambda \in \mathbb{F}_q(t)$ be the multiplier at ∞. Since ∞ is an attracting fixed point at the place ord_∞, we have $\mathrm{ord}_\infty(\lambda) > 0$. By the product formula, there exists a place v such that $v(\lambda) < 0$; that is, ∞ is v-adically repelling. Repelling fixed points belong to the Julia set $\mathcal{J}_v(f)$, and the backward orbit of any Julia point is dense in the Julia set. It follows that there is some $\alpha \in \overline{\mathbb{F}_q(t)}$ such that $f^n(\alpha) = \infty$ for some $n > 0$ and $v(\alpha) < 0$. This contradicts our assumption that all preperiodic points except ∞ are v-adically integral.

Let Z be a dynatomic curve with associated polynomial

$$H(X, Y) = H_0 X^\ell + H_1 X^{\ell-1} Y + \cdots + H_{\ell-1} X Y^{\ell-1} + H_\ell Y^\ell,$$

where each $H_i \in \mathbb{F}_q[t]$. The degree of the morphism $\varphi_Z \colon Z \to \mathbb{P}^1$ is ℓ. If $H_0 = 0$, then H is divisible by Y. As H is irreducible, this means that, after possibly dividing by an element of $\mathbb{F}_q^\times$, we have $H = Y$. The first three conclusions of the proposition are indifferent to the behavior of any one curve, and the final conclusion is clearly true for $\{Y = 0\}$. In what remains, we may suppose that $H_0 \neq 0$. Consider the univariate polynomial

$$h(z) = z^\ell + \frac{H_1}{H_0} z^{\ell-1} + \cdots + \frac{H_{\ell-1}}{H_0} z + \frac{H_\ell}{H_0} \in \mathbb{F}_q(t)[z].$$

The zeros of h are preperiodic points for the map f in $\mathbb{A}^1(\overline{\mathbb{F}_q(t)})$.

Let $v = \mathrm{ord}_\infty$, and view h as living in $\mathbb{F}_q(t)_v[z]$. By Lemma 3.2, the roots of h are defined over some finite extension $K/\mathbb{F}_q(t)_v$. Let r and e be the degree of the residue extension and ramification index of $K/\mathbb{F}_q(t)_v$, respectively. Observe that r, e depend only on f and not on Z. The points of $\varphi_Z^{-1}(\infty)$ are defined over $\mathbb{F}_{q^r}$ and have ramification index at most e. This completes the proof of conclusion (1).

Now we claim that the degrees of the H_i are uniformly bounded in terms of ℓ and the family f_t. As the H_i are polynomials over a finite field, this means there are only finitely many possibilities for the H_i, and hence finitely many dynatomic curves of degree ℓ over the t-line. The hypothesis that the finite preperiodic points of f are integral at all $v \neq \mathrm{ord}_\infty$ means $H_0 \in \mathbb{F}_q^\times$. Without loss of generality, we may assume $H_0 = 1$. Now view h as living in $\mathbb{F}_q(t)_v[z]$ with $v = \mathrm{ord}_\infty$. We may apply the final conclusion of Lemma 3.2 to obtain a nonnegative integer N, depending only on f, such that $\mathrm{ord}_\infty(x) \geq -N$ for all roots x of h. Since the coefficients H_i are symmetric polynomials in the roots of h, we find that

$$\mathrm{ord}_\infty(H_i) \geq -iN \text{ for all } 1 \leq i \leq \ell.$$

But $\mathrm{ord}_\infty = -\deg$, so we learn that $\deg(H_i) \leq iN \leq \ell N$. That is, the degrees of the coefficients H_i are uniformly bounded. Conclusion (2) is now proved.

Let Z be a dynatomic curve and consider the morphism $\varphi_Z \colon Z \to \mathbb{P}^1$. We have already shown that every point in the fiber over infinity is defined over $\mathbb{F}_{q^r}$ and has ramification index at most e. In particular, $\#Z(\mathbb{F}_{q^r}) \geq \deg(\varphi_Z)/e$. If γ is the gonality of Z, then we also know that $\#Z(\mathbb{F}_{q^r}) \leq \gamma(q^r + 1)$ since every point of $\mathbb{P}^1(\mathbb{F}_{q^r})$ has at most γ geometric points above it in Z. Combining these inequalities shows that

$$\gamma \geq \frac{\deg(\varphi_Z)}{e(q^r+1)}.$$

Since the degree of φ_Z tends to infinity as $Z \to \infty$, so does the gonality of Z, as desired in conclusion (3).

Finally, let $Z' = Z_{\mathbb{F}_{q^r}}$, the base extension of Z to $\mathbb{F}_{q^r}$. Let W be an irreducible component of Z'. If W is not geometrically irreducible, then the base extension $W_{\overline{\mathbb{F}}_q}$ has irreducible components $V \neq V'$ that are $\mathrm{Gal}(\overline{\mathbb{F}}_{q^r}/\mathbb{F}_{q^r})$-conjugate. Note that W projects onto $\mathbb{P}^1$ via the map $\varphi_{Z'}$. In particular, W has an $\mathbb{F}_{q^r}$-rational point since every point in the fiber of Z' over infinity is $\mathbb{F}_{q^r}$-rational. But then $V \cap V'$ contains each of these rational points, all of which must be singular on W. This is impossible since Z' is smooth. It follows that W is geometrically irreducible, and conclusion (4) holds. □

The connection between gonality and the Strong Uniform Boundedness Principle is given by the following proposition. The proof is a straightforward generalization of the argument for Theorem 1.7 in [9].

Proposition 3.7 *Let $f_t \in \mathbb{F}_q(t)(z)$ be a family of rational functions satisfying $\deg_z(f_t) > 1$, and let k be any field containing $\mathbb{F}_q$. If the gonalities of the dynatomic* *curves for f_t tend to infinity in any ordering, then the family f_t satisfies the Strong Uniform Boundedness Principle over any function field over k.*

Remark 3.8 The proof of Proposition 3.7 can be used to give an upper bound for the constant $B = B(\mathcal{F}, D)$ in the statement of the SUBP, which depends on q and D as well as the quantities r, e, and N appearing in the proof of Proposition 3.5. The bound obtained in this way, which is larger than $Dq^{(Dq^r)^2}$, is rather cumbersome and unlikely to be anywhere near optimal.

4 New families

As promised in the introduction, we now give sufficient conditions to be able to utilize Proposition 3.7.

Proposition 4.1 *Let $\mathbb{F}_q$ be a finite field, and let $f_t(z)$ be a family of rational functions of the form*

$$f_t(z) = \frac{a(z)}{b(z)},$$

where

(1) a and b are coprime monic polynomials with coefficients in $\mathbb{F}_q[t]$;
(2) $\deg(a) > \deg(b) + 1$; and
(3) a is separable.

For $v = \mathrm{ord}_\infty$, let R be the maximum v-adic absolute value of the roots of a. Then we further assume that

(4) $R > 1$;
(5) the roots of b all lie in the v-adic disk $D(0, R)$;
(6) there exists a root of $a(z) - zb(z)$ with v-adic absolute value R;
(7) for each root α of a, there is a disk $D(\alpha, r_\alpha)$ that maps onto $D(0, R)$; and
(8) the disks $D(\alpha, r_\alpha)$ and $D(\beta, r_\beta)$ are disjoint if α, β are distinct roots of a.

Then for any field k containing $\mathbb{F}_q$ and any function field K over k, the family f_t satisfies the Strong Uniform Boundedness Principle over K.

Proof By Proposition 3.7, it suffices to show that the gonalities of the dynatomic curves for f_t tend to infinity. We accomplish this by showing that the hypotheses of Proposition 3.5 are satisfied.

Let $v \neq \mathrm{ord}_\infty$ be a place of $\mathbb{F}_q(t)$. Let $x \in \overline{\mathbb{F}_q(t)}$ be an element with $|x|_v > 1$. Then x is larger than any root of a or b, as both polynomials are monic with v-adic integral coefficients. Thus,

$$|f(x)|_v = |x|_v^{\deg(a)-\deg(b)} \geq |x|_v^2 > |x|_v.$$

It follows that x is not preperiodic. That is, all finite preperiodic points of f are v-adically integral.

Next, take $v = \mathrm{ord}_\infty$. Write $A = \deg(a)$ and $B = \deg(b)$. Since a, b are coprime and $A - B \geq 2$, the point at infinity is a superattracting fixed point. Let $R > 1$ be the maximum absolute value of a root of a in $\overline{\mathbb{F}_q(t)}$. Set $U = \mathbb{P}^1 \smallsetminus D(0, R)$. We claim that $U = D_\infty$, the maximal open disk in the immediate basin of ∞. If $|x|_v > R$, then the fact that all roots of a and b lie in $D(0, R)$ shows that

$$|f(x)|_v = \frac{|a(x)|_v}{|b(x)|_v} = |x|_v^{A-B} \geq |x|_v^2 > R|x|_v.$$

So $U \subset D_\infty$. By hypothesis, the polynomial $a(z) - zb(z)$ has a root with absolute value R, which means f has a fixed point of absolute value R. That is, no disk larger than U lies in D_∞. Thus, $U = D_\infty$, as desired.

Let $Y = D(0, R)$. Hypotheses (3), (7), and (8) of the lemma say that $f^{-1}(Y)$ consists of $\deg(f) = \deg(a)$ pairwise disjoint disks. This completes the proof. □

Remark 4.2 The applicability of Proposition 4.1 may depend on the choice of coordinate. For example, the proposition does not apply to

$$f_t(z) = (z - t^2)(z - t^2 - t),$$

since condition (8) fails, though outside of characteristic 5 it does apply to the conjugate

$$g_t(z) := f_t(z + t^2) - t^2 = z^2 - tz - t^2.$$

We now show that the families from the introduction satisfy the hypotheses of Proposition 4.1. Let $\mathbb{F}_q$ be a fixed finite field throughout.

Example 4.3 Fix an integer $d > 1$, and let $\alpha_1, \ldots, \alpha_d \in \mathbb{F}_q[t]$ be distinct polynomials. Set

$$f_t(z) = (z - \alpha_1) \cdots (z - \alpha_d).$$

Assume that for each $i \neq j$, the following inequality is true:

$$\deg(\alpha_i - \alpha_j) + \sum_{\ell \neq j} \deg(\alpha_\ell - \alpha_j) > \max_\ell \deg(\alpha_\ell). \tag{1}$$

We now prove Theorem 1.1 by verifying the conditions of Proposition 4.1. Set $M = \max_\ell \deg(\alpha_\ell)$ for ease of notation.

(1-3, 5) Clear, since f_t is a polynomial in z of degree $d > 1$, and the α_i are distinct.

(4) Let $v = \mathrm{ord}_\infty$. Note that $M > 0$, for otherwise every α_i is constant and equation (1) does not hold. Since f is in factored form, it is immediate that the maximum absolute value of a root is $R = |t|_v^M > 1$.

(6) All roots of $f(z)$ have nonpositive v-adic valuation. As f is monic and at least one of the α_i is nonconstant, each segment of the Newton polygon lies below the x-axis. It follows that $f(z) - z$ and $f(z)$ have the same Newton polygon. In particular, $f(z) - z$ has a root with v-adic absolute value R.

(7) Fix j and define

$$r_j = \frac{R}{\prod_{\ell \neq j} |\alpha_\ell - \alpha_j|_v}.$$

Equation (1) is equivalent to the assertion that $r_j < |\alpha_i - \alpha_j|_v$ for all $i \neq j$. Set $x = \alpha_j + y$, where $|y|_v \leq r_j$. Then

$$\begin{aligned} |f(x)|_v &= |y|_v \cdot \prod_{\ell \neq j} |\alpha_j - \alpha_\ell + y|_v \\ &= |y|_v \cdot \prod_{\ell \neq j} |\alpha_j - \alpha_\ell|_v, \end{aligned}$$

since $|y|_v \leq r_j < |\alpha_j - \alpha_\ell|_v$ for all $\ell \neq j$. That is,

$$|f(x)|_v = \frac{|y|_v}{r_j} R.$$

As $|y|_v$ varies from 0 to r_j, we obtain elements $f(\alpha_j + y)$ with absolute value from 0 to R. It follows that f maps $D(\alpha_j, r_j)$ onto $D(0, R)$, as desired.

(8) If $i \neq j$, then the disks $D(\alpha_i, r_i)$ and $D(\alpha_j, r_j)$ are disjoint. Indeed, by the ultrametric inequality it suffices to check that $|\alpha_i - \alpha_j| > \max(r_i, r_j)$, and this was already observed in our proof of condition (7).

Example 4.4 Fix integers $d \geq 2$ and $e \leq d - 2$ such that $p = \mathrm{char}(\mathbb{F}_q)$ does not divide d. Define

$$f_t(z) = \frac{z^d - t}{z^e}.$$

We once again verify the conditions of Proposition 4.1 are met, thus proving Theorem 1.3.

(1-2) Clear.

(3) Since p does not divide d, the numerator $z^d - t$ is separable.

(4) Let $v = \mathrm{ord}_\infty$. The roots of the numerator of f are of the form $\varepsilon t^{1/d}$ for some d-th root of unity ε. These all have v-adic absolute value $R = |t|_v^{1/d} > 1$.

(5) Clear.

(6) The Newton polygon of $(z^d + t) - z \cdot z^e$ has a single segment, so that all fixed points of f have absolute value $R = |t|_v^{1/d}$.

(7) Let $x = \alpha + y$ for some root α of $z^d - t$ and some y with $|y|_v \leq R^{2+e-d}$. Then

$$f(x) = x^{-e} \sum_{i=1}^{d} \binom{d}{i} \alpha^{d-i} y^i.$$

Since p does not divide d, the $i = 1$ term in the sum strictly dominates the others, so we find

$$|f(x)|_v = R^{d-e-1}|y|_v = \frac{|y|_v}{R^{2+e-d}} R.$$

That is, as $|y|_v$ varies from 0 to R^{2+e-d}, we obtain elements $f(\alpha + y)$ with absolute value from 0 to R. It follows that f maps $D(\alpha, R^{2+e-d})$ onto $D(0, R)$.

(8) The fact that p does not divide d implies that any pair of distinct roots of $z^d - t$ are at distance R from each other. Since $R > 1 \geq R^{2+e-d}$, we find that the disks $D(\alpha, R^{2+e-d})$ are disjoint as α varies through the roots of $z^d - t$.

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