LOCAL MASS-CONSERVING SOLUTION FOR A CRITICAL COAGULATION-FRAGMENTATION EQUATION

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ABSTRACT. The critical coagulation-fragmentation equation with multiplicative coagulation and constant fragmentation kernels is known to not have global mass-conserving solutions when the initial mass is greater than 1. We show that for any given positive initial mass with finite second moment, there is a time $T^* > 0$ such that the equation possesses a unique mass-conserving solution up to T^*. The novel idea is to singularly perturb the constant fragmentation kernel by small additive terms and study the limiting behavior of the solutions of the perturbed system via the Bernstein transform.

1. Introduction

We study the local well-posedness of the following coagulation-fragmentation equation (C-F)

(1.1)
$$\begin{cases} \partial_t \rho(s,t) = Q_C(\rho)(s,t) + Q_F(\rho)(s,t) & \text{in } (0,\infty) \times (0,\infty), \\ \rho(s,0) = \rho_0(s) & \text{in } (0,\infty). \end{cases}$$

Here, $\rho(s,t)$ is the density of particles of size s at time $t \geqslant 0$. The coagulation and fragmentation terms are given by

$$Q_C(\rho)(s,t) = \frac{1}{2} \int_0^s a(s-\hat{s}, \hat{s}) \rho(s-\hat{s}, t) \rho(\hat{s}, t) \, d\hat{s} - \int_0^\infty a(s, \hat{s}) \rho(s,t) \rho(\hat{s}, t) \, d\hat{s},$$

$$Q_F(\rho)(s,t) = -\frac{1}{2} \int_0^s b(s-\hat{s}, \hat{s}) \rho(s,t) \, d\hat{s} + \int_0^\infty b(s, \hat{s}) \rho(s+\hat{s}, t) \, d\hat{s}.$$

The kernels of interest are

$$a(s, \hat{s}) = s\hat{s}, \qquad b(s, \hat{s}) = 1 \quad \text{for } s, \hat{s} > 0.$$

Let $m_k(t)$ be the k-th moment of $\rho(\cdot, t)$ for $k, t \geqslant 0$, that is,

$$m_k(t) = \int_0^\infty s^k \rho(s,t) \, ds.$$

In particular the first moment $m_1(t)$ represents the total mass of the system (1.1) at time $t \geqslant 0$.

It was conjectured that with this specific choice of kernels, if the system starts out with initial mass $m_1(0) \leqslant 1$, then there will be a unique mass-conserving solution (see [4, 12]). It

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has been proven that mass-conserving solutions cannot exist for all time when $m_1(0) > 1$ in [2, 11].

The history of C-F dates back more than a century ago with the work [9]. For more extensive discussions of the subject, we refer the readers to the following works [1–3, 11]. Although there has been a lot of advancement in the field, a lot still remains to be discovered, including the following question we set out to answer in this paper.

Question. *Fix $m_1(0) > 1$. Is there a time $T > 0$ such that the equation* (1.1) *still has a unique mass-conserving solution for $0 \leqslant t < T$?*

The answer to such question is not obvious as there are systems of coagulation-fragmentation equations that exhibit instantaneous mass loss, that is, $m_1(t) < m_1(0)$ for all $t > 0$.

For $t \geqslant 0$, let $\pi(\cdot, t)$ be the distribution corresponding to the density $\rho(\cdot, t)$, i.e.,

$$\pi(s, t) = \int_{(0,s]} \rho(r, t) \, dr \quad \text{for } s > 0 \, .$$

By probabilistic convention, we use the same notion $\pi(\cdot, t)$ to denote the measure on $(0, \infty)$ with this distribution function. We always use the following notion of weak solutions to (1.1).

Definition 1. We say that ρ is a weak solution of the equation (1.1) in the measure sense if

$$\frac{d}{dt} \int_0^\infty \phi(s) d\pi(s, t) = \frac{1}{2} \int_0^\infty \int_0^\infty \big(\phi(s + \hat{s}) - \phi(s) - \phi(\hat{s})\big) a(s, \hat{s}) \, d\pi(s, t) \, d\pi(\hat{s}, t)$$

$$- \frac{1}{2} \int_0^\infty \int_0^s \big(\phi(s) - \phi(s - \hat{s}) - \phi(\hat{s})\big) b(s - \hat{s}, \hat{s}) \, d\hat{s} \, d\pi(s, t) \, ,$$

for all test functions $\phi \in \mathrm{BC}([0, \infty)) \cap \mathrm{Lip}([0, \infty))$ with $\phi(0) = 0$. Here, $\mathrm{BC}([0, \infty))$ and $\mathrm{Lip}([0, \infty))$ are the classes of bounded continuous functions and Lipchitz continuous functions on $[0, \infty)$, respectively.

For $x \geqslant 0$, let $\phi_x(s) = 1 - e^{-xs}$ be a test function, and denote by $F(x, t)$ the Bernstein transform of $\rho(s, t)$, that is,

$$F(x, t) = \mathfrak{B}[\rho(\cdot, t)](x) \stackrel{\text{def}}{=} \int_0^\infty (1 - e^{-xs}) \, d\pi(s, t) \, .$$

Define $m \stackrel{\text{def}}{=} m_1(0)$. If $m_1(t) = m$ for $t \in [0, T)$ with some $T > 0$, then F satisfies the following singular Hamilton-Jacobi equation

(1.2)
$$\begin{cases} \partial_t F + \dfrac{1}{2}(\partial_x F - m)(\partial_x F - m - 1) + \dfrac{F}{x} - m = 0 & \text{in } (0, \infty) \times (0, T) \, , \\ 0 \leqslant F(x, t) \leqslant mx & \text{on } [0, \infty) \times [0, T) \, , \\ F(x, 0) = F_0(x) = \mathfrak{B}[\rho_0](x) & \text{on } [0, \infty) \, . \end{cases}$$

Our main result in this paper is the following.

Theorem 1.1. *Let ρ_0 be a density function such that $m = m_1(0) \geqslant 1/2$, and $m_2(0) > 0$. Then, there exists a unique mass-conserving weak solution to equation* (1.1) *for $t \in [0, T^*)$, where*

$$T^* = \frac{6m}{6m - 1} \frac{1}{m_2(0)} \, .$$

In the case when $a(s, \hat{s}) = s\hat{s}$ and $b(s, \hat{s}) = 0$, it was shown in [7] that the C-F loses mass exactly after $T = m_2(0)^{-1} > 0$. Therefore, our result is consistent with that in [7]. The reason for this is that fragmentation helps prevent gelation (mass loss by formation

of infinite-size particles). Equation (1.1) is more complicated because of the interaction between coagulation and fragmentation kernels. We note that T^* is not known to be sharp in Theorem 1.1.

We briefly summarize the progress of the conjecture in [4, 12] for $m = m_1(0) \in (0, 1]$. Under certain assumptions, global existence and uniqueness of mass-conserving solutions when $m \leqslant 1/(4 \log 2)$ was proven in [6] by the moment-bound method. By studying equation (1.2), the authors of this paper obtained the global well-posedness for $m < 1/2$ in [11], which means that $T^* = +\infty$ in this regime. Furthermore, while uniqueness of mass-conserving solutions for $m \in [1/2, 1]$ was established in [11], the existence question remains an outstanding open problem. For large time behavior results of (1.2), see [8, 11].

We continue this program of studying equation (1.2) to establish Theorem 1.1. The novel idea is to singularly perturb $b(s, \hat{s}) = 1$ by $b^\varepsilon(s, \hat{s}) = 1 + \varepsilon(s + \hat{s})$ for $\varepsilon > 0$ and study the limiting behavior of the solutions of the perturbed system via the Bernstein transform.

Outline of the paper. In Section 2, we introduce an approximating system to (1.1) and study various properties of it that is inherent to (1.1). In Section 3, we give a proof of Theorem 1.1 by taking the limit of the approximating system. Our arguments are based on studying viscosity solutions of (1.2). Finally, in Appendix, we give a heuristic argument explaining why one should expect that our approximating system possesses a mass-conserving solution.

2. **Approximating fragmentations**

In this section, we always assume the setting of Theorem 1.1. In order to study equation (1.1), we regularize it by adding a small additive term to the fragmentation kernel. More precisely, for $\varepsilon > 0$, we consider

$$(2.1) \qquad \begin{cases} \partial_t \rho^\varepsilon(s, t) = Q_C(\rho^\varepsilon)(s, t) + Q_F^\varepsilon(\rho^\varepsilon)(s, t) & \text{in } (0, \infty) \times (0, \infty), \\ \rho^\varepsilon(s, 0) = \rho_0(s) & \text{in } (0, \infty). \end{cases}$$

Here, the corresponding fragmentation kernel is

$$b^\varepsilon(s, \hat{s}) = 1 + \varepsilon(s + \hat{s}) \quad \text{for } s, \hat{s} > 0.$$

Equation (2.1) has been shown to have a unique mass-conserving solution by [4]. By studying the second moment of ρ^ε, we can show that the second moment of the solution to equation (1.1), if exists, is finite up to time $T^* = \frac{6m}{6m-1} \frac{1}{m_2(0)}$.

Let $F^\varepsilon(x, t) = \mathfrak{B}[\rho^\varepsilon(\cdot, t)](x)$, the Bernstein transform of ρ^ε. Then, F^ε satisfies the following equation.

$$(2.2) \qquad \begin{cases} \partial_t F^\varepsilon + \dfrac{1}{2}(\partial_x F^\varepsilon - m)(\partial_x F^\varepsilon - m - 1) + \dfrac{F^\varepsilon}{x} - m = \varepsilon G^\varepsilon(x, t) & \text{in } (0, \infty)^2, \\ 0 \leqslant F^\varepsilon \leqslant mx & \text{on } [0, \infty)^2, \\ F^\varepsilon(x, 0) = F_0(x) & \text{on } [0, \infty). \end{cases}$$

Here,

$$G^\varepsilon(x, t) \stackrel{\text{def}}{=} \frac{m_2^\varepsilon(t)}{2} - \frac{\partial_x^2 F^\varepsilon(x, t)}{2} - \frac{1}{x}\left(m - \partial_x F^\varepsilon(x, t)\right).$$

For derivations of equations (1.2) and (2.2), we refer the reader to [8, 11].

Remark 2.1. It is interesting to note that equation (2.2) is a backward parabolic equation. The well-posedness theory for this equation from the PDE viewpoint is a very interesting open question.

Lemma 2.2. *For $\varepsilon > 0$, let ρ^ε be the mass-conserving solution to equation (2.1). Then, for $0 \leqslant t < T^*$,*

$$m_2^\varepsilon(t) \leqslant \frac{1}{m_2(0)^{-1} - (6m)^{-1}(6m-1)t} \,.$$

Proof. We proceed by using the method in [4, Theorem 3.1]. The idea is to use a cut-off technique and the moment-bound method. For $n \in \mathbb{N}$, define

$$a^n(s, \hat{s}) \stackrel{\text{def}}{=} a(s, \hat{s}) \mathbf{1}_{[0,n]}(s + \hat{s}) \,,$$

$$b^{\varepsilon,n}(s, \hat{s}) \stackrel{\text{def}}{=} b^\varepsilon(s, \hat{s}) \mathbf{1}_{[0,n]}(s + \hat{s}) \,,$$

$$\rho_0^n \stackrel{\text{def}}{=} \rho_0 \mathbf{1}_{[0,n]} \,.$$

Denote the coagulation and fragmentation terms corresponding the above kernels as Q_C^n and $Q_F^{\varepsilon,n}$, respectively. Let $\rho^{\varepsilon,n}$ be the solution to the equation

$$(2.3) \qquad \begin{cases} \partial_t \rho^{\varepsilon,n}(s,t) = Q_C^n(\rho^{\varepsilon,n})(s,t) + Q_F^{\varepsilon,n}(\rho^{\varepsilon,n})(s,t) & \text{in } (0,\infty) \times (0,\infty) \,, \\ \rho^{\varepsilon,n}(s,0) = \rho_0^n(s) & \text{in } (0,\infty). \end{cases}$$

Then, $\operatorname{supp}(\rho^{\varepsilon,n}(\cdot,t)) \subseteq [0,n]$ for every $t \geqslant 0$. Let $\pi^{\varepsilon,n}(\cdot,t)$ be the distribution corresponding to the density $\rho^{\varepsilon,n}(\cdot,t)$. Take $\phi(s) = s^2 \mathbf{1}_{[0,n]}(s) + n^2 \mathbf{1}_{(n,\infty)}(s)$ as a test function in Definition 1. Then,

$$\begin{aligned}
\frac{d}{dt} m_2^{\varepsilon,n}(t) &= \int_0^\infty \int_0^\infty s^2 \hat{s}^2 \, \mathbf{1}_{[0,n]}(s + \hat{s}) \, d\pi^{\varepsilon,n}(s,t) \, d\pi^{\varepsilon,n}(\hat{s},t) \\
&\quad - \int_0^\infty \int_0^s (s - \hat{s}) \hat{s} \mathbf{1}_{[0,n]}(s) \, d\hat{s} \, (1 + \varepsilon s) \, d\pi^{\varepsilon,n}(s,t) \\
&\leqslant \int_0^\infty \int_0^\infty s^2 \hat{s}^2 \, d\pi^{\varepsilon,n}(s,t) \, d\pi^{\varepsilon,n}(\hat{s},t) - \int_0^\infty \frac{s^3}{6} \, \mathbf{1}_{[0,n]}(s) \, d\pi^{\varepsilon,n}(s,t) \\
&= m_2^{\varepsilon,n}(t)^2 - \frac{m_3^{\varepsilon,n}(t)}{6} \leqslant \left(1 - \frac{1}{6m}\right) m_2^{\varepsilon,n}(t)^2 \,,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last line

$$m m_3^{\varepsilon,n}(t) \geqslant m_1^{\varepsilon,n}(t) m_3^{\varepsilon,n}(t) \geqslant m_2^{\varepsilon,n}(t)^2.$$

Therefore, for $t < \frac{6m}{6m-1} \frac{1}{m_2^n(0)}$,

$$m_2^{\varepsilon,n}(t) \leqslant \frac{1}{m_2^n(0)^{-1} - (6m)^{-1}(6m-1)t} \,.$$

Letting $n \to \infty$ and using the compactness result in [4, Remark 3.10], we have

$$m_2^\varepsilon(t) \leqslant \frac{1}{m_2(0)^{-1} - (6m)^{-1}(6m-1)t} \,,$$

for $t < \frac{6m}{6m-1} \frac{1}{m_2(0)}$. The proof is finished by picking $T^* = \frac{6m}{6m-1} \frac{1}{m_2(0)}$. $\square$

We now let $\varepsilon \to 0$ in equation (2.2) to obtain a viscosity solution of (1.2).

Lemma 2.3. *For each $\varepsilon > 0$, let F^ε be the smooth solution to equation (2.2). Then, there exists $\bar{F} \in C([0, \infty) \times [0, T^*))$ such that, locally uniformly for $(x, t) \in [0, \infty) \times [0, T^*)$,*

$$\lim_{i \to \infty} F^{\varepsilon_i}(x, t) = \bar{F}(x, t),$$

for some sequence $\{\varepsilon_i\} \to 0$. Furthermore, $\bar{F}$ is a viscosity solution to (1.2).

Proof. We recall equation (2.2)

$$\begin{cases} \partial_t F^\varepsilon + \dfrac{1}{2}(\partial_x F^\varepsilon - m)(\partial_x F^\varepsilon - m - 1) + \dfrac{F^\varepsilon}{x} - m = \varepsilon G^\varepsilon(x, t) & \text{in } (0, \infty)^2, \\ 0 \leqslant F^\varepsilon \leqslant mx & \text{on } [0, \infty)^2, \\ F^\varepsilon(x, 0) = F_0(x) & \text{on } [0, \infty), \end{cases}$$

where

$$G^\varepsilon(x, t) = \frac{m_2^\varepsilon(t)}{2} - \frac{\partial_x^2 F^\varepsilon(x, t)}{2} - \frac{1}{x}\left(m - \partial_x F^\varepsilon(x, t)\right).$$

Recall that F^ε is the Bernstein transform of ρ^ε, i.e.,

$$F^\varepsilon(x, t) = \mathfrak{B}[\rho^\varepsilon(\cdot, t)](x).$$

Therefore, for $(x, t) \in [0, \infty)^2$, we have

$$0 \leqslant \partial_x F^\varepsilon(x, t) = \int_0^\infty s e^{-sx} d\pi^\varepsilon(s, t) \leqslant m,$$

$$|\partial_x^2 F^\varepsilon(x, t)| = \int_0^\infty s^2 e^{-sx} d\pi^\varepsilon(s, t) \leqslant m_2^\varepsilon(t).$$

Here, $\pi^\varepsilon(\cdot, t)$ is the distribution corresponding to the density $\rho^\varepsilon(\cdot, t)$. In particular,

$$\partial_x F^\varepsilon(0, t) = m.$$

By the mean value theorem, we have that for every $\varepsilon > 0$, $T < T^*$ and $(x, t) \in [0, \infty) \times [0, T]$, there exists $\theta \in (0, 1)$ such that

$$G^\varepsilon(x, t) = \frac{m_2^\varepsilon(t)}{2} - \frac{\partial_x^2 F^\varepsilon(x, t)}{2} + \partial_x^2 F^\varepsilon(\theta x, t).$$

Therefore, by Lemma 2.2,

$$(2.4) \qquad \sup_{(x,t)\in[0,\infty)\times[0,T]} |G^\varepsilon(x, t)| \leqslant \frac{5}{T^* - T}.$$

Furthermore, we also have

$$\left|\frac{1}{2}(\partial_x F^\varepsilon - m)(\partial_x F^\varepsilon - m - 1) + \frac{F^\varepsilon}{x} - m\right| \leqslant \frac{m(m+3)}{2}.$$

As a consequence, for $(x, t) \in [0, \infty) \times [0, T]$ and $\varepsilon \in (0, 1)$,

$$|\partial_t F^\varepsilon(x, t)| + |\partial_x F^\varepsilon(x, t)| \leqslant \frac{m(m+5)}{2} + \frac{5}{T^* - T}.$$

By the Arzelà-Ascoli theorem, there exist a function $\bar{F} \in C([0, \infty) \times [0, T^*))$ and a sequence $\{\varepsilon_i\} \to 0$ such that $F^{\varepsilon_i} \to \bar{F}$ locally uniformly on $[0, \infty) \times [0, T^*)$.

For $\varepsilon \geqslant 0$, write

$$H^\varepsilon(x, t, p, u) \overset{\text{def}}{=} \frac{(p - m)(p - m - 1)}{2} + \frac{u}{x} - m - \varepsilon G^\varepsilon(x, t).$$

Equations (2.2) can be rewritten as

$$\partial_t F^\varepsilon + H^\varepsilon(x, t, F^\varepsilon_x, F^\varepsilon) = 0 \quad \text{in } (0, \infty) \times (0, T^*) \,.$$

Furthermore, we have that by estimate (2.4),

$$H^\varepsilon \to H^0$$

locally uniformly in $(0, \infty) \times (0, T^*) \times [0, m] \times (0, \infty)$. Thus, by the stability of viscosity solutions (see, e.g., [10]), $\bar{F}$ is a viscosity solution of equation (1.2). $\quad\square$

Lemma 2.4. *Let $\bar{F}$ be a viscosity solution to equation* (1.2) *given by Lemma 2.3. Then, in the viscosity sense, in $(0, \infty) \times (0, T)$ for $T < T^*$, we have*

$$(2.5) \qquad\qquad\qquad\qquad 0 \leqslant \partial_x \bar{F} \leqslant m \,,$$

$$(2.6) \qquad\qquad\qquad\qquad |\partial_t \bar{F}| \leqslant \frac{m(m+5)}{2} + \frac{5}{T^* - T} \,,$$

and

$$(2.7) \qquad\qquad\qquad\qquad -\frac{2}{T^* - T} \leqslant \partial^2_x \bar{F} \leqslant 0 \,.$$

Proof. This is a consequence of $0 \leqslant \partial_x F^\varepsilon \leqslant m$, Lemmas 2.2 and 2.3. In particular, the inequalities (2.5) and (2.6) are straightforward.

Let us now prove (2.7). As F^ε is the Bernstein transform of ρ^ε, we have that for $(x, t) \in (0, \infty) \times (0, T)$,

$$-\frac{2}{T^* - T} \leqslant -m^\varepsilon_2(t) = \partial^2_x F^\varepsilon(0, t) \leqslant \partial^2_x F^\varepsilon(x, t) \leqslant 0 \,.$$

Furthermore, for $h > 0$, there exist $\theta, \tilde{\theta} \in (0, 1)$ such that

$$\begin{aligned}
0 &\geqslant \frac{F^\varepsilon(x + 2h, t) + F^\varepsilon(x, t) - 2F^\varepsilon(x + h, t)}{h^2} \\
&= \frac{\partial^2_x F^\varepsilon(x + h + \theta h, t)h^2 + \partial^2_x F^\varepsilon(x + \tilde{\theta} h, t)h^2}{2h^2} \\
&= \frac{1}{2}(\partial^2_x F^\varepsilon(x + h + \theta h, t) + \partial^2_x F^\varepsilon(x + \tilde{\theta} h, t)) \\
&\geqslant \partial^2_x F^\varepsilon(x, t) \geqslant -\frac{1}{T^* - T} \,.
\end{aligned}$$

In the above, we used the fact that $\partial^3_x F^\varepsilon \geqslant 0$. Letting $\varepsilon \to 0$, we obtain

$$0 \geqslant \frac{\bar{F}(x + 2h, t) + \bar{F}(x, t) - 2\bar{F}(x + h, t)}{h^2} \geqslant -\frac{2}{T^* - T} \,.$$

Inequality (2.7) follows immediately by letting $h \to 0$. $\quad\square$

In order to show the uniqueness of solutions to equation (1.2), we need the following comparison principle. A similar result was proven in [11]. We provide the details here for self-containment.

Lemma 2.5 (Comparison Principle for (1.2))**.** *For $T \in (0, T^*)$, let u be a sublinear viscosity subsolution and v be a sublinear viscosity supersolution to equation* (1.2)*, respectively. Then $u \leqslant v$.*

Proof. Since (1.2) is singular at $x = 0$, we cut off its singularity by introducing a sequence of function $\{\varphi_n\}$, where

$$\varphi_n(x) = \max\left\{\frac{1}{n}, x\right\} \quad \text{for all } x \in [0, \infty).$$

For each $n \in \mathbb{N}$, we consider the following approximating Hamilton-Jacobi equation

(2.8)
$$\begin{cases} \partial_t F + \frac{1}{2}(\partial_x F - m)(\partial_x F - m - 1) + \frac{F}{\varphi_n(x)} - m = 0 & \text{in } (0, \infty) \times (0, T), \\ F(x, 0) = F_0(x) & \text{on } [0, \infty), \\ F(0, t) = 0 & \text{on } [0, \infty), \end{cases}$$

We claim that u is a subsolution, and $v^n \overset{\text{def}}{=} v + \frac{m}{n}$ is a supersolution to equation (2.8), respectively. It is clear to see that u is a subsolution. To check that v is a supersolution, we note that,

$$\frac{v + \frac{m}{n}}{\varphi_n} - m = \begin{cases} \frac{v}{x} + \frac{m}{nx} - m \geqslant \frac{v}{x} - m, & \text{for } x \geqslant \frac{1}{n}, \\ nv + m - m \geqslant \frac{v}{x} - m, & \text{for } x < \frac{1}{n}. \end{cases}$$

Therefore,

$$\partial_t v^n + \frac{1}{2}(\partial_x v^n - m)(\partial_x v^n - m - 1) + \frac{v^n}{\varphi_n(x)} - m$$
$$\geqslant \partial_t v + \frac{1}{2}(\partial_x v - m)(\partial_x v - m - 1) + \frac{v}{x} - m \geqslant 0$$

in the viscosity sense. By the classical theory of viscosity solution applied to equation (2.8), we imply that

$$u \leqslant v^n.$$

As $v^n \to v$ uniformly when $n \to \infty$, we then conclude

$$u \leqslant v \quad \text{on } [0, \infty) \times [0, T),$$

as desired. $\qquad\square$

Corollary 2.6. *Let $\bar{F}$ be the function as in Lemma 2.3. Then $\bar{F}$ is the unique sublinear viscosity solution to equation (1.2) on $[0, \infty) \times [0, T)$ for $T < T^*$. As a consequence, the convergence in Lemma 2.3 is in full sequence, that is, locally uniformly on $[0, \infty) \times [0, T^*)$, we have*

(2.9)
$$\lim_{\varepsilon \to 0} F^\varepsilon = \bar{F}.$$

Proof. First, note that $\bar{F} \geqslant 0$ as it is a subsequential limit of $F^{\varepsilon_i} \geqslant 0$. By (2.6),

$$\bar{F}(x, t) \leqslant \left(\frac{m(m+5)}{2} + \frac{5}{T^* - T}\right)t + F_0(x),$$

for $(x, t) \in [0, \infty) \times [0, T)$. Therefore, $\bar{F}$ is sublinear on $[0, \infty) \times [0, T)$. By Lemma 2.5, $\bar{F}$ is the unique sublinear viscosity solution to equation (1.2). By this uniqueness, (2.9) follows immediately. $\qquad\square$

3. **Local existence of solutions to the C-F equation** (1.1)

We now prove our main result, Theorem 1.1. Throughout this section, we will always assume the setting of Theorem 1.1. Let $F = \bar{F}$ be the sublinear viscosity solution to equation (1.2) found in Section 2. By Lemma 2.4, we already have that $F \in C^{1,1}((0,\infty) \times (0,T^*))$. Let us now use this result to yield further that $F \in C^\infty((0,\infty) \times (0,T^*))$.

Proposition 3.1. *Let $F = \bar{F}$ be the sublinear viscosity solution to equation (1.2) for $T = T^*$. Then, $F \in C^\infty((0,\infty) \times (0,T^*))$.*

Proof. We proceed by using the method of characteristics (see [5, Chapter 3]). Fix $t \in (0,T^*)$ and denote by $X(x,s)$ the characteristic at time $s \in [0,t]$ starting from $x > 0$, that is, $X(x,0) = x$. Set $P(x,s) = \partial_x F(X(x,s),s))$, and $Z(x,s) = F(X(x,s),s)$. When there is no confusion, we just write $X(s), P(s), Z(s)$ instead of $X(x,s), P(x,s), Z(x,s)$, respectively. Then, $X(0) = x$, $P(0) = \partial_x F_0(x)$, $Z(0) = F_0(x)$. We have the following Hamiltonian system

$$\begin{cases} \dot{X} = \partial_p H(P(s), Z(s), X(s)) = P(s) - \left(m + \frac{1}{2}\right), \\ \dot{P} = -\partial_x H - (\partial_z H)P = \frac{Z(s)}{X(s)^2} - \frac{P(s)}{X(s)}, \\ \dot{Z} = P \cdot \partial_p H - H = \frac{P(s)^2}{2} - \frac{Z(s)}{X(s)} + \frac{m(1-m)}{2}. \end{cases}$$

Note first that $F \in C^{1,1}((0,\infty) \times (0,T^*))$, and also $0 \leqslant \partial_x F \leqslant m$ thanks to Lemma 2.4. Therefore,

$$(3.1) \qquad\qquad -\left(m + \frac{1}{2}\right) \leqslant \dot{X} \leqslant -\frac{1}{2}.$$

Besides, the concavity of F in x yields further that

$$\dot{P} = \frac{Z(s)}{X(s)^2} - \frac{P(s)}{X(s)} = \frac{1}{X(s)}\left(\frac{F(X(s),s)}{X(s)} - \partial_x F(X(s),s)\right) \geqslant 0.$$

Let us now show that $\{X(x,\cdot)\}_{x\in(0,\infty)}$ are well-ordered in $(0,\infty) \times (0,T^*)$, and none of these two characteristics intersect. Assume otherwise that $X(x,s) = X(y,s) > 0$ for some $x \neq y$ and $s \in (0,t]$. As $F \in C^{1,1}((0,\infty) \times (0,T^*))$, $\partial_x F(X(x,s),s)$ is uniquely defined, and therefore,

$$P(x,s) = P(y,s) = \partial_x F(X(x,s),s) \quad \text{and} \quad Z(x,s) = Z(y,s) = F(X(x,s),s).$$

Hence, $(X,P,Z)(x,s) = (X,P,Z)(y,s)$, and this contradicts the uniqueness of solutions to the Hamiltonian system on $[0,s]$ as we reverse the time.

By Lemma 2.4, we have that for $t < T^*$ and $(x,s) \in (0,\infty) \times [0,t]$,

$$-\frac{2}{T^* - t} \leqslant \partial_x^2 F(x,s) \leqslant 0$$

in the viscosity sense. We differentiate the first equation in the Hamiltonian system with respect to x and use the fact that $P(x,s) = \partial_x F(X(x,s),s)$ to yield that

$$\partial_x \dot{X}(x,s) = \partial_x P(x,s) = \partial_x^2 F(X(x,s),s) \cdot \partial_x X(x,s) \geqslant -\frac{2}{T^* - t}\partial_x X(x,s).$$

Thus, $\partial_x X(x,s)$ satisfies a differential inequality, and in particular,

$$s \mapsto e^{\frac{2s}{T^* - t}}\partial_x X(x,s) \qquad \text{is nondecreasing on } [0,t].$$

Therefore, $\partial_x X(x,s) > 0$ for all $(x,s) \in (0,\infty) \times [0,t]$ as $\partial_x X(x,0) = 1$. By the inverse function theorem, $X^{-1}(\cdot,s)$ is then locally smooth, and

$$F(x,s) = Z(X^{-1}(x,s),s)$$

is smooth as Z is also smooth. The proof is complete. $\qquad\square$

To show the absolute-monotone property of F, we exploit the approximating functions $\{F^\varepsilon\}_{\varepsilon>0}$, which are Bernstein functions themselves. This allows us to completely avoid the technical tour de force as in [11, Proposition 3.10].

Lemma 3.2. *Let $F = \bar{F}$ be the sublinear viscosity solution to equation* (1.2) *for $T = T^*$. Then, for every $(x,t) \in (0,\infty) \times (0,T^*)$ and $k \in \mathbb{N}$,*

$$(3.2) \qquad (-1)^{k-1}\partial_x^k F(x,t) \geqslant 0.$$

Proof. Fix $k \in \mathbb{N}$ and $t \in (0,T^*)$. For each $\varepsilon > 0$, let F^ε be the solution to equation (2.2). For every test function $\varphi \in C_c^\infty((0,\infty))$ with $\varphi \geqslant 0$, we have that

$$0 \leqslant \int_0^\infty (-1)^{k-1}\partial_x^k F^\varepsilon(x,t)\varphi(x)\,dx = \int_0^\infty (-1)F^\varepsilon(x,t)\partial_x^k\varphi(x)\,dx.$$

Letting $\varepsilon \to 0$, we have

$$\int_0^\infty (-1)F(x,t)\partial_x^k\varphi(x)\,dx \geqslant 0.$$

As $F \in C^\infty((0,\infty) \times (0,T^*))$, we integrate by parts once again to get

$$\int_0^\infty (-1)^{k-1}\partial_x^k F(x,t)\varphi(x)\,dx \geqslant 0,$$

from which (3.2) follows. $\qquad\square$

Proof of Theorem 1.1. Let $F = \bar{F}$ be the sublinear viscosity solution to equation (1.2) for $T = T^*$. By Proposition 3.1 and Lemma 3.2, we deduce that for each $t \in (0,T^*)$, $F(\cdot,t)$ is a Bernstein function. Theorem 1.1 follows from the fact that for each Bernstein function f, there exists a Borel measure μ so that $\mathfrak{B}[\mu] = f$ (see [11, Appendix]). $\qquad\square$

Appendix A. Some moment bounds for solutions to (2.1)

In this appendix, we will demonstrate a heuristic understanding on why equation (2.1) is well-posed for all $t > 0$, which was proven in [4]. This essentially comes from the ability to control all the moments based on certain differential inequalities. Here, the strong fragmentation term plays a crucial role. The argument here follows that in [4].

Assume that $m_k(0) < \infty$ for all $k \in \mathbb{N}$. Heuristically, we consider test functions $\phi(s) = s^k$ in Definition 1 to read off the information about $m_k^\varepsilon(t)$. We will demonstrate how this is done for $k = 2, 3$. Higher moments could be bounded in a similar manner inductively.

For $k = 2$, by using $\phi(s) = s^2$, we get that

$$\frac{d}{dt}m_2^\varepsilon(t) = m_2^\varepsilon(t)^2 - \frac{1}{6}m_3^\varepsilon(t) - \frac{\varepsilon}{6}m_4^\varepsilon(t).$$

We wish to control $m_2^\varepsilon(t)^2$ to prevent blow-up in finite time of $m_2^\varepsilon(t)$. By Hölder's inequality, we have

$$\left(\int_0^\infty \left(s^{4/3}\rho_\varepsilon^{1/3}\right)^3 ds\right)^{1/3}\left(\int_0^\infty \left(s^{2/3}\rho_\varepsilon^{2/3}\right)^{3/2} ds\right)^{2/3} \geqslant \int_0^\infty s^2\rho_\varepsilon\,ds.$$

Therefore,

$$m_4^\varepsilon(t) \geqslant \frac{m_2^\varepsilon(t)^3}{m^2}.$$

This implies that

$$\frac{d}{dt}m_2^\varepsilon(t) \leqslant m_2^\varepsilon(t)^2 - \frac{\varepsilon}{6}\frac{m_2^\varepsilon(t)^3}{m^2} \leqslant C_{\varepsilon,2},$$

for some $C_{\varepsilon,2} > 0$. Therefore, for $t \geqslant 0$,

$$m_2^\varepsilon(t) \leqslant m_2(0) + C_{\varepsilon,2} t \,.$$

For $k = 3$, using $\phi(s) = s^3$, we have

$$\frac{d}{dt} m_3^\varepsilon(t) = 3m_3^\varepsilon(t)m_2^\varepsilon(t) - \frac{1}{12} m_4^\varepsilon(t) - \frac{\varepsilon}{12} m_5^\varepsilon(t) \,.$$

By the Cauchy-Schwarz inequality, we have

$$m_5^\varepsilon(t)m_1^\varepsilon(t) \geqslant m_3^\varepsilon(t)^2 \,.$$

Therefore,

$$m_5^\varepsilon(t) \geqslant \frac{m_3^\varepsilon(t)^2}{m} \,,$$

and

$$\frac{d}{dt} m_3^\varepsilon(t) \leqslant 3(m_2(0) + C_{\varepsilon,2}t)m_3^\varepsilon(t) - \frac{\varepsilon}{12m} m_3^\varepsilon(t)^2 \leqslant C_{\varepsilon,3}(t+1)^2 \,,$$

for some $C_{\varepsilon,3} > 0$. Thus, for $t \geqslant 0$,

$$m_3^\varepsilon(t) \leqslant m_3(0) + C_{\varepsilon,3}(t+1)^3 \,.$$

References

1. Aldous, D. J. Deterministic and Stochastic Models for Coalescence (Aggregation and Coagulation): A Review of the Mean-Field Theory for Probabilists. *Bernoulli. Official Journal of the Bernoulli Society for Mathematical Statistics and Probability* **5,** 3–48. ISSN: 1350-7265 (1999).
2. Banasiak, J., Lamb, W. & Laurencot, P. *Analytic Methods for Coagulation-Fragmentation Models* (CRC Press, 2019).
3. Da Costa, F. P. in *Mathematics of Energy and Climate Change* 83–162 (Springer, Cham, 2015).
4. Escobedo, M., Laurençot, P., Mischler, S. & Perthame, B. Gelation and Mass Conservation in Coagulation-Fragmentation Models. *Journal of Differential Equations* **195,** 143–174. ISSN: 0022-0396 (2003).
5. Evans, L. C. *Partial Differential Equations* Second. xxii+749. ISBN: 978-0-8218-4974-3 (American Mathematical Society, Providence, RI, 2010).
6. Laurençot, P. Mass-conserving solutions to coagulation-fragmentation equations with balanced growth. *Colloquium Mathematicum* **159,** 127–155. ISSN: 0010-1354 (2020).
7. Menon, G. & Pego, R. L. Approach to Self-Similarity in Smoluchowski's Coagulation Equations. *Communications on Pure and Applied Mathematics* **57,** 1197–1232. ISSN: 0010-3640 (2004).
8. Mitake, H., Tran, H. V. & Van, T.-S. Large time behavior for a Hamilton-Jacobi equation in a critical Coagulation-Fragmentation model. *Communications in Mathematical Sciences* **19,** 495–512 (2021).
9. Smoluchowski, M. V. Drei Vortrage Uber Diffusion, Brownsche Bewegung Und Koagulation von Kolloidteilchen. *Zeitschrift fur Physik* **17,** 557–585 (1916).
10. Tran, H. V. *Hamilton-Jacobi equations—theory and applications* xiv+322. ISBN: 978-1-4704-6511-7 (American Mathematical Society, Providence, RI, Providence, Rhode Island, 2021).
11. Tran, H. V. & Van, T.-S. Coagulation-Fragmentation Equations with Multiplicative Coagulation Kernel and Constant Fragmentation Kernel. *Communications on Pure and Applied Mathematics* (Feb. 2021).

12. Vigil, R. D. & Ziff, R. M. On the Stability of Coagulation–Fragmentation Population Balances. *Journal of Colloid and Interface Science* **133,** 257–264 (Nov. 1989).

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