

Level-set forced mean curvature flow with the Neumann boundary condition

Jiwoong Jang^a, Dohyun Kwon^a, Hiroyoshi Mitake^b, Hung V. Tran^{a,*}

^a*Department of Mathematics, University of Wisconsin-Madison, Van Vleck Hall, 480 Lincoln Drive, Madison, Wisconsin, 53706, USA*

^b*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-Ku, Tokyo, 153-8914, Japan*

Abstract

Here, we study a level-set forced mean curvature flow with the homogeneous Neumann boundary condition. We first show that the solution is Lipschitz in time and locally Lipschitz in space. Then, under an additional condition on the forcing term, we prove that the solution is globally Lipschitz. We obtain the large time behavior of the solution in this setting and study the large time profile in some specific situations. Finally, we give two examples demonstrating that the additional condition on the forcing term is sharp, and without it, the solution might not be globally Lipschitz.

Résumé

Dans cet article on étudie, par la méthode des ensembles de niveau, le mouvement par courbure moyenne avec une condition de Neumann homogène au bord. On montre d'abord que la solution est Lipschitz en temps et localement Lipschitz en espace. Ensuite, sous une condition supplémentaire sur le terme de forçage, on montre que la solution est globalement Lipschitz. Dans ce contexte, on obtient le comportement en temps long de la solution et on étudie le profil asymptotique dans certaines situations. Enfin, on propose deux exemples pour montrer que la condition additionnelle sur le terme de forçage ne peut pas être améliorée et que sans celle-ci la solution peut perdre son caractère Lipschitz global.

*Corresponding author

Email addresses: jjang57@wisc.edu (Jiwoong Jang), dkwon7@wisc.edu (Dohyun Kwon), mitake@g.ecc.u-tokyo.ac.jp (Hiroyoshi Mitake), hung@math.wisc.edu (Hung V. Tran)

Preprint submitted to Journal de Mathématiques Pures et Appliquées *October 5, 2022*

Keywords: Level-set mean curvature flows, Neumann boundary problem, global Lipschitz regularity, large time behavior, the large time profile
2000 MSC: 35B40, 35B45, 35K20, 35K93, 49L25, 53E10

1. Introduction

In this paper, we study the level-set equation for the forced mean curvature flow

$$\begin{cases} u_t = |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) + c(x)|Du| & \text{in } \Omega \times (0, \infty), & (1.1) \\ \frac{\partial u}{\partial \bar{\mathbf{n}}} = 0 & \text{on } \partial\Omega \times [0, \infty), & (1.2) \\ u(x, 0) = u_0(x) & \text{on } \bar{\Omega}. & (1.3) \end{cases}$$

The domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is assumed to be bounded and $C^{2,\theta}$ for some $\theta \in (0, 1)$. Here, $c = c(x)$ is a forcing function, which is in $C^1(\bar{\Omega})$, and $\bar{\mathbf{n}}$ is the outward unit normal vector to $\partial\Omega$. Throughout this paper, we assume that $u_0 \in C^{2,\theta}(\bar{\Omega})$, and $\frac{\partial u_0}{\partial \bar{\mathbf{n}}} = 0$ on $\partial\Omega$ for compatibility.

We first notice that the well-posedness and the comparison principle for (1.1)–(1.3) are well established in the theory of viscosity solutions (see [2, 4, 10, 11] for instance). Our main interest in this paper is to go beyond the well-posedness theory to understand the Lipschitz regularity and large time behavior of the solution. The Lipschitz regularity for the solution is rather subtle because of the competition between the forcing term and the mean curvature term together with the constraint on perpendicular intersections of the level sets of the solution with the boundary of Ω . It is worth emphasizing that the geometry of $\partial\Omega$ plays a crucial role in the analysis.

We now describe our main results. First of all, we show that u is Lipschitz in time and locally Lipschitz in space.

Theorem 1.1. *Let u be the unique viscosity solution u of (1.1)–(1.3). Then, there exists a constant $M > 0$ and for each $T > 0$, there exists a constant $C_T > 0$ depending on T such that*

$$\begin{cases} |u(x, t) - u(x, s)| \leq M|t - s|, \\ |u(x, t) - u(y, t)| \leq C_T|x - y|, \end{cases} \quad \text{for all } x, y \in \bar{\Omega}, t, s \in [0, T].$$

We next show that if we put some further conditions on the forcing term c , then we have the global Lipschitz estimate in x of the solution. Denote by

$$\begin{cases} C_0 := \max\{-\lambda : \lambda \text{ is a principal curvature of } \partial\Omega \text{ at } x_0 \text{ for } x_0 \in \partial\Omega\} \in \mathbb{R}, \\ K_0 := \min\{d : d \text{ is the diameter of an open ball inscribed in } \Omega\} > 0. \end{cases}$$

Theorem 1.2. *Assume that there exists $\delta > 0$ such that*

$$\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > \max\left\{0, C_0|c(x)| + \frac{2nC_0}{K_0}\right\} \quad \text{for all } x \in \Omega. \quad (1.4)$$

Let u be the unique viscosity solution to (1.1)–(1.3). Then, there exist constants $M, L > 0$ depending only on the forcing term c and the initial data u_0 such that

$$\begin{cases} |u(x, t) - u(x, s)| \leq M|t - s|, \\ |u(x, t) - u(y, t)| \leq L|x - y|, \end{cases} \quad \text{for all } x, y \in \bar{\Omega}, t, s \in [0, \infty). \quad (1.5)$$

Let us now explain a bit the geometric meaning of K_0 . For each $x \in \partial\Omega$, let

$$K_x = \max\{2r > 0 : B(x - r\bar{\mathbf{n}}(x), r) \subset \Omega\}.$$

Then, $K_0 = \min_{x \in \partial\Omega} K_x$. We notice next that if Ω is convex in Theorem 1.2, then we clearly have $C_0 \leq 0$. In this case, (1.4) becomes $\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > 0$, a kind of coercive assumption, which often appears in the usage of the classical Bernstein method to obtain Lipschitz regularity (see [19] for instance).

In the specific case where $c \equiv 0$ and Ω is convex and bounded, the global Lipschitz estimate of the solution was obtained in [9]. See Remark 1. Moreover, a very interesting example was given in [9] to show that the solution is not globally Lipschitz continuous if Ω is not convex. Motivated by this example, we give two examples showing that u is not globally Lipschitz continuous if we do not impose (1.4). Furthermore, the examples demonstrate that condition (1.4) is sharp.

Let us note that the graph mean curvature flow with the Neumann boundary conditions has been studied much in the literature (see [13, 15, 20] and the references therein).

We next study the large time behavior of u under condition (1.4).

Theorem 1.3. Assume (1.4). Let u be the unique viscosity solution to (1.1)–(1.3). Then,

$$u(\cdot, t) \rightarrow v, \quad \text{as } t \rightarrow \infty,$$

uniformly on $\bar{\Omega}$ for some Lipschitz function v , which is a viscosity solution to

$$\begin{cases} -\left(\operatorname{div}\left(\frac{Dv}{|Dv|}\right) + c(x)\right) |Dv| = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \bar{\mathbf{n}}} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

We prove Theorem 1.3 by using a Lyapunov function, which is quite standard. We say that v is the large time profile of the solution u . It is important to note that the stationary problem (1.6) may have various different solutions, and thus, the question on how the large time profile v depends on the initial data u_0 is rather delicate and challenging. We are able to answer this question in the radially symmetric setting, and it is still widely open in the general settings.

Theorem 1.4. Suppose that, by abuse of notions,

$$\begin{cases} \Omega = B(0, R) \text{ for some } R > 0, \\ c(x) = c(r) \text{ for } |x| = r \in [0, R], \\ u_0(x) = u_0(r) \text{ for } |x| = r \in [0, R]. \end{cases} \quad (1.7)$$

Here, $c \in C^1([0, R], [0, \infty))$, and $u_0 \in C^2([0, R])$ with $u_0'(R) = 0$. Denote by

$$\begin{aligned} \mathcal{A} &:= \left\{ r \in (0, R] : c(r) = \frac{n-1}{r} \right\}, \\ \mathcal{A}_+ &:= \left\{ r \in (0, R] : c(r) > \frac{n-1}{r} \right\}, \\ \mathcal{A}_- &:= \left\{ r \in (0, R] : c(r) < \frac{n-1}{r} \right\}. \end{aligned}$$

Define $d : (0, R] \rightarrow (0, R]$ as

$$d(r) = \begin{cases} r & \text{if } r \in \mathcal{A}, \\ \max(\mathcal{A} \cap (0, r)) & \text{if } r \in \mathcal{A}_+, \\ \min(\mathcal{A} \cap (r, R]) & \text{if } r \in \mathcal{A}_- \text{ and } \mathcal{A} \cap (r, R] \neq \emptyset, \\ R & \text{if } r \in \mathcal{A}_- \text{ and } \mathcal{A} \cap (r, R] = \emptyset. \end{cases}$$

Write $u(x, t) = \phi(|x|, t)$ for $x \in \Omega = B(0, R)$ and $t \geq 0$. Then, the limiting profile $\phi_\infty(r) = \lim_{t \rightarrow \infty} \phi(r, t)$ can be written in terms of u_0 as: for each $r_0 \in (0, R]$,

$$\phi_\infty(r_0) = \max \{u_0(r) : r \geq d(r_0)\}. \quad (1.8)$$

As a by-product, Theorem 1.4 shows that the solution to (1.1)–(1.3) is not globally Lipschitz continuous with an appropriate choice of initial data u_0 .

Corollary 1.5. *Consider the setting in Theorem 1.4. Assume that there exist $0 < a < b < R$ such that $a, b \in \mathcal{A}$ and $(a, b) \subset \mathcal{A}_-$. Assume further that u_0 is a C^2 function on $[0, R]$ such that*

$$u_0(r) = \begin{cases} 1 & \text{for } r \leq a, \\ \in (0, 1) & \text{for } a < r \leq b, \\ 0 & \text{for } b < r \leq R. \end{cases}$$

Then, u is not globally Lipschitz, and

$$\phi_\infty(r) = \begin{cases} 1 & \text{for } r \leq a, \\ 0 & \text{for } a < r \leq R. \end{cases}$$

Lastly, we give another example to show the non global Lipschitz phenomenon in Theorem 6.1. Since we deal with the situation where Ω is unbounded there, we leave the precise statement of Theorem 6.1 and corresponding adjustments to Section 6.

Our problem (1.1)–(1.2) basically describes a level-set forced mean curvature flow with the homogeneous Neumann boundary condition. If a level set of the unknown u is a smooth enough surface, then it evolves with the normal velocity $V(x) = \kappa + c(x)$, where κ equals $(n - 1)$ times the mean curvature of the surface at x , and it perpendicularly intersects $\partial\Omega$ (if ever). What is really interesting and delicate here is the competition between the forcing term $c(x)$ and the mean curvature term κ coupled with the constraint on perpendicular intersections of the level sets with the boundary. It is worth emphasizing that we do not assume Ω is convex, and the geometry of $\partial\Omega$ plays a crucial role in the behavior of the solution here. Indeed, analyzing the competition between the two constraints, the force and the boundary condition subjected to $\partial\Omega$, as time evolves in viscosity sense is the main topic of this paper.

We now briefly describe our approaches to get the aforementioned results. We use the maximum principle and rely on the classical Bernstein method to establish *a priori* gradient estimates for the solution. The main difficulty is when a maximizer is located on the boundary, which we cannot apply the maximum principle directly. We deal with this difficulty by considering a multiplier that puts the maximizer, with the homogeneous Neumann boundary condition, inside the domain so that the maximum principle is applicable. To the best of our knowledge, the idea of handling a maximizer in the proof of Theorem [1.2](#) for the level-set equation for forced mean curvature flows under the Neumann boundary condition is new in the literature.

Once we get a global Lipschitz estimate for the solution, by using a standard Lyapunov function, we prove the convergence in Theorem [1.3](#). Next, the radially symmetric setting is considered, and [\(1.1\)–\(1.3\)](#) are reduced to a first-order singular Hamilton-Jacobi equation with the homogeneous Neumann boundary condition; see [\[7, 8\]](#) for a related problem on the whole space. By using the representation formula for the Neumann problem (see, e.g., [\[16\]](#)), we are able to obtain Theorem [1.4](#) and Corollary [1.5](#). The situation considered in Theorem [6.1](#) is related to that in [\[23, Section 4\]](#) with no forcing term. As we have a constant forcing c interacting with the boundary, the construction in the proof of Theorem [6.1](#) is rather delicate and involved. It is worth emphasizing that Corollary [1.5](#) and Theorem [6.1](#) demonstrate that condition [\(1.4\)](#), which is needed for the global Lipschitz regularity of u , is essentially optimal.

We conclude this introduction by giving a non exhaustive list of related works to our paper. There are several asymptotic analysis results on the forced mean curvature flows with Neumann boundary conditions [\[13, 21, 22, 24\]](#) or with periodic boundary conditions [\[3\]](#), but they are all for graph-like surfaces. The volume preserving mean curvature flow, which is a different type of forced mean curvature flows, was studied in [\[17, 18\]](#). Recently, the relation between the level set approach and the varifold approach for [\(1.1\)](#) with $c \equiv 0$ was investigated in [\[1\]](#). We also refer to [\[8, 14\]](#) for some recent results on the asymptotic growth speed of solutions to forced mean curvature flows with discontinuous source terms in the whole space.

Organization of the paper

The paper is organized as follows. In Section [2](#), we give the notion of viscosity solutions to the problem and some basic results. In Section [3](#), we

prove the local and global gradient estimates. Section 4 is devoted to the study on large time behavior of the solution and its large time profile. We give two examples that the spatial gradient of the solution grows to infinity as time tends to infinity in Sections 5 and 6 if we do not impose assumption (1.4) on the force c .

2. Preliminaries

In this section, we recall the notion of viscosity solutions to the Neumann boundary problem (1.1)–(1.3) and give some related results.

Let \mathcal{S}^n be the set of symmetric matrices of size n . Define $F : \bar{\Omega} \times (\mathbb{R}^n \setminus \{0\}) \times \mathcal{S}^n \rightarrow \mathbb{R}$ by

$$F(x, p, X) = \text{trace} \left(\left(I - \frac{p \otimes p}{|p|^2} \right) X \right) + c(x)|p|.$$

We denote the semicontinuous envelopes of F by, for $(x, p, X) \in \bar{\Omega} \times \mathbb{R}^n \times \mathcal{S}^n$,

$$F_*(x, p, X) = \liminf_{(y, q, Y) \rightarrow (x, p, X)} F(y, q, Y), \quad F^*(x, p, X) = \limsup_{(y, q, Y) \rightarrow (x, p, X)} F(y, q, Y).$$

Definition 2.1. *An upper semicontinuous function $u : \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{R}$ is said to be a viscosity subsolution of (1.1)–(1.3) if $u(\cdot, 0) \leq u_0$ on $\bar{\Omega}$, and, for any $\varphi \in C^2(\bar{\Omega} \times [0, \infty))$, if $(\hat{x}, \hat{t}) \in \bar{\Omega} \times (0, \infty)$ is a maximizer of $u - \varphi$, and if $\hat{x} \in \Omega$, then*

$$\varphi_t(\hat{x}, \hat{t}) - F^*(\hat{x}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})) \leq 0;$$

if $\hat{x} \in \partial\Omega$, then

$$\min \left\{ \varphi_t(\hat{x}, \hat{t}) - F^*(\hat{x}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})), \frac{\partial\varphi}{\partial\bar{\mathbf{n}}}(\hat{x}, \hat{t}) \right\} \leq 0.$$

Similarly, a lower semicontinuous function $u : \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{R}$ is said to be a viscosity supersolution of (1.1)–(1.3) if $u(\cdot, 0) \geq u_0$ on $\bar{\Omega}$, and, for any $\varphi \in C^2(\bar{\Omega} \times [0, \infty))$, if $(\hat{x}, \hat{t}) \in \bar{\Omega} \times (0, \infty)$ is a minimizer of $u - \varphi$, and if $\hat{x} \in \Omega$, then

$$\varphi_t(\hat{x}, \hat{t}) - F_*(\hat{x}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})) \geq 0;$$

if $\hat{x} \in \partial\Omega$, then

$$\max \left\{ \varphi_t(\hat{x}, \hat{t}) - F_*(\hat{x}, D\varphi(\hat{x}, \hat{t}), D^2\varphi(\hat{x}, \hat{t})), \frac{\partial\varphi}{\partial\vec{\mathbf{n}}}(\hat{x}, \hat{t}) \right\} \geq 0.$$

Finally, a continuous function u is said to be a viscosity solution of (1.1)–(1.3) if u is both its viscosity subsolution and its viscosity supersolution.

Henceforth, since we are always concerned with viscosity solutions, the adjective “viscosity” is omitted. The following comparison principle for solutions to (1.1)–(1.3) in a bounded domain is well known (see, e.g., [10]).

Proposition 2.2 (Comparison principle for (1.1)–(1.3)). *Let u and v be a subsolution and a supersolution of (1.1)–(1.3), respectively. Then, $u \leq v$ in $\bar{\Omega} \times [0, \infty)$.*

To obtain Lipschitz estimates, it is convenient to consider an approximate problem of (1.1)–(1.3) by considering, for $\varepsilon > 0$, $T > 0$,

$$\begin{cases} u_t^\varepsilon = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2} \operatorname{div} \left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} \right) + c(x) \sqrt{\varepsilon^2 + |Du^\varepsilon|^2} & \text{in } \Omega \times (0, T], \\ \frac{\partial u^\varepsilon}{\partial \vec{\mathbf{n}}} = 0 & \text{on } \partial\Omega \times [0, T], \\ u^\varepsilon(x, 0) = u_0(x) & \text{on } \bar{\Omega}. \end{cases} \quad (2.1)$$

Equation (2.1) describes the motion of the graph of $\frac{u^\varepsilon}{\varepsilon}$ under the forced mean curvature flow $V = \kappa + c$ in Ω with right contact angle condition on $\partial\Omega$. The following result on a priori estimates on the gradient of u^ε plays a crucial role in our analysis.

Theorem 2.3 (A priori estimates). *Assume that $\partial\Omega$ is smooth and $c \in C^\infty(\bar{\Omega})$. For each $\varepsilon \in (0, 1)$ and $T > 0$, assume that $u^\varepsilon \in C^\infty(\bar{\Omega} \times (0, T]) \cap C^1(\bar{\Omega} \times [0, T])$ is the unique solution of (2.1). Then, there exist a constant $M > 0$ and a constant $C_T > 0$ depending on T such that*

$$\|u_t^\varepsilon\|_{L^\infty(\bar{\Omega} \times [0, T])} \leq M \quad \text{and} \quad \|Du^\varepsilon\|_{L^\infty(\bar{\Omega} \times [0, T])} \leq C_T. \quad (2.2)$$

Here, M and C_T are independent of $\varepsilon \in (0, 1)$.

The proof of Theorem 2.3 is given in the next section. The a priori estimates then allow us to get the existence and uniqueness of solutions to (2.1).

Proposition 2.4. For each $\varepsilon \in (0, 1)$ and $T > 0$, equation (2.1) has a unique continuous solution u^ε . Furthermore, $u^\varepsilon \in C^{2,1}(\Omega \times (0, T]) \cap C^1(\bar{\Omega} \times [0, T])$ and (2.2) holds.

Proposition 2.4 can be obtained by the classical parabolic PDE theory. For instance, we refer to [22] for a similar form of Proposition 2.4. The proof of this proposition is quite standard, and hence, is omitted here.

Once we get (2.2), by the standard stability result of viscosity solutions, and the uniqueness of viscosity solutions to (1.1)–(1.3), we imply that

$$u^\varepsilon \rightarrow u \quad \text{as } \varepsilon \rightarrow 0 \quad \text{uniformly on } \bar{\Omega} \times [0, T]$$

for each $T > 0$. Moreover, Theorem 2.3 and Proposition 2.4 give us right away Theorem 1.1.

3. Lipschitz regularity

In this section, we prove Theorems 1.1, 1.2, and 2.3. As noted, it is actually enough to prove Theorems 1.2 and 2.3. First, we prove that the time derivative of u^ε is bounded.

Lemma 3.1. Assume that $\partial\Omega$ is smooth and $c \in C^\infty(\bar{\Omega})$. Suppose that $u^\varepsilon \in C^\infty(\bar{\Omega} \times (0, T]) \cap C^1(\bar{\Omega} \times [0, T])$ is the unique solution of (2.1) for each $\varepsilon \in (0, 1)$ and $T > 0$. Then, there exists $M > 0$ depending only on the forcing term c and the initial data u_0 such that, for $\varepsilon \in (0, 1)$,

$$\|u_t^\varepsilon\|_{L^\infty(\bar{\Omega} \times [0, T])} \leq \|u_t^\varepsilon(\cdot, 0)\|_{L^\infty(\bar{\Omega})} \leq M.$$

Proof. Set $b(p) = I_n - p \otimes p / (\varepsilon^2 + |p|^2)$. Then (2.1) is expressed as

$$u_t^\varepsilon - b^{ij}(Du^\varepsilon)u_{ij}^\varepsilon - c(x)\sqrt{\varepsilon^2 + |Du^\varepsilon|^2} = 0 \quad \text{in } \Omega \times (0, T]. \quad (3.1)$$

Here, we use the Einstein summation convention, and we write $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for $i, j = 1, \dots, n$, where $f = f(x, t)$ is a given function. We now show that

$$\|u_t^\varepsilon\|_{L^\infty(\bar{\Omega} \times [0, T])} \leq \|u_t^\varepsilon(\cdot, 0)\|_{L^\infty(\bar{\Omega})}. \quad (3.2)$$

To prove (3.2), it is enough to obtain the upper bound

$$\max_{\bar{\Omega} \times [0, T]} u_t^\varepsilon = \max_{\bar{\Omega}} u_t^\varepsilon(\cdot, 0)$$

as the lower bound can be obtained analogously.

Differentiating (3.1) with respect to t yields

$$(u_t^\varepsilon)_t - b^{ij}(u_t^\varepsilon)_{ij} - (b^{ij})_t u_{ij}^\varepsilon - c(x) \frac{(u_t^\varepsilon)_l u_l^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} = 0,$$

where

$$(b^{ij})_t = -\frac{(u_t^\varepsilon)_i u_j^\varepsilon}{\varepsilon^2 + |Du^\varepsilon|^2} - \frac{u_i^\varepsilon (u_t^\varepsilon)_j}{\varepsilon^2 + |Du^\varepsilon|^2} + \frac{2u_i^\varepsilon u_j^\varepsilon u_l^\varepsilon (u_t^\varepsilon)_l}{(\varepsilon^2 + |Du^\varepsilon|^2)^2}.$$

Suppose, on the contrary, that $u_t^\varepsilon(x, t) > \max_{\bar{\Omega}} u_t^\varepsilon(\cdot, 0)$ for some $(x, t) \in \bar{\Omega} \times (0, T]$. Then, there exist a small number $\delta > 0$ and $(x_0, t_0) \in \bar{\Omega} \times (0, T]$ such that $(x_0, t_0) \in \operatorname{argmax}_{\bar{\Omega} \times (0, T]} (u_t^\varepsilon - \delta t)$.

At (x_0, t_0) , we have $Du_t^\varepsilon = 0$, and note that the boundary case $x_0 \in \partial\Omega$ is included due to the homogeneous Neumann boundary condition. Thus,

$$(u_t^\varepsilon)_t - b^{ij}(u_t^\varepsilon)_{ij} = 0, \quad \text{at } (x_0, t_0). \quad (3.3)$$

On the other hand, $(u_t^\varepsilon - \delta t)_t \geq 0$, $-b^{ij}(u_t^\varepsilon)_{ij} \geq 0$ at (x_0, t_0) . Note that the Neumann boundary condition is used for $D^2 u_t^\varepsilon \leq 0$ at (x_0, t_0) as well. Since $(u_t^\varepsilon)_t \geq \delta > 0$, we arrive at a contradiction in (3.3). Thus, (3.2) holds. Choose

$$M = n^2 \|D^2 u_0\|_{L^\infty(\bar{\Omega})} + \|c\sqrt{1 + |Du_0|^2}\|_{L^\infty(\bar{\Omega})}$$

to complete the proof. \square

We are now ready to prove Theorems 1.2 and 2.3 using the classical Bernstein method. It is important emphasizing that the boundary behavior needs to be handled rather carefully. We first give a proof of Theorem 1.2.

Proof of Theorem 1.2. Assume first that $\partial\Omega$ is smooth and $c \in C^\infty(\bar{\Omega})$. For each $\varepsilon \in (0, 1)$ and $T > 0$, let $u^\varepsilon \in C^\infty(\bar{\Omega} \times (0, T]) \cap C^1(\bar{\Omega} \times [0, T])$ be the unique solution of (2.1).

Let $w^\varepsilon = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}$. In view of Lemma 3.1, we only need to show that

$$\max_{\bar{\Omega} \times [0, T]} w^\varepsilon \leq C \quad (3.4)$$

for some positive constant C depending only on $\|u_0\|_{C^2(\bar{\Omega})}$, $\|c\|_{C^1(\bar{\Omega})}$, the constants n , C_0 , K_0 , and δ from (1.4). The crucial point here is C does not depend on T and ε . Fix $(x_0, t_0) \in \operatorname{argmax}_{\bar{\Omega} \times [0, T]} w^\varepsilon$. If $t_0 = 0$, then

$$\max_{\bar{\Omega} \times [0, T]} w^\varepsilon \leq w^\varepsilon(x_0, 0) \leq \|Du_0\|_{L^\infty(\bar{\Omega})} + 1,$$

and (3.4) is valid. We next consider the case $t_0 > 0$.

We write $u = u^\varepsilon$, $w = w^\varepsilon$ in this proof for brevity. Differentiate (3.1) in x_k and multiply the result by u_k to get

$$u_k u_{kt} - (D_p b^{ij} \cdot Du_k) u_k u_{ij} - b^{ij} u_k u_{kij} - u_k c_k w - c \frac{u_k u_{lk} u_l}{w} = 0.$$

Substituting $ww_t = u_k u_{kt}$, $ww_k = u_l u_{kl}$ and $ww_{ij} = u_{kij} u_k + b^{kl} u_{ki} u_{lj}$, we get

$$ww_t - w(D_p b^{ij} \cdot Dw) u_{ij} - w b^{ij} w_{ij} + b^{ij} b^{kl} u_{ki} u_{lj} - w Du \cdot Dc - c Du \cdot Dw = 0. \quad (3.5)$$

We divide the proof into two cases: $x_0 \in \Omega$ and $x_0 \in \partial\Omega$.

Case 1: the interior case $x_0 \in \Omega$. We follow the computations of [6, Lemma 4.1]. At (x_0, t_0) , we have $w_t \geq 0$, $Dw = 0$, $D^2 w \leq 0$, and thus

$$w Du \cdot Dc \geq b^{ij} b^{kl} u_{ki} u_{lj}.$$

We then use the Cauchy-Schwarz inequality

$$(\text{tr} \alpha \beta)^2 \leq \text{tr}(\alpha^2) \text{tr}(\beta^2)$$

for all $\alpha, \beta \in \mathcal{S}^n$, and put $\alpha = A^{\frac{1}{2}} B A^{\frac{1}{2}}$, $\beta = I_n$, where $A = (b^{ij})$, $B = (u_{kl})$, I_n the n by n identity matrix to get $\text{tr}(AB)^2 \geq (\text{tr} AB)^2 / \text{tr}(I_n)$.

Therefore, at (x_0, t_0) ,

$$|Dc(x_0)| w^2 \geq w Du \cdot Dc \geq b^{ij} b^{kl} u_{ki} u_{lj} = \text{tr}(AB)^2 \geq \frac{(\text{tr} AB)^2}{\text{tr}(I_n)} = \frac{1}{n} (u_t - c(x_0) w)^2$$

Since $\frac{1}{n} c(x)^2 - |Dc(x)| \geq \delta > 0$ by (1.4), we imply that at (x_0, t_0) ,

$$\delta w^2 \leq \frac{2u_t c(x_0)}{n} w \implies w \leq \frac{2M \|c\|_{L^\infty(\bar{\Omega})}}{n\delta},$$

which confirms (3.4).

Case 2: the boundary case $x_0 \in \partial\Omega$. As $\partial\Omega$ is $C^{2,\theta}$, we assume that \mathbf{n} is defined as a C^1 function in a neighborhood of $\partial\Omega$. Note that the Neumann boundary condition $Du \cdot \mathbf{\bar{n}} = 0$ gives $(D^2 u \mathbf{\bar{n}} + D\mathbf{\bar{n}} Du) \cdot v = 0$ for all $v \in \mathbb{R}^n$ perpendicular to $\mathbf{\bar{n}}$ on $\partial\Omega \times [0, T]$. Thus, on $\partial\Omega \times [0, T]$,

$$\frac{\partial w}{\partial \mathbf{\bar{n}}} = \frac{D^2 u Du}{w} \cdot \mathbf{\bar{n}} = -\frac{D\mathbf{\bar{n}} Du \cdot Du}{w} \leq C_0 \frac{|Du|^2}{w},$$

where $C_0 = \sup\{-\lambda : \lambda \text{ is a principal curvature of } \partial\Omega \text{ at } x_0 \text{ for } x_0 \in \partial\Omega\}$.

If $C_0 < 0$, then $\frac{\partial w}{\partial \bar{\mathbf{n}}} < 0$ on $\partial\Omega \times [0, T]$, and hence w cannot attain its maximum on $\partial\Omega \times [0, T]$. Therefore, $C_0 \geq 0$. We consider the case when $C_0 > 0$ first, and deal with the case when $C_0 = 0$ later. We note that if $C_0 > 0$, then

$$\frac{\partial w}{\partial \bar{\mathbf{n}}} \leq C_0 \frac{|Du|^2}{w} < C_0 w.$$

Take $x_c \in \Omega$ so that $B := B(x_c, K_0/2)$ is inside Ω and tangent to the boundary $\partial\Omega$ at x_0 . Consider a multiplier

$$\rho(x) = -\frac{C_0}{K_0}|x - x_c|^2 + \frac{C_0 K_0}{4} + 1 \quad \text{for } x \in \bar{\Omega}.$$

Then, $\rho > 1$ in B , $\rho = 1$ on ∂B , and $\rho \leq 1$ on $\bar{\Omega} \setminus B$. Besides, $C_0 \rho(x_0) + \frac{\partial \rho}{\partial \bar{\mathbf{n}}}(x_0) = 0$.

Denote by $\psi = \rho w$. Then, at (x_0, t_0) ,

$$\frac{\partial \psi}{\partial \bar{\mathbf{n}}} = \frac{\partial(\rho w)}{\partial \bar{\mathbf{n}}} = \rho \frac{\partial w}{\partial \bar{\mathbf{n}}} + w \frac{\partial \rho}{\partial \bar{\mathbf{n}}} < w \left(C_0 \rho + \frac{\partial \rho}{\partial \bar{\mathbf{n}}} \right) = 0. \quad (3.6)$$

By the choice of ρ , it is clear that

$$\psi(z, t) \leq w(z, t) \leq w(x_0, t_0) = \psi(x_0, t_0) \quad \text{for } (z, t) \in (\bar{\Omega} \setminus B) \times [0, T],$$

and, by (3.6),

$$\max_{\bar{\Omega} \times [0, T]} \rho w = \max_{B \times [0, T]} \rho w > \psi(x_0, t_0) = w(x_0, t_0). \quad (3.7)$$

Let $(x_1, t_1) \in \operatorname{argmax}_{\bar{\Omega} \times [0, T]} \rho w$. If $t_1 = 0$, then for all $(x, t) \in \bar{\Omega} \times [0, T]$,

$$\begin{aligned} w(x, t) &\leq w(x_0, t_0) = \rho(x_0)w(x_0, t_0) \leq \rho(x_1)w(x_1, 0) \\ &\leq \left(\frac{C_0 K_0}{4} + 1 \right) \left(\|Du_0\|_{L^\infty(\bar{\Omega})} + 1 \right), \end{aligned}$$

and we are done. Thus, we may assume that $t_1 > 0$. In light of (3.6)–(3.7), we yield that $x_1 \in B \subset \Omega$. At this point (x_1, t_1) , we have $\psi_t \geq 0$, $D\psi = 0$, $D^2\psi \leq 0$. Consequently, as $\psi_t = \rho_t w + \rho w_t$, $D\psi = wD\rho + \rho Dw$, and $\psi_{ij} = w_{ij}\rho + w_i\rho_j + w_j\rho_i + w\rho_{ij}$, we have at (x_1, t_1) ,

$$\begin{cases} w_t \geq -\frac{\rho_t}{\rho}w = 0, \\ Dw = -\frac{w}{\rho}D\rho, \\ w_{ij} = \frac{1}{\rho}(\psi_{ij} - w_i\rho_j - w_j\rho_i - w\rho_{ij}). \end{cases}$$

Therefore, at (x_1, t_1) , by (3.5)

$$\begin{aligned} -\frac{\rho_t}{\rho}w^2 + \frac{w^2}{\rho}(D_p b^{ij} \cdot D\rho)u_{ij} + \frac{w}{\rho}b^{ij}(w_i\rho_j + w_j\rho_i + w\rho_{ij}) \\ + b^{ij}b^{kl}u_{ki}u_{lj} - wDu \cdot Dc + \frac{cw}{\rho}Du \cdot D\rho \leq 0. \end{aligned}$$

Now,

$$b_{pi}^{ij} = -\frac{\delta_{il}u_j}{\varepsilon^2 + |Du|^2} - \frac{\delta_{jl}u_i}{\varepsilon^2 + |Du|^2} + \frac{2u_iu_ju_l}{(\varepsilon^2 + |Du|^2)^2},$$

and thus,

$$\begin{aligned} w(D_p b^{ij} \cdot D\rho)u_{ij} &= w \left(-\frac{\rho_i u_j u_{ij}}{\varepsilon^2 + |Du|^2} - \frac{\rho_j u_i u_{ij}}{\varepsilon^2 + |Du|^2} + \frac{2u_i u_j u_l \rho_l u_{ij}}{(\varepsilon^2 + |Du|^2)^2} \right) \\ &= -2Dw \cdot D\rho + \frac{2(Du \cdot D\rho)(Du \cdot Dw)}{w^2}. \end{aligned}$$

Hence,

$$\begin{aligned} w(D_p b^{ij} \cdot D\rho)u_{ij} + b^{ij}w_i\rho_j + b^{ij}w_j\rho_i \\ = \frac{2(Du \cdot D\rho)(Du \cdot Dw)}{w^2} - \frac{u_i u_j w_i \rho_j}{w^2} - \frac{u_i u_j w_j \rho_i}{w^2} = 0. \end{aligned}$$

All in all, at $(x_1, t_1) \in \operatorname{argmax}_{\bar{\Omega} \times (0, T]} \rho w$ with $x_1 \in B \subset \Omega$, the inequality

$$-\frac{\rho_t}{\rho}w^2 + \frac{\rho_{ij}}{\rho}b^{ij}w^2 + b^{ij}b^{kl}u_{ki}u_{lj} - wDu \cdot Dc + \frac{cw}{\rho}Du \cdot D\rho \leq 0 \quad (3.8)$$

holds. Note that $\rho_t = 0$ here, but we keep this term in the above formula for the usage in the proof of Theorem 2.3 later.

Using the Cauchy-Schwarz type inequality as in the above, we obtain

$$\begin{aligned} \frac{1}{n}(u_t - c(x_1)w)^2 &\leq b^{ij}b^{kl}u_{il}u_{kj} \leq -\frac{w^2}{\rho}b^{ij}\rho_{ij} + wDu \cdot Dc - \frac{cw}{\rho}Du \cdot D\rho \\ &\leq \frac{2C_0}{K_0} \frac{w^2}{\rho} \left(n - \frac{|Du|^2}{\varepsilon^2 + |Du|^2} \right) + |Dc|w^2 + C_0|c|w^2 \end{aligned}$$

$$\leq \left(\frac{2nC_0}{K_0} + |Dc(x_1)| + C_0|c(x_1)| \right) w^2.$$

By (1.4),

$$\frac{1}{n}c(x)^2 - |Dc(x)| - C_0|c(x)| - \frac{2nC_0}{K_0} \geq \delta > 0 \quad \text{for all } x \in \bar{\Omega}$$

for some $\delta > 0$, we see that $w(x_1, t_1) \leq \frac{2M\|c\|_{L^\infty(\bar{\Omega})}}{n\delta}$. Thus,

$$w(x_0, t_0) \leq \rho(x_1)w(x_1, t_1) \leq \left(\frac{C_0K_0}{4} + 1 \right) \frac{2M\|c\|_{L^\infty(\bar{\Omega})}}{n\delta}.$$

Now, we handle the case when $C_0 = 0$. We consider a multiplier

$$\rho(x) = -\frac{\delta_1}{K_0}|x - x_c|^2 + \frac{\delta_1K_0}{4} + 1 \quad \text{for } x \in \bar{\Omega},$$

where

$$\delta_1 = \frac{\delta}{2(\|c\|_{L^\infty} + \frac{2n}{K_0})} > 0.$$

Then, at (x_0, t_0) ,

$$\frac{\partial w}{\partial \bar{\mathbf{n}}} \leq C_0 \frac{|Du|^2}{w} = 0,$$

and

$$\frac{\partial \psi}{\partial \bar{\mathbf{n}}} = \frac{\partial(\rho w)}{\partial \bar{\mathbf{n}}} = \rho \frac{\partial w}{\partial \bar{\mathbf{n}}} + w \frac{\partial \rho}{\partial \bar{\mathbf{n}}} \leq w \frac{\partial \rho}{\partial \bar{\mathbf{n}}} < 0.$$

Following the same argument as above with δ_1 in place of C_0 , we see that

$$\frac{1}{n}(u_t - c(x_1)w)^2 \leq \left(\frac{2n\delta_1}{K_0} + |Dc(x_1)| + C_0|c(x_1)| \right) w^2.$$

This inequality, together with the fact that

$$\frac{1}{n}c(x)^2 - |Dc(x)| - \delta_1|c(x)| - \frac{2n\delta_1}{K_0} \geq \delta - \frac{1}{2}\delta = \frac{1}{2}\delta > 0 \quad \text{for all } x \in \bar{\Omega},$$

implies (3.4).

By (3.4) and Lemma 3.1, Du^ε and u_t^ε are uniformly bounded in $\Omega \times [0, T]$ for all $\varepsilon \in (0, 1)$ and $T > 0$. Note that the bound depends only on $\|u_0\|_{C^2(\bar{\Omega})}$, $\|c\|_{C^1(\bar{\Omega})}$, the constants n , C_0 , K_0 , and δ from (1.4). By approximations, we see that the same result holds true in the case that $\partial\Omega \in C^{2,\theta}$ and $c \in C^1(\bar{\Omega})$. From the uniform convergence of u^ε to the unique viscosity solution u of (1.1)–(1.3), we conclude that u satisfies (1.5). \square

We remark for later usage that for any smooth function $\rho > 0$, (3.8) is valid at $(x_1, t_1) \in \operatorname{argmax}(\rho w) \cap (\Omega \times (0, T])$.

Remark 1. Let us discuss a bit the case where $c \equiv 0$ and Ω is convex and bounded. Then, w satisfies

$$ww_t - w(D_p b^{ij} \cdot Dw)u_{ij} - wb^{ij}w_{ij} + b^{ij}b^{kl}u_{ki}u_{lj} = 0.$$

And, on $\partial\Omega \times [0, T]$,

$$\frac{\partial w}{\partial \vec{\mathbf{n}}} = \frac{D^2 u Du}{w} \cdot \vec{\mathbf{n}} = -\frac{D\vec{\mathbf{n}} Du \cdot Du}{w} \leq 0.$$

By the usual maximum principle, we yield that

$$\max_{\overline{\Omega} \times [0, T]} w = \max_{\overline{\Omega}} w(\cdot, 0) \leq C.$$

We thus recover the gradient bound in [9]. It is worth to note that in this specific situation, condition (1.4) is not needed.

Proof of Theorem 2.3. Let $u = u^\varepsilon$ and $w = \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}$ as in the proof of Theorem 1.2. As above, we may assume $\partial\Omega$ is smooth and $c \in C^\infty(\overline{\Omega})$. Pick

$$M > \frac{2n(|C_0| + 1)}{K_0} + \|Dc\|_{L^\infty(\overline{\Omega})} + (|C_0| + 1)\|c\|_{L^\infty(\overline{\Omega})}$$

and $(x_0, t_0) \in \operatorname{argmax}_{\overline{\Omega} \times [0, T]} e^{-Mt} w(x, t)$. If $t_0 = 0$, then we have that for $(x, t) \in \overline{\Omega} \times [0, T]$,

$$w(x, t) \leq e^{MT} \left(\|Du_0\|_{L^\infty(\overline{\Omega})} + 1 \right).$$

Consider next the case that $t_0 > 0$. If $x_0 \in \Omega$, then by (3.8) with $\rho = e^{-Mt}$, at (x_0, t_0) ,

$$Mw^2 + b^{ij}b^{kl}u_{il}u_{kj} - wDu \cdot Dc \leq 0.$$

As $Mw^2 - wDu \cdot Dc > 0$ by the choice of M and $b^{ij}b^{kl}u_{il}u_{kj} \geq 0$, we arrive at a contradiction. Thus, $x_0 \in \partial\Omega$.

We repeat the proof of Theorem 1.2. Since $x_0 \in \operatorname{argmax}_{\overline{\Omega}} w(\cdot, t_0) \cap \partial\Omega$, we see as before that $C_0 \geq 0$. We use a new multiplier

$$\rho(x, t) = e^{-Mt} \left(-\frac{C_0 + 1}{K_0} |x - x_c|^2 + \frac{(C_0 + 1)K_0}{4} + 1 \right) \quad \text{for } (x, t) \in \overline{\Omega} \times [0, \infty).$$

Here, $B = B(x_c, K_0/2)$ is inside Ω and tangent to the boundary $\partial\Omega$ at x_0 .

Put $w_M = e^{-Mt}w$ and note that $w_M(x_0, t_0) = \max_{\bar{\Omega}} w_M$, $\frac{\partial w_M}{\partial \bar{\mathbf{n}}} \leq C_0 w_M$ on $\partial\Omega \times [0, T]$, and

$$\rho w = \left(-\frac{C_0 + 1}{K_0} |x - x_c|^2 + \frac{(C_0 + 1)K_0}{4} + 1 \right) w_M.$$

Observe as in the proof of Theorem 1.2 that $\frac{\partial(\rho w)}{\partial \bar{\mathbf{n}}}(x_0, t_0) < 0$, $\rho w \leq w_M$ on $(\bar{\Omega} \setminus B) \times [0, T]$, and therefore, $\operatorname{argmax}(\rho w) \subset B \times [0, T]$. Then, there is a point $(x_1, t_1) \in \operatorname{argmax}_{\bar{\Omega} \times [0, T]} \rho w$ with $(x_1, t_1) \in B \times [0, T]$. Consider the case $t_1 = 0$. For all $(x, t) \in \bar{\Omega} \times [0, T]$,

$$\begin{aligned} w_M(x, t) &\leq w_M(x_0, t_0) = (\rho w)(x_0, t_0) \leq (\rho w)(x_1, 0) \\ &\leq \left(\frac{(C_0 + 1)K_0}{4} + 1 \right) \left(\|Du_0\|_{L^\infty(\bar{\Omega})} + 1 \right). \end{aligned}$$

Thus, for $(x, t) \in \bar{\Omega} \times [0, T]$,

$$w(x, t) \leq e^{MT} \left(\frac{(C_0 + 1)K_0}{4} + 1 \right) \left(\|Du_0\|_{L^\infty(\bar{\Omega})} + 1 \right). \quad (3.9)$$

Next, we consider the case $t_1 > 0$. At (x_1, t_1) , thanks to (3.8), we have

$$Mw^2 + \frac{\rho_{ij}}{\rho} b^{ij} w^2 + b^{ij} b^{kl} u_{ki} u_{lj} - w Du \cdot Dc + \frac{cw}{\rho} Du \cdot D\rho \leq 0.$$

From this, recalling the choice of M , we obtain, as before,

$$0 \leq b^{ij} b^{kl} u_{ki} u_{lj} \leq \left(-M + \frac{2n(C_0 + 1)}{K_0} + |Dc(x_1)| + (C_0 + 1)|c(x_1)| \right) w^2 < 0,$$

which is absurd. Thus, the case $t_1 > 0$ does not occur, and (3.9) holds true. Lemma 3.1 and (3.9) then complete the proof. \square

Remark 2. We note that Theorems 1.1 and 1.2 are still valid when $\partial\Omega \in C^2$, $c \in C^1(\bar{\Omega})$, and $u_0 \in C^2(\bar{\Omega})$ by approximations as the Lipschitz bounds depend only on $\|u_0\|_{C^2(\bar{\Omega})}$, $\|c\|_{C^1(\bar{\Omega})}$, the constants n , C_0 , K_0 , and $T > 0$ in case of Theorem 1.1, and δ from (1.4) in case of Theorem 1.2.

4. Large time behavior of the solution

In this section, we prove the large time behavior of u , which is globally Lipschitz continuous thanks to Theorem [1.2](#). Let L be the spatial Lipschitz constant of u^ε for $\varepsilon \in (0, 1)$ given by the proof of Theorem [1.2](#).

Proof of Theorem [1.3](#). Although the proof is almost same as that of [\[12\]](#), Theorem 1.2], we give it for completeness.

We consider the following Lyapunov function

$$I^\varepsilon(t) = \int_{\Omega} \sqrt{\varepsilon^2 + |Du^\varepsilon|^2} dx.$$

By calculation,

$$\frac{d}{dt} \int_{\Omega} \sqrt{\varepsilon^2 + |Du^\varepsilon|^2} dx = \int_{\Omega} \frac{Du^\varepsilon \cdot Du_t^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} dx = - \int_{\Omega} u_t^\varepsilon \operatorname{div} \left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} \right) dx,$$

and thus,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \sqrt{\varepsilon^2 + |Du^\varepsilon|^2} dx &= - \int_{\Omega} u_t^\varepsilon \left(\frac{u_t^\varepsilon}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} - c(x) \right) dx \\ &= - \int_{\Omega} \left(\frac{(u_t^\varepsilon)^2}{\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}} - c(x)u_t^\varepsilon \right) dx \\ &\leq - \frac{1}{\sqrt{\varepsilon^2 + L^2}} \int_{\Omega} (u_t^\varepsilon)^2 dx + \int_{\Omega} c(x)u_t^\varepsilon dx. \end{aligned}$$

Rearranging the terms,

$$\frac{d}{dt} \left(\int_{\Omega} \sqrt{\varepsilon^2 + |Du^\varepsilon|^2} dx - \int_{\Omega} c(x)u^\varepsilon dx \right) \leq - \frac{1}{\sqrt{\varepsilon^2 + L^2}} \int_{\Omega} (u_t^\varepsilon)^2 dx.$$

Integrating the inequality above, we have

$$\begin{aligned} \int_0^T \int_{\Omega} (u_t^\varepsilon)^2 dx dt &\leq \sqrt{\varepsilon^2 + L^2} \int_{\Omega} c(x)(u^\varepsilon(x, T) - u^\varepsilon(x, 0)) dx \\ &\quad + \sqrt{\varepsilon^2 + L^2} \int_{\Omega} \left(\sqrt{\varepsilon^2 + |Du^\varepsilon|^2}(x, 0) - \sqrt{\varepsilon^2 + |Du^\varepsilon|^2}(x, T) \right) dx. \end{aligned}$$

Note that $\|u\|_{L^\infty(\bar{\Omega} \times [0, \infty))} \leq \|u_0\|_{L^\infty(\bar{\Omega})}$. Therefore,

$$\limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} (u_t^\varepsilon)^2 dx dt \leq C,$$

where C is a constant independent of $\varepsilon \in (0, 1)$ and $T > 0$. Hence, we get that $u_t^\varepsilon \rightharpoonup u_t$ weakly in $L^2(\bar{\Omega} \times [0, T])$ as $\varepsilon \rightarrow 0$ for each $T > 0$.

By weakly lower semi-continuity,

$$\int_0^T \int_{\Omega} (u_t)^2 dx dt \leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} (u_t^\varepsilon)^2 dx dt \leq C.$$

Since the constant C is independent of ε , T , we see that

$$\int_0^\infty \int_{\Omega} (u_t)^2 dx dt \leq C. \quad (4.1)$$

For every $\{t_k\} \rightarrow \infty$, by the Arzelà-Ascoli theorem, there exist a subsequence $\{t_{k_j}\}$ and a Lipschitz continuous function v such that

$$u_{k_j}(x, t) = u(x, t + t_{k_j}) \rightarrow v(x, t),$$

locally uniformly on $\bar{\Omega} \times [0, \infty)$. In particular,

$$u_{k_j}(x, t) = u(x, t + t_{k_j}) \rightarrow v(x, t), \quad (4.2)$$

uniformly on $\bar{\Omega} \times [0, T]$, for every $T > 0$. By stability results of viscosity solutions, v satisfies

$$\begin{cases} v_t = |Dv| \operatorname{div} \left(\frac{Dv}{|Dv|} \right) + c|Dv| & \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial \bar{\mathbf{n}}} = 0 & \text{on } \partial\Omega \times [0, \infty). \end{cases}$$

Thanks to [\(4.1\)](#), we have

$$\int_0^1 \int_{\Omega} (u_{k_j})_t^2 dx dt = \int_{t_{k_j}}^{1+t_{k_j}} \int_{\Omega} (u_t)^2 dx dt \rightarrow 0,$$

as $j \rightarrow \infty$. This shows that

$$(u_{k_j})_t \rightarrow 0,$$

weakly in $L^2(\overline{\Omega} \times [0, 1])$ as $j \rightarrow \infty$. On the other hand, (4.2) implies that

$$(u_{k_j})_t \rightharpoonup v_t,$$

weakly in $L^2(\overline{\Omega} \times [0, 1])$ as $j \rightarrow \infty$. Consequently, $v_t = 0$ weakly, and v is constant in t . Thus, v is a solution of (1.6), that is, v solves

$$\begin{cases} |Dv| \operatorname{div} \left(\frac{Dv}{|Dv|} \right) + c(x)|Dv| = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Equation (1.6) has many viscosity solutions in general. For example, as v is a solution, $v + C$ is also a solution for any $C \in \mathbb{R}$. Therefore, v may depend on the choice of subsequence $\{t_k\}_k$.

At last, we prove that v is independent of the choice of subsequence $\{t_k\}_k$. Since u_{k_j} converges uniformly to v on $\overline{\Omega} \times [0, 1]$, for every $\varepsilon > 0$ there exists j large enough such that

$$|u_{k_j}(x, t) - v(x)| < \varepsilon, \quad \text{for all } (x, t) \in \overline{\Omega} \times [0, 1].$$

In particular, $v(x) - \varepsilon < u_{k_j}(x, 0) = u(x, t_{k_j}) < v(x) + \varepsilon$ for all $x \in \overline{\Omega}$. By the comparison principle,

$$v(x) - \varepsilon \leq u(x, t) \leq v(x) + \varepsilon \quad \text{for } (x, t) \in \overline{\Omega} \times [t_{k_j}, \infty).$$

This implies that $u(\cdot, t)$ converges uniformly to v on $\overline{\Omega}$ without taking a subsequence. \square

5. The large time profile in the radially symmetric setting

In this section, we study the radially symmetric setting and illustrate some examples of multiplicity of solutions to the stationary problem (1.6). We always assume here (1.7), that is,

$$\begin{cases} \Omega = B(0, R) \text{ for some } R > 0, \\ c(x) = c(r) \text{ for } |x| = r \in [0, R], \\ u_0(x) = u_0(r) \text{ for } |x| = r \in [0, R]. \end{cases}$$

Here, $c \in C^1([0, R], [0, \infty))$, and $u_0 \in C^2([0, R])$ with $u_0'(R) = 0$ are given. In this setting, (1.6) reduces to the following Hamilton-Jacobi equation with Neumann boundary condition

$$\begin{cases} -\frac{n-1}{r}\phi_r - c(r)|\phi_r| = 0, & \text{in } (0, R), \\ \phi_r(R) = 0. \end{cases} \quad (5.1)$$

It is worth noting that no boundary condition is needed at $r = 0$, and that the Hamiltonian is concave and maybe noncoercive. Clearly, every constant is a solution to (5.1). Also, if ϕ is a solution to (5.1), then so is $C\phi$ for any given constant $C \geq 0$.

We have the following proposition.

Proposition 5.1. *Let $\mathcal{A} = \{r \in (0, R] : c(r) = \frac{n-1}{r}\}$. Denote by*

$$r_{\min} = \begin{cases} \min\{r : r \in \mathcal{A}\} > 0 & \text{if } \mathcal{A} \neq \emptyset, \\ R & \text{if } \mathcal{A} = \emptyset. \end{cases}$$

Let ϕ be a Lipschitz solution to (5.1). Then, ϕ is constant on each connected component of $(0, R) \setminus \text{int}(\mathcal{A})$. In particular, ϕ is constant on $[0, r_{\min}]$.

Proof. Factoring (5.1) into $(-\frac{n-1}{r} \pm c(r))\phi_r(r) = 0$, we see that either $-\frac{n-1}{r} \pm c(r) = 0$ or $\phi_r(r) = 0$ at each point of differentiability of ϕ .

Take $(a, b) \subset ((0, R) \setminus \text{int}(\mathcal{A}))$ for some $a < b$. By the above, we have that $\phi_r(r) = 0$ for a.e. $r \in (a, b)$, and thus, ϕ is constant on $[a, b]$. \square

Example 5.2 (A toy model). We consider the case that $c(r)$ is of the form

$$c(r) = \begin{cases} \frac{n-1}{a}, & 0 \leq r < a, \\ \frac{n-1}{r}, & a \leq r \leq b, \\ \frac{n-1}{b}, & b < r \leq R, \end{cases}$$

for some $0 < a < b < R$, then the stationary problem (5.1) admits multiple solutions of the form

$$\phi(r) = \begin{cases} c_1, & 0 \leq r \leq a, \\ g(r), & a \leq r \leq b, \\ c_2, & b \leq r \leq R, \end{cases}$$

where $c_1 \geq c_2$ are constants, $g(r)$ is any nonincreasing function on $[a, b]$ with $g(a) = c_1$, $g(b) = c_2$. Here, the function g can be discontinuous if we extend

the definition of viscosity solutions to discontinuous functions (see [5] for instance).

Example [5.2] shows further the multiplicity of solutions to (5.1) besides the constant functions noted above. Thus, it is important to address how the large-time limit ϕ_∞ depends on the initial data u_0 . In this radially symmetric setting, we are able to characterize the limiting profile and specify its dependence on the initial data.

Equations (1.1)–(1.3) become

$$\begin{cases} \phi_t - \frac{n-1}{r}\phi_r - c(r)|\phi_r| = 0 & \text{in } (0, R) \times (0, \infty), \\ \phi_r(R, t) = 0 & \text{for } t \geq 0, \\ \phi(r, 0) = u_0(r) & \text{for } r \in [0, R]. \end{cases}$$

Here, $u(x, t) = \phi(|x|, t)$ for $(x, t) \in B(0, R) \times [0, \infty)$. Note that this is a first-order Hamilton-Jacobi equation with a concave Hamiltonian. The associated Lagrangian $L = L(r, q)$ to the Hamiltonian $H(r, p) = -\frac{n-1}{r}p - c(r)|p|$ is

$$\begin{aligned} L(r, q) &= \inf_{p \in \mathbb{R}} \left\{ p \cdot q - \left(-\frac{n-1}{r}p - c(r)|p| \right) \right\} \\ &= \inf_{p \in \mathbb{R}} \left\{ \left(q + \frac{n-1}{r} \right) p + c(r)|p| \right\} \\ &= \begin{cases} 0, & \text{if } |q + \frac{n-1}{r}| \leq c(r), \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, we have the following representation formula for $\phi = \phi(r, t)$

$$\phi(r, t) = \sup \{ u_0(\gamma(0)) : (\gamma, v, l) \in \text{SP}(r, t) \},$$

where we denote by $\text{SP}(r, t)$ the Skorokhod problem. For a given $r \in (0, R]$, $v \in L^\infty([0, t])$, the Skorokhod problem seeks to find a solution $(\gamma, l) \in \text{Lip}((0, t)) \times L^\infty((0, t))$ such that

$$\begin{cases} \gamma(t) = r, & \gamma([0, t]) \subset (0, R], \\ l(s) \geq 0 & \text{for almost every } s > 0, \\ l(s) = 0 & \text{if } \gamma(s) \neq R, \\ \left| -v(s) + \frac{n-1}{\gamma(s)} \right| \leq c(\gamma(s)), \\ v(s) = -\dot{\gamma}(s) + l(s)n(\gamma(s)), \end{cases}$$

and the set $\text{SP}(r, t)$ collects all the associated triples (γ, v, l) . Here, $n(R) = 1$ is the outward normal vector to $(0, R)$ at R . See [16, Theorem 4.2] for the existence of solutions of the Skorokhod problem and [16, Theorem 5.1] for the representation formula. See [7] for a related problem on large time behavior and large time profile.

Example 5.3. Consider Example 5.2. To recall, $c(r)$ is defined in the following way

$$c(r) = \begin{cases} \frac{n-1}{a}, & 0 \leq r < a, \\ \frac{n-1}{r}, & a \leq r \leq b, \\ \frac{n-1}{b}, & b < r \leq R. \end{cases}$$

for some $0 < a < b < R$. We analyze the velocity condition $\left| \dot{\gamma}(s) + \frac{n-1}{\gamma(s)} \right| \leq c(\gamma(s))$. Note that $c(r)$ is less than $\frac{n-1}{r}$, equal to $\frac{n-1}{r}$, and greater than $\frac{n-1}{r}$ in the written order, respectively. In each case, then, the velocity condition becomes

$$\begin{cases} -\frac{n-1}{a} - \frac{n-1}{\gamma(s)} \leq \dot{\gamma}(s) \leq \frac{n-1}{a} - \frac{n-1}{\gamma(s)} < 0, & 0 < \gamma(s) < a, \\ -\frac{2(n-1)}{\gamma(s)} \leq \dot{\gamma}(s) \leq 0, & a \leq \gamma(s) \leq b, \\ -\frac{n-1}{b} - \frac{n-1}{\gamma(s)} \leq \dot{\gamma}(s) \leq \frac{n-1}{b} - \frac{n-1}{\gamma(s)}, & b \leq \gamma(s) < R. \end{cases}$$

Focusing the right hand side in each case, we see that the point $\gamma(s)$ must move left as time s increases, can stay still, and can go right in the written order, respectively. This point of view in terms of the Lagrangian $L(r, q)$ and Proposition 5.1 explain the limit $\phi_\infty(r)$ of $\phi(r, t)$ as $t \rightarrow \infty$ in the above illustration of Figure 1.

The description in Example 5.3 shows how to formulate and write the limit ϕ_∞ in terms of the initial data u_0 in full generality. We note one more thing on the boundary. If $c(h) < \frac{n-1}{h}$ for all $h \in (0, R]$, then the reversed curve $\eta(s) := \gamma(t - s)$ of an admissible curve γ must go right, and it stays on the boundary $r = R$ once it reaches there. This is where the effect of the Skorokhod problem comes in, and it means that the solution $\phi(r, t)$ needs to be understood in the sense of viscosity solutions. We also note that in this setting, we can prove that ϕ is same as the value function of the state constraint problem. Together with this observation on the boundary, analyzing curves $\gamma(s)$ explains how the limit ϕ_∞ depends on the initial data u_0 , and indeed the analysis of admissible curves yields the proof of Theorem 1.4.

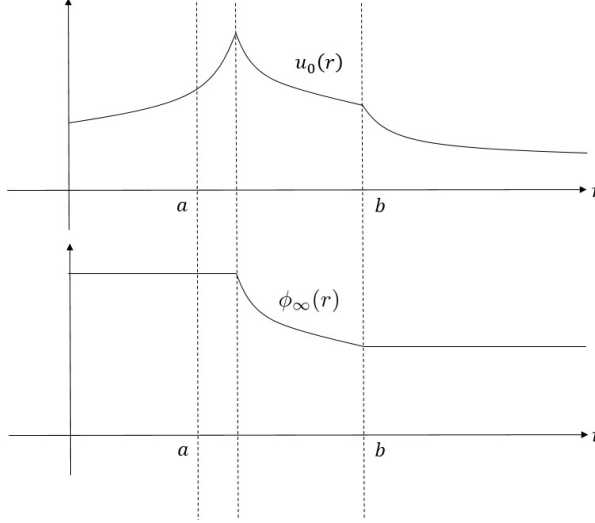


Fig. 1: Stationary solution of (5.1)

We now give some preparation steps in order to prove Theorem 1.4. Let $\eta(s) := \gamma(t - s)$, $s \in [0, t]$, be the reversed curve of a curve $\gamma \in AC([0, t], (0, R])$ with $(\gamma, v, l) \in SP(r, t)$. Then, we have the following velocity condition for η

$$-c(\eta(s)) + \frac{n-1}{\eta(s)} \leq \dot{\eta}(s) \leq c(\eta(s)) + \frac{n-1}{\eta(s)} \quad \text{for a.e. } s \in [0, t] \text{ with } \eta(s) \neq R. \quad (5.2)$$

The following lemma is a direct consequence of the comparison principle.

Lemma 5.4. *Let $r_0 \in (0, R)$. Let $\eta_1 \in AC([0, \infty), (0, R])$ be a curve satisfying*

$$\begin{cases} \dot{\eta}_1(s) = -c(\eta_1(s)) + \frac{n-1}{\eta_1(s)}, & \text{for } s > 0 \text{ provided that } \eta_1(s) < R, \\ \eta_1(0) = r_0. \end{cases}$$

If $\eta_1(s_0) = R$ for some $s_0 > 0$, then we set $\eta_1(s) = R$ for all $s \geq s_0$.

For each $t > 0$, let $\eta \in AC([0, t], (0, R])$ be the reversed curve given above with $\eta(0) \geq r_0$. Then, $\eta_1(s) \leq \eta(s)$ for all $s \in [0, t]$.

Lemma 5.5. *Assume the settings of Theorem 1.4 and Lemma 5.4. Then,*

$$\lim_{s \rightarrow \infty} \eta_1(s) = d(r_0). \quad (5.3)$$

Proof. If $r_0 \in \mathcal{A}$, then $\eta_1(s) = r_0$ for all $s \geq 0$, and hence (5.3) holds.

Next, we only need to consider the case that $r_0 \in \mathcal{A}_+$ as the proof of the case that $r_0 \in \mathcal{A}_-$ follows analogously. It is clear that η_1 is decreasing, and by Lemma 5.4, $\eta_1(s) \geq d(r_0)$ for all $s \geq 0$. Therefore, $\lim_{s \rightarrow \infty} \eta_1(s)$ exists, and

$$\lim_{s \rightarrow \infty} \eta_1(s) = r_1 \geq d(r_0).$$

This yields further that

$$\limsup_{s \rightarrow \infty} \dot{\eta}_1(s) = 0.$$

Hence,

$$-c(r_1) + \frac{n-1}{r_1} = 0,$$

which implies that $r_1 = d(r_0)$. □

Proof of Theorem 1.4. For $(r_0, t) \in (0, R) \times [0, \infty)$, we have

$$\phi(r_0, t) = \sup\{u_0(\eta(t)) : (\gamma, v, l) \in \text{SP}(r_0, t), \eta(s) = \gamma(t-s), s \in [0, t]\}.$$

We say that $\eta \in \text{AC}([0, t], (0, R])$ is admissible if $\eta(s) = \gamma(t-s)$, $s \in [0, t]$ for some γ with $(\gamma, v, l) \in \text{SP}(r_0, t)$. Let η_1 be the curve given in the statement of Lemma 5.4. By Lemma 5.4, $\eta(s) \geq \eta_1(s)$ for $s \in [0, t]$ for any admissible curve η . From this fact, we see that

$$\phi(r_0, t) \leq \sup\{u_0(r) : r \geq \eta_1(t)\},$$

and therefore, by Lemma 5.5,

$$\limsup_{t \rightarrow \infty} \phi(r_0, t) \leq \max\{u_0(r) : r \geq d(r_0)\}.$$

In order to complete the proof, it suffices to show the other direction

$$\liminf_{t \rightarrow \infty} \phi(r_0, t) \geq \max\{u_0(r) : r \geq d(r_0)\}. \quad (5.4)$$

To show this, let $r_1 \in [d(r_0), R]$ be such that

$$u_0(r_1) = \max\{u_0(r) : r \geq d(r_0)\}.$$

We consider first the case $r_0 \in \mathcal{A}$. Then, $r_1 \geq r_0$. Let η_2 solve

$$\begin{cases} \dot{\eta}_2(s) = c(\eta_2(s)) + \frac{n-1}{\eta_2(s)}, & \text{for } s > 0, \\ \eta_2(0) = r_0. \end{cases}$$

Note that $c(r) + (n-1)/r \geq (n-1)/R > 0$ for all $r \in (0, R]$. Then, there is a unique number $t_2 \geq 0$ such that $\eta_2(t_2) = r_1$. Now, for $t \geq t_2$, let η be defined as

$$\eta(s) = \begin{cases} r_0, & \text{if } s \leq t - t_2, \\ \eta_2(s - (t - t_2)), & \text{if } s \geq t - t_2. \end{cases}$$

Then, η is admissible, and $\phi(r_0, t) \geq u_0(\eta(t)) = u_0(r_1)$. Thus, (5.4) holds.

Next, we consider the case $r_0 \in \mathcal{A}_+$. If $r_1 \geq r_0$, then we repeat the above process to conclude. If $r_1 < r_0$, then $r_1 \in [d(r_0), r_0)$ necessarily, and in this case, we use the curve η_1 . We note that if $r_1 > d(r_0)$, then there is a unique number $t_1 \geq 0$ such that $\eta_1(t_1) = r_1$. Now, for $t \geq t_1$, let η be defined as

$$\eta(s) = \begin{cases} r_0, & \text{if } s \leq t - t_1, \\ \eta_1(s - (t - t_1)), & \text{if } s \geq t - t_1. \end{cases}$$

Then, the curve η is admissible, and $\phi(r_0, t) \geq u_0(\eta(t)) = u_0(r_1)$. If $r_1 = d(r_0)$, we take $\eta = \eta_1$ and recall that $\lim_{t \rightarrow \infty} \eta_1(t) = d(r_0)$, which gives $\phi(r_0, t) \geq u_0(\eta(t)) \rightarrow u_0(r_1)$ as $t \rightarrow \infty$. Therefore, (5.4) holds.

Finally, we study the case $r_0 \in \mathcal{A}_-$. Let η_2, t_2 be defined as above. There exists a unique $t_3 > 0$ such that $\eta_2(t_3) = d(r_0)$. In this case, $r_1 \geq d(r_0)$ and $t_2 \geq t_3$. For $t \geq t_2$, define

$$\eta(s) = \begin{cases} \eta_2(s), & \text{if } 0 \leq s \leq t_3, \\ d(r_0), & \text{if } t_3 \leq s \leq t - (t_2 - t_3), \\ \eta_2(s - (t - t_2)), & \text{if } t - (t_2 - t_3) \leq s \leq t. \end{cases}$$

Then, η is admissible, and $\eta(t) = r_1$, which yields (5.4). \square

Next, we prove Corollary 1.5, and discuss the sharpness of condition (1.4).

Proof of Corollary 1.5. The values of ϕ_∞ are computed directly from Theorem 1.4. This tells us the fact that the solution $u = u(r, t)$ is not globally Lipschitz because if it were globally Lipschitz, then the limit ϕ_∞ would be as well. \square

Corollary [1.5](#) realizes a jump discontinuity in the limit, which indicates that condition [\(1.4\)](#), which is needed for the globally Lipschitz continuity of u , is almost optimal. As the domain $\Omega = B(0, R)$ is convex, $C_0 \leq 0$, and [\(1.4\)](#) becomes $\frac{1}{n}c(x)^2 - |Dc(x)| - \delta > 0$. Let us now assume that $c(r)$ touches $\frac{n-1}{r}$ from below at a . Then,

$$c(a) = \frac{n-1}{a} \quad \text{and} \quad c'(a) = -\frac{n-1}{a^2}.$$

At $r = a$, we see that

$$\frac{1}{n}c(a)^2 - |c'(a)| = \frac{(n-1)^2}{na^2} - \frac{n-1}{a^2} = -\frac{n-1}{na^2} < 0.$$

Moreover, we see that condition [\(1.4\)](#) is essentially optimal if we seek to find sufficient conditions on the force c that are uniform in dimensions n and in R because the left hand side of the above goes to zero as $a \rightarrow \infty$.

6. The gradient growth as time tends to infinity in two dimensions

Let $n = 2$. Let the forcing term c be a positive constant in Ω , that is, $c(x) = c$ for all $x \in \bar{\Omega}$ for some $c > 0$. Consider the following nonconvex domain,

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < f(x_1)\}, \quad (6.1)$$

where $f(x) = \frac{m}{2}x^2 + k$ for fixed $m > 0$ and $k > 0$. Here, Ω is unbounded.

In this unbounded setting, let $R_0 > 0$ be a sufficiently large constant. Let $\tilde{\Omega} \subset \mathbb{R}^n$ be a bounded $C^{2,\theta}$ domain such that

$$\Omega \cap B(0, R_0) \subset \tilde{\Omega} \subset \Omega.$$

We say that u is a solution (resp., subsolution, supersolution) of [\(1.1\)](#)–[\(1.3\)](#) on $\bar{\Omega} \times [0, \infty)$ if there exists $\alpha \in \mathbb{R}$ such that

$$u - \alpha = u_0 - \alpha = 0 \quad \text{on } (\bar{\Omega} \setminus B(0, R_0)) \times [0, \infty), \quad (6.2)$$

and u is a solution (resp., subsolution, supersolution) of [\(1.1\)](#)–[\(1.3\)](#) with $\tilde{\Omega}$ in place of Ω .

Let u be the solution to [\(1.1\)](#)–[\(1.3\)](#). If a level set of u is a smooth curve, then it is evolved by the forced curvature flow equation $V = \kappa + c$, where V is the normal velocity and κ is the curvature in the direction of the normal.

Then, the classical Neumann boundary condition becomes the right angle condition for the level-set curves with respect to $\partial\Omega$, that is, if a smooth level curve and $\partial\Omega$ intersect, then their normal vectors are perpendicular at the points of intersections.

We show that if c is too small and fails to satisfy (1.4), then there exist discontinuous viscosity solutions to (1.6). In particular, we find that one such discontinuous solution of (1.6) is stable in the sense that the solution of (1.1)–(1.3) with a suitable choice of initial data converges to this discontinuous stationary solution as time goes to infinity. This implies that the global Lipschitz estimate for the solution of (1.1)–(1.3) does not hold. The following is the main result of this section.

Theorem 6.1. *Let Ω be the set given by (6.1), and $c(x) = c$ for all $x \in \bar{\Omega}$ for $c \in (0, r_{\min}^{-1})$, where r_{\min} is defined by (6.7). Let $u \in C(\bar{\Omega} \times [0, \infty))$ be the solution of (1.1)–(1.3) with the given initial data $u_0 \in C^{2,\theta}(\bar{\Omega})$ satisfying that $\frac{\partial u_0}{\partial \bar{\mathbf{n}}} = 0$ on $\partial\Omega$ and there exist constants l_1, l_2, α and β such that $l_1 \in (0, a_1)$, $l_2 \in (0, a_2 - a_1)$, $\alpha < \beta$,*

$$u_0(x) = \begin{cases} \beta & \text{for } x = (x_1, x_2) \in U(a_1 - l_1), \\ \alpha & \text{for } x = (x_1, x_2) \in \bar{\Omega} \setminus \overline{U(a_1 + l_2)}, \end{cases} \quad (6.3)$$

and $\alpha \leq u_0 \leq \beta$, where $U(a)$ is defined by (6.6) for $a > 0$, and $0 < a_1 < a_2$ is given in Theorem 6.4. Then,

$$\lim_{t \rightarrow \infty} u(x, t) = \begin{cases} \beta & \text{if } x \in U(a_1), \\ \alpha & \text{if } x \in \bar{\Omega} \setminus \overline{U(a_1)}. \end{cases}$$

6.1. Set-theoretic stationary solutions

For $a > 0$, consider a family of curves with constant curvature in Ω ,

$$X(a, \theta) = (X_1(a, \theta), X_2(a, \theta)) = p(a) + r(a)(\cos \theta, \sin \theta), \quad |\theta| < \arctan(ma), \quad (6.4)$$

where we choose $p(a)$, $r(a)$ so that the curve

$$\Gamma := \{(X_1(a, \theta), X_2(a, \theta)) : |\theta| < \arctan(ma)\} \\ \cup \{(-X_1(a, \theta), X_2(a, \theta)) : |\theta| < \arctan(ma)\}$$

has a constant curvature, and is perpendicular to the boundary $\partial\Omega$. Indeed, set

$$p(a) := \left(\frac{a}{2} - \frac{k}{ma}, 0 \right).$$

Then, we see that the tangent line for $\{(x_1, x_2) \mid x_2 = f(x_1)\}$ at $x_1 = a$ goes through $p(a)$. Moreover, setting

$$r(a) := \left| \left(a, \frac{ma^2}{2} + k \right) - p(a) \right| = \left(\frac{a}{2} + \frac{k}{ma} \right) \sqrt{m^2 a^2 + 1},$$

by elementary geometry, we can check that

$$\Gamma \perp \partial\Omega.$$

See Figure 2. The parameter a will be specified so that

$$c = \frac{1}{r(a)}$$

in Lemma 6.3.

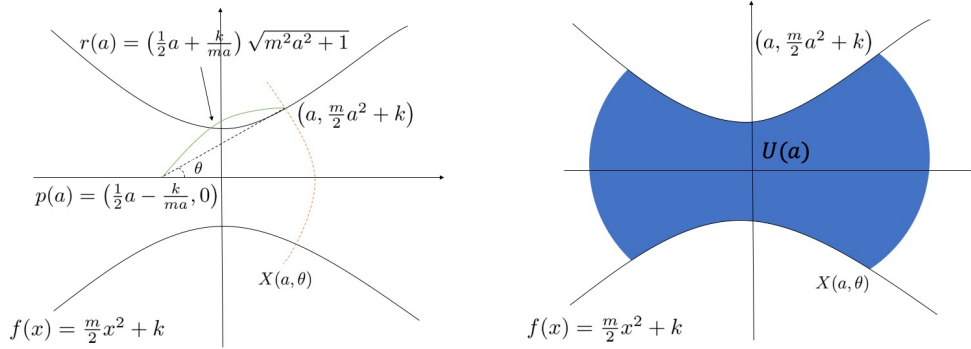


Fig. 2: Illustrations of (6.4) and (6.6)

The following definition is taken from [5, Definition 5.1.1].

Definition 6.2. *Let G be a set in $\mathbb{R}^n \times J$, where J is an open interval in $(0, T)$. We say that G is a set-theoretic subsolution (resp., supersolution) of*

$$V = \kappa + c \quad \text{on } \Gamma_t \quad \text{with } \Gamma_t \perp \partial\Omega \quad (6.5)$$

if χ_G^* is a viscosity subsolution (resp., $(\chi_G)_*$ is a viscosity supersolution) of (1.1)–(1.2) in $\mathbb{R}^n \times J$, where $\chi_G(x, t) = 1$ if $(x, t) \in G$, and $\chi_G(x, t) = 0$ if $(x, t) \notin G$, and χ_G^* and $(\chi_G)_*$ denote the upper semicontinuous envelope and the lower semicontinuous envelope of χ_G , respectively. If G is both a set-theoretic subsolution and supersolution of (6.5), G is called a set-theoretic solution of (6.5).

Set

$$U(a) := \{(x_1, x_2) \in \Omega : |x_1| < X_1(a, \theta), |x_2| < X_2(a, \theta), |\theta| < \arctan(ma)\}, \quad (6.6)$$

and

$$r_{\min} := \inf\{r(a) : a > 0\}. \quad (6.7)$$

Then, r_{\min} is positive since r is a continuous positive function in $(0, \infty)$ and

$$\lim_{a \rightarrow 0} r(a) = \lim_{a \rightarrow \infty} r(a) = \infty. \quad (6.8)$$

Moreover, by direct computation, we have

$$r'(a) = \frac{1}{\sqrt{m^2 a^2 + 1}} \left(m^2 a^2 + \frac{1}{2} - \frac{k}{ma^2} \right).$$

Therefore, r has only one critical point $a_* = \frac{1}{2m} \sqrt{-1 + \sqrt{1 + 16mk}}$ in $(0, \infty)$ and $r_{\min} = r(a_*)$. In addition,

$$r'(a) < 0 \text{ if } a < a_*, \text{ and } r'(a) > 0 \text{ if } a > a_*. \quad (6.9)$$

Lemma 6.3. *If $c = \frac{1}{r(a)}$ for some $a > 0$, then $U(a)$ is a set-theoretic stationary solution of (1.1)–(1.2).*

Proof. As a consequence of the nice characterization of set-theoretic solutions in [5, Theorem 5.1.2], $U(a)$ is a set-theoretic stationary solution of (6.5) if and only if $0 = \kappa + c$ on $\partial U(a) \cap \Omega$ and the right angle condition holds. The equality follows from the fact that $\partial U(a) \cap \Omega$ contains two arcs of two circles of the same radius $r(a)$ and curvature $\kappa = -r(a)^{-1} = -c$.

On the other hand, these arcs intersect with $\partial \Omega$ at four points $(a, \pm f(a))$, $(-a, \pm f(a))$. By symmetry, it suffices to prove the right angle condition at $(a, f(a))$. Notice that

$$(a, f(a)) = (X_1(a, \arctan(ma)), X_2(a, \arctan(ma)))$$

$$= p(a) + \frac{r(a)}{\sqrt{m^2 a^2 + 1}} \cdot (1, ma).$$

Therefore, the line joining $(a, f(a))$ and $p(a)$, the center of the arc, is tangent to $\partial\Omega$ at $(a, f(a))$. Thus, $\partial U(a) \cap \Omega$ satisfies the right angle condition at $(a, f(a))$. \square

Theorem 6.4. *If $c \in (0, \frac{1}{r_{\min}})$, then there exist two positive constants $a_1 < a_2$ such that $U(a_i)$ is a set-theoretic stationary solution of (6.5) for $i = 1, 2$.*

Proof. Thanks to (6.7)–(6.9), there exist two positive constants a_1, a_2 with $a_1 < a_* < a_2$ such that

$$r(a_1) = r(a_2) = \frac{1}{c}. \quad (6.10)$$

By Lemma 6.3, $U(a_i)$ is a set-theoretic stationary solution of (6.5) for $i = 1, 2$. \square

6.2. Stability

Let a_i be the constants given by Theorem 6.4 for $i = 1, 2$. In this section, we prove that $U(a_1)$ given by (6.6) is a set-theoretic solution which is stable in the sense of Theorem 6.1.

Lemma 6.5. *Let $l_1 \in (0, a_1)$, $l_2 \in (0, a_2 - a_1)$ and $\delta > 0$. Set $\underline{a}(t) := a_1 - l_1 e^{-\delta t}$ and $\bar{a}(t) := a_1 + l_2 e^{-\delta t}$. There exists $\delta_0 = \delta_0(m, k, l_1, l_2)$ such that $U(\underline{a}(t))$ and $U(\bar{a}(t))$ are a set-theoretic subsolution and supersolution to (6.5) for all $\delta \in (0, \delta_0)$, respectively.*

Proof. We only prove that $U(\underline{a}(t))$ is a set-theoretic subsolution, since we can similarly prove that $U(\bar{a}(t))$ is a set-theoretic supersolution. Let $\tilde{X}(t) := X(\underline{a}(t), \theta)$. From the characterization of set-theoretic solutions in [5, Theorem 5.1.2], it suffices to show that for $t \geq 0$,

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} \leq -\frac{1}{r(\underline{a}(t))} + c \quad \text{for all } t > 0, \quad (6.11)$$

where $\vec{\mathbf{n}}$ is the outward normal vector $\vec{\mathbf{n}}$ of $U(\underline{a}(t))$, that is, $\vec{\mathbf{n}} = (\cos \theta, \sin \theta)$.

Note that

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} = \frac{\partial \underline{a}}{\partial t} \frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} = \delta l_1 e^{-\delta t} \frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} = \delta (a_1 - \underline{a}(t)) \frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}}.$$

Also, for any constant $L > 0$, there exists $C = C(m, k, L) > 0$ such that

$$\begin{aligned} \frac{\partial X}{\partial a}(a, \theta) \cdot \vec{\mathbf{n}} &= p'(a) \cdot \vec{\mathbf{n}} + r'(a) \leq |p'(a)| + r'(a) \\ &= \frac{1}{2} + \frac{m^2 a^2 + \frac{1}{2}}{\sqrt{m^2 a^2 + 1}} + \frac{mk}{m^2 a^2 + 1 + \sqrt{m^2 a^2 + 1}} \leq C \end{aligned}$$

for all $a \in (0, L)$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore,

$$\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} = \delta(a_1 - \underline{a}(t)) \frac{\partial X}{\partial a} \cdot \vec{\mathbf{n}} \leq C \delta(a_1 - \underline{a}(t)).$$

The observation (6.9) implies that $r(\underline{a}(t)) > r(a_1) = c^{-1}$ for all $t \geq 0$, and thus we get

$$\left(\frac{d\tilde{X}}{dt} \cdot \vec{\mathbf{n}} \right) \left(-\frac{1}{r(\underline{a}(t))} + c \right)^{-1} \leq \delta C \frac{a_1 - \underline{a}(t)}{\frac{1}{r(a_1)} - \frac{1}{r(\underline{a}(t))}}.$$

Thus, (6.11) holds for $\delta \in (0, \delta_0)$, where

$$\delta_0 := \left(C \sup_{a \in [a_1 - l_1, a_1 + l_2]} h(a) \right)^{-1}.$$

Here the function $h : [a_1 - l_1, a_1 + l_2] \rightarrow \mathbb{R}$ is given by

$$h(a) := \begin{cases} \frac{a_1 - a}{\frac{1}{r(a_1)} - \frac{1}{r(a)}} & \text{for } a \in [a_1 - l_1, a_1 + l_2] \setminus \{a_1\}, \\ \frac{-r^2(a_1)}{r'(a_1)} & \text{for } a = a_1. \end{cases}$$

Since $a_1 + l_2 < a_2$, by (6.9) we have $r(a) \neq r(a_1)$ in $[a_1 - l_1, a_1 + l_2] \setminus \{a_1\}$ and $r'(a_1) < 0$. Therefore, h is well-defined and continuous in $[a_1 - l_1, a_1 + l_2]$. Thus, h is bounded in $[a_1 - l_1, a_1 + l_2]$, and hence, $\delta_0 > 0$ is well-defined, which implies that (6.11) holds for all $\delta \in (0, \delta_0)$. \square

Proof of Theorem 6.1. We let $\alpha = 0$ and $\beta = 1$ for simplicity. Set

$$\underline{u}(x, t) := \chi_{\overline{U(\underline{a}(t))}}(x) \quad \text{and} \quad \bar{u}(x, t) := \chi_{U(\bar{a}(t))}(x)$$

for $(x, t) \in \bar{\Omega} \times [0, \infty)$, where \underline{a} and \bar{a} are the functions defined in Lemma 6.5. By Lemma 6.5, we see that \underline{u} and \bar{u} are a subsolution and a supersolution of (1.1)–(1.2), respectively. Due to (6.3), we get

$$\underline{u}(\cdot, 0) = \chi_{\overline{U(\underline{a}(0))}} \leq u_0 \leq \chi_{U(\bar{a}(0))} = \bar{u}(\cdot, 0) \text{ on } \bar{\Omega}.$$

In addition, since

$$U(a) \subset V(a) := [-(|p(a)| + r(a)), |p(a)| + r(a)] \times [-f(a), f(a)]$$

by construction for $p(a)$ and $r(a)$ given in (6.4) and $f(a) = \frac{m}{2}a^2 + k$, we obtain

$$\text{supp}(\underline{u}) \subset \bigcup_{a \in [a_1 - l_1, a_1]} V(a) \times [0, \infty) \text{ and } \text{supp}(\bar{u}) \subset \bigcup_{a \in [a_1, a_1 + l_2]} V(a) \times [0, \infty).$$

As $|p(\cdot)| + r(\cdot)$ and f are continuous on $[a_1 - l_1, a_1 + l_2]$, there exists a constant $R_0 > 0$ satisfying (6.2).

By the comparison principle for (1.1)–(1.3), Proposition 2.2, we get

$$\underline{u}(\cdot, t) \leq u(\cdot, t) \leq \bar{u}(\cdot, t) \text{ on } \bar{\Omega} \text{ for all } t > 0.$$

On the other hand, since both $a_1 - l_1 e^{-\delta t}$ and $a_1 + l_2 e^{-\delta t}$ converge to a_1 as t goes to infinity,

$$\lim_{t \rightarrow \infty} \underline{u}(x, t) = \lim_{t \rightarrow \infty} \bar{u}(x, t) = 1 \text{ for } x \in U(a_1),$$

and

$$\lim_{t \rightarrow \infty} \underline{u}(x, t) = \lim_{t \rightarrow \infty} \bar{u}(x, t) = 0 \text{ for } x \in \bar{\Omega} \setminus \overline{U(a_1)},$$

which finish the proof. \square

Acknowledgements

The authors are extremely grateful to the anonymous referee for his/her careful reading and very constructive comments, which help much to improve the presentation of the paper.

The work of JJ was partially supported by NSF CAREER grant DMS-1843320. The work of HM was partially supported by the JSPS grants: KAKENHI #19H00639, #20H01816, #21H04431, #22K03382. The work of HT was partially supported by NSF CAREER grant DMS-1843320 and a Simons Fellowship.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest

There is no conflict of interest.

References

- [1] S. Aimi. Level set mean curvature flow with Neumann boundary conditions. arXiv preprint arXiv:2103.16386. *Hokkaido Mathematical Journal*, accepted.
- [2] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.* 33 (1991), no. 3, 749–786.
- [3] A. Cesaroni, and M. Novaga. Long-time behavior of the mean curvature flow with periodic forcing. *Comm. Partial Differential Equations*, 38 (2013), no. 5, 780–801.
- [4] L. C. Evans, and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.* , 33 (1991), no. 3, 635–681.
- [5] Y. Giga. *Surface evolution equations-A level set approach*, Monographs in Mathematics, Birkhäuser, 2006.
- [6] Y. Giga, H. Mitake, T. Ohtsuka, and H. V. Tran. Existence of asymptotic speed of solutions to birth and spread type nonlinear partial differential equations. *Indiana Univ. Math. J.*, 70 (2021), no. 1, 121–156.
- [7] Y. Giga, H. Mitake, and H. V. Tran. Remarks on large time behavior of level-set mean curvature flow equations with driving and source terms. *Discrete Contin. Dyn. Syst. Ser. B*, 25 (2020), no. 10, 3983–3999.
- [8] Y. Giga, H. Mitake, and H. V. Tran. On asymptotic speed of solutions to level-set mean curvature flow equations with driving and source terms. *SIAM J. Math. Anal.*, 48 (2016), no. 5, 3515–3546.

- [9] Y. Giga, M. Ohnuma, and M.-H. Sato. On the strong maximum principle and the large time behavior of generalized mean curvature flow with the Neumann boundary condition. *J. Differential Equations*, 154 (1999), no. 1, 107–131.
- [10] Y. Giga, and M.-H. Sato. Generalized interface evolution with the Neumann boundary condition. *Proc. Japan Acad. Ser. A Math. Sci.*, 67 (1991), no. 8, 263–266.
- [11] Y. Giga, and M.-H. Sato. Neumann problem for singular degenerate parabolic equations. *Differential and Integral Equations*, 6 (1993), no. 6, 1217–1230.
- [12] Y. Giga, H. V. Tran, and L. Zhang. On obstacle problem for mean curvature flow with driving force. *Geom. Flows*, 4 (2019), no. 1, 9–29.
- [13] B. Guan. Mean curvature motion of nonparametric hypersurfaces with contact angle condition. *Elliptic and parabolic methods in geometry*, 47–56, 1996.
- [14] N. Hamamuki, and K. Misu. Asymptotic shape of solutions to the mean curvature flow equation with discontinuous source terms, *in preparation*.
- [15] G. Huisken. Nonparametric mean curvature evolution with boundary conditions. *J. Differential Equations* 77 (1989), no. 2, 369–378.
- [16] H. Ishii. Weak KAM aspects of convex Hamilton-Jacobi equations with Neumann type boundary conditions. *J. Math. Pures Appl.* (9) 95 (2011), no. 1, 99–135.
- [17] I. Kim, and D. Kwon. Volume preserving mean curvature flow for star-shaped sets. *Calc. Var. Partial Differential Equations*, 59 (2020), no. 2, Paper No. 81, 40 pp.
- [18] I. Kim, D. Kwon, and N. Požár. On volume-preserving crystalline mean curvature flow. *Mathematische Annalen*, 384, 1–42 (2022).
- [19] P.-L. Lions, P. E. Souganidis. Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22 (2005), no. 5, 667–677.

- [20] X.-N. Ma, P.-H. Wang, W. Wei. Constant mean curvature surfaces and mean curvature flow with non-zero Neumann boundary conditions on strictly convex domains. *J. Funct. Anal.* 274 (2018), no. 1, 252–277.
- [21] H. Matano, K.-I. Nakamura, B. Lou. Periodic traveling waves in a two-dimensional cylinder with saw-toothed boundary and their homogenization limit, *Netw. Heterog. Media* 1 (2006), no. 4, 537–568.
- [22] M. Mizuno, and K. Takasao. Gradient estimates for mean curvature flow with Neumann boundary conditions. *NoDEA Nonlinear Differential Equations Appl.* , 24 (2017), no. 4, Paper No. 32, 24 pp.
- [23] T. Ohtsuka. Discontinuous stationary solution to generalized eikonal-curvature equation and its stability. *Comment. Math. Univ. St. Pauli* 63 (2014), no. 1-2, 233–260.
- [24] J. Xu. Mean curvature flow of graphs with Neumann boundary conditions. *Manuscripta Math.* 158 (2019), no. 1-2, 75–84.