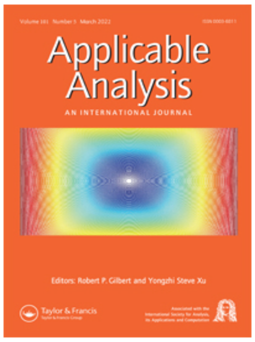

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The SMGT equation from the boundary: regularity and stabilization

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ABSTRACT
We consider the third-order (in time) linear equation known as SMGT-equation, as defined on a multidimensional bounded domain. Part A gives optimal interior and boundary regularity results from $L^2(0,T;L^2(\Gamma))$ – Dirichlet or Neumann boundary terms. Explicit representation formulas are given that can be taken to define the notion of solution in the canonical case ($\gamma = 0$), while the same regularity results hold for $\gamma \in L^\infty(\Omega)$. Part B considers the SMGT equation under Neumann dissipative boundary conditions and critical parameter $\gamma \in L^\infty(\Omega)$ and $\gamma(x) \geqslant 0$ a.e. in Ω. We provide two results: (i) uniform stabilization under minimal checkable geometric conditions, and (ii) strong stabilization in the absence of geometrical conditions.

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1. Introduction: the SMGTJ equation

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\Gamma = \partial\Omega$, as specified below. In this paper, we return to the equation, which should be called SMGTJ [for G. G. Stokes (1851), F. K. Moore & W. E. Gibson (1960), P. A. Thompson (1972) and P. M. Jordan (2004)], see [1–5]. As widely documented in the literature, it arises in a variety of physical contexts such as: effects of the radiation of heat on the propagation of sound; propagation of disturbances in a gas subject to relaxation effects; behavior of viscoelastic materials; propagation of acoustic waves, etc. In particular, if in classical models in nonlinear acoustics (Kuznetsov equation, Westervelt equation, Kokhlov–Zobolotskaya–Kuznetsov equation), one replaces the Fourier Law for the heat flux with the Maxwell–Cattaneo Law (to avoid the paradox of infinite speed of propagation), one obtains a third order in time PDE, whose linear part is the one considered in the present paper; that is [1,2]

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - b\Delta\psi_t = 0 \quad \text{in } (0,T] \times \Omega, \tag{1.1}$$

where, in the physical models of the literature, $\tau > 0, b > 0, c^2 > 0$ are fixed constants, whose physical meaning is not relevant here. See [6]. Hencefort we shall take $\tau = 1$ in (1.1) w.l.o.g. We are taking Ω in $\mathbb{R}^3$, as this is the physically significant setting. However, the mathematical analysis of this paper works on any $\mathbb{R}^d, d = 1,2,\dots$ (Dirichlet); $d = 2,3,\dots$ (Neumann).

The present paper is divided in two parts. Part A deals with optimal interior and boundary (trace) regularity of the mixed problem consisting of the third-order Equation (1.1), subject to Dirichlet or Neumann boundary control g of low regularity $L^2(0,T;L^2(\Gamma))$ (and zero I.C). Here we revisit the proofs of [7], which studied this problem. More on the literature below. With reference to problem

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(2.1a)–(2.1c), the interior and boundary regularity results of Part A hold true actually with coefficient γ in (2.1d) below satisfying $\gamma \in L^\infty(\Omega)$; the physically most interesting cases being $\alpha \in L^\infty(\Omega)$ and b, c^2 positive constants. In fact, in our approach, the coefficient γ is responsible for lower-order terms so that w.l.o.g we give explicit proofs in the text for the case $\gamma = 0$, which lead to explicit representation formulas. These may be taken as defining the notion of solution. The analysis for $\gamma \in L^\infty(\Omega)$ is reported in Appendix 1.

Part B considers the problem of uniform stabilization of the third-order equation under Neumann feedback control and critical parameter $\gamma \in L^\infty(\Omega), \gamma(x) \geqslant 0$ a.e. in Ω. Two results are given: (i) uniform stabilization under minimal checkable geometric conditions imposed only on the uncontrolled part of the boundary; and strong stabilization in the absence of geometric conditions. The uniform stabilization result for the SMGT feedback problem encounters several conceptual and technical challenges.

PART A.1: OPTIMAL INTERIOR AND BOUNDARY REGULARITY OF THE MIXED PROBLEM WITH DIRICHLET BOUNDARY TERM IN $L^2(0, T; L^2(\Gamma))$

2. Linear third-order SMGTJ-equation with non-homogeneous Dirichlet boundary term in $L^2(0, T; L^2(\Gamma))$

If the linear third-order Equation (1.1) is written in terms of the pressure, then Dirichlet non-homogeneous boundary terms are appropriate (I. Christov, private communication; P. Jordan, private communication). We then consider the following mixed problem in the unknown $y(t, x)$:

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2 \Delta y - b\Delta y_t = f \quad \text{in } Q = (0, T] \times \Omega & (2.1a) \\ y\big|_{t=0} = y_0; \quad y_t\big|_{t=0} = y_1; \quad y_{tt}\big|_{t=0} = y_2 \text{ in } \Omega & (2.1b) \\ y\big|_\Sigma = g \quad \text{in } \Sigma = (0, T] \times \Gamma & (2.1c) \end{cases}$$

The following quantity, introduced in [6,8] plays a critical role in stability

$$\gamma = \alpha - \frac{c^2}{b}, \tag{2.1d}$$

see more details following (2.10). By the principle of superposition, we shall consider separately two cases: $\{y_0, y_1, y_2\} \neq 0$ and $g = 0$ in Section 2.1, and $\{y_0, y_1, y_2\} = 0$ and $g \neq 0$ in Section 2.2, the key of Part A.1.

2.1. Case $g \equiv 0$.

A rather comprehensive study of this case was carried out in [6] in the constant coefficient case via semigroup/functional analytic techniques, later extended to variable $\alpha(x)$ [9, Theorem 2.1, p. 833]; and in [8,10] in the case of likewise variable $\alpha \in L^\infty(\Omega)$, via energy methods. Here we shall only report a subset of these results which are relevant to the present paper. In addition, we note explicitly s.c. group generation also in the space V_3 in (2.5d), which is obtained from U_3. The consequent regularity in $C([0, T]; V_3)$ for $\{y_0, y_1, y_2\} \in V_3$ is then consistent with the analysis of the problem for $\{y_0, y_1, y_2\} = 0, g \in L^2(0, T; L^2(\Gamma))$. Define the positive self adjoint operator on $H = L^2(\Omega)$:

$$\begin{aligned} A_0 h = -\Delta h, \quad \mathcal{D}(A_0) = H^2(\Omega) \cap H_0^1(\Omega) \\ \mathcal{D}(A_0^{1/2}) \equiv H_0^1(\Omega), \quad [\mathcal{D}(A_0^{1/2})]' = H^{-1}(\Omega), \end{aligned} \tag{2.2}$$

so that problem (2.1) (with $g = 0$) can be re-written abstractly as

$$u_{ttt} + \alpha u_{tt} + c^2 A_0 u + bA_0 u_t = f \quad \text{on } H = L^2(\Omega), \tag{2.3}$$

along with I.C. u_0, u_1, u_2. For $f = 0$ we re-write it as a first-order problem as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix} = G \begin{bmatrix} u \\ u_t \\ u_{tt} \end{bmatrix}, \quad G = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2 A_0 & -bA_0 & -\alpha I \end{bmatrix}. \tag{2.4}$$

Introduce the following spaces:

$$U_0 = H \times H \times H \tag{2.5a}$$

$$U_1 \equiv \mathcal{D}(A_0^{\frac{1}{2}}) \times \mathcal{D}(A_0^{\frac{1}{2}}) \times H; \quad U_2 \equiv \mathcal{D}(A_0) \times \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) \tag{2.5b}$$

$$U_3 \equiv \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) \times H; \quad U_4 \equiv \mathcal{D}(A_0^{\frac{3}{2}}) \times \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) \tag{2.5c}$$

$$U_5 \equiv V_3 \equiv H \times [\mathcal{D}(A_0^{1/2})]' \times [\mathcal{D}(A_0)]' \tag{2.5d}$$

Theorem 2.1 ([6, Section 2, constant coefficients],[9, Theorem 2.1, p. 833, variable $\alpha(x)$ i.e. $\in L^\infty(\Omega)$):]

(i) *Let $f = 0$. The operator G in (2.4) generates a s.c. group e^{Gt} on each of the spaces $U_1, \dots, U_5$ with appropriate domains so that*

$$\begin{bmatrix} u(t) \\ u_t(t) \\ u_{tt}(t) \end{bmatrix} = e^{Gt} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} \in C([0,T]; U_i), \quad i = 1,2,3,4,5 \tag{2.6a}$$

for $[u_0, u_1, u_2] \in U_i, i = 1, 2, 3, 4, 5$.

(ii) *Let $f \in L^1(0, T; U_i)$ be a forcing term acting on the RHS of (2.3) with $\{u_0, u_1, u_2\} = 0$. Then*

$$\begin{bmatrix} u(t) \\ u_t(t) \\ u_{tt}(t) \end{bmatrix} = \int_0^t e^{G(t-\tau)} f(\tau)\,\mathrm{d}\tau \in C([0,T]; U_i) \tag{2.6b}$$

continuously.

Below we shall emphasize the case U_3, whereby then

$$G : U_3 \supset \mathcal{D}(G) = \mathcal{D}(A_0) \times \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) \longrightarrow U_3 \tag{2.4 bis}$$

as well as the case V_3, whereby

$$G : V_3 \supset \mathcal{D}(G) = H \times H \times [\mathcal{D}(A_0^{\frac{1}{2}})]' \longrightarrow V_3 \tag{2.4 tris}$$

In both cases there is no smoothing of the first component space of $\mathcal{D}(G)$ and G has no compact resolvent in both cases [6]. This will make more challenging the analysis of Section 10. The group generation property points out that the third-order Equation (2.3) is strictly hyperbolic: its principal part $[\partial_{ttt} - b\Delta\partial_t]$ has symbol $[-i\tau^3 + ib|\xi|^2\tau]$ with three distinct roots: $0, \sqrt{b}|\xi|, -\sqrt{b}|\xi|$ [11, p. 23]. In fact, as in [6], rewrite (2.3) as

$$(u_t + \alpha u)_{tt} + bA_0\left(\frac{c^2}{b}u + u_t\right) = 0. \tag{2.7}$$

This suggests introducing a new variable, as in [6]

$$\text{either } z = \frac{c^2}{b}u + u_t, \quad \text{or else } \xi = u_t + \alpha u \tag{2.8}$$

(i) Thus,

$$\text{If } \alpha = \frac{c^2}{b} \text{ or } \gamma = 0, \text{ then (2.7)} \Longrightarrow z_{tt} + bA_0 z = 0 \ (z = \xi), \tag{2.9}$$

the pure abstract wave equation.

(ii) Otherwise,

$$z = \frac{c^2}{b}u + u_t = (\alpha u + u_t) - \gamma u, \quad \gamma = \alpha - \frac{c^2}{b} \tag{2.10}$$

$$(u_t + \alpha u)_{tt} = z_{tt} + \gamma u_{tt} = z_{tt} + \gamma \left(z - \frac{c^2}{b}u \right)_t. \tag{2.11}$$

Substituting (2.8), (2.10), (2.11) in (2.7) leads to the following hyperbolic system

$$\begin{cases} z_{tt} = -bA_0 z - \gamma u_{tt} = -bA_0 z - \gamma z_t + \gamma \dfrac{c^2}{b} z - \gamma \left(\dfrac{c^2}{b} \right)^2 u \text{ in } Q & \text{(2.12a)} \\ u_t = -\dfrac{c^2}{b} u + z, \ u(t) = e^{-\frac{c^2}{b}} u_0 + \displaystyle\int_0^t e^{-\frac{c^2}{b}(t-\tau)} z(\tau)\, \mathrm{d}\tau & \text{(2.12b)} \end{cases}$$

(models #1 and #2 in [6, Section 2]) coupling the hyperbolic z-equation with the scalar ODE in u. In the constant coefficient case, the constant $\gamma = \alpha - \frac{c^2}{b}$ plays a critical role in the stability of the s.c. group e^{Gt} on U_i. Indeed, e^{Gt} is uniformly stable on each U_i (with a sharp explicit decay rate [6, Theorem 3.3]) if and only if $\gamma > 0$ [6]. The case $\gamma = 0$, see (2.9) corresponds to the point spectrum $\sigma_p(G)$ of G being on the imaginary axis, while the point $-\frac{c^2}{b}$ is in its continuous spectrum [6]. Paper [12] claims that if $\gamma < 0$, and at least in the 1D case, the boundary homogeneous Equation (1.1) admits a chaotic and topologically mixing semigroup on Banach spaces of Herzog's type.

2.2. Case $y_0 = 0, y_1 = 0, y_2 = 0, f = 0, g \neq 0$. Statement of results

In this case, we seek to obtain sharp regularity of the map

$$g \longrightarrow \left\{ y, y_t, y_{tt}, \left. \frac{\partial y}{\partial \nu} \right|_\Sigma \right\} \tag{2.13}$$

from the Dirichlet boundary datum g of low regularity such as $L^2(0, T; L^2(\Gamma))$ to the interior solution $\{y, y_t, y_{tt}\}$ and the Neumann boundary trace $\frac{\partial y}{\partial \nu}|_\Sigma$.

Orientation: We seek optimal regularity results for the map in (2.13), initially for $g \in L^2(0, T; L^2(\Gamma))$, under the assumption $\gamma \in L^\infty(\Omega)$, the case $\alpha \in L^\infty(\Omega)$, and c^2, b positive constants being the most relevant case we wish to cover. We shall proceed in two steps.

Step 1. We assume at first that

$$\gamma = 0 \text{ or } \alpha = \frac{c^2}{b}, \quad y_{ttt} + \frac{c^2}{b} y_{tt} - c^2 \Delta y - b\Delta y_t = 0 \tag{2.14}$$

so that in view of the simplified version (2.14), the y-problem (2.1a)–(2.1c) with $\{y_0, y_1, y_2\} = 0$ can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[y_{tt} - b\Delta y \right] + \dfrac{c^2}{b}\left[y_{tt} - b\Delta y \right] = 0 \quad \text{in } Q = (0, T] \times \Omega & \text{(2.15a)} \\ \left[y_{tt} - b\Delta y \right]_{t=0} = y_2 - b\Delta y_0 = 0 \quad \text{in } \Omega & \text{(2.15b)} \\ y\big|_\Sigma = g \quad \text{in } \Sigma = (0, T] \times \Gamma & \text{(2.15c)} \end{cases}$$

Lemma 2.2: *y is a solution of problem* (2.15a)–(2.15c) *if and only if $y = w$ is a solution of*

$$\begin{cases} w_{tt} = b\Delta w \quad \text{in } Q = (0,T] \times \Omega & (2.16a) \\ w|_{t=0} = w_t|_{t=0} = 0 \quad \text{in } \Omega & (2.16b) \\ w\big|_{\Sigma} = g \quad \text{in } \Sigma = (0,T] \times \Gamma. & (2.16c) \end{cases}$$

Thus, in this canonical case $\gamma = 0$, the regularity of the map (2.13) coincides with the regularity of the by now well-known map $g \to \{w, w_t, w_{tt}, \frac{\partial w}{\partial \nu}\}$ for which we quote [13, p. 172], [14] and [15, Chapter 10, Section 5]. See Theorem 2.3.

Step 2. The claim is that $0 \neq \gamma \in L^\infty(\Omega)$ produces only lower-order terms in the analysis of the regularity of the map in (2.13). We choose to substantiate this claim by using the new variable z in (2.8) (or (2.38) below) that transforms the y-mixed problem in (2.1a)–(2.1c) into the z-mixed wave problem (2.39a)–(2.39c), which shows that $\gamma \neq 0$ is responsible for *lot*. We refer to Section 3, Step 1 and Appendix 1 for the relevant discussion. Moreover, in the case $\gamma = 0$, the z-analysis in Section 3 will likewise yield the conclusion: $y = w$ solution of (2.16a)–(2.16c), though in a few more steps, see Theorem 3.1.

Conclusion. The optimal regularity of the map (2.13) for the SMGT-mixed problem (2.1a)–(2.1c) with zero initial data and $\gamma \in L^\infty(\Omega)$ is the same as in the canonical case $\gamma = \alpha - \frac{c^2}{b} = 0$; in which case $y = w$ and all the desired quantities are given by the w-problem (2.16a)–(2.16c) as reported in Theorem 2.3 below. In this case, useful representation formulas are available [13,14,16,18] and [15, Chapter 10, Section 5]. □.

In order to provide explicit representation formulas we need to introduce a few quantities.

(i) recall A_0 from (2.2);

$$Af = -b\Delta f, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad \text{i.e. } A = bA_0 \tag{2.17}$$

(ii) The Dirichlet map

$$Dg = \varphi \iff \left\{\Delta\varphi = 0 \text{ in } \Omega, \quad \varphi\big|_\Gamma = g\right\}. \tag{2.18a}$$

$$D : L^2(\Gamma) \to H^{1/2}(\Omega) \subset H^{\frac{1}{2}-2\varepsilon}(\Omega) = \mathcal{D}(A^{\frac{1}{4}-\varepsilon}), \text{ or } A^{\frac{1}{4}-\varepsilon}D \in \mathcal{L}\left(L^2(\Gamma); L^2(\Omega)\right) \tag{2.18b}$$

by elliptic theory [16–18], with $\varepsilon > 0$ arbitrary, with A in (2.17). One cannot take $\varepsilon = 0$, see [15, Remark 3.1.4, p.186]. Moreover, [15, p. 181]

$$D^*Af = bD^*A_0 f = -b\frac{\partial f}{\partial \nu}, \quad f \in \mathcal{D}(A). \tag{2.18c}$$

(iii) the (strictly negative) self-adjoint operator $(-A)$ in (2.17) is the infinitesimal generator of a strongly continuous (self-adjoint) cosine operator family $\mathcal{C}(t)$ [19–21] with sine operator $\mathcal{S}(t)x = \int_0^t \mathcal{C}(\tau)x\,d\tau, x \in H$, with $A^{\frac{1}{2}}\mathcal{S}(t)$ strongly continuous:

$$\mathcal{S}(t-\tau) = \mathcal{S}(t)\mathcal{C}(\tau) - \mathcal{C}(t)\mathcal{S}(\tau) \tag{2.19a}$$

$$\mathcal{C}(t-\tau) = \mathcal{C}(t)\mathcal{C}(\tau) - A\mathcal{S}(t)\mathcal{S}(\tau), \quad \tau, t \in \mathbb{R} \tag{2.19b}$$

We have

$$\frac{d^2\mathcal{C}(t)x}{dt^2} = -A\mathcal{C}(t)x, x \in \mathcal{D}(A); \quad \frac{d\mathcal{C}(t)x}{dt} = -A\mathcal{S}(t)x, x \in \mathcal{D}(A^{\frac{1}{2}}), \tag{2.20}$$

$\mathcal{C}(t)$ is even on H, $\mathcal{C}(0) = I$; $\mathcal{S}(t)$ is odd on H, $\mathcal{S}(0) = 0$. The above formulae (2.20) on H with $H \supset \mathcal{D}(A) \to H$ can be extended to $[\mathcal{D}(A)]'$ with A now the extension $A_e : H \to [\mathcal{D}(A)]'$, which we still denote by A.

The solution of the wave problem (2.16a)–(2.16c) and hence of the y-problem (2.1a)–(2.1c) for $\gamma \equiv 0$ will be expressed by explicit representation formulas in terms of the above cosine and sine operators, generated by the operator $(-A)$ in (2.17).

Theorem 2.3 ([13, p. 172],[14,16]): *Consider the mixed problem* (2.16a)–(2.16c) *(without compatibility conditions), where Ω is a bounded domain in $\mathbb{R}^d, d \geq 1$, with sufficiently smooth boundary Γ. Then, continuously*

$$g \in L^2(0, T; L^2(\Gamma)) \rightarrow$$

$$w = A \int_0^t \mathcal{S}(t-\tau) Dg(\tau)\, \mathrm{d}\tau \in C([0,T]; L^2(\Omega)) \tag{2.21}$$

$$w_t = A \int_0^t \mathcal{C}(t-\tau) Dg(\tau)\, \mathrm{d}\tau \in C([0,T]; [\mathcal{D}(A^{1/2})]' = H^{-1}(\Omega)) \tag{2.22}$$

$$w_{tt} = b\Delta w \in C([0,T]; H^{-2}(\Omega)), \tag{2.23a}$$

as it follows from (2.16a) *and* (2.21) *[17, p. 85]. Additional versions may be obtained by differentiating* (2.22)*:*

$$w_{tt} = \begin{cases} (-A)\left[A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)\,\mathrm{d}\tau - Dg(t)\right] \in L^2(0,T;[\mathcal{D}(A)]') & (2.23b) \\ -A^2\int_0^t \mathcal{S}(t-\tau)Dg(\tau)\,\mathrm{d}\tau + ADg(t) & (2.23c) \\ \quad \in C([0,T];[\mathcal{D}(A)]') + L^2(0,T;[\mathcal{D}(A^{3/4+\varepsilon})]') \end{cases}$$

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma} = -\frac{1}{b} D^* A A \int_0^t \mathcal{S}(t-\tau) Dg(\tau)\,\mathrm{d}\tau \in H^{-1}(\Sigma), \tag{2.24}$$

recalling (2.18c) *in* (2.24)*, where $H^{-1}(\Sigma) =$ dual of $\{h \in H_0^1(\Sigma)\}$ i.e. with $h(\cdot,0) = 0$ and $h(\cdot,T) = 0$ on Γ (but actually, $h(\cdot,T) = 0$ is not needed).*

Moreover, from (2.23b)

$$g \in C([0,T]; L^2(\Gamma)) \rightarrow w_{tt} \in C([0,T]; [\mathcal{D}(A)]') \tag{2.25}$$

Remark 2.1: We recover (2.23a) from (2.23c) as follows as $[w - Dg] \in \mathcal{D}(A)$

$$(-A)\left[A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)\,\mathrm{d}\tau - Dg(t)\right] = b\Delta\left[w - Dg(t)\right] = b\Delta w \in C([0,T]; H^{-2}(\Omega)$$

Our main result in the present Part A.1 is the following

Theorem 2.4: (i) *With reference to problem* (2.1a)–(2.1c) *with zero I.C. $\{y_0, y_1, y_2\} = 0, f = 0$, and $\gamma \in L^\infty(\Omega)$ we have the following optimal interior regularity results:*

$$g \in L^2(0,T; L^2(\Gamma)) \Longrightarrow \begin{cases} y \in C([0,T]; L^2(\Omega)) & (2.26) \\ y_t \in C([0,T]; [\mathcal{D}(A^{\frac{1}{2}})]' = H^{-1}(\Omega)), & (2.27) \end{cases}$$

$$y_{tt} \begin{cases} \in C([0,T]; H^{-2}(\Omega)) & (2.28a) \\ \in L^2(0,T; [\mathcal{D}(A)]'), & (2.28b) \\ \in C([0,T]; [\mathcal{D}(A)]') + L^2(0,T; [\mathcal{D}(A^{3/4+\varepsilon})]') & (2.28c) \end{cases}$$

as well as the following boudary trace result:

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma} \in H^{-1}(\Sigma). \tag{2.29}$$

Moreover,

$$g \in C([0,T];L^2(\Gamma)) \implies y_{tt} \in C([0,T];[\mathcal{D}(A)]'), \tag{2.30}$$

all the maps being continuous.

(ii) *Let now $\gamma = 0$. Then (see also Step 1 in the Orientation Lemma 2.2)*

$$y = w = \text{ a solution of the problem (2.16a)–(2.16c)} \tag{2.31}$$

so that, in this case, the same representation formulas for $\{w, w_t, w_{tt}, \frac{\partial w}{\partial \nu}\} = \{y, y_t, y_{tt}, \frac{\partial y}{\partial \nu}\}$ of Theorem 2.3 hold for the y-problem with zero initial data.

An immediate corollary of Theorem 2.4(ii) yields an exact controllability result for the SMGT-equation (2.1a)–(2.1c) from the origin regarding the first two component $\{y(T), y_t(T)\}$ at the optimal time T, within the class of $L^2(0,T;L^2(\Gamma))$-Dirichlet controls.

Corollary 2.5: *Assume that the w-problem* (2.16a)–(2.16c) *is exact controllable (from the origin) at time T within the class of $L^2(0,T;L^2(\Gamma))$-Dirichlet boundary controls. That is, given an arbitrary pair $\{\phi, \psi\} \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a Dirichlet boundary control $g \in L^2(0,T;L^2(\Gamma))$ such that the solution of problem* (2.16a)–(2.16c) *satisfies $w(T) = \phi$, $w_t(T) = \psi$. Then the same control g used in problem* (2.1a)–(2.1c) *with zero initial data and $\gamma = 0$ satisfies*

$$y(T) = \phi \in L^2(\Omega), \quad y_t(T) = \psi \in H^{-1}(\Omega), \quad y_{tt}(T) = \Delta\phi \in H^{-2}(\Omega). \tag{2.32}$$

Of course, for the time reversible problems y and w, exact controllability from, or to, the origin, are equivalent statements.

To establish Theorem 2.4 for $\gamma \in L^\infty(\Omega)$, it remains to verify that γ produces only lower-order terms in the analysis of the regularity of the y-problem. This will be done in Section 3 and Appendix 1.

The proof of Theorem 2.3 is by PDE-techniques, either directly [14,16], or much more conveniently, by duality [13].

In fact, consider the following problem, dual of problem (2.16a)–(2.16c)

$$\begin{cases} \phi_{tt} = \Delta\phi + f & \text{in } Q \quad (2.33a)\\ \phi\big|_{t=T} = \phi_0; \quad \phi_t\big|_{t=T} = \phi_1 & \text{in } \Omega \quad (2.33b)\\ \phi\big|_{\Sigma} = 0 & \text{in } \Sigma \quad (2.33c) \end{cases}$$

Theorem 2.6 ([14],[13, Lemma 2.1, p. 154]): *The following (sharp, hidden) trace regularity holds true for problem* (2.33)

$$\int_0^T \int_\Gamma \left(\frac{\partial \phi}{\partial \nu}\right)^2 \mathrm{d}\Sigma = \mathcal{O}_T\left(\left\| \{\phi_0, \phi_1\} \right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} + \|f\|^2_{L^1(0,T;L^2(\Omega))} \right). \tag{2.34}$$

Since [13] it has been ascertained that a most convenient roadmap is to first show (by PDE-techniques) Theorem 2.6 and then obtain Theorem 2.3 on $\{w, w_t, w_{tt}\}$ by duality.

Analysis of smooth solutions of Equation (2.1a), *zero I.C.* $\{y_0, y_1, y_2\} = 0$, and $f = 0$.

Let $\{y, y_t, y_{tt}\}$ be a smooth solution of equation (2.1a) and I.C. $\{y_0, y_1, y_2\} = 0$. We let $y|_\Gamma$ and $y_t|_\Gamma$ be the corresponding Dirichlet traces at the boundary Γ. We return to Equation (2.1a) and re-write it, as usual [16,18], [15, Appendix 3B, p. 420–424], via (2.18a) as

$$y_{ttt} + \alpha y_{tt} - c^2\Delta(y - D\left(y|_\Gamma\right)) - b\Delta(y_t - D\left(y_t|_\Gamma\right)) = 0 \quad \text{in } Q \tag{2.35}$$

or abstractly, via (2.2), as

$$y_{ttt} + \alpha y_{tt} + c^2 A_0(y - D\left(y|_\Gamma\right)) + bA_0(y_t - D\left(y_t|_\Gamma\right)) = 0 \quad \text{in } H. \tag{2.36}$$

Extending, as usual [13,15,16], the original operator A_0 in (2.2): $L^2(\Omega) \supset \mathcal{D}(A_0) \to L^2(\Omega)$ to A_{0e} : $L^2(\Omega) \to [\mathcal{D}(A_0^*)]' = [\mathcal{D}(A_0)]'$; duality $[\quad]'$ w.r.t. $H = L^2(\Omega)$ by isomorphism, and retaining the symbol A_0 also for such extension, A_{0e} we re-write Equation (2.36) as

$$(y_t + \alpha y)_{tt} + bA_0\left(\frac{c^2}{b}y + y_t\right) = c^2A_0D(y|_\Gamma) + bA_0D(y_t|_\Gamma) \in [\mathcal{D}(A_0)]'. \tag{2.37}$$

Setting as in (2.10)

$$z = \frac{c^2}{b}y + y_t = (\alpha y + y_t) - \gamma y, \quad \gamma = \alpha - \frac{c^2}{b} \tag{2.38}$$

and proceeding as in going from (2.10)–(2.12), we re-write problem (2.1a) as the following hyperbolic system

$$\begin{cases} z_{tt} = -bA_0z - \gamma z_t + \gamma\dfrac{c^2}{b}z - \gamma\left(\dfrac{c^2}{b}\right)^2 y + c^2A_0D\left(y|_\Gamma\right) + bA_0D(y_t|_\Gamma) \in [\mathcal{D}(A_0)]' & \text{(2.39a)} \\ y_t = -\dfrac{c^2}{b}y + z,\ y(t) = \displaystyle\int_0^t e^{-\frac{c^2}{b}(t-\tau)}z(\tau)\,\mathrm{d}\tau & \text{(2.39b)} \end{cases}$$

along with the I.C. (recall that we are taking $y_0 = 0, y_1 = 0, y_2 = 0$)

$$z_0 = \frac{c^2}{b}y_0 + y_1 = 0, \quad z_1 = \frac{c^2}{b}y_1 + y_2 = 0. \tag{2.39c}$$

2.3. Literature review

Regularity We refer to the Orientation of Section 2.2 regarding the contributions of the present paper in the case of a non-smooth boundary term $g \in L^2(0, T; L^2(\Gamma))$ – the most desirable class. With both Dirichlet (Section 3) and Neumann (Section 5) boundary control, our proof of regularity of the map (2.13) or (5.2), respectively, is reduced (as in [7]) to the canonical model with $\gamma = 0$ in (2.14); in which case, the simplest observation of rewriting the resulting simplified equation (2.14) in the revealing form (2.15a) yields at once the most valuable conclusion $y \equiv w$ of Lemma 2.2 and $y \equiv \eta$ of Lemma 5.1, with w and η wave solutions with Dirichlet, respectively, Neumann boundary control. Thus, for $\gamma = 0$, the optimal regularity of the map (2.13) or (5.2), respectively, is obtained at once, with the additional advantage – exploited in Corollary 2.5 (Dirichlet) and a corresponding statement for Neumann – of having available the same representation formula of the wave mixed problem [13–16,18,22,23]. Next, as noted in [7], the case $\gamma \in L^\infty(\Omega)$ is responsible only for lower-order terms on the analysis of the regularity problem. This is perhaps best appreciated by using precisely the approach of [7] is Section 3, Appendix 1, which reduces the original problem (2.1a)–(2.1c) to the z-problem (2.39a)–(2.39c) following the change of variable (2.38) introduced in [6,10]. This approach reproves that $y = w$ or $y = \eta$, respectively, by the computations as in [7] (where a sign error occured).

The final results in Theorem 2.4 (Dirichet) and Theorem 5.3 (Neumann) refine those of [7]. A duality approach, in the vein of [13] is given in Section 4, following [7] with similar refinements. A full semigroup approach with a 3×3 matrix generator will be given for $\gamma \neq 0$ in a follow-up paper.

A different approach with low regularity of boundary datum g is presented in [24].

The method (for constant coefficients) 'embeds (2.1a) in a general class of integro-differential equations (depending on a parameter). The MGT equation is then a special instance of a wave equation with persistent memory [24, Equation (1.5) p. 839] which displays an affine term'.

Thus, in this paper the SMGT equation is reduced to a Volterra integral equation. The final interior regularity due to the Dirichlet boundary datum $g \in L^2(0, T; L^2(\Gamma))$ coincides with our Theorem 2.4, Equations (2.26), (2.27) and (2.28a). The interior regularity due to the Neumann boundary datum $g \in L^2(0, T; L^2(\Gamma))$ coincides with our Theorem 5.3, Equations (5.16), (5.17) and (5.18a). Our present paper has alternative (non-equivalent) versions: (2.28b)–(2.28c) and (5.18b)–(5.18c), respectively; again, a benefit of our representation formulas. [24] does not have boundary trace results with $0 \neq g \in L^2(0, T; L^2(\Gamma))$, such as (2.29) on $\frac{\partial y}{\partial \nu}|_\Sigma \in H^{-1}(\Sigma)$ in the Dirichlet case of Theorem 2.4 and $g|_\Sigma \in H^{2\hat{\alpha}-1}(\Sigma)$ in the Neumann case of Theorem 5.3. Again, in the present paper, these results follow at once from $y = w$ and [13–16] in the Dirichlet case; and $y = \eta$ and [15,25,34,35] in the Neumann case. In [24], the definition of solution is indirect via a solver of Volterra equation. In both approaches – [24] and Section 3 (Dirichlet) and 5 (Neumann) of the present paper – a critical role is played by the regularity of convolutions with Sine and Cosine operators, as given in [13,14,16], after such formulas were introduced, for boundary value problems, in [18].

Paper [26] provides a definition of solution of the y-problem by transposition via the adjoint equation. Transposition was used in the approach of [7, Sections 6 and 7]. In [26] however, transposition requires an upfront analysis of higher regularity of solutions to the SMGT equation, and thus, in this sense, is less direct. It requires the Dirichlet boundary term being regular ($g \in C([0, T]; H^{3/2}(\Gamma)) \cap H^2(0, T; L^2(\Gamma))$, $g_t \in C([0, T]; H^{1/2}(\Gamma))$) by combining a Volterra approach with Sakamoto's theory on first-order hyperbolic systems.

We note that the approach of the present paper naturally lends itself to an (optimal) analysis with more regular Dirichlet boundary term g, by employing the results of [13]. This analysis will be pursued in the future in a follow-up paper.

'Going up' with the regularity of the boundary datum is a less challenging problem than 'going down' [13]. We also note that the approaches of the present paper can be extended to the case where the Laplacian operator is replaced by a second-order elliptic operator with space variable coefficient of limited regularity, say C^2. To this end, one would use the Riemannian geometry-based/Carleman-type estimate approach for the 'wave' as in [13,27–32].

Stabilization: The present paper's main contribution is a uniform stabilization result for the SMGT equation with Neumann boundary dissipation, in the case where $\gamma \in L^\infty(\Omega)$ and $\gamma(x) \geqslant 0$ a.e. in Ω, under 'minimal' checkable geometric conditions imposed only on the uncontrolled part of the boundary Γ_1. In doing so, our contribution extends [33]. Reduction of geometrical conditions only in the part Γ_1 of the boundary introduces additional challenges to include Lemma 8.3 and the proof of Theorem 8.5 in Section 9. The corresponding uniform stabilization problem of the SMGT-equation with Dirichlet boundary dissipation is far more challenging and is left to future investigations. It will require to rely heavily in the treatment of the wave-equation with Dirichlet dissipation in [23]. In the present paper, in addition, we present in Section 10 a strong stabilization result in the absence of geometrical conditions.

3. Proof of theorem 2.4 (i) via a direct method: reduction to hyperbolic new-variable z

In this section, we shall pursue a different approach – from the one leading for $\gamma = 0$ from Lemma 2.2 to Theorem 2.4(ii) via Theorem 2.3 – based on the change of variable $z = \frac{c^2}{b} y + y_t$ in (2.38). When $\gamma = 0$, it will reproduce Lemma 2.2; that is: $y \equiv w$ solution of the wave-problem (2.16a)–(2.16c). In addition, such approach will establish that $0 \neq \gamma \in L^\infty(\Omega)$ is responsible only for lower-order terms in the analysis of the regularity of the map in (2.13), so that the regularity of $\{y, y_t, y_{tt}, \frac{\partial y}{\partial \nu}\}$ noted in Theorem 2.4(i) – and re-obtained in Theorem 3.1 in the present approach – will be the same as for $\gamma \equiv 0$.

Step 1 The coupling $\gamma(\frac{c^2}{b})^2 y = \gamma(\frac{c^2}{b})^2 \int_0^t e^{-\frac{c^2}{b}(t-\tau)} z(\tau)\, d\tau$ between the hyperbolic z-dynamics in (2.39a) and the ODE y-equation in (2.39b) is a mild (lower order) integral term. Thus, essentially w.l.o.g., we may take at first

$$\gamma = 0, \quad \text{i.e. } \alpha = \frac{c^2}{b}, \tag{3.1}$$

see (2.38), to simplify the computations. This will not affect the sought-after regularity of the map in (2.13). The terms z_t, z in (2.39a) that by taking $\gamma = 0$ disappear are benign terms for the argument that follows. We refer to Appendix 1 for their full treatment for $\gamma \in L^\infty(\Omega)$. Thus, we obtain the simplified problem (2.39a)

$$\begin{cases} z_{tt} = -bA_0 z + c^2 A_0 D(y|_\Gamma) + bA_0 D(y_t|_\Gamma) \in [\mathcal{D}(A_0)]' & (3.2a) \\ y_t = -\dfrac{c^2}{b} y + z & (3.2b) \end{cases}$$

along with zero I.C., where now under the (essentially w.l.o.g.) assumption (3.1), the z-problem is uncoupled; that is, explicitly, in PDE-form

$$\begin{cases} z_{tt} = b\Delta z \quad \text{in } Q = (0, T] \times \Omega & (3.3a) \\ z\big|_{t=0} = 0; \quad z_t\big|_{t=0} = 0 \quad \text{in } \Omega & (3.3b) \\ z\big|_\Sigma = \dfrac{c^2}{b}(y|_\Gamma) + (y_t|_\Gamma) \quad \text{in } \Sigma = (0, T] \times \Gamma & (3.3c) \end{cases}$$

$$\begin{cases} y_t = -\dfrac{c^2}{b} y + z,\ y(t) = \displaystyle\int_0^t e^{-\frac{c^2}{b}(t-\tau)} z(\tau)\, d\tau \quad \text{in } Q & (3.4a) \\ y\big|_{t=0} = 0 & (3.4b) \end{cases}$$

In the statement of Theorem 3.1, we deliberately list only the interior regularity of z, which is the one needed to obtain the regularity of $\{y, y_t, y_{tt}, \frac{\partial}{\partial \nu}|_\Sigma\}$. The regularity of z_t is given in Remark 3.1.

Theorem 3.1: *Consider the y-problem* (2.1a)–(2.1c) *with* $\{y_0, y_1, y_2\} = 0, f = 0$ *and* $g \in L^2(0, T; L^2(\Gamma))$ *and* $\gamma = 0$. *With* $z = \frac{c^2}{b} y + y_t$ *as in* (2.38), *consider the z-problem in PDE-form as in* (3.3a)–(3.3c), *or in abstract form as in* (3.2a)–(3.2b). *Then, the quantity* $z(t)$, *as well as the solution* $\{y, y_t, y_{tt}, \frac{\partial}{\partial \nu}|_\Sigma\}$ *can be expressed explicitly by the following representation formulas, with corresponding (optimal) regularity results in terms of g; continuously on g:*

$$g \in L^2(0,T; L^2(\Gamma)) \Longrightarrow z(t) = \frac{c^2}{b} A \int_0^t \mathcal{S}(t-\tau) Dg(\tau)\, d\tau + A \int_0^t \mathcal{C}(t-\tau) Dg(\tau)\, d\tau \tag{3.5a}$$

$$= \frac{c^2}{b} w(t) + w_t(t) \in C([0, T]; [\mathcal{D}(A^{1/2})]') \tag{3.5b}$$

hence (*as in Theorems* 2.3 *and* 2.4):

$$g \in L^2(0,T; L^2(\Gamma)) \Longrightarrow y(t) = w(t) = A \int_0^t \mathcal{S}(t-\tau) Dg(\tau)\, d\tau \in C([0, T]; L^2(\Omega)) \tag{3.6}$$

$$y_t(t) = w_t(t) = A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)\,d\tau \in C([0,T];[\mathcal{D}(A^{1/2})]') \qquad (3.7)$$

$$y_{tt} = \begin{cases} w_{tt} = b\Delta w \in C([0,T];H^{-2}(\Omega)), & (3.8a)\\ ADg(t) - A^2\int_0^t \mathcal{C}(t-\tau)Dg(\tau)\,d\tau \in L^2(0,T;[\mathcal{D}(A^{3/4+\varepsilon})]') + C([0,T];[\mathcal{D}(A)]') & (3.8b)\\ (-A)\left[A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)\,d\tau - Dg(t)\right] \in L^2(0,T;[\mathcal{D}(A)]') & (3.8c)\\ b\Delta\left[w - Dg\right] = b\Delta w & (3.8d)\end{cases}$$

continuously. In addition, we have, still continuously (from (3.8c)*)*

$$g \in C([0,T];L^2(\Gamma)) \to y_{tt} \in C([0,T];[\mathcal{D}(A)]'). \qquad (3.9)$$

3.1. Proof of Theorem 3.1

Step 1. As reported in Theorem 2.3 after [18],[13–16, p.172] – and used aready in (2.21) and (2.22) for the w-problem – the representation formulae for the solution of the Dirichlet-boundary problem (3.3a)–(3.3c), or its abstract version (3.2a) are given by ($\mathcal{S}(0) = 0$)

$$z(t) = A\int_0^t \mathcal{S}(t-\tau)D\left(\frac{c^2}{b}y(\tau)\big|_\Gamma\right)d\tau + A\int_0^t \mathcal{S}(t-\tau)D\left(y_t(\tau)\big|_\Gamma\right)d\tau \qquad (3.10a)$$

$$= z^{(1)}(t) + z^{(2)}(t) \qquad (3.10b)$$

Integrating by parts ($\mathcal{S}(0) = 0$) on $z^{(2)}(t)$ in (3.10a), we obtain

$$z^{(2)}(t) = A\int_0^t \mathcal{S}(t-\tau)D\left(y_t(\tau)\big|_\Gamma\right)d\tau$$

$$= \left[A\mathcal{S}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)\right]_{\tau=0}^{\tau=t} + A\int_0^t \mathcal{C}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau \qquad (3.11a)$$

$$= \cancel{A\mathcal{S}(0)D\left(y(t)\big|_\Gamma\right)} - A\mathcal{S}(t)D\left(y(0)\big|_\Gamma\right) + A\int_0^t \mathcal{C}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau. \qquad (3.11b)$$

At this point, we notice that since the component y of the solution of (2.1a) was taken to be smooth, then compatibility conditions apply and yield

$$y(0)\big|_\Gamma = y_0\big|_\Gamma = 0 \qquad (3.12)$$

as $y_0 = 0$ throughout. Then, by (3.12) used in (3.11b) we see that the second term in (3.11b) also vanishes and thus we obtain

$$z^{(2)}(t) = A\int_0^t \mathcal{C}(t-\tau)D(y(\tau)|_\Gamma)\,d\tau. \qquad (3.13)$$

Thus, combining (3.13) in (3.10a), we obtain

$$z(t) = \frac{c^2}{b}A\int_0^t \mathcal{S}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau + A\int_0^t \mathcal{C}(t-\tau)D\left(y(\tau)\big|_\Gamma\right)d\tau \qquad (3.14)$$

originally for smooth trace $y(\cdot)|_\Gamma$. Extending the integral term, by closedness and density, we finally obtain

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow z(t) = \frac{c^2}{b} A \int_0^t \mathcal{S}(t-\tau)Dg(\tau)\,d\tau + A\int_0^t \mathcal{C}(t-\tau)Dg(\tau)\,d\tau \tag{3.15a}$$

$$= \frac{c^2}{b} w(t) + w_t(t) \in C([0,T];H^{-1}(\Omega) \equiv [\mathcal{D}(A^{1/2})]') \tag{3.15b}$$

recalling w in (2.21) and w_t in (2.22) of the w-problem (2.16a)–(2.16c). Then (3.5) is established. Next recall that $z = \frac{c^2}{b} y + y_t$ from (2.38) and compare with (2.38). By subtraction we find

$$(y-w)_t = -\frac{c^2}{b}(y-w), \quad (y-w)(0) = 0, \tag{3.16}$$

and since $y(0) = w(0) = 0$, (3.16) implies

$$y(t) = w(t) = A\int_0^t \mathcal{S}(t-\tau)Dg(\tau)\,d\tau \in C([0,T];L^2(\Omega)) \tag{3.17}$$

and (3.6) is established. Then (3.7)–(3.9) follow at once. Theorem 3.1 is proved.

Remark 3.1: From (3.15) we obtain

$$g \in L^2(0,T;L^2(\Gamma)) \Longrightarrow z_t(t) = \frac{c^2}{b} w_t + w_{tt}(t) \tag{3.18}$$

with regularity and representation formula of w_t given by (2.22) and regularity and representation formula of w_{tt} given by (2.23a), (2.23b) or (2.23c). Statement (3.18) is not needed to obtain results (3.6)–(3.9) on the y-problem.

4. Second proof of Theorem 2.4 (interior regularity) by duality via the non-homogenous Dirichlet problem (2.1a)–(2.1c)

It was pointed out in [7, Section 6] that the dual problem of the boundary non-homogeneous Dirichlet problem (2.1a)–(2.1c) is actually the problem

$$\begin{cases} v_{ttt} - \alpha v_{tt} + c^2\Delta v - b\Delta v_t = f & \text{in } Q \quad (4.1a) \\ v\big|_{t=T} = v_0; \quad v_t\big|_{t=T} = v_1; \quad v_{tt}\big|_{t=T} = v_2 & \text{in } \Omega \quad (4.1b) \\ v\big|_{\Sigma} = 0 & \text{in } \Sigma \quad (4.1c) \end{cases}$$

$$\text{abstractly} \quad v_{ttt} - \alpha v_{tt} - c^2 Av + bAv_t = 0 \tag{4.2}$$

along with I.C. at $t = T$. For future use we report the following results from [7, Section 6]. For the first, refer also to Theorem 2.1.

Theorem 4.1: *With reference to problem* (4.1a)–(4.1c) *we have*

$$f \in L^1(0,T;H), \{v_0, v_1, v_2\} \in U_3 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}) \times H \Longrightarrow \{v, v_t, v_{tt}\} \in C([0,T];U_3) \tag{4.3}$$

and with $f = 0$, the solution is a s.c. group on U_3.

Theorem 4.2: *With reference to the v-problem* (4.1a)–(4.1c), *abstractly* (4.2), *we have with* $U_3 = \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times H$

$$\int_0^T \int_\Gamma \left(\frac{\partial v_t}{\partial \nu}\right)^2 \mathrm{d}\Sigma = \mathcal{O}_T \left(\left\| \{v_0, v_1, v_2\} \right\|_{U_3}^2 + \|f\|_{L^1(0,T;H)}^2 \right) \tag{4.4}$$

This is a sharp hidden regularity result, in the style of [13,14] with respect to the interior regularity result of v_t in (4.3) (a 'gain' of 1/2 derivative in the space variable). Next, by duality on the trace result in Theorem 4.2 for the v-problem (4.1a)–(4.1c), we shall re-obtain the basic interior regularity result of Theorem 2.4 for $\{y, y_t, y_{tt}\}$. While the proof of Theorem 2.4 in Section 3 was 'direct', the proof of Theorem 4.3 below is 'by duality', in the style of [13].

Theorem 4.3: *With reference to the Dirichlet problem* (2.1a)–(2.1c), *we have continuously*

$$g \in L^2(0,T; L^2(\Gamma)) \implies \left\{y, y_t, y_{tt}\right\} \in C([0,T]; H \times [\mathcal{D}(A^{\frac{1}{2}})]') \times L^2(0,T; [\mathcal{D}(A)]') \tag{4.5a}$$

$$g \in C([0,T]; L^2(\Gamma)) \implies y_{tt} \in C([0,T]; [\mathcal{D}(A)]'). \tag{4.5b}$$

This is a counterpart of [7, Theorem 6.4*]*

Proof of Theorem 4.3: **Step 1** ■

Lemma 4.4: *Under the appropriate regularity assumptions on the data:* $\{y_0, y_1, y_2\}, g$, *and* $\{v_0, v_1, v_2\}$, f *– to be made explicit in* (4.7) *below – the following identity holds true, where* $\langle , \rangle_\Omega$ *denotes the duality pairing with respect to* $H = L^2(\Omega)$*:*

$$\begin{aligned}
&\left\langle y_{tt}(T) + \alpha y_t(T), v(T)\right\rangle_\Omega - \left\langle y_t(T) + \alpha y(T), v_t(T)\right\rangle_\Omega + \left\langle y(T), v_{tt}(T)\right\rangle_\Omega \\
&\quad + \left\langle y_0, -v_{tt}(0) + \alpha v_t(0)\right\rangle_\Omega + \left\langle y_1, v_t(0) - \alpha v(0)\right\rangle_\Omega - \left\langle y_2, v(0)\right\rangle_\Omega \\
&\quad - b\langle \Delta y(T), v(T)\rangle_\Omega + b\langle \Delta y_0, v(0)\rangle_\Omega \\
&\quad - \langle y, f\rangle_{L^2(Q)} - b\left\langle g, \frac{\partial v_t}{\partial \nu}{}_{L^2(\Sigma)}\right\rangle + c^2 \left\langle g, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Sigma)} = 0.
\end{aligned} \tag{4.6}$$

This is a variation of [7, Appendix B, (B.3)] in that the term ④ in [7, Appendix B, (B.9)] is now first integrated in time before applying Green second theorem.

Step 2 (analysis of *y*) With reference to the y-problem (2.1a)–(2.1c) and the v-problem (4.1a)–(4.1c), we now take

$$\begin{cases} f \in L^1(0,T; L^2(\Omega)), v(T) = v_t(T) = v_{tt}(T) = 0 \\ g \in L^2(0,T; L^2(\Gamma)), y_0 = y_1 = y_2 = 0. \end{cases} \tag{4.7}$$

Then the duality indentity (4.6) specializes to

$$\int_0^T \int_\Omega yf \, \mathrm{d}Q = -b\left\langle g, \frac{\partial v_t}{\partial \nu}\right\rangle_{L^2(\Sigma)} + c^2 \left\langle g, \frac{\partial v}{\partial \nu}\right\rangle_{L^2(\Sigma)} \tag{4.8a}$$

$$= \mathcal{O}\left(\|\{g, f\}\|_{L^2(0,T;L^2(\Gamma)) \times L^1(0,T;L^2(\Omega))}\right) \tag{4.8b}$$

recalling $\frac{\partial v}{\partial \nu} \in C([0,T]; H^{1/2}(\Gamma))$ by (4.3) and trace theory and critically $\frac{\partial v_t}{\partial \nu} \in L^2(0,T; L^2(\Gamma))$ by (4.4). Hence estimate (4.8b) for all $f \in L^1(0,T; L^2(\Omega))$ implies $y \in L^\infty(0,T; L^2(\Omega)$, and hence

$$y \in C([0,T]; L^2(\Omega)) \text{ continuous w.r.t } g \in L^2(0,T; L^2(\Gamma)) \tag{4.9}$$

by extension by density.

Step 3 (analysis of y_t) We now take

$$\begin{cases} f \equiv 0, v(T) = 0; v_t(T) \in \mathcal{D}(A^{1/2}), v_{tt}(T) = 0 \\ g \in L^2(0, T; L^2(\Gamma)), y_0 = y_1 = y_2 = 0. \end{cases} \tag{4.10}$$

Then the duality indentity (4.6) specializes to

$$\langle y_t(T), v_t(T) \rangle_\Omega = -\alpha \langle y(T), v_t(T) \rangle_\Omega - b \left\langle g, \frac{\partial v_t}{\partial \nu} \right\rangle_{L^2(\Sigma)} + c^2 \left\langle g, \frac{\partial v}{\partial \nu} \right\rangle_{L^2(\Sigma)} \tag{4.11a}$$

$$= \mathcal{O}\left(\| \{g, v_t(T)\} \|_{L^2(0,T;L^2(\Gamma)) \times \mathcal{D}(A^{1/2})} \right) \tag{4.11b}$$

since $y(T) \in L^2(\Omega)$ by (4.9) in Step 2, and again using critically (4.4). Hence estimate (4.11b) for all datum $v_t(T) \in \mathcal{D}(A^{1/2})$ implies

$$y_t(T) \in [\mathcal{D}(A^{1/2})]', \quad \text{or } y_t(t) \in [\mathcal{D}(A^{1/2})]' \; 0 \leqslant t \leqslant T \tag{4.12a}$$

by having T as a general point, hence

$$y_t \in C([0, T]; [\mathcal{D}(A^{1/2})]' \text{ continuous w.r.t } g \in L^2(0, T; L^2(\Gamma)). \tag{4.12b}$$

Step 4 (analysis of y_{tt}) We now take

$$\begin{cases} f \equiv 0, v(T) \in \mathcal{D}(A), v_t(T) = v_{tt}(T) = 0 \\ g \in L^2(0, T; L^2(\Gamma)), y_0 = y_1 = y_2 = 0. \end{cases} \tag{4.13}$$

Then the duality indentity (4.6) specializes to

$$\langle y_{tt}(T) - b\Delta y(T), v(T) \rangle_\Omega = -\alpha \langle y_t(T), v(T) \rangle_\Omega + b \left\langle g, \frac{\partial v_t}{\partial \nu} \right\rangle_{L^2(\Sigma)} - c^2 \left\langle g, \frac{\partial v}{\partial \nu} \right\rangle_{L^2(\Sigma)} \tag{4.14a}$$

$$= \mathcal{O}\left(\| \{g, v(T)\} \|_{L^2(0,T;L^2(\Gamma)) \times \mathcal{D}(A)} \right) \tag{4.14b}$$

since $y_t(T) \in [\mathcal{D}(A^{1/2})]'$ by (4.12a) in Step 3. Hence estimate (4.14b) for all datum $v(T) \in \mathcal{D}(A)$, so that $v(T)|_\Gamma = 0$ by (2.2) implies

$$[y_{tt}(T) - b\Delta y(T)] \in [\mathcal{D}(A)]', \text{ or } [y_{tt} - b\Delta y] \in C([0, T]; [\mathcal{D}(A)]') \tag{4.15}$$

since T in the LHS of (4.15) is arbitrary. Moreover, by Green's Theorem

$$\langle \Delta y(T), v(T) \rangle_\Omega = \langle y(T), \Delta v(T) \rangle_\Omega + \cancel{\left\langle \frac{\partial y(T)}{\partial \nu}, v(T) \right\rangle_\Gamma} - \left\langle y(T), \frac{\partial v(T)}{\partial \nu} \right\rangle_\Gamma \tag{4.16}$$

where recalling $y(T) \in L^2(\Omega)$ by (4.9), $\Delta v(T) \in L^2(\Omega)$ as $v(T) \in \mathcal{D}(A) \in H^2(\Omega)$, we have a.e. in t in $[0, T]$:

$$\langle \Delta y(t), v(t) \rangle_\Omega = -\left\langle y(t), \frac{\partial v(t)}{\partial \nu} \right\rangle_\Gamma + \mathcal{O}\left(\| \{g, v\} \|_{L^2(0,T;L^2(\Gamma)) \times L^2(0,T;\mathcal{D}(A))} \right)$$

$$= \mathcal{O}\left(\| \{g, v\} \|_{L^2(0,T;L^2(\Gamma)) \times L^2(0,T;\mathcal{D}(A))} \right) \tag{4.17}$$

taking the datum $v \in L^2(0,T;\mathcal{D}(A))$ so that $\frac{\partial v}{\partial \nu}|_\Gamma \in L^2(0,T;H^{1/2}(\Gamma))$ and the boundary term with $y|_\Gamma = g$ in (4.17) is well-defined. Using (4.17) in (4.14b) (for a general t, $0 \leqslant t \leqslant T$) we obtain

$$\int_0^T \langle y_{tt}(t), v(t)\rangle_\Omega \, \mathrm{d}t = \mathcal{O}_T\left(\|\{g,v\}\|_{L^2(0,T;L^2(\Gamma))\times L^2(0,T;\mathcal{D}(A))}\right). \tag{4.18}$$

With datum $v \in L^2(0,T;\mathcal{D}(A))$, then (4.18) implies

$$y_{tt} \in L^2(0,T;[\mathcal{D}(A)]' \text{ continuous w.r.t } g \in L^2(0,T;L^2(\Gamma)). \tag{4.19}$$

Then (4.9) for y, (4.12b) for y_t and (4.19) for y_{tt} prove Theorem 4.3, (4.5a).

PART A.2: OPTIMAL INTERIOR AND BOUNDARY REGULARITY OF THE MIXED PROBLEM WITH NEUMANN BOUNDARY TERM IN $L^2(0,T;L^2(\Gamma))$

5. Linear third-order SMGT-equation with non-homogeneous Neumann boundary term in $L^2(0,T;L^2(\Gamma))$

If the SMGT Equation (2.1a) is written in terms of the scalar velocity potential, where pressure $= k\partial_t$ (velocity potential), then the Neumann non-homogeneous boundary terms are appropriate.

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t = 0 & \text{in } Q = (0,T] \times \Omega \quad (5.1\text{a}) \\ y\big|_{t=0} = y_0 = 0; \quad y_t\big|_{t=0} = y_1 = 0; \quad y_{tt}\big|_{t=0} = y_2 = 0 & \text{in } \Omega \quad (5.1\text{b}) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_\Sigma = g & \text{in } \Sigma = (0,T] \times \Gamma \quad (5.1\text{c}) \end{cases}$$

In this case, we seek to obtain optimal regularity of the map

$$g \longrightarrow \{y, y_t, y_{tt}, y|_\Sigma.\} \tag{5.2}$$

We proceed along the same approach as for Dirichlet boundary control.

Step 1. When $\gamma = \alpha - \frac{c^2}{b} = 0$, the argument in the Orientation below (2.13) yielding (2.15a)–(2.15c), ultimately Lemma 2.2, does not depend on the boundary conditions. Hence we likewise obtain that problem (5.1a)–(5.1c) can be rewritten for $\gamma = 0$ as∗

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[y_{tt} - b\Delta y\right] + \dfrac{c^2}{b}\left[y_{tt} - b\Delta y\right] = 0 & \text{in } Q = (0,T] \times \Omega \quad (5.3\text{a}) \\ \left[y_{tt} - b\Delta y\right]_{t=0} = y_2 - b\Delta y_0 = 0 & \text{in } \Omega \quad (5.3\text{b}) \\ \left.\dfrac{\partial y}{\partial \nu}\right|_\Sigma = g & \text{in } \Sigma = (0,T] \times \Gamma \quad (5.3\text{c}) \end{cases}$$

Lemma 5.1: *We have that y as a solution of problem* (5.3a)–(5.3c) *if and only if $y = \eta$ is a solution of*

$$\begin{cases} \eta_{tt} = b\Delta\eta & \textit{in } Q = (0,T] \times \Omega \quad (5.4\text{a}) \\ \eta|_{t=0} = \eta_t|_{t=0} = 0 & \textit{in } \Omega \quad (5.4\text{b}) \\ \dfrac{\partial \eta}{\partial \nu}\Big|_\Sigma = g & \textit{in } \Sigma = (0,T] \times \Gamma_0 \quad (5.4\text{c}) \end{cases}$$

Thus, in this canonical case $\gamma = 0$, the regularity of the map (5.2) coincides with the regularity of the well-known map $g \to \{\eta, \eta_t, \eta_{tt}, \eta|_\Sigma\}$ for which we quote [15, Vol II, Sect 8], [25,34,35]. To this end we introduce the parameter $\hat{\alpha}$:

$$\hat{\alpha} = \frac{2}{3} \text{ for a general sufficiently smooth domain } \Omega \subset \mathbb{R}^d, d \geqslant 2. \tag{5.5a}$$

$$\hat{\alpha} = \frac{3}{4} \text{ for a parallelepiped in } \mathbb{R}^d, d \geqslant 2. \tag{5.5b}$$

Moreover, we introduce the operators (not to be confused with those in Part A.1)

(i)

$$A_0 f = -\Delta f, \quad \mathcal{D}(A_0) = \left\{ h \in H^2(\Omega); \left.\frac{\partial h}{\partial \nu}\right|_{\Gamma} = 0 \right\} \quad \text{i.e. } A = bA_0 \tag{5.6}$$

A_0, A are strictly positive self-adjoint operators on $H = L^2(\Omega)/\mathbb{R}$, so that the fractional powers $A^\theta, 0 < \theta < 1$, are well-defined on H.

(ii) The Neumann map

$$Ng = \varphi \iff \left\{ \Delta\varphi = 0 \text{ in } \Omega, \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\Gamma} = g \right\}. \tag{5.7a}$$

$$N : L^2(\Gamma) \to H^{3/2}(\Omega) \subset H^{\frac{3}{2}-2\varepsilon}(\Omega) = \mathcal{D}(A^{\frac{3}{4}-\varepsilon}), \text{ or } A^{\frac{3}{4}-\varepsilon} N \in \mathcal{L}\left(L^2(\Gamma); L^2(\Omega)\right) \tag{5.7b}$$

by elliptic theory [16–18], Moreover,

$$N^* A f = bN^* A_0 f = -bf|_\Sigma \quad f \in \mathcal{D}(A). \tag{5.8}$$

(iii) Let (in this section) $\mathcal{C}(t)$ be the s.c. cosine operator generated by the operator $A = b\Delta$ (+BC) with corresponding sine operator $\mathcal{S}(t)$.

Theorem 5.2 ([13, p. 172],[14,16]): *With reference to the η-problem* (5.4a)–(5.4c) *we have, continuously*

$$g \in L^2(0, T; L^2(\Gamma)) \to$$

$$\eta(t) = A \int_0^t \mathcal{S}(t-\tau) Ng(\tau)\, d\tau \in C([0,T]; H^{\hat{\alpha}}(\Omega) \equiv \mathcal{D}(A^{\hat{\alpha}/2})) \tag{5.9}$$

$$\eta_t(t) = A \int_0^t \mathcal{C}(t-\tau) Ng(\tau)\, d\tau \in C([0,T]; H^{\hat{\alpha}-1}(\Omega) \equiv [\mathcal{D}(A^{(1-\hat{\alpha})/2})]') \tag{5.10}$$

$$\eta_{tt} = b\Delta\eta \in C([0,T]; H^{\hat{\alpha}-2}(\Omega)), \tag{5.11}$$

as it follows from (5.9). *Additional version may be obtained by differentiating* (5.10)*:*

$$\eta_{tt}(t) = \begin{cases} (-A)\left[A \int_0^t \mathcal{S}(t-\tau) Ng(\tau)\, d\tau - Ng(t) \right] \in L^2(0,T; [\mathcal{D}(A^{1-\hat{\alpha}/2})]') & (5.12a) \\ -A^2 \int_0^t \mathcal{S}(t-\tau) Ng(\tau)\, d\tau + ANg(t) & \\ \quad \in C([0,T]; [\mathcal{D}(A^{1-\hat{\alpha}/2})]') + L^2(0,T; [\mathcal{D}(A^{1/4+\varepsilon})]') & (5.12b) \end{cases}$$

$$\eta|_\Sigma = \frac{1}{b} N^* A\eta = \frac{1}{b} N^* AA \int_0^t \mathcal{S}(t-\tau) Ng(\tau)\, d\tau \in H^{2\hat{\alpha}-1}(\Sigma), \tag{5.13}$$

Moreover, from (5.12a)

$$g \in C([0,T]; L^2(\Gamma)) \to \eta_{tt} \in C([0,T]; [\mathcal{D}(A^{1-\hat{\alpha}/2})]') \tag{5.14}$$

Remark 5.1: We recover (5.11) from (5.12a) as follows

$$(-A)\left[A\int_0^t \mathcal{S}(t-\tau)Ng(\tau)\,\mathrm{d}\tau - Ng(t)\right] = b\Delta\left[\eta - Ng(t)\right] = b\Delta\eta \tag{5.15}$$

Our main result of Section 5 is

Theorem 5.3: (i) *With reference to problem* (5.1a)–(5.1c) *and* $\gamma \in L^\infty(\Omega)$ *we have the following optimal interior and boundary regularity results:*

$$g \in L^2(0,T; L^2(\Gamma)) \Longrightarrow \begin{cases} y \in C([0,T]; H^{\hat{\alpha}}(\Omega) \equiv \mathcal{D}(A^{\hat{\alpha}/2})) & (5.16) \\ y_t \in C([0,T]; H^{\hat{\alpha}-1}(\Omega) \equiv [\mathcal{D}(\mathcal{A}^{(1-\hat{\alpha})/2})]'), & (5.17) \end{cases}$$

$$y_{tt} \in \begin{cases} C([0,T]; H^{\hat{\alpha}-2}(\Omega)) & (5.18a) \\ L^2(0,T; [\mathcal{D}(A^{1-\hat{\alpha}/2})]'), & (5.18b) \\ C([0,T]; [\mathcal{D}(A^{1-\hat{\alpha}/2})]') + L^2(0,T; [\mathcal{D}(A^{1/4+\varepsilon})]') & (5.18c) \end{cases}$$

as well as the following boundary trace result:

$$\Longrightarrow y\big|_\Sigma \in H^{2\hat{\alpha}-1}(\Sigma). \tag{5.19}$$

Moreover, still continuously

$$g \in C([0,T]; L^2(\Gamma)) \Longrightarrow y_{tt} \in C([0,T]; [\mathcal{D}(A^{1-\hat{\alpha}/2})]'), \tag{5.20}$$

all the maps being continuous.

(ii) *Let now* $\gamma = 0$. *Then, by Lemma* 5.1 $y = \eta$ *is a solution of the problem* (5.4a)–(5.4c) *with corresponding representation formulas.*

PART B: Boundary uniform stabilization with neumann dissipative feedback. Main results

6. The case: $\gamma \in L^\infty(\Omega)$, $\gamma(x) \geqslant 0$ a.e. on Ω

As in [23], Ω is an open bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_i relatively open, $\Gamma_0 \neq \emptyset$, $\Gamma_1 \neq \emptyset$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. For the case $\Gamma_0 = \emptyset$ we refer to [36] for needed technical changes in the absence of the Poincaré's inequality. We consider the SMGT equation with Neumann dissipation on the boundary:

$$\begin{cases} y_{ttt} + \alpha(x)y_{tt} - c^2\Delta y - b\Delta y_t = 0 \text{ in } Q = (0,T]\times\Omega & (6.1a) \\ y\big|_{t=0} = y_0 \in \mathcal{D}(A_N^{1/2}); \quad y_t\big|_{t=0} = y_1 \in \mathcal{D}(A_N^{1/2}); \quad y_{tt}\big|_{t=0} = y_2 \in L^2(\Omega) \text{ in } \Omega & (6.1b) \\ \left[\dfrac{\partial y}{\partial \nu} + y_t\right]_{\Sigma_1} = 0 \text{ in } \Sigma_1 = (0,T]\times\Gamma_1; \quad y\big|_{\Sigma_0} = 0 \text{ in } \Sigma_0 = (0,T]\times\Gamma_0. & (6.1c) \end{cases}$$

where we introduce the positive ($\Gamma_0 \neq \emptyset$) self-adjoint operator

$$A_N f = -\Delta f, \quad \mathcal{D}(A_N) = \left\{ f \in H^2(\Omega) : f|_{\Gamma_0} = 0, \left.\frac{\partial f}{\partial \nu}\right|_{\Gamma_1} = 0 \right\} \tag{6.2a}$$

$$\mathcal{D}(A_N^{1/2}) \equiv H^1_{\Gamma_0}(\Omega) = \{f \in H^1(\Omega), f|_{\Gamma_0} = 0\}. \tag{6.2b}$$

Morever, we assume that the coefficient $\alpha(x) > 0$ in $L^\infty(\Omega)$ and positive a.e. in Ω so that

$$\gamma(x) = \alpha(x) - \frac{c^2}{b} \in L^\infty(\Omega) \quad \text{and} \quad \gamma(x) \geqslant 0 \text{ a.e. on } \Omega, \tag{6.3}$$

whereby $\gamma(x)$ can vanish on subsets of Ω, even of positive measure. The important special case $\gamma(x) \equiv 0$ will be noted in case Appendix 2. For problem (6.1a)–(6.1c) we shall establish the following results:

Theorem 6.1: *Consider the Neumann boundary problem (6.1a)–(6.1c). Then:*

(a) ***(well-posedness)*** *With $\gamma(x) \in L^\infty(\Omega)$ (not necesseraly nonnegative), the map $\{y_0, y_1, y_2\} \to \{y(t), y_t(t), y_{tt}(t)\}$ defines a s.c. semigroup $e^{G_{N,F}t}$ in the space*

$$U_1 \equiv \mathcal{D}(A_N^{1/2}) \times \mathcal{D}(A_N^{1/2}) \times H = H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Omega). \tag{6.4}$$

(b) ***(Strong Stabilization)*** *Assume* (6.3). *For any $\{y_0, y_1, y_2\} \in U_1$ we have*

$$\begin{bmatrix} y(t) \\ y_t(t) \\ y_{tt}(t) \end{bmatrix} = e^{G_{N,F}t} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \to 0 \text{ as } t \to +\infty \text{ in } U_1. \tag{6.5}$$

(c) ***(Uniform Stabilization)*** *Assume* (6.3). *Moreover, with $\Gamma_0 \neq \emptyset$, assume the following geometrical condition on the triple $\{\Omega, \Gamma_0, \Gamma_1\}$: there exists a coercive vector field $h(x) = [h_1(x), \dots, h_d(x)] \in C^2(\overline{\Omega})$ such that*

(iii)$_1$ *$h \cdot \nu \leqslant 0$ on Γ_0, $\nu =$ unit outward normal*

(iii)$_2$ *for some constant $\rho > 0$ and all vector $u(x) \in [L^2(\Omega)]^d$, we have*

$$\int_\Omega H(x)u(x) \cdot u(x)\, \mathrm{d}\Omega \geqslant \rho \int_\Omega |u(x)|^2\, \mathrm{d}\Omega,\ H(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_d}{\partial x_1} & \cdots & \frac{\partial h_d}{\partial x_d} \end{bmatrix}. \tag{6.6}$$

Then there exist a constant $\delta > 0$ and a constant $C = C_\delta \geqslant 1$, such that the semigroup solution of part (a) *satisfies*

$$\left\| \begin{bmatrix} y(t) \\ y_t(t) \\ y_{tt}(t) \end{bmatrix} \right\|_{U_1} = \left\| e^{G_{N,F}t} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \right\|_{U_1} \leqslant Ce^{-\delta t} \left\| \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \right\|_{U_1}, \quad t \geqslant 0. \tag{6.7}$$

Thus, the geometrical assumption is made only on the non-controlled part Γ_0 of the boundary, a contribution of [23] over prior literature.

7. Proof of wellposedness, Theorem 6.1 (a), $\gamma \in L^\infty(\Omega)$

Step 1 As in [6], introduce the new variable

$$z = \frac{c^2}{b}y + y_t; \quad y(t) = e^{-\frac{c^2}{b}t}y_0 + \int_0^t e^{-\frac{c^2}{b}(t-\tau)}z(\tau)\,d\tau. \tag{7.1}$$

As shown in [6, Model #2] and [33] regarding the boundary conditions, we then obtain a new dissipative problem for the wave equation in z, coupled with the ODE (7.1) in y:

$$\begin{cases} z_{tt} = b\Delta z - \gamma(x)y_{tt} = b\Delta z - \gamma(x)\left(z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}y\right) \quad \text{in } Q = (0,T]\times\Omega & (7.2a)\\ z\big|_{t=0} = \frac{c^2}{b}y_0 + y_1 \equiv z_0 \in \mathcal{D}(A_N^{1/2}); \quad z_t\big|_{t=0} = \frac{c^2}{b}y_1 + y_2 \equiv z_1 \in L^2(\Omega) \quad \text{in } \Omega & (7.2b)\\ y_t = -\frac{c^2}{b}y + z \quad \text{in } Q = (0,T]\times\Omega & (7.2c)\\ \left[\frac{\partial z}{\partial\nu} + z_t\right]_{\Sigma_1} = 0 \text{ in } \Sigma_1 = (0,T]\times\Gamma_1; \quad z\big|_{\Sigma_0} = 0 \text{ in } \Sigma_0 = (0,T]\times\Gamma_0 & (7.2d)\end{cases}$$

Remark 7.1 ([33]): Under the change of variables (7.1), then the y-problem (6.1a)–(6.1c) produces the z-problem (7.2a)–(7.2d). The converse likewise holds true if we assume, in addition, $\partial_\nu y(0) + y_t(0) = 0$ on Γ_1 and $y(0) = 0$ on Γ_0.

Step 2 We rewrite problem (7.2a)–(7.2d) as a first-order coupled system

$$\frac{d}{dt}\begin{bmatrix} z \\ z_t \\ y\end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ b\Delta + \gamma(x)\frac{c^2}{b}I & -\gamma(x)I & -\gamma(x)\frac{c^4}{b^2}I \\ I & 0 & -\frac{c^2}{b}I\end{bmatrix}\begin{bmatrix} z \\ z_t \\ y\end{bmatrix} = \mathcal{A}_{N,F}\begin{bmatrix} z \\ z_t \\ y\end{bmatrix} \tag{7.3}$$

where we have introduced the operator

$$\mathcal{A}_{N,F} = \left[\begin{array}{cc|c} 0 & I & 0 \\ b\Delta & -I & 0 \\ \hline 0 & 0 & -\frac{c^2}{b}I\end{array}\right] + \begin{bmatrix} 0 & 0 & 0 \\ \gamma(x)\frac{c^2}{b}I & (1-\gamma(x))I & -\gamma(x)\frac{c^4}{b^2}I \\ I & 0 & 0\end{bmatrix} = \mathcal{A}_{N,d} + P \tag{7.4a}$$

$$\mathcal{D}(\mathcal{A}_{N,F}) = \left\{[h_1,h_2,h_3] : h_2,h_3 \in \mathcal{D}(A_N^{1/2}); \Delta h_1 \in L^2(\Omega); \left[\frac{\partial h_1}{\partial\nu} + h_2\right]_{\Gamma_1} = 0, h_1|_{\Gamma_0} = 0\right\} \tag{7.4b}$$

on the space

$$\mathcal{H}_1 = \mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \times \mathcal{D}(A_N^{1/2}) \equiv \mathcal{H}_z \times \mathcal{D}(A_N^{1/2}), \quad \mathcal{H}_z = \mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \tag{7.5}$$

for the triple $\{z, z_t, y\}$. Set

$$\mathcal{W}_N \equiv \begin{bmatrix} 0 & I \\ b\Delta & -I\end{bmatrix} : \mathcal{H}_z \supset \mathcal{D}(\mathcal{W}_N) \to \mathcal{H}_z \tag{7.6a}$$

$$\mathcal{D}(\mathcal{W}_N) = \left\{[w_1,w_2] \in \mathcal{H}_z : \Delta w_1 \in L^2(\Omega), w_2 \in \mathcal{D}(A_N^{1/2})\left[\frac{\partial w_1}{\partial\nu} + w_2\right]_{\Gamma_1} = 0, w_1\big|_{\Gamma_0} = 0\right\}. \tag{7.6b}$$

Proposition 7.1: (i) *The operator $\mathcal{W}_N$ generates a s.c., uniformly stable group $e^{\mathcal{W}_N t}$ on the space $\mathcal{H}_z$ in* (7.5).

(ii) *The operator $\mathcal{A}_{N,d}$ (d = damped) in* (7.4a) *generates a s.c., uniformly stable semigroup $e^{\mathcal{A}_{N,d}t}$ on the space $\mathcal{H}_1$ in* (7.5).

(iii) *The operator $\mathcal{A}_{N,F}$ generates a s.c. semigroup $e^{\mathcal{A}_{N,F}}$ on the space $\mathcal{H}_1$ in* (7.5).

Proof: Part (i) is well-known and leads to part (ii) since the term $a_{33} = -\frac{c^2}{b}I$ is bounded on $\mathcal{D}(A_N^{1/2})$. Then, part (iii) follows since the terms $a_{21} = \gamma(x)\frac{c^2}{b}I$ and $a_{23} = -\gamma(x)\frac{c^4}{b^2}I$ in P are compact terms while the term $a_{22} = (1-\gamma(x))I$ is bounded, as $\gamma \in L^\infty(\Omega)$. ■

Corollary 7.2: *The wellposedness result of Theorem* 6.1(a) *holds true. More precisely, with reference to $\mathcal{A}_{N,F}$ in Proposition* 7.1 (iii), *we have:*

$$\begin{bmatrix} y(t) \\ y_t(t) \\ y_{tt}(t) \end{bmatrix} = e^{G_{N,F}t}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = Me^{\mathcal{A}_{N,F}t}M^{-1}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \tag{7.7}$$

with $e^{G_{N,F}t} = Me^{\mathcal{A}_{N,F}t}M^{-1}$ a s.c. semigroup on $U_1 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A_N^{1/2}) \times L^2(\Omega)$, with generator $G_{N,F} = M\mathcal{A}_{N,F}M^{-1}$. Here,

$$\begin{bmatrix} y \\ y_t \\ y_{tt} \end{bmatrix} = M\begin{bmatrix} z \\ z_t \\ y \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -\frac{c^2}{b} \\ -\frac{c^2}{b} & 1 & \frac{c^4}{b^2} \end{bmatrix} \tag{7.8}$$

as well as

$$\begin{bmatrix} z \\ z_t \\ y \end{bmatrix} = M^{-1}\begin{bmatrix} y \\ y_t \\ y_{tt} \end{bmatrix} \qquad M = \begin{bmatrix} \frac{c^2}{b} & 1 & 0 \\ 0 & \frac{c^2}{b} & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{7.9}$$

$$M: \ \text{bounded } \mathcal{H}_1 \equiv \mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \times \mathcal{D}(A_N^{1/2}) \to U_1 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \tag{7.10a}$$

$$M^{-1}: \ \text{bounded } U_1 \to \mathcal{H}_1 \tag{7.10b}$$

so that M is a homeomorphism between the spaces $\mathcal{H}_1$ and U_1. Thus, $e^{G_{N,F}t}$ is strongly stable (resp. uniformly stable) on U_1 if and only if $e^{\mathcal{A}_{N,F}t}$ is strongly stable (resp. uniformly stable) on $\mathcal{H}_1$.

8. Proof of uniform stabilization of theorem (6.1)(c) ($\gamma(x) \geqslant 0$)

It is based, as usual, on establishing two basic properties: (i) the energy dissipation identity of Lemma 8.1 in Step 1; (ii) the property that the 'energy' is dominated by the 'the dissipation' in Theorem 8.6 below, Step 5.

Step 1 (energy dissipation identity) We return to the z-component of system (7.2a)–(7.2d)

$$\begin{cases} z_{tt} = b\Delta z - \gamma(x)y_{tt} = b\Delta z - \gamma(x)\left(z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}y\right) & \text{in } Q = (0,T] \times \Omega \quad (8.1a) \\ z\big|_{t=0} = \frac{c^2}{b}y_0 + y_1 \equiv z_0 \in \mathcal{D}(A_N^{1/2}); \quad z_t\big|_{t=0} = \frac{c^2}{b}y_1 + y_2 \equiv z_1 \in L^2(\Omega) & \text{in } \Omega \quad (8.1b) \\ \left[\frac{\partial z}{\partial \nu} + z_t\right]_{\Sigma_1} = 0 \text{ in } \Sigma_1 = (0,T] \times \Gamma_1; \quad z\big|_{\Sigma_0} = 0 \text{ in } \Sigma_0 = (0,T] \times \Gamma_0 & \quad (8.1c) \end{cases}$$

Define the following 'energies'

$$E_z(t) = \int_\Omega \left[|\nabla z(t)|^2 + z_t^2(t)\right] d\Omega; \quad \mathbb{E}_\gamma(t) = E_z(t) + \frac{c^2}{b}\int_\Omega \gamma y_t^2(t)\, d\Omega. \tag{8.2}$$

Lemma 8.1: *Let* $\{y, y_t, y_{tt}\}$ *be a solution of problem* (6.1a)–(6.1c) *as asserted by Theorem* 6.1(a), *so* $\{z = \frac{c^2}{b}y + y_t, z_t = \frac{c^2}{b}y_t + y_{tt}\}$ *is a solution of problem* (8.1a)–(8.1c). *Then the following energy dissipation identity holds true:*

$$\mathbb{E}_\gamma(t) + 2\int_s^t \int_{\Gamma_1} z_t^2 \, d\Sigma_1 + 2\int_s^t \int_\Omega \gamma y_{tt}^2 \, dQ = \mathbb{E}_\gamma(s), \quad 0 \leqslant s \leqslant t. \tag{8.3}$$

Proof: Multiply Equation (8.1a) by z_t and integrate to obtain, as usual [23, Remark 7.1, p 218], using the B.C. (8.1c)

$$E_z(t) - E_z(s) = 2\int_s^t \int_{\Gamma_1} \frac{\partial z}{\partial \nu} z_t \, d\Sigma_1 - 2\int_s^t \int_\Omega \gamma y_{tt} z_t \, dQ \tag{8.4}$$

where by (7.1)

$$\begin{aligned} 2\int_\Omega \gamma \int_s^t y_{tt} z_t \, dt \, d\Omega &= \int_\Omega \gamma \int_s^t 2y_{tt}\left(\frac{c^2}{b}y_t + y_{tt}\right) dt \, d\Omega \\ &= \frac{c^2}{b}\left[\int_\Omega \gamma y_t^2 \, d\Omega\right]\Bigg|_s^t + 2\int_Q \gamma y_{tt}^2 \, dQ. \end{aligned} \tag{8.5}$$

Substituting (8.5) into (8.4), where $\frac{\partial z}{\partial \nu} = -z_t$ on Γ_1 by (8.1c) yields (8.3). ∎

From the dissipation identity (8.3) one obtains that the Energy $\mathbb{E}_\gamma(t)$ is decreasing as $t \to +\infty$, a property to be repeatedly invoked below e.g. in (8.6b).

Step 2: We let $\nabla_{\tan}$ to be the tangential gradient.

Theorem 8.2: *In the notation of Lemma* 8.1, *the following inequalities hold:*

(i)

$$\begin{aligned} &\int_0^T \mathbb{E}_\gamma(t) \, dt \\ &\leqslant C\left\{\int_Q \gamma y_{tt}^2 \, dQ + \int_{\Sigma_1}\left[z^2 + z_t^2 + |\nabla_{tan} z|^2\right] d\Sigma_1 + \int_0^T \int_\Omega z^2 \, dQ + \mathbb{E}_\gamma(T) + \mathbb{E}_\gamma(0)\right\} \end{aligned} \tag{8.6a}$$

for a constant $C > 0$ *independent of* T.

(ii) *In fact,* $\mathbb{E}_\gamma(0)$ *on the RHS of* (8.6a) *can be eliminated by using identity* (8.3) *with* $s = 0$, $t = T$, *thus obtaining*

$$\begin{aligned} T\mathbb{E}_\gamma(T) \leqslant \int_0^T \mathbb{E}_\gamma(t) \, dt \leqslant 3C\left\{\int_Q \gamma y_{tt}^2 \, dQ + \int_{\Sigma_1}\left[z^2 + z_t^2 + |\nabla_{\tan} z|^2\right] d\Sigma_1 \right. \\ \left. + \int_0^T \int_\Omega z^2 \, dQ\right\} + 2C\mathbb{E}_\gamma(T). \end{aligned} \tag{8.6b}$$

(iii) *Let* $T > 2C$, *then*

$$\mathbb{E}_\gamma(T) \leqslant \frac{3C}{T - 2C}\left\{\int_0^T \int_{\Gamma_1}\left[z^2 + z_t^2 + |\nabla_{\tan} z|^2\right] d\Sigma_1 + \int_0^T \int_\Omega \gamma y_{tt}^2 \, dQ + \int_0^T \int_\Omega z^2 \, dQ\right\}. \tag{8.7}$$

(iv) *Let $T > 2C$, then*

$$\int_0^T \mathbb{E}_\gamma(t)\,\mathrm{d}t \leqslant 3C\left(1+\frac{1}{T-2C}\right)\left\{\int_0^T\int_\Omega \gamma y_{tt}^2\,\mathrm{d}Q + \int_{\Sigma_1}\left[z^2+z_t^2+|\nabla_{tan}z|^2\right]\mathrm{d}\Sigma_1 + \int_0^T\int_\Omega z^2\,\mathrm{d}Q\right\}. \tag{8.8}$$

Proof of Theorem 8.2: The key estimate (8.6a) will be established below. Once (8.6a) is proved, then inequality (8.6b) in (ii) follows at once after substituting $\mathbb{E}_\gamma(0)$ in (8.3) in the RHS of (8.6a). The LHS inequality in (8.6b) is obtained by recalling that $\mathbb{E}_\gamma(t)$ is decreasing as $t \to +\infty$ from Lemma 8.1. In turn, estimate (8.6b) readily implies estimate (8.7) in (iii).

Finally, inequality (8.8) in (iv) follows from substituting estimate (8.7) for $\mathbb{E}_\gamma(t)$ on the RHS of estimate (8.6b). Thus, we need to establish the RHS of inequality (8.6a).

We return to Equation (7.2a) and use the two classical multipliers $h\cdot\nabla z$ and $z\mathrm{div}h$ in [13,36]. We obtain the following by now classical identity (with no use of B.C.): sum up the two identities in [36, Eq (2.18) p. 255] for $h\cdot\nabla z$ and in [36, Eq (2.19), p.255] for $z\mathrm{div}h$, and in addition we now need to account for the new terms $-\int_Q \gamma y_{tt} h\cdot\nabla z\,\mathrm{d}Q$ and $-\int_Q \gamma y_{tt} z\mathrm{div}h\,\mathrm{d}Q$, due to the RHS of (8.1a). The result is the following identity

$$\int_\Sigma \frac{\partial z}{\partial\nu}(h\cdot\nabla z)\,\mathrm{d}\Sigma + \frac12\int_\Sigma z_t^2 h\cdot\nu\,\mathrm{d}\Sigma - \frac12\int_\Sigma |\nabla z|^2 h\cdot\nu\,\mathrm{d}\Sigma + \frac12\int_\Sigma \frac{\partial z}{\partial\nu} z\mathrm{div}h\,\mathrm{d}\Sigma$$
$$= \int_Q H\nabla z\cdot\nabla z\,\mathrm{d}Q - \frac12\int_Q z\nabla(\mathrm{div}h)\cdot\nabla z\,\mathrm{d}Q + \beta_{0T} + \int_Q \gamma y_{tt} h\cdot\nabla z\,\mathrm{d}Q + \frac12\int_Q \gamma y_{tt} z\mathrm{div}h\,\mathrm{d}Q, \tag{8.9}$$

$$\beta_{0T} = \left[(z_t, h\cdot\nabla z) + \frac12(z_t, z\mathrm{div}h)\right]\Big|_0^T. \tag{8.10}$$

Identity (8.9) does not take into account the B.C. (7.2c). Next, we split $\Gamma = \Gamma_1 \cup \Gamma_0$. We use $z|_{\Gamma_0} = 0$, hence $h\cdot\nabla z = \frac{\partial z}{\partial\nu} h\cdot\nu$ in Σ_0 and $|\nabla z|^2 = (\frac{\partial z}{\partial\nu})^2$ in Σ_0; [14, (2.21b), p. 253]; while $|\nabla z|^2 = (\frac{\partial z}{\partial\nu})^2 + |\nabla_{\tan} z|^2$ on Σ_1. As to the boundary terms in the LHS of identity (8.9), we obtain:

$$\tilde{C}_h \int_{\Sigma_1}\left[z^2+z_t^2+|\nabla_{\tan}z|^2\right]\mathrm{d}\Sigma_1$$
$$\geqslant C_h\left\{\int_{\Sigma_0}\left(\frac{\partial z}{\partial\nu}\right)^2 h\cdot\nu\,\mathrm{d}\Sigma_0 + \int_{\Sigma_1}\left[\left(\frac{\partial z}{\partial\nu}\right)^2 + z^2 + z_t^2 + |\nabla_{\tan}z|^2\right]\mathrm{d}\Sigma_1\right\} \geqslant \text{LHS of (8.9)} \tag{8.11}$$

recalling the assumption $h\cdot\nu \leqslant 0$ on Γ_0 and $\frac{\partial z}{\partial\nu} = -z_t$ in Σ_1. As to the interior terms on the RHS of (8.9) we obtain

$$\beta_{0T} \geqslant -C_h\left[E_z(T) + E_z(0)\right] \tag{8.12}$$

by virtue of Poincaré inequality via $z|_{\Sigma_0} = 0$, with $\Gamma_0 \neq \emptyset$ as assumed. Moreover, by the geometrical assumption, see (6.6), we have using $\gamma^{1/2} \in L^\infty(\Omega)$:

$$\text{RHS of (8.9)} \geqslant \rho\int_0^T\int_\Omega |\nabla z|^2\,\mathrm{d}\Omega - \varepsilon\int_0^T\int_\Omega |\nabla z|^2\,\mathrm{d}\Omega - \frac{C_h}{\varepsilon}\int_Q z^2\,\mathrm{d}Q - \frac{C_{h,\gamma}}{\varepsilon}\int_Q \gamma y_{tt}^2\,\mathrm{d}Q$$
$$- \varepsilon\int_0^T\int_\Omega |\nabla z|^2\,\mathrm{d}\Omega - C_h[E_z(T)+E_z(0)] - c_h\int_Q z^2\,\mathrm{d}Q$$

$$\geqslant \frac{\rho - 2\varepsilon}{2}\int_0^T\int_\Omega |\nabla z|^2\,d\Omega + \frac{\rho - 2\varepsilon}{2}\int_0^T\int_\Omega |\nabla z|^2\,d\Omega$$
$$- \frac{C_{h,\gamma}}{\varepsilon}\int_Q \gamma y_{tt}^2\,dQ - \tilde{C}_h[E_z(T) + E_z(0)] - c_{h\varepsilon}\int_Q z^2\,dQ. \tag{8.13}$$

Combining (8.11) with (8.13), LHS of (8.9) $\geq$ RHS of (8.9), we obtain

$$\frac{C_h}{\varepsilon}\int_Q \gamma y_{tt}^2\,dQ + \tilde{C}_h\int_{\Sigma_1}\left[z^2 + z_t^2 + |\nabla_{\tan} z|^2\right]d\Sigma_1 + c_{h,\varepsilon}\int_Q z^2\,dQ$$
$$\geqslant \frac{\rho - 2\varepsilon}{2}\int_0^T\int_\Omega |\nabla z|^2\,d\Omega + \frac{\rho - 2\varepsilon}{2}\int_0^T\int_\Omega |\nabla z|^2\,d\Omega - \tilde{C}_h[E_z(T) + E_z(0)]. \tag{8.14}$$

Next on the second term of the RHS of (8.14) we recall the identity

$$\int_0^T\int_\Omega |\nabla z|^2\,d\Omega = \int_0^T\int_\Omega z_t^2\,d\Omega + \int_{\Sigma_1}\frac{\partial z}{\partial \nu} z\,d\Sigma_1 - \int_Q \gamma y_{tt} z\,dQ - [(z_t, z)]\Big|_0^T \tag{8.15}$$

obtained by multiplying Equation (7.2a) this time by z (see [36, Eq (2.20), p. 255]) plus the term $-\int_Q \gamma y_{tt} z\,dQ$. Substituting (8.15) in the second term on the RHS of (8.14) and recalling that $\frac{\partial z}{\partial \nu} = -z_t$ in Σ_1 by (7.2d) we obtain by use again of Poincaré inequality

$$C_{\gamma,h,\varepsilon}\int_Q \gamma y_{tt}^2\,dQ + C_{h,\varepsilon}\int_{\Sigma_1}\left[z^2 + z_t^2 + |\nabla_{\tan} z|^2\right]d\Sigma_1 + \tilde{c}\int_Q z^2\,dQ + C_h[E_z(T) + E_z(0)]$$
$$\geqslant \frac{\rho - 2\varepsilon}{2}\left[\int_0^T\int_\Omega \left[|\nabla z|^2 + z_t^2\right]d\Omega - \int_Q \gamma y_{tt} z\,dQ\right] \tag{8.16a}$$
$$= \frac{\rho - 2\varepsilon}{2}\left[\int_0^T \mathbb{E}_\gamma(t)\,dt + (A)\right] \tag{8.16b}$$

adding and subtracting $\frac{\rho-2\varepsilon}{2}\frac{c^2}{b}\int_0^T\int_\Omega \gamma y_t^2\,dQ$, where

$$(A) = -\left[\frac{c^2}{b}\int_0^T\int_\Omega \gamma y_t^2\,d\Omega + \int_0^T\int_\Omega \gamma y_{tt} z\,dQ\right]. \tag{8.17}$$

Recalling $z = \frac{c^2}{b}y + y_t$ we compute after integrating by parts on $y_{tt}y$:

$$-\int_0^T\int_\Omega \gamma y_{tt} z\,dQ$$
$$= -\int_\Omega \gamma\left[\int_0^T y_{tt}\left(\frac{c^2}{b}y + y_t\right)dt\right]d\Omega$$
$$= \frac{c^2}{b}\int_\Omega \gamma\int_0^T y_t^2\,dt\,d\Omega - \frac{c^2}{b}\int_\Omega \gamma\,[y_t(T)y(T) - y_t(0)y(0)]\,d\Omega + \int_\Omega \frac{\gamma}{2}[y_t^2(0) - y_t^2(T)]\,d\Omega. \tag{8.18}$$

Thus, substituting (8.18) into (8.17) yields after a cancellation of the term $\frac{c^2}{b}\int_Q \gamma y_t^2\,dQ$:

$$(A) = -\frac{c^2}{b}\int_\Omega \gamma\,[y_t(T)y(T) - y_t(0)y(0)]\,d\Omega + \int_\Omega \frac{\gamma}{2}[y_t^2(0) - y_t^2(T)]\,d\Omega. \tag{8.19a}$$

Next, recalling that $y = \frac{b}{c^2}(z - y_t)$ we obtain

$$-\frac{c^2}{b}\int_\Omega \gamma y_t(T)y(T)\,d\Omega = \int_\Omega \gamma\,[y_t^2(T) - y_t(T)z(T)]\,d\Omega \tag{8.19b}$$

$$\frac{c^2}{b}\int_\Omega \gamma y_t(0)y(0)\,d\Omega = -\int_\Omega \gamma\,[y_t^2(0) + y_t(0)z(0)]\,d\Omega. \tag{8.19c}$$

Substituting (8.19b)–(8.19c) in (8.19a) yields

$$(A) = \int_\Omega \left[\frac{\gamma}{2}y_t^2(T) - \frac{\gamma}{2}y_t^2(0)\right] d\Omega + \int_\Omega [-\gamma y_t(T)z(T) + \gamma y_t(0)z(0)]\,d\Omega \tag{8.20a}$$

$$\geqslant -k^2\int_\Omega [\gamma y_t^2(T) + \gamma y_t^2(0)]\,d\Omega - K^2\int_\Omega [|\nabla z(T)|^2 + |\nabla z(0)|^2]\,d\Omega \tag{8.20b}$$

$$\geqslant -(\text{const.})^2\left\{E_z(T) + \frac{c^2}{b}\int_\Omega \gamma y_t^2(T)\,d\Omega + E_z(0) + \frac{c^2}{b}\int_\Omega \gamma y_t^2(0)\,d\Omega\right\} \tag{8.20c}$$

where in going from (8.20a) to (8.20b) we have invoked once more the Poincaré inequality. The constant K^2 is such that $K^2 \sim \|\gamma\|^2_{L^\infty(\Omega)}$ and therefore the $(\text{const.})^2$ in (8.20c) is such that $(\text{const.})^2 \sim 1 + \|\gamma\|^2_{L^\infty(\Omega)}$. This along with the energy identity (8.2) yields:

$$(A) \geqslant -(\text{const.})^2\left[\mathbb{E}_\gamma(T) + \mathbb{E}_\gamma(0)\right]. \tag{8.21}$$

Substituting inequality (8.21) in (8.16b) we at the inequality

$$C_{\gamma\varepsilon}\int_Q \gamma y_{tt}^2\,dQ + C_{h\varepsilon}\int_{\Sigma_1}\left[z^2 + z_t^2 + |\nabla_{\tan} z|^2\right] d\Sigma_1 + c\int_0^T\int_\Omega z^2\,dQ$$
$$+ (\text{const.})^2\left[\mathbb{E}_\gamma(T) + \mathbb{E}_\gamma(0)\right] \geqslant \int_0^T \mathbb{E}_\gamma(t)\,dt. \tag{8.22}$$

Then, inequality (8.22) coincides with (8.6a). Theorem 8.2 is established. ■

Step 3 We shall need crucially the following result from [23, Lemma 7.2]

Lemma 8.3: *Let $\varepsilon > 0$ be arbitrarily small. Let z solve equation* (7.2a). *Then the following estimate holds true:*

$$\int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1} |\nabla_{\tan} z|^2\,d\Sigma_1$$
$$\leqslant C_{T\varepsilon}\left\{\int_0^T\int_{\Gamma_1}\left[\left(\frac{\partial z}{\partial\nu}\right)^2 + z_t^2\right] d\Sigma_1 + \|z\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))} + \|\gamma y_{tt}\|^2_{H^{-1/2+\varepsilon}(Q)}\right\} \tag{8.23}$$

Remark 8.1: Estimate (8.23) for the homogeneous equation $w_{tt} = \Delta w$ in Q, that is, for equation (7.2a) with RHS $f \equiv -\gamma y_{tt} \equiv 0$, is established by a microlocal argument in [23, Lemma 7.2]. If we trace the proof with, this time, a RHS term f, i.e. for the equation $w_{tt} = \Delta w + f$ in Q, we see that the final estimate requires the additional term $\|f\|^2_{H^{-1/2+\varepsilon}(Q)}$. This establishes (8.23).

Step 4 We continue with the proof of Theorem 6.1 (uniform stabilization).

We apply inequality (8.8) of Theorem 8.2 (iv) over the interval $[\varepsilon, T-\varepsilon]$ rather than over the interval $[0, T]$, and obtain, for T large

$$\int_\varepsilon^{T-\varepsilon}\mathbb{E}_\gamma(t)\,dt$$
$$\leqslant C\left\{\int_\varepsilon^{T-\varepsilon}\int_\Omega \gamma y_{tt}^2\,d\Omega\,dt + \int_\varepsilon^{T-\varepsilon}\int_{\Gamma_1}\left[z^2 + z_t^2 + |\nabla_{\tan} z|^2\right] d\Gamma_1\,dt + \int_\varepsilon^{T-\varepsilon}\int_\Omega z^2\,dQ\right\} \tag{8.24}$$

$$\leqslant C\left\{\int_0^T\int_\Omega \gamma y_{tt}^2\,\mathrm{d}Q+\int_0^T\int_{\Gamma_1}\left[z^2+z_t^2\right]\,\mathrm{d}\Sigma_1+\|z\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))}\right\}. \tag{8.25}$$

In passing from (8.24) to (8.25) we have invoked estimate (8.23) where, in the present setting, $\frac{\partial z}{\partial \nu}=-z_t$ in Σ_1 and where the $H^{-1/2+\varepsilon}(Q)$-norm of γy_{tt} is majorized by its $L^2(Q)$ -norm. Using that $\mathbb{E}_\gamma(t)$ is decreasing (Lemma 8.1) in the LHS of inequality (8.25) yields the following result.

Theorem 8.4: *Let* $\{y,y_t,y_{tt}\}$ *be a solution for problem* (6.1a)–(6.1c) *so that* $\{z=\frac{c^2}{b}y+y_t, z_t=\frac{c^2}{b}y_t+y_{tt}\}$ *is a solution of problem* (7.2a)–(7.2c). *Then there is a positive constant* $C_T>0$ *such that*

$$\left.\begin{array}{r}C_T\mathbb{E}_\gamma(T)\\ \tilde{C}_T\mathbb{E}_\gamma(0)\end{array}\right\}\leqslant\left\{\frac{c^2}{b}\int_0^T\int_\Omega \gamma y_{tt}^2\,\mathrm{d}Q+\int_0^T\int_{\Gamma_1} z_t^2\,\mathrm{d}\Sigma_1+\|z\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))}\right\}. \tag{8.26}$$

Estimate (8.26) with $\mathbb{E}_\gamma(T)$ produces the corresponding estimate (8.26) with $\mathbb{E}_\gamma(0)$ (and a different constant) by using the dissipative identity (8.3), and conversely.

Step 5 We next absorb the lower-order term $\|z\|^2$ in $L^2(0,T;H^{1/2+\varepsilon}(\Omega))$ through a compactness-uniqueness argument, that must account for both problems: the y-problem (6.1a)–(6.1c) and the resulting z-problem (8.1a)–(8.1c).

Theorem 8.5: *With reference to the solution of problem* (6.1a)–(6.1c) *satisfying inequality* (8.26), *there is a constant* κ_T *such that*

$$\|z\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))}\leqslant\kappa_T\left\{\frac{c^2}{b}\int_0^T\int_\Omega \gamma y_{tt}^2\,\mathrm{d}Q+\int_0^T\int_{\Gamma_1} z_t^2\,\mathrm{d}\Sigma_1\right\}. \tag{8.27}$$

The proof is given in Section 9. Once Theorem 8.5 is established, then Theorem 8.4 is refined as follows, to produce the critical result, anticipated at the outset, that the enerty $\mathbb{E}_\gamma(T)$ at time $t=T$ is dominated by the dissipation up to $t=T$.

Theorem 8.6: *In the setting of Theorem* 8.4 *we have: there are positive constants* $C_T,\tilde{C}_T$ *such that*

$$\left.\begin{array}{r}C_T\mathbb{E}_\gamma(T)\\ \tilde{C}_T\mathbb{E}_\gamma(0)\end{array}\right\}\leqslant\frac{c^2}{b}\int_0^T\int_\Omega \gamma y_{tt}^2\,\mathrm{d}Q+\int_0^T\int_{\Gamma_1} z_t^2\,\mathrm{d}\Sigma_1. \tag{8.28}$$

Estimate (8.28) with $\mathbb{E}_\gamma(T)$ produces the corresponding estimate (8.28) with $\mathbb{E}_\gamma(0)$ by using the dissipative identity (8.3), and conversely.

Step 6 So far we have established the two critical ingredients announced at the outset: (i) the energy dissipation identity (8.3) of Lemma 8.1; and (ii) control of the energy by the dissipation as in (8.28) of Theorem 8.6. Using these two ingredients we obtain:

$$\mathbb{E}_\gamma(0)=\mathbb{E}_\gamma(T)+2\left\{\int_0^T\int_\Omega \gamma y_{tt}^2\,\mathrm{d}Q+\int_0^T\int_{\Gamma_1} z_t^2\,\mathrm{d}\Sigma_1\right\} \tag{8.29}$$

$$\geqslant\mathbb{E}_\gamma(T)+C_T\mathbb{E}_\gamma(T)=(1+C_T)\mathbb{E}_\gamma(T) \tag{8.30}$$

or

$$\mathbb{E}_\gamma(T)\leqslant r_T\mathbb{E}_\gamma(0),\quad r_T=\frac{1}{1+C_T}<1. \tag{8.31}$$

Remark 8.2: If the energy $\mathbb{E}_\gamma(t)$ in (8.2) were equivalent to the norm $\|e^{G_{N,F}t}\|_{\mathcal{L}(U_1)}$ of the semigroup defining the solution of the Neumann-dissipative problem (6.1a)–(6.1c), then (8.31) would – as is well-known [37, p.178] – provide at once the sought–after exponential decay (6.7). However, the definition (8.2) of $\mathbb{E}_\gamma(t)$ penalizes $z = \frac{c^2}{b}y + y_t$ in $\mathcal{D}(A_N^{1/2})$, $z_t = \frac{c^2}{b}y_t + y_{tt}$ in $L^2(\Omega)$ and $\gamma^{1/2}y_t$ in $L^2(\Omega)$, from which is not possible to unscramble y in $\mathcal{D}(A_N^{1/2})$, the first space component of the U_1-norm in (6.4). If this were possible at this stage, we would be done, and (6.7) would follow.

Step 7 Our next step is inspired by Remark 8.2. The goal is to establish that the energy $\mathbb{E}_\gamma(t)$ in (8.2) is norm-equivalent to a well-defined semigroup-based dynamical system in the variables $\{z, z_t, \gamma^{1/2}y_t\}$ on the space

$$\mathcal{H}_0 = \mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \times L^2(\Omega) \equiv \mathcal{H}_z \times L^2(\Omega), \quad \mathcal{H}_z = \mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \tag{8.32}$$

Specifically in line with Remark 8.2. Specially, such system will define a s.c. semigroup e^{Lt} on such space $\mathcal{H}_0$ and so the energy $\mathbb{E}_\gamma(t)$ in (8.2) will be norm-equivalent to the norm $\|e^{Lt}\|_{\mathcal{L}(\mathcal{H}_0)}$ of the s.c. semigroup e^{Lt}. Moreover, because of the critical estimate (8.31), such s.c. semigroup e^{Lt} will be uniformly stable on $\mathcal{H}_0$ [37, p.178]. Such new dynamical system inspired by Remark 8.2, is provided by a new coupled $\{z, y\}$-problem, for which we now employ [6, Model #1]

$$\begin{cases} z_{tt} = b\Delta z - \gamma(x)y_{tt} = b\Delta z - \gamma(x)\left(z_t - \dfrac{c^2}{b}y_t\right) \quad \text{in } Q = (0,T] \times \Omega & (8.33a) \\ z\big|_{t=0} = \dfrac{c^2}{b}y_0 + y_1 \equiv z_0 \in \mathcal{D}(A_N^{1/2}); \quad z_t\big|_{t=0} = \dfrac{c^2}{b}y_1 + y_2 \equiv z_1 \in L^2(\Omega) \quad \text{in} & \\ \Omega\,(y_t)_t = -\dfrac{c^2}{b}(y_t) + z_t \quad \text{in } Q = (0,T] \times \Omega & (8.33b) \\ \left[\dfrac{\partial z}{\partial \nu} + z_t\right]_{\Sigma_1} = 0 \text{ in } \Sigma_1 = (0,T] \times \Gamma_1; \quad z\big|_{\Sigma_0} = 0 \text{ in } \Sigma_0 = (0,T] \times \Gamma_0 & (8.33c) \end{cases}$$

We rewrite it as follows

$$\frac{d}{dt}\begin{bmatrix} z \\ z_t \\ \gamma^{1/2}y_t \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ b\Delta & -\gamma(x)I & -\gamma^{1/2}(x)\frac{c^2}{b}I \\ 0 & \gamma^{1/2}(x)I & -\frac{c^2}{b}I \end{bmatrix}\begin{bmatrix} z \\ z_t \\ \gamma^{1/2}y_t \end{bmatrix} = L\begin{bmatrix} z \\ z_t \\ \gamma^{1/2}y_t \end{bmatrix} \tag{8.34a}$$

thereby introducing the operator L on the space $\mathcal{H}_0$ in (8.32) with domain

$$\mathcal{D}(L) = \left\{[\xi_1, \xi_2, \xi_3] : \xi_2 \in \mathcal{D}(A_N^{1/2}); \Delta\xi_1 \in L^2(\Omega); \xi_3 \in L^2(\Omega); \left[\frac{\partial \xi_1}{\partial \nu} + \xi_2\right]_{\Gamma_1} = 0, \xi_1|_{\Gamma_0} = 0\right\} \tag{8.34b}$$

for the triple $\{z, z_t, \gamma^{1/2}y_t\}$ with $\gamma \geqslant 0$ a.e. on Ω. One readly has the next result as $\gamma \in L^\infty(\Omega)$, and the terms a_{22}, a_{23} and a_{32} in (8.34a) are bounded on their respective component spaces.

Proposition 8.7: (i) *The operator L generates a s.c. semigroup e^{Lt} on the space $\mathcal{H}_0$.*
(ii) *The energy $\mathbb{E}_\gamma(t)$ in (8.2) is norm-equivalent to $\|e^{Lt}\|_{\mathcal{L}(\mathcal{H}_0)}$.*

By introducing suitable weights in the second and third component spaces of $\mathcal{H}_0$, L can be made maximal dissipative, see Section 10, and thus its corresponding semigroup becomes a contraction.

The next result – a corollary of Proposition 8.7 and estimate (8.31) – is critical in our concluding analysis.

Theorem 8.8: *Assume* (6.3) *and the geometrical assumption leading to* (6.6). *The s.c. semigroup* e^{Lt} *on* $\mathcal{H}_0$ *asserted by Proposition* 8.7(i) *is exponentially stable on* $\mathcal{H}_0$*: there exist constants* $\delta > 0$ *and* $M = M_\delta \geqslant 1$ *such that*

$$\left\| \begin{bmatrix} z(t) \\ z_t(t) \\ \gamma^{1/2} y_t(t) \end{bmatrix} \right\|_{\mathcal{H}_0} = \left\| e^{Lt} \begin{bmatrix} z_0 = \frac{c^2}{b} y_0 + y_1 \\ z_1 = \frac{c^2}{b} y_1 + y_2 \\ \gamma^{1/2} y_1 \end{bmatrix} \right\|_{\mathcal{H}_0} \leqslant M e^{-\delta t} \left\| \begin{bmatrix} z_0 \\ z_1 \\ \gamma^{1/2} y_1 \end{bmatrix} \right\|_{\mathcal{H}_0}, \quad t \geqslant 0. \tag{8.35}$$

Proof: By estimate (8.31) and proposition 8.7(ii) we have

$$\|e^{LT}\|_{\mathcal{L}(\mathcal{H}_0)} \leqslant r_T < 1,$$

for some $T > 0$ and then result follows from [37, p.178]. ■

Step 8 To conclude, as noted in Remark 8.2, we need to show the exponential decay of y in the $\mathcal{D}(A_N^{1/2})$-norm. To this end, we recall (7.1) and estimate with $\{y_0, y_1, y_2\} \in U_1 = \mathcal{D}(A_N^{1/2}) \times \mathcal{D}(A_N^{1/2}) \times L^2(\Omega)$ in (6.4)

$$A_N^{1/2} y(t) = e^{-\frac{c^2}{b}t} A_N^{1/2} y_0 + \int_0^t e^{-\frac{c^2}{b}(t-\tau)} A_N^{1/2} z(\tau)\, d\tau \tag{8.36}$$

$$\|y(t)\|_{\mathcal{D}(A_N^{1/2})} \leqslant e^{-\alpha t} \|y_0\|_{\mathcal{D}(A_N^{1/2})} + \int_0^t e^{-\alpha(t-\tau)} \|z(\tau)\|_{\mathcal{D}(A_N^{1/2})}\, d\tau \tag{8.37}$$

and invoking (8.35) for z and $\mathcal{H}_0 = \mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \times L^2(\Omega)$ in (8.32)

$$\begin{aligned} \int_0^t e^{-\frac{c^2}{b}(t-\tau)} \|z(\tau)\|_{\mathcal{D}(A_N^{1/2})}\, d\tau &\leqslant \int_0^t e^{-\frac{c^2}{b}(t-\tau)} M e^{-\delta\tau}\, d\tau\, \|\{z_0, z_1, \gamma^{1/2} y_1\}\|_{\mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \times L^2(\Omega)} \\ &\leqslant M \|\gamma^{1/2}\|_{L^\infty(\Omega)} \left[\frac{e^{-\delta t} - e^{-\frac{c^2}{b}t}}{\frac{c^2}{b} - \delta} \right] \|\{z_0, z_1, y_1\}\|_{\mathcal{D}(A_N^{1/2}) \times L^2(\Omega) \times L^2(\Omega)} \end{aligned} \tag{8.38}$$

$$= M_{1,\gamma} e^{-\min\left\{\delta, \frac{c^2}{b}\right\}} \|\{y_0, y_1, y_2\}\|_{U_1} \tag{8.39}$$

with $M_{1,\gamma}$ depending on $\|\gamma^{1/2}\|_{L^\infty(\Omega)}$, recalling z_0, z_1 from (8.35) and $U_1 = H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ from (6.4). Substituting (8.39) in (8.37) yields

$$\|y(t)\|_{\mathcal{D}(A^{1/2})} \leqslant M_{2,\gamma} e^{-at} \|\{y_0, y_1, y_2\}\|_{U_1}, \quad a = \min\left\{ \frac{c^2}{b}, \delta \right\}. \tag{8.40}$$

Next, by invoking again (8.35) for z as well as (8.40) we obtain by (7.2c)

$$\|y_t(t)\|_{\mathcal{D}(A^{1/2})} = \left\| z(t) - \frac{c^2}{b} y(t) \right\|_{\mathcal{D}(A^{1/2})} \leqslant M_{3,\gamma} e^{-at} \|\{y_0, y_1, y_2\}\|_{U_1} \tag{8.41}$$

and similarly, again by (8.35) this time on z_t and now by (8.41)

$$\|y_{tt}(t)\|_{L^2(\Omega)} = \left\| z_t(t) - \frac{c^2}{b} y_t(t) \right\|_{L^2(\Omega)} \leqslant M_{4,\gamma} e^{-at} \|\{y_0, y_1, y_2\}\|_{U_1}. \tag{8.42}$$

Thus, (8.40)–(8.42) prove (6.7).

9. Proof of Theorem 8.5: absorption of l.o.t

The absorption (8.27) of the l.o.t $\|z\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))}$ is done by applying to the case at hand the classical idea of a compactness-uniqueness proof by a contradiction argument.

Step 1 By contradiction, suppose there exists a sequence $\{y_n, \dot{y}_n, \ddot{y}_n\}$ of solutions of problem (6.1a)–(6.1c)

$$\begin{cases} \dddot{y}_n + \alpha\ddot{y}_n - c^2\Delta y_n - b\Delta\dot{y}_n = 0 \quad \text{in } Q & (9.1a)\\ y_n\big|_{t=0} = y_{0n}; \quad \dot{y}_n\big|_{t=0} = y_{1n}; \quad \ddot{y}_n\big|_{t=0} = y_{2n} \quad \text{in } \Omega & (9.1b)\\ \left[\dfrac{\partial y_n}{\partial \nu} + \dot{y}_n\right]\Big|_{\Sigma_1} = 0 \text{ in } \Sigma_1 = (0,T]\times\Gamma_1; \quad y_n\big|_{\Sigma_0} = 0 \text{ in } \Sigma_0 = (0,T]\times\Gamma_0 & (9.1c)\end{cases}$$

with related sequence $\{z_n = \frac{c^2}{b}y_n + \dot{y}_n, \dot{z}_n = \frac{c^2}{b}\dot{y}_n + \ddot{y}_n\}$ solution of the problem

$$\begin{cases} \ddot{z}_n = b\Delta z_n - \gamma\ddot{y}_n \quad \text{in } Q & (9.2a)\\ z_n\big|_{t=0} = z_{0n} = \dfrac{c^2}{b}y_{0n} + y_{1n}; \quad \dot{z}_n\big|_{t=0} = z_{1n} = \dfrac{c^2}{b}y_{1n} + y_{2n}; \quad \text{in } \Omega & (9.2b)\\ \left[\dfrac{\partial z_n}{\partial \nu} + \dot{z}_n\right]\Big|_{\Sigma_1} = 0 \text{ in } \Sigma_1 = (0,T]\times\Gamma_1; \quad z_n\big|_{\Sigma_0} = 0 \text{ in } \Sigma_0 = (0,T]\times\Gamma_0 & (9.2c)\end{cases}$$

such that

$$\begin{cases} \|z_n\|_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))} \equiv 1 & (9.3a)\\ \dfrac{c^2}{b}\displaystyle\int_0^T\int_\Omega \gamma(\ddot{y}_n)^2\,\mathrm{d}Q + \int_0^T\int_{\Gamma_1}(\dot{z}_n)^2\,\mathrm{d}\Sigma_1 \to 0, \text{ as } n \to +\infty. & (9.3b)\end{cases}$$

By assumption, each member of such sequence satisfies inequality (8.27) for $C_T > 0$

$$C_T\mathbb{E}^n_\gamma(0) \leqslant \left\{\frac{c^2}{b}\int_0^T\int_\Omega \gamma(\ddot{y}_n)^2\,\mathrm{d}Q + \int_0^T\int_{\Gamma_1}(\dot{z}_n)^2\,\mathrm{d}\Sigma_1 + \|z_n\|^2_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))}\right\} \tag{9.4}$$

so that from (9.3a) and (9.3b) we have from (9.4) recalling $\mathbb{E}_\gamma(0)$ in (8.2)

$$\begin{aligned}\mathbb{E}^n_\gamma(0) = E^n_z(0) + \frac{c^2}{b}\int_\Omega \gamma(y_{1n})^2\,\mathrm{d}\Omega &= \int_\Omega \left[|\nabla z_{0n}|^2 + z^2_{1n}\right]\mathrm{d}\Omega\\ &+ \frac{c^2}{b}\int_\Omega \gamma(y_{1n})^2\,\mathrm{d}\Omega \leqslant \text{Const., uniformly in } n\end{aligned} \tag{9.5}$$

Thus we can extract a subsequence $\{y_{0n}, y_{1n}, y_{2n}\}$ and corresponding $\{z_{0n}, z_{1n}\}$, still indexed by n, such that

$$z_{0n} \to \text{ some } \zeta_0, \text{ weakly in } H^1(\Omega); \tag{9.6a}$$

$$z_{1n} \to \text{ some } \zeta_1, \text{ weakly in } L^2(\Omega); \tag{9.6b}$$

$$\eta_{1n} = \gamma^{1/2}y_{1n} \to \text{ some } \eta_1, \text{ weakly in } L^2(\Omega); \tag{9.6c}$$

In line with (8.34a), we consider the dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_n(t) \\ \dot{z}_n(t) \\ \eta_n(t) \equiv \gamma^{1/2}\dot{y}_n(t) \end{bmatrix} = L\begin{bmatrix} z_n(t) \\ \dot{z}_n(t) \\ \eta_n(t) \equiv \gamma^{1/2}\dot{y}_n(t) \end{bmatrix} \tag{9.7a}$$

or by Proposition 8.7

$$\begin{bmatrix} z_n(t) \\ \dot{z}_n(t) \\ \eta_n(t) \equiv \gamma^{1/2}\dot{y}_n(t) \end{bmatrix} = e^{Lt}\begin{bmatrix} z_{0n} \\ z_{1n} \\ \eta_{1n} \equiv \gamma^{1/2}y_{1n} \end{bmatrix} \tag{9.7b}$$

as well as the corresponding dynamics originating from the limit points in (9.6a)–(9.6c)

$$\begin{bmatrix} \zeta(t) \\ \dot{\zeta}(t) \\ \eta(t) \end{bmatrix} = e^{Lt}\begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \eta_1 \end{bmatrix} \tag{9.8}$$

Thus, recalling (8.34a)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \zeta(t) \\ \dot{\zeta}(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ b\Delta & -\gamma(x)I & -\gamma^{1/2}(x)\frac{c^2}{b}I \\ 0 & \gamma^{1/2}(x)I & -\frac{c^2}{b}I \end{bmatrix}\begin{bmatrix} \zeta(t) \\ \dot{\zeta}(t) \\ \eta(t) \end{bmatrix} \tag{9.10}$$

or explicitly

$$\begin{cases} \ddot{\zeta} = b\Delta\zeta - \gamma^{1/2}\left[\gamma^{1/2}\dot{\zeta} - \dfrac{c^2}{b}\eta\right] = b\Delta\zeta - \gamma^{1/2}\dot{\eta} \quad \text{in } Q & (9.11\text{a}) \\ \gamma^{1/2}\dot{\zeta} = \dfrac{c^2}{b}\eta + \dot{\eta} & (9.11\text{b}) \\ \left[\dfrac{\partial\zeta}{\partial\nu} + \dot{\zeta}\right]\Big|_{\Sigma_1} = 0 \text{ in } \Sigma_1 = (0,T] \times \Gamma_1; \quad \zeta\big|_{\Sigma_0} = 0 \text{ in } \Sigma_0 = (0,T] \times \Gamma_0 & (9.11\text{c}) \end{cases}$$

plus respective initial data. We claim that

$$\{z_n, \dot{z}_n, \eta_n\} \to e^{Lt}\begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \eta_1 \end{bmatrix} = \{\zeta, \dot{\zeta}, \eta\} \text{ weak-star in } L^\infty(0,T;H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)) \tag{9.12}$$

Step 3 To show (9.11), let $f = \{f_1, f_2, f_3\} \in L^1(0,T;[H^1(\Omega)]' \times L^2(\Omega) \times L^2(\Omega))$. Then, by the Lebesgue dominated convergence Theorem compute

$$\int_0^T \left(\begin{bmatrix} z_n(t) \\ \dot{z}_n(t) \\ \eta_n(t) \end{bmatrix}, \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}\right)\mathrm{d}t = \int_0^T \left(e^{Lt}\begin{bmatrix} z_{0n} \\ z_{1n} \\ \eta_{1n} \end{bmatrix}, \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}\right)\mathrm{d}t = \int_0^T \left(\begin{bmatrix} z_{0n} \\ z_{1n} \\ \eta_{1n} \end{bmatrix}, e^{L^\# t}\begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}\right)\mathrm{d}t \tag{9.13}$$

$$\to \int_0^T \left(\begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \eta_1 \end{bmatrix}, e^{L^\# t}\begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}\right)\mathrm{d}t = \int_0^T \left(e^{Lt}\begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \eta_1 \end{bmatrix}, \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}\right)\mathrm{d}t, \tag{9.14}$$

recalling the weak convergence (9.6a)–(9.6c), where (,) denotes the duality between $H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ and $[H^1(\Omega)]' \times L^2(\Omega) \times L^2(\Omega)$. Then the convergence (9.12) → (9.13) proves the weak-star convergence in (9.11).

Step 4 It follows from the weak-star convergence results in (9.11) that there exists M independent of n such that

$$\|\{z_n, \dot{z}_n\}\|_{L^\infty(0,T;H^1(\Omega)\times L^2(\Omega))} \leqslant M, \; \forall n \tag{9.15}$$

Moreover, the injection

$$H^1(\Omega) \hookrightarrow H^{1/2+\varepsilon}(\Omega) \tag{9.16}$$

is compact. Thus, we are in the following situation: we have three spaces $B_0 \equiv H^1(\Omega)$, $B \equiv H^{1/2+\varepsilon}(\Omega)$, $B_1 = L^2(\Omega)$ such $B_0 \subset B \subset B_1$ (with continuous inclusion) and $B_0 \hookrightarrow B$ compact. Further we define the space the space

$$W \equiv \{v \in L^2(0,T;H^1(\Omega)); \dot{v} \in L^2(0,T;L^2(\Omega)\} \tag{9.17a}$$

equipped with the norm

$$\|v\|_W = \|v\|_{L^2(0,T;H^1(\Omega))} + \|\dot{v}\|_{L^2(0,T;L^2(\Omega))}. \tag{9.17b}$$

Then, we appeal to a well-known result [38,39] and conclude that the injection

$$W \hookrightarrow L^2(0,T;H^{1/2+\varepsilon}(\Omega)) \tag{9.18}$$

is compact. Thus, in view of (9.14) and (9.17), there exists a subsequence, still indexed by n, such that

$$z_n \to \zeta \text{ strongly in } L^2(0,T;H^{1/2+\varepsilon}(\Omega)) \tag{9.19}$$

Moreover, returning to (9.3a) and invoking here (9.18), we obtain

$$\|\zeta\|_{L^2(0,T;H^{1/2+\varepsilon}(\Omega))} = 1. \tag{9.20}$$

Step 5 Finally, our proof will be completed once we show that η and ζ satisfy the identity

$$\frac{c^2}{b}\int_0^T \int_\Omega \dot{\eta}^2 \, \mathrm{d}Q + \int_0^T \int_{\Gamma_1} (\dot{\zeta})^2 \, \mathrm{d}\Sigma_1 = 0. \tag{9.21}$$

In fact, (9.20) along with (9.10a)–(9.10c) imply

$$\begin{cases} \dot{\eta} \equiv 0 \text{ in } (0,T] \times \Omega \\ \dot{\zeta} \equiv 0 \text{ in } \Sigma_1 \end{cases} \Longrightarrow \begin{cases} \ddot{\zeta} = b\Delta\zeta \text{ in } Q \\ \frac{\partial \zeta}{\partial \nu}\Big|_{\Sigma_1} \equiv 0, \; \zeta\big|_{\Sigma_0} \equiv 0. \end{cases} \tag{9.22a}$$

$$\Longrightarrow \begin{cases} \dot{\zeta}_{tt} = b\Delta\dot{\zeta} \text{ in } Q \\ \frac{\partial \dot{\zeta}}{\partial \nu}\Big|_{\Sigma_1} \equiv 0, \; \dot{\zeta}\big|_{\Sigma} \equiv 0. \end{cases} \tag{9.22b}$$

where (9.21b) is obtained from (9.21a) by t-differentiation and recalling again $\dot{\zeta} \equiv 0$ on Σ_1 from (9.20). The overdetermined $\dot{\zeta}$-problem implies [40, Theorem 6.1, p.75]

$$\dot{\zeta} \equiv 0 \text{ in } Q \Longrightarrow \begin{cases} \Delta\zeta \equiv 0 \text{ in } Q \\ \frac{\partial \zeta}{\partial \nu}\Big|_{\Sigma_1} \equiv 0, \; \zeta\big|_{\Sigma_0} \equiv 0. \end{cases} \tag{9.23}$$

using $\dot{\zeta} \equiv 0$ in Q in (9.21a). Then the elliptic problem (9.22) (at each t) implies finally

$$\zeta \equiv 0 \text{ in } Q, \tag{9.24}$$

which is a contradiction of (9.19) and Theorem 8.5 is proved.

We finish the proof by establishing (9.20). To this end, recall from (9.3b) that

$$\frac{c^2}{b}\int_0^T\int_\Omega \gamma(\ddot{y}_n)^2\,\mathrm{d}Q + \int_0^T\int_{\Gamma_1}(\dot{z}_n)^2\,\mathrm{d}\Sigma_1 \to 0, \quad n\to\infty, \tag{9.25}$$

or as $\eta_n(t) \equiv \gamma^{1/2}\dot{y}_n(t)$,

$$\frac{c^2}{b}\int_0^T\int_\Omega \dot{\eta}_n^2\,\mathrm{d}Q + \int_0^T\int_{\Gamma_1}(\dot{z}_n)^2\,\mathrm{d}\Sigma_1 \to 0, \quad n\to\infty. \tag{9.26}$$

Recall that $\eta_n \to \eta$ weak-star in $L^\infty(0,T;L^2(\Omega))$ by (9.11) and that $z_n \to \zeta$ strongly in $L^2(0,T;H^{1/2+\varepsilon}(\Omega))$ by (9.18), the last implying (from continuity of the trace operator) that $z_n|_{\Gamma_1} \to \zeta|_{\Gamma_1}$ strongly in $L^2(0,T;L^2(\Gamma_1))$. Moreover, (9.25) imply that, at least for n large, $\{\dot{\eta}_n, \dot{z}_n|_{\Gamma_1}\}$ belongs to a fixed finite ball in $L^2(0,T;L^2(\Omega)\times L^2(\Gamma_1))$ for all such n.

Then, by maybe restricting to a further subsequence we have

$$\dot{\eta}_n \to \dot{\eta} \text{ weakly in } L^2(0,T;L^2(\Omega)) \tag{9.27}$$

$$\dot{z}_n|_{\Gamma_1} \to \dot{\zeta}|_{\Gamma_1} \text{ weakly in } L^2(0,T;L^2(\Gamma_1)) \tag{8.28}$$

In fact, if $\varphi \in \mathcal{D}(0,T;\Omega)$ we have by definition of distributional derivative:

$$\int \dot{\eta}_n\varphi\,\mathrm{d}Q = -\int \eta_n\dot{\varphi}\,\mathrm{d}Q \to -\int \eta\dot{\varphi}\,\mathrm{d}Q = \int \dot{\eta}\varphi\,\mathrm{d}Q \tag{9.29}$$

recalling $\eta_n \to \eta$ weak-star in $L^\infty(0,T;L^2(\Omega))$ and the uniqueness of the limits. Similarly, if $\varphi \in \mathcal{D}(0,T;\Gamma_1)$ then

$$\int \dot{z}_n|_{\Gamma_1}\varphi\,\mathrm{d}\Sigma_1 = -\int z_n|_{\Gamma_1}\dot{\varphi}\,\mathrm{d}\Sigma_1 \to -\int \zeta|_{\Gamma_1}\dot{\varphi}\,\mathrm{d}\Sigma_1 = \int \dot{\zeta}\varphi\,\mathrm{d}\Sigma_1 \tag{9.30}$$

recalling that $z_n|_{\Gamma_1} \to \zeta|_{\Gamma_1}$ strongly in $L^2(0,T;L^2(\Gamma_1))$.

In our last step, we invoke the weak convergence in (9.26) and (9.27) along with the weak lower semicontinuity of a convex functional f, in particular the norm $f(x) = \|x\|$ [37, Corollary 1.8.3, p.30], to conclude via (9.25) that

$$\begin{aligned} 0 &\leqslant \frac{c^2}{b}\int_0^T\int_\Omega \dot{\eta}^2\,\mathrm{d}Q + \int_0^T\int_{\Gamma_1}(\dot{\zeta})^2\,\mathrm{d}\Sigma_1 = \int_0^T\left[\frac{c^2}{b}\|\dot{\eta}\|^2_{L^2(\Omega)} + \|\dot{\zeta}\|^2_{L^2(\Gamma_1)}\right]\mathrm{d}t \\ &\leqslant \liminf_{n\to\infty}\int_0^T\left[\frac{c^2}{b}\|\dot{\eta}_n\|^2_{L^2(\Omega)} + \|\dot{z}_n\|^2_{L^2(\Gamma_1)}\right]\mathrm{d}t = 0. \end{aligned} \tag{9.31}$$

Thus (9.30) establishes (9.20) and this concludes the proof of Theorem (8.5).

10. Proof of Theorem 6.1(b): strong stabilization (without geometrical conditions on $\{\Omega, \Gamma_0, \Gamma_1\}$) for, $\gamma \in L^\infty(\Omega)$, $\gamma(x) \geqslant 0$ a.e. in Ω

The strategy is to use the Arendt-Batty [41–43] result, the version of which presently needed is [41, Theorem 2.3, p. 35]: Assume that $\{T(t), t \geqslant 0\}$ is a bounded s.c. semigroup with generator A. If $\sigma(A) \cap i\mathbb{R} = \emptyset$, then $\lim_{t\to\infty} T(t)x = 0$ for all $x \in X$.

Step 1 We return to problem (8.33a)–(8.33b), or (8.34a), and introduce the following operator $\hat{L}$ which is the same as the operator L in (8.34a), except that the space $\mathcal{H}_0$ in (8.32) for L is replaced by the space $\hat{\mathcal{H}}_0$ introduced below. For it, we introduce the (benign) weight $\frac{1}{\sqrt{b}}$ in its second component space, and the (benign) weight $\frac{c}{b}$ on its third component space. The goal is to get $\hat{L}$ dissipative in $\hat{\mathcal{H}}_0$. This space is defined by

$$\hat{\mathcal{H}}_0 \equiv H^1_{\Gamma_0}(\Omega) \times L^2_{1/\sqrt{b}}(\Omega) \times L^2_{c/b}(\Omega). \tag{10.1}$$

Here, $H^1_{\Gamma_0}(\Omega) = \{f \in H^1(\Omega), f|_{\Gamma_0} = 0\}, \Gamma_0 \neq \emptyset$, is topologized by the gradient norm (by Poincare's inequality). Moreover,

$$(u, v)_{L^2_{1/\sqrt{b}}(\Omega)} = \left(\frac{1}{\sqrt{b}}u, \frac{1}{\sqrt{b}}v\right)_{L^2(\Omega)}, \quad (u, v)_{L^2_{c/b}(\Omega)} = \left(\frac{c}{b}u, \frac{c}{b}v\right)_{L^2(\Omega)} \tag{10.2}$$

The operator $\hat{L}$ is defined by

$$\hat{L}\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ b\Delta & -\gamma I & -\gamma^{1/2}\frac{c^2}{b}I \\ 0 & \gamma^{1/2}I & -\frac{c^2}{b}I \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix} : \hat{\mathcal{H}}_0 \supset \mathcal{D}(\hat{L}) \to \hat{\mathcal{H}}_0 \tag{10.3a}$$

$$\mathcal{D}(\hat{L}) = \mathcal{D}(L) = \left\{ \{z_1, z_2, \eta\} \in \hat{\mathcal{H}}_0 : z_2 \in \mathcal{D}(A_N^{1/2}) \equiv H^1_{\Gamma_0}(\Omega); \Delta z_1 \in L^2(\Omega) \right.$$
$$\left. \frac{\partial z_1}{\partial \nu}\bigg|_{\Gamma_1} = -z_2|_{\Gamma_1} \in H^{1/2}(\Gamma_1),\ z_1|_{\Gamma_0} = 0, \text{ so that } z_1 \in H^2(\Omega) \right\}. \tag{10.3b}$$

We note that for $\{z_1, z_2, \eta\} \in \mathcal{D}(\hat{L})$, the third component $\eta \in L^2(\Omega)$ is no smoother than the third space component of $\hat{\mathcal{H}}_0$. Thus, $\hat{L}$ does <u>not</u> have compact resolvent in $\hat{\mathcal{H}}_0$.

Lemma 10.1: *The operator $\hat{L}$ is dissipative in $\hat{\mathcal{H}}_0$:*

$$Re\left\{\left(\hat{L}\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix}\right)_{\hat{\mathcal{H}}_0}\right\} = -\int_{\Gamma_1} z_2^2 d\Gamma_1 - \frac{1}{b}\|\gamma^{1/2}z_2\|^2_{L^2(\Omega)} - \frac{c^2}{b}\left(\frac{c}{b}\right)^2\|\eta\|^2_{L^2(\Omega)} \leqslant 0. \tag{10.4}$$

Proof: For $\{z_1, z_2, \eta\} \in \mathcal{D}(\hat{L})$ we compute via (10.2), Green's Theorem and (10.3b)

$$\text{Re}\left\{\left(\begin{bmatrix} 0 & I \\ b\Delta & -\gamma I \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)_{H^1_{\Gamma_0}(\Omega)\times L^2_{1/\sqrt{b}}(\Omega)}\right\}$$
$$= \text{Re}\left\{(\nabla z_2, \nabla z_1)_{L^2(\Omega)} + (\Delta z_1, z_2)_{L^2(\Omega)} - \frac{1}{b}(\gamma z_2, z_2)_{L^2(\Omega)}\right\} \tag{10.5}$$
$$= \text{Re}\left\{(\nabla z_2, \nabla z_1)_{L^2(\Omega)} - \int_{\Gamma_1} z_2^2 d\Gamma_1 - (\nabla z_1, \nabla z_2) - \frac{1}{b}(\gamma z_2, z_2)_{L^2(\Omega)}\right\} \tag{10.6}$$
$$= -\int_{\Gamma_1} z_2^2 d\Gamma_1 - \frac{1}{b}\|\gamma^{1/2}z_2\|^2_{L^2(\Omega)}. \tag{10.7}$$

Moreover, again by (10.2)

$$\text{Re}\left\{\left(-\frac{1}{\sqrt{b}}\gamma^{1/2}\frac{c^2}{b}\eta, \frac{1}{\sqrt{b}}z_2\right)_{L^2(\Omega)} + \left(\frac{c}{b}\gamma^{1/2}z_2, \frac{c}{b}\eta\right)_{L^2(\Omega)}\right\} = 0. \tag{10.8}$$

Thus, (10.4) is proved by (10.7), (10.8) returning to $\hat{L}$. ■

Step 2

Lemma 10.2: *With reference to* $\hat{L}$ *in* (10.3a) *we have* $0 \in \rho(\hat{L})$ = *the resolvent set of* $\hat{L}$.

Proof: Given $\{h_1, h_2, h_3\} \in \hat{\mathcal{H}}_0$, the unique solution $\{z_1, z_2, \eta\} \in \mathcal{D}(\hat{L})$ of

$$\hat{L}\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ b\Delta & -\gamma I & -\gamma^{1/2}\frac{c^2}{b}I \\ 0 & \gamma^{1/2}I & -\frac{c^2}{b}I \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \tag{10.9}$$

is

$$z_2 = h_1 \in H^1_{\Gamma_0}(\Omega), \quad \eta = \frac{b}{c^2}\left[\gamma^{1/2}h_1 - h_3\right] \in L^2(\Omega) \tag{10.10}$$

while $z_1 \in H^2(\Omega)$ is the unique solutions of the problem

$$\begin{cases} b\Delta z_1 = h_2 + 2\gamma h_1 - \gamma^{1/2}h_3 \in L^2(\Omega) & \text{(10.11a)} \\ \left.\dfrac{\partial z_1}{\partial \nu}\right|_{\Gamma_1} = -z_2|_{\Gamma_1} \in H^{1/2}(\Gamma_1), \; z_1|_{\Gamma_0} = 0 & \text{(10.11b)} \end{cases}$$

■

Corollary 10.3: $\hat{L}$ *is maximal dissipative in* $\hat{\mathcal{H}}_0$ *and thus generates a s.c. contraction semigroup* $e^{\hat{L}t}$ *on* $\hat{\mathcal{H}}_0$.

Proof: Lemma 10.2 implies maximal dissipativity since a small disk of $\mathbb{C}$ containing the origin is still contained in $\rho(\hat{L})$ (the resolvent set is open) and the Lummer–Phillips theorem yields the conclusion. ■

Step 3 As a consequence of Corollary 10.3, we have that the spectrum $\sigma(\hat{L})$ of $\hat{L}$ in contained in the closed half-plane $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leqslant 0\}$. Since $\hat{L}$ does not have compact resolvent, it is not enough to show that $\sigma_p(\hat{L}) \cap i\mathbb{R} = \emptyset$, where σ_p = point spectrum. We shall show directly that

Proposition 10.4: *With reference to* $\hat{L}$ *in* (10.3a) *we have* $i\mathbb{R} \in \rho(\hat{L})$.

Proof: in view of Lemma 10.2 we need to show that $i\omega \in \rho(\hat{L})$ for all $0 \neq \omega \in \mathbb{R}$. To this end, let $\{h_1, h_2, h_3\} \in \hat{\mathcal{H}}_0$. We seek a unique solution $\{z_1, z_2, \eta\} \in \mathcal{D}(\hat{L})$ such that:

$$\begin{bmatrix} 0 & I & 0 \\ b\Delta & -\gamma I & -\gamma^{1/2}\frac{c^2}{b}I \\ 0 & \gamma^{1/2}I & -\frac{c^2}{b}I \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix} - i\omega\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \tag{10.12}$$

or

$$\begin{cases} z_2 - i\omega z_1 = h_1; & \text{(10.13a)} \\ b\Delta z_1 - \gamma z_2 - \gamma^{1/2}\dfrac{c^2}{b}\eta - i\omega z_2 = h_2; & \text{(10.13b)} \\ \gamma^{1/2}z_2 - \dfrac{c^2}{b}\eta - i\omega\eta = h_3; \quad \left[i\omega + \dfrac{c^2}{b}\right]\eta = i\omega\gamma^{1/2}z_1 + \gamma^{1/2}h_1 - h_3 & \text{(10.13c)} \end{cases}$$

Next, multiply (10.13a) by γ, sum to (10.13b), replace z_2 from (10.13a) and obtain

$$b\Delta z_1 - i\omega\gamma z_1 + \omega^2 z_1 = \frac{c^2}{b}\gamma^{1/2}\eta + i\omega h_1 + \gamma h_1 + h_2. \tag{10.14}$$

Rationalizing the fraction for η in (10.13c) one finds

$$\eta = \frac{\left(\omega^2 + i\omega\frac{c^2}{b}\right)\gamma^{1/2}z_1 + \left(\frac{c^2}{b} - i\omega\right)\gamma^{1/2}h_1 + \left(i\omega - \frac{c^2}{b}\right)h_3}{\left(\frac{c^2}{b}\right)^2 + \omega^2} \tag{10.15}$$

which substituted as the RHS of (10.14) yields the final problem

$$\begin{aligned} b\Delta z_1 + (\omega^2 - i\omega\gamma)z_1 - \frac{\frac{c^2}{b}\left[\omega^2 + i\omega\frac{c^2}{b}\right]}{\left(\frac{c^2}{b}\right)^2 + \omega^2}\gamma z_1 \\ = \gamma + i\omega + \frac{\frac{c^2}{b}\left[\frac{c^2}{b} - i\omega\right]}{\left(\frac{c^2}{b}\right)^2 + \omega^2}\gamma h_1 + h_2 + \frac{i\omega - \left(\frac{c^2}{b}\right)^2\gamma^{1/2}}{\left(\frac{c^2}{b}\right)^2 + \omega^2}h_3 \end{aligned} \tag{10.16a}$$

with B.C

$$\left[\frac{\partial z_1}{\partial \nu} + i\omega z_1\right]_{\Gamma_1} = -h_1|_{\Gamma_1} \in H^{1/2}(\Gamma_1); \quad i\omega z_1|_{\Gamma_0} = -h_1|_{\Gamma_0} \in H^{1/2}(\Gamma_0). \tag{10.16b}$$

Thus, for $\omega \neq 0$, given $\{h_1, h_2, h_3\} \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, one obtains $\Delta z_1 \in L^2(\Omega)$, yielding in turn $z_2 = i\omega z_1 + h_1 \in H^1(\Omega)$, and finally

$$\eta = \frac{\gamma^{1/2}z_2 - h_3}{i\omega + \frac{c^2}{b}} \in L^2(\Omega).$$

It remains to show that the above solution is unique in the class of solutions with the same regularity. ∎

Proof of uniqueness: Suppose that there exist two such solutions z_1, z_1' of problem (10.16a)–(10.16b). Hence $\zeta = z_1 - z_1'$ solves the following problem

$$\begin{cases} b\Delta\zeta + (\omega^2 - i\omega\gamma)\zeta - \dfrac{\frac{c^2}{b}\left[\omega^2 + i\omega\frac{c^2}{b}\right]}{\frac{c^2}{b} + \omega^2}\gamma\zeta & (10.18a) \\ \left[\dfrac{\partial\zeta}{\partial\nu} + i\omega\zeta\right]_{\Gamma_1} = 0 \quad i\omega\zeta|_{\Gamma_0} = 0 & (10.18b) \end{cases}$$

Multiply (10.17a) by ζ and integrate by parts. Separate the resulting identity into real and imaginary parts. The imaginary part is

$$-i\omega\left\{b\int_{\Gamma_1}\zeta^2 d\Gamma_1 + (\gamma\zeta, \zeta) + \frac{\left(\frac{c^2}{b}\right)^2}{\left(\frac{c^2}{b}\right)^2 + \omega^2}(\gamma\zeta, \zeta)\right\} = 0. \tag{10.18}$$

Thus, for $\omega \neq 0$ we obtain a.e.

$$\zeta \equiv 0 \text{ on } \Gamma_1 \quad \gamma^{1/2}\zeta = 0 \text{ in } \Omega. \tag{10.19}$$

Return to problem (10.17a)–(10.17b) and apply the conditions (10.19). We obtain the over determined problem

$$\begin{cases} b\Delta\zeta = -\omega^2\zeta \text{ in } \Omega & (10.20a) \\ \left.\dfrac{\partial\zeta}{\partial\nu}\right|_{\Gamma_1} = 0 \quad \zeta|_{\Gamma_0} = 0 & (10.20b) \end{cases}$$

which implies [40, Theorem 6.1, p.75]

$$\zeta \equiv 0 \tag{10.21}$$

and uniqueness of z_1 is established, from which uniqueness of z_2, η follows by (10.13a) and (10.13c). Thus we have shown that given $\{h_1, h_2, h_3\} \in \hat{\mathcal{H}}_0$, we have found a unique triple $\{z_1, z_2, \eta\} \in \mathcal{D}(\hat{L})$ such that

$$\begin{bmatrix} z_1 \\ z_2 \\ \eta \end{bmatrix} = (\hat{L} - i\omega)^{-1} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \tag{10.22}$$

for $\omega \neq 0$. The case $\omega = 0$ was shown in Lemma 10.2. Thus Proposition 10.4 is established. ■

Step 4 Thus $\hat{L}$ is the generator of a s.c contraction semigroup $e^{\hat{L}t}$ in $\hat{\mathcal{H}}_0$ and $i\mathbb{R} \in \rho(\hat{L})$. It follows by the [42,43] theorem that

Theorem 10.5: *The s.c. contraction semigroup $e^{\hat{L}t}$ in $\hat{\mathcal{H}}_0$ is strongly stable*

$$e^{\hat{L}t} \begin{bmatrix} z_{10} \\ z_{20} \\ \eta_0 \end{bmatrix} = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \eta(t) \end{bmatrix} \to 0 \textit{ in } \hat{\mathcal{H}}_0 \textit{ as } t \to \infty. \tag{10.23}$$

Step 5 We next employ (part of Theorem 10.5) to show that Theorem 6.1(b) holds true

$$\begin{bmatrix} y(t) \\ y_t(t) \\ y_{tt}(t) \end{bmatrix} = e^{G_{N,F}t} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \to 0 \text{ as } t \to +\infty \text{ in } U_1 = \mathcal{D}(A_N^{1/2}) \times \mathcal{D}(A_N^{1/2}) \times H \tag{10.24}$$

where $\mathcal{D}(A_N^{1/2}) = H^1_{\Gamma_0}(\Omega)$ and $H = L^2(\Omega)$. Let $\{y_0, y_1, y_2\} \in U_1$, $z_0 = \frac{c^2}{b}y_0 + y_1 \in \mathcal{D}(A_N^{1/2}), z_1 = \frac{c^2}{b}y_1 + y_2 \in L^2(\Omega)$. By (10.23) of Theorem 10.5 we have

$$\|z(t; z_1, z_2)\|_{\mathcal{D}(A_N^{1/2})} + \|z_t(t; z_1, z_2)\|_{L^2(\Omega)} \to 0 \text{ as } t \to \infty. \tag{10.25}$$

We shall see that it will suffices to show that

$$\|y(t)\|_{\mathcal{D}(A_N^{1/2})} \to 0 \quad \text{as } t \to \infty \tag{10.26}$$

for then strong stability of $\|y_t(t)\|_{\mathcal{D}(A_N^{1/2})} = \|z(t) - \frac{c^2}{b}y(t)\|_{\mathcal{D}(A_N^{1/2})} \to 0$ would then follow from (10.26) and (10.25); and this is turn via again (10.25) will imply strong stability of $\|y_{tt}(t)\|_{L^2(\Omega)} =$

$\|z_t(t) - \frac{c^2}{b}y_t\|_{L^2(\Omega)} \to 0$. To establish (10.26) we return to (3.4a). For $0 < T < t$, we rewrite it as

$$y(t) = e^{-\frac{c^2}{b}t}y_0 + e^{-\frac{c^2}{b}t}\int_0^T e^{\frac{c^2}{b}\tau}z(\tau)\,\mathrm{d}\tau + e^{-\frac{c^2}{b}t}\int_T^t e^{\frac{c^2}{b}\tau}z(\tau)\,\mathrm{d}\tau. \tag{10.27}$$

For fixed I.C., given $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $\|z(\tau)\|_{\mathcal{D}(A_N^{1/2})} \leqslant \varepsilon$, for all $\tau > T_\varepsilon$. Thus, with $T = T_\varepsilon$ identity (10.27) yields for all $t \gg T_\varepsilon$:

$$\|y(t)\|_{\mathcal{D}(A_N^{1/2})} \leqslant e^{-\frac{c^2}{b}t}\|y_0\|_{\mathcal{D}(A_N^{1/2})} + \frac{b}{c^2}e^{-\frac{c^2}{b}t}\|z\|_{L^\infty(0,T_\varepsilon;\mathcal{D}(A_N^{1/2}))}(e^{\frac{c^2}{b}T_\varepsilon} - 1) + \frac{\varepsilon b}{c^2}\left(1 - e^{-\frac{c^2}{b}(t-T_\varepsilon)}\right) = \mathcal{O}(\varepsilon). \tag{10.28}$$

Thus (10.26) is proved, and so is Theorem 6.1(b).

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Appendices

Appendix 1. The general case $0 \neq \gamma \in L^\infty(\Omega)$ in the Proof of Theorem 2.4. Handling lower order terms

We return to Step 1 of the (first) proof of the Theorem 2.4 in Section 3.1, where we have explained that we took at first $\gamma \equiv 0$ to simplify the computations. In this appendix we indicate the (benign) modifications over the content of Section 3.1, needed to handle the general case $0 \neq \gamma \in L^\infty(\Omega)$ in (2.39a). We may proceed in two ways:

First way (minimal re-writing). We now call (initially) $\tilde{C}(t)$ the cosine operator corresponding to the operator $b\Delta - \gamma\partial_t + \gamma\frac{c^2}{b}$ plus B.C or

$$\begin{bmatrix} 0 & I \\ b\Delta + \frac{c^2}{b} & -\gamma \end{bmatrix} = \text{generator of a group } \tilde{G}(t) = \begin{bmatrix} \tilde{C}(t) & \tilde{S}(t) \\ \tilde{C}'(t) & \tilde{C}(t) \end{bmatrix}.$$

Then the only extra term in (2.39a) that needs to be handled is $-\gamma(\frac{c^2}{b})y$ where $y(t) = \int_0^t e^{-\frac{c^2}{b}(t-\tau)}z(\tau)\,d\tau$. We shall only argue for the regularity of $z(\cdot)$, which in turn leads to the regularity of $z_t(\cdot)$. We obtain the variations of (3.7a)–(3.7b), (3.8a)–(3.8b)

$$z(t) = z^{(1)}(t) + z^{(2)}(t) - \gamma\left(\frac{c^2}{b}\right)\int_0^t \tilde{\mathcal{S}}(t-\tau)y(\tau)\,d\tau \tag{A1.1}$$

$$z_t(t) = z_t^{(1)}(t) + z_t^{(2)}(t) - \gamma\left(\frac{c^2}{b}\right)\int_0^t \tilde{C}(t-\tau)y(\tau)\,d\tau \tag{A1.2}$$

Equations (A1.1) and (A1.2) are the new versions corresponding to (3.10b) and (3.18),respectively, for $\gamma = 0$. Compute the new term, changing the order of integration

$$\int_0^t \tilde{\mathcal{S}}(t-\tau)y(\tau)\,d\tau = \int_0^t \tilde{\mathcal{S}}(t-\tau)\int_0^\tau e^{-\frac{c^2}{b}(\tau-s)}z(s)\,ds\,d\tau = \int_0^t\int_s^t \tilde{\mathcal{S}}(t-\tau)e^{-\frac{c^2}{b}(\tau-s)}z(s)\,d\tau\,ds$$

thus

$$-\gamma\left(\frac{c^2}{b}\right)\int_0^t \tilde{\mathcal{S}}(t-\tau)y(\tau)\,d\tau = -\gamma\left(\frac{c^2}{b}\right)\int_0^t K_1(t,s)z(s)\,ds$$

with smooth kernel

$$K_1(t,s) = e^{\frac{c^2}{b}s}\int_s^t \tilde{\mathcal{S}}(t-\tau)e^{-\frac{c^2}{b}\tau}\,d\tau$$

Thus, the new version of $z(t)$ for $\gamma \neq 0$ is

$$z(t) = z^{(1)}(t) + z^{(2)}(t) - \gamma\left(\frac{c^2}{b}\right)\int_0^t K_1(t,s)z(s)\,ds \tag{A1.3}$$

an integral equation in $z(\cdot)$. Equation (A1.3) is the definitive counterpart of Equation (3.10b) for $\gamma = 0$. Then, either by integral equation theory or by Gronwall inequality, we find that

$$z^{(1)}(t) + z^{(2)}(t) \in C([0,T];H^{-1}(\Omega)) \implies z(t) \in C([0,T],H^{-1}(\Omega)) \tag{A1.4}$$

as the regularity of $z^{(i)}(t)$ under the new $\tilde{C}(\cdot)$ is the same as the regularity of $z^{(i)}(t)$ under the original $C(\cdot)$, established in (3.15b), as corresponding to $\gamma = 0$. If one wishes to appeal to a Grownwall inequality, recall from [19, p. 27, 28] that $\|\tilde{\mathcal{C}}(t)\| \leqslant Ce^{\omega|t|}$ and thus $\|\tilde{\mathcal{S}}(t)\| \leqslant \frac{C(e^{\omega|t|}-1)}{\omega}$, $-\infty < t < +\infty$. This yields $K_1(t,s) \leqslant Ce^{\omega(t-s)}$, $s < t$.

Second way In (2.39a) with $\gamma \in L^\infty(\Omega)$, we keep $\mathcal{C}(t)$ to be the cosine operator generated only by the operator $b\Delta$ (as in the proof of Theorem 2.2) in which case the new form of $z(t)$ is now

$$z(t) = z^{(1)}(t) + z^{(2)}(t) + \gamma\left(\frac{c^2}{b}\right)\int_0^t \mathcal{S}(t-\tau)z(\tau)\,d\tau - \gamma\int_0^t \mathcal{S}(t-\tau)z_t(\tau)\,d\tau - \gamma\left(\frac{c^2}{b}\right)\int_0^t \mathcal{S}(t-\tau)y(\tau)\,d\tau$$

while

$$\int_0^t \mathcal{S}(t-\tau)z_t(\tau)\,d\tau = -\int_0^t \mathcal{C}(t-\tau)z(\tau)\,d\tau$$

integrating by parts with $z(0) = 0$. In conclusion:

$$\begin{aligned} z(t) &= z^{(1)}(t) + z^{(2)}(t) + \text{a new integral term in } z(t) \\ &= \text{integral equation.} \end{aligned}$$

Appendix 2. Direct proof of Theorem 6.1(c) in the canonical case $\gamma = 0$.

We shaw assume the same hypothesis of Theorem 6.1(c) and in addition, that $\gamma = 0$. In this case, a short proof can be given by applying directly to known results [23].

Step 1 With reference to problem (7.2a)–(7.2d) with $\gamma = 0$ [23, Theorem 1.2], (under the noted geometrical condition on $\{\Omega, \Gamma_0, \Gamma_1\}$) yields the uniform decay

$$\left\| \begin{bmatrix} z(t) = \alpha y(t) + y_t(t) \\ z_t(t) = \alpha y_t(t) + y_{tt}(t) \end{bmatrix} \right\|_{\mathcal{D}(A_N^{1/2}) \times L^2(\Omega)} \leqslant Me^{-\delta t} \left\| \begin{bmatrix} z_0 = \alpha y_0 + y_1 \\ z_1 = \alpha y_1 + y_2 \end{bmatrix} \right\|_{\mathcal{D}(A_N^{1/2}) \times L^2(\Omega)}, \quad t \geqslant 0. \tag{A2.1}$$

Step 2 We finally need to show that on the trajectories $\{z(t), z_t(t)\}$ and $\{y(t), y_t(t), y_{tt}(t)\}$ related by (7.1), the decay (A2.1) implies the decay (6.6) for I.C. $\{y_0, y_1, y_2\} \in U_1$ in (6.4). In fact, in this case, by (7.1)

$$\|y(t)\|_{\mathcal{D}(A_N^{1/2})} \leqslant e^{-\alpha t} \|y_0\|_{\mathcal{D}(A_N^{1/2})} + \int_0^t e^{-\alpha(t-\tau)} \|z(\tau)\|_{\mathcal{D}(A_N^{1/2})} \, d\tau \tag{A2.2}$$

and invoking (A2.1) and $U_1 \equiv \mathcal{D}(A_N^{1/2}) \times \mathcal{D}(A_N^{1/2}) \times H$ from (6.4)

$$\int_0^t e^{-\alpha(t-\tau)} \|z(\tau)\|_{\mathcal{D}(A_N^{1/2})} \, d\tau \leqslant \int_0^t e^{-\alpha(t-\tau)} Me^{-\delta\tau} \, d\tau \, \|\{z_0, z_1\}\|_{\mathcal{D}(A_N^{1/2}) \times L^2(\Omega)}$$

$$\leqslant M \left[\frac{e^{-\delta t} - e^{-\alpha t}}{\alpha - \delta} \right] \|\{z_0, z_1\}\|_{\mathcal{D}(A_N^{1/2}) \times L^2(\Omega)} \tag{A2.3}$$

$$= M_1 e^{-\min\{\delta, \alpha\}} \|\{y_0, y_1, y_2\}\|_{U_1} \tag{A2.4}$$

recalling $z_0 = \alpha y_0 + y_1$ and $z_1 = \alpha y_1 + y_2$ from (7.2b). Substituting (A2.4) in (A2.2) yields

$$\|y(t)\|_{\mathcal{D}(A^{1/2})} \leqslant M_2 e^{-at} \|\{y_0, y_1, y_2\}\|_{U_1}, \quad a = \min\{\alpha, \delta\}. \tag{A2.5}$$

Next, by invoking again (A2.1) as well as (A2.5) we obtain by (7.1)

$$\|y_t(t)\|_{\mathcal{D}(A^{1/2})} = \|z(t) - \alpha y(t)\|_{\mathcal{D}(A^{1/2})} \leqslant M_3 e^{-at} \|\{y_0, y_1, y_2\}\|_{U_1} \tag{A2.6}$$

and similarly, again by (A2.4) and now by (A2.6)

$$\|y_{tt}(t)\|_{L^2(\Omega)} = \|z_t(t) - \alpha y_t(t)\|_{L^2(\Omega)} \leqslant M_4 e^{-at} \|\{y_0, y_1, y_2\}\|_{U_1}. \tag{A2.7}$$

Thus, (A2.5)–(A2.7) prove (6.7).