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BOUNDARY STABILIZATION OF THE LINEAR MGT EQUATION WITH PARTIALLY ABSORBING BOUNDARY DATA AND DEGENERATE VISCOELASTICITY

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Abstract. The Jordan{Moore{Gibson{Thompson (JMGT) equation is a well-established and recently widely studied model for nonlinear acoustics (NLA). It is a third{order (in time) semilinear Partial Dierential Equation (PDE) with a distinctive feature of predicting the propagation of ultrasound waves at nite speed. This is due to the heat phenomenon known as second sound which leads to hyperbolic heat-wave propagation. In this paper, we consider the problem in the so called \critical" case, where free dynamics is unstable. In order to stabilize, we shall use boundary feedback controls supported on a portion of the boundary only. Since the remaining part of the boundary is not \controlled", and the imposed boundary conditions of Neumann type fail to saitsfy Lopatinski condition, several mathematical issues typical for mixed problems within the context o boundary stabilizability arise. To resolve these, special geometric constructs along with sharp trace estimates will be developed. The imposed geometric conditions are motivated by the geometry that is suitable for modeling the problem of controlling (from the boundary) the acoustic pressure involved in medical treatments such as lithotripsy, thermotherapy, sonochemistry, or any other procedure involving High Intensity Focused Ultrasound (HIFU).

1. Introduction.

It was not until the rst decade of the XXI century that third{order in time models became central in the study of the propagation of acoustic waves. Fattorini [14] points out that models with three time derivatives are, in general, ill-posed. Nevertheless, from a modelling point of view, the appearance of a third derivative in time seems unavoidable. For once, if one seeks to understand the eects of (thermal) relaxation in the propagation of sound, a (by now) well{known strategy is the use of hyperbolic models for the heat ux (also known as second{sound phenomena), which introduce one extra time derivative [15, 16] in explicit models. Even more essential for controlling medical or engineering (acoustic) phenomena is the fact that the presence of a third time derivative predicts nite speed

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of propagation of the waves, a novelty in comparison to the classic parabolic mod-els where heat uxes are modeled through diusion (Fourier's law). The issue of wellposedness is naturally remedied in modern formulations of nonlinear acoustics { in particular models leading to the so called HIFU eld { due to the structural damping eect caused by sound diusivity in a given tissue or group of tissues [20]. This pleasant feature allows for a better understanding of the role of sound diu-sion and propagation in the acoustic environment. Instead, classical second{order in time models (see (2) below) lead to strong smoothing of solutions exhibited by the analyticity of the underlying dynamics.

Studies toward more accurate calibration of HIFU eld generator devices are plenty, specially in the last few decades. Such devices are pivotal for several types of thermal therapy as treatment of ablating solid tumors of the prostate, liver, breast, kidney, brain, among others. The feature of raising the temperature of a focal region very rapidly with minimal damage to the biological material around it comes at the price of very high (sometimes even with formation of shocks) acoustic pressure [7]. It is, therefore, of paramount interest the study of models that provide suitable (optimal) proles for the HIFU devices ensuring that the acoustic pressure will remain within safety range. In fact, in recent years we have witnessed a large body of work dealing with the questions of wellposedness and stability of third order dynamics, in both linear and nonlinear versions [21, 17] and on bounded [27] and unbounded (R^n) domains [29]. However, very little is known regarding how the third order model responds to the inputs from the boundary-particularly with low regularity. This particular interest needs no defense, given prevalence of boundary control problems (imaging, HIFU) associated with acoustic waves that can be ac-tuated just on the boundary of the spatial region. Since the model itself can be seen as a hyperbolic system [4] { which is however characteristic { one may expect mathematical interest and the associated challenges. Of great physical and mathe-matical interest are issues such as wellposedness with low regularity boundary data and a potential stabilizing eect of boundary damping. The latter is particularly of interest in the case when natural viscoelastic damping (strongly compromised by the second sound phenomenon) is either very weak or even non{existent.

This paper accomplishes an important step towards the described goal, namely a boundary stabilizability property for the linearized third{order in time acoustic wave models with degenerated viscoelastic eects and with boundary dissipation located on a suitable portion of the boundary. One of the salient feature is the fact that part of the boundary subject to Neumann boundary conditions is not ob-served/dissipated - in line with the conguration expected from applications to boundary control. Unobserved Neumann part of the boundary (rather then Dirich-let where suitable methods have been well developed) is known as causing major challenges in the derivation of observability estimates { even in the case of the wave equation [24]. This diculty is dealt with by using suitable geometric and microlocal analysis constructions applicable to the third{order in time models.

1.1. PDE model and motivation. We assume that the acoustic pressure $u = u(t;x)$ at the material point x 2 R^d (d = 2 or 3) and instant t 2 R_+ obeys the Jordan{Moore{Gibson{Thompson equation

$$u_{ttt} + (\quad 2ku)u_{tt} \quad c^2 u \quad (+ c^2)u_t = 2ku^2; \qquad {}_t \qquad (1)$$

where $c; ; k >$ 0 are constants representing the speed and diusivity of sound and a nonlinearity parameter, respectively. The function :
! R represents the natural

frictional damping provided by the medium. The parameter > 0 represents the thermal relaxation time and its presence allows for a more precise distinction within propagation of sound in dierent media. Indeed, the semilinear equation is a (singular perturbation) renement of the classical quasilinear Westervelt's equation ($= 0$):

$$(\quad 2ku)u_{tt} \quad c^2 u \quad u_t = 2ku^2_t : \tag{2}$$

Although not unique, one interesting way of obtaining (1) from a similar procedure as the one to obtain (2) is simply to use Maxwell{Cattaneo law [13, 5, 6] in place of Fourier's law. The advantage of this strategy (which is by no means physics{proof [31, 8]) is that it provides a suitable model for studying relaxation eects. Since waves propagate at a nite speed, it allows the construction of optimal policies for controlling the HIFU eld. Overall, in its simplicity, (1) catches most of the key features that would be present in a more detailed model.

The mathematical study of (1) as well as the dierences (and similarities) when compared to (2) started around 2010 with the works of I. Lasiecka, R. Triggiani and B. Kaltenbacher [18, 19, 27] where the issues of wellposedness and stability of solutions under homogeneous Dirichlet and Neumann boundary data were addressed for both nonlinear and linearized dynamics. The obtained results depend critically on the positivity of the stability parameter

$$(x) \quad (x) \quad \frac{c^2}{b} \quad {}_0 > 0 \text{ a.e. in } \quad : \tag{3}$$

In addition to ensuring uniform exponential decays of solutions for the linearized ($k = 0$) problem, condition (3) allows for the construction of nonlinear ows via \barrier's" method. In face of such results, the natural question is: what if (x) is no longer positive? It is known that if < 0 one may have chaotic solutions [10]. If 0 then the energy is conserved [19, 20]. This raises an interesting question on how to ensure stability of the dynamics when the frictional parameter degenerates (x) 0. It has been recently shown that adding viscoelastic eects produces in some cases the asymptotic decay of the energy, cf. e.g. [25, 11, 12]. In this work we concentrate on boundary stabilization. This is also motivated by recent consideration of control problems dened for MGT dynamics [9, 3]. By actuation { say on the boundary { one aims at obtaining a desired outcome measured by certain functional cost. It is well known that control problems { particularly on innite horizon { are strongly linked with stabilizability properties of the linearized model. One interesting problem , considered by Clason-Kaltenbacher in [9] is that of actuating the external part of the boundary through a transducer[1] with the aim at targeting acoustic signal on a given area inside the domain, cf. Figure 1. Such conguration will call for stabilizing eects emanating from uncontrolled part of the boundary, say $_1$, while the actuation itself will take place on the remaining { accessible to the user { part of the boundary, say $_0$, which is not subjected to dissipation or absorption. This conguration leads to the following model.

[1] A transducer is a device that takes power from one source and supplies power usually in another form to a second system. In the particular case of HIFU processes, the transducer concentrates the energy generated by the vibration of sound in a given medium and delivers it to a targeted area in form of heat.

$$
\begin{cases}
u_{ttt} + \alpha(x)u_{tt} - c^2\Delta u - b\Delta u_t = f & \text{in } Q = (0,T]\times\Omega \quad \text{(4a)}\\
u(0) = u_0; \quad u_t(0) = u_1; \quad u_{tt}(0) = u_2 & \text{in } \Omega \quad \text{(4b)}\\
\partial_\nu u + \gamma_0(x)u = 0 & \text{in } \Sigma_0 = (0,T]\times\Gamma_0 \quad \text{(4c)}\\
\partial_\nu u + \gamma_1(x)u_t = 0 & \text{in } \Sigma_1 = (0,T]\times\Gamma_1 \quad \text{(4d)}
\end{cases}
$$

with $\alpha, c, b > 0$; $\alpha \in L^\infty(\Omega)$, $\gamma_0 \in L^\infty(\Gamma_0)$ and $\gamma_1 \in L^\infty(\Gamma_1)$; $\gamma_1(x) \geq \gamma_1 > 0$; $\gamma_0 > 0$ a.e.

Assuming for the time being that a solution $u = u(t;x)$ for (4a)–(4d) exists in a suitable topology, and critical parameter admits degeneracy $\alpha(x) \geq 0$, our goal is to study its asymptotic properties as $t \to \infty$. More precisely, we want to show that for large times the acoustic pressure will be small, i.e., $\lim_{t\to\infty} u(t;\cdot) = 0$, hopefully at exponential rate.

It should be noted that boundary stabilization of linear MGT has been studied recently. However, the existing results [1] and [2] do not allow for un-dissipated Γ_0 with Neuman-Robin boundary conditions and degenerate viscoelasticity. The latter provides for major mathematical challenge (even in the case of wave equation). This is due to the fact that boundary conditions on Γ_0 fail to satisfy strong Lopatinski conditions. On the other hand, control problems under consideration call for Γ_0 to be an active (rather than passive) wall where control actuation takes place. From the mathematical point of view, this new scenario requires drastically different strategies and constructions.

This brings us up to the main topic of this paper: stabilization problem closely related to optimal control problems for MGT in infinite time horizon and with Neumann boundary feedback supported only partially on Γ. In order to motivate our assumptions on the geometry we look at Figure 1 where a schematic transducer is represented. The only needed (and realistic) assumption is the convexity of the red part, which we will call Γ_0: The other portion of the boundary, Γ_1, will be assumed to be "smooth". The schematic representation is given in Figure 2.

From the practical point of view, the quantity $\alpha(x)$ is interpreted as the viscoelasticity at the material point $x \in \Omega$ and, in particular in the medical field, is not expected to be known for all points of Ω. By making the more physically relevant assumption that $\alpha \in L^\infty(\Omega)$, $\alpha(x) \geq 0$ a.e. in Ω (allowing the critical case $\alpha \equiv 0$, or the case where measurements can only me made at isolated points of the domain), we ask ourselves whether a non–invasive (boundary) action can drive the pressure to zero at large times regardless of the particular knowledge of α (as long as it is nonnegative). This question was answered in [1, 2] with the final conclusion that in the case Dirichlet zero boundary conditions are assumed on Γ_0, which is also star-shaped, the dissipative boundary effects assumed on Γ_1 is strong enough to stabilize the system regardless of the particular structure of $\alpha \geq 0$

The present paper addresses the problem: what happens when boundary conditions on Γ_0 are of Neuman-Robin type [Lopatinski condition fails]? This allows to place actuators on Γ_1. Thus we keep the dissipative Neumann boundary condition on Γ_1 and supplement Γ_0 with a homogeneous Robin boundary data (see (4c)). Our result states that uniform stability still holds provided, however, that Γ_0 is convex in addition to being star-shaped. If one considers a "benchmark" optimal control

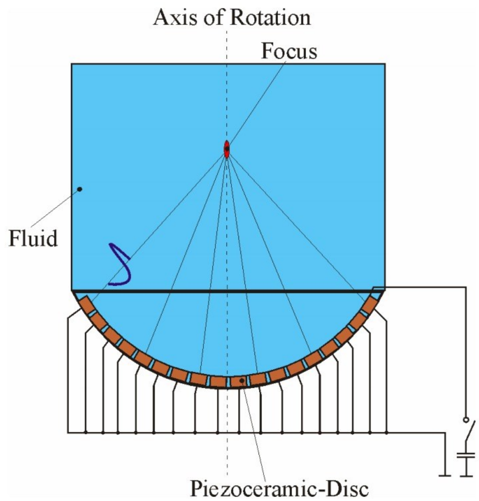

Figure 1. Illustration of the domain. The "red" convex portion of the boundary, in the context of HIFU, represents a device called transducer and its role is to concentrate the sound waves in the direction of the focus. The remaining part of the boundary represents an absorption area. (Font: B. Kaltenbacher)

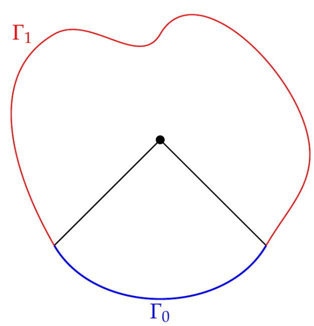

Figure 2. Representation of the domain

problem:

$$\begin{cases} \min_g J(u) := \int_0^\infty \|\nabla u\|^2 + \|g\|^2_{\Gamma_0} & \text{(5a)} \\ \text{subject to (4a); (4b); (4d) and replacing (4c) by } \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_0} = g & \text{(5b)} \end{cases}$$

then showing that (4a)–(4d) is uniform exponentially stable proves a stabilizability property for the control problem introduced above. Indeed, one takes $g = u|_{\Gamma_0} \in L^2(\Gamma_0)$ as a stabilizing feedback. This means that at least one strategy (control) exists capable of stabilizing the system on infinite horizon. We note that related optimal control problem subject to "smooth" controls and finite time horizon has been considered in [9]. Our point is to address the case of nonsmooth controls –

just L^2 controls – defined on infinite time horizon. This leads to new mathematical developments in the area of boundary stabilizability.

2. **Main results.** We begin by introducing a phase (finite energy) space, the abstract version of (4a)–(4d) along with some notation. We will work on a phase space $\mathbb{H}$ given by

$$\mathbb{H} := H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \tag{6}$$

To proceed, let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator defined as

$$\mathcal{A} = -\Delta, \quad \mathcal{D}(\mathcal{A}) = \left\{ \psi \in H^2(\Omega);\ \partial_\nu \psi|_{\Gamma_1} = 0; [\partial_\nu \psi + \gamma_0 \psi]_{\Gamma_0} = 0 \right\} \tag{7}$$

In this setting, $\mathcal{A}$ is a positive, self-adjoint operator with compact resolvent and $\mathcal{D}\left(\mathcal{A}^{1/2}\right) = H^1(\Omega)$ (equivalent norms). In addition, with some abuse of notation we (also) denote by $\mathcal{A} : L^2(\Omega) \to [\mathcal{D}(\mathcal{A})]'$ the extension (by duality) of the operator $\mathcal{A}$.

Next, we write (4a)–(4d) as a first–order abstract system on $\mathbb{H}$. To this end we need to introduce Harmonic (boundary $\to$ interior) extensions for the Neumann data on Γ_1. We proceed as follows: for $\varphi \in L^2(\Gamma_1)$, let $h := N(\varphi)$, be the unique solution of the elliptic problem

$$\begin{cases} \Delta h = 0 & \text{in } \Omega \\ \partial_\nu h = \varphi|_{\Gamma_1} & \text{on } \Gamma_1 \\ \partial_\nu h + \gamma_0 h = 0 & \text{on } \Gamma_0. \end{cases} \tag{8}$$

It follows from elliptic theory that $N \in \mathcal{L}(H^s(\Omega); H^{s+3/2}(\Omega))$[2] $(s \in \mathbb{R})$ and

$$N^*\mathcal{A}\psi = \begin{cases} \psi & \text{on } \Gamma_1 \\ 0 \text{ on } & \Gamma_0, \end{cases} \tag{9}$$

for all $\psi \in \mathcal{D}(\mathcal{A})$, where N^* represents the adjoint of N when the latter is considered as an operator from $L^2(\Gamma_1)$ to $L^2(\Omega)$. For the reader's convenience, we present a short proof of (9) since it will be used critically for the proof of Theorem 2.1(i). For $\psi \in \mathcal{D}(\mathcal{A})$ and $\varphi \in L^2(\Gamma_1)$, we first use the definition of adjoint followed by Green's second formula to obtain

$$(N^*\mathcal{A}\psi, \varphi) = (\mathcal{A}\psi, N\varphi) = -(\Delta\psi, N\varphi) = -(\psi, \Delta(N\varphi)) - (\partial_\nu\psi, N\varphi)_\Gamma + (\psi, \partial_\nu N\varphi)_\Gamma.$$

To complete we use the definition of $N\varphi$ as the solution of problem (8) and the fact that $\psi \in \mathcal{D}(\mathcal{A})$. This gives

$$\begin{aligned} (N^*\mathcal{A}\psi, \varphi) &= -(\psi, \Delta(N\varphi)) - (\partial_\nu\psi, N\varphi)_\Gamma + (\psi, \partial_\nu N\varphi)_\Gamma \\ &= (\gamma_0\psi, N\varphi)_{\Gamma_0} + (\psi, \partial_\nu N\varphi)_{\Gamma_0} + (\psi, \varphi)_{\Gamma_1} \\ &= (\psi, \gamma_0 N\varphi + \partial_\nu N\varphi)_{\Gamma_0} + (\psi, \varphi)_{\Gamma_1} = (\psi, \varphi)_{\Gamma_1}, \end{aligned} \tag{10}$$

~~which is preci~~sely (9).

Thus, the u–problem can be written (distributionally with the values in $[\mathcal{D}(\mathcal{A})]'$) as

$$u_{ttt} + \alpha(x)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t + c^2\mathcal{A}N(\gamma_1 N^*\mathcal{A}u_t) + b\mathcal{A}N(\gamma_1 N^*\mathcal{A}u_{tt}) = f. \tag{11}$$

Next, we introduce the operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \subset \mathbb{H} \to \mathbb{H}$ with the action (on $\Phi \cong (\varphi_1, \varphi_2, \varphi_3)^\top$):

$$\tilde{A} := (_2; _3; \quad _3 \quad c^2 A(_1 + N(_1 N A_2)) \quad bA(_2 + N(_1 N A_3))) \tag{12}$$

[2] $L(X; Y)$ denote the space of linear bounded operators from X to Y

and the domain

$$D(A) = \Big\{ \tilde{} 2 H; \ {}_3 2 D \ A^{1=2} \ ; \ {}_i + N({}_1 N A_{i+1}) \ 2 \ D(A); \ \text{for } i = 1;2 \Big\}$$

$$= \tilde{2} \ H^2(\ )^2 \ D \ A^{1=2} \ ; \ \frac{@ {}_1}{@} + {}_{01} = {}_0 \ \frac{@ {}_2}{@} + {}_{02} = 0_0$$

$$\frac{@ {}_1}{@} + {}_{12} = {}_1 \ \frac{@ {}_2}{@} + {}_{13} = 0_1 \ ) : \quad (13)$$

The rst order abstract version of u{problem is thus given by

$$\begin{cases} {}_t = A + F \\ (0) = {}_0 = (u_0; u_1; u_2)^{>}; \end{cases} \quad (14)$$

with $A : D(A) \ H \ ! \ H$ dened in (12) and $F^{>} = (0;0;f)$:

We are ready for our rst result.

Theorem 2.1. [Wellposedness and Regularity]

(i) The operator A generates a strongly continuous semigroup on H.

(ii) Let $f \ 2 \ L^1(0;T;L^2(\ ))$, ${}_{1;2} \ 0$ a.e and $2 \ L^1(\ )$. Then, for every initial data ${}_0 := (u_0;u_1;u_2)$ in H, there exists a unique (semigroup) solution $= (u;u_t;u_{tt})$ of (4a)-(4d) such that $2 \ C([0;T];H)$ for every $T > 0$: Moreover, if the initial datum belongs to $D(A)$ and $f \ 2 \ C^1([0;T];L^2(\ ))$ the corresponding solution is in $C((0;T];D(A)) \setminus C^1([0;T];H)$.

Our main result pertains to exponential decay of solutions asserted by Theorem 2.1 and requires geometric assumptions on the undissipated part of the boundary ${}_0$. We assume that ${}_0$ is convex, in the sense of being described by the level set of a convex function. In addition, we require that the following \star shaped" condition holds:

$$(x \ x_0) \ 0 \quad (15)$$

on ${}_0$ for some $x_0 \ 2 \ R^n$:

Theorem 2.2. [Uniform stability] Let $(x) \ 0$ and the geometric condition stated above holds true. The semigroup generated by A is exponentially stable, i.e., there exist constants $M \ 1; ! > 0$ such that

$$k(t)k_H \ Me^{\ !t} k_0 k_H \text{ for}$$

all ${}_0 \ 2 \ H$.

Remark 1. We note critical role being played by the fact that boundary conditions imposed are of Neumann type and there is nontrivial part of the boundary ${}_0$ which is not dissipated. It is known from the observability theory for wave equation, that standard techniques do not apply to uncontrolled Neumann parts of the boundary, see [24, 23] . The reason is that known multipliers do not collect the energy from this part of the boundary due to conicting sign of the vector eld on the undissipated part of the boundary. Recently, new geometric methods have been introduced in order to handle this diculty. We shall adapt these methods to the present system.

Conclusion. The result of Theorem 2.2 provides a positive answer to the question of exponential stabilizability of MGT equation in the critical case (0) via a boundary feedback supported only partially on with the requirement of convexity imposed on ${}_0$.

The remainder of this paper is devoted to the proofs of our main results.

3. Semigroup generation. We recall that, topologically, the space H introduced in the previous section is equivalent to

$$D \quad A^{1=2} \quad D \quad A^{1=2} \quad L^2(\quad) \tag{16}$$

with the topology induced by the inner product dened, for all $=\tilde{}\,(_1;_2;_3)^>;\tilde{} = ('_1;'_2;'_3)^>$ 2 H, as

$$;\tilde{}\ _H = (A^{1=2}{}_1; A^{1=2}{'_1}) + b(A^{1=2}{}_2; A^{1=2}{'_2}) + (_3;'_3): \tag{17}$$

Because of this equivalence we will be using the same H to denote both spaces. It is useful to notice that

$$(A^{1=2}u; A^{1=2}v) = (ru; rv) + {}_0 \int_0 uvd\ {}_0 \tag{18}$$

We need to show that A : D(A) H ! H generates a strongly continuous semigroup on H: It is convenient to introduce the following change of variables bz = $bu_t + c^2u$ (see [27]) which reduces the problem to a PDE{abstract ODE coupled system.

Let M 2 L(H) dened by

$$M\tilde{} = \quad _1;_2 + \frac{c^2}{b}{}_1;_3 + \frac{c^2}{b}{}_2$$

which has inverse $M^{\ 1}$ 2 L(H) given by

$$M^{\ 1}\tilde{} = \quad _1;_2 \quad \frac{c^2}{b}{}_1;_3 \quad \frac{c^2}{b}{}_2 + \frac{c^4}{b^2}{}_1$$

and therefore is an isomorphism of H. The next lemma makes precise the translation of u{problem to a dierent system involving a component of a suitable wave equation labeled by z:

Lemma 3.1. Assume that the compatibility conditions

$$\frac{@}{@}u_0 + {}_0u_0 = 0 \text{ on } {}_0; \qquad \frac{@}{@}u_0 + {}_1u_1 = 0 \text{ on } {}_1 \tag{19}$$

hold. Then 2 $C^1(0;T;H) \setminus C(0;T;D(A))$ is a strong solution for (14) if, and only if, = M 2 $C^1(0;T;H) \setminus C(0;T;D(A))$ is a strong solution for

$$\begin{cases} {}_t = A \quad + G \\ (0) = {}_0 = M_0 = \quad u_0; u_1 + \frac{c^2}{b}u_0; u_2 + \frac{c^2}{b}u_1 \quad ^> ; \end{cases} \tag{20}$$

where G = MF and A = MAM ^{1} with

$$D(A) = \Big(\tilde{}\,2\ H^2(\ )^2 \ D\ A^{1=2}\ ; \frac{@_2}{@} + {}_{02}\ {}_0 = 0; \ \frac{@_2}{@} + {}_{13}\ {}_1 = 0 \Big) \tag{21}$$

Proof. The only non-trivial step is to prove that boundary conditions (going from z to u{problem) match. To this end, assume that = $(u; z; z_t)$ 2 $C^1(0; T; H) \setminus C(0; T; D(A))$ is a strong solution for (20). Let

$$(t) := \frac{@u(t)}{@} + {}_0 u(t) \quad ; \; t \; 0_0$$

and notice that $b_t + c^2 = 0$ for all t. This along with the compatibility condition $(19)_1$ ((0) = 0) implies that 0. The same argument mutatis mutandis recovers the boundary condition for u on $_1$: The proof is then complete. □

For $\tilde{} = ({}_1; {}_2; {}_3)^>$ 2 $D(A)$ a basic algebraic computation yields the explicit formula for A:

$$A\tilde{} = \quad {}_2 \quad \frac{c^2}{b}{}_1; {}_3; \quad {}_3 \quad b \; {}_2 + \frac{c^2}{b^2}{}_1 \quad \frac{c^4}{} \qquad bA_2 \quad b_1 ANN \quad A_3 \tag{22}$$

where $= \frac{c^2}{b} \, 2 \, L^{1}($):

We are ready for our generation result.

Theorem 3.2. *The operator A generates a strongly continuous semigroup on H.*

Proof. Equivalently, we show that A generates a strongly continuous semigroup on H. If $fS(t)g_{t0}$ is the said semigroup then $fT(t)g_{t0}$, $T(t) := M^{\;1}S(t)M; t \; 0$, will be the semigroup generated by A :

Write $A = A_d + P$ where

$$P\tilde{} = \quad {}_2; 0; \; \frac{c^2}{b} \; {}_2 \quad \frac{c^2}{b}{}_1 \quad + (1 \quad){}_3 \quad ; \quad 2 \; \tilde{H}$$

is bounded in H and

$$A_d\tilde{} = \quad \frac{c^2}{b}{}_1; {}_3; \quad {}_3 \quad bA({}_2 + N({}_1 N A_3)) \quad ; \; 2 \; D(A_d\tilde{}); \tag{23}$$

where $D(A_d) := D(A)$: It then suces to prove generation of A_d on H, see [28, Page 76]

We start by showing dissipativity: for $\tilde{} = ({}_1; {}_2; {}_3)^>$ 2 $D(A)$ we have

$$\begin{aligned}
A_d\tilde{}; \tilde{}_H &= \frac{c^2}{b} kA^{1=2}{}_1 k^2_{L^2(\,)} + b \; A^{1=2}{}_3; A^{1=2}{}_2 \qquad p \; \text{—} \\
&\quad k_3 k^2_{L^2(\,)} \qquad 2 \\
&\quad b(A^{1=2}{}_2; A^{1=2}{}_3) \quad bk \quad {}_1{}_3 k_{L^2(\,{}_1)} \\
&= \frac{c}{b} kA^{1=2}{}_1 k_{L^2(\,)} \qquad 2 \qquad \text{—} \quad 2 \\
&\quad k_3 k_{L^2(\,)} \\
&\quad bk^p{}_1{}_3 k_{L^2(\,{}_1)} \quad 0;
\end{aligned}$$

hence, A_d is dissipative in H.

For maximality in H, given any $L = (f; g; h)$ 2 H we need to show that there exists $= ({}_1; {}_2; {}_3)^>$ 2 $D(A)$ such that (A_d) = L, for some > 0: This leads to a solvability of the system of equations:

$$\begin{cases} {}_1 + \frac{c^2}{b}{}_1 = f; \\ {}_2 \quad {}_3 = g; \\ {}_3 + {}_3 + bA({}_2 + N({}_1 N A_3)) = h; \end{cases} \tag{24}$$

which implies $_1 = \quad + \frac{c^{2 \quad 1}}{b} \quad f \; 2 \; D(A^{1=2})$: Moreover, since $A \;\; ^1 \; 2 \; L(L^2($
$))$ a combination of the second and third equations above yields

$$K_3 = A \;\; ^1 h \quad bg \tag{25}$$

where $K \;\; : L^2($
$) \; ! \;\; L^2($
$)$ acts on an element $\;2 \; L^2($
$)$ as

$$K \; = \; (^2 + \;)A \;\; ^1 + b(I + N(_1 N A) \;\; :$$

We now notice the restriction $Kj_{D(A^{1=2})}$ is strictly positive. Indeed it follows by (9) that, given $\;2 \; D(A^{1=2})$ we have

$$(K;\,)_{D(A^{1=2})} = (^2 + \;)kk^2 + bkA_2^{1=2}k^2 + b(A^{1=2}_{\;2}N(_1 N A); A^{1=2}) = (^2 + \;)kk^2 + bkA^{1=2}k^2 + bk_{2\;1} N A k^2_{\;\;1} > 0: \quad p \;\_\_ \; 2$$

Among the consequences of positivity, is the fact that $K_{D(A^1_{1=2})} \; 2 \; L(D(A^{1=2}))$. Therefore, since $A \;\; ^1 h \quad bg \; 2 \; D(A^{1=2})$ we have that

$$_3 := K \;^1(A \;\; ^1 h \quad bg) \; 2 \; D(A^{1=2}) \text{ is}$$

the solution of (25). Finally,

$$_2 = \quad ^1(_3 + \; g) \; 2 \; D(A^{1=2}):$$

For the nal step to conclude membership of $(_1;\,_2;\,_3)$ in $D(A)$ we look at the abstract version of the description of $D(A)$:

$$D(A) = \; \overset{n}{\tilde{\;}} 2 \; H; \;\; _2 + N(_1 N A_3) \; 2 \; D(A) \qquad ^{o}$$

whereby one only needs to check that $_2 + \; N(_1 N A_3) \; 2 \; D(A)$ The desired regularity will follow from (24), which implies

$$b(_2 + N(_1 N A_3)) = \quad (^2 + \;)A \;\; ^1{}_3 + \; A \;\; ^1 h \; 2 \; D(A);$$

since $_3; h \; 2 \; L^2($
$)$:

The proof is complete. □

Theorem 2.1(b) then follows as a straightforward corollary of Theorem 3.2 and Lemma (3.1).

4. Stabilization. Our stability results rely on a chain of estimates developed through the process of the proof whose main ingredient is to propagate dissipation from a portion of the boundary $\;_1$ into the entire domain. Let us rst outline the main conceptual ideas.

(i) In order to handle the estimates on $\;_0$ { the undissipated part of the boundary { a typical radial vector eld leads to conicting signs in front of tangential boundary-time derivative. In order to handle this, special vector elds are introduced which are constructed locally by \bending" tangentially the radial eld on the undissipated part of the boundary. This can be accomplished by exploiting convexity of $\;_0$ along with a general star shaped requirement. Having a vector eld which is tangential to the boundary allows us to annihilate the normal component of this vector eld { taking care of the tangential derivatives on the undissipated part of the boundary (note that in the Dirichlet case, the contribution on the undissipated part of the boundary is just zero).

(ii) On the absorbing part of the boundary $_1$ we use the fact that the time derivative of the solution is given through the energy relation. By applying microlocal analysis argument, one estimates the space-time tangential contribution of the solution in terms of the time time derivatives and some lower order terms.

(iii) The resulting lower order terms are eliminated by a suitable compactness-uniqueness argument.

We start by introducing the energy functional. We work with smooth (classical solutions) guaranteed by Theorem 2.1 and then Theorem 2.2 is obtained via density argument along with the convexity of the energy functional.

Let $(u; u_t; u_{tt}) \in H$ be a classical solution of (14) and recall the corresponding z{problem determined via (20), from which follows that $z = u_t + \frac{c^2}{b} u$ solves the equation

$$z_{tt} + bA(z_t + N(_1 N Az)) = u_{tt} + f; \tag{26}$$

with initial conditions described in (20).

With this notation, we dene the functional $E(t) = E_0(t) + E_1(t)$ where $E_i : [0; T] \to R_+$ $(i = 0; 1)$ are dened by

$$E_1(t) := \frac{b}{2}\|A^{1=2} z\|_2^2 + \frac{1}{2}\|z_t\|_2^2 + \frac{c^2}{2b}\|{}^{1=2} u_t\|_2^2 \tag{27}$$

and

$$E_0(t) := \frac{1}{2}\|{}^{1=2} u_t\|_2^2 + \frac{c^2}{2}\|A^{1=2} u\|_2^2 : \tag{28}$$

The next lemma guarantees that stability of solutions in H is equivalent to uniform exponential decay of the function $t \to E(t)$: One thing to notice is that $E_1(t)$ is dissipative along the unforced solution.This no longer holds for the full energy $E(t)$.

Lemma 4.1. Let $= (u; u_t; u_{tt})$ be a weak solution for the u{ problem in H and assume that (19) is in force. Then the following statements are equivalent:

a) $t \to \|(t)\|_H^2$ decays exponentially.
b) $t \to \|M(t)\|_H^2 = \|(u; z; z_t)\|_H^2$ decays exponentially.
c) $t \to E(t)$ decays exponentially.

Proof. Proof relies on algebraic manipulations. Details can be found in [1]. □

Remark 2. The purpose of Lemma (4.1) is that it allows us to use both the expression of the energy $E(t) = E_0(t) + E_1(t)$ and the norm of the solution $\|(u; z; z_t)\|_H^2$ interchangeably. The specic structure of the energy contributes to a discovery of certain invariances and dissipative laws. However, from the topological point of view, it is essential that the following three quantities $\|A^{1=2} z\|_2$, $\|z_t\|_2$ and $\|ru\|_2$ display the appropriate decays.

The next proposition provides the set of main identities for the linear stabilization in H:

Proposition 4.2. Let $T > 0$. If $(u; z; z_t)$ is a classical solution of (20) then the following holds

(i) (Energy Identity) For $0 \le t \le T$,

$$E_1(T) + b\int_t^T\int_{_1} z_t^2 \, d_1 \, ds + \int_t^T\int u_{tt}^2 \, d \, ds = E_1(t) + \int_t^T\int f z_t \, d \, ds: \tag{29}$$

(ii) Energy (L^1–norm)–Reconstruction. For $0 < s < T/2$,

$$\int_s^{T-s} E_1(t)dt \lesssim [E_1(s) + E_1(T-s)] + C_T\left(b\int_{\Sigma_1} z_t^2 d\Sigma_1 + \int_Q u_{tt}^2 dQ + \text{lot}(z)\right) + C_T\int_Q f^2 dQ. \tag{30}$$

where $\text{lot}(z) \leq C \sup_{t\in[0,T]}\left(\|z\|^2_{H^{1/2+\epsilon}(\Omega)} + \|z_t\|^2_{H^{-1/2+\epsilon}(\Omega)}\right)$, for $\epsilon > 0$.

Remark 3. Notice that the energy identity (29) involves only partial information on the dynamics. In fact $E_1(t)$ does not reconstruct $\|u\|_2$, a critical ingredient of the MGT system. The second inequality (30) is an "almost" reconstruction of partial energy in terms of the dissipation and lower order terms.

Proof. 1. Proof of (29). Let, on $\mathcal{H}$, the bilinear form $\langle\cdot,\cdot\rangle$ be given by

$$\langle\Phi,\tilde\varphi\rangle = b\left(A^{1/2}\phi_2, A^{1/2}\varphi_2\right) + (\phi_3,\varphi_3) + \frac{c^2}{b}\left(\phi_2 - \frac{c^2}{b}\phi_1, \varphi_2 - \frac{c^2}{b}\varphi_1\right) \tag{31}$$

for all $\tilde\phi = (\phi_1,\phi_2,\phi_3)^\top, \tilde\varphi = (\varphi_1,\varphi_2,\varphi_3)^\top \in \mathcal{H}$, which is continuous. Moreover, recalling that $\Phi(t) = (u(t), z(t), z_t(t))$ it follows that $2E_1(t) = \langle\Phi(t),\Phi(t)\rangle$. Therefore,

$$\begin{aligned}
\frac{dE_1(t)}{dt} &= \left\langle \frac{d\Phi(t)}{dt}, \Phi(t)\right\rangle_* = \langle \mathcal{A}\Phi(t) + G, \Phi(t)\rangle \\
&= \left\langle \left(z - \frac{c^2}{b}u, z_t, -z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}u - bA(z_t + N(\gamma_1 N^* Az)) + f\right)^\top, \Phi(t)\right\rangle \\
&= \left(-z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}u - bA(z_t + N(\gamma_1 N^*Az)) + f, z_t\right) \\
&\quad + b\left(A^{1/2}z_t, A^{1/2}z\right) + \frac{c^2}{b}\left(z_t - \frac{c^2}{b}\left(z - \frac{c^2}{b}u\right), z - \frac{c^2}{b}u\right) \\
&= \left(-z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}u, z_t\right) - b(A(z_t + N(\gamma_1 N^* Az)), z_t) \\
&\quad + b\left(A^{1/2}z_t, A^{1/2}z\right) + (f, z_t) + \frac{c^2}{b}\left(z_t - \frac{c^2}{b}\left(z - \frac{c^2}{b}u\right), z - \frac{c^2}{b}u\right) \\
&= -\int_\Omega \left(z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}u\right)\left(z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}u\right)d\Omega - b\int_{\Gamma_1} z_t^2 d\Gamma_1 + (f, z_t) \\
&= -\int_\Omega u_{tt}^2 d\Omega - b\int_{\Gamma_1} z_t^2 d\Gamma_1 + \int_\Omega f z_t d\Omega
\end{aligned}$$

since $z_t - \frac{c^2}{b}z + \frac{c^4}{b^2}u = u_{tt}$. Identity (29) then follows by an integration in time on (t, T).

2. Proof of (30) This second part will be established via multipliers technique making strong use of the geometrical conditions preceding the statement of Theorem 2.2. However, due to the fact that Neumann boundary conditions are imposed on the acoustic pressure u and the absorbtion of the energy (dissipation) occurs only on a portion (Γ_1) of it, standard radial multipliers used in observability theory of waves do not apply. There is a conflicting sign requirement for the vector field to be constructed [24, 22]. To resolve this issue one needs to construct a different

multiplier with the property that its Jacobian generates a positive metric, and at the same time complies with the conicting sign on $_1$. This has been accomplished (see for instance [24]) under the condition that the non-dissipative part of the boundary is convex and leads to a construction of a special vector eld $h \; 2 \; C^1$ which enjoys the following properties [22]

$$h \quad = 0 \text{ on } \quad _0; \qquad J(h(x)) \quad c_0 > 0; x \; 2$$

where $J(h)$ denotes the Jacobian matrix of the vector eld h. Such vector eld has been constructed in [24] based on the idea introduced in [32] and further generalised in [22] for domains with the properties: $_0$ is a convex part of which also satises star shaped condition:

$$(x \quad x_0) \quad 6 \; 0; \text{ on } \quad _0; \text{ for some } x_0 \; 2 \; R^n \tag{32}$$

The condition (32) guarantees that a suciently large portion of the boundary is under absorbtion. This is typical condition required by Moravetz-Strauss theory. However, convexity of $_0$ is a new requirement. This allows for a construction of suitable vector eld with the postulated properties. The construction is based on a perturbation [bending tangentially] of the radial vector eld. With that eld h in hand, we rst multiply equation (26) by $h \quad rz$ integrate by parts in $(s; T \quad s)$. This gives

$$\frac{b}{2}\int_s^{T} \int J(h)jrzj^2 d \; dt + \int \int z^2 \quad bjrzj^2 \quad div(h)d \; dt = \frac{1}{2}\int_s^{T} \int_t \ldots$$

$$\frac{1}{2}\int_s^{T}\int \ldots \; u_{tt}(h \quad rz)d \; dt \quad \int z_t(h \quad rz)d \; \ldots \quad \frac{1}{2}\int_s^{T} \int_s \; 2$$

$$+ \int_s^{T} \int_1 z_t \quad bjrzj^2 \quad (h \quad)d \quad _1 dt + \int_s^{T} \int_0 z_t \quad bjrzj^2 \quad (h \quad)d \quad _0 dt \tag{33}$$

$$+ \; b \int_s^{T} \int @z(h \quad rz)d \quad dt + \int_s^{T} \int f(h \quad rz)d \quad dt: \tag{34}$$

where we notice that the second term in (33) vanishes since $h \quad = 0$ on $_0$:

Next, we multiply equation (26) by $z \, div(h)$ and integrate by parts in $(s; T \quad s)$. This leads to

$$\frac{1}{2}\int_s^{T}\int bjrzj^2 \quad z_t^2 \quad div(h)d \; dt = \frac{b}{2}\int_s^{T}\int @zzdiv(h)d \quad dt \; \int_s^{T} \ldots$$

$$\frac{1}{2}\int \int u_{tt}z\,div(h)d \quad \frac{1}{2}\int \ldots \; dt \; \frac{b}{2} \int \int z_t z\,div(h)d \quad \frac{1}{2}\int \int$$

s

s

T s T s

z r z r(div(h))d

dt + f zdiv(h)d

dt: (35) s

s

Adding (34) with (35) we have

$$\frac{b}{2}\int_s^{T-s}\int_\Omega J(h)|\nabla z|^2 d\Omega\, dt = -\int_s^{T-s}\int_\Omega u_{tt}A_h(z)\,d\Omega\, dt - \left[\int_\Omega z_t A_h(z)\,d\Omega\right]_s^{T-s} + \int_s^{T-s}\int_\Omega f A_h(z)\,d\Omega\, dt \tag{36}$$

$$+\frac{1}{2}\int_s^{T-s}\int_{\Gamma_1}\left[z_t^2 - b|\nabla z|^2\right](h\cdot\nu)\,d\Gamma_1 dt + b\int_s^{T-s}\int_{\Gamma}\partial_\nu z A_h(z)\,d\Gamma\, dt - \frac{b}{2}\int_s^{T-s}\int_\Omega z\nabla z\cdot\nabla(\mathrm{div}(h))\,d\Omega\, dt \tag{37}$$

where $A_h(z) = h\cdot\nabla z + \frac{1}{2}z\,\mathrm{div}(h)$. Notice now that, we have the upper estimate $\|A_h z\|^2_{L^2(\Omega)} \lesssim \|\nabla z\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(\Omega)}$[3], which combined with Peter-Paul's inequality implies

$$\int_s^{T-s}\int_\Omega u_{tt}A_h(z)\,d\Omega\, dt \lesssim \varepsilon\frac{b}{2}\int_s^{T-s}\int_\Omega |\nabla z|^2 d\Omega\, dt + C_\varepsilon\left[\int_s^{T-s}\int_\Omega |u_{tt}|^2 d\Omega\, dt\right] + lot(z) \tag{38}$$

for $\varepsilon > 0$ to be precised later. Analogously, we deal with the last integral in the RHS of (37) as follows

$$\frac{b}{2}\int_s^{T-s}\int_\Omega z\nabla z\nabla(\mathrm{div}(h))\,d\Omega\, dt \lesssim \varepsilon\frac{b}{2}\int_s^{T-s}\int_\Omega |\nabla z|^2 d\Omega\, dt + C_\varepsilon lot(z). \tag{39}$$

Plugging (38) and (39) into (37) and choosing $\varepsilon < J(h)/2$ we conclude the following upper estimate for the potential energy of z

$$\frac{b}{2}\int_s^{T-s}\int_\Omega |\nabla z|^2 d\Omega\, dt \lesssim [E_1(s) + E_1(T)] + b\int_s^{T-s}\int_{\Gamma_1}|z_t|^2 d\Gamma_1 dt + \int_Q |u_{tt}|^2 dQ + \frac{b}{2}\int_s^{T-s}\int_{\Gamma_1}|\nabla z|^2(h\cdot\nu)\,d\Gamma_1 dt$$
$$+ b\int_s^{T-s}\int_{\Gamma}\partial_\nu z A_h(z)\,d\Gamma\, dt + \int_Q f^2 dQ + lot(z). \tag{40}$$

In order to obtain the estimate for the kinetic part of the energy, we multiply (26) by z and integrate by parts over $(s, T-s)\times\Omega$ to obtain

$$
Z\ {}_{T}\ Z\ \ b|rz|^{2}\quad z_t^2\ d\ \ \ Z\ {}_{T}\ Z_{s}\ \ \ Z\ {}_{T}\ Z_{s}
$$

$$
dt + \quad {}_0 z^2 d\ {}_0 dt = \quad {}_s\ f z d
$$

$$
dt\ {}^{s} Z\ {}^{s}_{T} \qquad \frac{b}{2} \qquad {}_s
$$

$$
Z \qquad Z\ {}^{s}_{T}\ {}_{0}\ Z\ {}^{s}_{T}
$$

$$
dt \quad u_{tt} z d \quad z_t z d + \quad {}_1 z^2 d\ {}_1 \quad : \qquad (41)\ {}_s
$$

$$
{}_s \qquad {}_1 \qquad {}_s
$$

[3] By $a\ .\ b$ we mean that there exists a constant $C > 0$ { possibly depending on any xed quantity of the problem: $c; b;{}_0;{}_1$ and $\max_{x\,2} |h(x)|$ { but independent of time and .

Identity (41) implies the following upper estimate for the kinetic energy

$$\frac{1}{2}\int_s^{T-s}\int_\Omega |z_t|^2 d\Omega\, dt \lesssim \frac{b}{2}\int_s^{T-s}\int_\Omega |\nabla z|^2 d\Omega\, dt + \frac{b}{2}\int_s^{T-s}\int_{\Gamma_0} |z|^2 d\Gamma_0 \tag{42}$$
$$+ [E_1(s) + E_1(T)] + b\int_{\Sigma_1} |z_t|^2 d\Sigma_1 + \int_Q u_{tt}^2 dQ + \int_Q f^2 dQ + lot(z).$$

Combining (40) with (42) and accounting for (18) we conclude

$$\begin{aligned}\int_s^{T-s} E_1(t)dt \lesssim{}& [E_1(s) + E_1(T-s)] + b\int_{\Sigma_1} |z_t|^2 d\Sigma_1 \\ &+ \int_Q u_{tt}^2 dQ + lot(z) + \Phi(s;T-s) + \int_Q f^2 dQ\end{aligned} \tag{43}$$

where

$$\Phi(s;T-s) = \frac{b}{2}\int_s^{T-s}\int_{\Gamma_1} |\nabla z|^2 (h\cdot\nu) d\Gamma_1 dt + b\int_s^{T-s}\int_\Gamma \partial_\nu z A_h(z) d\Gamma\, dt$$

and the boundary integral $\int_{\Gamma_0} |z|^2 d\Gamma_0$ resulting from (18) is included in $lot(z)$. The latter is by virtue of trace estimate and compact embedding $H^1(\Omega) \subset H^{1/2}(\Omega)$. The boundary integrals above are estimated next. Recalling the notation $A_h(z) = h\cdot\nabla z + \frac{1}{2} z\, \mathrm{div}(h)$, we have

$$\begin{aligned}\Phi(s;T-s) ={}& \frac{b}{2}\int_s^{T-s}\int_{\Gamma_1} |\nabla z|^2 (h\cdot\nu) d\Gamma_1 dt + b\int_s^{T-s}\int_\Gamma \partial_\nu z(h\cdot\nabla z) d\Gamma dt \\ &+ \frac{b}{2}\int_s^{T-s}\int_\Gamma \partial_\nu z z\, \mathrm{div}(h) d\Gamma dt \\ \equiv{}& \frac{b}{2} I_1 + b I_2 + \frac{b}{2} I_3.\end{aligned} \tag{44}$$

Moreover, we write the gradient at the boundary as

$$\nabla z = \underbrace{(\nabla z\cdot\nu)}_{\partial_\nu z}\nu + \sum_{i=1}^{n-1} \underbrace{(\nabla z\cdot\tau_i)}_{\partial_{\tau_i} z}\tau_i = (\partial_\nu z)\nu + \sum_{i=1}^{n-1} (\partial_{\tau_i} z)\tau_i;$$

which implies $|\nabla z|^2 = |\partial_\nu z|^2 + |\partial_\tau z|^2$. These along with the boundary conditions for the z–equation allow us to estimate the first integral in the RHS of (44) as follows

$$|I_1| \lesssim \int_{\Sigma_1} |z_t|^2 d\Sigma_1 + \int_s^{T-s} \|\partial_\tau z\|^2_{L^2(\Gamma_1)} dt. \tag{45}$$

The second integral I_2 is more involved. We first rewrite it as

$$\begin{aligned}I_2 ={}& \int_s^{T-s}\int_\Gamma \partial_\nu z(h\cdot\nabla z) d\Gamma_1 dt \\ ={}& \int_s^{T-s}\int_{\Gamma_1} |\partial_\nu z|^2 (h\cdot\nu) d\Gamma_1 dt + \int_s^{T-s}\int_{\Gamma_1} + \int_{\Gamma_0} \partial_\nu z\partial_\tau z(h\cdot\tau) d\Gamma dt \\ \equiv{}& I_{2;1} + I_{2;2} + I_{2;3}\end{aligned} \tag{46}$$

and then estimate the three resulting boundary integral terms. We notice that the boundary conditions for z on Γ_1 allow us to estimate $I_{2;1}$ as

$$I_{2;1} \lesssim \int_{\Sigma_1} |z_t|^2 d\Sigma_1. \tag{47}$$

and $I_{2;2}$ as

$$I_{2;2} = \int_s^{T-s}\int_{\Gamma_1} \partial_\nu z\,\partial_\tau z(h\cdot\tau)\,d\Gamma_1 dt \lesssim \int_{\Sigma_1} |z_t|^2 d\Sigma_1 + \int_s^{T-s} \|\partial_\tau z\|^2_{L^2(\Gamma_1)} dt. \tag{48}$$

For the remaining integral $I_{2;3}$ we use the boundary condition for z on Γ_0: $\partial_\nu z = -z$ and the trace theorem, which implies that $z|_{\Gamma_0} \in H^{1/2}(\Gamma_0)$. On the other hand, since $z \in H^1(\Omega)$ we have $\partial_\tau z \in H^{-1/2}(\Gamma)$. Hence, for $\varepsilon > 0$ we have

$$
\begin{aligned}
I_{2;3} &= \int_s^{T-s}\int_{\Gamma_0} \partial_\nu z\,\partial_\tau z(h\cdot\tau)\,d\Gamma_0 dt \\
&\lesssim \int_0^T \left(C_\varepsilon \|\partial_\nu z\|^2_{\frac{1}{2},\Gamma_0} + \varepsilon \|\partial_\tau z\|^2_{-\frac{1}{2},\Gamma_0} \right) dt \\
&\lesssim C_\varepsilon \int_0^T \|z\|^2_{\frac{1}{2},\Gamma_0} dt + \varepsilon \int_0^T \|z\|^2_{3/2,\Omega} dt \\
&\lesssim C_\varepsilon \int_0^T \|z\|^2_{H^{1/2+\varepsilon}(\Omega)} dt + \varepsilon \int_0^T \|z\|^2_{H^1(\Omega)} dt \\
&\lesssim \varepsilon \int_0^T \|\nabla z\|^2_{L^2(\Omega)} dt + C_T\, lot(z)
\end{aligned} \tag{49}
$$

for $\varepsilon > 0$ to be determined. Here also, the boundary term in (18) is included in a lower order term.

Plugging (47), (48) and (49) into (46) we estimate I_2 as

$$|I_2| \lesssim \int_{\Sigma_1} |z_t|^2 d\Sigma_1 + \int_s^{T-s} \|\partial_\tau z\|^2_{L^2(\Gamma_1)} dt + \varepsilon \int_0^T \|\nabla z\|^2_{L^2(\Omega)} dt + C_T\, lot(z) \tag{50}$$

Finally, integral I_3 is estimated as

$$|I_3| \lesssim \int_s^{T-s}\int_{\Gamma_1} \left(|\partial_\nu z|^2 + |z|^2\right) d\Gamma\, dt \lesssim \int_{\Sigma_1} |z_t|^2 d\Sigma_1 dt + lot(z). \tag{51}$$

Collecting (45), (50) and (51) and returning to (44) we conclude

$$
\begin{aligned}
|\Phi(s;T-s)| \lesssim{}& \int_{\Sigma_1} |z_t|^2 d\Sigma + \int_s^{T-s} \|\partial_\tau z\|^2_{L^2(\Gamma_1)} dt \\
&+ \varepsilon \int_0^T \|\nabla z\|^2_{L^2(\Omega)} dt + C_T\, lot(z).
\end{aligned} \tag{52}
$$

The tangential derivative above is estimated using an adaptation of Lemma 2.1 in [23], which was obtained for the homogeneous case,

$$\int_s^{T-s} \|\partial_\tau z\|^2_{0,\Gamma_1} dt \leq \ldots \tag{53}$$

$$C_T \quad {}_0 \quad k_@ z k_{0;} + k z_t k_{0;\ 1} dt + lot(z) + k u_{tt} + f k_H \ {}_{1=2+s(Q)} \quad :$$

Combining (52) and (53) we arrive at

$$|(s;T-s)| \lesssim \varepsilon \int_0^T \|\nabla z\|^2_{L^2(\Omega)} dt + C_T \Big[\int_{\Sigma_1} |z_t|^2 d\Sigma_1 + lot(z) + \|u_{tt} + f\|^2_{H^{-1/2+s}(Q)} \Big] \tag{54}$$

Returning with (54) to (43) and choosing $\varepsilon > 0$ properly we conclude (30). □

Our next result deals with lot(z), which can be absorbed by the damping using a compactness uniqueness argument.

Proposition 4.3. For $T > 0$ there exists a constant $C_T > 0$ such that the following inequality holds:

$$lot(z) \le C_T \Big[\int_{\Sigma_1} |z_t|^2 d\Sigma_1 + \int_Q |u_{tt}|^2 dQ \Big]. \tag{55}$$

Proof. As pointed out in (30), we have

$$lot(z) \le C \sup_{t \in [0,T]} \Big\{ \|z\|^2_{H^{1-\theta}(\Omega)} + \|z_t\|^2_{H^{-\theta}(\Omega)} \Big\};$$

for $\theta \in (0;1/2)$. Then we prove Proposition (4.3) as a corollary of the following Lemma

Lemma 4.4. For every $\theta \in (0;1/2)$, there exists a constant $C_{T,\theta} > 0$ such that

$$\|(z;z_t)\|^2_{L^2(0;T;H^{1-\theta}(\Omega)\times H^{-\theta}(\Omega))} \leqslant C_T \Big[\int_{\Sigma_1} |z_t|^2 d\Sigma_1 + \int_Q |u_{tt}|^2 dQ \Big] \tag{56}$$

Proof. Using the notation of [30], let $X = H^1(\Omega)$, $B = H^{1-\theta}(\Omega)$ and $Y = H^{-\theta}(\Omega)$: Then it follows from [26, Theorem 16.1] that the injection of X in B is compact. Moreover, since $\theta \in (0;1/2)$, [26, Theorem 12.4] allows us to write

$$Y = H^{-\theta}(\Omega) = [L^2(\Omega); H^{-1}(\Omega)]_\theta;$$

and then the injection of B in Y is continuous (even dense). Introduce the space as

$$\{v \in L^2(0;T;X); v_t \in L^2(0;T;Y)\}$$

equipped with the norm

$$\|v\| = \|v\|_{L^2(0;T;X)} + \|v_t\|_{L^2(0;T;Y)}:$$

Then it follows from [30] that the injection of W into $L^2(0;T;B)$ is compact. We are then ready for proving (56)

By contradiction, suppose that there exists a sequence of initial data $\{u_{0n}; u_{1n}; u_{2n}\}$ with corresponding $E_1^n(0)$ energy uniformly (in n) bounded generating a sequence $\{u_n; u_{t,n}; u_{tt,n}\}$ of solutions of problem (14) with related sequence

$$z_n = \frac{c^2}{b} u_n + u_{t,n};\; z_{t,n} = \frac{c^2}{b} u_{t,n} + u_{tt,n}$$

solutions of problem 20 such that

$$\begin{cases} \|z_n\|^2_{L^2(0;T;H^{1-\theta}(\Omega)} \ldots^2 \end{cases}$$

)) + $k\underline{z}_n k_{L^2(0;T;H}$ (

)) 1 (57a) $_2$ Z $_T$ Z Z $_T$ Z

: (u_n) dQ + $_1(\underline{z}_n)$ d_1 ! 0; as n ! + 1 : (57b)

0

0 1

From idenity (29) (with f = 0) we see that the uniform boundedness $E_1^n(0)$ implies uniform boundedness of $E_1^n(t)$, t 2 [0; T]. Therefore, one might choose a (non{relabeled) subsequence satisfying

$$z_n \; ! \quad \text{some} \; ; \; \text{weak in } L^1(0; T; H^1($$

$$)) \qquad (58a)$$

$$\underline{z}_n \; ! \quad \text{some} \;_1; \; \text{weak in } L^1(0; T; L^2($$

$$)) \; , ! \; L^2(0; T; H \quad ($$

$$)); \qquad (58b)$$

$$^{1=2}\underline{u}_n \; ! \quad \text{some} \; ; \; \text{weak in } L^1(0; T; L^2($$

$$)); \qquad (58c)$$

It easily follows from distributional calculus that $\_ = \;_1$ and, in the limit, the functions and satisfy the equation

$$8 \quad \bullet = b \quad ^{1=2}\_ \qquad \text{in } Q \qquad (59a)$$

$$> \;_{1=2} = \frac{c^2}{b} + \_ \qquad (59b)$$

$$? \quad \frac{@}{@} + \;_1 \;_1 = 0; \qquad \frac{@}{@} + \;_0 \;_0 = 0: \qquad (59c)$$

plus respective initial data.

It follows from the weak convergence that there exist M independent of n such that

$$k(z_n; \underline{z}_n)k_{L^1(0;T;H^1()H \; ())} = kz_nk \; 6 \quad M; \qquad (60)$$

for all n. Then, by compactness (of in $L^2(0; T; H^1$ ( )) there exists a subse-quence, still indexed by n, such that

$$z_n \; ! \quad \text{strongly in } L^2(0; T; H^1 \; (\;)): \qquad (61)$$

Next we show that and are zero elements. Indeed, from (57b) we obtain that $^{1=2}u_n$! 0 in $L^2(0; T; L^2($) and $\underline{z}_n j$! 0 in $L^2(0; T; L^2(\;_1))$: This implies that $\_ = 0$ and j = 0. Indeed, the last claim follows from $^{1=2}u_n$! $\_$ in $H^1(0; T; L^2($) where by the uniqueness of the limit one must have $\_$ 0 . Similar argument applies to infer $j_{\;_1} = 0$:

Next, passing to the limit as n ! 1 yields the following over determined (on $_1$) problem:

$$8 \quad \bullet = b \qquad \text{in } Q \qquad (62a)$$

$$> \;_{1=2} = \frac{c^2}{b} \qquad (62b)$$

$$? \quad \frac{@}{@ \;_1} = 0; \qquad \frac{@}{@} + \;_0 \;_0 = 0; \qquad t j_{\;_1} = 0 \qquad (62c)$$

plus respective initial data.

The overdetermined {problem implies in particular with v $_t$

$$\bullet = bv$$

with the overdetermined boundary conditions

$$@_1 = 0; \qquad vj_1 = 0$$

which yields overdetermination of boundary data on $_1$ for the wave operator. This gives v 0, hence $_t$ 0 and $_{tt}$ = 0 distributionally . Using this information in (59a) yields

$$= 0; \quad \frac{@}{@}\Big|_{1} = 0; \quad \frac{@}{@} + {}_0 \Big|_{0} = 0:$$

Standard elliptic estimate, along with $_0 >$ 0 gives 0 in Q:

Finally, weak convergence of z_n in $L^1(0;T;L^2($) and the compacity of $\_$ $L^2($) into H ( ) (see [26, Theorem 16.1 with s = 0 and " =])[so $z_n(t)$! (t) strongly in H ( ) for a.e. t 2 [0; T]] allow us to compute (due to Lebesgue dominated convergence theorem):

$$\lim_{n!1} kz_n k^2_{L^2(0;T;H\ (\ ))} = \lim_{n!1} \int_0^T kz_n(t)k^2_{H\ (\ )}dt = \int_0^T \lim_{n!1} kz_n(t)k^2_{H\ (\ )}dt = kk^2_{L^2(0;T;H\ (\ ))} = 0$$

since 0 in Q: Then, passing with the limit as n ! 1 in (57a) we have

$$0 = kzk_{L^2(0;T;H^1(\ ))} = 1;$$

which is a contradiction. The Lemma is proved. □

Lemma 4.4 implies in a straightforward way the result of the proposition 4.3. □

We are ready to establish the exponential decay of the the energy functional E_1.

Theorem 4.5. Assume that f = 0. Hence, the energy functional E_1 is exponentially stable, i.e. there exists T > 0 and constants M; ! > 0 such that

$$E_1(t) \quad Me^{\ !t}E_1(0); \qquad \text{for } t > T: \tag{63}$$

Proof. Using identity (29) we have

$$\left(\int_0^s + \int_{T\ s}^T\right) E_1(t)dt \quad 2sE_1(0):$$

Since s < T=2 can be taken arbitrarily small, we x s < 1=2 in the above inequality and use it to completethe L^1{norm of the energy E_1 in (30). We obtai

$$\begin{aligned}\int_0^T E_1(t)dt &\lesssim E_1(0) + E_1(s) + E_1(T\ \ s) \\ &+ C_T\left(\int_1 b_1 z_t\, d^2_1 + \int_Q u_{tt} dQ^2 + lot(z)\right):\end{aligned} \tag{64}$$

The remaining terms in s are estimated using the dissipativity of E_1 (see identity (29) for f = 0). In fact we rewrite the above as follows

$$\int_0^T E_1(t)dt \lesssim E_1(T) + C_T\left(\int_1 b_1 z_t\, d^2_1 + \int_Q u_{tt} dQ^2 + lot(z)\right):$$

The lot (z) is \absorved" using Lemma 4.3, thus

$$\int_0^T E_z(t)dt \lesssim E_1(T) + C_T\left(\int_1 b_1 z_t\, d^2_1 + \int_Q u_{tt} dQ^2\right): \tag{65}$$

On the other hand, using identity (29) (with f = 0) once more, we deduce

$$TE_1(T) \lesssim \int_0^T E_1(t)dt + C_T\left(\int_1 b_1 z_t\, d^2_1 + \int_Q u_{tt} dQ^2\right): \tag{66}$$

Combining (65) and (66) we arrive at

$$(T - C)E_1(T) + \int_0^T E_1(t)dt \le C_T\left(\int_{\Gamma_1} b_1 z^2 d\Gamma_1 + \int_Q u_{tt}^2\, dQ\right),$$

for some $C > 0$. Choosing $T = 2C$ and replacing the "damping" term using identity (29) (with $f = 0$) we rewrite the above estimate as follows

$$E_1(T) + \int_0^T E_1(t)dt \lesssim C_T\,[E_1(0) - E_1(T)]$$

which implies

$$E_1(T) \le \underbrace{\frac{C_T}{1 + C_T}}_{:=\gamma} E_1(0),$$

where $0 < \gamma < 1$ does not depend on the solution. Repeating the process on the interval $[mT, (m+1)T]$ and we obtain $E_1((m+1)T) \le \gamma E_1(mT)$, for every $m \ge 0$. This implies

$$E_1(mT) \le \gamma^m E_1(0),$$

for every $m \ge 1$. Thus, for $t > T$ we write $t = mT + s$, with $s \in (0, T]$ and $m \ge 1$, which implies

$$E_1(t) \le E_1(mT) \le \gamma^m E_1(0) = e^{-|\ln\gamma| m} E_1(0) = e^{-|\ln\gamma|\frac{t-s}{T}} E_1(0) = e^{-\frac{|\ln\gamma|}{T}t}\frac{1}{\gamma}E_1(0),$$ which

implies (63) with $\omega = |\ln\gamma|/T$ and $M = 1/\gamma$. □

The previous result is key to establish the exponential stability of $E(t)$, which is given next.

Proof of Theorem 2.2. Notice that the exponential decay for E_1 obtained in Theorem (4.5) implies exponential decay of the quantities $\|z\|_{D(A^{1/2})}, \|z_t\|_{L^2(\Omega)}$, and we will show that this implies exponential decay of E. In view of Lemma 4.1, the only remaining quantity we need to show exponential decay is $\|u\|_{D(A^{1/2})}$ and this follows from the fact that $bu_t + c^2 u = z$. Indeed, the variation of parameter formula implies that

$$u(t) = e^{-\frac{c^2}{b}t}u_0 + \int_0^t e^{-\frac{c^2}{b}(t-\tau)}z(\tau)d\tau, \tag{67}$$

then, computing the $D(A^{1/2})$–norm both sides we estimate

$$\|u(t)\|_{D(A^{1/2})} \leqslant e^{-\frac{c^2}{b}t}\|u_0\|_{D(A^{1/2})} + \int_0^t e^{-\frac{c^2}{b}(t-\tau)}\|z(\tau)\|_{D(A^{1/2})}d\tau \tag{68}$$

hence it follows from (63) that

$$\begin{aligned}\|u(t)\|_{D(A^{1/2})} &\leqslant e^{-\frac{c^2}{b}t}\|u_0\|_{D(A^{1/2})} + ME_1(0)\int_0^t e^{-\frac{c^2}{b}(t-\tau)-\omega\tau}d\tau\\ &\leqslant e^{-\frac{c^2}{b}t}E(0) + \frac{(c^2 - b\omega)(e^{-\omega t} - e^{-\frac{c^2}{b}t})}{\omega c^2}ME(0) \leqslant \overline{M}e^{-\overline{\omega}t}E(0),\end{aligned}$$

where we have made the benign assumption that $\frac{c^2}{b} > \omega$ from (63), as if $\omega > \frac{c^2}{b}$ we use formula (63) with $\omega_1 := \frac{c^2}{b} - \varepsilon$ so $\omega > \omega_1$ and $\frac{c^2}{b} > \omega_1$.

The proof is complete. □

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