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BOUNDARY FEEDBACK STABILIZATION OF A CRITICAL NONLINEAR JMGT EQUATION WITH NEUMANN–UNDISSIPATED PART OF THE BOUNDARY

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Abstract. Boundary feedback stabilization of a *critical* third–order (in time) semilinear Jordan–Moore–Gibson–Thompson (JMGT) is considered. The word *critical* here refers to the usual case where media–damping effects are non–existent or non–measurable and therefore cannot be relied upon for stabilization purposes. Motivated by modeling aspects in high-intensity focused ultrasound (HIFU) technology, the boundary feedback under consideration is supported only on a portion of the boundary. At the same time, the remaining part is undissipated and subject to Neumann/Robin boundary conditions. As such, unlike Dirichlet, it fails to satisfy the Lopatinski condition, a fact which compromises tangential regularity on the boundary [37]. In such a configuration, the analysis of uniform stabilization from the boundary becomes subtle and requires careful geometric considerations and microlocal analysis estimates. The nonlinear effects in the model demand construction of suitably small solutions which are invariant under the dynamics. The assumed smallness of the initial data is required only at the lowest energy level topology, which is sufficient to construct sufficiently smooth solutions to the nonlinear model.

1. Introduction.

1.1. PDE Model and an Overview.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ denote a bounded domain with sufficiently smooth boundary $\Gamma := \partial\Omega$ within which a sound wave propagates. In HIFU technology, as well as in other related areas, one is interested in tracking – and often controlling – the evolution of an acoustic pressure $u = u(t, x)$ $(t \in \mathbb{R}_+, x \in \Omega)$ triggered by wave propagation. In media within which heat propagates hyperbolically (which is the case of most biological tissues), the

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evolution of the acoustic pressure can be assumed to obey the semilinear JMGT equation which is given by the *third order in time abstract evolution* equation

$$\tau u_{ttt} + (\alpha - 2ku)u_{tt} - c^2\Delta u - (\delta + \tau c^2)\Delta u_t = 2ku_t^2, \tag{1}$$

where $c, \delta, k > 0$ are constants representing the speed and diffusivity of sound and a nonlinearity parameter, respectively, while the function $\alpha : \overline{\Omega} \to \mathbb{R}^+$ accounts for a natural frictional damping with the quantitative properties depending on the media. The parameter $\tau > 0$ – also media–dependent – accounts for time relaxation and essentially transfers the hyperbolicity of the heat to the acoustic wave.

The semilinear equation (1.1) can be viewed as a singular perturbation and, to some extent, a *refinement* of the classical quasilinear Westervelt's equation

$$(\alpha - 2ku)u_{tt} - c^2\Delta u - \delta\Delta u_t = 2ku_t^2. \tag{2}$$

obtained by setting $\tau = 0$. Physically, the main difference between (1.1) and (1.1) is that the latter accounts for finite speed of propagation of the heat waves. From the modeling point of view, this results from using Maxwell-Cattaneo Law – rather than Fourrier's Law – to model the heat flux for the acoustic heat waves. More details regarding the physical interpretation of the model (1.1), its derivation and overall discussion see [18, 9, 10, 40, 12, 19]. An analysis of asymptotic behavior of solutions when the parameter of relaxation tends to zero can be found in: [24, 25, 3].

The issues of wellposedness and stability of solutions under *homogeneous* Dirichlet and Neumann boundary data were first addressed for both nonlinear and linearized ($k = 0$) dynamics around 2010, see [22, 33, 23]. For the analysis of long–time dynamics of (1.1), in both linear and nonlinear cases, the function

$$\gamma : \overline{\Omega} \to \mathbb{R}, \qquad \gamma(x) \equiv \alpha(x) - \frac{\tau c^2}{b}$$

plays a central role. In fact, the existence of a positive constant γ_0 such that $\gamma(x) \geqslant \gamma_0 > 0$ a.e. in Ω ensures both: that the linear dynamics is uniformly exponentially stable and that stable nonlinear flows can be constructed via a *barrier* method [23]. A natural question to ask appears to be: what about other profiles of γ? It is known that if $\gamma < 0$ one may have chaotic solution [13] and if $\gamma \equiv 0$ then the energy is conserved [22]. This raises an interesting question: which mechanisms could be employed to ensure the stability of the dynamics when γ degenerates, i.e., $\gamma(x) \geqslant 0$? Here "criticality" is used within the context of stability rather than nonlinear parameters related to the validity of Sobolev's embeddings.

From a practical point of view, the quantity $\gamma(x)$ is interpreted as the viscoelasticity of the material point $x \in \Omega$ and, in particular, in the medical field, is not expected to be known for all points of Ω. By making the physically relevant assumption that $\gamma \in L^\infty(\Omega)$, $\gamma(x) \geqslant 0$ a.e. in Ω (allowing the critical case $\gamma \equiv 0$, or the case where measurements can only be made at isolated points of the domain), we ask the question of whether a non–invasive (*boundary*) action can drive the acoustic pressure asymptotically to zero, regardless of the particular knowledge of γ (as long as it is nonnegative). This question, besides being of independent interest in stability theory, is critical in ensuring *global* wellposedness of nonlinear dynamics. Otherwise, the nonlinearity may cause blow-up of solutions [11].

It has been recently shown that added viscoelastic effects produce, in some cases, the asymptotic decay of the energy. The type of the results obtained depend on the properties of the viscoelastic kernel, which requires rather special structural properties in order to produce the exponential decays in the critical case [30, 31, 14,

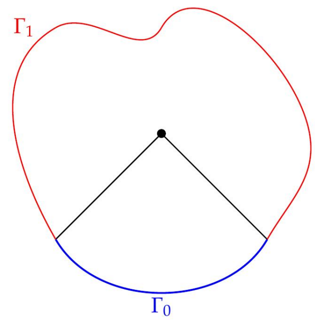

FIGURE 1. Representation of the domain

15, 17, 16]. Even more, unlike a wave equation, it is known [17] that viscous effects acting upon pressure only $u(t, x)$ can not produce *uniform* decays of the energy in the critical case. Because of the above, the question of identifying other mechanisms forcing the energy of critical MGT to decay uniformly to zero is of paramount interest. Boundary dissipation is a natural, physically attractive candidate since the control action is applied on the boundary only. Hence it is easily accessible to external manipulations. As always in the case of boundary dissipation in hyperbolic dynamics, geometric configuration for the damping plays a pivotal role.

Of particular interest is a configuration arising in the ultrasound technology, where an acoustic medium is excited on one part of the boundary while the remaining part is subject to absorbing boundary conditions, see Fig 1. This control model was introduced in [21, 20] in the case of Westervelt–Kuznetsov equation and later pursued in [8] for a finite time horizon MGT equation. This corresponds to the following boundary conditions

$$\begin{aligned} \lambda\partial_\nu u + \kappa_0(x)u = 0 \text{ on } \Sigma_0 \coloneqq (0,T)\times\Gamma_0 \\ \partial_\nu u + \kappa_1(x)u_t = 0 \text{ on } \Sigma_1 \coloneqq (0,T)\times\Gamma_1 \end{aligned} \tag{3}$$

with $\Gamma_0, \Gamma_1 \subset \Gamma$ relatively open, $\Gamma_0 \neq \emptyset$, $\overline{\Gamma_0}\cup\overline{\Gamma_1} = \Gamma$, $\Gamma_0\cap\Gamma_1 = \emptyset$, $\lambda > 0$, $\kappa_0 \in L^\infty(\Gamma_0)$, $\kappa_1 \in L^\infty(\Gamma_1)$, $\kappa_i \geq 0$.

Remark 1.1. Note that the boundary conditions imposed on Γ change the structure from Γ_0 to Γ_1. This particular model is motivated by applications [21, 20] where only one part of the boundary is subject to dissipation (Γ_1), while the other part is left free or subject to some control actions. When $\lambda = 0$, the boundary conditions in (1.1) are of *mixed* type-involving both Dirichlet and Neumann boundary conditions. It is known that mixed boundary conditions, imposed on the same part of the boundary and not separated, lead to singularities of the corresponding elliptic solutions [38]. Maximal amount of regularity in the case $\kappa_0 = \kappa_1 \equiv 0$, as shown in [38], is up to $B^{3/2}_{2,\infty}(\Omega)$, the latter stands for Besov's space. Some regularity improvement is possible, assuming that the two boundaries meet under a certain angle. For this reason, when considering boundary stabilization problems for Dirichlet–Neumann problem, it is typically assumed that Γ_0 and Γ_1 do not intersect.

However, the situation is different when $\lambda > 0$ say, without loss of generality, $\lambda = 1$. This Neumann-Robin case is under consideration in the present paper. The principal symbols associated with the boundary conditions on both parts Γ_0 and Γ_1

are the same, so the regularity of the elliptic solutions is dictated by the regularity of the forcing on the boundary, in line with standard elliptic theory [32]. More on this will be given later.

1.2. **Past results.** The boundary stabilization of MGT dynamics, in the critical case, has been considered only very recently and for *linear* models only. In [4], *linear* dynamics is considered in the Dirichlet–Neumann case, i.e., $\lambda = 0$ and $\kappa_0 \equiv \kappa_1 \equiv 1$ in (1.1). However, this configuration, due to the regularity issues associated with *mixed* boundary value problems, as noted before, requires the two parts of the boundary to be *separated*. Stabilization estimates require the domain Ω to be star-shaped. This latter restriction has been removed in [6], still for the *linear* dynamics, by resorting to a microlocal analysis argument. The final result in [6] holds without any geometric conditions on the controlled part of the boundary, and only the uncontrolled part is subject to the star-shaped condition. This is in line with the physics of the problem.

In the case of boundary conditions in (1.1) with $\lambda > 0$, both parts of the boundary are adjacent, thus touching, and we are dealing with the Neumann–Robin problem. This allows the propagation of higher-order regularity, up to $H^2(\Omega)$. However, the question of propagating stability through suitable flux multipliers now becomes problematic. This is due to the failure of the Lopatinski condition [37] on the non-dissipated part Γ_0 of the boundary and is discussed below.

It is well known that the failure of the strong Lopatinski condition [42, 41] leads to new challenges at the level of proving controllability or stabilization even for the wave equation in dimensions higher than one. The mathematical-technical reason for this is that the presence of tangential boundary derivatives on Γ_0 cannot be handled by the standard flux multiplier methods suitable for studying controllability or stabilization from the boundary. Note that in the Dirichlet case the corresponding tangential traces on the non-dissipated part are simply zero.

The first progress in solving this open problem in the MGT case was made in [5], where *linear model* is considered. There it is shown that the energy decays exponentially at the low (base) level. This result is obtained by a suitable construction of flux multipliers under certain geometrical constraints imposed on the boundary. The imposed geometric conditions require *convexity* of the level sets of the part of the boundary that is not subject to dissipation. For the *nonlinear* case, one needs to "boost" these estimates to higher topological levels. However, nonlinear effects force a different functional environment in which the higher-level energy functionals and the higher topology of the solutions must be controlled in time. Moreover, quasilinear effects force one to consider appropriately small solutions. Thus, obtaining stability estimates, at several topological levels, which are also invariant subject to the dynamics lies at the heart of the matter of the nonlinear problem.

1.3. **New Challenge.** The main challenge of the present work is to study the problem for the *nonlinear* dynamics. This is non-trivial for at least three reasons: first, the nonlinear effects require a higher degree of regularity, which automatically forces one to raise the stabilization estimates to higher topological levels. Combined with the dynamics at the boundary, it is challenging to lift the linear estimates to higher levels. This is due to the geometry and the fact that the usual multipliers are not commutative with the generator. The second reason is that the initial data for nonlinear problems must be *well prepared*. This includes compatibility conditions and also suitable *smallnesss* of the energy solutions. To deal with the latter problem, the smallness requirement is imposed only for the lower topology of the initial data. The analysis performed shows that this smallness is propagated by the dynamics at the lower topological level so that the higher derivatives can remain large. This is the crucial point for the claim that the presented framework preserves the fully nonlinear properties of the model and the resulting acoustic waves remain genuinely nonlinear. Finally, the third reason is that geometric configurations are necessary for "trapping" the energy rays. The estimates must be boosted to higher energy levels while maintaining smallness only at the lowest energy level, so this is *not* a perturbation argument.

For other relatively recent references related to the questions of boundary regularity for linear MGT equation, the interested reader is referred to: [7, 36, 43].

2. **Main Results and discussion.** Let $\Omega \in \ BR^3$ be a bounded domain with C^2 boundary $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0, Γ_1 are relatively open, nonempty subsets of Γ. Consider the following PDE system:

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - (\delta + \tau c^2)\Delta u_t = u_t^2 + u u_{tt} & \text{in } Q := (0,T) \times \Omega \\ \partial_\nu u + \kappa_0(x) u = 0 & \text{on } \Sigma_0 := (0,T) \times \Gamma_0 \\ \partial_\nu u + \kappa_1(x) u_t = 0 & \text{on } \Sigma_1 := (0,T) \times \Gamma_1 \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ u_{tt}(0,x) = u_2(x) & \text{in } \Omega \end{cases} \tag{4}$$

where we assume without loss of generality that $\lambda = 1$ and $2k = 1$ in (1.1).

Notation. Here and throughout the paper we denote by $L^2(\Omega)$ and $L^2(\Gamma)$ the sets of measurable (in the Lebesgue and Hausdorff measures, respectively) functions whose squares are integrable on Ω and Γ respectively, equipped with the norms given by the inner products

$$(u, v) = \int_\Omega uv d\Omega \quad \text{and} \quad (u, v)_\Gamma = \int_\Gamma uv d\Gamma.$$

and denoted respectively by $\|\cdot\|_2$ and $\|\cdot\|_\Gamma$. The remaining $L^p(\Omega)$– spaces $(1 \leqslant p \leqslant \infty)$ will also have norms denoted by $\|\cdot\|_p$. In addition, we denote by $H^s(\Omega)$ the (L^2–based) Sobolev space of order $s \in \mathbb{R}$ [32]. Let $B^s_{p,q}(D)$ denote Besov spaces of order $s \geqslant 0$, $1 \leqslant p \leqslant q \leqslant \infty$ defined on a domain D (be it Ω or a boundary Γ), see [34, Section 3.3].

2.1. **Functional Analytic Setting.** Let $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator defined as

$$A\xi = -\Delta\xi, \ \mathcal{D}(A) = \left\{ \xi \in L^2(\Omega); \Delta u \in L_2(\Omega) \text{ and } \partial_\nu \xi|_{\Gamma_1} = 0, [\partial_\nu \xi + \kappa_0 \xi]_{\Gamma_0} = 0 \right\}.$$

In this case A is a positive self–adjoint operator with compact resolvent. With $\kappa_0(x) > 0$, on an open set of Γ_0, $\mathcal{D}\left(A^{1/2}\right) = H^1(\Omega)$ with the – equivalent to $H^1(\Omega)$ – topology of $\mathcal{D}(A^{1/2})$ given by

$$\|u\|^2_{\mathcal{D}(A^{1/2})} := \|\nabla u\|_2^2 + \int_{\Gamma_0} \kappa_0 |u|^2 d\Gamma_0.$$

Moreover, with some abuse of notation, we (also) denote by $A : L^2(\Omega) \to [\mathcal{D}(A)]'$ the extension (by duality) of the operator A.

The phase space $\mathbb{H}$ is given by

$$\mathbb{H} := \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega) \sim H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega).$$

Since the main emphasis in the paper is on nonlinear dynamics, one needs to work with more regular solutions. For this, an additional regularity of the coefficients κ_i will be required. In what follows, we shall assume (conservatively) that $\kappa_i \in W^1_p(\Gamma_i), p > 2$ .The above regularity implies that κ_i are the multipliers on $H^s(\Gamma_i), i = 0, 1$ and $0 < s < 1/2$. Indeed, for the latter it suffices that $\kappa_i \in L_\infty(\Gamma_i) \cap B^{1/2}_{4,\infty}(\Gamma_i), i = 0, 1$, [34, p. 126].

Under such assumption one has the following regularity of the domain $\mathcal{D}(A)$,

$$\mathcal{D}(A) \subset H^\theta(\Omega);\ \theta < 2$$

Remark 2.1. Note that the restriction $\theta < 2$ is due to possible singularity caused by Robin–Neumann boundary conditions. Indeed, $\kappa_i \in W^1_p(\Gamma_i)$, $p > 2$, yields $\kappa_i u \in H^s(\Gamma_i), s \leq 1/2$ for $u \in H^1(\Omega)$ [34]. However, the singularity of the characteristic function across the interface (from Γ_0 to Γ_1) propagates only $H^{1/2-\varepsilon}(\Gamma)$ regularity for the normal derivative on Γ. If one assumes that κ_i's are also of compact support in Γ_i, then θ can be taken equal to two. For analysis in this paper $\theta < 2$ provides sufficient regularity.

Next, we rewrite (2) as a first–order abstract system on $\mathbb{H}$. For this, we introduce the classical boundary $\to$ interior harmonic extension for the Neumann data on Γ_1 as follows: for $\varphi \in L^2(\Gamma_1)$, let $\psi := N(\varphi)$, be the unique solution of the elliptic problem

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \\ \partial_\nu \psi = \varphi|_{\Gamma_1} & \text{on } \Gamma_1 \\ \partial_\nu \psi + \kappa_0 \psi = 0 & \text{on } \Gamma_0. \end{cases}$$

From elliptic theory, it follows that that $N \in \mathcal{L}(H^s(\Gamma_1), H^{s+3/2}(\Omega))$, $(0 \leq s < 1/2)$ and

$$N^* A\xi = \begin{cases} \xi \text{ on } & \Gamma_1 \\ 0 \text{ on } & \Gamma_0, \end{cases}$$

for all $\xi \in \mathcal{D}(A)$, where N^* represents the adjoint of N when it is considered as an operator from $L^2(\Gamma_1)$ to $L^2(\Omega)$ [28, 32].

The u–problem can be written (via duality on $[\mathcal{D}(A)]'$) as

$$\tau u_{ttt} + \alpha u_{tt} + c^2 Au + bAu_t + c^2 AN(\kappa_1 N^* Au_t) + bAN(\kappa_1 N^* Au_{tt}) = u_t^2 + uu_{tt} \tag{5}$$

Next, we introduce the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ with the action:

$$\mathcal{A}\begin{bmatrix}\xi_1 \\ \xi_2 \\ \xi_3\end{bmatrix} := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\dfrac{c^2}{\tau}A & -\dfrac{c^2}{\tau}AN(\kappa_1 N^* A) - \dfrac{b}{\tau}A & -\dfrac{b}{\tau}AN(\kappa_1 N^* A) - \dfrac{\alpha}{\tau}I \end{bmatrix}\begin{bmatrix}\xi_1 \\ \xi_2 \\ \xi_3\end{bmatrix} \tag{6}$$

and the domain (with $\vec{\xi} \equiv (\xi_1, \xi_2, \xi_3)^\top$)

$$\mathcal{D}(\mathcal{A}) = \left\{\vec{\xi} \in \mathbb{H};\ \xi_3 \in \mathcal{D}\left(A^{1/2}\right),\ \xi_i + N(\kappa_1 N^* A\xi_{i+1}) \in \mathcal{D}(A),\ \text{for } i = 1, 2\right\}$$

This gives (see Remark 2.1 and the regularity of the Neumann-Robin map)

$$\mathcal{D}(\mathcal{A}) \subset H^\theta(\Omega) \times H^\theta(\Omega) \times H^1(\Omega),\ \theta < 2$$

with a proper injection.

The first order abstract version of the u–problem is thus given by

$$\begin{cases} \Phi_t = \mathcal{A}\Phi + \mathcal{F}(\Phi) \\ \Phi(0) = \Phi_0 = (u_0, u_1, u_2)^\top, \end{cases} \tag{7}$$

in the variable $\Phi = (u, u_t, u_{tt})^\top$ with $\mathcal{A}$ defined in (2.1) and $\mathcal{F}(\Phi)^\top \equiv (0, 0, \tau^{-1}(u_t^2 + uu_{tt}))$.

To treat the nonlinear problem one needs to consider "smoother" solutions than those generated by the topology of $\mathbb{H}$. This leads to the following construction of the second phase space denoted by $\mathbb{H}_1$, which is"tighter" than $\mathbb{H}$ but strictly larger than $\mathcal{D}(\mathcal{A})$. The new phase space $\mathbb{H}_1$ is defined as

$$\mathbb{H}_1 = \{\vec{\xi} \in \mathbb{H}; \Delta\xi_1 \in L^2(\Omega); [\partial_\nu \xi_1 + \kappa_0 \xi_1]_{\Gamma_0} = 0; [\partial_\nu \xi_1 + \kappa_1 \xi_2]_{\Gamma_1} = 0\}$$

and endowed with the norm

$$\|\vec{\xi}\|_{\mathbb{H}_1}^2 = \|\vec{\xi}\|_{\mathbb{H}}^2 + \|\Delta\xi_1\|_2^2$$

We remark that the boundary conditions in the definition of the space $\mathbb{H}_1$ are well defined due to the property: $\Delta\xi_1 \in L^2(\Omega)$ and $\xi_1 \in H^1(\Omega)$ then $\partial_\nu \xi_1 \in H^{-1/2}(\Gamma)$ – the latter allowing to define the boundary conditions as a distribution. We also note that since $\xi_1, \xi_2 \in H^1(\Omega)$ we have $\xi_i|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$ $(i = 1, 2)$ and therefore $\partial_\nu \xi_i \in H^{1/2}(\Gamma_i)$, the latter a consequence of regularity of $\kappa_i \in B^{1/2}_{4,\infty}(\Omega)$.This, along with the elliptic regularity implies:

$$\mathbb{H}_1 \subset H^\theta(\Omega) \times H^1(\Omega) \times L^2(\Omega), \theta < 2$$

with a proper injection, see also Remark 2.1. Note also the inequality

$$\|\vec{\xi}\|_{\mathbb{H}_1}^2 \leq ||\vec{\xi}\|_{\mathbb{H}}^2 + \|\Delta\xi_1\|_2^2 + \|\partial_\nu \xi_1\|_{H^{1/2}(\Gamma)}^2 \sim ||\vec{\xi}\|_{\mathbb{H}}^2 + ||\xi_1||_{H^2(\Omega)}^2$$

The important property is that the nonlinear term in (2.1) is invariant under $\mathbb{H}_1$ topology in dimensions up to 3. This will be extensively used throughout the paper.

2.2. **Formulation of Main Results.** We begin with a preliminary result on the generation of linear semigroups within the framework of spaces $\mathbb{H}$ and $\mathbb{H}_1$. The operator $\mathcal{A}$ generates a C_0 semigroup $S(t)$ on $\mathbb{H}$, as shown in [5] . We shall show that the action defined by $\mathcal{A}$ with its natural domain also generates a C_0–semigroup $\{T(t)\}_{t\geqslant 0}$ on $\mathbb{H}_1$.

Theorem 2.2 (**Generation).** *Let $S(t)$, $t \geq 0$, denote the C_0 semigroup generated by $\mathcal{A}$ on the space $\mathbb{H}$. Then, the family $T(t) := S(t)|_{\mathbb{H}_1}$, $t \geqslant 0$, is also a C_0–semigroup with the generator $\mathcal{A}$ and its realization on $\mathbb{H}_1$.*

The second result deals with an exponential stability of the semigroups on the phase space $\mathbb{H}$ and $\mathbb{H}_1$. For this, one needs to introduce the following geometric condition.

Assumption 2.3. *The boundary Γ_0 is star–shaped and convex. This is to say: there exists $x_0 \in \mathbb{R}^n$ such that $(x - x_0) \cdot \nu(x) \leqslant 0$ for all $x \in \Gamma_0$ where $\nu(x)$ is the outwards normal vector to the boundary at x. In addition, there exists a convex level set function which defines Γ_0, see [29]. We shall also assume that the coefficient κ_0 has its support nonempty and compact in Γ_0, while $\kappa_1(x) \geq \kappa_1 > 0, x \in \Gamma_1$.*

As shown in [5], Assumption 2.3 is sufficient for the semigroup $S(t)$ to be exponentially stable on $\mathbb{H}$. We shall show that exponential stability also holds on $\mathbb{H}_1$. This will be critical for the study of the nonlinear problem. The corresponding result is formulated below.

Theorem 2.4 (**Two level uniform stability**). *Let Assumption 2.3 on Γ be in force and let $\gamma(x) \geqslant 0$. Then the semigroup $\{T(t)\}_{t\geqslant 0}$ generated by $\mathcal{A}$ in $\mathbb{H}_1$ is uniformly exponentially stable with decay rate $\omega_1 > 0$, where $\omega_1 < \omega_0$ with ω_0 the decay rate corresponding to the semigroup $S(t)$.*

Once linear wellposedness and uniform stability of the linear ($k = 0$) problem are established with respect to the appropriate topologies, our next task is to prove the wellposedness of nonlinear dynamics on $\mathbb{H}_1$. To accomplish this, initial data need to be assumed sufficiently small. *How small?* This is an important question as argued in [3]. We will be able to show that some smallness will be imposed only at the lowest level of regularity $\mathbb{H}$, while higher derivatives can remain large. As a consequence, in the following theorem, we establish the existence of $\mathbb{H}_1$–valued solutions with given $\mathbb{H}_1$ small initial data in $\mathbb{H}$ only. The proof, given in Section 5, relies on estimates derived via interpolation inequalities which allow exhibiting certain "invariance" of a $\mathbb{H}$-small ball under the nonlinear dynamics in $\mathbb{H}_1$.

We start specifying the notion of solution for the semilinear problem (2). We denote the initial data here by $\Phi_0 = (u_0, u_1, u_2)^\top$. Given $T > 0$, we say that

$$\Phi(t) = (u(t), u_t(t), u_{tt}(t))$$

is a **mild solution** for the system (2) provided $\Phi \in C([0,T], \mathbb{H}_1)$ and

$$\Phi(t) = T(t)\Phi_0 + \int_0^t T(t-\tau)\mathcal{F}(\Phi)(\tau)d\tau. \tag{8}$$

It is important to notice that the notion of mild solution given above *cannot* be extended to the base topological level $\mathbb{H}$. The reason is that the nonlinearity, here described by the function $\mathcal{F}$, is not invariant under $\mathbb{H}$, hence justifying the need for a smoother phase space.

Before stating the theorem, we denote by $\mathbb{H}^\rho$ (for $\rho > 0$) the set

$$\mathbb{H}^\rho := \left\{\Phi \in \mathbb{H}_1; \|\Phi\|_{\mathbb{H}} < \rho\right\}.$$

Theorem 2.5 (**Global Solutions**). *Let Assumption 2.3 be imposed on Γ. Then, there exists $\rho > 0$ sufficiently small (depending on the parameters in the equation), such that, given any $\Phi_0 \in \mathbb{H}^\rho$ the formula (2.2) defines a continuous $\mathbb{H}_1$–valued mild solution for the system (2). Moreover, for such $\rho > 0$, there exists $R = R(\|\Phi_0\|_{\mathbb{H}_1})$ such that all trajectories starting in $B_{\mathbb{H}^\rho}(0,R)$[1] remain in $B_{\mathbb{H}^\rho}(0,R_1)$ for all $t \geqslant 0$, for some $R_1 > R$.*

Once global solutions are shown to exist, we take on the issue of asymptotic (in time) stability. The final result is positive and it and holds uniformly (w.r.t γ) as long as $\gamma \in L^\infty(\Omega)$ and $\gamma(x) \geqslant 0$ a.e. in Ω.

Theorem 2.6 (**Nonlinear Uniform Stability**). *Let Assumption 2.3 imposed on Γ be in force and assume $\gamma \in L^\infty(\Omega)$ and $\gamma(x) \geqslant 0$. Then, there exists $\rho > 0$ sufficiently small and $M(\rho), \omega > 0$ such that if $\Phi_0 \in \mathbb{H}^\rho$ then*

$$\|\Phi(t)\|_{\mathbb{H}_1} \leqslant M(\rho)e^{-\omega t}\|\Phi_0\|_{\mathbb{H}_1}, \qquad t \geqslant 0$$

[1]The $\mathbb{H}^\rho$–ball centered at the origin and with radius R.

where Φ is the mild solution given by Theorem 2.5.

2.3. **Discussion.** The main novelty of the present paper is the study of stabilizability of a *nonlinear critical* JMGT equation with Neumann–Robin undissipated portion of the boundary. Should the problem be subcritical (i.e. $\gamma(x) > \gamma_0 > 0$ for $x \in \Omega$), the difficulty created by the failure of the Lopatinski condition would not enter the picture, simply because there would be no need to propagate stability from the boundary into the interior. As already mentioned before, linear dynamics with absorbing boundary conditions on Γ_1 and zero Dirichlet data on Γ_0 subject to star–shaped conditions have been considered in [4, 6]. Mathematical difficulties in propagating stability through the undissipated part of the boundary are not present in this case, since the tangential traces are null on Γ_0. To cope with the new challenge we shall employ geometric constructs developed earlier in [29]. These results allow for the construction of suitable–non–radial–vector fields from the tangential bending of the radial and star–shaped fields. The newly constructed fields propagate the needed estimates through the undissipated part of the boundary. This method has been already used in [5] for the *linear* model. However, handling the nonlinear effects, as in the present manuscript, brings another layer of difficulties. First, stability estimates need to be boosted to the higher topological levels. And this also involves "cooperation" of the new multipliers, in addition to topological issues of keeping invariance of suitably small solutions. It should also be noted that the nonlinear approach used in the past (for subcritical cases) relied on the so-called *barrier* method which is based on a contradiction argument. However, the application of this method in the boundary critical case meets several technical difficulties even at the level of low frequencies (lower-order terms). In this paper, we exploit another technique that, to the best of our knowledge, is new and makes strong use of the fact that we *only* require initial data to be small in $\mathbb{H}$. One of the advantages of such construction (for JMGT) was already exploited by the authors in [3] in allowing extension by density in the nonlinear environment. In the present paper, we discovered that it also allows to

a) prove global existence and exponential stability by the representation of the solution and two-level stability of linear flows. Here, the smallness interplay comes to the picture through a nonlinear propagation of an estimate of the type
$$\|\mathcal{F}(\Phi)\|_{\mathbb{H}_1} \leqslant C_1(\|\Phi\|_{\mathbb{H}})C_2(|\Phi\|_{\mathbb{H}_1}.)$$
where the size of $C_1(\|\Phi\|_{\mathbb{H}})$ can be controlled by ρ. See Theorem 2.5;

b) obtain a continuity property of the decay rate with respect to the $\mathbb{H}$–size of the initial data and the decay rate of the linear flow, ω_1. In general, we prove that if ε is the $\mathbb{H}$–size of the initial data and $\omega(\varepsilon)$ is the corresponding decay rate, then there exists $\underline{\omega}(\varepsilon)$ such that $\omega(\varepsilon) \geqslant \underline{\omega}(\varepsilon)$ and $\underline{\omega}(\varepsilon) \to \omega_1$ as $\varepsilon \to 0^+$.

The rest of the paper is devoted to the proofs.

3. **Generation of linear semigroups.**

3.1. **Preliminaries.** The proof of generation of the linear semigroup on $\mathbb{H}$ is given in [5]. Thus, the main task of this section is to prove generation on the space $\mathbb{H}_1$. To be reasonably self-contained, we shall be repeating a few – mainly notational – details from [5]. These will be needed within the context of proving the generation of higher regularity solutions.

First note that $\mathbb{H}$ can be topologised by the inner product

$$\left(\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \right)_{\mathbb{H}} = (A^{1/2}\xi_1, A^{1/2}\varphi_1) + \frac{b}{\tau}(A^{1/2}\xi_2, A^{1/2}\varphi_2) + (\xi_3, \varphi_3),$$

for all $(\xi_1, \xi_2, \xi_3)^\top, (\varphi_1, \varphi_2, \varphi_3)^\top \in \mathbb{H}$.

For notational convenience and future use, we introduce the following change of variables $bz = bu_t + c^2 u$ which reduces the problem to a PDE–abstract ODE coupled system. The change from the coordinates (u, u_t, u_{tt}) to (u, z, z_t) is described through the isomophism $M \in \mathcal{L}(\mathbb{H})$ given by (see [33])

$$M = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{c^2}{b} & 1 & 0 \\ 0 & \dfrac{c^2}{b} & 1 \end{bmatrix}.$$

The next lemma makes the above topological statement precise.

Lemma 3.1. *Assume that the compatibility conditions*

$$\partial_\nu u_0 + \kappa_0 u_0 = 0 \text{ on } \Gamma_0, \qquad \partial_\nu u_0 + \kappa_1 u_1 = 0 \text{ on } \Gamma_1 \tag{9}$$

hold. Then $\Phi \in C^1(0,T;\mathbb{H}) \cap C(0,T;\mathcal{D}(\mathcal{A}))$ *is a strong solution of*

$$\begin{cases} \Phi_t = \mathcal{A}\Phi \\ \Phi(0) = \Phi_0 \end{cases}$$

if, and only if, $\Psi = M\Phi \in C^1(0,T;\mathbb{H}) \cap C(0,T;\mathcal{D}(\mathbb{A}))$ *is a strong solution for*

$$\begin{cases} \Psi_t = \mathbb{A}\Psi \\ \Psi(0) = \Psi_0 = M\Phi_0 \end{cases} \tag{10}$$

where $\mathbb{A} = M\mathcal{A}M^{-1}$ *with*

$$\mathcal{D}(\mathbb{A}) = \left\{ \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \in \left[H^\theta(\Omega)\right]^2 \times \mathcal{D}\left(A^{1/2}\right); [\partial_\nu \xi_2 + \kappa_0 \xi_2]_{\Gamma_0} = 0, [\partial_\nu \xi_2 + \kappa_1 \xi_3]_{\Gamma_1} = 0 \right\}$$

Proof. We only check the matching of the boundary conditions. Assume that $\Psi = (u, z, z_t) \in C^1(0,T;\mathbb{H}) \cap C(0,T;\mathcal{D}(\mathbb{A}))$ is a strong solution for (3.1). Let

$$\Upsilon(t) := (\partial_\nu u(t) + \kappa_0 u(t))|_{\Gamma_0}, \; t \geqslant 0$$

and notice that $b\Upsilon_t + c^2\Upsilon = 0$ for all t. This along with the compatibility condition $(3.1)_1$ ($\Upsilon(0) = 0$) implies that $\Upsilon \equiv 0$. The same argument *mutatis mutandis* recovers the boundary condition for u on Γ_1. The loss of differentiability in $\mathcal{D}(\mathbb{A})$ is due to the fact that the two parts of the boundary are not separated, see Remark 2.1. ☐

For convenience, we explicitly write a formula for the new operator $\mathbb{A} = M\mathcal{A}M^{-1}$.

$$\mathbb{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -\dfrac{c^2}{b}I & I & 0 \\ 0 & 0 & I \\ -\gamma\dfrac{c^4}{\tau b^2}I & \gamma\dfrac{c^2}{\tau b}I - \dfrac{b}{\tau}A & -\gamma\dfrac{1}{\tau}I - \dfrac{b}{\tau}AN(\kappa_1 N^* A) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

where $\gamma = \alpha - \dfrac{\tau c^2}{b} \in L^\infty(\Omega)$.

3.2. **Proof of Theorem 2.2.** In the first step, it is shown that $\mathbb{A}$ generates a C_0–semigroup on $\mathbb{H}$, from which the semigroup is generated by $\mathcal{A}$ can be recovered via M. The semigroup on $\mathbb{H}_1$ will then be obtained by a restriction argument. The details are below.

We write $\mathbb{A} = \mathbb{A}_d + P$ where

$$P := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\gamma\dfrac{c^4}{\tau b^2} & \gamma\dfrac{c^2}{\tau b} & I - \gamma\dfrac{I}{\tau} \end{bmatrix}$$

is bounded in $\mathbb{H}$ and $\mathcal{D}(\mathbb{A}_d) := \mathcal{D}(\mathbb{A})$. It then suffices to prove generation of $\mathbb{A}_d$ on $\mathbb{H}$, see [35, p. 76] and this is done by verifying the hypothesis of Lummer–Philips Theorem: dissipativity and maximality.The details of this argument are in [5]. This yields that A_d is maximal dissipative, therefore generates a C_0– semigroup of contractions due to Lummer–Phillips Theorem. Since P is bounded, $A = \mathbb{A}_d + P$ generates a C_0–semigroup on $\mathbb{H}$.

For generation in $\mathbb{H}_1$, one applies an argument inspired by the one presented in ([33], p. 26) with the needed modifications to account for different boundary conditions.. Since we already know that $\mathcal{A}$ generates a C_0 semigroup $\{S(t)\}_{t\geqslant 0}$ on a larger space $\mathbb{H}$, we only show that

$$\{T(t)\}_{t\geqslant 0} := \{S(t)|_{\mathbb{H}_1}\}_{t\geqslant 0}$$

is also a semigroup and that its infinitesimal generator is $\mathcal{A}$ when considered as an operator in $\mathbb{H}_1$.

This entails to the proof of two facts: $\{T(t)\}_{t\geqslant 0}$ satisfies the semigroup property – which follows from the fact that the problem is autonomous – and invariance: $T(t)(\mathbb{H}_1) \subset \mathbb{H}_1$ for all $t \geqslant 0$.

If $\Phi_0 = (u_0, u_1, u_2)^\top \in \mathbb{H}_1$ then $\partial_\nu u_0 + k_1 u_1 = 0$ on Γ_1. We then need to show that this condition is invariant under the dynamics and, in addition, the regularity $\Delta u \in C\left([0,T); L^2(\Omega)\right)$ holds true. This, along with the boundary conditions and regularity of elliptic problems would lead to $u \in C([0,T); H^\theta(\Omega))$ with $\theta < 2$. Recall that an incremental loss of differentiability is due to change of the boundary conditions from Γ_0 to Γ_1 with the normal direction having $H^{1/2-\varepsilon}(\Gamma)$ regularity across the interface.

In order to show that $\Delta u \in C\left([0,T); L^2(\Omega)\right)$, we appeal to the change of variables $z = u_t + \frac{c^2}{b}u$. By the variation of parameters formula we have

$$u(t) = e^{-\frac{c^2}{b}t}u_0 + \int_0^t e^{-\frac{c^2}{b}(t-\sigma)}z(\sigma)d\sigma$$

and since $\Delta u_0 \in L^2(\Omega)$ ($\Phi_0 \in \mathbb{H}_1$), it suffices to verify that

$$\int_0^t e^{-\frac{c^2}{b}(t-\sigma)}\Delta z(\sigma)d\sigma \in L^2(\Omega),\ \forall t \geqslant 0.$$

To this end we recall that $(z, z_t) \in C([0,T); H^1(\Omega) \times L^2(\Omega))$. Writing the linear solution of (1.1) (with $k = 0$) in the z–variable yields

$$\int_0^t e^{-\frac{c^2}{b}(t-\sigma)}\Delta z(\sigma)d\sigma = \frac{\tau}{b}\int_0^t e^{-\frac{c^2}{b}(t+\sigma)}\left[z_{tt}(\sigma) + \gamma u_{tt}(\sigma)\right]d\sigma$$

$$
\begin{aligned}
&= \frac{\tau}{b}\left[z_t(t) + \gamma u_t(t) - e^{-\frac{c^2}{b}t}[z_t(0) + \gamma u_1]\right] \\
&+ \frac{c^2}{b^2}\int_0^t e^{-\frac{c^2}{b}(t-\sigma)}[z_t(\sigma) + \gamma u_t(\sigma)]d\sigma \in L^2(\Omega),
\end{aligned}
$$

as needed.

We show next that the boundary conditions are also invariant under the dynamics. For this we again use the variation of parameters formula for u (and its time derivative) to write (for continuous $\mathcal{D}(\mathcal{A})$–valued solutions):

$$
\begin{aligned}
\partial_\nu u(t) + \kappa_1 u_t(t) &= e^{-\frac{c^2}{b}t}\partial_\nu u_0 + \int_0^t e^{-c^2b^{-1}(t-\sigma)}\partial_\nu z(\sigma)d\sigma \\
&+ \kappa_1\left(-\frac{c^2}{b}e^{-\frac{c^2}{b}t}u_0 + e^{-\frac{c^2}{b}t}z(0) + \int_0^t e^{-\frac{c^2}{b}(t-\sigma)}z_t(\sigma)d\sigma\right) \\
&= e^{-\frac{c^2}{b}t}\partial_\nu u_0 + \int_0^t e^{-c^2b^{-1}(t-\sigma)}\partial_\nu z(\sigma)d\sigma \\
&+ \kappa_1\left[-\frac{c^2}{b}e^{-\frac{c^2}{b}t}u_0 + e^{-\frac{c^2}{b}t}\left(u_1 + \frac{c^2}{b}u_0\right) + \int_0^t e^{-\frac{c^2}{b}(t-\sigma)}z_t(\sigma)d\sigma\right] \\
&= e^{-\frac{c^2}{b}t}\left[\partial_\nu u_0 + \kappa_1 u_1\right] + \int_0^t e^{-c^2b^{-1}(t-\sigma)}\left[\partial_\nu z(\sigma) + \kappa_1 z_t(\sigma)\right]d\sigma = 0,
\end{aligned}
$$

where the conclusion follows from the fact that the initial conditions for u satisfy the absorbing boundary conditions and the variable z satisfies the absorbing boundary conditions along the trajectory. This completes the proof of Theorem 2.2.

4. **Exponential decays – Proof of Theorem 2.4.**

4.1. **Preliminaries.** In this section we work with the linearized version of (1.1) – i.e., we take $k = 0$ – in the z–variable. Moreover, since τ is fixed, we lose no generality by setting $\tau = 1$ to be assumed for the rest of the paper. Recall the change of variables $z = u_t + \frac{c^2}{b}u$ transforming (1.1) into

$$
z_{tt} + bA(z_t + N(\kappa_1 N^* Az)) = -\gamma u_{tt} + f. \tag{11}
$$

We assume smooth initial conditions and a u–independent – $L^2(\Omega)$–valued and C^1 in time – forcing term f. This ensures the existence and uniqueness of strong solutions which are, in addition, continuously dependent on the initial data. We can eventually extend the results that follow to semigroup solutions by density.

The energy for a solution $\Phi = (u, u_t, u_{tt}) \in \mathcal{D}(\mathcal{A})$ will be computed at two levels. We define the **lower energy** functional by $E(t) = E_0(t) + E_1(t)$ where

$$
E_1(t) := \frac{b}{2}\|A^{1/2}z\|_2^2 + \frac{1}{2}\|z_t\|_2^2 + \frac{c^2}{2b}\|\gamma^{1/2}u_t\|_2^2
$$

$$
E_0(t) := \frac{1}{2}\|\alpha^{1/2}u_t\|_2^2 + \frac{c^2}{2}\|A^{1/2}u\|_2^2
$$

and the **higher energy** functional $\mathcal{E}(t) = E(t) + E_2(t)$ where

$$
E_2(t) = \frac{b}{2}\|\Delta u\|_2^2
$$

where α and γ may depend on x. We also note that

$$
\|A^{1/2}u\|_2^2 = \|\nabla u\|_2^2 + \|\sqrt{\kappa_0}u\|_{\Gamma_0}^2
$$

By Poincare- Wirtinger inequality we have $\|A^{1/2}u\|_2 \sim \|u\|_{H^1(\Omega)}$.

It is standard to see that $E(t) \sim \|\Phi(t)\|^2_{\mathbb{H}}$, see [4] for details. The following lemma follows from classical elliptic theory [32] and provides the estimate which is necessary for justifying our choice of *higher* energy functional. As a remark, here and hereafter we use the notation $a \lesssim b$ to say that $a \leqslant Cb$ where C is a constant possibly depending on the physical parameters of the model $(\tau, c, b > 0)$ but independent of space, time and $\gamma \in L^\infty(\Omega)$.

Lemma 4.1. *Let Ω be a smooth domain and consider a function $u : \Omega \to \mathbb{R}$ such that $\Delta u \in L^2(\Omega)$ and $\partial_\nu u|_\Gamma \in H^s(\Gamma)$, $0 \le s \le 1/2$, then $u \in H^{3/2+s}(\Omega)$ and*

$$\|u\|_{H^{3/2+s}(\Omega)} \lesssim \|\Delta u\|_2 + \|\partial_\nu u\|_{H^s(\Gamma)}.$$

4.2. **Propagation of boundary dissipation – Flow multipliers.** We begin with energy identity for E_1. Since the problem is linear we work with smooth solutions (in the domain of the generator) which can be extended by density to the phase space solutions.

Proposition 4.2 (**Energy Identity**)**.** *Let $T > 0$. If $\Psi = (u, z, z_t)$ is a weak solution of* (4.1) *then*

$$E_1(T) + \int_t^T D_\Psi(s)ds = E_1(t) + \int_t^T \int_\Omega f z_t d\Omega ds \tag{12}$$

holds for $0 \le t \le T$, where D_Ψ represents the interior/boundary damping and is given by

$$D_\Psi := b\int_{\Gamma_1} \kappa_1 z_t^2 d\Gamma_1 + \int_\Omega \gamma u_{tt}^2 d\Omega ds \tag{13}$$

Proof. The energy identity (4.2) is first derived for strong solutions and then extended by density to weak solutions. The details are the same as in [5]. We repeat some of the calculations for the reader's convenience. Consider the bilinear form $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ given by

$$\left\langle \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \right\rangle := b\Big(A^{1/2}\xi_2, A^{1/2}\varphi_2\Big) + (\xi_3, \varphi_3) + \frac{c^2}{b}\left(\gamma\left(\xi_2 - \frac{c^2}{b}\xi_1\right), \varphi_2 - \frac{c^2}{b}\varphi_1\right)$$

which is continuous. Moreover, recalling that $\Psi = (u, z, z_t)$ it follows that $2E_1(t) = \langle \Psi(t), \Psi(t) \rangle$.Therefore, with $G = (0, 0, f)^\top$, after straightforward calculations (details in [5])

$$\begin{aligned} 2\frac{dE_1(t)}{dt} &= \left\langle \frac{d\Psi(t)}{dt}, \Psi(t) \right\rangle = \langle \mathbb{A}\Psi(t) + G, \Psi(t) \rangle \\ &= -\int_\Omega \gamma u_{tt}^2 d\Omega - b\int_{\Gamma_1} \kappa_1 z_t^2 d\Gamma_1 + \int_\Omega f z_t d\Omega. \end{aligned}$$

Identity (4.2) then follows by an integration in time on $[t, T]$. □

In the next step, we reconstruct the integral of the full energy on a truncated time interval $(s, T - s)$ for $0 < s < T/2$, in terms of the dissipation and lower order terms. This step requires a transfer of dissipation from the boundary into the entire spatial domain. Some comments are in order.

We study stability properties of both $S(t)$ and $T(t)$ assuming the general degenerated case for γ, i.e., $\gamma \in L^\infty(\Omega)$ and $\gamma(x) \geqslant 0$ a.e. in Ω. This includes a completely degenerate (critical) case when $\gamma = 0$ and the uncontrolled dynamics is unstable. The stability of $S(t)$ has been already considered in [5], however, to make the arguments self-contained, some of the constructs will be repeated below. With a feedback boundary control, it will be shown that the semigroups can be stabilized. For this to happen, additional geometric conditions are needed which are stronger than the ones typically assumed in a boundary stabilization theory of hyperbolic dynamics. It is clear that if $\Gamma_0 = \emptyset$ then the entire boundary Γ is dissipated and therefore stability results would hold without any additional geometric restrictions. Assuming that $\Gamma_0 \neq \emptyset$, and Γ_0 is star-shaped (standard condition), classical stability methods (multipliers) do not work due to conflicting signs of radial vector fields on the boundary Γ_0, when acting on the tangential derivatives. This fact has been recognized already in [29].

However, the convexity of Γ_0 and Assumption 2.3 save the situation due to the following construction [29]: there exists a vector field $h(x) = [h_1(x), \cdots, h_d(x)] \in C^2(\overline{\Omega})$ such that

$$h \cdot \nu = 0 \text{ on } \Gamma_0 \tag{14}$$

with ν being the unit outward normal, and that for some constant $\delta > 0$ and all vector $v(x) \in [L^2(\Omega)]^n$, we have

$$\int_\Omega J(h)|v(x)|^2 d\Omega \geqslant \delta \int_\Omega |v(x)|^2 d\Omega, \tag{15}$$

where $J(h)$ represents the Jacobian matrix of the vector field h.

Remark 4.3. We note that the more general typical star–shaped condition $h \cdot \nu \leqslant 0$ on Γ_0 is not sufficient. This is due to the presence of tangential derivatives on uncontrolled part of the boundary which can not be "absorbed" via dissipation by the microlocal argument [27, 41, 26]. The latter requires time derivatives of the Γ_0 traces to be controlled according to the inequality:

$$\|\partial_\tau u\|_{\Sigma_0} \leq C\|u_t\|_{\Sigma_0} + C\|\partial_\nu u\|_{\Sigma_0} + lot_Q$$

valid on solutions. Above, lot_Q mean lower order terms on $Q = \Omega \times [0, T]$. By "bending" on the boundary Γ_0 the radial vector field allows to eliminate its contribution of the tangential derivatives. See Remark 4.7.

Remark 4.4. Convexity of Γ_0 is only one sufficient condition. Several examples where the construction in (4.2) holds for other types of domains are given in [29].

Proposition 4.5. *Let $T > 0$ and assume that Assumption 2.3 holds. For any strong solution of (3.1) (u, z, z_t) the following inequality is valid*

$$\int_s^{T-s} E_1(t)dt \lesssim [E_1(s) + E_1(T-s)] + \\ C_T \left[\int_0^T D_\Psi(s)ds + \int_Q f^2 dQ + lot_\delta(z) \right]$$

for $0 < s < T/2$. Here, $lot_\delta(z)$ is a collection of lower order terms satisfying

$$lot_\delta(z) \leq C_\delta \sup_{t\in[0,T]} \left\{ \|z(t)\|^2_{H^{1-\delta}(\Omega)} + \|z_t(t)\|^2_{H^{-\delta}(\Omega)} \right\},$$

for some $0 < \delta < 1/2$.

Proof. We start with energetic calculations performed first on regular (strong) solutions [with at least $H^{3/2+}(\Omega)$ regularity]. Let's multiply equation (4.1) by $h \cdot \nabla z$ and integrate by parts in $(s, T-s) \times \Omega$, for $s \in [0, T/2)$. This gives

$$
\begin{aligned}
&\frac{1}{2}\int_s^{T-s}\!\!\int_\Gamma \left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma dt + b\int_s^{T-s}\!\!\int_\Gamma \partial_\nu z(h\cdot\nabla z)d\Gamma dt \\
&= \int_s^{T-s}\!\!\int_\Omega \gamma u_{tt}(h\cdot\nabla z)d\Omega dt + \int_\Omega z_t(h\cdot\nabla z)d\Omega\Big|_s^{T-s} \\
&+ \frac{b}{2}\int_s^{T-s}\!\!\int_\Omega J(h)|\nabla z|^2 d\Omega dt + \frac{1}{2}\int_s^{T-s}\!\!\int_\Omega \left(z_t^2 - b|\nabla z|^2\right)\mathrm{div}(h)d\Omega dt \\
&- \int_s^{T-s}\!\!\int_\Omega f(h\cdot\nabla z)d\Omega dt, \qquad (16)
\end{aligned}
$$

where $J(h)$ is the Jacobian with the properties in (4.2). Now notice that equipartition of kinetic and potential energy appears in the above identity via the term

$$\int_\Omega \left(z_t^2 - b|\nabla z|^2\right)\mathrm{div}(h)d\Omega,$$

We next multiply (4.1) by $z\mathrm{div}(h)$, integrate by parts to obtain the following identity

$$
\begin{aligned}
&\frac{b}{2}\int_s^{T-s}\!\!\int_\Gamma \partial_\nu z z\mathrm{div}(h)d\Gamma dt = \frac{1}{2}\int_s^{T-s}\!\!\int_\Omega \gamma u_{tt} z\mathrm{div}(h)d\Omega dt \\
&+ \frac{1}{2}\int_\Omega z_t z\mathrm{div}(h)d\Omega\Big|_s^{T-s} + \frac{1}{2}\int_s^{T-s}\!\!\int_\Omega \left(b|\nabla z| - z_t^2\right)\mathrm{div}(h)d\Omega dt \\
&- \frac{b}{2}\int_s^{T-s}\!\!\int_\Omega z\nabla z\cdot\nabla(\mathrm{div}(h))d\Omega dt - \frac{1}{2}\int_s^{T-s}\!\!\int_\Omega fz\mathrm{div}(h)d\Omega dt \qquad (17)
\end{aligned}
$$

where we have, as in (4.2), kept the boundary terms on the left–hand–side. Adding (4.2) and (4.2) yields

$$
\begin{aligned}
&\frac{1}{2}\int_s^{T-s}\!\!\int_\Gamma \left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma dt \\
&+ b\int_s^{T-s}\!\!\int_\Gamma \partial_\nu z(h\cdot\nabla z)d\Gamma dt + \frac{b}{2}\int_s^{T-s}\!\!\int_\Gamma \partial_\nu z z\mathrm{div}(h)d\Gamma dt \\
&= \int_s^{T-s}\!\!\int_\Omega \gamma u_{tt}(h\cdot\nabla z)d\Omega dt + \frac{1}{2}\int_s^{T-s}\!\!\int_\Omega \gamma u_{tt} z\mathrm{div}(h)d\Omega dt \\
&+ \frac{b}{2}\int_s^{T-s}\!\!\int_\Omega J(h)|\nabla z|^2 d\Omega dt + \int_\Omega z_t(h\cdot\nabla z)d\Omega\Big|_s^{T-s} \\
&+ \frac{1}{2}\int_\Omega z_t z\mathrm{div}(h)d\Omega\Big|_s^{T-s} - \frac{b}{2}\int_s^{T-s}\!\!\int_\Omega z\nabla z\cdot\nabla(\mathrm{div}(h))d\Omega dt \\
&- \int_s^{T-s}\!\!\int_\Omega f(h\cdot\nabla z)d\Omega dt - \frac{1}{2}\int_s^{T-s}\!\!\int_\Omega fz\mathrm{div}(h)d\Omega dt,
\end{aligned}
$$

The boundary terms can be written more compactly in terms of the interior terms as

$$\int_s^{T-s} B(\Gamma)(t)dt$$

$$= \int_s^{T-s}\int_\Omega (\gamma u_{tt} - f)\, M_h(z) d\Omega dt + \int_\Omega z_t M_h(z) d\Omega\Big|_s^{T-s}$$
$$+ \frac{b}{2}\int_s^{T-s}\int_\Omega J(h)|\nabla z|^2 d\Omega dt - \frac{b}{2}\int_s^{T-s}\int_\Omega z\nabla z\cdot\nabla(\mathrm{div}(h)) d\Omega dt. \tag{18}$$

by defining $M_h(z) := h\cdot\nabla z + \frac{1}{2}z\mathrm{div}(h)$ and

$$B(\Gamma) := \frac{1}{2}\int_\Gamma \left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu) d\Gamma + b\int_\Gamma \partial_\nu z M_h(z) d\Gamma.$$

Now, the second part of geometrical condition (4.2) allow us to obtain an estimate for the potential z–energy. Since $M_h(z)$ is controlled by the potential energy

$$\|M_h(z)\|_2 \leqslant \sup_{x\in\overline{\Omega}}(|h(x)|{+}\mathrm{div}(h)(x)|)\left(\|\nabla z\|_2 + \frac{1}{2}\|z\|_2\right) \lesssim \|z\|_{H^1(\Omega)} \lesssim \|A^{1/2}z\|_2, \tag{19}$$

due to Robin boundary condition imposed on Γ_0 which allows to control L^2 norms by the gradient. Moreover, the last term in (4.2) can be estimated as

$$\left|\int_s^{T-s}\int_\Omega z\nabla z\nabla(\mathrm{div}(h)) d\Omega dt\right| \lesssim \varepsilon\int_s^{T-s}\int_\Omega |\nabla z|^2 d\Omega dt + C_\varepsilon\|z\|^2_{L^2(s,T-s;L^2(\Omega))}, \tag{20}$$

for any $\varepsilon > 0$, due to Young's inequality and boundedness of D^2h in $\overline{\Omega}$. Similarly, for any $\varepsilon > 0$

$$\left|\int_s^{T-s}\int_\Omega \gamma u_{tt} M_h(z) d\Omega dt\right| \lesssim \varepsilon\int_s^{T-s}\int_\Omega \|A^{1/2}z\|^2 d\Omega dt + C_\varepsilon\int_s^{T-s}\int_\Omega \gamma|u_{tt}|^2 d\Omega dt$$

and finally

$$\int_\Omega z_t M_h(z) d\Omega\Big|_s^{T-s} \lesssim E_1(s) + E_1(T-s). \tag{21}$$

Combining (4.2), (4.2), (4.2) and the second part of assumption (4.2) yields, for $\varepsilon > 0$ sufficiently small,

$$\int_s^{T-s}\int_\Omega |\nabla z|^2 d\Omega dt \lesssim E_1(s) + E_1(T-s) + \int_0^T D_\Psi(s) ds$$
$$+ \int_s^{T-s} B(\Gamma)(t) dt + \int_Q f^2 dQ + \|z\|^2_{L^2(s,T-s;L^2(\Omega))}. \tag{22}$$

The reconstruction of kinetic energy comes next. To this end we multiply (4.1) by z and again integrate by parts over $(s, T-s)\times\Omega$ to obtain

$$\int_s^{T-s}\int_\Omega \left[b|\nabla z|^2 - z_t^2\right] d\Omega dt + b\int_s^{T-s}\int_\Gamma z\partial_\nu d\Gamma dt =$$
$$-\int_s^{T-s}\int_\Omega \gamma u_{tt} z d\Omega dt - \int_\Omega z_t z d\Omega\Big|_s^{T-s} + \int_s^{T-s}\int_\Omega f z d\Omega dt. \tag{23}$$

Identity (4.2) implies the following upper estimate for the kinetic energy

$$\int_s^{T-s}\int_\Omega |z_t|^2 d\Omega dt \lesssim [E_1(s)+E_1(T-s)] + \int_s^{T-s}\int_\Omega |\nabla z|^2 d\Omega dt + \int_s^{T-s}\int_\Gamma z\partial_\nu z d\Gamma dt + \int_0^T D_\Psi(s)ds + \int_Q f^2 dQ + \|z\|^2_{L^2(s,T-s;L^2(\Omega))} \tag{24}$$

and then combining (4.2) and (4.2) (adding and subtracting the boundary term defining $\|A^{1/2}z\|_2^2$ from the gradient), and noticing that the term $\|\gamma^{1/2}u_t\|_2$ in $E_1(t)$ can be estimates from the first two inequalities, we conclude

$$\int_s^{T-s} E_1(t)dt \lesssim E_1(s)+E_1(T-s)+\int_0^T D_\Psi(s)ds + \int_s^{T-s}\tilde{B}(\Gamma)(t)dt + \int_Q f^2 dQ + \|z\|^2_{L^2(s,T-s;L^2(\Omega))}. \tag{25}$$

where

$$\tilde{B}(\Gamma) := \frac{1}{2}\int_\Gamma \left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma + b\int_\Gamma \partial_\nu z M_h(z)d\Gamma + \int_\Gamma z\partial_\nu z d\Gamma + \int_{\Gamma_0}\kappa_0|z|^2 d\Gamma_0 \tag{26}$$

We notice that all the computations carried up to now required only coercivity of the Jacobian (4.2). Only at this point in the analysis of the boundary term $\tilde{B}(\Gamma)$ will we use the property (4.2),

Lemma 4.6 (Key Lemma). *Let Assumption 2.3 be valid. The boundary term $\tilde{B}(\Gamma)$ satisfies the following estimate*

$$\int_s^{T-s}\tilde{B}(\Gamma)(t)dt \lesssim E_1(s) + \varepsilon\int_s^{T-s}\|\nabla z\|_2^2 dt + C_T\int_0^T D_\Psi(s)ds + C_T\|f\|^2_{H^{-1/2+\delta}(Q)} + C_{T,\varepsilon}lot_\delta(z),$$

where $lot_\delta(z)$ has the properties stated in Proposition 4.5.

Proof. We need to estimate all the terms in (4.2). We immediately notice that the first boundary term contains $|z_t|^2 - |\nabla_\tau z|^2$ evaluated on the boundary. However, on Γ_0 we have no information on either tangential derivative – which provides contribution unbounded with respect to the energy level. And it is at this point that we will be using orthogonality of the constructed vector field h with respect to the normal direction to the boundary. The details are given below.

First note that

$$\begin{aligned}
&\int_s^{T-s}\int_\Gamma z\partial_\nu z d\Gamma dt \\
&= -\int_s^{T-s}\int_{\Gamma_0} z\left(\kappa_0(x)z\right)d\Gamma_0 dt + \int_s^{T-s}\int_{\Gamma_1} z\left(-\kappa_1(x)z_t\right)d\Gamma_1 dt \\
&= -\int_s^{T-s}\int_{\Gamma_0}\kappa_0 z^2 d\Gamma_0 - \frac{1}{2}\int_{\Gamma_1}\kappa_1(x)z^2(T-s)d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1}\kappa_1(x)z^2(s)d\Gamma_1 \\
&\leqslant \frac{1}{2}\int_{\Gamma_1}\kappa_1(x)z^2(s)d\Gamma_1 \lesssim \|A^{1/2}z(s)\|_2^2 \lesssim E_1(s),
\end{aligned} \tag{27}$$

due to trace inequality on Γ_1 and $k_1 \in L^\infty(\Gamma_1)$. Next, we notice that

$$\begin{aligned}
&\int_s^{T-s}\int_\Gamma \left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma dt\\
&= \int_s^{T-s}\left(\int_{\Gamma_0}+\int_{\Gamma_1}\right)\left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma dt\\
&\lesssim \int_0^T D_\Psi(s)ds - b\int_s^{T-s}\int_{\Gamma_1}|\nabla z|^2(h\cdot\nu)d\Gamma dt,
\end{aligned} \tag{28}$$

due to $h\cdot\nu = 0$ on Γ_0 and the definition of the damping term (4.2).

Remark 4.7. Notice that assuming only $h\cdot\nu \leqslant 0$, would allow to dispense with the term

$$\int_s^{T-s}\int_{\Gamma_0} z_t^2(h\cdot\nu) \leqslant 0.$$

However, the gradient term

$$\int_s^{T-s}\int_{\Gamma_0}|\nabla z|^2(h\cdot\nu) = \int_s^{T-s}\int_{\Gamma_0}(|\partial_\nu z|^2 + |\partial_\tau z|^2)(h\cdot\nu)$$

where ∂_τ indicates derivative in the tangential direction, poses difficulties. Boundary condition on Γ_0 provide good estimate for the first part. However, for the second no estimate is available unless $z_t|_{\Gamma_0}$ is under control, which may be given through the dissipation or geometry $h\cdot\nu = 0$.

Back to (4.2), for Γ_1 a tangential–trace estimate is available. In fact, using microlocal analysis estimate in [29, 26] one obtains

$$\begin{aligned}
&\int_s^{T-s}\int_{\Gamma_1}|\partial_\tau z|^2 d\Gamma_1 dt\\
&\leqslant C_T\int_0^T\int_{\Gamma_1}(|\partial_\nu z|^2 + z_t^2)d\Gamma_1 dt\\
&+ C_T\left[\|\gamma u_{tt} + f\|^2_{H^{-1/2+\delta}(Q)} + lot_\delta(z)\right]\\
&\lesssim C_T\int_0^T D_\Psi(s)ds + C_T\|f\|^2_{H^{-1/2+\delta}(Q)} + C_T lot_\delta(z),
\end{aligned} \tag{29}$$

with $lot_\delta(z)$ complying with the condition stated in Proposition 4.5. With the above we then improve estimate (4.2) as follows

$$\begin{aligned}
&\int_s^{T-s}\int_\Gamma\left(z_t^2 - b|\nabla z|^2\right)(h\cdot\nu)d\Gamma dt\\
&\lesssim \int_0^T D_\Psi(s)ds + \int_s^{T-s}\int_{\Gamma_1}|\partial_\tau z|^2 d\Gamma dt\\
&\lesssim C_T\int_0^T D_\Psi(s)ds + C_T\|f\|^2_{H^{-1/2+\delta}(Q)} + C_T lot_\delta(z),
\end{aligned}$$

due to (4.2). Finally, we tackle the tangential derivatives on the boundary. We notice first that,

$$\int_s^{T-s}\int_\Gamma \partial_\nu z M_h(z)d\Gamma dt = \int_s^{T-s}\int_\Gamma \partial_\nu z\left(h\cdot\nabla z + \frac{1}{2}z\,\mathrm{div}(h)\right)d\Gamma dt \tag{30}$$

$$\lesssim \int_s^{T-s} \int_\Gamma \partial_\nu z\,(h\cdot\nabla z)\,d\Gamma dt + E_1(s),$$

where we have used the fact that the second integral in (4.2) is exactly the one in (4.2) up to an uniformly bounded term. For the first integral in (4.2), we use the identity

$$\partial_\nu z(h\cdot\nabla z) = |\partial_\nu z|^2(h\cdot\nu) + \partial_\nu z\partial_\tau z(h\cdot\tau)$$

which is obtained by writing the coordinates of the vector ∇z in the basis $\{\tau,\nu\}$. This allows us to write, recalling the damping terms (4.2) and the tangential trace inequality (4.2):

$$\begin{aligned}\int_s^{T-s}\!\!\int_\Gamma \partial_\nu z\,(h\cdot\nabla z)\,d\Gamma dt \lesssim \int_s^{T-s}\!\!\int_{\Gamma_0} \partial_\nu z\partial_\tau z(h\cdot\tau)\\ + C_T\left[\int_0^T D_\Psi(s)ds + \|f\|^2_{H^{-1/2+\delta}(Q)} + lot_\delta(z)\right].\end{aligned}$$

Now notice that the κ_0 in Assumption 2.3 and the trace theorem applied to tangenial derivatives imply that, at each time $t>0$, $\partial_\nu z = -\kappa_0 z \in H^{1/2}(\Gamma_0), \partial_\tau z \in H^{-1/2}(\Gamma_0)$, hence $\int_{\Gamma_0}\partial_\nu z\partial_\tau z(h\cdot\tau)$ is well defined via duality. Thus, taking into consideration the fact that support of κ_0 is compact in Γ_0 and again the regularity of κ_0 in Assumption 2.3, we estimate

$$\begin{aligned}\int_s^{T-s}\int_{\Gamma_0} \partial_\nu z\partial_\tau z(h\cdot\tau)d\Gamma_0 dt = -\int_s^{T-s}\int_{\Gamma_0}\kappa_0 z\partial_\tau z(h\cdot\tau)d\Gamma_0 dt\\ = -1/2\int_s^{T-s}\int_{\Gamma_0}\kappa_0\partial_\tau|z|^2(h\cdot\tau)d\Gamma_0 \lesssim C_{\kappa_0}\int_s^{T-s}\int_{\Gamma_0}|z|^2 d\Gamma_0 dt\\ \lesssim \varepsilon\int_s^{T-s}\|z\|^2_{H^1(\Omega)}dt + C_{T,\varepsilon}\|z\|^2_{L_2(\Omega)}. \qquad (31)\end{aligned}$$

Estimate (4.2) finishes the proof of Lemma 4.6. □

Finally, Lemma 4.6 and the inequality (4.2) yield Proposition 4.5 after taking ε small enough. □

The next result aims at improving Lemma 4.5 by absorbing $lot_\delta(z)$ via the damping. For the linear problem, this is done by a compactness uniqueness argument as in [5]. The corresponding result is stated below.

Proposition 4.8. *For $T>0$ there exists a constant $C_T>0$ such that the following inequality holds:*

$$lot_\delta(z) \le C_T\int_0^T D_\Psi(s)ds$$

Compactness follows from the compactness of Sobolev's embeddings implicated in the definition of lower order terms with respect to the finite energy space for variables (z,z_t) which are $H^1(\Omega)\times L^2(\Omega)$. Uniqueness, instead, follows from the overdetermination of the wave equation with overdetermined Neuman–Dirichlet data on the boundary Γ_1. The details are in [5], 4.3. **Completion of the Proof of Theorem 2.4.** We are ready to establish the exponential decay of the the energy functional E_1.

Theorem 4.9. *Let Assumption 2.3 be valid. Assume that $f = 0$. The energy functional E_1 is exponentially stable, i.e. there exists $T > 0$ and constants $M, \omega > 0$ such that*

$$E_1(t) \leq Me^{-\omega t}E_1(0), \quad \textit{for } t \geqslant 0. \tag{32}$$

Proof. For $f = 0$, identity (4.2) implies

$$\left(\int_0^s + \int_{T-s}^T\right) E_1(t)dt \leq 2sE_1(0).$$

Since $s < T/2$ can be taken arbitrarily small, we fix $s < 1/2$ in the above inequality. Then by dissipativity of E_1 (for $f = 0$) along with Propositions 4.5 and 4.8 we infer

$$\int_0^T E_1(t)dt \lesssim E_1(T) + C_T\int_0^T D_\Psi(s)ds. \tag{33}$$

On the other hand, using identity (4.2) (with $f = 0$) once more, we deduce

$$TE_1(T) \lesssim \int_0^T E_1(t)dt + C_T\int_0^T D_\Psi(s)ds. \tag{34}$$

Combining (4.3) and (4.3) we arrive at

$$(T - C)E_1(T) + \int_0^T E_1(t)dt \leq C_T\int_0^T D_\Psi(s)ds$$

for some $C > 0$. Choosing $T = 2C$ and replacing the "damping" term using identity (4.2) (with $f = 0$) we rewrite the above estimate as follows

$$E_1(T) + \int_0^T E_1(t)dt \lesssim C_T[E_1(0) - E_1(T)]$$

which implies

$$E_1(T) \leqslant \frac{C_T}{1 + C_T}E_1(0) = \mu E_1(0),$$

where $0 < \mu < 1$ does not depend on the solution. This implies (4.9) with $\omega = |\ln\mu|/T$ and $M = 1/\mu$. $\square$

The result of Theorem 4.9 is the key to establishing the exponential stability of the semigroup $S(t)$, generated by $\mathbb{A}$ on $\mathbb{H}$.

Notice that the exponential decay for E_1 implies exponential decay of the quantities $\|z\|_{\mathcal{D}(A^{1/2})}, \|z_t\|_{L^2(\Omega)}$, and we will show that this implies exponential decay for the total energy $E(t)$, provided that the initial data u_0 is controlled with respect to the topology induced by $A^{1/2}$. For this, the only remaining quantity we need to show exponential decay is $\|u\|_{\mathcal{D}(A^{1/2})}$ and this follows from the abstract ODE $bu_t + c^2u = z$.The details for this step are the same as in [5].

To complete the proof of Theorem 2.4, the exponential stability on $\mathbb{H}_1$ is the main task remained. This is argued below. The first step is to derive energy estimate for the higher order energy functional E_2. We start with the following multiplier identity.

Proposition 4.10. *Let $\Psi = (u, z, z_t)$ be a weak solution for (4.1). Then for all $0 \leqslant s < t \leqslant T$ the following identity holds*

$$b\int_s^t (\Delta z, \Delta u) d\sigma = \left[(z_t, \Delta u) + \frac{1}{2}\|\nabla z\|_2^2\right]\Big|_s^t + \int_s^t \int_\Gamma z_t \partial_\nu z d\Gamma d\sigma$$
$$+ \int_s^t \left[\left(\frac{c^2}{b} z_t + \gamma u_{tt}, \Delta u\right)\right] d\sigma - \int_s^t (f, \Delta u) d\sigma \tag{35}$$

Proof. Arguing first for "strong" solutions, we have $\Delta u, \Delta u_t \in L^2(\Omega)$ -the latter allows to justify the formalism of calculations. We compute

$$\begin{aligned}
(z_{tt}, \Delta u) &= \frac{d}{dt}(z_t, \Delta u) - (z_t, \Delta u_t) \\
&= \frac{d}{dt}(z_t, \Delta u) - \left(z_t, \Delta\left(z - \frac{c^2}{b} u\right)\right) \\
&= \frac{d}{dt}(z_t, \Delta u) + \frac{1}{2}\frac{d}{dt}\|\nabla z\|_2^2 \\
&- \frac{1}{\lambda}\int_{\Gamma_0} z_t(\kappa_0(x) z)\Gamma_0 - \int_{\Gamma_1} z_t(\kappa_1(x) z_t) d\Gamma_1 + \left(\frac{c^2}{b} z, \Delta u\right) \\
&= \frac{d}{dt}(z_t, \Delta u) + \frac{1}{2}\frac{d}{dt}\|\nabla z\|_2^2 - \frac{1}{2\lambda}\frac{d}{dt}\|\kappa_0^{1/2} z\|_{\Gamma_0}^2 - \|\kappa_1^{1/2} z_t\|_{\Gamma_1}^2 + \left(\frac{c^2}{b} z, \Delta u\right).
\end{aligned}$$

Thus, taking the L^2–inner product of $z_{tt} - b\Delta z = -\gamma u_{tt} + f$ with $\Delta u \in L^2(\Omega)$ gives

$$\begin{aligned}
b\int_s^t (\Delta z, \Delta u) d\sigma &= \int_s^t (z_{tt} + \gamma u_{tt} - f, \Delta u) d\sigma \\
&= \int_s^t (z_{tt}, \Delta u) d\sigma + \int_s^t (\gamma u_{tt} - f, \Delta u) d\sigma \\
&= \left[(z_t, \Delta u) + \frac{1}{2}\|\nabla z\|_2^2 - \frac{1}{2\lambda}\|\kappa_0^{1/2} z\|_{\Gamma_0}^2\right]\Big|_s^t - \int_s^t \|\kappa_1^{1/2} z_t\|_{\Gamma_1}^2 d\sigma \\
&+ \int_s^t \left[\left(\frac{c^2}{b} z_t + \gamma u_{tt}, \Delta u\right)\right] d\sigma - \int_s^t (f, \Delta u) d\sigma.
\end{aligned}$$

We now derive the estimate for E_2. We take the L^2–inner product of

$$\Delta(bu_t + c^2 u) = b\Delta z$$

with Δu and integrate in time. By connecting it with (4.10) one obtains that

$$b\|\Delta u\|_2^2 + c^2\int_0^T \|\Delta u\|_2^2 = b\|\Delta u_0\|_2^2 + b\int_0^T (\Delta z, \Delta u) = b\|\Delta u_0\|_2^2$$
$$+ \left[(z_t, \Delta u) + \frac{1}{2}\|\nabla z\|_2^2 - \frac{1}{2\lambda}\|\kappa_0^{1/2} z\|_{\Gamma_0}^2\right]\Big|_0^T - \int_0^T \|\kappa_1^{1/2} z_t\|_{\Gamma_1}^2 d\sigma$$
$$+ \int_0^T \left[\left(\frac{c^2}{b} z_t + \gamma u_{tt}, \Delta u\right)\right] d\sigma - \int_0^T (f, \Delta u) d\sigma \tag{36}$$

Since all terms in (4.3) are benign in the sense that all (but f and Δu) are either subordinated by $E(t)$ or they are bounded above by the damping, it follows that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$b\|\Delta u(T)\|_2^2 + c^2\int_0^T \|\Delta u\|_2^2 \lesssim \mathcal{E}(0) + \varepsilon\left(\|\Delta u\|_2^2 + \int_0^T \|\Delta u\|_2^2 d\sigma\right)$$

$$+ C_\varepsilon \left(E_1(t) + \int_0^T E_1(\sigma)d\sigma + \int_Q f^2 dQ \right).$$

Then, taking ε small and using (4.9) we have

$$b\|\Delta u(T)\|_2^2 + c^2 \int_0^T \|\Delta u\|_2^2 \lesssim \mathcal{E}(0) + \int_0^T E_1(\sigma)d\sigma + \int_Q f^2 dQ, \tag{37}$$

From (4.3) we have obtained that $\Delta u(t) \in L^2(\Omega)$. In addition $(u, u_t, u_{tt}) \in \mathbb{H}$ implies that $u(t) \in H^1(\Omega)$ and $u_t(t) \in H^1(\Omega)$. By standard duality argument one this obtains that $\partial_\nu u(t) \in H^{-1/2}(\Gamma)$. We will be able to improve this regularity by appealing to $\mathbb{H}$ regularity already obtained in the previous section. On the other hand, by using invariance of boundary conditions along with the fact that κ_i are the multipliers on $H^{1/2-\varepsilon}(\Gamma_i)$ (recall $\kappa_i \in B^{1/2}_{4,\infty}(\Gamma_i)$ [34]) , we also have

$$\partial_\nu u(t)|_{\Gamma_0} = -\kappa_0 u(t) \in H^{1/2-\varepsilon}(\Gamma_0) \qquad \partial_\nu u(t)|_{\Gamma_1} = -\kappa_1 u_t(t) \in H^{1/2-\varepsilon}(\Gamma_1).$$

By the definition of the norm in $\mathbb{H}_1$, the above implies that $(u, u_t, u_{tt}) \in \mathbb{H}_1$, as desired. Moreover we have a control of the norms:

$$\begin{aligned} \|(u, u_t, u_{tt})\|_{\mathbb{H}_1} \leqslant C [\|(u, u_t, u_{tt})\|_{\mathbb{H}} &+ \|\Delta u(t)\|_2 \\ &+ \|\sqrt{\kappa_0} u(t)\|_{H^{1/2-\varepsilon}(\Gamma_0)} + \|\sqrt{\kappa_1} u_t(t)\|_{H^{1/2-\varepsilon}(\Gamma_1)} \end{aligned}$$

which proves the desired regularity in $\mathbb{H}_1$. We are ready to complete the proof of Theorem 2.4.

Recall that $\mathcal{E}(t) \coloneqq E(t) + E_2(t)$ and let $f = 0$. Adding $E(T) + \int_0^T E(\sigma)d\sigma$ to both sides of (4.3) we obtain,

$$\begin{aligned} \mathcal{E}(T) + \int_0^T \mathcal{E}(\sigma)d\sigma \lesssim \mathcal{E}(0) + E(T) + \int_0^T E(\sigma)d\sigma \\ &\leqslant \mathcal{E}(0) + ME(0)e^{-\omega t} + ME(0)\int_0^t e^{-\omega\sigma}d\sigma \\ &= \mathcal{E}(0) + ME(0)e^{-\omega t} - \omega^{-1}ME(0)\left[e^{-\omega t} - 1\right] < +\infty, \end{aligned}$$

for all $t \geqslant 0$, for some $\omega, M > 0$. By making $T \to \infty$ we see that

$$\int_0^\infty \mathcal{E}(\sigma)d\sigma < +\infty,$$

and the result follows by Pazy–Datko's Theorem [35]. □

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5. **Proof of Theorem 2.5 – Construction of Global $\mathbb{H}_1$– valued Solutions.**

Our goal now is to prove that fixed–point solutions can be constructed for the nonlinear problem in $\mathbb{H}_1$. To this end, fix $r > 0$ such that $\|\Phi_0\|_{\mathbb{H}_1} \leqslant r$ and let X_r^β be the set defined as

$$X_r^\beta = \left\{ \Psi = \begin{bmatrix} w \\ w_t \\ w_{tt} \end{bmatrix} \in C([0,T];\mathbb{H}_1);\ \sup_{t\in[0,T]} \|\Psi(t)\|_{\mathbb{H}_1} \lesssim r+1 \text{ and } \sup_{t\in[0,T]} \|\Psi(t)\|_{\mathbb{H}} < \beta \right\}$$

where $\beta > 0$ is for the time being a given positive number but we will take it to be sufficiently small later. Moreover, the condition $\sup\limits_{t\in[0,T]} \|\Psi(t)\|_{\mathbb{H}_1} \lesssim r+1$ simply means that solutions will exist in bounded sets of $C([0,T];\mathbb{H}_1)$ with respect to $\mathbb{H}_1$

but this introduces no small size restriction on the data in $\mathbb{H}_1$. The number 1 could, then, be replaced by any other positive number. Let's equip X_r^β it with the norm

$$\|\Psi\|^2_{X_r^\beta} := \sup_{t\in[0,T]} \|\Psi(t)\|^2_{\mathbb{H}_1}; \Psi \in X_r^\beta$$

We start with a following regularity lemma.

Lemma 5.1. *For $\Psi = (w, w_t, w_{tt})^\top$, let the action $\mathcal{F}$ on Ψ be given by*

$$\mathcal{F}(\Psi) = \begin{bmatrix} 0 \\ 0 \\ w_t^2 + ww_{tt} \end{bmatrix}.$$

Then the following assertions hold true:

(i) *$\mathcal{F}$ defines a continuous map $\mathcal{F} : X_r^\beta \to C([0, T]; \mathbb{H}_1)$ and for each t the inequality*

$$\|\mathcal{F}(\Psi(t))\|_{\mathbb{H}_1} \leqslant C\beta\|\Psi(t)\|_{\mathbb{H}_1}, \qquad \Psi \in X_r^\beta \tag{38}$$

holds for some $C > 0$ fixed.

(ii) *Stronger than continuity, the following estimate holds:*

$$\|\mathcal{F}(\Phi)\|_{C([0,T];\mathbb{H}_1)} \lesssim \beta^2 + \beta^{1+\alpha}(r + 1)^{1-\alpha}, \alpha \in (0, 1/2). \tag{39}$$

Proof. Recall that $w_t \in H^1_{\Gamma_1}(\Omega) \hookrightarrow L^6(\Omega)$ and then $w_t^2 \in C([0, T]; L^2(\Omega))$. Moreover, since $H^\theta(\Omega) \hookrightarrow L^\infty(\Omega)$ for $\theta > 3/2$, it follows that $ww_{tt} \in C([0, T]; L^2(\Omega))$. We shall frequently invoke the following interpolation inequalities.

$$\begin{gathered} \|w\|_4 \lesssim \|w\|_2^{1/4}\|w\|^{3/4}_{H^1} \\ \|w\|_\infty \lesssim \|w\|^\alpha_{H^1}\|w\|^{1-\alpha}_{H^\theta}, \alpha = \frac{\theta - 3/2 - \varepsilon}{\theta - 1}, \varepsilon > 0. \end{gathered} \tag{40}$$

where we recall $\mathbb{H}_1 \subset H^\theta(\Omega) \times H^1(\Omega) \times L_2(\Omega)$ with $3/2 < \theta < 2$. So that with $\alpha \in (0, 1/2)$,

$$\begin{aligned} \|\mathcal{F}(\Phi)(t)\|_{\mathbb{H}_1} &= \|w_t^2(t) + w(t)w_{tt}(t)\|_2 \lesssim \|w_t(t)\|_4^2 + \|w(t)\|_\infty\|w_{tt}(t)\|_2 \\ &\lesssim \left\|A^{1/2}w_t(t)\right\|_2^2 + \left\|A^{1/2}w(t)\right\|_2^\alpha \|w(t)\|^{1-\alpha}_{H^\theta(\Omega)}\|w_{tt}(t)\|_2 \\ &\lesssim \|\Psi(t)\|^2_{\mathbb{H}} + ||\Psi(t)||^{1+\alpha}_{\mathbb{H}}\|\Psi(t)\|^{1-\alpha}_{\mathbb{H}_1} \lesssim \beta\|\Psi(t)\|_{\mathbb{H}_1} \end{aligned} \tag{41}$$

which yields (5.1) and, by taking the supremum over time t on both sides, (5.1) follows. Moreover, returning to the intermediate estimate (5), we further notice

$$\begin{aligned} \|\mathcal{F}(\Phi)(t)\|_{\mathbb{H}_1} &\lesssim \left\|A^{1/2}w_t\right\|_2^2 + \left\|A^{1/2}w\right\|_2^\alpha \|w\|^{1-\alpha}_{H^\theta(\Omega)}\|w_{tt}\|_2 \\ &\lesssim \left[\sup_{t\in[0,T]} \|\Psi(t)\|_{\mathbb{H}}\right]^2 + \left[\sup_{t\in[0,T]} \|\Psi(t)\|_{\mathbb{H}}\right]^{1+\alpha} \left[\sup_{t\in[0,T]} \|\Psi(t)\|_{\mathbb{H}_1}\right]^{1-\alpha} \\ &\lesssim \beta^2 + \beta^{1+\alpha} \sup_{t\in[0,T]} \|\Psi(t)\|^{1-\alpha}_{\mathbb{H}_1}, \end{aligned}$$

which yields (5.1) and completes the proof. $\square$

The validity of the previous Lemma along with the fact that $\mathcal{A}$ generates C_0-semigroups $T(t)$ and $S(t)$ on $\mathbb{H}_1$ and $\mathbb{H}$ respectively, guarantees that, for each $\Psi \in X$ there exists a unique $\Phi = (u, u_t, u_{tt})^\top =: \Theta(\Psi) \in C([0, T]; \mathbb{H}_1)$ solution of (2.1) characterized as the variation of parameters formula with forcing term $\mathcal{F}(\Psi)$ and initial condition $\Phi_0 = (u_0, u_1, u_2) \in \mathbb{H}_1$, i.e.,

$$\Theta(\Psi)(t) = T(t)\Phi_0 + \int_0^t T(t-\sigma)\mathcal{F}(\Psi)(\sigma)d\sigma \tag{42}$$

Note that the same formula is valid if we replace $T(t)$ by $S(t)$. Moreover, uniform exponential stability implies the existence of positive constants $\omega_0, \omega_1, M_0, M_1 > 0$ such that

$$\|T(t)\Phi_0\|_{\mathbb{H}_1} \leqslant M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1} \text{ and } \|T(t)\Phi_0\|_{\mathbb{H}} \leqslant M_0 e^{-\omega_0 t}\|\Phi_0\|_{\mathbb{H}} \tag{43}$$

for all $t \geqslant 0$. Among other properties, the exponential stability of the linear problem implies invariance of the map Θ in X_r^β, as we make precise below.

Lemma 5.2. *Given $\Phi_0 \in \mathbb{H}_1$ such that $\|\Phi_0\|_{\mathbb{H}_1} \leq \frac{r}{2M}$, $M \equiv \max\{M_0, M_1\}$, there exist $\beta > 0$ and $\rho_\beta > 0$ with the property that if $\|\Phi_0\|_{\mathbb{H}} < \rho_\beta$ then the map Θ is X_r^β–invariant.*

Proof. Proving this claim is equivalent to prove that there exists $\beta > 0$ for which $\|\Theta(\Psi)(t)\|_{\mathbb{H}_1} \lesssim r + 1$ and $\|\Theta(\Psi)(t)\|_{\mathbb{H}} < \beta$ for all $t \in [0, T)$ and each $\Psi \in X_r^\beta$, provided $\|\Phi_0\|_{\mathbb{H}} < \rho_\beta$, with ρ_β conveniently chosen. From (5) and (5) it follows, for each $t \in [0, T)$,

$$\begin{aligned}
\|\Theta(\Psi)(t)\|_{\mathbb{H}_1} &\leqslant \|T(t)\Phi_0\|_{\mathbb{H}_1} + \int_0^t \|T(t-\sigma)\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}_1} d\sigma \\
&\leqslant M_1 \left(\|\Phi_0\|_{\mathbb{H}_1} + \int_0^t e^{-\omega_1(t-\sigma)}\|\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}_1} d\sigma \right) \\
&\lesssim M_1[\|\Phi_0\|_{\mathbb{H}_1} + C_{\omega_1} \sup_{t\in[0,T]} \|\mathcal{F}(\Psi)(t)\|_{\mathbb{H}_1}] \\
&\lesssim M_1 C + M_1 C_{\omega_1} \left(\beta^2 + \beta^{\alpha+1}(r+1)^{1-\alpha}\right) \lesssim r + 1,
\end{aligned} \tag{44}$$

provided β is sufficiently small. Moreover, by Lemma 5.1 (and again (5) and (5))

$$\begin{aligned}
\|\Theta(\Psi)(t)\|_{\mathbb{H}} &\leqslant \|T(t)\Phi_0\|_{\mathbb{H}} + \int_0^t \|T(t-\sigma)\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}} d\sigma \\
&\leqslant M_0 \left(\|\Phi_0\|_{\mathbb{H}} + \int_0^t e^{-\omega_0(t-\sigma)}\|\mathcal{F}(\Psi)(\sigma)\|_{\mathbb{H}_1} d\sigma \right) \\
&\lesssim M_0\|\Phi_0\|_{\mathbb{H}} + M_0 C_{\omega_0} \sup_{t\in[0,T]} \|\mathcal{F}(\Psi)(t)\|_{\mathbb{H}_1} \\
&\lesssim \rho_\beta + \left(\beta^2 + \beta^{\alpha+1}(r+1)^{1-\alpha}\right) < \beta,
\end{aligned} \tag{45}$$

provided β and $\rho_\beta < 1/2\beta$ are sufficiently small. ☐

We are then ready to prove that for a (possibly smaller) β, the map Θ is a contraction.

Lemma 5.3. *There exist $\beta > 0$ and $\rho_\beta > 0$ with the property that if $\|\Phi_0\|_{\mathbb{H}} < \rho_\beta$ then Θ is a contraction.*

Proof. Let $\Psi_1, \Psi_2 \in X_r^\beta$, $\Psi_1 = (v, v_t, v_{tt})^\top$ and $\Psi_2 = (w, w_t, w_{tt})^\top$. The key point of this proof is to estimate $\|\mathcal{F}(\Psi_1) - \mathcal{F}(\Psi_2)\|_{C([0,T];\mathbb{H}_1)}$, which is where we start.

First notice that, since the first two coordinates of both $\mathcal{F}(\Psi_1)$ and $\mathcal{F}(\Psi_2)$ are zero, we just care about the third one, whose difference, for each t, is given by

$$\begin{aligned} v_t^2 + vv_{tt} - w_t^2 - ww_{tt} &= \underbrace{(v_t + w_t)(v_t - w_t)}_{=I_1} + \underbrace{(v - w)v_{tt}}_{=I_2} + \underbrace{w(v_{tt} - w_{tt})}_{=I_3} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Now we estimate the supremmum of the L^2-norm of I_1. For this we notice that a combination of Holder's inequality with the Sobolev embedding $H^1_{\Gamma_1}(\Omega) \hookrightarrow L^4(\Omega)$ yields

$$\begin{aligned} \|I_1\|_2 = \|(v_t + w_t)(v_t - w_t)\|_2 &\leqslant (\|v_t\|_4 + \|w_t\|_4)\, \|v_t - w_t\|_4 \\ &\lesssim (\|\nabla v_t\|_2 + \|\nabla w_t\|_2)\, \|\nabla(v_t - w_t)\|_2 \lesssim \beta \|\Psi_1 - \Psi_2\|_{X_r^\beta}, \end{aligned}$$

for each t. Then $\sup\limits_{t\in[0,T]} \|I_1\|_2 \leqslant \beta\|\Psi_1 - \Psi_2\|_{X_r^\beta}$. Next, for estimating the suppremum of the L^2–norm of I_2 we notice that the Sobolev embedding $H^\theta_{\Gamma_1} \hookrightarrow L^\infty(\Omega)$ yields

$$\begin{aligned} \|I_2\|_2 = \|v_{tt}(v - w)\|_2 &\leqslant \|v_{tt}\|_2 \|v - w\|_\infty \\ &\lesssim \|v_{tt}\|_2 \|(v - w)\|_{H^\theta(\Omega)} \lesssim \beta \|\Psi_1 - \Psi_2\|_{X_r^\beta}, \end{aligned}$$

for each $t \in [0, T)$. Then $\sup\limits_{t\in[0,T]} \|I_2(t)\|_2 \leqslant \beta \sup\limits_{t\in[0,T]} \|\Psi_1(t) - \Psi_2(t)\|_{X_r^\beta}$. Finally, for estimating the supremum of the L^2–norm of I_3 we will use the (in addition to the Sobolev emdedding $H^\theta_{\Gamma_1}(\Omega) \hookrightarrow L^\infty(\Omega)$) the interpolation inequality (5) which holds for all $w \in H^\theta_{\Gamma_1}(\Omega)$. We have

$$\begin{aligned} \|I_3\|_2 = \|w(v_{tt} - w_{tt})\|_2 &\leqslant \|w\|_\infty \|v_{tt} - w_{tt}\|_2 \\ &\lesssim \|\nabla w\|_2^\alpha \|\ w\|^{1-\alpha}_{H^\theta(\Omega)} \|v_{tt} - w_{tt}\|_2 \\ &\lesssim \beta^\alpha \|\Psi_2\|^{1-\alpha}_{\mathbb{H}_1}\ \|\Psi_1 - \Psi_2\|_{X_r^\beta} \lesssim \beta^\alpha (r+1)^{1-\alpha} \|\Psi_1 - \Psi_2\|_{X_r^\beta}, \end{aligned}$$

for each $t \in [0, T)$. Then $\sup\limits_{t\in[0,T]} \|I_3(t)\|_2 \lesssim \beta^\alpha (r+1)^{1-\alpha} \sup\limits_{t\in[0,T]} ||\Psi_1(t) - \Psi_2(t)\|_{X_r^\beta}$.

The above allows to complete the proof of contractivity:

$$\begin{aligned} \|\Theta(\Psi_1) - \Theta(\Psi_2)\|_{X_r^\beta} &\leq \sup_{t\in[0,T]} \left\| \int_0^t T(t-\sigma)\left[\mathcal{F}(\Psi_1) - \mathcal{F}(\Psi_2)\right] d\sigma \right\|_{\mathbb{H}_1} \\ &\leqslant \frac{C_{\omega_1}}{\tau} \sup_{t\in[0,T]} \tau \|\mathcal{F}(\Psi_1)(t) - \mathcal{F}(\Psi)(t)\|_{\mathbb{H}_1} \\ &\lesssim \sup_{t\in[0,T]} (\|I_1(t)\|_2 + \|I_2(t)\|_2 + \|I_3(t)\|_2) \\ &\lesssim \left(2\beta + \beta^{1-\alpha}(r+1)^{1-\alpha}\right) \|\Psi_1 - \Psi_2\|_{X_r^\beta} \\ &= C_\beta \|\Psi_1 - \Psi_2\|_{X_r^\beta} \end{aligned} \tag{46}$$

owning the property $C_\beta < 1$ achieved through the smallness of β. □

Notice that exponential stability of the linear problem in $\mathbb{H}$ and $\mathbb{H}_1$ allows we to obtain the estimates (5), (5) and (5) with right hand side time–independente, which allows us to take $T = \infty$, yielding global in time solutions. The proof of Theorem 2.5 is completed by taking $\rho = \rho_\beta$.

6. **Proof of Theorem 2.6-Uniform decay rates for the nonlinear problem.** In this section we show that one can easily obtain that the solution of the nonlinear problem decay exponentially to zero as $t \to \infty$ by taking advantage of three facts established in this paper.

(i) The fact that the solution is a fixed point of the map Θ defined in (5), and therefore can be implicitly represented as

$$\Phi(t) = T(t)\Phi_0 + \int_0^t T(t-\sigma)\mathcal{F}(\Phi)(\sigma)d\sigma \tag{47}$$

(ii) The fact that our existence of global solution result requires smallness of initial data *only* in the lower topology and the use of this along with interpolation inequalities allowed us to obtain the key estimate (5.1).

(ii) The fact that the semigroup $T(t)$ in (6) is uniformly exponentially stable in both $\mathbb{H}$ and $\mathbb{H}_1$.

The final result of this section is the following.

Theorem 6.1. *There exists $\rho > 0$ such that the solution Φ constructed in (2.5) is such that*

$$\|\Phi(t)\|_{\mathbb{H}_1} \leqslant 2M_1 e^{-\frac{\omega_1}{2}t}\|\Phi_0\|_{\mathbb{H}_1}$$

for all $t \geqslant 0$, where M_1, ω_1 are the constants describing the uniform stability of the linear semigroup $T(t)$.

The proof of this result relies heavily on the facts (i)–(ii) outlined above and a Grownwall type inequality. This inequality seems to have been originally introduced in [1], but here we are using [2, Corollary 1, p. 389]. We state the inequality here for convenience, but in a version which is suitable for our use in what follows. We invite the reader to consult [1, 2] and references therein for more details

Lemma 6.2 (Grownwall–Beesack Inequality)**.** *Let $u, f, g, h : \mathbb{R} \to \mathbb{R}$ measurable functions such that fh, gh and uh are integrable. If u, f, g, h are nonnegative and*

$$u(t) \leqslant f(t) + g(t)\int_0^t h(\sigma)u(\sigma)d\sigma$$

then

$$u(t) \leqslant f(t) + g(t)\int_0^t f(\sigma)h(\sigma)\exp\left\{\int_\sigma^t g(s)h(s)ds\right\}d\sigma$$

Proof of Theorem 6.1. We use the same constants as in (5), that is, we use that

$$\|T(t)\|_{\mathcal{L}(\mathbb{H}_1)} \leqslant M_1 e^{-\omega_1 t} \tag{48}$$

for all t. Moreover, we know that the solution Φ exists in some X_r^β for $\beta > 0$ small and that the whole argument of the proof for the existence of global solution would still be true if one decreased β. Therefore, by possibly taking it smaller, we assume

$$\beta < \frac{\omega_1}{2M_1 C} \tag{49}$$

where ω_1 is the rate of exponential decay of the semigroup $T(t)$ in $\mathbb{H}_1$ for a fixed $\tau = 1$. As in the proof of global wellposedness, we take $\rho = \rho_\beta$. We then compute, via (6) and (5.1)

$$\|\Phi(t)\|_{\mathbb{H}_1} \leqslant \|T(t)\Phi_0\|_{\mathbb{H}_1} + \int_0^t \|T(t-\sigma)\mathcal{F}(\Phi)(\sigma)\|_{\mathbb{H}_1}d\sigma$$

$$\begin{aligned}&\leqslant M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1} + \int_0^t M_1 e^{-\omega_1 (t-\sigma)}\|\mathcal{F}(\Phi)(\sigma)\|_{\mathbb{H}_1} d\sigma\\ &\leqslant M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1} + M_1 C\beta e^{-\omega_1 t}\int_0^t e^{\omega_1 \sigma}\|\Phi(\sigma)\|_{\mathbb{H}_1} d\sigma.\end{aligned}$$

We then apply the Grownwall–Beesack inequality with

$$u(t) = \|\Phi(t)\|_{\mathbb{H}_1},\quad f(t) = M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1},\quad g(t) = M_1 C\beta e^{-\omega_1 t},\quad h(t) = e^{\omega_1 t}$$

to obtain

$$\begin{aligned}\|\Phi(t)\|_{\mathbb{H}_1} &\leqslant M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1} + M_1^2 C\beta\|\Phi_0\|_{\mathbb{H}_1} e^{-\omega_1 t}\int_0^t \exp\{M_1 C\beta(t-\sigma)\}\, d\sigma\\ &= M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1} + M_1\|\Phi_0\|_{\mathbb{H}_1}\exp\{(M_1 C\beta - \omega_1)\, t\}\,(1 - \exp\{-M_1 C\beta t\})\\ &\leqslant M_1 e^{-\omega_1 t}\|\Phi_0\|_{\mathbb{H}_1} + M_1\|\Phi_0\|_{\mathbb{H}_1}\exp\{(M_1 C\beta - \omega_1)\, t\} \leqslant 2M_1 e^{-\frac{\omega_1}{2}t}\|\Phi_0\|_{\mathbb{H}_1},\end{aligned}$$

and we observe that due to (6) we have

$$M_1 C\beta - \omega_1 < -\frac{\omega_1}{2} < 0.$$

The proof is complete. □

Corollary 1. *Let β_0 be the largest number such that the map Θ has a fixed point in X^{β_0} which is, moreover, uniformly exponentially stable as in Theorem 2.6. Let $\omega : (0, \beta_0] \to \mathbb{R}_+$ be the function that maps each $\beta > 0$ to the decay rate $\omega(\beta)$. Then there exists another function $\underline{\omega} : (0, \beta_0] \to \mathbb{R}_+$ such that $\omega(\beta) \geqslant \underline{\omega}(\beta)$ for all feasible β and*

$$\lim_{\beta\to 0}\underline{\omega}(\beta) = \omega_1,$$

where ω_1 is the decay rate of the linear semigroup $T(t)$.

Proof of Corollary 1. The proof of Theorem 2.6 already provides a proof of Corollary 1. Indeed, it suffices to define $\underline{\omega} : (0, \beta_0] \to \mathbb{R}_+$ by

$$\underline{\omega}(\beta) = \omega_1 - M_1 C\beta > 0.$$

□

As a final note, a question of constructing decay rates which are *independent* on the relaxation parameter is relegated to future work. Such result ,in the case of internal dissipation, has been obtained in [3].

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