Longest cycles in 3-connected hypergraphs and bipartite graphs

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Abstract

In the language of hypergraphs, our main result is a Dirac-type bound: we prove that every 3-connected hypergraph $\mathcal{H}$ with $\delta(\mathcal{H}) \geq \max\{|V(\mathcal{H})|, \frac{|E(\mathcal{H})|+10}{4}\}$ has a hamiltonian Berge cycle. This is sharp and refines a conjecture by Jackson from 1981 (in the language of bipartite graphs). Our proofs are in the language of bipartite graphs, since the incidence graph of each hypergraph is bipartite.

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1 Introduction

1.1 Long cycles in bipartite graphs

For positive integers n, m, and δ with $\delta \leq m$, let $\mathcal{G}(n, m, \delta)$ denote the set of all bipartite graphs with a partition (X, Y) such that $|X| = n \geq 2, |Y| = m$ and for every $x \in X$, $d(x) \geq \delta$. In 1981, Jackson [3] proved that if $\delta \geq \max\{n, \frac{m+2}{2}\}$, then every graph $G \in \mathcal{G}(n, m, \delta)$ contains a cycle of length $2n$, i.e., a cycle that covers X. This result is sharp. Jackson also conjectured that if $G \in \mathcal{G}(n, m, \delta)$ is 2-connected, then the upper bound on m can be weakened.

Conjecture 1.1 (Jackson [3, 4])**.** *Let m, n, δ be integers. If $\delta \geq \max\{n, \frac{m+5}{3}\}$, then every 2-connected graph $G \in \mathcal{G}(n, m, \delta)$ contains a cycle of length $2n$.*

Recently, the conjecture was proved in [7]. The restriction $\delta \geq \frac{m+5}{3}$ cannot be weakened because of the following example.

Construction 1.2. *Let $n_1 \geq n_2 \geq n_3 \geq 1$ be such that $n_1 + n_2 + n_3 = n$. Let $G_3(n_1, n_2, n_3; \delta) \in \mathcal{G}(n, 3\delta - 4, \delta)$ be the bipartite graph obtained from $K_{\delta-2,n_1} \cup K_{\delta-2,n_2} \cup K_{\delta-2,n_3}$ by adding two vertices a and b that are both adjacent to every vertex in the parts of size n_1, n_2, and n_3. Then a longest cycle in $G_3(n_1, n_2, n_3; \delta)$ has length $2(n_1 + n_2) \leq 2(n - 1)$.*

The goal of this paper is to find a best lower bound on δ guaranteeing the existence of a $2n$-cycle in a graph $G \in \mathcal{G}(n, m, \delta)$ if G is not only 2-connected, but 3-connected. The following simple extension of Construction 1.2 shows that the bound could not be larger than $\frac{m+10}{4}$.

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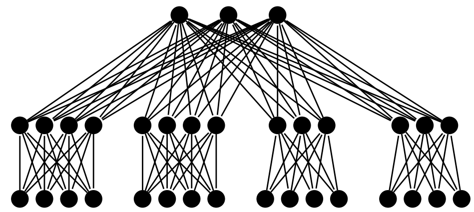

Figure 1: An example of Construction 1.3.

Construction 1.3. *Let $n_1 \geq n_2 \geq n_3 \geq n_4 \geq 1$ be such that $n_1 + n_2 + n_3 + n_4 = n$. Let $G_4(n_1, \ldots, n_4; \delta) \in \mathcal{G}(n, 4\delta - 9, \delta)$ be the bipartite graph obtained from $\bigcup_{j=1}^4 K_{\delta-3,n_j}$ by adding 3 vertices a_1, a_2, a_3, all of which are adjacent to every vertex in the parts of size n_1, n_2, n_3, and n_4. Then a longest cycle in $G_4(n_1, \ldots, n_4; \delta)$ has length $2(n_1 + n_2 + n_3) \leq 2(n - 1)$.*

The main result of the paper is that Construction 1.3 is indeed extremal for 3-connected graphs:

Theorem 1.4. *Let m, n, δ be integers. If $\delta \geq \max\{n, \frac{m+10}{4}\}$, then every 3-connected graph $G \in \mathcal{G}(n, m, \delta)$ contains a cycle of length $2n$.*

We discuss possible extensions of Theorem 1.4 to k-connected bipartite graphs and hypergraphs in concluding remarks. We will apply this theorem in a forthcoming paper on so-called *super-pancyclic* bipartite graphs and hypergraphs. This notion was introduced and discussed in [7].

In the next section, we discuss how Theorem 1.4 can be translated into the language of hamiltonian Berge cycles.

1.2 Hamiltonian Berge cycles in hypergraphs

A *hypergraph* $\mathcal{H}$ is a set of vertices $V(\mathcal{H})$ and a set of edges $E(\mathcal{H})$ such that each edge is a subset of $V(\mathcal{H})$.

We consider hypergraphs with edges of any size. The *degree*, $d(v)$, of a vertex v is the number of edges that contain v. The *minimum degree* of a hypergraph $\mathcal{H}$ is $\delta(\mathcal{H}) := \min_{v \in V(\mathcal{H})} d(v)$. The *co-degree* of a vertex set A is the number of edges that contain A.

A *Berge cycle* of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \ldots, v_\ell\}$ and ℓ distinct edges $\{e_1, \ldots, e_\ell\}$ such that $v_i, v_{i+1} \in e_i$ for every $i \in [\ell]$ (indices are taken modulo ℓ). The vertices $\{v_1, \ldots, v_\ell\}$ are the *base vertices* of the cycle.

Naturally, a *Berge hamiltonian cycle* in a hypergraph $\mathcal{H}$ is a Berge cycle whose set of base vertices is $V(\mathcal{H})$.

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph. The *incidence graph* of $\mathcal{H}$ is the bipartite graph $I(\mathcal{H})$ with parts (X, Y) where $X = V(\mathcal{H})$, $Y = E(\mathcal{H})$ such that for $e \in Y, v \in X$, $ev \in E(I(\mathcal{H}))$ if and only if the vertex v is contained in the edge e in $\mathcal{H}$.

If $\mathcal{H}$ has n vertices, m edges and minimum degree at least δ, then $I(\mathcal{H}) \in \mathcal{G}(n, m, \delta)$. There is a simple relation between the cycle lengths in a hypergraph $\mathcal{H}$ and its incidence graph $I(\mathcal{H})$: If $\{v_1, \ldots, v_\ell\}$ and $\{e_1, \ldots, e_\ell\}$ form a Berge cycle of length ℓ in $\mathcal{H}$, then $v_1 e_1 \ldots v_\ell e_\ell v_1$ is a cycle of length 2ℓ in $I(\mathcal{H})$, and vice versa.

For a positive integer k, call a hypergraph k**-connected** if its incidence graph is k-connected.

If one would like to prove an analog of Dirac's theorem on hamiltonian cycles in graphs for hamiltonian Berge cycles in hypergraphs, then the bound on the minimum degree would be exponential in n. One of the examples is the following construction from [7].

Construction 1.5 ([7]). *Let $V(\mathcal{H}) = V_1 \cup V_2$ where $|V_1| = \lceil (n+2)/2 \rceil$, $|V_2| = \lfloor (n-2)/2 \rfloor$, $V_1 \cap V_2 = \emptyset$, and let $E(\mathcal{H}) = E_1 \cup E_2$, where E_1 is the set of all subsets A of $V(\mathcal{H})$ of size $\lceil n/4 \rceil$ such that $|V_1 \cap A| = 1$ (and $|V_2 \cap A| = \lceil n/4 \rceil - 1$), and $E_2 = \{V_1\}$. Then $\mathcal{H}$ has an exponential in n minimum degree, high connectivity and positive codegree of each pair of the vertices. But $\mathcal{H}$ has no Berge hamiltonian cycle.*

On the other hand, rephrasing Theorem 1.4 in terms of hypergraphs, we get a reasonable and sharp bound on the minimum degree in terms of the number of vertices and edges that provides the existence of hamiltonian Berge cycles in 3-connected hypergraphs.

Theorem 1.6. *Let positive integers n, m, δ be such that*

$$\delta \geq \max\{n, \tfrac{m+10}{4}\}. \tag{1}$$

Then every 3-connected n-vertex hypergraph with m edges and minimum degree at least δ has a hamiltonian Berge cycle.

1.3 Notation and outline of the proof of Theorem 1.4

For a graph G, a cycle C in G, and a vertex x not appearing in C, let $t(x, C)$ denote the size of a largest $x, V(C)$-*fan* in G, i.e. the largest number of $x, V(C)$-paths such that any two of them share only x. Since G is 3-connected, $t(x, C) \geq 3$.

Our proof is by contradiction. We assume that for some positive integers m, n, δ with $\delta \geq \max\{n, \frac{m+10}{4}\}$, there is a counter-example: a 3-connected (X, Y)-bigraph $G \in \mathcal{G}(n, m, \delta)$ with no $2n$-cycles. We study the properties of G.

We consider each cycle C in G equipped with a clockwise direction. For every vertex u of C, $x^+_C(u)$ denotes the closest to u clockwise vertex of X distinct from u. For every vertex u of C, $x^-_C(u)$ denotes the closest to u counterclockwise vertex of X distinct from u. For a set $U \subset V(C)$, $X^+_C(U) = \{x^+_C(u) \ : \ u \in U\}$. When C is clear from the context, the subscripts could be omitted. The vertices $y^+(u), y^-(u)$ and the sets $X^-(U), Y^+(U), Y^-(U)$ are defined similarly.

We consider triples (C, x, F) where C is a cycle, $x \in X - V(C)$ and F is an x, C-fan. By $D(C, x)$ we will denote the component of $G - C$ containing x. By definition, $V(F) - V(C) \subseteq D(C, x)$.

Definition 1.7. *A triple (C, x, F)* **is better than** *a triple (C', x', F') if*

(a) $|C| > |C'|$, *or*

(b) $|C| = |C'|$ *and* $t(x, C) > t(x', C')$, *or*

(c) $|C| = |C'|$, $t(x, C) = t(x', C')$, *and* $|V(F) \cap V(C) \cap Y| > |V(F') \cap V(C') \cap Y|$, *or*

(d) $|C| = |C'|$, $t(x, C) = t(x', C')$, $|V(F) \cap V(C) \cap Y| = |V(F') \cap V(C') \cap Y|$, *and* $|V(F)| < |V(F')|$, *or*

(e) $|C| = |C'|$, $t(x, C) = t(x', C')$, $|V(F) \cap V(C) \cap Y| > |V(F') \cap V(C') \cap Y|$, $|V(F)| = |V(F')|$ *and* $|V(D(C, x))| < |V(D(C', x))|$.

Choose a best triple (C, x, F). Let

$$2\ell = |C|, \quad t = t(x, C), \quad T = T(C, x, F) = V(F) \cap V(C),$$
$$t_X = |T \cap X|, \quad t_Y = |T \cap Y|.$$

Similarly, let $\widetilde{T} = \widetilde{T}(C, x)$ be the set of all vertices of C adjacent to a vertex of $D(C, x)$, and let $\widetilde{t} = \widetilde{t}(C, x) = |\widetilde{T}|$. By definition, $\widetilde{T} \supseteq T$ and $\widetilde{t} \geq t$. Viewing F as a tree (spider) with root x, any two vertices $u, v \in V(F)$ define the unique u, v-path $F[u, v]$ in F. For $u, v \in V(C)$, let $C[u, v]$ be the clockwise u, v-path in C and let $C^-[u, v]$ be the counterclockwise u, v-path in C. If $D = D(C, x)$ and $u, v \in D \cup \widetilde{T}(C, x)$, then let $P_D[u, v]$ be a longest u, v-path all of whose internal vertices are in D.

We will analyze the properties of best triples (C, x, F) and in all cases will come to a contradiction, either by finding a better triple or by proving that $m \geq 4\delta - 9$. For this, we will try to construct so called *good subsets* W of $X \cap T$, defined later, such that total neighborhood of $W \cup \{x\}$ will be too large. One feature of a good set will be that no two members of such set have a common neighbor outside of C, **CON** for short.

In the next section we prove basic properties of our best triple (C, x, F). Then in Section 3 we show that $t = \widetilde{t} = 3$. Since G is 3-connected, this means that *for every* $x' \in X - C$, $t(x', C) = 3$. In Section 3.1, we discuss special types of components of $G - C$ and possibilities to choose a triple (C, x, F) with x in such a component. After that we consider $T = T(C, x, F)$ and try to find a 4-element good subset of the set $A = X^+(T) \cup X^-(T)$. The main obstacles will be that some members of A have many common neighbors, in particular, CONs. Section 4 is devoted to the case analysis of different types of such CONs. We conclude the paper with some comments.

2 Preliminary lemmas

Lemma 2.1. *The following inequalities always hold:*

$$\text{(i) } \ell \geq t + t_X; \qquad \text{(ii) } |X| - \ell + t_X \geq 3; \qquad \text{(iii) } |X| \geq t + 3.$$

Proof. If $w \in T \cap X$ and $y^+(w) \in T$, then the cycle $wF[w, y^+(w)]y^+(w)C[y^+(w), w]w$ is longer than C, a contradiction. Similarly, $y^-(w), x^+(w), x^-(w) \notin T$. Thus, $t_X \leq \ell/2$ and $t_Y \leq \ell - 2t_X$. This proves (i).

Since $\delta \geq |X| \geq \ell + 1 \geq t + 1 = d_F(x) + 1$, there is $y \in N(x) - N_F(x)$. By (d) in the definition of (C, x, F), $y \notin V(F)$. By the maximality of t, $y \notin V(C) - V(F)$. Since G is 3-connected, $G - x$ has a y, C-fan F' of size 2. Let x', x'' be the neighbors of y in F'. If, say $x' \in V(C)$, then by the maximality of t, $x' \in T$. Thus $\{x, x', x''\} \subset (X - V(C)) \cup (T \cap X)$. This yields (ii). Now (i) and (ii) together imply (iii). □

Lemma 2.2. *If* $w \in \widetilde{T} \cap X$, *then*

(i) $y^+(w) \notin \widetilde{T}$ *and*

(ii) $y^+(w)$ *has no neighbors in* $X^+(\widetilde{T}) - x^+(w)$.

Proof. If $y^+(w)$ has a neighbor in $D = D(C, x)$, then the cycle $wP_D[w, y^+(w)]y^+(w)C[y^+(w), w]w$ is longer than C. This contradiction proves (i).

Suppose $y^+(w)u \in E(G)$ for some $u \in X^+(\widetilde{T}) - x^+(w)$. Let $u = x^+(v)$ for $v \in \widetilde{T} - w$. Consider the cycle $C' = wC^-[w, u]uy^+(w)C[y^+(w), v]vP_D[v, w]w$. Then C' is longer than C, unless $v \in X$ and v and w have a common neighbor y in D. In the last case, $|C'| = |C|$ and the only vertex in $V(C) - V(C')$ is $y^+(v)$ which by (i) does not have neighbors in D. Define an x, C'-fan F' as follows. If $y \notin V(F)$, then let $F' = F$. If $y \in V(F)$, say $y \in F[x, u_i]$ for some $u_i \in T$, then let $F' = F - E(F[y, u_i])$. In both cases, since $y^+(v)$ does not have neighbors in $D(C', x) \subset D$, the triple (C', x, F') is better than (C, x, F): if $y \notin V(F)$, then by (e), otherwise either by (c) or by (d). □

Lemma 2.3. *If* $x_1 \in X^+(\widetilde{T})$, *then* x_1 *cannot have a neighbor in* $D = D(C, x)$, *i.e.,* $x_1 \notin \widetilde{T}$.

Proof. Suppose x_1 has a neighbor y' in D. Let $u_1 \in \widetilde{T}$ be such that $x_1 = x^+(u_1)$ and z be a neighbor of u_1 in D. Let P be a z, y'-path in D and the cycle C' be defined by $C' = x_1C[x_1, u_1]u_1zPy'x_1$. If $y' \neq z$, then C' is longer than C and we are done. Thus $z = y'$ and hence $u_1 \in X$. In this case C' and C have the same length and $t(x, C') = t(x, C)$. As in the proof of Lemma 2.2(ii), if $y' \notin V(F)$, then let $F' = F$. If $y' \in V(F)$, say $y' \in F[x, u_i]$ for some $u_i \in T$, then let $F' = F - E(F[y', u_i])$. In both cases, since by Lemma 2.2(i), $y^+(u_1)$ does not have neighbors in $D(C', x) \subset D$, the triple (C', x, F') is better than (C, x, F): if $y' \notin V(F)$, then by (e), otherwise either by (c) or by (d). □

Given a cycle C and distinct $x_1, x_2, x_3 \in X \cap V(C)$, we say that x_1 *and* x_2 *cross at* x_3 if the cyclic order is x_1, x_3, x_2 and $x_1y^+(x_3), x_2y^-(x_3) \in E(G)$ or if the cyclic order is x_1, x_2, x_3 and $x_1y^-(x_3), x_2y^+(x_3) \in E(G)$. In this case, we also say that x_3 *is crossed by* x_1 *and* x_2.

Lemma 2.4. *Suppose that* $x_1, x_2 \in X^+(\widetilde{T})$, *cross at* $x_3 \in X \cap V(C)$. *Then* $x_3 \notin \widetilde{T}$.

Proof. Suppose that the cyclic order is x_1, x_3, x_2 and $x_1y^+(x_3), x_2y^-(x_3) \in E(G)$ (the other case is symmetric). Let y be a neighbor of x_3 in D. Let $u_1 \in \widetilde{T}$ be such that $x_1 = x^+(u_1)$ and z be a neighbor of u_1 in D. Let P be a z, y-path in D and the cycle C' be defined by

$$C' := x_1y^+(x_3)C[y^+(x_3), u_1]u_1zPyx_3C^-[x_3, x_1]x_1.$$

If $y \neq z$, then C' is longer than C and we are done. Thus $z = y$. In this case, C' and C have the same length and $t(x, C') = t(x, C)$. As in the proof of Lemma 2.2(ii), if $y \notin V(F)$, then let $F' = F$. If $y \in V(F)$, say $y \in F[x, u_i]$ for some $u_i \in T$, then let $F' = F - E(F[y, u_i])$. Again as in the proof of Lemma 2.2, the triple (C', x, F') is better than (C, x, F). □

Recall that for two vertices in G, CON means “a common neighbor outside of C.”

Lemma 2.5. *Suppose that* $x_1, x_2 \in X^+(\widetilde{T})$. *Then*

(i) x_1 *and* x_2 *have no CON;*

(ii) neither of x_1 *and* x_2 *has a CON with* x.

Proof. Part (ii) follows from Lemma 2.3. So, suppose x_1 and x_2 have a CON y, and $u_1, u_2 \in \widetilde{T}$ are such that $x_1 = x^+(u_1)$ and $x_2 = x^+(u_2)$. By Lemma 2.3, $y \notin D$. Consider the cycle

$$C' := x_1C[x_1, u_2]u_2P_D[u_2, u_1]u_1C^-[u_1, x_2]x_2yx_1.$$

Cycle C' is longer than C, unless $u_1, u_2 \in X$ and have a common neighbor y' in D. In the last case, $|C'| = |C|$ and the only vertices in $V(C) - V(C')$ are $y^+(u_1)$ and $y^+(u_1)$ which by Lemma 2.3(i) do not have neighbors in D. Define an x, C'-fan F' as follows. If $y' \notin V(F)$, then let $F' = F$. If $y' \in V(F)$, say $y' \in F[x, u_i]$ for some $u_i \in T$, then let $F' = F - E(F[y', u_i])$. In both cases, since $y^+(u_1)$ and $y^+(u_2)$ do not have neighbors in $D(C', x) \subset D$, the triple (C', x, F') is better than (C, x, F): if $y \notin V(F)$, then by (e), otherwise either by (c) or by (d). □

Lemma 2.6. *Suppose $u_1, u_2 \in \widetilde{T}$ are such that the path $P_D[u_1, u_2]$ contains an internal vertex in X. If $x_1 = x^+(u_1)$ and $x_2 = x^+(u_2)$ cross at $x_3 \in X \cap V(C)$, then*

(i) $x_3 \notin \widetilde{T}$ and if $x_3 = x^+(u)$ where $u \in \widetilde{T}$, then $u \in Y$;

(ii) G has a cycle C' containing $(X \cap V(C) - x_3) \cup (X \cap P_D[u_1, u_2])$ such that $|C'| \geq |C|$;

(iii) x_3 has no CON with any vertex in the set $\{x\} \cup X^+(T)$;

(iv) x_3 has at most t neighbors on C.

Proof. Part (i) follows from Lemmas 2.2 and 2.4. The cycle

$$C_1 := x_1y^+(x_3)C[y^+(x_3), u_2]u_2P_D[u_2, u_1]u_1C^-[u_1, x_2]x_2y^-(x_3)C^-[y^-(x_3), x_1]x_1$$

proves (ii).

To prove (iii), assume that y is a CON of x_3 with a vertex in $\{x\} \cup X^+(T)$, and consider all cases. First note that by Lemma 2.4, $y \notin D$; in particular, x_3 has no CON with x. If $u_j \in \widetilde{T}$, $x_j = x^+(u_j)$, $yx_j \in E(G)$, and $x_j \in C[y^+(x_3), u_1]$, then the cycle

$$C' := x_1C[x_1, x_3]x_3yx_jC[x_j, u_1]u_1P_D[u_1, u_j]u_jC^-[u_j, y^+(x_3)]y^+(x_3)x_1$$

is longer than C, unless $P_D[u_1, u_j] = u_1y'u_j$ for some $y' \in D$. If $P_D[u_1, u_j] = u_1y'u_j$, then $|C'| = |C|$ and the only vertices in $V(C) - V(C')$ are $y^+(u_1)$ and $y^+(u_j)$ which by Lemma 2.3(i) do not have neighbors in D. Define an x, C'-fan F' as at the end of the proof of Lemma 2.3, and see that (C', x, F') is better than (C, x, F) exactly as there. Similarly, if $x_j \in C[u_1, y^-(x_3)]$, then the cycle

$$C' := x_2C[x_2, u_j]u_jP_D[u_j, u_2]u_2C^-[u_2, x_3]x_3yx_jC[x_j, y^-(x_3)]y^-(x_3)x_2$$

is longer than C, unless $P_D[u_j, u_2] = u_jy'u_2$ for some $y' \in D$. Again, defining F' as above, we get a triple (C', x, F') better than (C, x, F), a contradiction. This proves (iii).

By the choice of (C, x, F) and (ii), x_3 has at most t neighbors on C_1. The only vertices in $Y \cap V(C) - V(C_1)$ are $y^-(x_1)$ and $y^-(x_2)$. If $x_3y^-(x_1) \in E(G)$, then the cycle

$$y^-(x_1)C[y^-(x_1), y^-(x_3)]y^-(x_3)x_2C[x_2, u_1]u_1P_D[u_1, u_2]u_2C^-[u_2, x_3]x_3y^-(x_1)$$

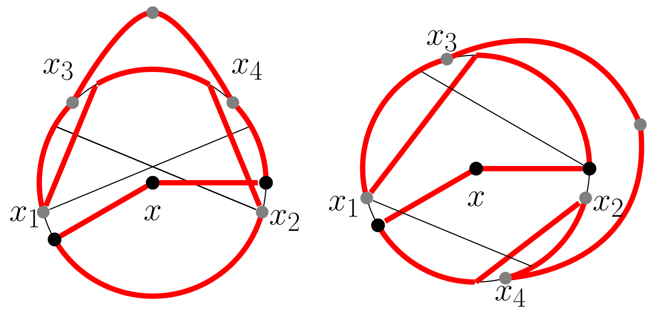

Figure 2: Longer cycles when x_1 and x_2 in Lemma 2.7 have multiple crossings.

is longer than C. If $x_3y^-(x_2) \in E(G)$, then the cycle

$$x_1C[x_1,x_3]x_3y^-(x_2)C[y^-(x_2),u_1]u_1P_D[u_1,u_2]u_2C^-[u_2,y^+(x_3)]y^+(x_3)x_1$$

is longer than C. This proves (iv). □

Lemma 2.7. *Suppose $u_1, u_2 \in \widetilde{T}$ are such that the path $P_D[u_1,u_2]$ contains an internal vertex in X, $x_1 = x^+(u_1)$, and $x_2 = x^+(u_2)$. Then at most one vertex in C is crossed by x_1 and x_2.*

Proof. Suppose vertices $x_3, x_4 \in V(C) \cap X$ are crossed by x_1 and x_2. We will show first that x_3 and x_4 have no CON. Suppose there is some $y \in (N(x_3) \cap N(x_4)) - V(C)$. By Lemma 2.6, $y \notin V(D)$.

We consider two cases. If x_3 and x_4 both are on $C[x_1,x_2]$ or both are on $C[x_2,x_1]$, then we may assume that their cyclic order is x_1, x_3, x_4, x_2. In this case, the cycle

$$x_1C[x_1,x_3]x_3yx_4C[x_4,u_2]u_2P_D[u_2,u_1]u_1C^-[u_1,x_2]x_2y^-(x_4)C^-[y^-(x_4),y^+(x_3)]y^+(x_3)x_1$$

(see Figure 2, left) is longer than C.

If one of x_3 and x_4 is on $C[x_1,x_2]$ and the other is on $C[x_2,x_1]$, then we may assume that their cyclic order is x_1, x_3, x_2, x_4. In this case, the cycle

$$x_1C[x_1,x_3]x_3yx_4C^-[x_4,x_2]x_2y^+(x_4)C[y^+(x_4),u_1]u_1P_D[u_1,u_2]u_2C^-[u_2,y^+(x_3)]y^+(x_3)x_1$$

(see Figure 2, right) is longer than C. This proves that x_3 and x_4 have no CON.

Let $A = X^+(T) \cup \{x, x_3, x_4\}$ (possibly, $X^+(T) \cap \{x_3, x_4\} \neq \emptyset$), and $A' = A - \{x, x_3, x_4\}$. Note that $|A'| \geq t - 2$.

By definition, $|N(x) - C| \geq \delta - t_Y$. By Lemma 2.6(iv), $|N(x_3) - C| \geq \delta - t$ and $|N(x_4) - C| \geq \delta - t$. By Lemma 2.2,

$$\sum_{u \in A'} |N(u) \cap V(C)| \leq \ell|A'| - t_X|A'| + \min\{t_X, |A'|\}. \tag{2}$$

By Lemmas 2.5 and 2.6(iii), no two distinct vertices in A have a CON. Thus, using (2) and remembering about the ℓ vertices in $Y \cap V(C)$, we get

$$|Y| \geq \ell + \sum_{u \in A} |N(u) - V(C)|$$

$$
\begin{aligned}
&= \ell + |N(x) - V(C)| + |N(x_3) - V(C)| + |N(x_4) - V(C)| + \sum_{u \in A'} |N(u) - V(C)| \\
&\geq \ell + (\delta - t_Y) + (\delta - t) + (\delta - t) + (\delta|A'| - \sum_{u \in A'} |N(u) \cap V(C)|) \\
&\geq \ell + (|A'| + 3)\delta - 2t - t_Y - (\ell - t_X)|A'| - \min\{t_X, |A'|\} \\
&\geq \ell + (t - 2 + 3)\delta - 2t - (t - t_X) - (\ell - t_X)(t-2) - \min\{t_X, t-2\} \\
&= \ell + (t+1)\delta - (3t - t_X) - (\ell - t_X)(t-2) - \min\{t_X, t-2\} \\
&\geq \ell + (t+1)\delta - 3t - (\ell - t_X)(t - 3 + 1) \\
&= (t+1)\delta - 3t - (\ell - t_X)(t-3) + t_X.
\end{aligned}
$$

Since by Lemma 2.1, $\delta \geq \ell - t_X + 3$, this yields

$$|Y| \geq (t+1)\delta - 3t - \delta(t-3) + 3(t-3) + t_X = 4\delta - 9 + t_X.$$

This contradiction proves the lemma. □

The following lemma holds for any bipartite graph G (no restrictions on minimum degree or connectivity).

Lemma 2.8. *Let C be a cycle of G, and let $u, v \in V(C) \cap X$. If u and v have at most a crossings, then $d_C(u) + d_C(v) \leq |V(C)|/2 + 2 + a$.*

Proof. We induct on a. Suppose $a = 0$. Consider the two paths $P_1 = C[u,v]$ and $P_2 = C^-[u,v]$. In $P_1 = v_1 \dots v_k$ ($v_1 = u, v_k = v$), each $v_i \in X$ satisfies at most one of the following: $v_{i+1}u \in E(G)$ or $v_{i-1}v \in E(G)$. So $d_{P_1}(u) + d_{P_1}(v) \leq |V(P_1) \cap X|$. Similarly, $d_{P_2}(u) + d_{P_2}(v) \leq |V(P_2) \cap X|$. Since $(X \cap V(P_1)) \cap (X \cap V(P_2)) = \{u, v\}$ and $V(P_1) \cup V(P_2) = V(C)$, we get $d_C(u) + d_C(v) \leq |V(C)|/2 + 2$.

For $a \geq 1$, delete an edge incident to u that is used in a crossing, and apply induction. □

3 Bounds on t and $\widetilde{t}$ in best triples

Recall that (C, x, F) is a best triple, $D = D(C, x)$ is the component of $G - V(C)$ containing x, $T = V(F) \cap V(C)$, and $\widetilde{T} = N_C(D)$.

A set of vertices $W = \{x_1, \dots, x_k\} \subseteq X \cap V(C)$ is **good** if

(i) $d_C(x) \leq k$,

(ii) the vertices of $\{x\} \cup W$ pairwise have no CON, and

(iii) we can partition W into sets $W_1, \dots, W_s$ such that for each $j \in [s]$, $|W_j| \geq 2$ and any two distinct vertices in W_j cross at no more than one vertex in C.

Lemma 3.1. *If W is a good set, then $|W| < \max\{4, t\}$.*

Proof. Suppose $k \geq \max\{4, t\}$ and $W = \{x_1, \dots, x_k\}$ is a good set. Note that $\delta \geq |X| \geq |W| \geq 4$. Let $(W_1, \dots, W_s)$ be a partition of W satisfying (iii) in the definition of a good set. By Lemma 2.8, if x_i and x_j have at most one crossing, then $d_C(x_i) + d_C(x_j) \leq \ell + 3$. Hence

$$\sum_{i=1}^{k} d_C(x_i) = \sum_{j=1}^{s} \sum_{w \in W_j} d_C(w) \leq k(\ell+3)/2.$$

Since $|Y \cap V(C)| = \ell$, $\delta(G) \geq \delta$ and $k \geq t$, we get

$$|Y| \geq \ell + (k+1)\delta - t - k\frac{\ell+3}{2} \geq \ell\left(1 - \frac{k}{2}\right) + k\left(\delta - \frac{5}{2}\right) + \delta.$$

Since the net coefficient at ℓ is negative and $\ell \leq |X| - 1 \leq \delta - 1$, this is at least $k\left(\frac{\delta}{2} - 2\right) + 2\delta - 1$. Now the net coefficient at k is nonnegative, so the minimum is attained at $k = 4$. Hence $|Y| \geq 4\delta - 9$, a contradiction. □

Next, we show that both t and $\widetilde{t}$ are small.

Lemma 3.2. $t = 3$.

Proof. Since G is 3-connected, $t = |T| \geq 3$. Suppose $t \geq 4$. We claim that $X^+(T)$ is a good set.

Since F is a largest x, C-fan, x has at most t neighbors in C. By Lemma 2.7, for any $x_i, x_j \in X^+(T)$, x_i and x_j have at most one crossing in C. By Lemma 2.5, no two distinct vertices in $X^+(T) \cup \{x\}$ have a CON. This certifies that $X^+(T)$ is good, a contradiction to Lemma 3.1. □

Lemma 3.3. $|\widetilde{T}| = 3$.

Proof. We have $T \subseteq \widetilde{T}$. Suppose $|\widetilde{T}| \geq 4$. Choose a set $U = \{u_1, \ldots, u_4\} \subseteq \widetilde{T}$ so that $T = \{u_1, u_2, u_3\}$, and $u_4 \in \widetilde{T} - T$. Let P be a shortest path from u_4 to $F - C$ in $G[D + u_4]$. Let $j \in [3]$ be such that the end, p, of P distinct from u_4 belongs to the x, u_j-path in F. Assume $[3] = \{j, j', j''\}$. The path $u_{j'}F[u_{j'}, p]pPu_4$ contains an internal vertex in X (namely, x). Partition U into $U' = \{u_4, u_{j'}\}$ and $U'' = \{u_j, u_{j''}\}$.

By Lemma 2.7, each of the pairs U' and U'' has at most one crossing in C. Since F is a largest x, C-fan, x has at most t neighbors in C. By Lemma 2.5, no two distinct vertices in $X^+(U) \cup \{x\}$ have a CON. This certifies that $X^+(U)$ is good, a contradiction to Lemma 3.1. □

Remark 3.4. *Lemma 3.3 implies that $T = \widetilde{T}$, i.e., the only vertices in C with neighbors in D are the vertices of T. In particular, no vertex in $V(C) - T$ has a CON with x.*

3.1 More structure and fewer crossings

One of the results of this section is that for any best triple (C, x, F), no vertices in $X^+(T)$ cross in C. Recall that by Lemma 3.2, $|T| = |V(F) \cap V(C)| = 3$.

A component D of $V(G) - C$ is **2-rich** if there is a set $U = \{u_1, u_2, u_3\} = V(C) \cap N(D)$ such that for all distinct i, j, D contains a u_i, u_j-path with at least two internal vertices in X.

Lemma 3.5. *If $|T \cap X| \leq 1$, then D is 2-rich.*

Proof. Suppose $T = \{u_1, u_2, u_3\}$ where $u_1, u_2 \in Y$. If some $y \in D \cap Y$ is not adjacent to u_3, then all y, C-paths contain internal vertices in X, and hence D is 2-rich. Thus we may assume that each $y \in D \cap Y$ is adjacent to u_3. In particular, $u_3 \in X$.

By Rule (d) of Definition 1.7, $d_F(x) = t = 3$, so because $\delta \geq |X| + 1 \geq t + 3 + 1 \geq 7$, there is $y' \in N(x)$ with $y'x \notin E(F)$. Since G is 3-connected, it contains a y', C-fan F' with 3 paths. Recall that $y'u_3$ is one of such paths. For $i = 1, 2$, let P_i be the y', u_i-path in F' and $v_iy' \in E(P_i)$. Suppose that for $i = 1, 2$, there is $y_i \in N(v_i) - C - y' - P_{3-i}$ (possibly, $y_2 = y_1$). Then D is 2-rich: $P_1 \cup P_2$ connects u_1 with u_2, and for $i \in \{1, 2\}$, path $u_3y_iv_iy'P_{3-i}$ connects u_3 with u_{3-i}; and each of these three paths contains $\{v_1, v_2\} \subset X$. Hence by symmetry we may assume that every neighbor of v_1 is in $V(C) \cup P_2$. Note $N(v_1) \cap V(C) \subseteq \{u_1, u_2, u_3\}$, since $|\widetilde{T}| = 3$. Then the cycle $v_1y'P_2C[u_2, u_1]v_1$ has at least 2δ vertices, a contradiction. □

Lemma 3.6. *Suppose D is not 2-rich. For any $x' \in X \cap V(C)$, $G - x'$ has no cycle C' such that*

(i) $X \cap V(C') \supseteq X \cap V(C) - x' + x$, and

(ii) C' contains the neighbors $y^+(x')$ and $y^-(x')$ of x' on C.

Proof. Suppose we have C' satisfying (i) and (ii). If we have strict containment in (i), then $|C'| > |C|$, contradicting (a) in the choice of (C, x, F). Thus $X \cap V(C') = X \cap V(C) - x' + x$.

Let D' be the component of $G - V(C')$ containing x'. Let M be the set of neighbors of D' on C'. By (ii), $\{y^+(x'), y^-(x')\} \subset M$. Since G is 3-connected, $D' - \{y^+(x'), y^-(x')\}$ contains an x', C'-path P. Then P together with the edges $x'y^+(x')$ and $x'y^-(x')$ forms an x', C'-fan F' with $|V(F') \cap V(C') \cap Y| \geq 2$. Moreover since D was not 2-rich, by Lemma 3.5, $|V(F) \cap V(C) \cap Y| \leq 1$. So (C', x', F') is a better triple than (C, x, F), a contradiction. □

Lemma 3.7. *No two vertices in $X^+(T)$ cross in C.*

Proof. Suppose $x_i = x^+(u_i)$ and $x_j = x^+(u_j)$ cross at some vertex $x_0 \in V(C) \cap X$. By symmetry, we may assume that their cyclic order is x_i, x_0, x_j. Let

$$C' := x_iC[x_iy^-(x_0)]y^-(x_0)x_jC[x_j, u_i]u_iP_D[u_i, u_j]u_jC^-[u_j, y^+(x_0)]y^+(x_0)x_i.$$

If D is 2-rich, then $P_D[u_i, u_j]$ has at least 2 internal vertices in X, and so C' is longer than C. If D is not 2-rich, then C' satisfies conditions (i) and (ii) of Lemma 3.6, a contradiction. □

Let e be an edge of C, let $u, v \in V(C)$, and let P be any u, v-path containing e, which we orient from u to v. We say that P and C *agree on* the edge e if the orientation of e (oriented from u to v) in the u, v-segment of C containing e is the same as the orientation of e in P.

Lemma 3.8. *Let $u, v \in X \cap V(C)$. Suppose that there is a u, v-path P with $(X \cap V(C)) \cup \{x\} \subseteq V(P)$ and there exists some $z, z' \in V(P)$ such that $V(P) \cap V(D) = V(P[z, z'])$, i.e., P enters and leaves D exactly once. Then*

(i) u and v have no common neighbor outside of P, and

(ii) if P and C agree on an edge e, then u and v cannot have a crossing at an endpoint of e.

Proof. Note that $x \in P[z, z']$. If u and v had a common neighbor outside P, then we could extend P to a cycle longer than C, so (i) holds.

To prove (ii), suppose that P and C agree on an edge e which lies on $C[u, v]$, $w \in X \cap V(C)$ is an endpoint of e, and u and v cross at w. Suppose that the edges of $C[u, v]$ incident to w are $y'w$ and wy'', so that uy'' and vy' are the two edges forming u and v's crossing on w. Without loss of generality, $e = y'w$. The condition that P and C agree on e guarantees that $P[u, w]$ contains y'.

There are two cases to consider: either both $y'w$ and wy'' are edges of P, or just $y'w$.

In the first case, let $C' := uP[u, y']y'vP[v, y'']y''u$. Then $V(C') \supseteq V(C) - \{w\} + \{x\}$. If we have strict containment, then $|C'| > |C|$, a contradiction. So we may assume $V(C') \cap X = (V(C) \cap X) - \{w\} + \{x\}$. Observe that C' satisfies Lemma 3.6 for $x' = w$. So D is 2-rich. Let a be the vertex in P preceding z and a' the vertex in P succeeding z' (so $a, a' \in V(C)$). Let P' be a a, a'-path internally disjoint from C that contains at least 2 internal vertices in X. Let C'' be obtained by replacing in C' the segment $P[a, a']$ with P'. We have $|V(C'') \cap X| > |X + \{x\} - \{w\}|$. Therefore $|C''| > |C|$, a contradiction.

In the second case, the cycle $uP[u, y']y'vP[v, w]wy''u$ is longer than C, since it contains all of $X \cap V(C)$ as well as x, a contradiction. □

4 Handling the case $\widetilde{t} = 3$

4.1 Short, medium, and long-type configurations

We continue to study properties of a best triple (C, x, F). Recall that by Lemma 3.3, $\widetilde{t} = |\widetilde{T}| = 3$, so we will assume that $N(D) \cap V(C) = \{u_1, u_2, u_3\}$. Partition $V(C) - \{u_1, u_2, u_3\}$ into U_1, U_2 and U_3, where for $i \in [3]$, $U_i = V(C[u_i, u_{i+1}]) - \{u_i, u_{i+1}\}$, i.e. U_i is the set of vertices on C from u_i to u_{i+1} not including either endpoint. Here and in the remainder of the paper, we let the indices on D's neighbors wrap around modulo 3, so that, for example, $u_0 = u_3$ and $u_4 = u_1$.

Let $X_i = U_i \cap X$ and $Y_i = U_i \cap Y$. For $j > 0$, let $x_{i,j}$ be the j^{th} vertex in X_i clockwise; let $x_{i,-j}$ be the j^{th} vertex in X_{i-1} counterclockwise. For example, $x_{i,1} = x^+(u_i)$ and $x_{i,-1} = x^-(u_i)$. Define $y_{i,j}$ similarly.

One of the lines of attack in this section is trying to find a 4-element good subset of $X^+(T) \cup X^-(T)$, which will contradict Lemma 3.1. This will not work if several of these vertices have many CONs. We will classify the obstacles to this approach into three types. For each $i \in [3]$, we say that:

- i has *short type* if $x_{i,-1}$ and $x_{i,1}$ have a CON.
- i has *medium type* if $x_{i,1}$ and $x_{i+1,-1}$ have a CON.
- i has *long type* if $x_{i,-1}$ and $x_{i+1,1}$ have a CON.

These three configurations are shown in Figure 3.

We first prove that each segment U_i contains at least two vertices in X.

Lemma 4.1. *For any $x' \in X^+(T)$, $d_C(x) + d_C(x') \geq 8$. In particular, $d_C(x') \geq 5$.*

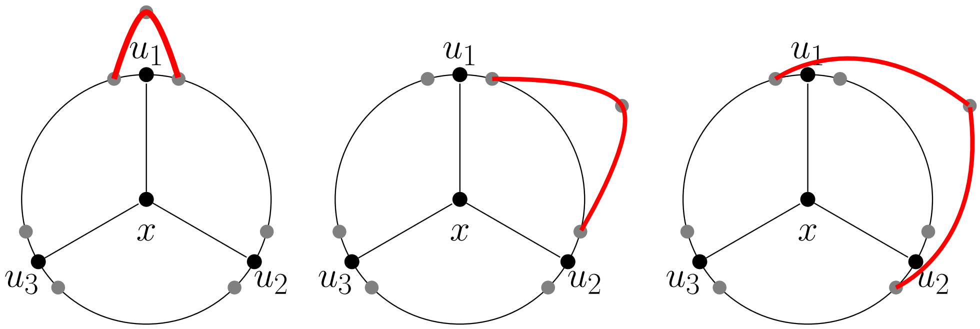

Figure 3: Short-type, medium-type, and long-type configurations.

Proof. Suppose $x' = x_{1,1}$ and $d_C(x) + d_C(x') \leq 7$. No two vertices in the set $X^+(T) \cup \{x\}$ have a CON or cross in C. By Lemmas 3.7 and 2.8, $d_C(x_{2,1}) + d_C(x_{3,1}) \leq \ell + 2$. Therefore

$$|Y| \geq 4\delta - (d_C(x) + d_C(x_{1,1})) - (d_C(x_{2,1}) + d_C(x_{3,1})) + \ell \geq 4\delta - 7 - (\ell + 2) + \ell = 4\delta - 9.$$

This contradiction proves $d_C(x) + d_C(x') \geq 8$. Since $d_C(x) \leq t = 3$, $d_C(x') \geq 5$. □

Lemma 4.2. *For each* $i \in [3]$, $x_{i,1} \neq x_{i+1,-1}$.

Proof. Let $C' = u_i F[u_i, u_{i+1}]u_{i+1}C[u_{i+1}, u_i]u_i$. Then $|C'| \geq |C|$. If the component D of $G - C$ containing x is 2-rich, then $|C'| > |C|$. So by Lemma 3.5, $|T \cap Y| \leq 1$, and hence $d_C(x) \leq 1$. By the choice of (C, x, F) as a best triple, $d_{C'}(x_{i,1}) \leq 1$ as well. Since $V(C) - \{y^-(x_{i,1}), y^+(x_{i,1}), x_{i,1}\} \subseteq V(C')$, $N_C(x_{i,1}) \subseteq N_{C'}(x_{i,1}) \cup \{y^-(x_{i,1}), y^+(x_{i,1})\}$, and therefore $d_C(x_{i,1}) \leq 1 + 2$. This contradicts Lemma 4.1. □

It is possible that some segments U_i contain only two vertices of X, but in that case, we can deduce some additional structure we will use later.

Lemma 4.3. *For each* $i \in [3]$, *if* $x_{i,2} = x_{i+1,-1}$, *then* $i + 1$ *does not have short or long type.*

Proof. If $i + 1$ has short or long type, we can find a cycle C' such that $X \cap C'$ includes x but leaves out $x_{i,1}$.

If $i + 1$ has short type and y is a CON of $x_{i+1,-1}$ and $x_{i+1,1}$, then

$$C' := u_{i+1}C^-[u_{i+1}, x_{i+1,-1}]x_{i+1,-1}yx_{i+1,1}C[x_{i+1,1}, u_i]u_iF[u_i, u_{i+1}]u_{i+1}.$$

Note that C includes at most three vertices of Y which are not in C': $y^+(x_{i,1})$, possibly $y^-(x_{i,1})$ (if $u_i \in X$), and possibly $y^+(u_{i+1})$ (if $u_{i+1} \in X$).

If $i + 1$ has long type and y is a CON of $x_{i+1,-1}$ and $x_{i-1,1}$, then

$$C' := x_{i+1,-1}C[x_{i+1,-1}, u_{i-1}]u_{i-1}F[u_{i-1}, u_i]u_iC^-[u_i, x_{i-1,1}]x_{i-1,1}yx_{i+1,-1}.$$

Again, C includes at most three vertices of Y which are not in C': $y^+(x_{i,1})$, possibly $y^-(x_{i,1})$ (if $u_i \in X$), and possibly $y^+(u_{i-1})$ (if $u_{i-1} \in X$).

In both cases, $|C'| \geq |C|$, with strict inequality if D is 2-rich. So we may assume D is not 2-rich. By Lemma 3.5, $|T \cap Y| \leq 1$, and hence $d_C(x) \leq 1$. Therefore by the choice of (C, x, F) as a best triple, $d_{C'}(x_{i,1}) \leq 1$. Then in either case $d_C(x_{i,1}) \leq 1 + 3$, contradicting Lemma 4.1. □

Lemma 4.4. *For each $i \in [3]$, one of the following configurations must appear:*

(i) i has short type, or

(ii) one of $i-1$ or i has medium type, or

(iii) $i+1$ has long type.

Proof. Suppose for some $i \in [3]$ none of (i)–(iii) holds. Let $W = \{x_{i-1,1}, x_{i,-1}, x_{i,1}, x_{i+1,-1}\}$. By Lemma 3.7 (applied to C and also to the backward orientation of C), the vertices inside the sets $W_1 = \{x_{i-1,1}, x_{i,1}\}$ and $W_2 = \{x_{i,-1}, x_{i+1,-1}\}$ have no crossings. By Lemma 2.5, no vertex in W can have a CON with x. Since by Lemma 3.1, W is not a good set, some two vertices in W have a CON. By Lemma 2.5 again, $x_{i-1,1}$ and $x_{i,1}$ have no CONs, and $x_{i,-1}$ and $x_{i+1,-1}$ have no CONs. This leaves the configurations in the statement of this lemma. □

The plan of the remainder of this paper is as follows:

1. In the next subsection we define abundant indices and show that not all $i \in [3]$ are abundant. This will help to handle medium-type and short-type configurations.
2. In Subsection 4.3 we show that at most one $i \in [3]$ has long type.
3. In Subsection 4.4 we prove that no $i \in [3]$ has medium type. An important part of this proof is Lemma 4.5 from Subsection 4.2.
4. In Subsection 4.5 we show that none of $i \in [3]$ has long type. So, by Lemma 4.4, every $i \in [3]$ has short type.
5. Subsection 4.6 finishes the proof of the main theorem by handling the case that every $i \in [3]$ has short type.

4.2 On abundant indices

Call an $i \in [3]$ *abundant* if each of the vertices $x_{i,2}, x_{i,3}, \ldots, x_{i+1,-2}$ has a CON with $x_{i,1}$ and a CON with $x_{i+1,-1}$.

Lemma 4.5. *At least one $i \in [3]$ is not abundant.*

Proof. Suppose all $i \in [3]$ are abundant. For $i \in [3]$, let $w_i = y^+(x^-(u_i))$. In other words, $w_i = u_i$ if $u_i \in Y$, and $w_i = y_{i,-1}$ if $u_i \in X$. Define $W = \{w_1, y^-(w_1), w_2, y^-(w_2), w_3, y^-(w_3)\}$. We claim that for all $i \in [3]$,

$$N_C(x_{i,1}) \subseteq Y_i \cup W. \tag{3}$$

Suppose that $x_{i,1}$ has a neighbor $y_{j,k}$ where $j \neq i$ and $y_{j,k} \in Y_j - \{w_{j+1}, y^-(w_{j+1})\}$. By Lemma 2.2, if $u_j \in X$, then $y_{j,k} \neq y_{j,1}$. So $y_{j,k}$ lies strictly between $x_{j,1}$ and $x_{j+1,-2}$. Since j is abundant,

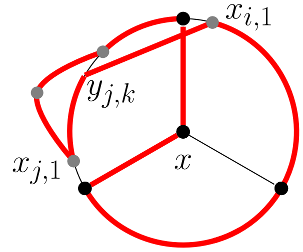

Figure 4: A longer cycle when $x_{i,1}$ has a neighbor $y_{j,k}$.

$x^+(y_{j,k})$ and $x_{j,1}$ have a CON, say y. Then the cycle

$$C' := x_{i,1}C[x_{i,1},u_j]u_jF[u_j,u_i]u_iC^-[u_i,x^+(y_{j,k})]x^+(y_{j,k})yx_{j,1}C[x_{j,1},y_{j,k}]y_{j,k}x_{i_1}$$

(see Figure 4) is longer than C, a contradiction. This proves (3).

Next we show that

$$\textit{if } |X_j| = 2 \textit{ and } x_{i,1}y^+(x_{j,1}) \in E(G), \textit{ then } N(x_{j,1}) \cap W = \{y^+(x_{j,1})\}. \tag{4}$$

Indeed, let P_1 be a longest u_j,u_i-path all internal vertices of which are in $D = D(C,x)$. Consider the cycle

$$C'' := x_{i,1}C[x_{i,1},u_j]u_jP_1u_iC^-[u_i,y^+(x_{j,1})]y^+(x_{j,1})x_{i_1}.$$

If D is 2-rich, then C'' is longer than C, a contradiction. Thus D is not 2-rich, and hence by Lemma 3.5, $|Y \cap T| \le 1$. In this case, $|C''| \ge |C|$. Let F'' be a best $x_{j,1},C''$-fan. Since the triple $(C'',x_{j,1},F'')$ is not better than (C,x,F), $|C''| = |C|$ and $|N(x_{j,1}) \cap V(C'')| \le 1$. Since $y^+(x_{j,1}) \in N(x_{j,1})$ by definition, and $W \subseteq V(C'')$, (4) follows.

Now we show that similarly to (4),

$$\textit{if } |X_j| \ge 3 \textit{ and } x_{i,1}y^-(w_{j+1}) \in E(G), \textit{ then } |N(x_{j,1}) \cap W| \le 1. \tag{5}$$

Indeed, let P_1 be a longest u_j,u_i-path all internal vertices of which are in $D = D(C,x)$. Since $|X_j| \ge 3$ and j is abundant, $x_{j+1,-1}$ and $x_{j,2}$ have a CON, say y. Consider the cycle

$$C''' := x_{i,1}C[x_{i,1},u_j]u_jP_1u_iC^-[u_i,x_{j+1,-1}]x_{j+1,-1}yx_{j,2}C[x_{j,2},y^-(w_{j+1})]y^-(w_{j+1})x_{i_1}.$$

If D is 2-rich, then C''' is longer than C, a contradiction. Thus D is not 2-rich, and by Lemma 3.5, $|Y \cap T| \le 1$. In this case, $|C''| \ge |C|$. Let F''' be a best $x_{j,1},C'''$-fan. Since the triple $(C''',x_{j,1},F''')$ is not better than (C,x,F), $|C'''| = |C|$ and $|N(x_{j,1}) \cap V(C''')| \le 1$. Since $W \subseteq V(C''')$, (5) follows.

If there are no distinct $i,j \in [3]$ such that $x_{i,1}y^-(w_{j+1}) \in E(G)$, then by (3), $\sum_{i\in[3]} |N_C(x_{i,1})| \le \sum_{i\in[3]}(|Y_i|+2)$, and hence

$$\sum_{i\in[3]} N_C(x_{i,1}) \le \ell + 6. \tag{6}$$

If there is only one $j \in [3]$ such that $y^-(w_{j+1})$ is adjacent to $x_{j-1,1}$ or to $x_{j+1,1}$ (say, $x_{i,1}y^-(w_{j+1}) \in E(G)$), then by (3), $|N_C(x_{i,1})| \le |Y_i| + 3$ for $i \ne j$, but by (4) and (5), $|N_C(x_{j,1})| \le |Y_j|$. So again (6) holds.

Finally, if there are distinct $j_1, j_2 \in [3]$ such that $x_{i_s,1}y^-(w_{j_s+1}) \in E(G)$ for $s \in [2]$ and some i_s, then by (4) and (5), $|N_C(x_{j_s,1})| \leq |Y_{j_s}|$, and by (3), $|N_C(x_{i,1})| \leq |Y_i| + 4$ for $i \in [3] - \{j_1, j_2\}$. Thus (6) holds in all cases.

By Lemma 2.5, no two vertices in the set $A = \{x, x_{1,1}, x_{2,1}, x_{3,1}\}$ have a CON. Therefore, by (6), $|Y| \geq \ell + 4\delta - 3 - (\ell - 6) = 4\delta - 9$, a contradiction. □

4.3 Eliminating multiple long-type configurations

Lemma 4.6. *At most one $i \in [3]$ has long type.*

Proof. Suppose the lemma does not hold. By symmetry, we may assume that $x_{3,-1}$ and $x_{1,1}$ have a CON a, and $x_{1,-1}$ and $x_{2,1}$ have a CON b. Since $x_{1,1}$ and $x_{2,1}$ cannot have a CON, $a \neq b$. Consider the cycle

$$C' := u_3C[u_3, x_{1,-1}]x_{1,-1}bx_{2,1}C[x_{2,1}, x_{3,-1}]x_{3,-1}ax_{1,1}C[x_{1,1}, u_2]u_2F[u_2, u_3]u_3$$

formed as shown in Figure 5.

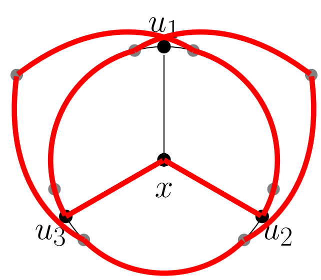

Figure 5: The cycle C' formed by two long-type configurations.

Cycle C' includes x and all vertices of $X \cap V(C)$, except possibly u_1, hence $|C'| \geq |C|$. If $u_1 \in Y$, C' is longer than C, which is a contradiction. Moreover, if $F[u_2, u_3]$ contains at least 2 internal X vertices, then $|C'| > |C|$.

If $u_1 \in X$, let yu_1 be the last edge of the x, u_1-path of F. As G is 3-connected, there is a path P from y to $V(C) \cap V(C')$ not containing x or u_1. Since by definition, deleting $\{u_1, u_2, u_3\}$ disconnects x, and therefore y, from C, path P must go from y to some vertex u' on either the x, u_2-path or the x, u_3-path in F. Without loss of generality, assume u' is on the x, u_2-path.

Consider the cycle

$$C'' := u_2C^-[u_2, x_{1,1}]x_{1,1}ax_{3,-1}C^-[x_{3,-1}, x_{2,1}]x_{2,1}bx_{1,-1}C^-[x_{1,-1}, u_3]u_3F[u_3, y]yPu'F[u', u_2]u_2$$

shown in Figure 6, obtained from C' by replacing the segment $C'[u', x]$ contained in F by the union of P and $F[x, y]$. This is longer than C' (and therefore longer than C) except in one case: when each of P and the $F[x, y]$ is a single edge, and $u' = u_2$ (which must then be in X). In this case,

$$C'' := u_2 C^-[u_2, x_{1,1}]x_{1,1}ax_{3,-1}C^-[x_{3,-1}, x_{2,1}]x_{2,1}bx_{1,-1}C^-[x_{1,-1}, u_3]u_3F[u_3, x]xyu_2.$$

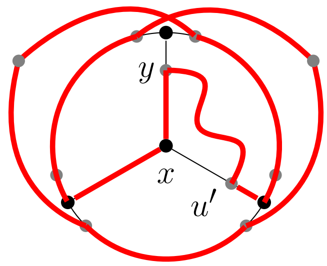

Figure 6: The cycle C'' formed using the path P.

Let F'' be the u_1, C''-fan formed by the paths $C[x_{1,-1}, u_1]$ and $C[u_1, x_{1,1}]$, and the edge u_1y. The triple (C'', u_1, F'') has $|C''| = |C|$ and $t(u_1, C'') = t(x, C)$, so by our choice of the triple (C, x, F), we must have $|V(F'') \cap V(C'') \cap Y| \leq |V(F) \cap V(C) \cap Y|$. Since $V(F'') \cap V(C'') \cap Y = \{y\}$, $|V(F) \cap V(C) \cap Y| \geq 1$, which can only happen if $u_3 \in Y$. Therefore the x, u_3-path in F consists of a single edge xu_3, and the only vertices of $V(C'') - V(C)$ are x, y, a, and b.

Let y' be the vertex of F between x and u_2 on the x, u_2-path of F. Since G is 3-connected, there is a path P' from y' to $V(C) \cup V(C'')$ not containing x or u_2. However, we know that deleting $\{u_1, u_2, u_3\}$ disconnects x, and therefore y', from C. Therefore either P' goes from y' to a vertex in $V(C'') - V(C)$, which can only be y, or else P' goes from y' to one of the vertices u_1, u_3.

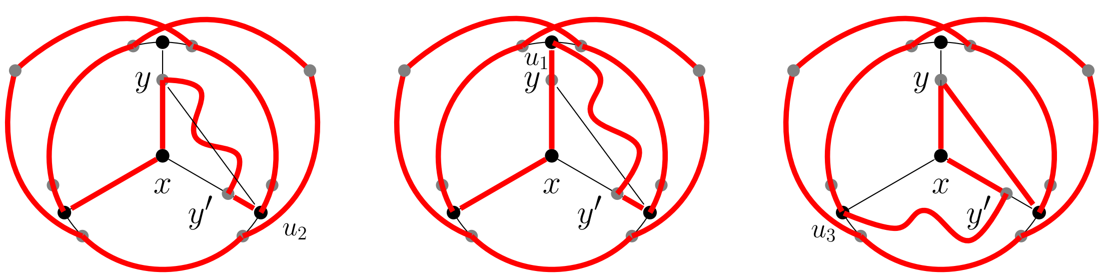

Figure 7: Three ways to extend C'' to a longer cycle

In each of these cases, we obtain a longer cycle. If P' goes from y' to y, we can extend C'' by replacing edge u_2y with u_2y' followed by P' to get the cycle

$$u_2C^-[u_2, x_{1,1}]x_{1,1}ax_{3,-1}C^-[x_{3,-1}, x_{2,1}]x_{2,1}bx_{1,-1}C^-[x_{1,-1}, u_3]u_3F[u_3, x]xyP'y'u_2,$$

as shown on the left in Figure 7. If P' goes from y' to u_1, we can extend C'' by replacing edge u_2y with u_2y', P', and u_1y to get the cycle

$$u_2C^-[u_2, x_{1,1}]x_{1,1}ax_{3,-1}C^-[x_{3,-1}, x_{2,1}]x_{2,1}bx_{1,-1}C^-[x_{1,-1}, u_3]u_3F[u_3, u_1]u_1P'y'u_2,$$

as shown in the middle of Figure 7. Finally, if P' goes from y' to u_3, we can extend C'' by replacing edge xu_3 with xy' followed by P' to get the cycle

$$u_2C^-[u_2, x_{1,1}]x_{1,1}ax_{3,-1}C^-[x_{3,-1}, x_{2,1}]x_{2,1}bx_{1,-1}C^-[x_{1,-1}, u_3]u_3P'y'F[y', y]yu_2,$$

as shown on the right in Figure 7. □

Thus, no more than one $i \in [3]$ can have long type.

4.4 Eliminating medium-type configurations

In this subsection, our goal is to show that no $i \in [3]$ has medium type.

Recall that $i \in [3]$ is *abundant* if each of the vertices $x_{i,2}, x_{i,3}, \ldots, x_{i+1,-2}$ has a CON with $x_{i,1}$ and a CON with $x_{i+1,-1}$.

Lemma 4.7. *If $i \in [3]$ has medium type, then i is abundant.*

Proof. Without loss of generality, we will assume that $i = 1$ has medium type. We will show that for all $j \geq 1$, $x_{2,-1}$ and $x_{2,-j}$ share a CON. This is the same as showing $x_{2,-1}$ and $x_{1,a}$ share a CON for all $a \geq 1$ such that $x_{1,a} \neq x_{2,-1}$. Showing that $x_{1,1}$ and $x_{1,j}$ have a CON is symmetric.

Suppose there is an a such that $x_{1,a}$ shares no CON with $x_{2,-1}$, but $x_{1,a'}$ does for all $1 \leq a' < a$. Our goal is to show $\{x_{1,-1}, x_{2,-1}, x_{3,-1}, x_{1,a}\}$ is a good set. Let y' be the common neighbor of $x_{2,-1}$ and $x_{1,a-1}$. Note that $x_{1,-1}, x_{2,-1}, x_{3,-1}$ can have no CON by Lemma 2.5. Additionally, by Lemma 3.7, $x_{2,-1}, x_{3,-1}$ have no crossings.

By our choice of a, vertices $x_{1,a}$ and $x_{2,-1}$ have no CON. By Lemma 3.8 via the path

$$P := x_{1,a}C[x_{1,a}, x_{2,-1}]x_{2,-1}y'x_{1,a-1}C^-[x_{1,a-1}, u_1]u_1F[u_1, u_2]u_2C[u_2, x_{1,-1}]x_{1,-1}$$

shown in Figure 8, $x_{1,a}$ and $x_{1,-1}$ have no crossings and no CON outside P. However, y' is the only possible CON of $x_{1,a}$ and $x_{1,-1}$ on P, and if $x_{1,-1}y' \in E(G)$, $x_{1,-1}$ and $x_{2,-1}$ would have a CON, which also is impossible.

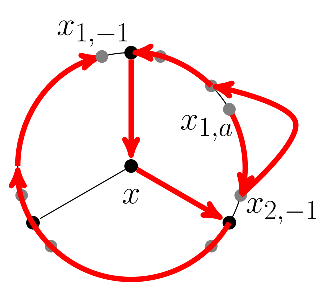

Figure 8: The path P from $x_{1,a}$ to $x_{1,-1}$.

Finally, we argue $x_{1,a}$ and $x_{3,-1}$ have no CON. Suppose y is such a CON; then the cycle

$$u_3C[u_3, x_{1,a-1}]x_{1,a-1}y'x_{2,-1}C^-[x_{2,-1}, x_{1,a}]x_{1,a}yx_{3,-1}C^-[x_{3,-1}, u_2]u_2F[u_2, u_3]u_3$$

is a longer cycle than C. So $x_{1,a}$ has no CONs with any of $x_{1,-1}, x_{2,-1}, x_{3,-1}$; $x_{1,a}$ and x_{1-1} have no crossings, and neither do $x_{2,-1}$ and $x_{3,-1}$. This certifies that $\{x_{1,-1}, x_{2,-1}, x_{3,-1}, x_{1,a}\}$ is a good set, a contradiction to Lemma 3.1. □

Lemma 4.8. *If i has medium type, then for $x_{i,j} \in \{x_{i,1}, \ldots, x_{i+1,-2}\}$,*

(i) $x_{i,j}$ and $x_{i+1,1}$ have no CONs and no crossings, and

(ii) $x_{i,j}$ and $x_{i-1,1}$ have no CONs.

Symmetrically, $x_{i,j} \in \{x_{i,2}, \ldots, x_{i+1,-1}\}$ and $x_{i,-1}$ have no CONs and no crossings, and $x_{i,j}$ and $x_{i-1,-1}$ have no CONs.

Proof. Without loss of generality, let $i = 1$. Suppose $x_{1,j}$ and $x_{2,1}$ have a common neighbor y (the $x_{1,-1}$ case is symmetric). By Lemma 4.7, $x_{1,1}$ and $x_{1,j+1}$ have a CON y'. By Lemma 3.8 and the path

$$P := x_{1,j}C^{-}[x_{1,j}, x_{1,1}]x_{1,1}y'x_{1,j+1}C[x_{1,j+1}, u_2]u_2F[u_2, u_1]u_1C^{-}[u_1, x_{2,1}]x_{2,1},$$

shown in Figure 9, $x_{1,j}$ and $x_{2,1}$ share no CONs (otherwise $x_{1,1}$ and $x_{2,1}$ share a CON) and no crossings.

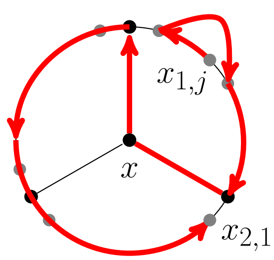

Figure 9: The path P from $x_{1,j}$ to $x_{2,1}$.

Suppose that $x_{1,j}$ has a CON y with $x_{3,1}$. By Lemma 4.7, $x_{1,1}$ and $x_{1,j+1}$ have a CON y'. Moreover, by Lemma 2.5, $x_{1,1}$ and $x_{3,1}$ can have no CON, so $y \neq y'$. In this case, we obtain a longer cycle than C: the cycle

$$x_{1,1}C[x_{1,1}, x_{1,j}]x_{1,j}yx_{3,1}C[x_{3,1}, u_1]u_1F[u_1u_3]u_3C^{-}[u_3, x_{1,j+1}]x_{1,j+1}y'x_{1,1}.$$

This is a contradiction, so $x_{1,j}$ and $x_{3,1}$ have no CON. The $x_{3,-1}$ case is symmetric. □

Lemma 4.9. *If $j \in [3]$ does not have medium type, then every $i \in [3]$ that has medium type also has long type.*

Proof. Without loss of generality, suppose i has medium type but $j = i - 1$ does not. The case where $j = i + 1$ is symmetric, after reorienting C. It suffices to show that in such a case, i has long type.

By Lemma 4.2, we may assume $x_{i,1} \neq x_{i+1,-1}$. Let $A = \{x_{i,2}, x_{i+1,1}, x_{i,-1}, x_{i-1,1}\}$. By Lemma 4.8, $x_{i,-1}$ and $x_{i,2}$ have no CONs or crossings; by Lemma 2.5 and Lemma 3.7, $x_{i-1,1}$ and $x_{i+1,1}$ have no CONs or crossings.

If $x_{i,2} \neq x_{i+1,-1}$, then Lemma 4.8 further tells us that $x_{i,2}$ has no CONs with $x_{i-1,1}$ or $x_{i+1,1}$. If $x_{i,2} = x_{i+1,-1}$ then Lemma 4.3 gives the same conclusion.

By assumption, $i-1$ does not have medium type, so $x_{i-1,1}$ and $x_{i,-1}$ have no CONs. If $x_{i,-1}$ and $x_{i+1,1}$ also have no CONs, then A is a good set, contradicting Lemma 3.1. Therefore $x_{i,-1}$ and $x_{i+1,1}$ must have a CON; in other words, i has long type. □

The three previous lemmas help us to prove the main result of this subsection:
Lemma 4.10. *No $i \in [3]$ has medium type.*

Proof. Suppose the lemma does not hold. If all $i \in [3]$ have medium type, then by Lemma 4.7, all of them are abundant, a contradiction to Lemma 4.5. Thus there is a $j \in [3]$ that does not have medium type. Then by Lemma 4.9, each $i \in [3]$ that has medium type also has long type. Now Lemma 4.6 yields that only one i can have medium type. Suppose by symmetry that this i is 1.

Let b be the smallest integer such that $x_{2,1}$ and $x_{1,-b}$ have no CON, and consider instead the set $X' = \{x_{1,2}, x_{2,1}, x_{3,-1}, x_{1,-b}\}$. Let y be the CON of $x_{1,-b+1}$ and $x_{2,1}$. By Lemma 3.8 and the path

$$x_{1,-b}C^{-}[x_{1,-b}, u_3]u_3F[u_3, u_2]u_2C^{-}[u_2, x_{1,-b+1}]x_{1,-b+1}yx_{2,1}C[x_{2,1}, x_{3,-1}]x_{3,-1}$$

shown in Figure 10 (left), $x_{3,-1}$ and $x_{1,-b}$ have no CON (otherwise $x_{3,-1}$ and $x_{2,1}$ have a CON, making 2 medium-type) and can only cross at a vertex $x_{1,j}$ for $j \geq 1$ or a vertex $x_{1,-a}$ where $a < b$. Note by Lemma 2.6 they cannot cross at u_1.

In the first case, if $j > 1$, let $y^- = y^-(x_{1,j})$. Note that $x_{2,-1}$ and $x^-(y^-)$ share a CON y'. We get a contradiction by the cycle

$$u_3C[u_3, x^-(y^-)]x^-(y^-)y'x_{2,-1}C^{-}[x_{2,-1}, y^-]y^-x_{3,-1}C[x_{3,-1}, u_2]u_2F[u_2, u_3]u_3.$$

If $j = 1$, then let y' be a CON of $x_{1,1}$ and $x_{1,2}$, and let y be a CON of $x_{2,1}$ and $x_{1,-b+1}$. Then we get the longer cycle

$$x_{1,1}y'x_{1,2}C[x_{1,2}, u_2]u_2F[u_2, u_1]u_1C^{-}[u_1, x_{1,-b+1}]x_{1,-b+1}yx_{2,1}C^{-}[x_{2,1}, x_{1,-b}]x_{1,-b}y^+(x_{1,1})x_{1,1}.$$

In the second case, let y' be a CON of $x_{2,1}$ and $x_{1,-a}$. Then we get a longer cycle

$$u_3C[u_3, y^-(x_{1,-a})]y^-(x_{1,-a})x_{3,-1}C^{-}[x_{3,-1}, x_{2,1}]x_{2,1}y'x_{1,-a}C[x_{1,-a}, u_2]u_2F[u_2, u_3]u_3.$$

By Lemma 4.8, $x_{1,2}$ and $x_{2,1}$ have no CONs and no crossings, and $x_{1,2}$ shares no CONs with $x_{3,-1}$.

Suppose y' is a CON of $x_{1,2}$ and $x_{1,-b}$. By the choice of b, $x_{2,1}$ and $x_{1,-b+1}$ have a CON y. The cycle

$$C' := x_{1,-b+1}yx_{2,1}C[x_{2,1}, x_{1,-b}]x_{1,-b}y'x_{1,2}C[x_{1,2}, u_2]u_2F[u_2, u_1]u_1C^{-}[u_1, x_{1,-b+1}]x_{1,-b+1}$$

shown in Figure 10(right) excludes $x_{1,1}$ but contains the rest of $X \cap V(C) - \{x_{1,1}\}$. Moreover, C' contains all but at most four vertices in $Y \cap C$: $y^+(x_{1,-b}), y^+(x_{1,1})$, and possibly $y^-(x_{1,1})$ or $y^-(x_{2,1})$, if $u_1 \in X$ or $u_2 \in X$ respectively. If D is 2-rich, then $|C'| > |C|$, so we may assume that D is not 2-rich, and $d_C(x) \leq 1$ by Lemma 3.5. By the choice of (C, x, F) as a best triple, $d_{C'}(x_{1,1}) \leq 1$

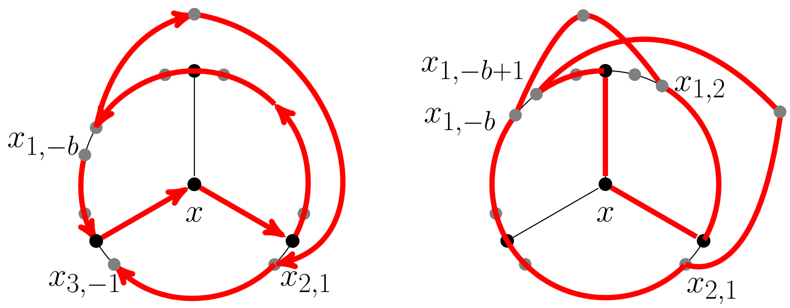

Figure 10: An $x_{1,-b}, x_{3,-1}$-path, and a longer cycle obtained when $x_{1,2}$ and $x_{1,-b}$ have a CON.

as well. Then $d_C(x) + d_C(x_{1,1}) \leq 1 + 1 + 4$. This contradicts Lemma 4.1, which shows that $x_{1,2}$ and $x_{1,-b}$ share no CONs.

Since 2 does not have medium type, $x_{2,1}$ and $x_{3,-1}$ share no CONs. By the definition of b, $x_{1,-b}$ and $x_{2,1}$ share no CONs. Thus, X' is good, a contradiction to Lemma 3.1. □

4.5 Eliminating long-type configurations

Lemma 4.11. *No $i \in [3]$ has long type.*

Proof. Suppose some $i \in [3]$ has long type. By Lemma 4.6, there is only one such i. By symmetry, assume $x_{3,-1}$ and $x_{1,1}$ have a CON a, i.e., only 3 has long type. Then by Lemma 4.4, since no j has medium type, 1 has short type, which means $x_{1,-1}$ and $x_{1,1}$ have a CON b.

Let $W = \{x_{1,-1}, x_{1,2}, x_{2,1}, x_{3,1}\}$. We will show that W is a good set.

By Lemma 2.5 and Lemma 3.7, $x_{2,1}$ and $x_{3,1}$ have no CON or crossings. Also, $x_{1,-1}$ and $x_{1,2}$ have no CON or crossings: This follows from Lemma 3.8, as shown on the left in Figure 11, where the path

$$P := x_{1,-1}C^{-}[x_{1,-1}, u_3]u_3F[u_3, u_1]u_1C[u_1, x_{1,1}]x_{1,1}ax_{3,-1}C^{-}[x_{3,-1}, x_{1,2}]x_{1,2}$$

agrees with the cycle C on all edges.

We now show that the remaining pairs in W do not have CONs. If $x_{1,-1}$ and $x_{2,1}$ have a CON, then we have a second long-type configuration. If $x_{1,-1}$ and $x_{3,1}$ have a CON, the we have a medium-type configuration.

If $x_{1,2}$ and $x_{2,1}$ have a CON c, then the cycle

$$u_3C[u_3, x_{1,1}]x_{1,1}ax_{3,-1}C^{-}[x_{3,-1}, x_{2,1}]x_{2,1}cx_{1,2}C[x_{1,2}, u_2]u_2F[u_2, u_3]u_3$$

is longer than C, as shown in the middle of Figure 11. Finally, if $x_{1,2}$ and $x_{3,1}$ have a CON c then the cycle

$$x_{3,1}C[x_{3,1}, x_{1,-1}]x_{1,-1}bx_{1,1}C^{-}[x_{1,1}, u_1]u_1F[u_1, u_3]u_3C^{-}[u_3, x_{1,2}]x_{1,2}cx_{3,1}$$

is longer than C, as shown on the right in Figure 11.

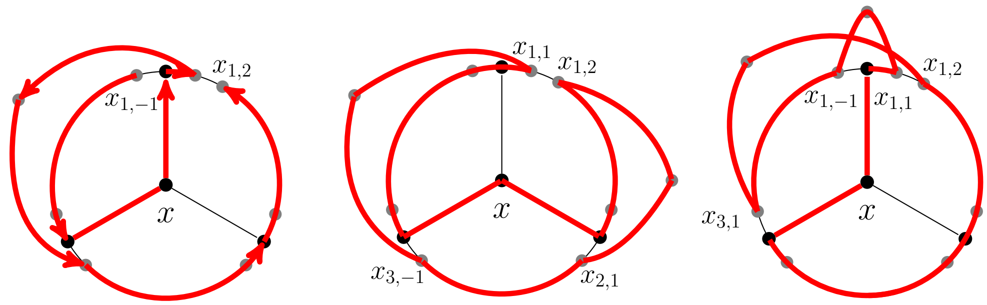

Figure 11: An $x_{1,-1}, x_{1,2}$-path, and longer cycles obtained if $x_{1,2}$ has a CON with $x_{2,1}$ or $x_{3,1}$.

Therefore W is a good set, contradicting Lemma 3.1. □

4.6 Eliminating short-type configurations and finishing the proof of Theorem 1.4

Lemma 4.12. *If there are no long-type configurations and no medium-type configurations, then every $i \in [3]$ is abundant.*

Proof. By Lemma 4.4, every $i \in [3]$ has short type and no other types.

For definiteness, consider $i = 1$. By the definition of short type, $x_{1,-1}$ and $x_{1,1}$ have a CON. Let $b > 1$ be the least integer such that $x_{1,-1}$ has no CON with $x_{1,b}$. Some such b exists, because $x_{2,-1}$ has no CON with $x_{1,-1}$. Moreover, if $x_{1,b} = x_{2,-1}$, then we find a cycle C' longer than C: if y_1 is a CON of $x_{1,-1}$ and $x_{2,-2}$, and y_2 is a CON of $x_{2,-1}$ and $x_{2,1}$, then $y_1 \neq y_2$ (since $x_{1,-1}$ has no CON with $x_{2,-1}$) and therefore

$$u_1 C[u_1, x_{2,-2}] x_{2,-2} y_1 x_{1,-1} C^-[x_{1,-1}, x_{2,1}] x_{2,1} y_2 x_{2,-1} C[x_{2,-1}, u_2] u_2 F[u_2, u_1] u_1$$

is a cycle longer than C. So b exists and $x_{1,b} \neq x_{2,-1}$. Note that this implies $x_{1,2} \neq x_{2,-1}$.

Consider the set $W_b = \{x_{1,-1}, x_{1,b}, x_{2,-1}, x_{3,1}\}$. We will show that it is *almost* a good set.

By Lemma 2.5 and Lemma 3.7, $x_{1,-1}$ and $x_{2,-1}$ have no CON or crossing. A CON of $x_{1,b}$ and $x_{3,1}$ is distinct from any CON of $x_{1,-1}$ and $x_{1,b-1}$ because $x_{1,-1}$ and $x_{3,1}$ have no CON. Let c be the CON of $x_{1,-1}$ and $x_{1,b-1}$. By applying Lemma 3.8 to the path

$$x_{3,1} C[x_{3,1}, x_{1,-1}] x_{1,-1} c x_{1,b-1} C^-[x_{1,b-1}, u_1] u_1 F[u_1, u_3] u_3 C^-[u_3, x_{1,b}] x_{1,b},$$

as on the left in Figure 12, we see that they can have no other CON, and can only cross at a vertex $x_{1,a}$ with $a < b$.

If such a crossing existed, however, then in particular $x_{3,1}$ would be adjacent to a neighbor of $x_{1,a}$ and letting c be the CON of $x_{1,-1}$ and $x_{1,a+1}$ we would obtain a longer cycle

$$x_{3,1} C[x_{3,1}, x_{1,-1}] x_{1,-1} c x_{1,a+1} C[x_{1,a+1}, u_3] u_3 F[u_3, u_1] u_1 C[u_1, y^-(x_{1,a+1})] y^-(x_{1,a+1}) x_{3,1}$$

as shown on the right of Figure 12. In the special case $a = b - 1$, the cycle looks only slightly different. Letting c be the CON of $x_{1,-1}$ and $x_{1,b-1}$, it is

$$x_{3,1}C[x_{3,1},x_{1,-1}]x_{1,-1}cx_{1,b-1}C^{-}[x_{1,b-1},u_1]u_1F[u_1,u_3]u_3C^{-}[u_3,y^{+}(x_{1,b-1})]y^{+}(x_{1,b-1})x_{3,1}.$$

We conclude that $x_{1,b}$ and $x_{3,1}$ have no CON or crossings.

By the choice of b, $x_{1,-1}$ and $x_{1,b}$ have no CON. The pair $x_{1,-1}$ and $x_{3,1}$ have no CON, otherwise a medium-type configuration would be formed. The pair $x_{2,-1}$ and $x_{3,1}$ have no CON, otherwise a long-type configuration would be formed.

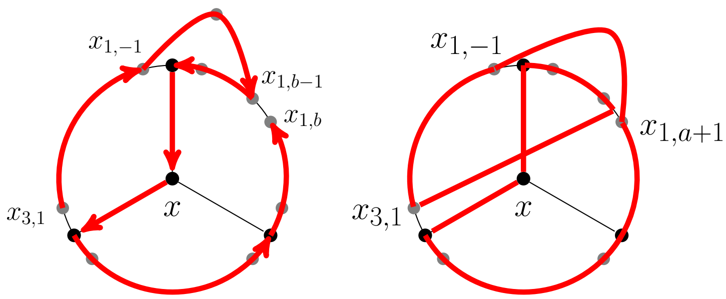

Figure 12: An $x_{3,1}, x_{1,b}$-path, and a longer cycle obtained if $x_{1,b}$ and $x_{3,1}$ have a crossing at $x_{1,a}$.

If $x_{1,b}$ and $x_{2,-1}$ have no CON, then W_b is a good set, a contradiction to Lemma 3.1. Thus, $x_{1,b}$ and $x_{2,-1}$ have a CON.

We now prove that

$$\textit{for each } c \geq b \textit{ such that } x_{1,c} \in C[x_{1,b}, x_{2,-2}], \textit{ vertices } x_{1,c} \textit{ and } x_{2,-1} \textit{ have a CON.} \tag{7}$$

Indeed, suppose (7) does not hold and c is the least integer such that $c \geq b$ and $x_{1,c}$ has no CON with $x_{2,-1}$. By the previous paragraph, $c > b$. Consider the set $W_c = \{x_{1,-1}, x_{1,c}, x_{2,-1}, x_{3,-1}\}$. We will show that this is a good set.

Indeed, $x_{1,-1}$ and $x_{2,-1}$ have no CON or crossing, by Lemma 2.5 and Lemma 3.7. Any CON of $x_{1,c}$ and $x_{3,-1}$ is distinct from any CON of $x_{1,c-1}$ and $x_{2,-1}$, since $x_{3,-1}$ and $x_{2,-1}$ have no CON. They have no other CON or crossings, as shown by the path

$$x_{1,c}C[x_{1,c},x_{2,-1}]x_{2,-1}qx_{1,c-1}C^{-}[x_{1,c-1},u_3]u_3F[u_3,u_2]u_2C[u_2,x_{3,-1}]x_{3,-1}$$

(see the left in Figure 13) and Lemma 3.8, where q is the CON of $x_{2,-1}$ and $x_{1,c-1}$.

We show that the remaining pairs have no CONs. Indeed, $x_{1,c}$ and $x_{2,-1}$ have no CON by our choice of c. The pairs $\{x_{1,-1}, x_{3,-1}\}$ and $\{x_{2,-1}, x_{3,-1}\}$ have no CONs, by Lemma 2.5. Finally, suppose r is a CON of $x_{1,-1}$ and $x_{1,c}$ Let q be a CON of $x_{2,-1}$ and $x_{1,c-1}$. Then the cycle

$$u_2C[u_2,x_{1,-1}]x_{1,-1}rx_{1,c}C[x_{1,c},x_{2,-1}]x_{2,-1}qx_{1,c-1}C^{-}[x_{1,c-1},u_1]u_1F[u_1,u_2]u_2$$

shown on the right of Figure 13 is longer than C.

Therefore we have a good set of size 4, a contradiction to Lemma 3.1. This proves (7). In other words, $x_{1,1}, x_{1,2}, \ldots, x_{1,b-1}$ all have a CON with $x_{1,-1}$ while $x_{1,b}, x_{1,b+1}, \ldots, x_{2,-2}$ all have a CON

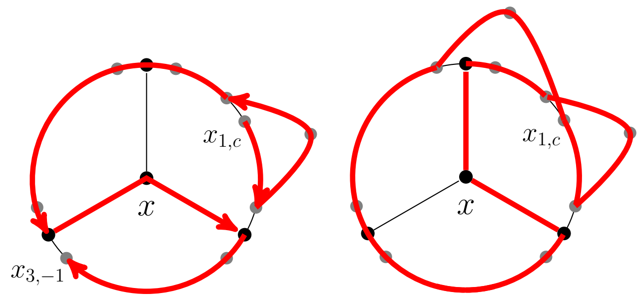

Figure 13: An $x_{1,c}, x_{3,-1}$-path, and a longer cycle obtained when $x_{1,-1}$ and $x_{1,c}$ have a CON.

with $x_{2,-1}$. Moreover, in this case, $x_{2,1}$ and $x_{2,-2}$ can have no CON, or else we obtain a longer cycle,

$$x_{2,1}C[x_{2,1}, x_{1,-1}]x_{1,-1}rx_{1,b-1}C^{-}[x_{1,b-1}, u_1]u_1F[u_1, u_2]u_2C^{-}[u_2, x_{2,-1}]x_{2,-1}sx_{1,b}C[x_{1,b}, x_{2,-2}]x_{2,-2}tx_{2,1},$$

where r is the CON of $x_{1,-1}$ and $x_{1,b-1}$, s is the CON of $x_{2,-1}$ and $x_{1,b}$, and t is the CON of $x_{2,-2}$ and $x_{2,1}$, as shown in Figure 14.

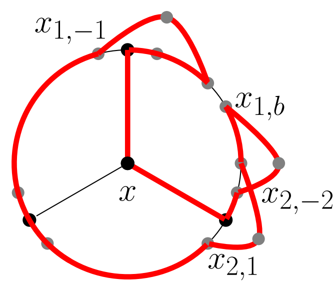

Figure 14: A longer cycle obtained when $x_{2,1}$ and $x_{2,-2}$ have a CON.

We can apply the argument in this subsection in six possible ways: we can swap the roles of $x_{1,1}$ and $x_{1,-1}$ in the argument above, and we can choose any of the three short-type configurations in place of the one formed by $x_{1,1}$ and $x_{1,-1}$. All six of these arguments must terminate in the same case. In particular, just as we concluded that $x_{2,1}$ and $x_{2,-2}$ can have no CON, we also conclude that $x_{1,-1}$ and $x_{1,2}$ can have no CON. This means that in the argument above (and in all variations of the argument), we must have $b = 2$.

Therefore, for each i, the vertices $x_{i,2}, x_{i,3}, \ldots, x_{i+1,-2}$ all have a CON with both $x_{i,1}$ and with $x_{i+1,-1}$. In other words, all $i \in [3]$ are abundant. $\square$

By Lemma 4.10 and Lemma 4.11, no $i \in [3]$ has medium or long type. Therefore by Lemma 4.12, every $i \in [3]$ is abundant. This contradicts Lemma 4.5, completing the proof of Theorem 1.4.

5 Concluding remarks

1. Theorem 1.4 is a natural 3-connected strengthening of Conjecture 1.1 for 2-connected graphs. Consider the following family of k-connected graphs.

   Construction 5.1. *Let k be a positive integer, and let $n_1 \geq \ldots \geq n_{k+1} \geq 1$ be such that $n_1 + \ldots + n_{k+1} = n$. Let $G_k(n_1, \ldots, n_{k+1}; \delta) \in \mathcal{G}(n, (k+1)(\delta - k) + k, \delta)$ be the bipartite graph obtained from $K_{\delta-k,n_1} \cup \ldots \cup K_{\delta-k,n_{k+1}}$ by adding k vertices $a_1, \ldots, a_k$ that are each adjacent to every vertex in the parts of size $n_1, \ldots, n_{k+1}$. Let $\mathcal{G}_k(n, \delta)$ be the collection of the graphs $G_k(n_1, \ldots, n_{k+1}; \delta)$ for all suitable choices of $n_1, \ldots, n_{k+1}$.*

   When $k = 2$ or $k = 3$, $\mathcal{G}_k$ is the family of all graphs in Construction 1.2 or Construction 1.3 respectively.

   Question 5.2. *Let m, n, k, δ be integers. Suppose $k \geq 4$, $\delta \geq n$ and $m \leq (k+1)(\delta - k) + k - 1$. Is it true that every k-connected graph $G \in \mathcal{G}(n, m, \delta)$ contains a cycle of length $2n$? Moreover, if $k \geq 3$, are the graphs in the family $\mathcal{G}_k(n, \delta)$ the only extremal examples with $m = (k+1)(\delta - k) + k$?*

   If the answer is negative, it would also be interesting to find the value(s) of k at which other extremal examples occur.

2. Jackson also made the following conjecture.

   Conjecture 5.3 (Jackson [3])**.** *Let m, n, δ be integers with $n > \delta$. If a graph $G \in \mathcal{G}(n, m, \delta)$ is 2-connected and satisfies*

   $$m \leq \left\lfloor \frac{2(n-\alpha)}{\delta - 1 - \alpha} \right\rfloor (\delta - 2) + 1$$

   where $\alpha = 1$ if δ is even and $\alpha = 0$ if δ is odd, then G contains a cycle of length at least $2\min(n, \delta)$.

   This conjecture remains open. A weaker version is proved in [6] in the language of hypergraphs.

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