PAPER

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Algebraic area enumeration for open lattice walks

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Abstract

We calculate the number of open walks of fixed length and algebraic area on a square planar lattice by an extension of the operator method used for the enumeration of closed walks. The open walk area is defined by closing the walks with a straight line across their endpoints and can assume half-integer values in lattice cell units. We also derive the length and area counting of walks with endpoints on specific straight lines and outline an approach for dealing with walks with fully fixed endpoints.

Keywords: random walks, Hofstadter model, quantum torus, Dyck paths, algebraic area enumeration

(Some figures may appear in colour only in the online journal)

1. Introduction

The enumeration of walks of a given length and enclosed algebraic area on various lattices is a fascinating problem [1]. It is of interest in pure combinatorics, where it has its own justification in terms of counting of objects among pre-defined ensembles, but also in physics, as it maps to various models in quantum and statistical physics. The most famous among these models is the Hofstafdter Hamiltonian [2], which describes the dynamics of a quantum particle hopping on a lattice in a perpendicular magnetic field and leads to the celebrated 'butterfly' energy spectrum. In addition, the lattice walk enumeration problem is related [3] to even more exotic quantum concepts such as exclusion statistics [4], which generalizes standard Fermi statistics to particles obeying a stronger exclusion principle parametrized by an integer $g \geqslant 2$ (usual Fermi statistics is $g = 1$; $g = 2$ is relevant for square lattice walks).

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The connection to quantum statistics offers an approach for addressing the enumeration problem of more general classes of walks on various lattices. For example, chiral walks on a triangular lattice, with a nonHermitian Hofstadter-like quantum Hamiltonian generating the biased hopping on the lattice, have been analyzed [3] in terms of particles obeying $g=3$ exclusion statistics. Walks on the honeycomb lattice can also be examined in this framework [5] and involve particles obeying a mixture of Fermi and $g=2$ exclusion statistics. These developments have also born new results in quantum statistics, leading to a full description of exclusion statistics particles in a nondegenerate discrete 1-body quantum spectrum and to explicit expressions for their cluster coefficients and ensuing thermodynamics [6]. In addition, this approach establishes a connection [7] between planar lattice walk enumeration and specific one-dimensional walks ('paths') of generalized Dyck or Motzkin form (also known as Lukasiewicz paths [8]). The various algebraic enumeration formulae obtained for walks admit a combinatorial interpretation in terms of paths [9], which leads to cross-pollination between the two subjects.

The algebraic area enumeration was up to now restricted to closed walks that return to their starting point on the lattice. In that case the algebraic area is the area enclosed by the walk weighted by the winding number of the walk around each area patch. However, it is also possible to define the algebraic area of an *open* walk using a 'closing' prescription, the most natural one being joining its two endpoints by a straight line. In the case of square lattice walks, it is easy to see that the algebraic area defined this way is always an integer or half-integer in units of elementary lattice cells.

Once the algebraic area of an open walk is defined, the enumeration of open walks of given length and algebraic area starting from a fixed point on the lattice becomes an interesting issue. In principle, the tools developed earlier for the case of closed walks [1, 3] are not directly applicable to open walks, and their enumeration for fixed length and area was an open problem. Still, we will demonstrate in this work that the problem of open walks on the square lattice can be recast in terms quite similar to the closed walk case through the introduction of a new operator σ, in addition to the usual u, v hopping lattice operators that were useful for closed walk enumeration, thus enlarging the usual 'quantum torus' algebra $uv = \mathrm{Q}\,vu$ to a 'reflection quantum torus' algebra. This algebra also appeared, in a different guise and representation, in the enumeration of *closed* walks on the honeycomb lattice [5], and it is remarkable that it reappears in the seemingly different context of open square lattice walks.

In the following sections we will calculate the generating function of planar open walks weighted by their algebraic area and will derive explicit formulae for the multiplicity of open walks of given length and area. The case of walks with endpoints fixed on a straight line is also readily tractable with our method, and we will derive the corresponding generating functions. We will also briefly address the problem of walks with *fully* fixed endpoints and will highlight possible approaches for a complete algebraic area enumeration.

2. Open walks

We consider open random walks of fixed length **n** (number of steps) on the square lattice starting at the origin and ending at an arbitrary lattice point (k, l). We assign to such open walks an algebraic area by closing them with a straight line from (k, l) to the origin [10]. With this 'radial' definition the algebraic area measured in units of lattice plaquettes can be half-integer.

We note that other closing prescriptions can be defined, such as, e.g. a 'rectangular' prescription of first returning vertically from the endpoint to the horizontal axis where the starting point lies and then returning to the origin horizontally. The algebraic areas for the two

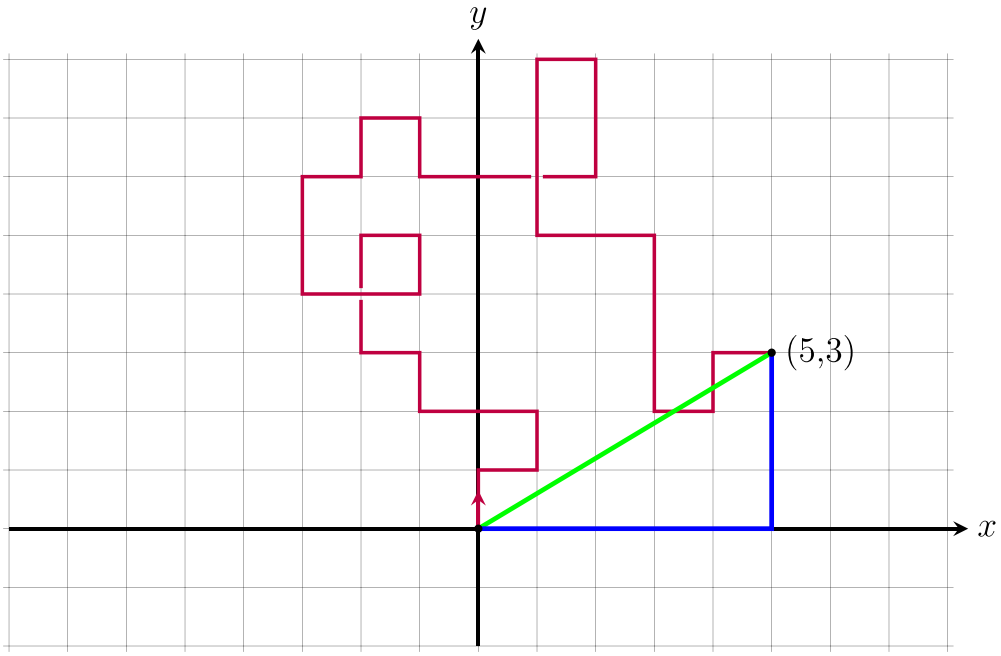

Figure 1. A square lattice walk (purple) starting at the origin and ending at the point (5,3), closed radially with a (green) straight line from its endpoint to the origin. If it were closed with a rectangular (blue) line from its endpoint to the horizontal axis and then to the origin, the area of the closed walk in plaquette units would be an integer. The difference of the areas between the two closings is the area of the right triangle with vertices at (0,0), (5,0) and (5,3), that is $5 \times 3/2 = 7.5$, making the area of the radially closed walk half-integer.

prescriptions are trivially related by the area of the orthogonal triangle on the lattice with vertices on the two endpoints of the walk, but they still group the walks differently in terms of their area and lead to different counting formulae. This also explains the fact that the radially closed walk area can be half-integer (see figure 1). We adopt the radial definition as more natural and symmetric, although the rectangular one can also be examined with minor modifications.

2.1. Algebraic construction

Similarly to closed walks, we define the algebraic area generating function of open walks as

$$G_{\mathbf{n}}(\mathrm{Q}) = \sum_{A} C_{\mathbf{n}}(A)\mathrm{Q}^{2A} \tag{1}$$

with $C_{\mathbf{n}}(A)$ the number of walks of length $\mathbf{n}$ and area A and Q a parameter dual to the area. The exponent of Q was chosen to be $2A$ to avoid fractional powers arising from half-odd-integer values of A.

The calculation of $G_{\mathbf{n}}(\mathrm{Q})$ can be achieved by establishing an algebraic framework similar to the one for closed walks, with an additional twist. Define operators u, v, σ satisfying the defining relations

$$vu = \mathrm{Q}^2 uv\,, \quad u\sigma = \sigma u^{-1}\,, \quad v\sigma = \sigma v^{-1} \tag{2}$$

and a formal trace operation $\mathrm{Tr}\,(\,\cdot\,)$ on their algebra such that

$$\mathrm{Tr}\,\sigma = \mathrm{Tr}\,(u\sigma) = \mathrm{Tr}\,(v\sigma) = 1\ ,\ \ \mathrm{Tr}\,(vu\sigma) = \mathrm{Q}. \tag{3}$$

We define the Hamiltonian for the random walk as

$$H = u + u^{-1} + v + v^{-1}. \tag{4}$$

Then the area generating function is obtained as

$$G_{\mathbf{n}}(\mathrm{Q}) = \mathrm{Tr}\,(H^{\mathbf{n}}\sigma). \tag{5}$$

The proof is along similar lines as in the closed walk case (where σ is absent). Expanding $H^{\mathbf{n}}$ produces $4^{\mathbf{n}}$ monomials of the form $v^{l_i}u^{k_i}\dots v^{l_1}u^{k_1}$, each corresponding to a walk with k_1 horizontal steps followed by l_1 vertical steps etc concluding with k_i horizontal and l_i vertical steps, and representing all possible walks with $\mathbf{n}$ steps. Using $vu = \mathrm{Q}^2uv$ to rearrange the terms brings $H^{\mathbf{n}}$ to the form

$$H^{\mathbf{n}} = \sum_{k,l} g_{k,l}(\mathrm{Q})\, v^l u^k\ ,\ \ |k \pm l| \leqslant \mathbf{n} \tag{6}$$

reducing each walk ending at lattice point (k,l) to a rectangular walk with k horizontal steps followed by l vertical steps and produces a coefficient $\mathrm{Q}^{2A'}$, with A' the area between the original walk and the corresponding rectangular walk. $g_{k,l}(\mathrm{Q})$ accounts for all paths ending on (k,l) weighted by the corresponding area factors. Finally, using commutation and trace relations (2) and (3) we can show that

$$\mathrm{Tr}\,(v^l u^k \sigma) = \mathrm{Q}^{kl} \tag{7}$$

$kl/2$ is the area of the rectangular walk closed with a straight line to the origin. Overall,

$$\mathrm{Tr}\,(H^{\mathbf{n}}\sigma) = \sum_{k,l} g_{k,l}(\mathrm{Q})\mathrm{Tr}\,(v^l u^k \sigma) = \sum_{k,l} g_{k,l}(\mathrm{Q})\mathrm{Q}^{kl} \tag{8}$$

gives the full sum over walks of all possible endpoints weighted by Q^{2A}, $A = A' + \frac{kl}{2}$ being their total area, reproducing $G_{\mathbf{n}}(\mathrm{Q})$.

2.2. *Representation of u, v, σ*

The main task is to evaluate the trace $\mathrm{Tr}\,(H^{\mathbf{n}}\sigma)$. This is made possible by finding an explicit matrix representation for the operators u, v, σ, for which Tr would become the usual matrix trace tr.

The subalgebra generated by u, v is the standard clock-shift (or quantum torus) algebra and has finite dimensional irreducible representations (irreps) for $\mathrm{Q}^2 = \exp(2i\pi p/q)$ with p, q mutually prime positive integers. The full u, v, σ algebra (2) has been analyzed in [5] with the extra condition $\sigma^2 = 1$. Since σ^2 is a central element of the algebra, it becomes a constant in an irrep and can be absorbed by an algebra-preserving redefinition $\sigma \to \lambda\sigma$, so the irreps found in [5] also apply to our case. Note that the algebra has two additional central elements (Casimirs), $A = u^q + u^{-q}$ and $B = v^q + v^{-q}$.

In general, irreps are of size $2\,q$. In block form:

$$u = \begin{pmatrix} u_o & 0 \\ 0 & u_o^{-1} \end{pmatrix},\ \ v = \begin{pmatrix} v_o & 0 \\ 0 & v_o^{-1} \end{pmatrix},\ \ \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{9}$$

with u_o, v_o the q-dimensional irrep of the u, v algebra and Casimirs $A = 2\cos qk_x$ and $B = 2\cos qk_y$. This representation, however, does not fulfill the trace conditions (3), giving vanishing traces. The remaining possibility is the reduced, q-dimensional irrep that exists when

the quantum torus Casimirs $u^q = e^{iqk_x}$ and $v^q = e^{iqk_y}$ become ± 1 ($k_x, k_y \in \{0, \pi/q\}$), and is given by the action on periodically defined basis states $|j\rangle$

$$\begin{aligned} u|j\rangle &= \mathrm{e}^{\mathrm{i}(k_x + 2\pi pj/q)}|j\rangle, \quad |j\rangle \equiv |j \ (\mathrm{mod}\ q)\rangle \\ v|j\rangle &= \mathrm{e}^{\mathrm{i}k_y}|j-1\rangle \\ \sigma|j\rangle &= \mathrm{e}^{\mathrm{i}k_y(2j-r)}|r-j\rangle, \quad rp + qk_x/\pi = 0 \ (\mathrm{mod}\ q). \end{aligned} \tag{10}$$

The 'pivot' r in the inversion action of σ is $r = 0$, if $k_x = 0$, or the primary solution of the Diophantine equation $kq - rp = 1$, if $k_x = \pi/q$. Imposing the trace conditions (3) further restricts q to an *odd* integer $q = 2s+1$ (an even q gives vanishing traces). It is convenient to fix the Casimirs $k_x = k_y = 0$ and take j in the range $-s \leqslant j \leqslant s$, thus placing the pivot state $|0\rangle$ in the middle. We obtain the specific realization

$$\begin{aligned} &u|j\rangle = \mathrm{Q}^{2j}|j\rangle, \ v|j\rangle = |j-1\rangle, \ \sigma|j\rangle = |-j\rangle \\ &\mathrm{tr}\,\sigma = \mathrm{tr}\,(u\sigma) = \mathrm{tr}\,(v\sigma) = 1, \ \ \mathrm{tr}\,(vu\sigma) = \mathrm{Q} \\ &\text{with } \mathrm{Q} = e^{i\frac{2\pi p(s+1)}{2s+1}}, \ \ |-s-1\rangle \equiv |s\rangle, \ |s+1\rangle \equiv |-s\rangle \end{aligned} \tag{11}$$

Note that Q is a specific square root of the quantum torus algebra parameter $\mathrm{Q}^2 = e^{i\frac{2\pi p}{2s+1}}$ and that $\mathrm{Q}^{2s+1} = 1$. This corresponds to the $(2s+1)$-dimensional matrix realization

$$u = \begin{pmatrix} \mathrm{Q}^{-2s} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Q}^{-2s+2} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \mathrm{Q}^{2s-2} & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \mathrm{Q}^{2s} \end{pmatrix}, \ v = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

2.3. Calculation of traces

In the realization (11) $\sigma^2 = 1$, and the $(2s+1)$-dimensional space decomposes into an $(s+1)$-dimensional subspace with $\sigma = 1$ and an s-dimensional subspace with $\sigma = -1$. The Hamiltonian $H = u + u^{-1} + v + v^{-1}$ commutes with σ, therefore

$$\mathrm{tr}\,(H^{\mathbf{n}}\sigma) = \mathrm{tr}\,H_+^{\mathbf{n}} - \mathrm{tr}\,H_-^{\mathbf{n}}, \quad H_\pm = H\frac{1 \pm \sigma}{2} \tag{12}$$

and we can evaluate the trace separately in each subspace.

To further facilitate the calculation, we use the trick [11] of adopting a realization of u, v, σ that eliminates the diagonal terms in H. This is achieved by the redefinition

$$u \to \mathrm{Q}uv, \quad v \to v. \tag{13}$$

One can check that the algebra and trace relations (2) and (3) as well as the Casimirs $u^q = v^q = 1$ remain invariant under this redefinition. The Hamiltonian becomes

$$H = (1+\mathrm{Q}u)\,v + \left(1+\mathrm{Q}u^{-1}\right)v^{-1}. \tag{14}$$

Choosing the basis $|j\rangle_\pm$ for the subspaces

$$\begin{aligned} &|j\rangle_\pm = \frac{1}{\sqrt{2}}\left(|j\rangle \pm |-j\rangle\right)\ ,\ j\neq 0\\ &|0\rangle_+ = |0\rangle,\ |0\rangle_- = 0\ ;\quad |-j\rangle_\pm = \pm|j\rangle,\ |s+1\rangle_\pm = \pm|s\rangle \end{aligned} \tag{15}$$

the action of H on these subspaces becomes

$$\begin{aligned} H|j\rangle_\pm &= \left(1+\mathrm{Q}^{-2j-1}\right)|j+1\rangle_\pm + \sqrt{2}^{\,\delta_{j,1}}\left(1+\mathrm{Q}^{2j-1}\right)|j-1\rangle_\pm\ ,\ j\neq 0,s\\ H|0\rangle_+ &= \sqrt{2}\left(1+\mathrm{Q}^{-1}\right)|1\rangle_+\\ H|s\rangle_\pm &= \pm 2\,|s\rangle_\pm + (1+\mathrm{Q}^{2s-1})|s-1\rangle_\pm \end{aligned} \tag{16}$$

we note that there is a single remaining diagonal term ± 2 for $j=s$.

We can view both H_+ and H_- as acting on the same states $|j\rangle, j=0,1,\ldots,s$ with common matrix elements connecting states j and $j\pm 1$, differing only on $0 \leftrightarrow 1$ and $s\to s$ transitions:

$$(H_+)_{01} = (H_+)^*_{10} = \sqrt{2}(1+\mathrm{Q})\ ,\quad (H_-)_{01} = (H_-)_{10} = 0\ ,\quad (H_\pm)_{ss} = \pm 2. \tag{17}$$

Traces can be expressed in terms of periodic 'paths' of indices $i_1, i_2, \ldots, i_{\mathbf{n}}, i_1$

$$\mathrm{tr}\,H_\pm^{\mathbf{n}} = \sum_{i_1,i_2,\ldots i_{\mathbf{n}}} (H_\pm)_{i_1 i_2}(H_\pm)_{i_2 i_3}\ldots(H_\pm)_{i_{\mathbf{n}} i_1} \tag{18}$$

The contribution of paths not going through $j=0$ and not containing $s\to s$ steps is the same for $H_\pm$ and will cancel in (12). Therefore, only paths that touch 0 or 'creep' on s will contribute. Further, all paths will have an equal number of steps $i-1\ \to i$ and $i\to i-1$, and each pair of such transitions will contribute an amplitude

$$\begin{aligned} s_i &= \ \left(1+\mathrm{Q}^{-2i+1}\right)\left(1+\mathrm{Q}^{2i-1}\right) = \left(\mathrm{Q}^{-i+\frac{1}{2}}+\mathrm{Q}^{i-\frac{1}{2}}\right)^2\quad i>1\\ s_1 &= 2\ \left(2+\mathrm{Q}^{-1}+\mathrm{Q}\right) = 2\ \left(\mathrm{Q}^{-\frac{1}{2}}+\mathrm{Q}^{\frac{1}{2}}\right)^2 \end{aligned} \tag{19}$$

the above observations allow us to evaluate the trace in a combinatorial way, by examining separately the cases of walks of even and odd length. We will assume, for the moment, that $\mathbf{n}\leqslant 2s$, which eliminates paths that would span the full width of states $(0,s)$, and will extend the results to arbitrary $\mathbf{n}$ in the next section.

(a) **Even length** $\mathbf{n}=2n$: Paths must have an even number of steps $s\to s$, since all remaining steps come in pairs, and these steps will contribute the *same* factor in H_+^{2n} and H_-^{2n}. Therefore, such paths will cancel unless they touch 0. Only H_+ contributes for such paths. Assuming $n\leqslant s$, these paths cannot have any steps $s\to s$. (Paths of vertical increments ± 1, are known as Dyck paths in the mathematics literature.) A typical such path is depicted in figure 2.
Assuming the path reaches maximum level j, and calling l_i $(1\leqslant i\leqslant j)$ the number of (up or down) transitions between levels $i-1$ and i, the trace is expressed as

$$\mathrm{tr}\,(H^{2n}\sigma) = \mathrm{tr}\,H_+^{2n} = \sum_{\sum_i l_i = n} 2n\,c(l_1,l_2,\ldots,l_j)\,s_1^{l_1}s_2^{l_2}\ldots s_j^{l_j} \tag{20}$$

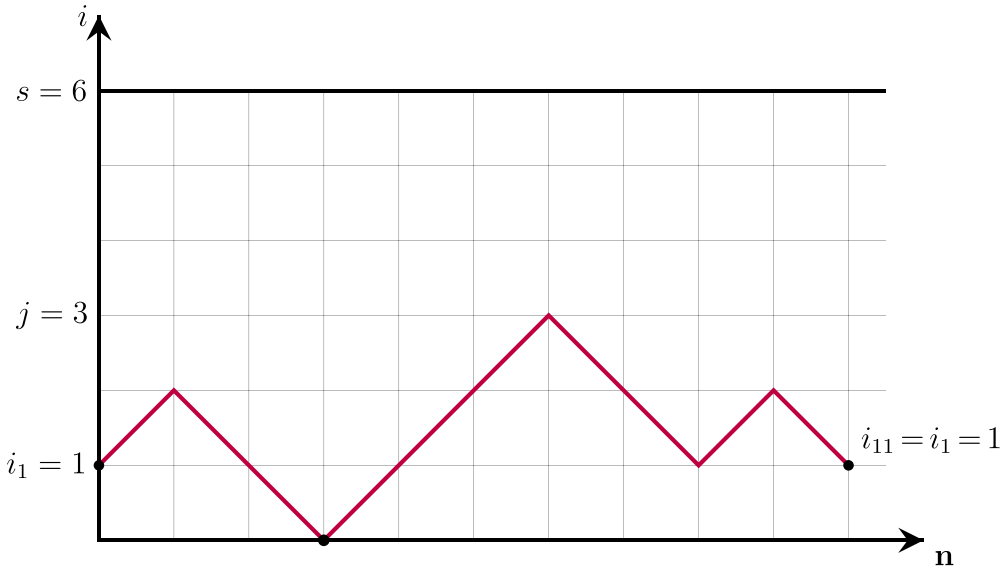

Figure 2. A typical path contributing to $\mathrm{tr}\,(H^{\mathbf{n}}\sigma)$ for $s=6$, with even length $\mathbf{n}=2n=10$, starting and ending at index $i=1$, and touching 0 at $i_4=0$. It reaches a maximal level $j=3$ and has transitions $l_1=1, l_2=3, l_3=1$.

the sum is over all nonvanishing integers summing to n, that is, over all compositions of n, and $2n\,c(l_1,\ldots,l_j)$ is the numbers of distinct periodic paths with the given number l_i of transitions per level (the prefactor $2\,n$ is introduced to conform with previous conventions[4]). The counting of these paths is known, derived combinatorially or via a secular determinant method, and $c_2(l_1,\ldots,l_j)$ takes the form

$$c(l_1,l_2,\ldots,l_j)=\frac{1}{l_1}\prod_{i=1}^{j-1}\binom{l_i+l_{i+1}-1}{l_{i+1}}=\frac{1}{l_j}\prod_{i=1}^{j-1}\binom{l_i+l_{i+1}-1}{l_i}. \tag{21}$$

combining (20), (19) and (21), we obtain

$$\begin{aligned} G_{2n}(\mathrm{Q}) &= 2n\sum_{\substack{l_1,l_2,\ldots,l_j\\ \text{composition of } n}} 2^{l_1}\,c(l_1,l_2,\ldots,l_j)\prod_{i=1}^{j}\left(\mathrm{Q}^{-i+\frac{1}{2}}+\mathrm{Q}^{i-\frac{1}{2}}\right)^{2l_i} \qquad (22)\\ &= 2n\sum_{\substack{l_1,l_2,\ldots,l_j\\ \text{composition of } n}} \frac{2^{l_1}}{l_1}(2+\mathrm{Q}^{-1}+\mathrm{Q})^{l_1}\prod_{i=2}^{j}\binom{l_{i-1}+l_i-1}{l_i}\\ &\quad\times\left(2+\mathrm{Q}^{-2i+1}+\mathrm{Q}^{2i-1}\right)^{l_i} \end{aligned}$$

(b) **Odd length $\mathbf{n}=2n-1$**: Paths must have an odd number of steps $s\to s$, so these steps will contribute *opposite* factors for H_+ and H_-. Assuming, again, $n\leqslant s$, such paths never

[4] $c(l_1,\ldots,l_j)$ is related to the nth cluster coefficient of identical particles with quantum exclusion statistics and was denoted $c_2(l_1,\ldots,l_j)$ in [3], the index 2 referring to exclusion of order 2.

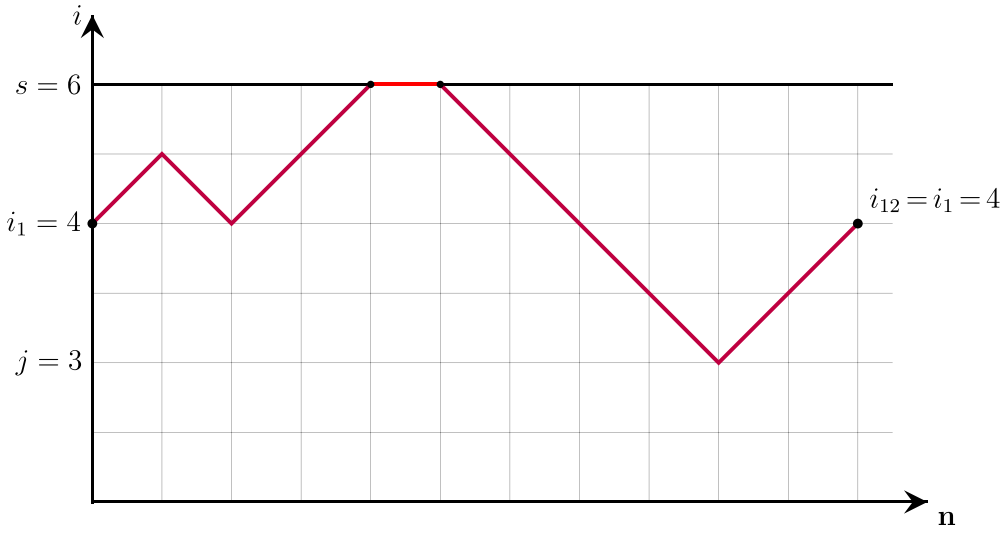

Figure 3. A typical path contributing to $\mathrm{tr}\,(H^{\mathbf{n}}\sigma)$ for $s=6$, with odd length $\mathbf{n}=2n-1\ =11$, starting and ending at index $i=4$, with one step on $i=s$ between $i_5=i_6=6$. It dips to a minimal level $s-j=2$ ($j=4$) and has transitions $l_0=1, l_1=1, l_2=2, l_3=1, l_4=1$.

touch 0 and thus the total amplitude for H_-^{2n-1} is the opposite of that for H_+^{2n-1}. Such a path is depicted in figure 3.

Assuming a path dips down to minimum level $s-j$, we call l_i ($i\geqslant 1$) the number of (down or up) transitions between level $s-i+1$ and $s-i$, and $2l_0-1$ ($l_0>0$) the (odd) number of $s\to s$ steps. The total number of steps is $2l_0\ -1+2\sum_{i=1}^{j} l_i = 2n-1$, so $\sum_{i=0}^{j} l_i = n$. The total trace can be expressed combinatorially as

$$\mathrm{tr}\,(H^{2n-1}\sigma) = 2\,\mathrm{tr}\,H_+^{2n-1} = 2\sum_{\sum_i l_i=n} (2n-1)\,\bar{c}(l_0,l_1,\ldots,l_j)\,2^{2l_0-1} s_s^{l_1}\ldots s_{s-j+1}^{l_j} \tag{23}$$

where $(2n-1)\,\bar{c}(l_0,l_1,\ldots,l_j)$ denotes the number of discrete periodic paths with the given number of $s\to s$ steps and transitions. By taking each $s\to s$ step and extending it to an $s\to s+1\ \to s$ set of transitions by adding a fictitious $s+1$ level, such paths become the mirror-image of paths touching 0 upon mapping levels $i\to s+1-i$, so

$$\bar{c}(l_0,l_1,\ldots,l_j) = c(2l_0-1,l_1,\ldots,l_j) = \frac{1}{2l_0-1}\binom{2l_0+l_1-2}{l_1}\prod_{i=1}^{j} \times \binom{l_{i-1}+l_i-1}{l_i}. \tag{24}$$

(The fact that the promotion of $s\to s$ to two transitions $s\to s+1$ and $s+1\ \to s$ increases the length of the chain by $2l_0-1$ is compensated by the fact that paths cannot start at $s+1$.) Noting also that

$$s_{s-i+1} = 2 + \mathrm{Q}^{-2s+2i-1} + \mathrm{Q}^{2s+2i+1} = 2 + \mathrm{Q}^{-2i} + \mathrm{Q}^{2i} = (\mathrm{Q}^{-i} + \mathrm{Q}^{i})^2 \quad (25)$$

we obtain the final result

$$G_{2n-1}(\mathrm{Q}) = (2n-1) \sum_{\substack{l_0,l_1,\ldots,l_j \\ \text{composition of } n}} c(2l_0-1, l_1, \ldots, l_j) \prod_{i=0}^{j} (\mathrm{Q}^{-i} + \mathrm{Q}^{i})^{2l_i} \quad (26)$$

$$= (2n-1) \sum_{\substack{l_0,l_1,\ldots,l_j \\ \text{composition of } n}} 4^{l_0} \frac{(2l_0+l_1-2)!(l_0-1)!}{(l_0+l_1-1)!(2l_0-1)!} \prod_{i=1}^{j} \times \binom{l_{i-1}+l_i-1}{l_i} \left(\mathrm{Q}^{-i} + \mathrm{Q}^{i}\right)^{2l_i}.$$

2.4. Generalization for all lengths and specific examples

We finally address the assumption made so far that $\mathbf{n} \leqslant 2s$. In general, for $\mathbf{n} > 2s$ states near both $|0\rangle$ and $|s\rangle$ need be considered and would lead to 'umklapp' effects[5]. However, formulae (22) and (26) do not involve s explicitly, Q being the only parameter. Consequently, we can simply ignore the constraint $\mathbf{n} \leqslant 2s$ and treat Q as a formal expansion parameter as in the original defining relation (1). Therefore, (22) and (26) are valid for all values of $\mathbf{n}$ *without* restriction.

It is reassuring to give a few examples of the generating function formulae for low values of the length:

- for length $\mathbf{n} = 1$, setting $n = 1$ in (26) only the term $l_0 = 1$ survives and we obtain $G_1(\mathrm{Q}) = 4$.
- For length $\mathbf{n} = 2$, setting $n = 1$ in (22) only the $l_0 = 1$ term survives and we obtain $G_2(\mathrm{Q}) = 8 + 4(\mathrm{Q}^{-1} + \mathrm{Q})$.
- For length $\mathbf{n} = 3$, setting $n = 2$ in (26) only the $l_0 = 2$ and $l_0 = l_1 = 1$ terms survive and we obtain $G_3(\mathrm{Q}) = 40 + 12(\mathrm{Q}^{-2} + \mathrm{Q}^2)$.
- For length $\mathbf{n} = 4$, setting $n = 2$ in (22) only the $l_0 = 2$ and $l_0 = l_1 = 1$ terms survive and we obtain $G_4(\mathrm{Q}) = 80 + 48(\mathrm{Q}^{-1} + \mathrm{Q}) + 16(\mathrm{Q}^{-2} + \mathrm{Q}^2) + 16(\mathrm{Q}^{-3} + \mathrm{Q}^3) + 8(\mathrm{Q}^{-4} + \mathrm{Q}^4)$.

It can be checked that these reproduce the correct number of open walks with the corresponding length and area (exponent of Q^2), and that the total number of walks, obtained by setting $\mathrm{Q} = 1$ in the generating function $G_{\mathbf{n}}(\mathrm{Q})$, is $4^{\mathbf{n}}$ as required.

2.5. Walk enumeration

From the generating functions (22) and (26) we can infer the number of paths of given length and algebraic area $A/2$ by expanding in powers of Q and isolating the coefficient of the term Q^A. It is already clear from the form of (22) that the expansion in powers of Q will involve both even and odd powers, reflecting the fact that paths of even length can have half-integer algebraic area, while (26) clearly involves only even powers, consistent with the fact that paths of odd length can only have an integer area.

[5] By 'umklapp' effects we mean walks with algebraic areas differing by multiples of $s + \frac{1}{2}$ being counted together, since $\mathrm{Q}^{2s+1} = 1$. Their algebraic counterparts are paths of indices that wind around the periodic states $|i\rangle = |i + q\rangle$, which in the $H_{\pm}$ formulation manifest as paths that both touch 0 and creep over s.

A binomial expansion of the upper expression in (22) gives

$$C_{2n}(A) = 2n \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} 2^{l_1}\, c(l_1,\ldots,l_j) \sum_{k_2=-l_2}^{l_2} \cdots \sum_{k_j=-l_j}^{l_j} \times \binom{2l_1}{l_1 + \sum_{r=2}^{j}(2r-1)k_r - 2A} \prod_{s=2}^{j} \binom{2l_s}{l_s + k_s} \tag{27}$$

while a similar expansion of the upper expression in (26) gives

$$C_{2n-1}(A) = (2n-1) \sum_{\substack{l_0,l_1,\ldots,l_j \\ \text{composition of } n}} 4^{l_0}\, c(2l_0-1,l_1,\ldots,l_j) \sum_{k_2=-l_2}^{l_2} \cdots \sum_{k_j=-l_j}^{l_j} \times \binom{2l_1}{l_1 + \sum_{r=2}^{j} r k_r - A} \prod_{s=2}^{j} \binom{2l_s}{l_s + k_s}. \tag{28}$$

In all expressions, binomial coefficients with entries outside of their range vanish and products with lower term rank higher than the upper one become unity.

2.6. Paths with fixed endpoints

We conclude with a brief discussion of the most general situation, namely, the enumeration of walks of given length and area and with a fixed endpoint (the starting point is always placed at the origin). As before, we will focus on evaluating the algebraic area generating function for such walks.

To fix the endpoint of the walks there are two possible approaches. One approach would be to consider a generating function that assigns specific weights to the endpoint of the walk. This can be achieved by considering operators u, v with nontrivial Casimirs u^q, v^q through the substitution

$$u \to e^{ik_x} u\ , \quad v \to e^{ik_y} v \tag{29}$$

and evaluating $\mathrm{tr}\,(H^{\mathbf{n}}\sigma)$ as before. Note that, for such u, v,

$$\sigma u = e^{2ik_x} u^{-1}\sigma\ , \quad \sigma v = e^{2ik_y} v^{-1}\sigma\ , \quad \mathrm{tr}\,(v^l u^k \sigma) = e^{ikk_x + ilk_y} \mathrm{Q}^{kl} \tag{30}$$

so

$$G_{\mathbf{n}}(\mathrm{Q}; k_x, k_y) = \mathrm{tr}\,(H^n \sigma) \tag{31}$$

reproduces the generating function of open walks, weighted by phase factors $e^{ikk_x + ilk_y}$ depending on their endpoint. Walks ending at k, l can then be isolated by

$$\tilde{G}_{\mathbf{n}}(\mathrm{Q}; k, l) = \frac{1}{4\pi^2} \int_0^{2\pi} dk_x dk_y e^{-ikk_x - ilk_y}\, G_{\mathbf{n}}(\mathrm{Q}; k_x, k_y). \tag{32}$$

The explicit evaluation of the trace in (31), however, is nontrivial, as H does not commute with σ any more.

An alternative approach would be to evaluate a general *matrix element* of the function $H^{\mathbf{n}}$, which, as we shall explain, yields the generating function of walks ending on a straight line weighted by their endpoint on this line. Specifically, consider walks starting at the origin and ending on the even-parity sublattice $k+l=2I$, with I a fixed integer, and define the quantity $G_{2n}^{(2I,J)}(\mathrm{Q})$ depending on I and another integer J

$$G_{2n}^{(2I,J)}(\mathrm{Q}) := \langle J+I|H^{2n}|J-I\rangle = \big\langle |J+I|\big|(H_+^{2n}+\mathrm{sgn}(J^2-I^2)H_-^{2n})\big||J-I|\big\rangle \tag{33}$$

with H as in (14), referring to the modified realization $u\to \mathrm{Q}uv$, and states as defined in (11). Monomials $v^l u^k$ become

$$v^l u^k \to v^l(\mathrm{Q}uv)^k = \mathrm{Q}^{-k^2} v^{l+k}u^k \tag{34}$$

and their $J+I, J-I$ matrix elements become

$$\langle J+I|\mathrm{Q}^{-k^2}v^{l+k}u^k|J-I\rangle = \mathrm{Q}^{-k^2+2(J+I)k}\,\delta_{k+l,2I} = Q^{2Jk+lk}\,\delta_{l,2I-k} \tag{35}$$

therefore, $G_{2n}^{(2I,J)}(\mathrm{Q})$ gives the area-weighted sum of walks ending on even-parity points $k+l=2I$, weighted by the factor $\mathrm{Q}^{2\,Jk}$ depending on their final position $(k,2I-k)$. Multiplying by $\mathrm{Q}^{-2\,Jk_0}$ and summing over J would isolate the term $k=k_0\,(\mathrm{mod}(2s+1))$, thus reproducing the generating function of walks ending at an even-parity sublattice point $(k_0,2I-k_0)$ up to an 'umklapp' periodicity $k_0\ \sim k_0+2s+1$. The umklapp effect becomes relevant for walks of length long enough to reach more than one periodic copies, and can be eliminated by assuming

$$\sum_J \mathrm{Q}^{2\,Jx} = q\,\delta_{x,0} \tag{36}$$

thus ignoring the finiteness of $q=2s+1$. A similar construction generalizing (33) would work for walks ending on an odd-parity sublattice point. The full calculation of these matrix elements and corresponding sums is yet to be done. In appendices A and B we provide the evaluation of $G_{2n}^{(2I,0)}(\mathrm{Q})$ for $J=0$, which gives the area-weighted generating function of walks with endpoints on the line $k+l=2I$ without additional weights for their endpoint.

3. Conclusions and closed walks

We conclude with some comments on closed walks of necessarily even length $\mathbf{n}=2n$, for which an expression for their generating function and the corresponding algebraic area counting formula are known [1]. The methods in the present work offer an alternative way of calculating these closed walk quantities, and one could hope to obtain alternative equivalent expressions. The main reason for this hope is that the method based on traces involving σ, or corresponding general matrix elements as in section 2.6, seems to at least partially evade 'umklapp' effects, as stressed in section 2.4.

In more detail, the first approach exposed here, based on k_x, k_y, has *no* umklapp effects whatsoever. The second approach, relying on matrix elements for states $|J\pm I\rangle$, has a *partial* umklapp effect relating to the position of the endpoint on the paradiagonal, but has *no* umklapp effect with the position of the paradiagonal. The complexity of calculating $G_{2n}^{2I,J}(\mathrm{Q})$, and especially $G_{2n}(\mathrm{Q};k_x,k_y)$, is the main impediment in deriving expressions for closed walks, and their evaluations remains a task for the future.

Note that, putting $I=0$ in (33) and summing over J amounts to calculating the trace $\mathrm{tr}\,H^{2n}$, reproducing the known trace expression for the generating function of closed walks. Similarly,

the integral in (32) for $x = y = 0$ would isolate terms v^0u^0 in H^{2n} and would reduce $\mathrm{tr}\,(H^{2n}\sigma)$ to $(1/q)\mathrm{tr}\,H^{2n}$, again reproducing the known closed walk counting formula in terms of the trace. It is the evaluation of $G_{2n}(\mathrm{Q};k_x,k_y)$ as an explicit function of k_x,k_y, using the techniques of the present work, that might yield alternative formulae. This is yet to be accomplished.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. 'Diagonal' walks

The simplest case is walks that start at the origin and end anywhere along the line of lattice points with coordinates $k+l=0$. Such walks necessarily have an even length. The evaluation of their generating function $G^{(0,0)}$ is achieved by simply calculating the 00 matrix element

$$G_{2n}^{(0,0)}(\mathrm{Q}) = \langle 0|H^{2n}|0\rangle \tag{37}$$

with H as in (14), referring to the modified representation $u \to \mathrm{Q}uv$, and states as defined in (11). Monomials $v^l u^k$ become in this representation

$$v^l u^k \to v^l(\mathrm{Q}uv)^k = \mathrm{Q}^{-k^2} v^{l+k} u^k \tag{38}$$

and the 00 matrix element becomes

$$\langle 0|\mathrm{Q}^{-k^2} v^{l+k} u^k|0\rangle = \mathrm{Q}^{-k^2}\delta_{k+l} \tag{39}$$

constraining the walks to the $k+l=0$ subset and reproducing the radial area of such walks. We note that walks ending on the diagonal $k=l$ trivially have the same generating function, due to the invariance of the algebraic area under lattice $\pi/2$ rotations.

The evaluation of the matrix element is done in analogy to the trace of $H^n\sigma$: since $\sigma|0\rangle = |0\rangle$, only the part $\langle 0|H_+^{2n}|0\rangle$ will contribute. We obtain an expression similar to (20), with the difference that now the combinatorial prefactor counts only paths that start from 0 and end at 0. This counting differs from the full counting $2nc(l_0,\ldots,l_j)$ by a factor of $l_1/(2n)$. We obtain

$$\begin{aligned} G_{2n}^{(0,0)}(\mathrm{Q}) &= \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} 2^{l_1}\, l_1\, c(l_1,l_2,\ldots,l_j) \prod_{i=1}^{j}\left(\mathrm{Q}^{-i+\frac{1}{2}} + \mathrm{Q}^{i-\frac{1}{2}}\right)^{2l_i} \\ &= \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} 2^{l_1}\,(2+\mathrm{Q}^{-1}+\mathrm{Q})^{l_1} \prod_{i=2}^{j}\binom{l_{i-1}+l_i-1}{l_i} \\ &\quad \times \left(2+\mathrm{Q}^{-2i+1}+\mathrm{Q}^{2i-1}\right)^{l_i} \end{aligned} \tag{40}$$

and a corresponding expression for the enumeration of walks

$$C_{2n}^{(0,0)}(A) = \sum_{\substack{l_1,l_2,\ldots,l_j \\ \text{composition of } n}} 2^{l_1} \sum_{k_2=-l_2}^{l_2} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + \sum_{r=2}^{j}(2r-1)k_r - 2A} \prod_{i=2}^{j} \times \binom{l_{i-1}+l_i-1}{l_i} \binom{2l_i}{l_i+k_i}. \tag{41}$$

Appendix B. 'Paradiagonal' walks

With a similar reasoning, we can consider walks that end in an even-parity paradiagonal $k+l = 2I$, $I \geqslant 0$. (The cases $I<0$, and $k-l=2I$, are, again, trivially related to the present one by π or $\pi/2$ lattice rotations.) These have as generating function

$$G_{2n}^{(2I,0)}(\mathrm{Q}) = \langle -I|H^{2n}|I\rangle = \langle I|H_+^{2n}|I\rangle - \langle I|H_-^{2n}|I\rangle \tag{42}$$

only index paths that touch 0 contribute, through $\langle I|H_+^{2n}|I\rangle$.

The evaluation of the matrix element process similarly to the case $I=0$: we call again l_i ($1 \leqslant i \leqslant j$) the number of up or down transitions between levels $i-1$ and i, where now necessarily $j \geqslant I$. The number of paths $P(I;l_1,\ldots,l_j)$ starting and ending at level I with steps $l_1,\ldots,l_j$ is given by

$$P(I;l_1,\ldots,l_j) = (l_I + l_{I+1})\, c(l_1,l_2,\ldots,l_j) \quad (l_0 = l_{j+1} \equiv 0) \tag{43}$$

with $c(l_1,\ldots,l_j)$ as in (21). (The term proportional to l_I above is the number of paths starting downwards at I, and the term proportional to l_{I+1} is the number of paths starting upwards.) Note that summing over all I gives

$$\begin{aligned}\sum_{I=0}^{j} P(I;l_1,\ldots,l_j) &= [l_1 + (l_1+l_1) + \ldots + (l_{j-1}+l_j) + l_j]\, c(l_1,\ldots,l_j) \\ &= 2(l_1+\ldots+l_j)\, c(l_1,\ldots,l_j) = 2n\, c(l_1,\ldots,l_j)\end{aligned} \tag{44}$$

reproducing the full counting of paths for unrestricted walks. Overall we obtain

$$G_{2n}^{(2I,0)}(\mathrm{Q}) = \sum_{\substack{l_1,l_2,\ldots,l_j;\, j\geqslant I \\ \text{composition of } n}} 2^{l_1}\, (l_I + l_{I+1})\, c(l_1,l_2,\ldots,l_j) \prod_{i=1}^{j} \left(\mathrm{Q}^{-i+\frac{1}{2}} + \mathrm{Q}^{i-\frac{1}{2}}\right)^{2l_i} \tag{45}$$

clearly (40) is a special case of (45) with $I=0$.

Odd-parity paradiagonal walks ending at $x+y = 2I+1$ ($I \geqslant 0$) necessarily have an odd length $2n-1$. Their generating function can be expressed as

$$G_{2n-1}^{(2I-1,0)} = \langle s-I|H^{2n-1}|-s+I\rangle = \langle s-I|(H_+^{2n-1} - H_-^{2n-1})|s-I\rangle. \tag{46}$$

Index paths for such walks will have an odd number of transitions $s \to s$, so the contributions of H_+ and H_- are equal. Assuming an index path that dips down to minimum level $s-j$, where necessarily $j \geqslant I$, we call l_i ($i \geqslant 1$) the number of (down or up) transitions between level $s-i+1$ and $s-i$, and $2l_0-1$ ($l_0 > 0$) the (odd) number of $s \to s$ steps. Using the trick of

adding a fictitious $s+1$ level and mapping levels $i \to s+1-i$, the number of index paths that start and end at level $s-I$ becomes

$$\begin{aligned}\bar{c}(I;l_0,l_1,\ldots,l_j) &= (l_I + l_{I+1})\, c(2l_0-1,l_1,\ldots,l_j)\ , \quad I \geqslant 1 \\ &= (2l_0 - 1 + l_1)\, c(2l_0-1,l_1,\ldots,l_j)\ , \ I=0.\end{aligned} \tag{47}$$

Note that summing over all I we have

$$\sum_{I=0} \bar{c}(I;l_0,l_1,\ldots,l_j) = 2l_0 - 1 + 2l_1 + \ldots + 2l_j = 2n-1 \tag{48}$$

reproducing the full counting $(2n-1)\,\bar{c}(l_0,\ldots,l_j)$ of (24). Overall we obtain

$$G_{2n}^{(2I-1,0)}(\mathrm{Q}) = \sum_{\substack{l_0,l_1,\ldots,l_j;j\geqslant I \\ \text{composition of } n}} (l_I + l_{I+1})\, c(2l_0-1,l_1,\ldots,l_j) \prod_{i=0}^{j} \left(\mathrm{Q}^{-i} + \mathrm{Q}^{i}\right)^{2l_i}. \tag{49}$$

The number of paradiagonal walks of fixed length and area can be found by isolating the term Q^{2A} in expressions (45) and (49), as in (27), (28), and (41) and we will not write the explicit formulae.

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