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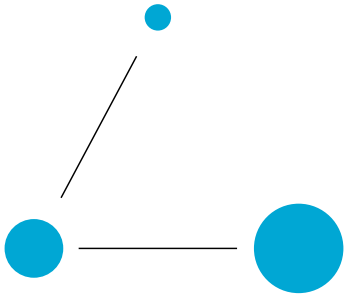

ANNALES
HENRI LEBESGUE

DAVID FISHER

SEBASTIAN HURTADO

A NEW PROOF OF FINITENESS OF MAXIMAL ARITHMETIC REFLECTION GROUPS

UNE NOUVELLE PREUVE DE FINITUDE DES GROUPES DE RÉFLEXION ARITHMÉTIQUES MAXIMAUX

Abstract. — We give a new proof of the finiteness of maximal arithmetic reflection groups. Our proof is novel in that it makes no use of trace formulas or other tools from the theory of automorphic forms and instead relies on the arithmetic Margulis lemma of Fraczyk, Hurtado and Raimbault.

Résumé. — Nous donnons une nouvelle preuve de la finitude des groupes de réflexion arithmétiques maximaux. Notre preuve est nouvelle en ce qu'elle n'utilise pas de formules de traces ou d'autres outils de la théorie des formes automorphes et s'appuie à la place sur le lemme arithmétique de Margulis de Fraczyk, Hurtado et Raimbault.

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1. Introduction

In this note, we offer a new proof of the following result:

THEOREM 1.1. — *For every $n \geqslant 2$, there exist only finitely many maximal arithmetic reflection groups $\Gamma \subset Isom(\mathbb{H}^n)$.*

Nikulin proved in [Nik80] that there were at most finitely many maximal arithmetic reflection groups in each dimension n where the degree of the adjoint trace field is fixed. Therefore Theorem 1.1 follows from bounding the degree of the number field and our main contribution is to give a new proof of the following:

THEOREM 1.2. — *For every $n \geqslant 2$, there exists $d_n > 0$, such that if Γ is arithmetic reflection group in $\mathbb{H}^n$, then degree of the adjoint trace field of Γ over $\mathbb{Q}$ is at most d_n.*

The fact that Theorem 1.1 implies finiteness of all maximal arithmetic reflection groups can be deduced in a variety of ways from prior work of Nikulin, Prokhorov and Vinberg, see the introduction to [ABSW08] or the survey [Bel16]. We note here that we do not use that the reflection group is maximal in our proof of Theorem 1.2.

We briefly discuss the prior history of finiteness of maximal arithmetic reflection groups, referring the reader to Belolipetsky's excellent survey for more details on this and other related points [Bel16]. In [LMR06], Long, Maclachlan and Reid prove finiteness of arithmetic surfaces of genus zero, which implies the desired result. Shortly afterwards, Agol proved finiteness of maximal arithmetic Kleinian reflection groups in [Ago06]. Following that work there were two independent proofs of finiteness in higher dimensions, one by Nikulin and one by Agol, Belolipetsky, Storm and Whyte [ABSW08, Nik07]. Roughly [ABSW08] generalizes the proofs of [Ago06, LMR06] to higher dimensions using some new inputs, while [Nik07] uses older reflection group technology from Nikulin's prior work on finiteness to prove the case of general dimension by induction where the base case depends on the results in dimension 2 and 3. In either case, the proofs relies in the end in a central way on deep results in the theory of automorphic forms that provide an absolute lower bound on the first eigenvalue of the Laplacian on the relevant orbifolds. Our main motivation in writing this note is to give a proof that is independent of the spectral bounds and the theory of automorphic forms.

Our main tool will be the following:

LEMMA 1.3 (Arithmetic Margulis Lemma). — *For every $n > 0$, there exists ϵ_n such that if $\Gamma \subset Isom(\mathbb{H}^n)$ is an arithmetic group whose trace field has degree d over $\mathbb{Q}$, then for every $x \in \mathbb{H}^n$, the group:*

$$\Gamma_x := \langle \{\gamma \in \Gamma \mid d_{\mathbb{H}^n}(\gamma x, x) \leqslant \epsilon_n d\} \rangle$$

is virtually solvable.

This lemma is proven in [MHR22] using Breuillard's height gap theorem from [Bre11]. Breuillard's proof of the height gap theorem is also far from elementary, but a recent elementary proof was obtained by the second author with Chen and Lee [CHL21]. In particular, our proof of Theorem 1.1 does not rely on heavy machinery from number

theory or representation theory in any way. We thank Ian Agol for suggesting that it might be interesting to find such a proof.

We mention briefly here that once Theorem 1.2 was known, further work has been done on explicitly computing d_n. For a fairly up to date account of these developments see [Bel16, Section 5]. Only d_2 has been computed exactly in more recent work of Linowitz [Lin18]. To achieve anything in this direction by our methods would require an effective version of Lemma 1.3 and so an effective version of the height gap theorem. This seems quite difficult in general, but could perhaps be easier in the explicit settings required for the study of hyperbolic reflection groups.

We finish this introduction by giving an outline of the Proof of Theorem 1.2 in the case $n = 2$. Assume Γ is an arithmetic reflection group generated by reflections in the sides of an acute-angled polygon P in $\mathbb{H}^2$. Let $\mathcal{E} = \{e_1, e_2, \ldots, e_k\}$ be the edges of P and consider the collection of balls $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$ of radius $\frac{\epsilon_2 d}{2}$ centered at the midpoints of the edges of P, where ϵ_2 is the constant in Lemma 1.3 and d is the degree of the trace field of Γ.

If a ball $B_i \in \mathcal{B}$ intersects three edges (or intersect two other balls in $\mathcal{B}$), the group generated by reflections in these three sides will be typically a non-virtually solvable group[(1)], therefore if d is sufficiently large, Lemma 1.3 implies the balls in B typically cannot have triple intersections and intersect at most two edges of P. To illustrate the argument, assume there are no triple intersections of balls of $\mathcal{B}$ and that each ball in $\mathcal{B}$ intersects only one edge, and so half of each ball is contained in P. This implies that the area of P has to be greater than $\frac{k}{4}\mathrm{Vol}(B_{\mathbb{H}^2}(\epsilon_2 d))$, but elementary hyperbolic geometry shows that P has volume at most $(k-2)\pi$. If d is sufficiently large both inequalities are not possible. In the case $n \geqslant 3$, we will apply a similar argument to a two-dimensional face of P, in this case we will also need to make use of the simplicity of such polyhedra due to Vinberg and a theorem of Andreev.

Remark. — Very shortly after we shared a draft of this paper, Jean Raimbault replied with an even shorter proof of Theorem 1.1. Raimbault's proof uses [MHR22, Theorem D] and a remarkable new result: for any reflection hyperbolic manifold in dimension n, the thin part has at least a fixed proportion of the volume [Rai22]. Since [MHR22, Theorem D] uses the trace formula, we remark here that it is also possible to produce a proof of Theorem 1.1 using [MHR22, Theorem C], Raimbault's lower bound on the volume of the thin part, and Nikulin's results from [Nik80]. While Raimbault's proof is shorter and more conceptual than ours, it seems interesting to publish our approach as the proof is more elementary and the ideas might prove useful for a variety of problems concerning real and complex hyperbolic lattices generated by torsion elements.

(1) unless the three edges are adjacent and meet at right angles, a problematic case that requires further considerations

2. Proof of Theorem 1.2

Definition 2.1. — *For a hyperplane H in $\mathbb{H}^n$, let I_H be the involution in $\mathbb{H}^n$ with respect to H. Let us define* $\mathrm{Fix}(H)$ *to be the set of fixed points of I_H in the boundary $\partial\mathbb{H}^n$.*

Let P be the acute-angled hyperbolic polytope defining Γ, and so there exists a collection of oriented hyperplanes $\{H_\alpha\}_{\alpha\in I}$, such that $P = \bigcap H_\alpha^+$ and $\Gamma = \langle\{I_{H_\alpha}\}_{\alpha\in I}\rangle$. We will argue by contradiction and suppose that Γ is defined over a number field of degree $d \geqslant d_n$, where d_n is the smallest positive integer such that the ball B of radius $d_n\epsilon_n/4$ in $\mathbb{H}^2$ satisfies:

$$\mathrm{Vol}_{\mathbb{H}^2}\left(B_{d_n\epsilon_n/4}\right) \geqslant 12\pi \tag{2.1}$$

where ϵ_n is the constant in the Arithmetic Margulis Lemma 1.3.

For now, we work with any acute angled polyhedron P, where the associated group Γ is not necessarily arithmetic or even cofinite volume. Our other main tool, Lemma 2.6, holds in this generality.

We will make use of the following results of Andreev [And71] and Vinberg about the polyhedron P.

Theorem 2.2 (Vinberg [Vin85, Cor. Thm 3.1]). — *Let P be an acute-angled polytope in $\mathbb{H}^n$. Then P is* simple*, meaning that for every k, each face of codimension k is contained in exactly k codimension-one faces.*

We note for clarity that the polytope here is contained in $\mathbb{H}^n$ and so *face* in this statement does not include vertices at infinity. In [Vin85], Vinberg proves this result for the polygon before defining vertices at infinity and never claims the analogous result for vertices at infinity. The statement is not true for vertices at infinity. This is easily seen by considering the all right dodecahedron in dimension 3 where some vertices at infinity have links that are squares not triangles or the all right 24-cell in dimension four which has vertices at infinity with links that are cubes and not simplices.

Theorem 2.3 (Andreev). — *Let P be an acute-angled polyhedron and suppose $\{F_i\}$ is a (finite or countable) collection of codimension one faces of P and $\{H_i\}$ is the corresponding collection of hyperplanes in $\mathbb{H}^n$. Then*

$$\dim\left(\cap_{i\in I}\overline{F_i}\right) = \dim\left(\cap_{i\in I}\overline{H_i}\right).$$

Here the closures occurring in the statement occur in $\overline{\mathbb{H}^n}$ and we assume a point in the boundary of $\mathbb{H}^n$ has dimension -1 and that the empty set has dimension $-\infty$. One can read Andreev's theorem as saying that the faces of the polygon do not have "extra" intersections not already occurring in $\overline{P}$.

We will use repetitively the following elementary facts:

Proposition 2.4. — *If Γ is a virtually solvable discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, then there exists $x \in \mathbb{H}^n \cup \partial\mathbb{H}^n$ such that $|\Gamma x| \leqslant 2$. Moreover if Γ is not finite, then $x \in \partial\mathbb{H}^n$.*

Proof. — If Γ is finite then Γ fixes a point in $\mathbb{H}^n$, so assume Γ is infinite. Let $\Gamma_0 < \Gamma$ be a finite index solvable subgroup which we can choose to be a normal subgroup. Then $\Gamma_0 < P$ a parabolic subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, as can be seen, for example, by considering its Zariski closure. By definition P is the stabilizer of a point x in $\partial\mathbb{H}^n$ and Γ_0 fixes this point. Either Γ also fixes x or Γ_0 has more than one fixed point. It is easy to see that the later case only occurs when Γ_0 is the stabilizer of a geodesic and that in this case it fixes exactly two points in $\partial\mathbb{H}^n$ which are permuted by Γ. □

Proposition 2.5. — *Suppose H_1 and H_2 are two hyperplanes in $\mathbb{H}^n$, and suppose that $H_1 \cap H_2 \neq \emptyset$, then if $x \notin \mathrm{Fix}(H_1) \cup \mathrm{Fix}(H_2)$ and $\Gamma := \langle I_{H_1}, I_{H_2} \rangle$ is the subgroup generated by reflections in H_1, H_2, then $|\Gamma x| \geqslant 3$.*

Proof. — The hypotheses imply that $\mathbb{H}^n \setminus \mathrm{Fix}(H_1) \cup \mathrm{Fix}(H_2)$ has four components. If one picks x in any of those four components, then Γx is easily seen to visit all four components. □

We begin the proof of Theorem 1.2. Choose a 2-dimensional face F of P, and cyclically enumerate its collection of edges $e_0, e_1, \ldots, e_m$ in such a way that e_i is adjacent to e_{i+1}. By the simplicity of P we have that F is contained in exactly $H_1, H_2, \ldots, H_{n-2}$ hyperplanes determining faces of P and so $F = (\cap_{j=1}^{n-2} H_j) \bigcap P$, and for each edge e_i there exists a unique hyperplane H_{e_i} such that $e_i = F \cap H_{e_i}$.

Our main technical tool in the proof is:

Lemma 2.6. — *Let e_i, e_j, e_k be three different edges of F, let*

$$\Gamma_0 := \left\langle I_{H_{e_i}}, I_{H_{e_j}}, I_{H_{e_k}}, I_{H_1}, \ldots, I_{H_{n-2}} \right\rangle$$

then at least one of the following holds:

1. *Γ_0 is not virtually solvable.*
2. *The edges e_i, e_j, e_k are all adjacent, and so up to reordering $j = i+1, k = i+2$. Moreover the angles at $e_i \cap e_{i+1}$, and $e_{i+1} \cap e_{i+2}$ are both $\frac{\pi}{2}$.*

Proof. — Observe that Γ_0 is not finite, otherwise Γ_0 has a fixed point in the interior that must be in the intersection of $H_1, \ldots H_{n-2}, H_{e_i}, H_{e_j}, H_{e_k}$, but that would contradict the simplicity of P (Theorem 2.2).

We will show that if item 2) does not hold then Γ_0 is not virtually solvable. By Proposition 2.4 we have to show that if $x \in \partial\mathbb{H}^n$, we must have $|\Gamma_0 x| \geqslant 3$

Let

$$\mathbb{H}^2 := H_1 \cap H_2 \cdots \cap H_{n-2}$$

and consider the following cases:

Case 1: Suppose $x \notin \mathbb{H}^2$, and so up to reordering suppose that $I_{H_{n-2}}(x) \neq x$. If either $I_{H_{e_i}}(x), I_{H_{e_j}}(x), I_{H_{e_k}}(x)$ are different than x, as H_{n-2} intersect the hyperplanes $H_{e_i}, H_{e_j}, H_{e_k}$, then we can apply 2.5 to show $|\Gamma_0 x| \geqslant 3$. Therefore $I_{H_{e_i}}(x) = I_{H_{e_j}}(x) = I_{H_{e_k}}(x) = x$, moreover as $I_{H_{n-2}}(x)$ is also not fixed by $I_{H_{n-2}}$, the same argument shows that both x and $I_{H_{n-2}}(x)$ are fixed by $I_{H_1}, \ldots I_{H_{n-3}}, I_{H_{e_i}}, I_{H_{e_j}}, I_{H_{e_k}}$, but then the line $l := \overline{x I_{H_{n-2}}(x)}$ must be fixed by $I_{H_1}, \ldots I_{H_{n-3}}, I_{H_{e_i}}, I_{H_{e_j}}, I_{H_{e_k}}$, and by Andreev's Theorem we must have that l contains a one dimensional face e of P, contradicting the simplicity of P at e.

Case 2: Suppose $x \in \mathbb{H}^2$. If $I_{H_{e_i}}(x) \notin \mathbb{H}^2$, then $I_{H_{e_i}}(x)$ is not fixed by some I_{H_l}, and as H_{e_i} intersects H_l, Lemma 2.5 implies that $|\Gamma_0(x)| \geqslant 3$. Similarly we can assume that $I_{H_{e_j}}(x), I_{H_{e_k}}(x) \in \mathbb{H}^2$. Let l_i, l_j, l_k be the lines containing e_i, e_j, e_k respectively.

Suppose $x \notin \mathrm{Fix}(H_{e_i}) \cup \mathrm{Fix}(H_{e_j}) \cup \mathrm{Fix}(H_k)$ and $|\Gamma_0 x| = 2$, then the line $l := \overline{xI_{H_{e_i}}(x)}$ must be orthogonal to all three of H_{e_i}, H_{e_j}, H_{e_k} and each of them contains a codimension one face of P, and this contradicts the convexity of P. Therefore we can assume that $I_{H_{e_i}}(x) = x$. If l_j (similarly l_k) intersects l_i and the angle at the intersection is not $\frac{\pi}{2}$, it follows that all $x, I_{H_{e_j}}(x)$ and $I_{H_{e_i}} I_{H_{e_j}}(x)$ are all distinct and we are done. Therefore we have (up to renaming) that either:

Case 2a: The line l_i intersects l_k at angle $\frac{\pi}{2}$ but l_i does not intersect l_j (in the interior of $\mathbb{H}^2$). In this case the set $\{x, I_{H_{e_k}}(x), I_{H_{e_j}}(x), I_{H_{e_j}}(I_{H_{e_k}}(x))\}$ contains at least three different points and so $|\Gamma_0(x)| \geqslant 3$.

Case 2b: The line l_i does not intersect $l_j \cup l_k$ in the interior of $\mathbb{H}^2$. By Andreev's theorem 2.3 again, if l_i intersects either l_j or l_k at infinity, it does so in a vertex at infinity of P and then by convexity only one of the two intersections can occur at x. In this case, we can suppose that $I_{H_{e_k}}(x) \neq x$, and we have that $I_{H_{e_k}}(x)$ is not fixed by $I_{H_{e_i}}$ therefore $x, I_{H_{e_k}}(x), I_{H_{e_k}} I_{H_{e_i}}(x)$ are all distinct. We remark that this case is not actually needed for the proof of our main results, since non-compact arithmetic reflection groups all have trace field $\mathbb{Q}$.

Case 2c: The line l_i intersects both l_j and l_k perpendicular. In this case Andreev's Theorem 2.3 implies that e_j, e_i, e_k are adjacent and perpendicular as in item 2) and we are done. □

To illustrate Lemma 2.6 and for ease of use, we state the following easy corollary.

COROLLARY 2.7. — *Let $\mathcal{E} = \{e_{i_1}, \ldots, e_{i_k}\}$ be a set of k different edges of F, let*

$$\Gamma_{\mathcal{E}} := \left\langle I_{H_{e_1}}, \cdots, I_{H_{e_{i_k}}}, I_{H_1}, \ldots, I_{H_{n-2}} \right\rangle.$$

If $k > 3$ then $\Gamma_{\mathcal{E}}$ is not virtually solvable. Moreover if $k = 3$ and F has finite volume and three sides, then $\Gamma_{\mathcal{E}}$ is also not virtually solvable.

Proof. — If $k > 4$, we can simply pick three non-adjacent edges in $\mathcal{E}$, and apply Lemma 2.6, noting that the second conclusion cannot hold. If $k = 4$, then if $\Gamma_{\mathcal{E}}$ is virtually solvable so is the group generated by omitting any edge from $\mathcal{E}$. This implies that each triple of edges is adjacent with all angles right angles. In other words F has four sides with all right angles, which is impossible in hyperbolic geometry. The last statement is just the fact that hyperbolic triangle groups are not virtually solvable. This can be deduced from Lemma 2.6, since the configuration in the second conclusion of that lemma cannot be a hyperbolic triangle even with the last vertex at infinity, since the angle sum is at least π. □

We now restrict our attention to polygons P for which Γ is an arithmetic lattice.

DEFINITION 2.8. — *We say that an edge e_i of F is small (or large) if the hyperbolic length $l(e_i) \leqslant \frac{\epsilon d}{2}$ ($l(e_i) > \frac{\epsilon d}{2}$), where ϵ is the constant in the Arithmetic Margulis Lemma 1.3.*

PROPOSITION 2.9. — *There are no two consecutive small edges e_i, e_{i+1} in F.*

Proof. — Let's argue by contradiction, suppose that $i = 2$ and that e_2, e_3 are small, and assume that F has at least four edges. Let

$$\Gamma_4 := \left\langle I_{H_{e_1}}, I_{H_{e_2}}, I_{H_{e_3}}, I_{H_{e_4}}, I_{H_1}, \dots I_{H_{n-2}} \right\rangle.$$

By the arithmetic Margulis lemma Γ_4 applied to the vertex $e_2 \cap e_3$, Γ_4 is virtually solvable, but then this contradicts Corollary 2.7. For the case of F having 3 sides, the argument is similar and simpler. One considers the vertex where the two consecutive short edges meet and notices that this vertex is close to all hyperplanes in F. The Margulis Lemma then implies that the group generated by the reflections giving all faces of F is virtually solvable and this is impossible by Corollary 2.7. □

So from now on we can assume that there are no two small consecutive edges in F, and so at least $m/2$ among $e_1, e_2, \dots, e_m$ are large.

Proposition 2.10. — *If e_i is large, there exists $q_i \in e_i$ such that the ball B_i in the plane containing F with center q_i and radius $\epsilon d/4$ satisfies that half of B_i is contained in F, more precisely* $\mathrm{Vol}(B_i \cap F) = \frac{1}{2}\mathrm{Vol}(B_i)$.

Proof. — Consider a closed ball B_i of radius $\epsilon d/4$ centered at the midpoint p_i of s_i. We will slide the ball B_i along s_i (moving the center p_i in the edge e_i) until half of B_i is totally contained in F as follows:

Let x_i, y_i be the vertices of e_i. If B_i does not intersect another edge of F we are done. Suppose that B_i intersects another edge $e_j \neq e_i$, we start moving B_i as follows:

Case 1: If e_j is adjacent to e_i, say $e_j = e_{i-1}$, we slide B_i towards y_i, until either the interior of half of B_i is totally contained in F and we are done, or until at some point B_i intersects three different edges including e_{i-1}(tangencies count), so we have that B_i intersects e_i, e_{i-1} and e_k, $k \neq i, i-1$, and moreover the angle at x_i is less than 90 degrees. By the Arithmetic Margulis Lemma applied to the center of B_i, we have that $\Gamma_0 := \langle I_{H_{e_i}}, I_{H_{e_{i-1}}}, I_{H_{e_k}}, I_{H_1}, \dots I_{H_{n-2}} \rangle$ is virtually solvable, contradicting Lemma 2.6, and so this triple intersection is not allowed.

Case 2: The idea is the same. If e_j is not adjacent to e_i, we slide B_i towards x_i until half of B_i is totally contained in F, or until half of B_i intersects another edge e_k. By the Arithmetic Margulis Lemma applied to the center of B_i, we have that $\Gamma_0 := \langle I_{H_{e_i}}, I_{H_{e_j}}, I_{H_{e_k}}, I_{H_1}, \dots I_{H_{n-2}} \rangle$ is virtually solvable, and so by Lemma 2.6 e_j, e_k, e_i are consecutive and the angles at $e_j \cap e_k, e_k \cap e_i$ are $\frac{\pi}{2}$, we now slide B_i towards y_i until half of B_i is either totally contained in P or we have that B_i intersects e_i, e_j and another edge e_l (different that e_k). In this case, again by the Arithmetic Margulis Lemma and Lemma 2.6, we have e_i, e_l, e_j are adjacent and perpendicular. This then implies that F has four edges and all angles $\frac{\pi}{2}$, which is impossible. □

We can now finish the proof of Theorem 1.2 by the following volume considerations.

For every large edge e_i consider the half ball $B_i \cap F$, observe that by the Arithmetic Margulis Lemma and Lemma 2.6, no four of these half balls intersect non-trivially. If four balls did intersect non-trivially, the group generated by the reflections in the corresponding sides would be virtually solvable by the Arithmetic Margulis Lemma, which easily contradicts Lemma 2.6 or Corollary 2.7. Now recalling that F has m edges and at least $m/2$ large edges, we have:

$$3\mathrm{Vol}(F) \geqslant \sum_{i \text{ large}} \mathrm{Vol}(B_i \cap F) \geqslant \frac{m}{2}\frac{\mathrm{Vol}(B_1)}{2}$$

Subdividing F into triangles, elementary hyperbolic geometry shows $(m-2)\pi \geqslant \mathrm{Vol}(F)$, which implies that $12\pi > vol(B_1)$ and contradicts that $d \geqslant d_n$ by equation (2.1).

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David FISHER
Rice University Math Department
MS 136 P.O. Box 1892
Houston, TX 77005-1892 (USA)
df32@rice.edu

Sebastian HURTADO
Department of Mathematics
PO Box 208283 New Haven,
CT 06520-8283 Mailcode: 376 (USA)
sebastian.hurtado-salazar@yale.edu