Authors	Jennifer A. Czocher Hamilton L. Hardison Sindura Subanemy Kandasamy
Title	A Bridging Study Analyzing Mathematical Model Construction through a Quantities-Oriented Lens
Affiliation	Texas State University, (Mathematics), San Marcos, (Texas), USA
Contact	Jennifer Czocher Czocher.1@txstate.edu +1 512 245 3414
Funding	This material is based upon work supported by the National Science Foundation under Grant No. 1750813.

A Bridging Study Analyzing Mathematical Model Construction through a Quantities-Oriented Lens

Jennifer A. Czocher
Texas State University
Czocher.1@txstate.edu

Hamilton L. Hardison
Texas State University
hhardison@txstate.edu

Sindura S. Kandasamy
Texas State University
s_k252@txstate.edu

Abstract: Mathematical modelling is endorsed as both a means and an end to learning mathematics. Despite its utility and inclusion as a curricular objective, one of many questions remaining about learners' modelling regards how modelers choose relevant situational attributes and express mathematical relationships in terms of them. Research on quantitative reasoning has informed the field on how individuals quantify attributes and conceive of covariational relationships among them. However, this research has not often attended to modelers' mathematization in open modelling tasks, an endeavour that invites further attention to theoretical and methodological details. To this end, we offer a synthesis of existing theories to present a cognitive constructivist account of mathematical model construction through a quantity-oriented lens. Second, we use empirical data to illustrate why it is productive for theories of modelling to attend to and account for students' quantitative reasoning during modelling activity. Finally, we identify remaining theoretical and methodological challenges to reconciling theories of model construction with a cognitive constructivist view of quantity.

Keywords: mathematical modelling, modelling process, quantitative reasoning, post-secondary students, qualitative research

1 Introduction

Mathematical modelling is a foundational skill for students to learn, regardless of age or career aspiration. The value of mathematical modelling experiences lies in their potential to develop students' capacity for more knowledgeable participation in society and their mathematical conceptual knowledge (Julie & Mudaly, 2007). For this reason, it enjoys prominence as a named learning objective in curricular documents around the world (Kaiser, 2017). Thus, research on mathematical modelling is typically conducted with an eye to promoting it within classrooms, improving its instruction, task design, and ultimately, fostering students' capacity to create models of real-world scenarios (Cai et al., 2014).

Researchers have distinguished between modelers' overall capacity to create models (competence) and the skills and abilities needed to address a modelling problem (competencies) (Blomhøj & Jensen, 2003). Modelling competencies include forming an understanding of a real-world scenario, simplifying, mathematizing (producing a representation), solving, interpreting, and validating (Maaß, 2006). Students at all mathematical levels struggle to carry out these competencies. The competencies associated with model construction—simplifying and mathematizing—are known to be especially challenging (Galbraith & Stillman, 2006; Stillman et al., 2010). Over the years, research has pointed to many sources of students' cognitive difficulties, including problematic transfer of knowledge (Carraher et al., 1985), preservation of real-world complexities (Czocher, 2019), naïve explanations for real-world phenomena (Posner et al., 1982), and the disparate domains of knowledge required (Stillman, 2000).

Despite progress the field has made in articulating reasons for why students struggle to carry out modelling competencies (and develop overall competence), "no worldwide accepted research evidence exists on the effects of short- and long-term mathematical modelling examples and courses in school and higher education on the development of modelling competencies" (Cevikbas et al., 2021, p. 206). In their comprehensive review of published literature on modelling competencies, Cevikbas et al. (2021) argued that fostering and assessing modelling competencies depend on how those competencies are conceptualized and further claimed that the field's existing strategies were too uniform. They called for more innovative approaches to studying modelling competencies, encouraging both theoretical and empirical studies to strengthen the epistemologies of modelling.

Our study responds to this call by operationalizing the *simplifying (specifying)* and *mathematizing* competencies in terms of constructs from theories of quantitative reasoning. Theoretically and methodologically bridging modelling and quantitative reasoning lines of inquiry is promising. From one side, model construction is often characterized as beginning with identifying (ir)relevant variables, an account that could be enriched by attending to how those variables are cognitively constituted. From the other, scholars of quantitative reasoning have long characterized modelling as establishing quantitative structures expressed through mathematical representations (Larson, 2013; O'Bryan; Sherin, 2001; Stroup, 2002; Thompson, 2011), a theory-building move that does not yet have an empirical counterpart in research settings where mathematical model construction (rather than quantitative reasoning) is the subject of inquiry. Establishing empirical compatibility of the two lines of inquiry would offer explanatory mechanisms for successful, and especially, unsuccessful attempts at model construction that are based in students' prior mathematical constructions. A deeper account of model construction would be valuable to facilitators seeking to elicit and extend students' reasoning during model modelling.

In this paper, we address an important aspect of mathematical modelling: modelers' expression of mathematical relationships in terms of their choice of relevant quantities. Our study bridges research on modelling to scholarship of quantitative reasoning through a post-hoc analysis of three students' quantitative reasoning during an open-ended modelling task. Our contribution is three-fold. First, we synthesize perspectives on model construction and quantitative reasoning to present a cognitive constructivist account of mathematical modelling. Second, we illustrate with empirical data compelling reasons why it is productive to account for individuals' quantitative reasoning when studying model construction and making recommendations for fostering students' capacity for modelling. Finally, we identify challenges to operationalizing mathematical model construction through the lens of quantitative reasoning.

2 Literature Review

2.1 Mathematical model construction from a cognitive perspective

Our study falls within the larger agenda of building cognitive models to explain how individuals construct mathematical models. By *mathematical model*, we mean a conceptual system *expressed* using a variety of representations (e.g., concrete materials, written symbols, spoken language) "for constructing, describing, explaining, manipulating, predicting or controlling systems that occur in the world" (Lesh et al., 2003, p. 213). We focus on the process of modelling which is typically conceptualized as an iterative cycle consisting of a series of phases and cognitive or behavioral actions connecting those stages. Visualizations of the process, called modelling cycles (MCs) (e.g., Blum & Leiß, 2007; Zbiek & Conner, 2006), are used as research lenses and each enumerates, nominalizes, and defines the phases differently. We are concerned primarily with *model construction*, which roughly corresponds with the upper arc of many MC's. It begins when the modeler identifies a real-world problem to solve[1] and terminates with a mathematical representation.

We adopt Zbiek and Connor's (2006) decomposition of model construction. In their MC, *specifying* comprises identifying conditions and assumptions of the real-world scenario to prioritize. It includes identifying variables, parameters, their interdependencies, as well as any restrictions or conditions that could be placed upon the modelling products. It includes de-prioritizing or ignoring information the modeler deems irrelevant to solving the problem. Not all assumptions effectively simplify the problem; some preserve complexity (Czocher, 2019). Next is *mathematizing*, which "creates or acknowledges mathematical properties and parameters that correspond to the situational conditions and assumptions that have been specified" (Zbiek & Conner, 2006, p. 99). For example, in a projectile motion problem, one condition could be that the projectile stays above ground level, corresponding to the mathematical property $h > 0$. Mathematizing organizes a correspondence between among conditions, assumptions, properties, and parameters through construction of relations as well as combining them to produce a mathematical representation of those relations. The properties, parameters, and attendant mathematical concepts become components of the eventual mathematical representation of the scenario. Representations can be tables, graphs, charts, gestures, or words, when the signifying elements are mathematical notation.

Mathematizing may happen nearly instantaneously when the modeler has established ways of reasoning about a familiar scenario, such as when an algebra instructor immediately conceives of the speed a steadily moving object travels in terms of time and distance, writing $r = \Delta x/\Delta t$. In contrast, conceiving of a ratio as a model capable of answering the question "how fast?" may be challenging for a learner (Stroup, 2002). Throughout the process, modelers validate and verify their emergent models by reflecting on their anticipated, intermediate outcomes (Stillman & Brown, 2014), resulting in modifications to the representation or to the modeler's conception of the real-world scenario (Czocher, 2018). Satisfactory (to the modeler) model construction depends on *aligning* conceived real-world conditions and assumptions with acknowledged mathematical properties and parameters (Zbiek & Conner, 2006).

2.2 Model construction – state of the field

The main goals of research from a cognitive perspective on modelling have been to characterize individuals' routes through the modelling process as a series of stages, difficulties students encounter as they engage in mathematical modelling activities, or individuals' metacognitive acts as they progress through the process (Kaiser, 2017, p. 274). A common approach is to document the order of steps and sub- or meta-processes, such as identifying variables, creating representations, working mathematically, or validating as they pass through an MC (Ärlebäck & Bergsten, 2010; Borromeo Ferri, 2006; Czocher, 2016; Hankeln, 2020; Hankeln et al., 2019; Sol et al., 2011). The body of work primarily supports claims about idiosyncrasy of individuals' modelling routes. The second theme concerns characterizing specific cognitive obstacles resulting from students' current ways of reasoning that lead toward or away from normative solutions (Carreira et al., 2011; Kaiser, 2017; Stillman, 2000). Findings indicate that knowledge acquired through lived experience is often prioritized when interpreting cues

[1] Traditionally, the cognitive perspective on modelling maintains a distinction between the situation-as-perceived-in-the-world and the modeler's mental presentation of the situation-as-conceived-by-the-modeler. For simplicity of vocabulary, we use "real-world" to refer to either, but recognize that the modeler only models the latter.

about problem constraints and that the nature of the knowledge activated is task-specific as well as student-specific (Czocher, 2019; Manouchehri & Lewis, 2015; Stillman, 2000). Other work has pointed clearly to the limitations of a mathematical knowledge-based approach to explaining students' modelling processes. For example, Roorda et al. (2007) studied students' concepts of derivative in conjunction with application of these concepts during a modelling task and experienced difficulty applying Zandieh (1997)'s derivatives framework for explaining transitions between the task's real-world scenario and the requisite mathematical concepts of the framework. That is, conceptual analyses of students' mathematics insufficiently captured their mathematization of the scenario. A third theme within the cognitive approach focuses on the role of metacognitive processes in modelling (Stillman & Galbraith, 1998; Vorhölter, 2018). Galbraith and Stillman (2006) made some progress in explaining students' cognitive blockages in terms of metacognitive processes and Niss (2010) argued that reflection on the idealization of the task scenario, while anticipating the outcome, and relevant mathematical representations for the scenario, would beget successful mathematization. Introducing such forward-looking meta-processes encourages viewing students' modelling work relative to the goals they set for themselves within the task-scenario. Collectively, this body of work suggests that individuals' modelling routes, applications of knowledge, and self-monitoring of solution strategies are neither regular nor, across individuals, predictable.

The theoretical and methodological commitments adopted in studies from the cognitive perspective support claims about where in the modelling process impediments to progress are located and suggest how to remediate the impediment. For example, to remove a blockage in the process, theory would recommend a teacher direct a student to make an assumption, ignore a variable, correct a representation, recall prior mathematics knowledge, or reflect on inconsistencies. These suggestions are often devised with normatively correct models for the scenario in mind (Kaiser, 2017). Yet, deeper questions remain about student reasoning underlying the blockages, apparent inconsistencies within students' models, or how to extend modelers' thinking (c.f. Stillman, 2011, p. 174). A cognitive theory of model construction ought to account for the ways the various conceptual, representational, and real-world systems align *for the modeler*.

We emphasize that a mathematical model is a *conceptual product* coordinating relations among objects and their properties with mathematical systems *expressed* using mathematical conventions. Relevant to how these systems align and are represented for the modeler, Vergnaud (1998) observed that "the relationship between signifier and signified is not usually a one-to-one correspondence." We infer two further consequences: there is ambiguity in representation for an object and a signifying element (e.g., an inscription like x) signifies the modeler's meaning for it rather than a real-world object or a mathematical entity.

2.3 Quantitative reasoning – theory and lines of inquiry

The above synthesis intimates a need for attending to how modelers' mathematical conceptual systems and their expressions acquire situationally relevant meanings and how those meanings are coordinated. To address this theoretical need, we leverage theories of quantitative and covariational reasoning[2] (QRT) to conjecture sources for the meanings of variables and correspondences underlying representations. A quantity is a measurable attribute an individual

[2] The phrase "quantitative reasoning" can refer to cognitive activities, to theories involving constructs for analyzing those activities, or entire lines of inquiry seeking to understand or develop students' cognitive activities. We will use QRT to indicate the theories and lines of inquiry and QR to indicate the mathematical reasoning done by a human being.

imputes to an object or situation (Thompson, 1994a, 2011). Quantities consist of three interdependent conceptual components: (a) an object or entity (b) an attribute or quality of the object, and (c) a quantification. Quantification entails a conceived measurement process for a specified attribute associated with a specified object such that "the attribute's measure entails a proportional relationship…with its unit" (Thompson, 2011, p. 37). Quantities are mental constructions, existing in the mind of an individual, rather than in the world (Thompson, 1994b), and therefore vary from person to person. Quantitative reasoning involves conceiving of situations as consisting of quantities and establishing relationships among quantities. Covariational reasoning entails an individual's sustained image of two quantities' values or magnitudes varying simultaneously (Carlson et al., 2002; Saldanha & Thompson, 2002; Thompson & Carlson, 2017).

Piaget (1965) characterized different types of quantification (e.g., gross and extensive), which have been adapted by later scholars. Gross quantification entails comparative operations, such as those permitting a determination of which of two lengths is longer. Extensive quantification entails operations that introduce units, which supports mathematical activities like measuring and counting (Steffe, 1991; Thompson 1994). New quantities can be established by applying quantitative operations to extant quantities (Thompson, 2011). For example, the amount by which the angularity of one trajectory exceeds another's, relative to horizontal, is a quantitative difference (new quantity) constructed through making an additive comparison (quantitative operation).

As a line of inquiry, QRT studies explore how individuals conceive of specific types of relationships including linear (Ellis, 2007a), quadratic (Ellis, 2007b), exponential (Castillo-Garsow, 2013; Ellis et al., 2015), and trigonometric (Moore, 2014) functions to construct targeted covariational relationships within specific contexts. However, the applicability of these studies to studying model construction is limited. Because quantification processes are the focus, the learning environments are designed to target quantities the researcher desires the participants to engage with and abstract invariant (functional) relationships from, under the conditions laid out in the task statement (e.g., Ellis et al., 2015; Moore, 2014). Other studies include graphing activities with specified coordinate systems or focus on isolated single linear (or bi-linear) relationships among two quantities being related or combined to form a third (e.g., Thompson, 1994a). In contrast, a modelling task would leave *specifying* to the participant, including choosing which attributes to quantify and establish relationships for.

Despite the surface-level similarities and compatibility, we note there are few studies of mathematical modelling – those emphasizing how mathematics is employed to solve a real-world problem – that account for quantitative reasoning. In this vein, a recent study of undergraduate business calculus students' covariational reasoning as they addressed optimization problems involving marginal revenue and marginal cost found that students' conflation of change and rate-of-change persist, and can be explained by attending to students' introduction of quantities besides those provided in the problem statement (Mkhatshwa & Doerr, 2018). Thus, examining model construction as a process that generates and exploits such relations among quantities is a promising way to connect scholarship in the two areas and advance the field's understanding. Larson (2013) addressed exactly this issue in a study of students' development of a ranking algorithm for a data set. She broadly characterized students' models as systems comprised of quantities, relationships among quantities, and operations that describe how those quantities interact. Larson (2013) argued that the operations students chose "to invoke on quantities are reflective of the relationships they perceive among those quantities" (p. 117). Although she claimed that quantitative reasoning was central to the iterative refinement of real-world problems into mathematical problems, the study was limited to examining students' ways for combining numerical values given in the task. In other words, numerical operations were foregrounded rather than quantitative ones. Neither study detailed how the students' quantities were identified or provided indicators of students' engagement in quantitative reasoning absent numerical values.

The literature suggests that leveraging QRT can move studies of model construction beyond observational descriptions of impediments to modelling processes and towards a productive characterization of the cognitive sources (rather than content-knowledge sources) of idiosyncrasies in students' modelling routes. Meanwhile, our review of QRT literature suggests realizing its potential to yield insights into model construction would require attending more closely to quantification processes as the participant derives her model from first principles and to the interplay among quantities, specifying and mathematizing. The purpose of this study was to investigate the empirical feasibility and theoretical generativity of enhancing the cognitive approach to studying mathematical model construction with theories of quantitative reasoning. We address the following questions: *How does augmenting a study of model construction from the cognitive perspective with theories of quantitative reasoning bear out empirically? What considerations remain for improving this approach?*

3 Methodology

In our view, descriptions of model construction processes may be enhanced through attending to students' in-the-moment and potentially unstable quantitative meanings for their inscriptions. When a modeler produces a normatively correct (or self-validated) mathematical model, we would anticipate she produced suitable (to herself) consistency among attributes and explanations of the scenario, mathematical concepts, and representations, observable as a smooth flow through phases described by modelling cycles. Our theoretical framework attributes a perturbation or standstill to a modeler's recognition of inconsistency among meanings she attributed to aspects of the emerging model. Thus, seeking cases of such misalignment in a modelers' work should induce an explanatory account of their modelling in terms of their conceptions of the quantities imputed to the scenario.

3.1 Research Setting and Dataset Constitution

Data were generated as part of a larger study of the characteristics of tasks and facilitator interventions that could elicit specific mathematical modelling competencies. The project took place at a large university in the southern United States. Data were generated through a series of cognitive, clinical interviews that enabled analysis to focus on generating viable explanatory hypotheses regarding participants' reasoning for their observable activities (Clement, 2000). The project sample comprised 15 volunteers from a cross-section of mathematical levels (pre-algebra, algebra, calculus, and differential equations) who worked on tasks ranging from simple word problems to more complex modelling problems. One goal of the broader project was to explore the feasibility and consequences of attending to quantitative reasoning during model construction, and so we administered the Monkey Problem (below) to the three advanced students:

A wildlife veterinarian is trying to hit a monkey in a tree with a tranquilizing dart. The monkey and the veterinarian can change their positions. Create scenarios where the veterinarian aims the tranquilizing dart to shoot the monkey.

Iseult's, Safi's, and Merik's work on the Monkey Problem constitutes the dataset for the present study. We selected their work not due to their demographic characteristics but rather to showcase the interplay between their conceptions of the task and quantitative reasoning that informed their model construction. All three were mathematics majors intending to teach secondary mathematics, were recruited from a course on ordinary differential equations, and had completed or were enrolled in multivariable and vector calculus, probability, statistics, analysis, abstract and linear algebras. Their advanced mathematics backgrounds increased the likelihood they had previous experiences deriving and using mathematical representations as models of real-world phenomena.

The Monkey Problem presents a conceivably authentic real-world scenario couching a familiar problem of projectile motion appearing in courses on mechanics, pre-calculus, calculus, or differential equations. The task is open and ill-defined (see Yeo, 2007), enabling observation of how participants conceived and structured the scenario, goals they established, and activities they pursued. Depending on assumptions made during the specifying phase of modelling, concepts from right-triangle trigonometry, quadratics, or differential equations can yield satisfactory models. That is, the problem elicited *specifying* and *quantitative reasoning*. Our participants' work on the Monkey Problem was adequate to our purpose for the following reasons: all three made multiple attempts at resolving the mathematical problems they identified within the task scenario, made both implicit and explicit assumptions about the task scenario leading to distinct formulations of the task, articulated their reasoning aloud, and created many inscriptions documenting their work.

Participants were given unlimited time to address the task in a manner satisfying to them. During the interview, we provisionally accepted all student productions without actively correcting, leading, or removing ambiguity (Goldin, 2000). Our theoretical framework necessitated we assume each modeler's interpretation of the task scenario, meanings for inscriptions, or mathematical knowledge brought to bear was different from our own. Follow-up questions and interviewer interventions aimed to clarify the students' statements or inscriptions with the purpose of documenting nascent or in-the-moment conjectures about the participants' thinking. Safi and Iseult received a contingent prompt requesting they develop a way to guide the veterinarian to accurately aim and hit the monkey. The audio/video recorded interview sessions lasted 26 minutes for Iseult, 34 for Safi, and 46 for Merik.

3.2 Data Analysis

The overarching research design was to constitute theoretical cases that "embody causal processes operating in a microcosm." (Walton, 1992, p. 122)." The theoretical cases we sought were of the mutual influence of *specifying* and *quantitative reasoning* during model construction. This approach to *case-study* stands in contrast to constituting a case as a complete account of an individual's reasoning; we do not claim that ways of reasoning were uniform across nor within individuals. Our results are a set of carefully curated vignettes showcasing the main hypothesized causal elements for misalignments we observed in participants' work. To arrive at the vignettes, we coordinated multiple sequential analyses of the a/v recordings using MaxQDA, described below.

We first addressed *specifying*. According to the theoretical framework, individuals' model construction activity is driven by their interpretations of the task scenario. Therefore, for each participant, we asked *How is the modeler conceiving the task?* We documented the imagery we inferred was immediately available to them, goals they set, and the mathematical concepts, procedures, and real-world explanations they appealed to.

Next, we addressed *quantifying*. The theoretical framework prescribes that a quantity is an individual's conception of a measurable attribute of an object along with a conceived measurement process. We applied the quantification criteria framework (see Table 1) developed by Czocher and Hardison (2021) to catalogue the attributes each participant quantified. The eight criteria serve as indicators the participant engaged in mental operations necessary for, or indicative of, a conceived measurement process for each situational attribute, through considering variation, measurement, and relationships among already-quantified situational attributes. A *situational attribute* is one for which we were able to infer a situational referent within the task scenario (e.g., "the tree's height"). Instances in which the modeler mentioned generic attributes (e.g., "velocity is distance over time") for which we were unable to infer situational referents were not considered situational attributes. The criteria are generous in inferring participants' treatment of units. They include a quantified attribute when the participant mentions a standard dimensional unit in the sense of Schwartz (1988) (e.g., meters) or indicates mental operations producing units, in the sense of Steffe (1991) (e.g., iteration, partitioning, etc.[3]). Three coders independently and systematically coded the interview records according to these criteria. When a participant's work met any of the eight criteria, we documented a quantified attribute for that participant. For each situational attribute, we noted the inclusion criteria that were initially satisfied by participants' words, writing, and actions. The result was a cumulative list of quantified attributes each modeler imputed to their image of the scenario (see Table 3 and Table 4).

Table 1 Criteria indicating a modeler has quantified an attribute (Czocher & Hardison, 2021). Enumeration indicates neither a chronological progression nor order of priority.

QC	Description	Justification for Criteria Inclusion from QRT
	Variation Criteria	Quantities are conceived to have values that can potentially vary
1	Discussing variation of a situational attribute	Quantities' values may vary independently as their objects or attributes undergo change. Variational reasoning refers to how an individual conceives of changes for a single quantity's value or magnitude. For example, a quantity's value may vary discretely or continuously (Castillo-Garsow, 2012).
	Measurement Criteria	Quantities are conceived as measurable attributes of objects
2	Substituting, assuming, or deducing a numerical value for a symbol with a situational referent	Numerical values can be assigned to extents of an attribute (Schwartz, 1996).

[3] Although the usage of a standard unit in the sense of Schwartz (1988) does not necessarily imply mental operations that produce units as described by Steffe (1991), our goal was to account for all potential quantified attributes participants imputed to the scenario. Because it is possible for individuals to assimilate standard units in terms of such mental operations, we included both kinds of units in our criteria.

QC	Description	Justification for Criteria Inclusion from QRT
3	Expressing a desire to measure a situational attribute	Quantification can be motivated by a desire to measure (Schwartz, 1988).
4	Interpreting a value in context	Numerical values for magnitudes measure attributes of objects
5	Specifying a situational reference object (e.g., line or point from which to measure; situational 0)	Extensive quantification supports measuring and counting relative to some initial position (Steffe, 1991)
6	Specifying a (potentially non-standard) unit of measure for a situational attribute	Unit-producing operations such as partitioning and iterating (Steffe, 1991) are indicative of quantification; dimensional units (Schwartz, 1988) can be associated with such operations.
	Relationship Criteria	Quantitative reasoning entails conceiving of quantities and relationships among quantities.
7	Explicitly expressing a quantitative relationship, a dependence or causal relationship among already-introduced quantities, describing one quantity in terms of other quantities	Quantitative operations include comparing or combining two quantities multiplicatively or additively(Thompson, 1994a)
8	Nominalizing an attribute via verbally labeling, symbolically labeling, or describing its relation to other attributes of objects	Attributes are associated with objects

Finally, we addressed *alignment* of meaning with representations. We assumed that "what we ultimately observe are the external components (representations), but these cannot be disengaged from the conceptual systems" (Lesh & Doerr, 2003, p. 213), We operationalized *representation* not as a static thing, but rather as a dynamic process shaped by individuals' quantitative reasoning (Vergnaud, 1998). We catalogued the students' written inscriptions by attending to spatial and temporal organization of the students' writing (see Czocher & Hardison, 2019 for details on this process). Through coordinating participants' utterances and gestures with their inscriptions, we inferred the quantitative meanings ascribed to the inscriptions. Our approach enabled examination of the interplay among the modelers' conceptions of the task scenario, the attributes they quantified, and the consequences of both on model construction.

3.3 Sample Data Analysis

Table 2 contains an excerpt of the first 3 minutes of Merik's interview, where he produced an illustration reflecting his conception of the scenario and imposed a right triangle upon it (Figure 1). Figure 2 shows a snapshot of MaxQDA coding for the excerpt. Because coding relied primarily on video, including speech, writing, gesture, documentation of which inscriptions Merik's attention was on, and our own experiences conducting the interviews, we style the excerpt as rows of transcript enriched with descriptive field notes and justifications for applying quantification criteria from Table 1.

Time	Merik's Speech	Researcher Description	Criteria & Justification
0:45	Let's pretend we have a tree. This one, I really wanna draw this one. And you got some sort of monkey coming out, just hanging out in that tree and usually, something like this	Merik sketched a monkey and a vet in a hat. A straight line passed through his stick hands to a small dot, indicating where the tip of the rifle would be.	As he assimilated the task scenarios he created, Merik set a mathematical problem-solving goal to identify the angle at which the vet should shoot the dart to hit the monkey.
1:10	And you got a veterinarian. He has his gun, so I can set this like this. Triangle. We'll take it from shooting, it's gonna aim right at the heart of the monkey. And you get the tranquilizer stuff, flows around, so that you can make the triangle here.	The dot signified the initial position of the dart. A small dot "right at the heart of the monkey" indicated the dart's final position. He drew the straight line connecting the two dots, dropped a vertical line to the same horizontal level as the gun-dot and connected the gun-dot to the foot of that vertical line, forming a horizontal segment "so you can make the triangle here."	Neither dot was coded as indicating a quantity, since no quantification criteria were met. The triangle's hypotenuse was not coded as a quantity (until 4:40) since he did not indicate an object, attribute, nor unit associated with it in this exchange.
2:19	He just wants to make a straight shot, uh, that won't work out cause gravity is a real thing. Hmm. It's weird …	He assigned symbols to represent the vet's distance from the tree measured from the vet to the base of the tree (x), the monkey's height in the tree, as measured from the base of the tree (y), and the angle formed by the gun to the ground as measured from the path emanating from the gun-dot relative to the horizontal (θ).	The immediacy of the sketching, imposing, and labeling after reading the prompt suggest that the right-angle configuration, the three quantities, and their labels were available to him upon assimilation of the task scenario. All three quantified attributes met QC#8.
2:33	Okay. I don't really like this model cause it's not gonna fly in that path, it's gonna make more of a arc with destination. Cause you're gonna have some initial velocity which is	Below the tree diagram, he sketched a curved arc connecting two dots, one corresponding to the gun-dot and one corresponding to the monkey-dot.	Merik acknowledged that the dart would travel along a curved path, rather than a straight one, as a consequence of gravity being "a real thing." We did not credit Merik with conceiving

Time	Merik's Speech	Researcher Description	Criteria & Justification
	just distance over time. Yeah.		of gravity as a quantity in this instance since he gave evidence only of its effect on the dart's path (object, attribute). Similarly, we did not credit Merik as quantifying either initial velocity or time because there was no clear, dedicated situational referent. He used the words "initial velocity" but then described with the mantra "distance over time." We marked both "initial velocity" and "time" in black codeline (Figure 2) but did not associate a quantification criterion with them.
3:03	Taking away gravity which is 9.8 meters per, squared, this is just in terms of nearest per second and then, how would that flight path really work? Cause it doesn't really matter how tall the, you can just take where it's leaving from is the zero. Cause, like an origin, just work from there, trying to hit the monkey or whatever, whatever height he is at and the distance.	Below the curved arc, he wrote $v - 9.8$. subsequently writing in the units "m/s" and "m/s^2" next to the inscriptions v and 9.8.	Referring to the configuration of objects represented in the tree diagram, Merik evidenced he was able to conceptualize the monkey's position and the vet's position as free to move but constrained to the vertical (tree) and horizontal (along the ground) legs of the triangle, respectively. This indicated variation of the positions relative to their respective zeros (base of the tree, starting position of dart). QC#2 for both distances was indicated. Merik referenced units for gravity and assigned a numerical value, meeting QC#2 and QC#6.

Table 2 Excerpt of the first 3 minutes of Merik's interview. Column 2 indicates the time that speech started, which does not always coincide with timing of writing or gestures.

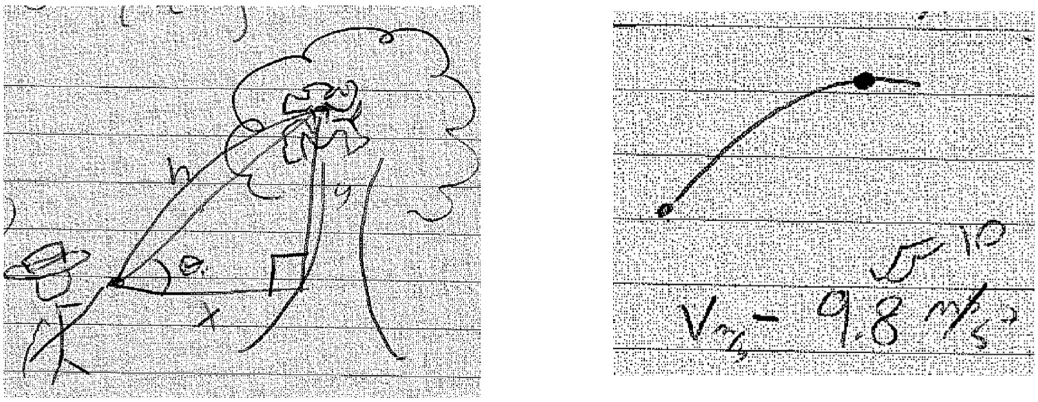

Figure 1 Merik's written work. Cumulative inscriptions on his tree diagram (left) and his trajectory diagram (right)

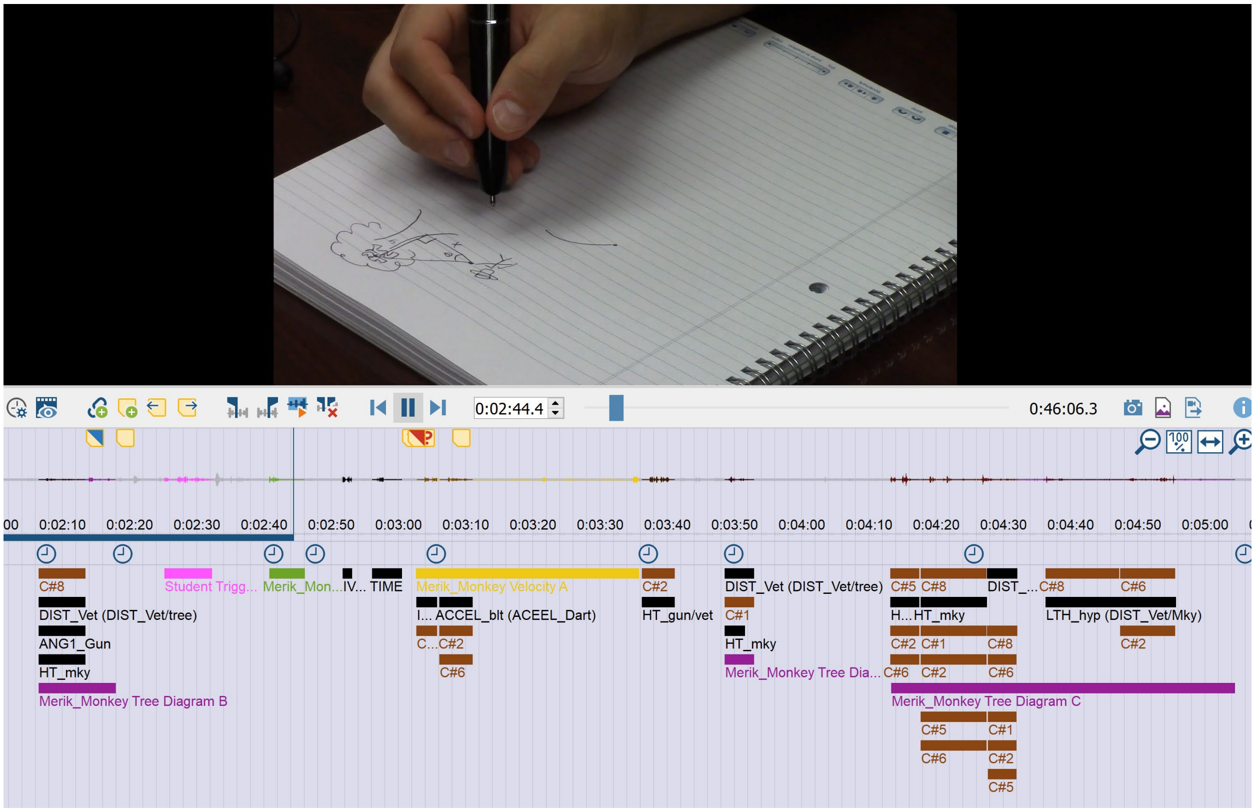

Figure 2 Screenshot from MaxQDA analysis of Merik's work. Black codeline corresponds to potential quantity imputed, brown codeline corresponds to quantification criteria met, other color codelines correspond to distinct inscriptions Merik's attention is on.

4 Results

We first provide a brief overview of the modelers' initial conceptions and progress. We elaborate on the quantified attributes we inferred they imputed to their scenarios and contrast quantifications of particular situational attributes. We then discuss misalignments rooted in the modelers' quantitative reasoning that influenced their model construction.

4.1 Modelers' initial conceptions

All three participants spontaneously considered two scenarios: one without gravitational force and one with. In the first scenario, each modeler sketched a straight path from the vet to the tree (see Figure 3). They attended to base angle determined by the horizontal distance from the vet to the tree, the vertical height of the monkey in the tree, and distance between the monkey and the vet. The modelers indicated relationships among the lengths and angle given by the Pythagorean Theorem and (inverse) trigonometric formulae. All three asserted that the distance between the monkey and the vet could be determined given the other lengths. Merik and Safi explained that the inverse trigonometric formulae would yield the angle the gun should be fired at, presuming a straight path, even when the veterinarian and the monkey were positioned arbitrarily. In contrast, Iseult concluded that the veterinarian should aim 45° above horizontal if he and the monkey were each 10 feet from the base of the tree. She maintained these fixed values throughout the interview.

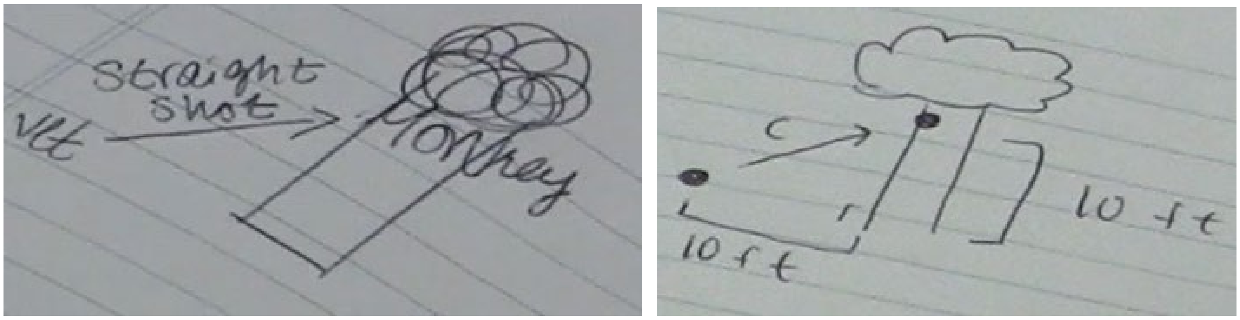

Figure 3 Safi's (left) and Iseult's (right) tree diagrams, considering straight trajectories

For their second scenario, all three modelers inscribed curved paths beginning at the vet's position and passing through the monkey at their apexes, each implicitly assuming that the monkey and the maximum height of the dart coincided (see Figure 4). Each modeler observed that the path traveled by the dart would be influenced by its initial velocity. Although each made progress, toward providing a mathematical model of their chosen scenarios, none succeeded (from their perspectives nor ours).

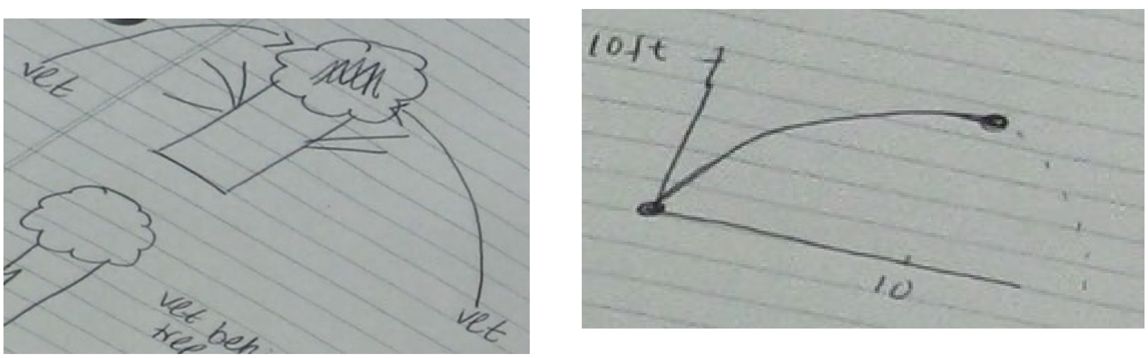

Figure 4 Safi's (left) and Iseult's (right) parabolic trajectories

4.2 Catalogue of quantified attributes

After analyzing each interview according to the indicators in Table 1 we inferred that Merik, Safi, and Iseult quantified 14, 9, and 10 attributes, respectively, during their *specifying* activity (see Table 3 and Table 4). We recorded the quantity using the same notation (e.g., ANG_{STR}) if we inferred that the participants were indicating the same attribute of the same object from our perspective, even if their quantifications or nominalization differed. For example, ANG_{STR} denotes the object *gun* and attribute *angle of ascension relative to horizontal*, regardless of the measurement process or units used by a specific participant. Of the 19 unique quantified attributes, only 6 were imputed by all three participants: ANG_{STR}, ANG_{PAR}, $DIST_{VET/TREE}$, $DIST_{VET/MKY}$, $HT_{MKY/GUN}$ $IVEL_{DART}$. When considering a scenario without gravitational force, each modeler inscribed a right triangle and considered four quantities: ANG_{STR}, $DIST_{VET/TREE}$, $DIST_{VET/MKY}$, and $HT_{MKY/GUN}$. Each modeler observed that, subject to gravity, the curved path traveled by the dart would be influenced in some way by the dart's initial velocity ($IVEL_{DART}$). We further analyze distinctions in quantifications in the next section.

Table 3 Collective list of quantified attributes imputed by the modelers.

Quantified Attribute	Type	Description
ANG_{STR}	Angle	Measure of angle gun is aimed relative to the horizontal, for straight path
ANG_{PAR}	Angle	Measure of angle gun is aimed relative to the horizontal, for parabolic path
$ANG_{VET,3D}$	Angle	Measure of the plane angle formed by a designated axis and the line through the tree & veterinarian in 3-space.
$FOR_{GUN/DART}$	Force	Force the gun applies to the dart
$DIST_{VET/TREE}$	Length	Horizontal distance from vet to the tree/under the monkey.
$HT_{TREE/GRD}$	Length	Height of the tree
$DIST_{VET/MKY}$	Length	Length of the straight path from the vet's gun to the monkey.
$HT_{GUN/GRD}$	Length	Height of gun (or vet) relative to ground.
$HT_{MKY/GUN}$	Length	Height of the monkey relative to the vet's gun.
$DIST_{DART}$	Length	Distance traveled by the dart
HT_{DART}	Length	Height of the dart
$TALL_{PATH}$	Length	Tallness (vertex height) of the parabolic path
VEL_{DART}	Rate	Velocity of the dart
SPD_{DART}	Rate	Speed of the dart
$IVEL_{DART}$	Rate	Initial linear velocity of the dart.
$VVEL_{DART-I}$	Rate	Initial vertical velocity of the dart
$HVEL_{DART-I}$	Rate	Initial horizontal velocity of the dart
ACC_{DART}	Rate	(Vertical) acceleration of dart due to gravity
TIME	Time	Elapsed (figurative) time

Table 4 Time of first evidence of the quantified attribute, along with criteria observed, for each modeler

Quantified Attribute	Safi		Iseult		Merik	
	Criteria	Time	Criteria	Time	Criteria	Time
ANG_{STR}			2,8	5:28	8	2:08
ANG_{PAR}	7,8	10:30	1	7:12	1	6:04
$ANG_{VET,3D}$					5,7,8	24:38
$FOR_{GUN/DART}$			3,8	11:05		
$DIST_{VET/TREE}$	5,7,8	10:50	2,5,6,8	3:20	8, 6	2:09
$HT_{TREE/GRD}$	3,8	24:08			2,5,6	4:13
$DIST_{VET/MKY}$	7	14:48	8	3:35	8	4:37
$HT_{GUN/GRD}$					2	3:36
$HT_{MKY/GUN}$			2,5,6,8	3:30	8	2:10
$DIST_{DART}$	7,8	30:50				
HT_{DART}					1	15:35

Quantified Attribute	Safi		Iseult		Merik	
	Criteria	Time	Criteria	Time	Criteria	Time
$\text{TALL}_{\text{PATH}}$	1	5:04				
VEL_{DART}			1,3,7,8	18:12		
SPD_{DART}	7,8	10:42	8	4:59		
$\text{IVEL}_{\text{DART}}$	8,1	33:07	7	20:35	8	3:02
$\text{VVEL}_{\text{DART-I}}$					8	40:23
$\text{HVEL}_{\text{DART-I}}$					8	25:42
ACC_{DART}			2,6,8	12:50	2,6	3:06
TIME	8	27:30			8	16:08

4.3 Participants' quantifications of attributes differed

Table 3 and Table 4 show that the modelers imputed distinct quantities into their scenarios and prioritized them differently, often exhibiting differing sets of criteria. Distinct attributes were sometimes associated with the same object (e.g., height of the dart, speed of the dart), and we also found evidence the modelers conceived of quantities in subtly different ways (e.g., time). Some situational features were attended-to across modelers (e.g., gravity), but not all modelers treated the feature in a way that met at least one quantification indicator. We discuss examples below.

Safi treated gravity as an actor generating effects within the scenario; her work did not reveal evidence meeting any quantification criteria for gravity. She claimed that the dart would eventually "start to curve" even if shot straight due to "the law of gravity" but indicated that gravity's effect would be mitigated by the dart's speed. She argued that a faster (slower) dart would have less (more) time for gravity to affect its path, a mental action indicative of directional covariational reasoning (Carlson et al., 2002). In contrast, Merik and Iseult indicated gravity as a quantified attribute. For example, Merik noted that the dart would "make more like an arc to its destination" due to gravity and wrote $v - 9.8\text{m}/\text{s}^2$. Because Merik chose a magnitude and a dimensional unit for ACC_{DART}, and he incorporated it arithmetically, we credited him with quantifying ACC_{DART}. Though the expression's units were not consistent, we view his representation as an instance of the symbolic form $\blacksquare - \blacksquare$ (Sherin, 2001), which Merik employed to represent the impact of gravity on the dart's velocity.

In the modelers' treatment of TIME, we were able to infer non-equivalent quantifications. Iseult did not indicate imputing TIME as a quantity during the interview; she conceived the scenario as static. Safi and Merik both evidenced conceiving of time elapsed since the dart was fired as a quantity. Safi considered the speed of the dart and the total amount of time the dart would be airborne, evidencing at least gross quantification of TIME because she evidenced directional covariation of time and speed. Merik explained his meaning for a Cartesian graph, "as time moves this [path] is just tracking the [dart's] height." We infer Merik had an image of time passing continuously, though we were unable to infer whether it entailed the rhythmic segmentation characteristic of operative conceptual time (Thompson & Carlson, 2017). Therefore, we consider Merik to have indicated imputing at least elapsed figurative time to the scenario. Additionally, Merik substituted a particular value (1 second) for time while conducting dimensional analysis of a quadratic equation. When asked whether knowing the time at which the bullet hit the monkey might help him solve the rest of the problem, Merik replied, "No, I don't think that time is what I need to be concerning myself with cause…the distances are the variables." Although Merik considered when the dart was in the gun and when the dart reached the monkey, and that time elapsed as the dart traveled from one location to the other, we were unable to infer that Merik considered specific instantiations of TIME between these two instantiations[4]. In each case, the modeler's conception of TIME may have constrained the models generated. We further analyze the role of TIME in Merik's model construction below.

4.4 Quantitative reasoning may be necessary, but does not guarantee model construction

In this section, we examine the interplay between the quantified attributes and the modelers' progress constructing a mathematical model satisfactory to them. Safi's vignette demonstrates that despite attending to quantities, and specifying their interdependencies, she did not produce a representation inclusive of arithmetic operations. Merik's vignette demonstrates how attending to quantities occasioned reflection on the meanings of the arithmetic representations he produced.

4.4.1 Directional covariation without algebraic representations

Safi conceived the path of the dart as an object and associated its tallness as an attribute that could vary. First, she hypothesized conditions to achieve contact between the dart and the monkey. She stated: "the highest point is where the dart is at the monkey," but she indicated $\text{TALL}_{\text{PATH}}$ and $\text{HT}_{\text{MKY/GUN}}$ were distinct quantities. She observed that the vet would be a "certain distance away from the tree, and based on that, he must angle the tranquilizer in such a way such that at the highest point of the dart['s path, it] would curve." Second, she explained that were there no monkey, the dart would continue through to complete a parabola. Safi set the goal of determining the angle to aim such that the apex of the curve coincided with the monkey. The interviewer prompted her to explicitly consider how the parabola's shape might depend on the angle at which the vet aimed. Safi introduced a right triangle with base angle ANG_{STR} and indicated she would need to know magnitudes of $\text{HT}_{\text{TREE/GRD}}$ and $\text{DIST}_{\text{VET/TREE}}$. She acknowledged that the dart's initial velocity would impact the shape of the parabola because it would influence the time needed to reach apex, and in turn would be influenced by gravity. For Safi, the quantities she conceived could vary and were interdependent. However, gross quantification and recognition of directional covariation among subsets of quantities were not sufficient to support her in producing or populating a template to serve as a mathematical representation of the dart's trajectory.

4.4.2 Model validation can be occasioned by conflicting quantitative meanings

Approximately nine minutes into the session, Merik established the goal to seek a quadratic equation because "that is the path [the dart] is going to follow." He wrote $f(x) = Ax^2 + Bx + C$, which we interpret as a template (Sherin, 2001). Merik hinted at situational referents such as when he treated Ax^2 as a placeholder for the effects of gravity and Bx as a placeholder for the effects of velocity of the dart. However, he referred to C directly as the y-intercept of the graph of the expression and the image of 0 as being $0 + C$ without clearly imbuing a quantitative referent from the scenario. Merik's equation included the symbol x, which lacked a dedicated situational referent. For Merik, the symbol x at times explicitly represented the horizonal position of the dart (implicitly at a given moment in conceptual time) while at other times he used it to represent elapsed time (at one or the other of the two locations for the dart). The ambiguity of referents for x was not initially observably problematic for Merik.

[4] Substituting 1s involved enactment of numerical operations that were not situationally quantitative. Specifically, Merik did not link the 1s to the bullet's position or time traveled. Thus, considering a unit of TIME (1s) did not constitute sufficient evidence an extensive conception of TIME

In fact, the shifting interpretations were not realized by Merik until the interviewer prompted him to share what $A, B, C, x,$ and $f(x)$ represented. Merik responded that A was the rate of change in velocity and that x would be time. Consequently, he realized that substituting 30 for x and 40 for y (specific values he selected for triangle leg lengths) was not compatible with this interpretation of the quadratic expression.

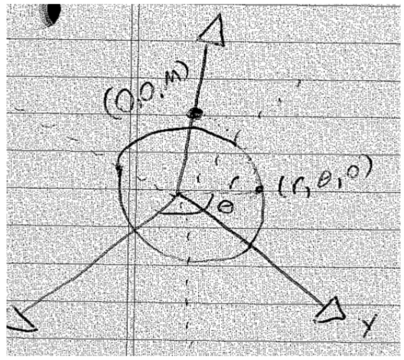

Figure 5 Merik's coordinate axes, support to and evidence of his reasoning about the monkey and vetrinarian in 3-space.

4.4.3 Quantitative reasoning coincides with aspects of modelling sub-processes

Around 23 minutes, Merik set the goal of finding ANGPAR and introduced a 3-axis coordinate system to record his work. The system aligned the tree with the vertical axis, located the monkey at $(0,0,m)$, and the veterinarian "somewhere" in the x, y-plane. He assumed the location would be "sort of radius distance away and angle from" arriving at $(r, \theta, 0)$, with a parabolic path between the two points. He conveyed the new quantity m to his quadratic equation, writing $m + 100x - 10x^2$ and setting parameters values -10m/s^2 for gravity and 100m/s for initial velocity. He crossed out the expression because it combined m, a vertically oriented distance, with x, a horizontally oriented distance. Merik worked for several minutes to convert his rectilinear coordinates to polar ones before asserting that "the monkey is somewhere up and down the z-axis and the veterinarian is along the x, y-plane and so no matter where they move…it doesn't really matter." He explained his conclusion in terms of quantities he previously treated as varying but which he could instead assume to be constant: the tree would stay the same height, the monkey's height would be measured "straight down" regardless of which side of the tree he hung from, and the veterinarian's distance would always "be the r in this particular situation" because "wherever he goes around will just be another θ." Throughout this vignette, Merik made assumptions, introduced variables, specified conditions and assumptions, and converted those to properties and parameters. Thus, attending to Merik's spontaneous quantitative reasoning still affords insights into his model construction activity.

5 Discussion

According to our theoretical approach, quantities are measurable attributes of objects conceived by individuals. The relevant objects and attributes are a consequence of how modelers assimilate a scenario and the goals they formulate. Thus, theory predicts that modelers would identify differing (across modelers) and multiple (within modelers) quantities to associate with the same objects or with the same attributes. The prediction held empirically in the context of addressing an open modelling task. The participants imputed non-equivalent sets of quantified attributes and those attributes held in common were not necessarily quantified in the same way (i.e., through the same implied measurement processes).

We observed similar figures (e.g., right-triangles, parabolas) and mathematical representations (e.g., trigonometry, quadratic formula) across modelers. The initial task prompt, being the same for all modelers, still occasioned differing mathematical activities across the modelers. In particular, the modelers' quantitative meanings for representations and their interpretations of relations among relevant quantities varied substantively. The participants reasoned differently about and with their models because the meanings of the models differed in terms of the situationally-relevant quantities imputed. There are two implications of this finding.

First, we argue that dichotomously evaluating students' work regarding the presence or absence of variables during an open modelling task has limited diagnostic value for a facilitator. Our quantification criteria operationalize a researcher's attribution of a *quantity* to an individual, which we generously applied to give credit to potentially quantified attributes. Our analysis revealed that some attributes clearly met (or failed to meet) our criteria for quantity, that some attributes entailed at least gross variation (or covariation with other attributes) absent indications of extensive quantifications, and that other attributes may have been quantified in students' previous experiences but lacked observable indicators of situational referents during the interviews. If analyzing the data from a modelling cycles perspective, a codebook would indicate giving credit to Safi for mentioning gravity (a parameter). However, when considering quantitative meanings Safi indicated in her work, she had not quantified ACC_{DART} and we did not observe a place for it in her representations; instead, for Safi, gravity determined only the shape of the dart's trajectory. Similarly, the $+C$ in Merik's quadratic expression would have received credit as a parameter from a modelling perspective, but it did not carry situationally relevant quantitative meaning since it indicated only the y-intercept on a coordinate plane. Future research should acknowledge that modelers can introduce symbols or qualities during model construction that may not carry situationally relevant quantitative meanings.

Second, these distinctions among modelers' quantifications for situationally relevant attributes imply they may respond differently to facilitator prompts and interventions. Thus, it is important to carefully formulate scaffolds for students' reasoning during model construction. For example, when we asked Iseult to consider TIME, she dismissed the suggestion because it was not relevant to her conception of the scenario as a completed trajectory. In comparison, TIME was, in a way, relevant for Merik in that he indicated quantifying it, though it varied only implicitly for him (a complication identified by Mkhatshwa and Doerr (2018)). In both cases, drawing the modelers' attention to TIME was insufficient for scaffolding their progress towards suitable (to them) models; instead, conceiving of time operationally might have supported both modelers in mathematically characterizing the motion of the dart. For example, a more productive approach for Merik may have been to encourage him to parameterize displacement of the dart at arbitrary, intermediate moments of elapsed time between the instant when the dart left the gun and the instant the dart struck the monkey. Quantifying $\text{DIST}_{\text{DART}}$ through parametrization by and covariation with TIME may have resolved tension he experienced as he attempted to relate his two (spatial and temporal) interpretations of the quadratic expression. Further strengthening our inference is its compatibility with recent studies of student sense-making arguing that the efficacy of instructional actions is not uniform across students (Cengiz et al., 2011). That is, naming an omitted quantity may be productive for some students, but attending to how a modeler has quantified an attribute may be necessary to others.

When facilitating modelling or word problems, it is common to observe a student "lose track" of the situational referent signified by a symbol in an equation. According to Radford et al. (2011), meaning is ever-evolving as an individual engages in goal-oriented activity and the quantitative reasoning perspective offers deeper insight into this phenomenon in the context of modelling. It is possible for modelers to (implicitly) hold an instance of symbol Z to represent quantity W and another instance of Z to represent quantity U. Sometimes this simultaneity of meaning is productive, such as when y can represent both a length (distance above ground) and a magnitude (number of units above a horizontal axis). However, a single instantiation of the same symbol may signify incompatible quantitative referents at the same time for a modeler. In Merik's work, we infer that his quadratic template held a mix of situationally-relevant quantitative referents associated with objects and their attributes in the scenario and situation-general quantitative referents (Moore et al., 2019) that were associated with his conception of quadratics, equations, graphs, and coordinates upon Cartesian planes.

When considering two complementary (or competing) theories of modelers' reasoning – in this case, descriptive modelling cycles and quantitative reasoning -- we must reflect on both the extent to which they overlap and the extent to which they diverge in their accounts. On the one hand, there is some overlap in our definition of quantities and the model construction phase *specifying* (*simplifying*) because both treat the core aspect of identifying variables to be used in subsequent modelling activity. For example, we reported that Merik introduced and removed quantities when working in his 3-D representation. These same instances would be identified by a codebook derived from a process-based view of modelling. Thus, imputing a situationally-relevant quantity (QRT) can be viewed as identifying a (ir)relevant variable (modelling perspective). On the other hand, the converse does not seem to be true. Our analysis revealed cases where nominalizing an important factor, object, or attribute (e.g., time or gravity) did not provide sufficient evidence that an individual had conceived of it as a quantity with attendant quantitative operations. That is, naming something often considered a variable according to MC codebooks may not be sufficient evidence to claim that a student has meaningfully engaged in *specifying* (*simplifying*) activities.

With regards to *mathematizing*, the data support Thompson's (2011) position that quantitative reasoning should be the basis of mathematical modelling. However, we found that some kinds of quantitative reasoning may not be sufficient for successful mathematization – and so something more than quantitative or covariational reasoning is needed for successful mathematization. Analysis with the quantification criteria permitted a close examination of the spontaneous quantitative and covariational reasoning occasioned by the Monkey Task. We found evidence of modelers leveraging gross quantification or directional covariation, consistent with prior research (Carlson et al., 2002; Piaget, 1965). This finding foregrounds the salience of gross quantity and covariation from the student perspective. For example, Safi coordinated quantities and attended to variant and invariant relationships among quantities, seeking to represent those relationships. Yet, she primarily evidenced gross quantification of relevant quantities and indicated conceiving directional covariation. We maintain that conceiving of quantities in terms of measurable attributes with units is critical, ultimately, since gross quantification and directional covariation were insufficient for determining the angle the veterinarian should aim. When addressing open modelling tasks absent numerical values, these intermediate kinds of quantification and covariation are likely important for associating how modelers envision the task scenario with mathematical representations. However, it is premature to claim that students will conceive a direct correspondence between a pair of covarying quantities and the arithmetic operations that are the building blocks of equations. Thus, it is yet an open question how to scaffold students towards formally representing the covariational relationships they conceive among quantified attributes. We conjecture such scaffolding to move modelers from conceiving coordination of quantities to articulating arithmetic operations among them would involve aiding the student in conceiving a quantitative relation between quantities.

6 Limitations

One limitation of our retrospective analysis is that interviewer probes did not systematically explore (a) whether for particular attributes, the participants were limited to gross quantification or covariation nor (b) whether superseding ways of reasoning about quantity and covariation might have mitigated cognitive obstacles students encountered during model construction. Thompson remarked, “persons limited to gross quantification are blocked from conceiving” scenarios in ways amenable to mathematization (Thompson, 1994a, p. 185). Nevertheless, that gross quantification and covariation were salient in participants’ work necessitates that researchers anticipate such conceptions in modelling tasks and consider ways of interacting with students that acknowledge the affordances of these ways of reasoning (Stroup, 2002).

A limitation to the overall approach is the extent to which a facilitator is able to apply QRT in-the-moment during model construction. In particular, the retrospective methods we used here are intensive and may complicate data collection (or classroom instruction). First, the quantities students spontaneously impute to a scenario portrayed through a given prompt are *a priori* unknown. Neither is it possible to predict what operations particular quantities will permit for modelers, nor therefore, what quantitative relationships might be conceived among them. Future studies might investigate task-specific clusters of quantities students tend to impute; however, open modelling contexts are legion. Thus, we advocate work to develop epistemic students, meaning well-articulated, commonly occurring, quantifications students may conceive, which may be attribute-specific (e.g., distinct ways of conceiving of time or angularity). Thus, we recommend shifting focus from task- or context-specific attributes and towards the the operations permitted by quantities in order to support mathematization. Some work, of course, has already been completed in QRT studies which could be tested in open modelling contexts. However, the manner facilitators might gain insight into students’ in-the-moment reasoning raises a second methodological issue: how can a facilitator grok students’ quantitative and covariational reasoning without interrupting their modelling process? We view a trade-off between the kind of probing that supports strong inferences about the ways students think about particular quantities and the constraints that probing may place on students’ autonomous model construction. Future work should attend to which distinctions in quantitative reasoning, established at which grain-sizes of analyses, are most crucial for facilitators wishing to understand and support students’ modelling activities in-the-moment.

7 Conclusions

Our study demonstrates that applying theories of quantitative and covariational reasoning through analysis of attendant constructs does enrich cognitive accounts of mathematical modelling, with potential to move the latter genre forward in understanding how individuals construct mathematical models. We found evidence supporting the claim that the complexities of quantification and quantitative reasoning influence individuals’ model construction through the meanings they attribute to and express in mathematical representations. We argue that attending to quantitative reasoning during model construction, specifically, allows glimpses into “the gap between the way learners intuitively think about a phenomenon and the formalisms used to represent it in expert practice" (Quintana et al., 2004, p. 345). Further, attending to the quantitative meanings students ascribe to representations within the context of open modelling problems would add to the growing body of scholarship making parallel claims about the role of quantities in development of mathematical reasoning (Bishop et al., 2014; Leslie, 2013; Moore et al., 2019; Moore & Thompson, 2015). Finally, we have identified two directions for future work. First, future research can undertake the question of whether mathematization is productively viewed as representing quantitative relationships via symbolic arithmetic operations. Second, we raised questions about how quantitative reasoning theories can be parlayed into scaffolding for mathematization, since doing so may entail modelers' gross quantitative reasoning and directional covariation among some variables while other variables may need to be constituted by extensive quantitative operations (e.g., segmenting, iterating, recursion).

Funding
This work was supported by the National Science Foundation (Grant number 1750813).

Compliance with Ethical Standards
The authors have no relevant financial or non-financial interests to disclose.

Informed consent was obtained from all human subject participants.

The study was approved by the Texas State University IRB.

Data availability
The datasets generated and/or analyzed during the current study are not publicly available due to the fact that participants have not consented to the public release of data.

8 References

Ärlebäck, J., & Bergsten, C. (2010). On the Use of Realistic Fermi Problems in Introducing Mathematical Modelling in Upper Secondary Mathematics. In (pp. 597-609). https://doi.org/10.1007/978-1-4419-0561-1_52

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and Affordances for Integer Reasoning: An Analysis of Children's Thinking and the History of Mathematics. *Journal for Research in Mathematics Education, 45*(1), 19-61. https://doi.org/10.5951/jresematheduc.45.1.0019

Blomhøj, M., & Jensen, T. H. (2003). Developing Mathematical Modelling Competence: Conceptual Clarification and Educational Planning. *Teaching Mathematics and its Applications, 22*(3), 123-139.

Blum, W., & Leiß, D. (2007). How do Students and Teachers Deal with Modelling Problems? In (pp. 222-231). https://doi.org/10.1533/9780857099419.5.221

Borromeo Ferri, R. (2006). Theoretical and Empirical Differentiations of Phases in the Modelling Process. *ZDM, 38*(2), 86-95.

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V., Stillman, G., English, L. D., Geoff Wake, Kaiser, G., & Kwon, O. (2014). Mathematical modelling in school education mathematical, cognitive, curricular, instructional, and teacher education perspectives.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modelling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the Streets and in Schools. *British Journal of Developmental Psychology, 3*, 21-29.

Carreira, S., Amado, N., & Lecoq, F. (2011). Mathematical Modelling of Daily Life in Adult Education: Focusing on the Notion of Knowledge. In *Trends in Teaching and Learning of Mathematical Modelling* (pp. 199-209). https://doi.org/10.1007/978-94-007-0910-2_21

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes, R. Bonilla, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning: current state of understanding.* (Vol. 2, pp. 55-73). University of Wyoming.

Castillo-Garsow, C. (2013, Oct-Dec). The role of multiple modelling perspectives in students' learning of exponential growth. *Math Biosci Eng, 10*(5-6), 1437-1453. https://doi.org/10.3934/mbe.2013.10.1437

Cengiz, N., Kline, K., & Grant, T. J. (2011). Extending students' mathematical thinking during whole-group discussions. *Journal of Mathematics Teacher Education, 14*(5), 355-374. https://doi.org/10.1007/s10857-011-9179-7

Cevikbas, M., Kaiser, G., & Schukajlow, S. (2021). A systematic literature review of the current discussion on mathematical modelling competencies: state-of-the-art developments in conceptualizing, measuring, and fostering. *Educational Studies in Mathematics, 109*(2), 205-236. https://doi.org/10.1007/s10649-021-10104-6

Clement, J. (2000). Analysis of Clinical Interviews: Foundations and Model Viability. In *Handbook of research methodologies for science and mathematics education* (pp. 341-385). Lawrence Erlbaum.

Czocher, J. A. (2016, 2016/04/02). Introducing Modelling Transition Diagrams as a Tool to Connect Mathematical Modelling to Mathematical Thinking. *Mathematical Thinking and Learning, 18*(2), 77-106. https://doi.org/10.1080/10986065.2016.1148530

Czocher, J. A. (2018, 2018/10/01). How Does Validating Activity Contribute to the Modelling Process? *Educational Studies in Mathematics, 99*(2), 137-159. https://doi.org/10.1007/s10649-018-9833-4

Czocher, J. A. (2019). Precision, Priority, and Proxies in Mathematical Modelling. In *Lines of Inquiry in Mathematical Modelling Research in Education* (pp. 105-123). https://doi.org/10.1007/978-3-030-14931-4_6

Czocher, J. A., & Hardison, H. (2019). *Characterizing Evolution of Mathematical Models* Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics
Education, St Louis, MO.

Czocher, J. A., & Hardison, H. L. (2021). Attending to Quantities through the Modelling Space. In F. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical Modelling Education in East and West*. Springer.

Ellis, A. B. (2007a). Connections between Generalizing and Justifying: Students' Reasoning with Linear Relationships. *Journal for Research in Mathematics Education, 38*(3), 194-229. https://doi.org/10.2307/30034866

Ellis, A. B. (2007b). The Influence of Reasoning with Emergent Quantities on Students' Generalizations. *Cognition and Instruction, 25*(4), 439-478. www.jstor.org/stable/27739866

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39*, 135-155. https://doi.org/10.1016/j.jmathb.2015.06.004

Galbraith, P., & Stillman, G. (2006). A Framework For Identifying Student Blockages During Transitions In The Modelling Process. *ZDM, 38*(2), 143-162.

Goldin, G. A. (2000). A Scientific Perspective on Structured, Task-Based Interviews in Mathematics Education Research. *Handbook of research design in mathematics and science education*, 517-545.

Hankeln, C. (2020). Mathematical modelling in Germany and France: a comparison of students' modelling processes. *Educational Studies in Mathematics, 103*(2), 209-229. https://doi.org/10.1007/s10649-019-09931-5

Hankeln, C., Adamek, C., & Greefrath, G. (2019). Assessing Sub-competencies of Mathematical Modelling—Development of a New Test Instrument. In *Lines of Inquiry in Mathematical Modelling Research in Education* (pp. 143-160). https://doi.org/10.1007/978-3-030-14931-4_8

Julie, C., & Mudaly, V. (2007). Mathematical Modelling of Social Issues in School Mathematics in South Africa. In P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 503-510). Springer. https://doi.org/10.1007/978-0-387-29822-1

Kaiser, G. (2017). The Teaching and Learning of Mathematical Modelling. In *Compendium for Research in Mathematics Education* (pp. 267-291). The National Council of Teachers of Mathematics, Inc.

Larson, C. (2013). Modelling and quantitative rasoning: the summer jobs problem. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modelling Students' Mathematical Modelling Competencies* (pp. 111-118). Springer. https://doi.org/10.1007/978-94-007-6271-8_

Lesh, R., Doerr, H. M., Carmona, G., & Hjalmarson, M. (2003). Beyond Constructivism. *Mathematical Thinking and Learning, 5*(2&3), 211-233.

Leslie, P. S. (2013). On Children's construction of quantification. In R. L. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning in mathematics and science education: papers from an international STEM research symposium* (pp. 13-41). University of Wyoming.

Maaß, K. (2006). What are Modelling Competencies? *ZDM, 38*(2), 113-142.

Manouchehri, A., & Lewis, S. T. (2015). *Reconciling Intuitions and Conventional Knowledge: The Challenge of Teaching and Learning Mathematical Modelling*.

Mkhatshwa, T. P., & Doerr, H. M. (2018). Undergraduate Students' Quantitative Reasoning in Economic Contexts. *Mathematical Thinking and Learning, 20*(2), 142-161. https://doi.org/10.1080/10986065.2018.1442642

Moore, K. (2014). Quantitative Reasoning and the Sine Function: The Case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138. https://doi.org/10.5951/jresematheduc.45.1.0102

Moore, K. C., Liang, B., Stevens, I. E., & Tasova, H. I. (2019). *A Quantitative Reasoning Framing of Concept Construction* Proceedings of the Forty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, St. Louis, MO.

Moore, K. C., & Thompson, P. W. (2015). Shape Thinking and Students' Graphing Activity. 18th Annual Conference on Research in Undergraduate Mathematics Education,

Niss, M. (2010). Modelling a Crucial Aspect of Students' Mathematical Modelling. In R. Lesh, P. L. Galbraith, C. Haines, & A. Hurford (Eds.), *Modelling Students' Mathematical Modelling Competencies* (pp. 43-59). Springer. https://doi.org/10.1007/978-1-4419-0561-1

O'Bryan, A. E. (2019). *Quantitative reasoning and symbolization activity: do individuals expect acalculations and expressions to have quantitative significance?* Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education, Boston, MA.

Piaget, J. (1965). *The Child's Conception of Number*. W. W. Norton & Company.

Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of a Scientific Conception: Toward a Theory of Conceptual Change. *Science Education, 66*(2), 211-227.

Quintana, C., Reiser, B. J., Davis, E. A., Krajcik, J., Fretz, E., Duncan, R. G., Kyza, E., Edelson, D., & Soloway, E. (2004). A Scaffolding Design Framework for Software to Support Science Inquiry. *The Journal of the Learning Sciences, 13*(4), 337-386.

Radford, L., Schubring, G., & Seeger, F. (2011). Signifying and meaning-making in mathematical thinking, teaching, and learning. *Educational Studies in Mathematic, 77*(2/3), 149-156. https://doi.org/https://www.jstor.org/stable/41485922

Roorda, G., Vos, P., & Goedhart, M. (2007). The concept of the derivative in modelling and applications. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical Modelling: Education, Engineering, and Economics*. Woodhead Publishing. https://doi.org/10.1533/9780857099419.5.288

Saldanha, L., & Thompson, P. (2002). Conceptions of Sample and Their Relationship to Statistical Inference. *Educational Studies in Mathematics, 51*, 257-270.

Schwartz, J. L. (1988). Intensive Quantity and Referent Transforming Arithmetic Operations. In *Number concepts and operations in the middle grades* (pp. 41-52). National Council of Teachers of Mathematics.

Schwartz, J. L. (1996). *Semantic Aspects of Quantity*. https://sites.google.com/site/mathmindhabits/on-th-teaching-of-mathematics/semantic-aspects-of-quantity

Sherin, B. L. (2001). How Students Understand Physics Equations. *Cognition and Instruction, 19*(4), 479-541. www.jstor.org/stable/3233857

Sol, M., Giménez, J., & Rosich, N. (2011). Project Modelling Routes in 12–16-Year-Old Pupils. In *Trends in Teaching and Learning of Mathematical Modelling* (pp. 231-240). https://doi.org/10.1007/978-94-007-0910-2_24

Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences, 3*(1), 61-82. https://doi.org/10.1016/1041-6080(91)90004-k

Stillman, G. (2000). Impact of Prior Knowledge of Task Context on Approaches to Applications Tasks. *Journal of Mathematical Behavior, 19*, 333-361.

Stillman, G. (2011). Applying Metacognitive Knowledgeand Strategies in Applications and Modelling Tasks at Secondary School. In G. Kaiser, W. Blum, R. B. Ferri, & G. Stillman

(Eds.), *Trends in Teaching and Learning of Mathematical Modelling* (pp. 165-180). Springer Science+Business Media B.V. https://doi.org/10.1007/978-94-007-0910-2

Stillman, G., Brown, J., & Galbraith, P. (2010). Identifying Challenges within Transition Phases of Mathematical Modelling Activities at Year 9. In R. Lesh, P. L. Galbraith, C. Haines, & A. Hurford (Eds.), *Modelling Students' Mathematical Modelling Competencies* (pp. 385-398). Springer. https://doi.org/10.1007/978-1-4419-0561-1

Stillman, G., & Brown, J. P. (2014). Evidence of Implemented Anticipation in Mathematising by Beginning Modellers. *Mathematics Education Research Journal, 26*, 763-789. https://doi.org/10.1007/s13394-014-0119-6

Stillman, G. A., & Galbraith, P. L. (1998). Applying Mathematics With Real World Connections: Metacognitive Characteristics of Secondary Students. *Educational Studies in Mathematics, 36*, 157-189.

Stroup, W., M. (2002). Understanding Qualitative Calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning, 7*, 167-215.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179). State University of new York press.

Thompson, P. W. (1994b). Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.

Thompson, P. W. (2011). Quantitative Reasoning and Mathematical Modelling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 33-57). University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 421-456). National Council of Teachers of Mathematics.

Vergnaud, G. (1998). Comprehensive Theory of Representations. *Journal of Mathematical Behavior, 17*(2), 167-181.

Vorhölter, K. (2018). Conceptualization and Measuring of Metacognitive Modelling Competencies: Empirical Verification of Theoretical Assumptions. *ZDM, 50*, 343-354. https://doi.org/10.1007/s11858-017-0909-x

Walton, J. (1992). Making the Theoretical Case. In C. C. Ragin & H. S. Becker (Eds.), *What is a case? : exploring the foundations of social inquiry* (pp. 121-137). Cambridge Univeristy Press.

Yeo, J. B. W. (2007). *Mathematical tasks: Clarification, classification and choice of suitable tasks for different types of learning and assessment*.

Zandieh, M. J. (1997). *The Evolution of Student Understanding of the Concept of Derivative* (Publication Number 9805485) Oregon State University.

Zbiek, R. M., & Conner, A. (2006). Beyond Motivation: Exploring Mathematical Modelling as a Context for Deepening Students' Understandings of Curricular Mathematics. *Educational Studies in Mathematics, 63*, 89-112. https://doi.org/10.1007/s10649-005-9002-4