On the Regularity of Non-Scattering Anisotropic Inhomogeneities

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Abstract

In this paper we examine necessary conditions for an anisotropic inhomogeneous medium to be non-scattering at a single wave number and for a single incident field. These conditions are expressed in terms of the regularity of the boundary of the inhomogeneity. We assume that the coefficients, characterizing the constitutive material properties of the medium, are sufficiently smooth, and the incident wave is appropriately non-degenerate. Our analysis utilizes the Hodograph transform as well as regularity results for nonlinear elliptic partial differential equations. Our approach requires that the boundary a-priori is of class $C^{1,\alpha}$ for some $0 < \alpha < 1$.

1 Introduction

Spectral problems with the wave number as spectral parameter play a central role in mathematical scattering theory. A particularly noteworthy development has been the theory of scattering resonances, which correspond to the poles of the scattering operator viewed as a meromorphic operator valued function of the wave number [14]. Quite related is the question of injectivity of the relative scattering (incident-to-outgoing [13, 23]) operator for inhomogeneous media, which has led to the study of the transmission eigenvalues [7]. To be precise, given an inhomogeneous medium of compact support, the question is if there are wave numbers for which there exist incident waves that are not scattered by the medium, i.e., the medium is rendered invisible to this particular probing experiment. Such wave numbers are referred to as *non-scattering wave numbers*, and the corresponding incident waves as non-scattering incident fields. An inhomogeneous medium that admits non-scattering wave numbers is called a *non-scattering inhomogeneity*.

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If the given inhomogeneity does not scatter an incident wave at a fixed wave number, then inside the support of the media this incident wave and the corresponding total field solve the Helmholtz equation governing wave propagation in the background and in the medium, respectively, and share the same Cauchy data on the boundary. This homogeneous boundary value problem is known as the transmission eigenvalue problem, and its eigenvalues (the wave numbers) are referred to as *transmission eigenvalues*. The transmission eigenvalue problem is non-selfadjoint and has an interesting mathematical structure. Its formulation involves the governing equations for the medium and background only inside the support of the inhomogeneity, i.e., it does not depend on the external probing wave. Transmission eigenvalues play a central role in inverse scattering theory for inhomogeneous media; we refer the reader to [7] for an up-to-date discussion of transmission eigenvalues, their applications to inverse scattering, and the vast literature on the subject. In particular, non-scattering wave numbers form a subset (possibly empty) of the real transmission eigenvalues. The existence of infinitely many real transmission eigenvalues is proven for a large class of (not necessarily regular) inhomogeneities [8, 9]. The question then arises, whether these inhomogeneities are non-scattering, in other words whether any of the corresponding real transmission eigenvalues is a non-scattering wave number. It turns out that the existence of a non-scattering wave number (unlike the existence of real transmission eigenvalues) generically implies a certain regularity of the inhomogeneity. Such regularity results are proven in [10] and [27] for the case of isotropic media with contrast only in the lower order term of the Helmholtz equation. In this paper we deal with scattering by an anisotropic medium governed by the Helmholtz equation, with variable coefficients in the principal operator and the lower order term. We establish necessary conditions for such a medium to be non-scattering, or equivalently by negation, sufficient conditions for it to be scattering. The investigation of non-scattering anisotropic media presents additional mathematical difficulties and leads to interesting open questions.

Consider the following time-harmonic scattering problem

$$\begin{aligned} \nabla \cdot A\nabla u + k^2 n u = 0 \quad &\text{in } \mathbb{R}^d, \\ u = u^{\mathrm{in}} + u^{\mathrm{sc}} \quad &\text{in } \mathbb{R}^d, \\ \lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|} u^{\mathrm{sc}} - \mathrm{i}k u^{\mathrm{sc}} \right) = 0, \ &\text{uniformly for all } \hat{x} := \tfrac{x}{|x|} \in \mathcal{S}^{d-1}, \end{aligned} \tag{1}$$

where $k > 0$ is the wave number proportional to interrogating frequencies, and $d \geq 2$. Here, u^{in} is a given incident field which satisfies the Helmholtz equation $\Delta u^{\mathrm{in}} + k^2 u^{\mathrm{in}} = 0$ in $\mathbb{R}^d$ and u^{sc} is the corresponding scattered field. We assume that n is a real-valued function in $L^\infty(\mathbb{R}^d)$, and $A = (a_{ij})$ is a real, $d \times d$ symmetric matrix-valued function with $L^\infty(\mathbb{R}^d)$ entries, satisfying

$$c_0^{-1} |\xi|^2 \leq \xi^\top A(x) \xi \leq c_0 \, |\xi|^2 \qquad \text{for almost all } x \in \mathbb{R}^d \text{ and all } \xi \in \mathbb{R}^d, \tag{2}$$

for some positive constant c_0. We further assume that $A - I$ and $n - 1$ are supported in $\overline{\Omega} \subset \mathbb{R}^d$, where Ω is a bounded Lipschitz domain. Let ν denote the outward unit normal vector to $\partial\Omega$. In this model $A(x), n(x)$, for $x \in \Omega$, characterize the constitutive material properties of an anisotropic dielectric inhomogeneous medium occupying the region Ω and

located in an isotropic homogeneous background with constitutive material properties scaled to one. In what follows (A, n, Ω) refers to this inhomogeneous medium. It is known that the scattering problem (1) admits a *unique solution* $u \in H^1_{\rm loc}(\mathbb{R}^d)$ (see e.g. [7]). Note: the fact that $u \in H^1_{\rm loc}(\mathbb{R}^d)$ solves (1) implies

$$u^- = u^+, \qquad \nu^\top A \nabla u^- = \partial_\nu u^+, \qquad \text{on } \partial\Omega \ , \tag{3}$$

where $+$ and $-$ indicate traces on the boundary $\partial\Omega$ from outside and inside Ω, respectively. Additional regularity conditions on the inhomogeneity (A, n, Ω) will be imposed later as required. Of main interest to us is the case when A has a jump across $\partial\Omega$. In the following, when A (or n) is continuous on $\overline{\Omega}$, and we talk about the value of A (or n) on $\partial\Omega$, we always mean the limit value from inside Ω.

The main focus of this paper is to investigate the implied regularity on $\partial\Omega$ if it happens that $u^{\rm in}$ is a non-scattering wave for the medium (A, n, Ω), in other words if $u^{\rm sc}$ vanishes identically outside Ω. In this case, with $v := u^{\rm in}$, for simplicity of notation, we have that $u|_\Omega$ along with v becomes a solution to the following homogeneous problem

$$\begin{aligned} \nabla \cdot A \nabla u + k^2 n u = 0, \quad \Delta v + k^2 v = 0, \qquad & \text{in } \Omega \ , \\ u = v, \quad \nu^\top A \nabla u = \partial_\nu v, \qquad & \text{on } \partial\Omega \ , \end{aligned} \tag{4}$$

with v actually being a solution to

$$\Delta v + k^2 v = 0 \quad \text{in all of } \mathbb{R}^d. \tag{5}$$

We note that v is real analytic in $\mathbb{R}^d$. In our earlier terminology, the wave number $k > 0$ is a non-scattering wave number for (A, n, Ω). Considering only the set of homogeneous equations (4), also known as the transmission eigenvalue problem, we conclude that $k > 0$ is a *real* transmission eigenvalue. The anisotropic transmission eigenvalue problem has been subject of extensive investigation in the past decade [7]. Despite its deceptively simple formulation, it is a non-selfadjoint eigenvalue problem even for real valued coefficients A and n. Nevertheless, for A, n real valued, and in $L^\infty(\Omega)$ and $\partial\Omega$ Lipschitz, the existence of an infinite discrete set of real transmission eigenvalues accumulating only at $+\infty$ is proven in [8] provided $A - I$ is one-sign definite uniformly in Ω and $n \equiv 1$, and in [9] provided both $A - I$ and $n - 1$ are one-sign definite (the same or opposite sign) uniformly in Ω. The state-of the-art of the spectral analysis for the transmission eigenvalue problem, including discreteness of real and complex eigenvalues, completeness of generalized eigenfunctions in $(L^2(\Omega))^2$, and Weyl asymptotics for the eigenvalue counting function, can be found in [25]. This spectral analysis is conditional on some "ellipticity" assumptions on the coefficients at the boundary $\partial\Omega$. More specifically, it requires that $\partial\Omega$ is of class C^2, and that A and n are continuous on $\overline{\Omega}$ and satisfy for all $x \in \partial\Omega$,

$$(A(x)\nu \cdot \nu)(A(x)\xi \cdot \xi) - (A(x)\nu \cdot \xi)^2 \neq 1 \qquad \text{and} \qquad (A(x)\nu \cdot \nu) n(x) \neq 1,$$

for all unit vectors $\xi \in \mathbb{R}^d$ perpendicular to the normal vector ν (the first condition is equivalent to the complementing condition, due to Agmon, Douglis and Nirenberg [1]). However, under the above less restrictive assumptions on A and n, it is not known whether real

transmission eigenvalues exist. We refer the reader to [28] for results about the location of transmission eigenvalues in the complex plane.

Whereas real transmission eigenvalues exist for a broad class of (not necessarily smooth) anisotropic inhomogeneities, the main result of this paper states that the existence of a non-scattering wave number associated with an appropriate incident field, for regular A and n implies a certain regularity of the inhomogeneity. In particular we prove that if A and n are real analytic in $\overline{\Omega}$, $A \neq I$ on $\partial\Omega$, and $\partial\Omega$ is a-priori of class $C^{1,\alpha}$ for some $0 < \alpha < 1$, and if $\partial\Omega$ is not analytic in any neighborhood of a point $P \in \partial\Omega$, then (A, n, Ω) scatters all incident fields having a non-vanishing $(A - I)$ conormal derivative at that point (see (9)). In terms of the transmission eigenvalue problem, formulated only in Ω, our result provides a necessary condition, that the part v of the corresponding eigenfunction can be extended as a solution of the Helmholtz equation in the exterior of Ω (*i.e.*, v is real analytic up to the boundary). We establish similar results for less regular A and n. To prove these results we employ the Hodograph transform to locally straighten the boundary and transform (4) to a strongly elliptic second order nonlinear partial differential equation in divergence form, accompanied with a nonlinear oblique derivative boundary condition. The regularity result is then obtained by appealing to the work by Agmon, Douglis and Nirenberg [1]. The idea to use the Hodograph transform for $C^{1,\alpha}$ boundaries is inspired by the work of Kinderlehrer and Nirenberg [19] (see also the work by Kinderlehrer, Nirenberg and Spruck [20] and Alessandrini and Isakov [2]). Scattering from inhomogeneities (A, n, Ω) where $\partial\Omega$ contains corner singularities is investigated in [11] and [33] by a different approach based on the so-called Complex Geometric Optics (CGO) solutions combined with asymptotic analysis in a neighborhood of the corner. For the readers' convenience, we summarize these results on corner scattering in Section 3, where we also provide a simple example of a non-scattering inhomogeneity of the form (A, n, Ω) with corners. If Ω is a ball of radius R centered at the origin, and $A := a(r)I$, $n := n(r)$ with scalar functions depending only on the radial variable r, satisfying

$$\frac{1}{R}\int_0^R \left(\frac{n(r)}{a(r)}\right)^{1/2} dr \neq 1 \ ,$$

then it is possible to show by separation of variables the existence of infinitely many non-scattering wave numbers [9, 13]. In fact, for this spherically stratified medium the set of transmission eigenvalues and the set of non-scattering wave numbers coincide. Furthermore, the non-scattering incident waves are superpositions of plane waves, otherwise known as Herglotz wave functions with particular densities, and each density is associated with an infinite set of non-scattering wave numbers.

The fact that we in this paper assume that $A - I \neq 0$ on the boundary is essential, and it makes the analysis more challenging. Regularity results for non-scattering inhomogeneities with $A \equiv I$ in $\mathbb{R}^d$ are established in [10] and [27] (see also [4]). In this case the starting regularity of the boundary is Lipschitz, and the incident wave can not vanish on boundary points of interest. These results can be extended to the case when $A(x) = a(x)I$ with a scalar function $a(x)$ which is at least $C^{2,\alpha}$ up to and across the boundary, by use of a standard Liouville transformation. However, they do not extend to the case when a has a jump across the boundary, nor to the anisotropic case. Scattering from corners has been investigated in [3, 11, 15, 18, 26, 33]. It turns out that for geometries with corners and smooth A across the

boundary (in particular $A = I$ on $\partial\Omega$) the non-vanishing of the incident wave is not necessary to achieve non-zero scattered field. These results provide the foundation for proving that a convex polygonal inhomogeneity with smooth A in a neighborhood of all corners is uniquely determined from scattering data corresponding to one single incident wave [11, 15, 18]. Our example in Section 3 show that such results cannot be expected in general for (A, n, Ω), when A has a jump across the boundary, without additional assumptions on A and n. Finally let us mention, that the non-scattering phenomenon of spherically symmetric media is very unstable. More precisely, if a disk in $\mathbb{R}^2$, with $A := I$ and $n > 0$ constant, is perturbed using a very broad class of perturbations, then there exist at most finitely many positive wave numbers for which Herglotz wave functions, with smooth non-trivial densities in an appropriately compact set, can be non-scattering (see [29, 30]).

2 The Hodograph Transform and the Main Result

We start by rewriting (4)-(5) in a different form more convenient for our analysis, that is, we write it in terms of the scattered field instead of the total field. To this end, we define $w := u - v$. Then w satisfies

$$\nabla \cdot A\nabla w + k^2 n w = -\nabla \cdot (A - I)\nabla v - k^2(n-1)v \qquad \text{in } \Omega \ , \tag{6}$$

with

$$w = 0 \ \text{ and } \quad \nu^\top A\nabla w = -\nu^\top (A - I)\nabla v \qquad \text{on } \partial\Omega \ , \tag{7}$$

with v being a solution to (5), and hence real analytic in $\mathbb{R}^d$. As it will become clear later in the paper, what matters to our results is that v is sufficiently regular on $\overline{\Omega}$.
From now on we shall assume that Ω is a $C^{1,\alpha}$ domain and that the matrix valued function A has entries in $C^{1,\alpha}(\overline{\Omega})$[1]. Due to the regularity of v, a standard regularity result for elliptic equations (see Corollary 8.36 in [17]) gives that $w \in C^{1,\alpha}(\overline{\Omega})$. Moreover, since $w = 0$ on $\partial\Omega$, we have $\nu = \pm\nabla w/|\nabla w|$, provided $\nabla w \neq 0$. The second condition in (7) therefore leads to

$$(\nabla w)^\top A\nabla w + (\nabla w)^\top (A - I)\nabla v = 0 \qquad \text{on } \partial\Omega \ . \tag{8}$$

We are interested in the case when A has a jump across $\partial\Omega$. Now let P be a (fixed) point on $\partial\Omega$ with $A(P) \neq I$. We also assume that

$$\nu^\top (A - I)\nabla v(P) \neq 0 \ . \tag{9}$$

Then, recalling the boundary condition (7) we also have

$$\nu^\top A\nabla w \neq 0 \qquad \text{at } P.$$

[1] By $C^{k,\alpha}(\overline{\Omega})$, k an integer ≥ 0, and $0 < \alpha < 1$, we understand the set of functions that may be extended as $C^{k,\alpha}$ functions in an open neighborhood of $\overline{\Omega}$. The analogue of Whitney's Extension Theorem [31] for $C^{k,\alpha}$ functions, $0 < \alpha < 1$, asserts that this definition of $C^{k,\alpha}(\overline{\Omega})$ amounts to requiring that all derivatives of order less than or equal to k are α-Hölder continuous in Ω, and up to the boundary $\partial\Omega$, with constants that are uniformly valid in Ω.

Up to a rigid change of coordinates we may assume that $\nu^\top A = (-c_1, 0, \dots, 0)$ with $c_1 = |A\nu| > 0$ at P, and that $P = 0$. Without loss of generality we may also assume that $\nu^\top(P)A(P)\nabla w(P) = -c_1\partial_1 w(0) < 0$. Then, by the regularity of w we can find positive constants r and c_2 such that

$$0 < c_2^{-1} < \partial_1 w(x) < c_2, \qquad \text{for all } x \in \overline{\Omega} \cap B_r(0) \ , \tag{10}$$

and hence

$$0 < c_3^{-1} < -\frac{(\nabla w)^\top (A - I)\nabla v}{|\nabla w|} < c_3 \ , \qquad \text{for all } x \in \partial\Omega \cap B_r(0) \tag{11}$$

for some constant c_3. Denoting $x = (x_1, x_2, \dots, x_d) = (x_1, x')$, we consider the mapping

$$H: \ x \mapsto y = (w(x), x') \ , \qquad x \in \overline{\Omega} \cap B_r(0) \ .$$

Due to the regularity of w, the map H is in $C^{1,\alpha}(\overline{\Omega} \cap B_r(0))$. It can be calculated directly that the Jacobian of H is

$$DH(x) = \begin{bmatrix} \partial_{x_1} w & \nabla_{x'}^\top w \\ 0 & I \end{bmatrix} \ .$$

Therefore, H is invertible on $\overline{\Omega} \cap B_r(0)$, thanks to the non-vanishing of $\partial_{x_1} w$. We may conclude that H is bijective from $\Omega \cap B_r(0)$ to $V^+ = V \cap \{(y_1, y') \ : \ y_1 > 0\}$, and also from $\partial\Omega \cap B_r(0)$ to $\Sigma = V \cap \{(y_1, y') \ : \ y_1 = 0\}$, where V is an open neighborhood of 0. The inverse map of H can be expressed as $H^{-1}(y) = (z(y), y')$ for some function $z \in C^{1,\alpha}(V^+ \cup \Sigma)$. In terms of z the Jacobian of H can be re-written as

$$DH(x) = \begin{bmatrix} \partial_{x_1} w & \nabla_{x'}^\top w \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1/\partial_{y_1} z & -\nabla_{y'}^\top z/\partial_{y_1} z \\ 0 & I \end{bmatrix} \circ H(x) \ .$$

Next, we derive equations for z. By a direct calculation we obtain for $y = H(x)$ that

$$\nabla_x^\top = \widetilde{\nabla}^\top z \ \partial_{y_1} + (0, \nabla_{y'}^\top) \ , \qquad \text{where } \widetilde{\nabla}^\top z = \frac{1}{\partial_{y_1} z}(1, -\nabla_{y'}^\top z) \ .$$

In the following, all the differential operators are with respect to y, unless they carry explicit subscripts of x. We then have

$$(\nabla_x w) \circ H^{-1} = \widetilde{\nabla} z \qquad \text{and} \qquad (\nabla_x \cdot A\nabla_x w) \circ H^{-1} = \left(\widetilde{\nabla}^\top z \ \partial_1 + (0, \nabla_{y'}^\top)\right)(A_H \widetilde{\nabla} z) \ ,$$

with $A_H = A \circ H^{-1}$. A direct calculation gives that

$$2\,\widetilde{\nabla}^\top z \ \partial_{y_1}(A_H \widetilde{\nabla} z) = \partial_{y_1}\left((\widetilde{\nabla}^\top z)\, A_H \widetilde{\nabla} z\right) + \widetilde{\nabla}^\top z \ (\partial_{y_1} A_H) \ \widetilde{\nabla} z \ ,$$

and that

$$\partial_{y_1} A_H(y) = \partial_{y_1} A(H^{-1}(y)) = \partial_{y_1} A(z(y), y') = (\partial_{y_1} z)\, \frac{\partial A}{\partial x_1}(H^{-1}(y)) \ .$$

Therefore, we deduce from (6) that $z(y)$ satisfies

$$\sum_{j=1}^{d} \partial_{y_j} a_j(y, z, \nabla z) + a_0(y, z, \nabla z) = 0 \qquad \text{in } V^+, \tag{12}$$

where a_j, $j = 0, 1, \ldots, d$, are functions of $2d + 1$ arguments given by

$$a_1 = \frac{1}{2}(\widetilde{\nabla}^\top z) A_H \widetilde{\nabla} z \ , \qquad a_j = (A_H \widetilde{\nabla} z)_j \ , \quad j = 2, \ldots, d \ ,$$

and

$$a_0 = \frac{1}{2}\,(\partial_1 z)\,\widetilde{\nabla}^\top z\,(\partial_{x_1} A)\,\widetilde{\nabla} z + k^2 n y_1 + \nabla_x \cdot (A - I)\nabla_x v + k^2(n-1)v \ .$$

Here, in the lower order term a_0, with an abuse of notation, we still use n to denote $n \circ H^{-1}(y) = n(z(y), y')$; the same applies to A and v, as well as their derivatives with respect to x. Similarly, from the boundary condition (8) for w we obtain that

$$b(y', z, \nabla z) := (\widetilde{\nabla} z)^\top A_H \widetilde{\nabla} z + (\widetilde{\nabla} z)^\top (A_H - I)\nabla_x v = 0 \qquad \text{on } \Sigma \ , \tag{13}$$

where $A_H = A_H(0, y')$ and $\nabla_x v = (\nabla_x v) \circ H^{-1}(0, y')$.

We claim that (12) is a uniformly strongly (nonlinear) elliptic equation and (13) is an associated proper (nonlinear) oblique derivative boundary condition. This result heavily depends on the fact that $\partial_1 z = 1/\partial_{x_1} w$ and thus

$$0 < c_2^{-1} < \partial_1 z(y) < c_2 \qquad \text{for all } y \in V^+ \cup \Sigma \ . \tag{14}$$

We shall present the proof of this claim in the Appendix.

Remark 2.1. In the special case when $A = aI$ in $\overline{\Omega}$ with a being a positive constant, the problem for z becomes

$$\begin{aligned} -\frac{1}{2}\,a\,\partial_1 \left(\frac{1}{(\partial_1 z)^2}(1 + |\nabla_{y'} z|^2) \right) + a \nabla_{y'} \cdot \left(\frac{1}{\partial_1 z} \nabla_{y'} z \right) &= k^2 n y_1 + k^2 (n - a) v \qquad \text{in } V^+, \\ a\,\frac{1 + |\nabla_{y'} z|^2}{(\partial_1 z)^2} + (a-1)\frac{\partial_{x_1} v - (\nabla_{y'} z) \cdot \nabla_{x'} v}{\partial_1 z} &= 0 \qquad \text{on } \Sigma \ . \end{aligned}$$

Moreover, the non-degeneracy condition (9) simplifies to

$$a \neq 1 \qquad \text{and} \qquad \partial_\nu v(P) \neq 0 \ .$$

We now assume that $n \in C^{1,\alpha}(\overline{\Omega})$ and $A \in \left(C^{2,\alpha}(\overline{\Omega})\right)^{d \times d}$. We are then in a position to use Theorem 11.2 of [1], with $l = m = m_1 = 1$ and $p = 2$, to conclude that a solution to the boundary value problem (12)-(13) which belongs to $C^{1,\alpha}(V^+ \cup \Sigma)$ is indeed in $C^{2,\alpha}(V^+ \cup \Sigma)$. Here we use (among other things) that the functions a_j in (12), $j = 0, 1, \ldots, d$, are $C^{1,\alpha}$ in all the arguments, and so is b in (13). Once we have established that z is $C^{2,\alpha}$ near $P = 0$, it follows that $\partial\Omega$ and w are $C^{2,\alpha}$ near 0. Furthermore, if A and n admit higher regularity up to C^∞, by a bootstrap procedure we can establish higher regularity results for $\partial\Omega$ (as well as for w and u) up to C^∞ . In addition, if A and n are analytic, we can also prove analyticity of $\partial\Omega$ near $P = 0$, using results of Morrey [24]. To be quite precise we have established

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, of class $C^{1,\alpha}$. Suppose for some integer $\ell \geq 1$ that $n \in C^{\ell,\alpha}(\overline{\Omega})$ and that $A \in \left(C^{\ell+1,\alpha}(\overline{\Omega})\right)^{d\times d}$ with the condition (2) satisfied. Let (u, v) be a solution to (4), with v a solution to $\Delta v + k^2 v = 0$ in a neighborhood of Ω. Let P be a point on $\partial\Omega$ at which the non-degeneracy condition (9) is satisfied. Then $\partial\Omega$ is of class $C^{\ell+1,\alpha}$ near P. Moreover, if both A and n are C^∞ on $\overline{\Omega}$, then $\partial\Omega$ is C^∞ near P. If A and n are real analytic on $\overline{\Omega}$, then $\partial\Omega$ is also analytic near P.*

Remark 2.2. We note that we never specifically use in the proof of Theorem 2.1 that v solves the Helmholtz solution in the exterior of Ω, it suffices that it has a smooth extension. In fact, we need only sufficient regularity of v, that is $C^{\ell+2,\alpha}$ up to the boundary. Furthermore, the higher regularity assumptions on the coefficients in Theorem 2.1 are only needed locally in $\overline{\Omega} \cap B_R(P)$ for some $R > 0$.

Remark 2.3. Theorem 2.1 in particular implies $C^{\ell+1,\alpha}$ regularity of a non-scattering inhomogeneity, provided that the associated incident field v satisfies (9), since in that case the incident field v is an analytic solution to $\Delta v + k^2 v = 0$ in all of $\mathbb{R}^d$.

Of course Theorem 2.1 only adds insight if the wave number k is a real transmission eigenvalue (which is a necessary condition for the incident field to produce a vanishing scattered field). If $k > 0$ is a transmission eigenvalue, a variant of Theorem 2.1 also sheds light onto regularity up to the boundary of the v part of the corresponding transmission eigenfunction. In particular we have the following result.

Corollary 2.1. *Assume $k > 0$ is a real transmission eigenvalue with eigenfunction (u, v) satisfying (4). Assume that $\partial\Omega$ is $C^{1,\alpha}$, $A \in \left(C^{1,\alpha}(\overline{\Omega})\right)^{d\times d}$, $0 < \alpha < 1$ satisfying (2). Let $P \in \partial\Omega$, and assume that $A(P) - I \neq 0$. The following assertions hold:*

1. *If A and n are real analytic in a neighborhood of P, and $\partial\Omega \cap B_r(P)$ is not real analytic for any ball $B_r(P)$, then v can not be real analytic in any neighborhood of P, unless $\nu^\top (A - I)\nabla v(P) = 0$.*
2. *If $n \in C^{\ell,\alpha}(\overline{\Omega}) \cap B_R(P)$, $A \in C^{\ell+1,\alpha}(\overline{\Omega} \cap B_R(P))$ for some integer $\ell \geq 1$, $0 < \alpha < 1$ and some ball $B_R(P)$, and $\partial\Omega \cap B_r(P)$ is not $C^{\ell+1,\alpha}$ for any ball $B_r(P)$ then v cannot be in $C^{\ell+1,\alpha}(\overline{\Omega} \cap B_r(P))$ for any ball $B_r(P)$, unless $\nu^\top (A - I)\nabla v(P) = 0$.*

The idea to use the Hodograph transform, as an essential tool when starting from a $C^{1,\alpha}$ boundary, is one we have borrowed from Kinderlehrer and Nirenberg [19] (see also Kinderlehrer, Nirenberg and Spruck [20], and Alessandrini and Isakov [2]). For the problem treated in [2] in the two dimensional case there is a direct approach using conformal mapping, which works when starting only with a Lipschitz assumption on $\partial\Omega$. That approach does not work for our problem even in $\mathbb{R}^2$, and so it remains an open and very interesting problem to resolve whether the initial $C^{1,\alpha}$ assumption on $\partial\Omega$ can be relaxed to a Lipschitz assumption.

In the case when $A = I$ but n differs from 1, a result similar to that in Theorem 2.1 was established in [10] (see also [18]), starting from a Lipschitz domain, and under the assumption that the incident wave does not vanish (locally). In that case, techniques of Caffarelli [6]

and Williams [32] were used to proceed from Lipschitz to $C^{1,\alpha}$. The result in [10] is entirely local in character, and therefore also applies if $A = I$ in a neighborhood of $P \in \partial\Omega$ with $n(P) \neq 1$. In that case it only guarantees local regularity of $\partial\Omega$ near P. For $A = I$ the non-vanishing of the incident wave turns out not to be necessary for scattering from corners; see e.g. [11, 15, 26, 33].

3 Concluding Remarks

3.1 Examples of Non-Scattering Inhomogeneities

Scattering properties of inhomogeneities (A, n, Ω) in $\mathbb{R}^2$ with support Ω containing a corner are studied in [11, 33]. In the remark below we summarize the main results, obtained in [33] in the case when A is isotropic and is discontinuous across $\partial\Omega$.

Remark 3.1. Let $\Omega \in \mathbb{R}^2$ and $P \in \partial\Omega$ be the vertex of a corner with aperture 2θ, $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$. Assume that $A := aI$ with a scalar function $a \in C^{1,\alpha}(\overline{\Omega}) \cap B_R(P)$, for some ball $B_R(P)$, such that $a(P) \neq 1$. Then this inhomogeneity (A, n, Ω) scatters any incident field v (which is a solution to the Helmholtz equation in a neighborhood of Ω), provided that:

(a) $v(P) \neq 0$ and $\nabla v(P) \neq 0$, or

(b) $v(P) = 0$ and $2\theta \neq p\pi/N$ for any integer $p \geq 1$, with N being the vanishing order of v at P (in particular if $2\theta \in (0, 2\pi)/(\mathbb{Q}\pi)$), or

(c) $v(P) \neq 0$ and $\nabla v(P) = 0$, and either $2\theta \notin \{\pi/2, 3\pi/2\}$ with v admitting a specific expansion, or $2\theta \in \{\pi/2, 3\pi/2\}$ and $a(P) \neq n(P)$.

In the next example we show that (exceptional) non-scattering may indeed happen for non-smooth inhomogeneities.

Example (Non-Scattering Waves and Non-Scattering Isotropic Inhomogeneities with Corners)**.** Consider the case when $A = aI$ and $n = a$, with a a constant different from 1. In that case any k^2 for which (4) has a non-trivial solution is either a Dirichlet eigenvalue for $-\Delta$ or a Neumann eigenvalue for $-\Delta$ on Ω [12]. To see this, define $w_1 = u - v$ and $w_2 = au - v$. Then $\Delta w_j + k^2 w_j = 0$ in Ω, $j = 1, 2$, and $w_1 = 0$ on $\partial\Omega$ while $\partial_\nu w_2 = 0$ on $\partial\Omega$. The functions w_1 and w_2 cannot both vanish identically, since (u, v) is a non-trivial solution to (4), and so it follows that k^2 is *either* a Dirichlet eigenvalue *or* a Neumann eigenvalue for $-\Delta$. Conversely, if $\lambda = k^2$ is a Dirichlet or a Neumann eigenvalue for $-\Delta$ on Ω with associated eigenfunction w, then $(u, v) = (w, aw)$ or $(u, v) = (w, w)$, respectively, is a nontrivial solution to

$$\begin{aligned} \nabla \cdot a\nabla u + k^2 a u = 0, \quad \Delta v + k^2 v = 0, \qquad & \text{in } \Omega\ , \\ u = v, \quad a\partial_\nu u = \partial_\nu v, \qquad & \text{on } \partial\Omega\ . \end{aligned}$$

Whether w corresponds to a non-scattering incident wave at wave number k, depends on whether w can be extended to a (smooth) solution to $\Delta w + k^2 w = 0$ in *all of* $\mathbb{R}^d$.

Let us now take $\Omega \subset \mathbb{R}^2$ be to the unit square $\Omega = Q = (0,1) \times (0,1)$. In that case all the Dirichlet eigenvalues of $-\Delta$ are given by $(p^2+q^2)\pi^2$, where p and q are positive integers; corresponding Dirichlet eigenfunctions are given by $w(x_1,x_2) = \sin p\pi x_1 \sin q\pi x_2$. All the Neumann eigenvalues of $-\Delta$ are given by $(p^2+q^2)\pi^2$, where p and q are non-negative integers, and corresponding Neumann eigenfunctions are given by $w(x_1,x_2) = \cos p\pi x_1 \cos q\pi x_2$. Notice that all these eigenfunctions are smooth functions on all of $\mathbb{R}^2$. As a consequence it follows that, as an incident wave for the medium $0 < a = n \neq 1$, $u^{\text{in}} := v = \cos p\pi x_1 \cos q\pi x_2$ is non-scattering (at wave number $k = \sqrt{(p^2+q^2)\pi^2})$ for any $p, q \in \mathbb{Z}$. The same holds for the incident wave $\sin p\pi x_1 \sin q\pi x_2$ for any $p, q \in \mathbb{Z} \setminus \{0\}$. We note that ∇w vanishes at all the corners of Ω for any of these w, which violates the non-degeneracy condition (9). This example of non-scattering waves in fact falls into the possible set of non-scattering waves in Remark 3.1.

This very simple example illustrates two points: (i) It furnishes an example of a Lipschitz domain and isotropic A for which there exist plenty of non-scattering incident waves, however, notice that these non-scattering incident waves all have vanishing gradients at the irregular boundary points. (ii) In doing so, it indicates the need for a non-degeneracy condition of the form (9) (or $\nabla v \neq 0$ at points, where the normal is not well defined) if one wants to prove regularity of $\partial\Omega$, starting with a Lipschitz assumption.

Example (Non-Scattering Inhomogeneities with Anisotropic A)**.** If one allows anisotropic A, there is a natural way to construct a media which is non-scattering for any incident wave. Simply let Φ be a sufficiently smooth ($C^{3,\alpha}$) diffeomorphism from Ω onto Ω, with $\Phi(x) = x$ on $\partial\Omega$, and define

$$A = \Phi_* I = \frac{D\Phi D\Phi^\top}{|\det D\Phi|} \circ \Phi^{-1} \quad \text{and} \quad n = \Phi_* 1 = \frac{1}{|\det D\Phi|} \circ \Phi^{-1} \ .$$

(A, n, Ω) is the so-called pushforward of $(I, 1, \Omega)$ under Φ. It is well known that the operator $\nabla\cdot(A\nabla\cdot) + k^2 n\cdot$ has the same Dirichlet-to-Neumann data map (or Cauchy data) as $\Delta + k^2$ for any $k > 0$ (see [21, 22]). Therefore (A, n, Ω) is non-scattering for any wave number and any incident wave. We note that $\Phi_* I$ is anisotropic, unless $\Phi = id$, in which case $A = I$ and $n = 1$. To understand why this construction does not contradict Theorem 2.1, one must understand that if $\partial\Omega$ is not of class $C^{l+1,\alpha}$ near P, then either $\text{Range}(A(P) - I) = \text{Range}(\Phi_* I(P) - I)$ is orthogonal to the normal to $\partial\Omega$ at P (remember, the gradient field of the incident wave at P can be arbitrary) or $(A, n) = (\Phi_* I, \Phi_* 1)$ fails to be $(C^{l+1,\alpha}, C^{l,\alpha})$ near P. To further illustrate this: if one takes $\Omega \subset \mathbb{R}^2$ be to be a polygon, with vertices P_i, $i = 1, \ldots, N$, and if $\Psi(x)$ is a nontrivial C^∞ vector field that vanishes on $\partial\Omega$, then $\Phi(x) = Ix + \epsilon\Psi(x)$ is a C^∞ diffeomorphism of Ω onto Ω for ϵ sufficiently small, with $\Phi(x) = x$ on $\partial\Omega$ and with $D\Phi(P_i) = I$ for $i = 1, \ldots, N$. The corresponding medium $(A, n, \Omega) = (\Phi_* I, \Phi_* 1, \Omega)$ is anisotropic and non-scattering for any wave number and any incident wave; it has C^∞ coefficients A and n, however, it has $A(P_i) = I$, $n(P_i) = 1$ for all i.

3.2 Incident Fields satisfying the Non-Degeneracy Condition (9)

A natural question to ask is whether there are incident fields that satisfy the non-degeneracy condition (9) at any point on $\partial\Omega$, i.e., whether one can find v that satisfies

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^d \qquad \text{and} \qquad \nu^\top (A - I)\nabla v \neq 0 \quad \text{on } \partial\Omega \ , \tag{15}$$

provided that $A \neq I$ on $\partial\Omega$. For (isotropic) inhomogeneities with $A = aI$ and $a(x) \neq 1$ for $x \in \partial\Omega$ this reduces to finding v that solves

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^d \qquad \text{and} \qquad \frac{\partial v}{\partial \nu} \neq 0 \quad \text{on } \partial\Omega \ , \tag{16}$$

A plane wave $v(x) = \cos(k\,\xi\cdot x)$ (or $v(x) = \sin(k\,\xi\cdot x)$) where $\xi \in \mathcal{S}^{d-1}$ is the unit propagation direction, is one of the most commonly used incident waves in scattering theory. It doesn't satisfy the non-vanishing condition in (16) on all of $\partial\Omega$. In fact, for a smooth boundary, by continuity of $\nu(x)$ there always exists a point $P \in \partial\Omega$ where $\nu^\top(P)\cdot\xi = 0$ and hence $\partial v/\partial\nu(P) = 0$. This applies to point sources as well. However, there are plenty of Herglotz wave functions (superpositions of plane waves) given by

$$v_g(x) := \int_{\mathcal{S}^{d-1}} g_1(\xi)\cos(k\,\xi\cdot x)\,ds_\xi \ + \ \int_{\mathcal{S}^{d-1}} g_2(\xi)\sin(k\,\xi\cdot x)\,ds_\xi \ , \tag{17}$$

which satisfy (15), and thus (16) for $A = aI$, if we exclude some values of wave number $k > 0$. Such values of $k > 0$ correspond to k^2 being an eigenvalue of the oblique Neumann eigenvalue problem

$$\Delta q + k^2 q = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \nu^\top (A - I)\nabla q = 0 \quad \text{on } \partial\Omega \ .$$

If we assume the regular oblique derivative condition, i.e.,

$$(A(x)\nu\cdot\nu)(A(x)\xi\cdot\xi) - (A(x)\nu\cdot\xi)^2 \neq 1 \qquad \text{for all} \quad \xi \in \mathcal{S}^{d-1} \quad \text{such that } \nu\cdot\xi = 0 \ ,$$

then the oblique Neumann eigenvalues form a discrete set that accumulate at ∞ (see e.g. [17]). Now if k^2 is not an oblique Neumann eigenvalue, then one can find a $v \in C^{1,\alpha}(\overline{\Omega})$ that satisfies the Helmholtz equation $\Delta v + k^2 v = 0$ in Ω such that $\nu^\top(A - I)\nabla v = \gamma > 0$ on $\partial\Omega$. Therefore any Herglotz wave function (17) that approximates v sufficiently well in the $C^1(\overline{\Omega})$ norm[2] satisfies the non-degeneracy condition (15) on all of $\partial\Omega$. Since Theorem 2.1 is meaningful only if k is a real transmission eigenvalue, the excluded values are real transmission eigenvalues $k > 0$ such that k^2 is simultaneously a (oblique) Neumann eigenvalue. It is an open problem if such a set of wave numbers is finite or not.

[2]It is possible to slightly modify the duality argument in the proof of Preposition 3.4 in [27] to show that Herglotz wave functions (17) with $g_1, g_2 \in C^\infty(\mathcal{S}^{d-1})$ are dense in the space $\{v :\in W^{1+\sigma,p}(\Omega) : \Delta v + k^2 v = 0\}$ for some $0 < \sigma < 1$ and any $p > 1$, with respect to the $W^{1+\sigma,p}(\Omega)$-norm. Then the approximation property in $C^1(\overline{\Omega})$ follows from the Sobolev Imbedding Theorem.

Appendix

Here we show that the first variation of the nonlinear partial differential equations (12)–(13) is uniformly strongly elliptic with a proper oblique derivative boundary condition; this allows us to apply Theorem 11.2 in [1]. The first variation

$$\mathcal{L}_z(y, \phi, \nabla\phi) = 0$$

of the set of nonlinear equations (12)–(13), in shorthand written as

$$\mathcal{F}(y, z, \nabla z) = 0 \ ,$$

is defined by means of

$$\mathcal{L}_z = \mathcal{I}'(0) \qquad \text{where} \qquad \mathcal{I}(\tau) = \mathcal{F}(y, z + \tau\phi, \nabla(z + \tau\phi)) \ .$$

See [16, Chapter 8] and also [1, Page 684]. Assuming that z is a function of d variables, ∇z is regarded as d arguments, namely $\partial_j z$, $j = 1, \ldots, d$, of $\mathcal{F}$.
A straightforward calculation shows that the first variation of (12) has the following matrix coefficient $\widetilde{A}$ in the principal (divergence form second order) part of the operator at each point $y \in V^+$,

$$\widetilde{A} = -\frac{1}{\partial_1 z}\begin{bmatrix} (\widetilde{\nabla}^\top z)\, A\, \widetilde{\nabla} z & \frac{1}{\partial_1 z}(\vec{a}' - A_{d-1}\nabla_{y'} z)^\top \\ \frac{1}{\partial_1 z}(\vec{a}' - A_{d-1}\nabla_{y'} z) & A_{d-1} \end{bmatrix},$$

where $\vec{a}'^\top = (a_{12}, a_{13}, \ldots, a_{1d})$ and $A_{d-1} = (a_{ij})_{i,j=2}^d$, that is,

$$A = \begin{bmatrix} a_{11} & \vec{a}'^\top \\ \vec{a}' & A_{d-1} \end{bmatrix} \ .$$

For simplicity of notations, we have again used $A = (a_{ij})$ to denote $A_H = A \circ H^{-1} = (a_{ij} \circ H^{-1})$. We calculate for $\xi = (\xi_1, \xi'^\top)^\top \in \mathbb{R}^d$ that

$$-(\partial_1 z)\, \xi^\top \widetilde{A}\, \xi = \xi_1^2\ (\widetilde{\nabla}^\top z)\, A\, \widetilde{\nabla} z + 2\, \xi_1 \frac{1}{\partial_1 z}(\vec{a}' - A_{n-1}\nabla_{y'} z)^\top \xi' + \xi'^\top A_{n-1} \xi' = \widetilde{\xi_z}^\top A\, \widetilde{\xi}_z \ ,$$

with

$$\widetilde{\xi_z}^\top = \left(\frac{1}{\partial_1 z}\xi_1 \ , \ (\xi' - \frac{\nabla_{y'} z}{\partial_1 z}\, \xi_1)^\top\right) = (0, \xi'^\top) + \xi_1 \widetilde{\nabla}^\top z \ .$$

It then follows by continuity (and compactness) that, for all $y \in V^+$,

$$(c_4)^{-1}|\xi|^2 \ \leq \ -\,\xi^\top \widetilde{A}\, \xi \ \leq \ c_4 |\xi|^2,$$

with some positive constants $c_4 > 0$ depending on c_0 in (2), c_2 in (10), and $\|\widetilde{\nabla} z\|_{V^+ \cap \Sigma} = \|\nabla_x w\|_{\overline{\Omega} \cap B_r(0)}$. This verifies the uniform, strong ellipticity condition.
In regards to the boundary condition (13), the principal part of the first variation at each boundary point $y = (0, y')$ is given by $\sum_{j=1}^d b_j \partial_j$ with

$$b_1 = -\frac{1}{\partial_1 z}\left(2\,(\widetilde{\nabla}^\top z)\,A\,\widetilde{\nabla} z + (\widetilde{\nabla}^\top z)\,(A-I)\,\nabla_x v\right) = -\frac{1}{\partial_1 z}(\widetilde{\nabla}^\top z)\,A\,\widetilde{\nabla} z\ ,$$

and

$$b_j = -\frac{1}{\partial_1 z}\left(2A\,\widetilde{\nabla} z + (A-I)\,\nabla_x v\right)_j\,, \qquad j = 2,\dots,d\ ,$$

where again, by abuse of notation, we use A in place of $A \circ H^{-1}(0,y')$ and $\nabla_x v$ in place of $(\nabla_x v)\circ H^{-1}(0,y')$. From (2) and (14) we get that

$$0 < c_0^{-1}c_2^{-3} < c_0^{-1}c_2^{-1}|\widetilde{\nabla} z|^2 < -b_1 \qquad \text{for all } y = (0,y') \text{ on } \Sigma\ .$$

This ensures that the linearized boundary condition (on Σ) is a proper oblique derivative condition, and as a consequence it is "covering" for the linearized second order elliptic differential operator.

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