Journal of Combinatorics
Volume 13, Number 4, 561–574, 2022

Proper permutations, Schubert geometry, and randomness*

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We define and study *proper* permutations. Properness is a geometrically natural necessary criterion for a Schubert variety to be Levi-spherical. We prove the probability that a random permutation is proper goes to zero in the limit.

AMS 2000 subject classifications: Primary 14M15; secondary 05E14, 20P05.
Keywords and phrases: Schubert varieties, spherical varieties, proper permutations.

1. Introduction

Let X denote the variety of complete flags

$$\langle 0\rangle \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n,$$

where F_i is a subspace of dimension i. The general linear group GL_n of invertible $n \times n$ complex matrices acts naturally on X by basis change. Let $B \subset GL_n$ be the Borel subgroup of upper triangular invertible matrices. B acts on X with finitely many orbits; these are the *Schubert cells* X_w° indexed by permutations w in the symmetric group S_n on $[n] := \{1, 2, \ldots, n\}$. Their closures

$$X_w := \overline{X_w^\circ}$$

are the *Schubert varieties*; these objects are of significant interest in combinatorial algebraic geometry. A standard reference is [3] and we also point the reader to the expository papers [6, 8].

*The project was completed as part of the ICLUE (Illinois Combinatorics Lab for Undergraduate Experience) program, which was funded by the NSF RTG grant DMS 1937241.

†Funded by an AMS Simons Travel grant.

‡Funded by a Simons Collaboration grant, and UIUC's Center for Advanced Study.

Now, $\dim X_w = \ell(w)$ where

$$\ell(w) = \#\{1 \leq i < j \leq n : w(i) > w(j)\}$$

counts *inversions* of w. Also, let

$$J(w) = \{1 \leq i \leq n-1 : w^{-1}(i+1) < w^{-1}(i)\}$$

be the set of *left descents* of w. Assume $I \subseteq J(w)$ and let

$$D := [n-1] - I = \{d_1 < d_2 < \ldots < d_k\};$$

also, $d_0 := 0, d_{k+1} := n$. Let $L_I \subseteq GL_n$ be the Levi subgroup of invertible block diagonal matrices

$$L_I \cong GL_{d_1-d_0} \times GL_{d_2-d_1} \times \cdots \times GL_{d_k-d_{k-1}} \times GL_{d_{k+1}-d_k}.$$

As explained in, *e.g.*, [7, Section 1.2], L_I acts on X_w. Moreover, X_w is said to be *L_I-spherical* if X_w has a dense orbit of a Borel subgroup of L_I. If in addition, $I = J(w)$, we say X_w is *maximally spherical*. We refer the reader to *ibid.*, and the references therein, for background and motivation about this geometric condition on a Schubert variety.

Definition 1. Let $d(w) = \#J(w)$. $w \in S_n$ is *proper* if $\ell(w) - \binom{d(w)+1}{2} \leq n$.

For $1 \leq n \leq 10$, proper permutations are not rare; the enumeration is:

$$1, 2, 6, 24, 120, 684, 4348, 30549, 236394, 2006492, \ldots$$

Proposition 2.1 shows that if X_w is L_I-spherical for some $I \subseteq J(w)$, then w is proper. The proof explains the Lie-theoretic origins of the condition. We study proper permutations using standard probabilistic considerations, but towards our geometric application.

Theorem 1.1. *If $w \in S_n$ is chosen uniformly at random,*

$$\lim_{n\to\infty} \Pr[w \textit{ is proper}] = 0.$$

Proposition 2.1 and Theorem 1.1 combined imply our main result:

Theorem 1.2.

$$\lim_{n\to\infty} \Pr[w \in S_n, X_w \textit{ is } L_I\textit{-spherical for some } I \subseteq J(w)] = 0.$$

In particular,

$$\lim_{n\to\infty} \Pr[w \in S_n, X_w \textit{ is maximally spherical}] = 0.$$

While it is a "common-sense expectation" that specific geometric conditions on Schubert varieties are rarely met, Theorem 1.2 gives the first proof of this fact for the sphericality condition.

We also resolve a conjecture from [7]. In *ibid.*, the second and third authors introduced the notion of permutation $w \in S_n$ being I*-spherical*; in the case $I = J(w)$ we call $w \in S_n$ *maximally spherical*. This combinatorial definition is recapitulated in Section 2. Proposition 2.2 shows that if $w \in S_n$ is I-spherical, then w is proper. That proposition, together with Theorem 1.1, gives the first proof of [7, Conjecture 3.7]:

Theorem 1.3.

$$\lim_{n\to\infty} \Pr[w \in S_n, w \textit{ is } I\textit{-spherical for some } I \subseteq J(w)] = 0.$$

Therefore,

$$\lim_{n\to\infty} \Pr[w \in S_n, w \textit{ is maximally spherical}] = 0.$$

Since this work was announced in December 2020, the second and third author, together with Y. Gao [5] proved that if $I \subseteq J(w)$ then $w \in S_n$ is I-spherical if and only if X_w are L_I-spherical; this was [7, Conjecture 1.9]. Therefore, *post facto*, Theorems 1.2 and 1.3 are equivalent (earlier, we had regarded Theorem 1.3 as evidence of [7, Conjecture 1.9]). Additionally, using this characterization proved in [5], C. Gaetz [4] solved [7, Conjecture 3.8] which, combined with the Marcus–Tardos theorem (Stanley–Wilf conjecture), implies a strengthened form of [7, Conjecture 3.7] and thus Theorem 1.2. In contrast, we do not use [5] nor the Marcus–Tardos theorem. Rather our proof is essentially self-contained, relying instead on the notion of properness.

In Section 2 we prove Theorems 1.2 and 1.3, assuming Theorem 1.1. We then prove Theorem 1.1 in Section 3.

2. Properness is necessary for sphericality; proof of Theorems 1.2 and 1.3

Let T be the maximal torus of diagonal matrices in GL_n. For $I \subseteq J(w)$, define

$$B_I = L_I \cap B.$$

Hence B_I is the Borel subgroup of upper triangular matrices in L_I. For a positive integer j, let U_j be the maximal unipotent subgroup of GL_j consisting of upper triangular matrices with 1's on the diagonal. Then

$$\dim U_j = \binom{j}{2}. \tag{1}$$

Let U_I be the maximal unipotent subgroup of B_I. It is basic (see, *e.g.*, [1, Chapter IV]) that

$$U_I \cong U_{d_1-d_0} \times U_{d_2-d_1} \times \cdots \times U_{d_k-d_{k-1}} \times U_{d_{k+1}-d_k}. \tag{2}$$

Proposition 2.1. *If X_w is L_I-spherical then w is proper.*

Proof. Since L_I acts spherically on X_w, by definition, there is a Borel subgroup $K \subset L_I$ such that K has a dense orbit $\mathcal{O}$ in X_w. Thus

$$\dim X_w = \dim \mathcal{O}.$$

Let $x \in \mathcal{O}$. By [2, Proposition 1.11],

$$\mathcal{O} = K \cdot x$$

is a smooth, closed subvariety of X_w of dimension $\dim K - \dim K_x$, where K_x is the isotropy group of x. Hence

$$\dim X_w = \dim \mathcal{O} = \dim K - \dim K_x \leq \dim K. \tag{3}$$

All Borel subgroups of a connected algebraic group are conjugate [1, §11.1], and so

$$\dim K = \dim B_I.$$

The fact that L_I acts on X_w implies $I \subseteq J(w)$, and hence $L_I \subseteq L_{J(w)}$ [7, Section 1.2]. This implies $B_I \subseteq B_{J(w)}$. By [1, Theorem 10.6.(4)],

$$B_I = T \ltimes U_I.$$

Combining all this we have

$$\dim K = \dim B_I \leq \dim B_{J(w)} = \dim T + \dim U_{J(w)}. \tag{4}$$

Let

$$D = [n-1] - J(w) = \{d_1 < d_2 < \ldots < d_k\}.$$

It follows from (1) and (2) that

$$\dim U_{J(w)} = \binom{d_1 - d_0}{2} + \binom{d_2 - d_1}{2} + \cdots + \binom{d_{k+1} - d_k}{2}.$$

The right hand side is maximized when there exists a t such that $d_t - d_{t-1} = n - k$ and $d_j - d_{j-1} = 1$ for all $j \neq t$. Thus

$$\dim U_{J(w)} \leq \binom{n-k}{2} = \binom{n - ((n-1) - d(w))}{2} = \binom{d(w)+1}{2}.$$

Combining this with (3), (4), and the fact that $\ell(w) = \dim X_w$, we see

$$\ell(w) \leq n + \binom{d(w)+1}{2},$$

that is, w is proper. □

Next, we recall the definition of I-spherical permutations in S_n [7]. Let $s_i = (i\ i+1)$ denote the simple transposition interchanging i and $i+1$. An expression

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$$

for $w \in S_n$ is *reduced* if $\ell = \ell(w)$. Let $\mathsf{Red}(w)$ be the set of all reduced expressions for w.

Definition 2 (Definition 3.1 of [7])**.** $w \in S_n$ is *I-spherical* if

$$R = s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}} \in \mathsf{Red}(w)$$

exists such that

(I) s_{d_i} appears at most once in R
(II) $\#\{m : d_{t-1} < i_m < d_t\} \leq \binom{d_t - d_{t-1}+1}{2} - 1$ for $1 \leq t \leq k+1$.

This is a combinatorial analogue of Proposition 2.1:

Proposition 2.2. *Let $w \in S_n$ and $I \subseteq J(w)$. If w is I-spherical then w is proper.*

Proof. First suppose $I = J(w)$. Consider a reduced word $R \in \mathsf{Red}(w)$. By Definition 2(I), at most $n - 1 - d(w)$ of the factors of R are of the form s_x

where $x \notin J(w)$. Thus, at least $\ell(w) - (n-1-d(w))$ factors are of the form s_x where $x \in J(w)$. Clearly, if $j_1, \ldots, j_k$ are positive integers then

$$\sum_{i=1}^{k+1} \binom{j_i+1}{2} \leq \binom{j_1+\cdots+j_{k+1}+1}{2}.$$

Equivalently,

$$\sum_{i=1}^{k+1} \binom{j_i+2}{2} - 1 = \sum_{i=1}^{k+1} \binom{j_i+1}{2} + j_i \leq \binom{j_1+\cdots+j_{k+1}+1}{2} + (j_1+\cdots+j_{k+1}).$$

Set $j_i = d_i - d_{i-1} - 1$ (for $1 \leq i \leq k+1$). Then

$$j_1 + \ldots + j_{k+1} = d_{k+1} - d_0 - (k+1) = n - 1 - k = d(w).$$

Thus, by Definition 2(II), at most $\binom{d(w)+1}{2} + d(w)$ factors are of the from s_x where $x \in J(w)$. Therefore,

$$\binom{d(w)+1}{2} + d(w) \geq \ell(w) - (n-1-d(w)).$$

Rearranging,

$$\ell(w) \leq n - 1 - d(w) + \binom{d(w)+1}{2} + d(w) \Longleftrightarrow \ell(w) < n + \binom{d(w)+1}{2}.$$

So, w is proper.

For $I \neq J(w)$, we use that if w is I-spherical then w is $J(w)$-spherical [7, Proposition 2.12]. □

Conclusion of proof of Theorems 1.2 and 1.3. These claims follow immediately from Theorem 1.1 combined with Proposition 2.1 and Proposition 2.2, respectively. □

Although we chose not to pursue it, using similar techniques, it should be possible to prove analogues of our results for the other classical Lie types.

3. Proof of Theorem 1.1

We apply standard methods in probabilistic combinatorics (see, *e.g.*, D. Zeilberger's [10]). However, we are not aware of the result itself in the literature. In any case, our argument is derived from first principles.

For $w \in S_n$, define

$$\mathcal{E}_{ij} = \text{ the event } \{w^{-1}(i) > w^{-1}(j)\}.$$

Let X_{ij} be the indicator for $\mathcal{E}_{ij}$; that is, $X_{ij} = 1$ if event $\mathcal{E}_{ij}$ happens and $X_{ij} = 0$ otherwise. Then if w is chosen from S_n uniformly at random, then:

$$\mathbb{E}\left[X_{ij}\right] = \Pr\left[X_{ij} = 1\right] = \frac{1}{2!}(1 - \delta_{i,j}) = 1 - \Pr\left[X_{ij} = 0\right].$$

Since $\ell(w) = \ell(w^{-1})$ and $\#J(w) = \#\{i : w^{-1}(i+1) < w^{-1}(i)\}$, the random variable (r.v.) $\ell(w) - \binom{d(w)+1}{2}$ can be modeled as the r.v.

$$X := L - \binom{D+1}{2},$$

where:

$$L = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij},$$

$$\text{and } D = \sum_{i=1}^{n-1} X_{i,i+1}.$$

Notice that if $i_1, i_2, i_3, i_4 \in [n]$ are distinct, then X_{i_1,i_2} and X_{i_3,i_4} are independent.

Lemma 3.1. *For $n \geq 2$,*

$$\mathbb{E}\left[X\right] = \frac{3n^2 - 7n + 2}{24}.$$

Proof. It is true that:

(a) $(X_{i,j})_{i<j}$ are identically distributed,
(b) $\mathbb{E}\left[X_{i,i+1}X_{i,i+1}\right] = \mathbb{E}\left[X_{i,i+1}^2\right] = \mathbb{E}\left[X_{i,i+1}\right] = 1/2$ since $X_{i,i+1}$ is an indicator r.v.,
(c) $\mathbb{E}\left[X_{i,i+1}X_{i+1,i+2}\right] = \Pr\left[w^{-1}(i) > w^{-1}(i+1) > w^{-1}(i+2)\right] = \frac{1}{3!}$,
(d) $X_{i,i+1}$ and $X_{j,j+1}$ are independent if $i + 1 < j$.

With this, the expression $\mathbb{E}\left[L\right]$ can be expanded as:

$$\mathbb{E}\left[L\right] = \mathbb{E}\left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}\right]$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}\left[X_{ij}\right] \qquad \text{lin. of expectation}$$
$$= \frac{1}{2}\binom{n}{2} \qquad \text{identically distributed.}$$

Similarly,

$$\mathbb{E}\left[D\right] = \frac{n-1}{2}.$$

Next, the expression $\mathbb{E}\left[D^2\right]$ can be expanded as:

$$\begin{aligned}
\mathbb{E}\left[D^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n-1} X_{i,i+1}\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n-1} X_{i,i+1}^2 + \sum_{i=1}^{n-1}\sum_{j\neq i} X_{i,i+1}X_{j,j+1}\right] \\
&= \sum_{i=1}^{n-1} \mathbb{E}\left[X_{i,i+1}^2\right] + \sum_{i=1}^{n-1}\sum_{j\neq i} \mathbb{E}\left[X_{i,i+1}X_{j,j+1}\right] && \text{lin. of expectation} \\
&= \frac{n-1}{2} + \sum_{i=1}^{n-1}\sum_{j\neq i} \mathbb{E}\left[X_{i,i+1}X_{j,j+1}\right] && \text{by (b)} \\
&= \frac{n-1}{2} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n-1} \mathbb{E}\left[X_{i,i+1}X_{j,j+1}\right] \\
&= \frac{n-1}{2} + 2\left(\sum_{i=1}^{n-2} \mathbb{E}\left[X_{i,i+1}X_{i+1,i+2}\right] + \sum_{i=1}^{n-1}\sum_{j=i+2}^{n-1} \mathbb{E}\left[X_{i,i+1}X_{j,j+1}\right]\right) \\
&= \frac{n-1}{2} + 2\left(\frac{n-2}{3!} + \frac{1}{2^2}\left(\binom{n-1}{2} - (n-2)\right)\right) && \text{by (c) and (d))} \\
&= \frac{n-1}{2} + \frac{n-2}{3} + \frac{1}{2}\left(\binom{n-1}{2} - (n-2)\right).
\end{aligned}$$

Thus by linearity of expectation,

$$\mathbb{E}\left[\binom{D+1}{2}\right] = \frac{1}{2}\mathbb{E}\left[D^2 + D\right] = \frac{n-1}{2} + \frac{n-2}{6} - \frac{n-2}{4} + \frac{1}{4}\binom{n-1}{2}$$

and

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}\left[L - \binom{D+1}{2}\right] \\ &= \mathbb{E}[L] - \mathbb{E}\left[\binom{D+1}{2}\right] \\ &= \frac{3n^2 - 7n + 2}{24}.\end{aligned}$$

□

Lemma 3.2.

$$\mathbb{E}\left[X^2\right] = \frac{n^4}{64} + o(n^4).$$

Proof. Notice that:

$$\begin{aligned}\mathbb{E}\left[X^2\right] &= \mathbb{E}\left[L^2\right] + \mathbb{E}\left[\binom{D+1}{2}^2\right] - 2\mathbb{E}\left[L\binom{D+1}{2}\right] \\ &= \mathbb{E}\left[L^2\right] + \frac{1}{4}\left(\mathbb{E}\left[D^4\right] + 2\mathbb{E}\left[D^3\right] + \mathbb{E}\left[D^2\right]\right) - \mathbb{E}\left[LD^2\right] - \mathbb{E}[LD].\end{aligned}$$

Now, $0 \leq D^3, D^2, LD \leq n^3$, so

$$\mathbb{E}\left[D^3\right], \mathbb{E}\left[D^2\right], \mathbb{E}[LD] = o(n^4).$$

Thus it suffices to study the asymptotics of $\mathbb{E}\left[L^2\right], \mathbb{E}\left[D^4/4\right], \mathbb{E}\left[LD^2\right]$.

We will repeatedly use the following observation. For a set S with $|S| = o(f(n))$:

$$\sum_{(i_1,j_1,\ldots,i_c,j_c)\in S} \mathbb{E}\left[\prod_{k=1}^{c} X_{i_k,j_k}\right] \leq |S| = o(f(n)). \tag{5}$$

Expanding $\mathbb{E}\left[L^2\right]$ gives:

$$\mathbb{E}\left[L^2\right] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{i'=1}^{n} \sum_{j'=i'+1}^{n} \mathbb{E}\left[X_{i,j}X_{i',j'}\right].$$

There are $\binom{n}{2}^2 = n^4/4 + o(n^4)$ many terms in this summation. Further, there are $\binom{n}{2}\binom{n-2}{2} = n^4/4 + o(n^4)$ many terms in this summation such that i, j, i', j' are distinct. Therefore, there must be $o(n^4)$ terms where i, j, i', j'

are *not* distinct. Now,

$$\begin{aligned}
&\sum_{\text{distinct } i<j,i'<j'\in[n]} \mathbb{E}\left[X_{i,j}X_{i',j'}\right] \\
&= \sum_{\text{distinct } i<j,i'<j'\in[n]} \mathbb{E}\left[X_{i,j}\right]\mathbb{E}\left[X_{i',j'}\right] \quad \text{(independence when indices are distinct)} \\
&= \left(\frac{1}{2}\right)^2 \binom{n}{2}\binom{n-2}{2} \\
&= \left(\frac{1}{2}\right)^2 (n^4/4 + o(n^4)).
\end{aligned}$$

Combining this with (5) gives

$$\mathbb{E}\left[L^2\right] = \frac{1}{16}n^4 + o(n^4). \tag{6}$$

To expand $\mathbb{E}\left[D^4/4\right]$, first we have

$$\mathbb{E}\left[D^4\right] = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{i'=1}^{n-1}\sum_{j'=1}^{n-1} \mathbb{E}\left[X_{i,i+1}X_{j,j+1}X_{i',i'+1}X_{j',j'+1}\right].$$

There are $(n-1)^4 = n^4 + o(n^4)$ many terms in this summation. Further, there are $4!\binom{n-4}{4} = n^4 + o(n^4)$ many terms in this summation such that $i, i+1, j, j+1, i', i'+1, j', j'+1$ are distinct. Here we have used the fact that there are $\binom{n-k}{k}$ ways to choose k non-consecutive numbers from $[n-1]$. Therefore, there must be $o(n^4)$ terms where $i, i+1, j, j+1, i', i'+1, j', j'+1$ are *not* distinct. We compute

$$\begin{aligned}
&\frac{1}{4} \cdot \sum_{\substack{i,j,i',j'\in[n] \\ i,i+1,j,j+1,i',i'+1,j',j'+1 \text{ are distinct}}} \mathbb{E}\left[X_{i,i+1}X_{j,j+1}X_{i',i'+1}X_{j',j'+1}\right] \\
&= \frac{1}{4} \cdot \sum_{\substack{i,j,i',j'\in[n] \\ i,i+1,j,j+1,i',i'+1,j',j'+1 \text{ are distinct}}} \mathbb{E}\left[X_{i,i+1}\right]\mathbb{E}\left[X_{j,j+1}\right]\mathbb{E}\left[X_{i',i'+1}\right]\mathbb{E}\left[X_{j',j'+1}\right] \\
&= \frac{1}{4} \cdot \left(\frac{1}{2}\right)^4 \cdot 4!\binom{n-4}{4} \\
&= \frac{1}{4} \cdot \left(\frac{1}{2}\right)^4 (n^4 + o(n^4)).
\end{aligned}$$

Hence by (5),

$$\mathbb{E}\left[D^4/4\right] = \frac{1}{64}n^4 + o(n^4). \tag{7}$$

Expanding $\mathbb{E}\left[LD^2\right]$ gives:

$$\mathbb{E}\left[LD^2\right] = \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{i'=1}^{n-1}\sum_{j'=1}^{n-1}\mathbb{E}\left[X_{i,j}X_{i',i'+1}X_{j',j'+1}\right].$$

There are $\binom{n}{2}(n-1)^2 = n^4/2+o(n^4)$ many terms in this summation. Further, there are $2!\binom{n-2}{2}\binom{n-4}{2} = n^4/2 + o(n^4)$ many terms such that $i, j, i', i'+1, j', j'+1$ are distinct. This can be seen by first choosing i' and j', and then choosing the pair (i, j) such that $i < j$. Therefore, there must be $o(n^4)$ terms where $i, j, i', i'+1, j', j'+1$ are *not* distinct. We have:

$$\begin{aligned}
&\sum_{\substack{i<j,i',j'\in[n]\\ i,j,i',i'+1,j',j'+1 \text{ are distinct}}} \mathbb{E}\left[X_{i,j}X_{i',i'+1}X_{j',j'+1}\right]\\
&= \sum_{\substack{i<j,i',j'\in[n]\\ i,j,i',i'+1,j',j'+1 \text{ are distinct}}} \mathbb{E}\left[X_{i,j}\right]\mathbb{E}\left[X_{i',i'+1}\right]\mathbb{E}\left[X_{j',j'+1}\right]\\
&= \left(\frac{1}{2}\right)^3 \cdot 2!\binom{n-2}{2}\binom{n-4}{2}\\
&= \left(\frac{1}{2}\right)^3 \cdot (n^4/2 + o(n^4)).
\end{aligned}$$

Therefore by (5),

$$\mathbb{E}\left[LD^2\right] = \frac{1}{16}n^4 + o(n^4). \tag{8}$$

Summarizing, we have shown that

$$\mathbb{E}\left[X^2\right] = \mathbb{E}\left[L^2\right] + \mathbb{E}\left[D^4/4\right] - \mathbb{E}\left[LD^2\right] + o(n^4).$$

Now the result follows from (6), (7), (8). □

Lemma 3.3. $\lim_{n\to\infty}\Pr\left[X \le n\right] = 0$.

Proof. The event $\{X \le n\}$ is contained in the event $\{|X - \mathbb{E}[X]| \ge t\}$ when $t = \mathbb{E}[X] - n$ because $|X - \mathbb{E}[X]| \ge t$ implies that either

(A) $X - \mathbb{E}[X] \geq t$, or
(B) $\mathbb{E}[X] - X \geq t$,

and the above choice of t causes inequality (B) to be $X \leq n$. Now, we can apply Chebyshev's Inequality to X and $t = \mathbb{E}[X] - n$ to get:

$$\begin{aligned}\Pr[X \leq n] &\leq \Pr[|X - \mathbb{E}[X]| \geq \mathbb{E}[X] - n] \\ &\leq \frac{\operatorname{Var}[X]}{(\mathbb{E}[X] - n)^2} \\ &= \frac{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}{(\mathbb{E}[X] - n)^2}.\end{aligned}$$

The result follows from the fact that, by Lemma 3.2,

$$\mathbb{E}\left[X^2\right] = \frac{n^4}{64} + o(n^4)$$

and by Lemma 3.1, both

$$(\mathbb{E}[X])^2 = \frac{n^4}{64} + o(n^4) \quad \text{and} \quad (\mathbb{E}[X] - n)^2 = \Omega(n^4). \qquad \square$$

This completes the proof of Theorem 1.1. □

Acknowledgements

We thank Mahir Can and Yibo Gao for helpful discussions. We also thank the anonymous referees for their useful suggestions.

References

[1] A. Borel, *Linear algebraic groups.* Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991. MR1102012

[2] M. Brion, *Introduction to actions of algebraic groups.* Notes of the course "Actions hamiltoniennes: invariants et classification" (CIRM, Luminy, 2009), available on-line at https://www-fourier.ujf-grenoble.fr/~mbrion/notes_luminy.pdf.

[3] W. Fulton, *Young tableaux. With applications to representation theory and geometry.* London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997. x+260 pp. MR1464693

[4] C. Gaetz, *Spherical Schubert varieties and pattern avoidance*, preprint, 2021. arXiv:2104.03264.

[5] Y. Gao, R. Hodges, and A. Yong, *Classification of Levi-spherical Schubert varieties*, preprint, 2021. arXiv:2104.10101.

[6] M. Gillespie, *Variations on a theme of Schubert calculus*. In: Barcelo H., Karaali G., Orellana R. (eds.) Recent Trends in Algebraic Combinatorics. Association for Women in Mathematics Series, vol 16. Springer, 2019. MR3969573

[7] R. Hodges and A. Yong, *Coxeter combinatorics and spherical Schubert geometry*, preprint, 2020. arXiv:2007.09238.

[8] S. L. Kleiman and D. Laksov, *Schubert calculus*. Amer. Math. Monthly 79 (1972), 1061–1082. MR0323796

[9] A. Marcus and G. Tardos, *Excluded permutation matrices and the Stanley–Wilf conjecture*. J. Combin. Theory Ser. A 107 (2004), no. 1, 153–160. MR2063960

[10] D. Zeilberger, *Symbolic moment calculus. I. Foundations and permutation pattern statistics*. Ann. Comb. 8 (2004), no. 3, 369–378. MR2161644

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Received September 10, 2021