POLYNOMIAL HULLS OF ARCS AND CURVES II

ALEXANDER J. IZZO

Abstract. We prove that if a compact set E in $\mathbb{C}^N$ is contained in an arc J, then there is a choice of J whose polynomial hull $\widehat{J}$ is $J \cup \widehat{E}$. This strengthens an earlier result of the author. We also correct an inaccuracy in the statement, and fill a gap in the proof, of that earlier result.

1. Introduction

The purpose of this paper is to strengthen results in the author's earlier paper [5] and to address a gap in that paper. Our main result shows that every compact set E that is contained in an arc in $\mathbb{C}^N$ is contained in an arc whose polynomial hull is no larger than it is obviously forced to be by virtue of containing the set E.

Theorem 1.1. *Let E be a compact set that is contained in an arc in $\mathbb{C}^N$. Then there exists an arc J in $\mathbb{C}^N$ that contains E and is such that $\widehat{J} = J \cup \widehat{E}$. Furthermore, J can be chosen to lie in an arbitrary connected neighborhood of E and such that each component of $J \setminus E$ is a C^∞-smooth open arc. In addition, J can be taken to be a simple closed curve rather than an arc provided $N \geq 2$.*

The special case when the set E is polynomially convex yields the following.

Corollary 1.2. *Each compact polynomially convex set E that is contained in an arc in $\mathbb{C}^N$ is contained in a polynomially convex arc J that can be chosen to lie in an arbitrary connected neighborhood of E. Furthermore, J can be chosen so that each component of $J \setminus E$ is a C^∞-smooth open arc. With that choice, if $P(E) = C(E)$, then $P(J) = C(J)$. In addition, J can be taken to be a simple closed curve rather than an arc provided $N \geq 2$.*

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In the statements of Theorem 1.1 and Corollary 1.2, we have tacitly assumed that the set E contains at least two points. In case E consists of a single point or is empty, the results are still true and rather trivial, except that obviously the components of $J \setminus E$ will not be open arcs.

We include the case $N = 1$ in the above results mainly for completeness. In that case the polynomial convexity assertions are trivial since every subset of an arc in the complex plane is polynomially convex. Nevertheless, the results are not trivial when $N = 1$; in fact, there are topological difficulties in that case that are not present when $N \geq 2$.

We recall here some standard terminology and notation already used above. By definition an *arc* is a space homeomorphic to the closed unit interval, and a *simple closed curve* is a space homeomorphic to the unit circle. An *open arc* is a space homeomorphic to the open unit interval. For convenience we will also use the term *arc* to refer to a topological embedding whose domain is an interval. A mapping that is referred to as a C^∞-*smooth arc* will be required not only to be injective and of class C^∞ but also to be an immersion, i.e., to have nowhere vanishing derivative. The supremum of a function f over a set S will be denoted by $\|f\|_S$. For a C^∞-smooth function f and a positive integer n, the C^n-norm of f will be denoted by $\|f\|_{C^n}$. Throughout the paper, N will be a positive integer whose value is arbitrary except where noted otherwise. We denote by m the $2N$-dimensional Lebesgue measure on $\mathbb{C}^N$. Neighborhoods will be assumed to be open. For a compact set X in $\mathbb{C}^N$, we denote by $C(X)$ the space of all continuous complex-valued functions on X and by $P(X)$ the uniform closure in $C(X)$ of the polynomials in the complex coordinate functions $z_1, \ldots, z_N$. The *polynomial hull* $\widehat{X}$ of X is defined by

$$\widehat{X} = \{z \in \mathbb{C}^N : |p(z)| \leq \max_{x \in X} |p(x)| \text{ for all polynomials } p\}.$$

The set X is said to be *polynomially convex* if $\widehat{X} = X$.

For the special case in which the set E is totally disconnected, a result along the lines of Theorem 1.1 was presented in [5, Theorem 1.3] and used there to establish the existence of arcs with certain properties. However, the proof given there seems to have a gap. It was asserted that given the existence of an arc that contains the totally disconnected, compact set E, "one can show that there is such an arc J_0 with the additional property that the closure of each component of $J_0 \setminus E$ is a C^∞-smooth arc". We will prove in the present paper that there is an arc J_0 through E such that each component of $J_0 \setminus E$ is a C^∞-smooth open arc, but the author does not know whether these open arcs can be chosen so that their closures are smooth (closed) arcs. That stronger condition was used in the proof given for [5, Theorem 1.3] in that it gave that $J_0 \setminus E$ was contained in a countable union of disjoint *compact* one-dimensional smooth manifolds-with-boundary and thus made it possible to use the stability of smooth embeddings in the C^1-topology.

The (flawed) approach used in [5] can be adapted to give a correct proof using the stability of smooth embeddings in the strong topology (also known as the fine topology or the Whitney topology). (See [4, p. 35] for the definition.) This, however, makes the details somewhat more complicated. We will instead use a different approach that uses results from [7]. Theorem 1.1 is closely related to [7, Theorem 1.1] and can, in fact, be regarded as a generalization of that result which we state here for the reader's convenience.

Theorem 1.3. *A polynomially convex arc λ in $\mathbb{C}^N$, $N \geq 2$, is contained in a polynomially convex simple closed curve γ that can be chosen to lie in an arbitrarily small neighborhood of the given arc. Furthermore, γ can be chosen such that the open arc $\gamma \setminus \lambda$ is C^∞-smooth. With this choice, if $P(\lambda) = C(\lambda)$, then $P(\gamma) = C(\gamma)$.*

Fortunately the gap in the proof of [5, Theorem 1.3] has very little effect on the applications in [5]. The proof of [5, Theorem 1.1], which gives the existence of arcs and simple closed curves in $\mathbb{C}^3$ having "hull with dense invertibles" goes through unchanged except for invoking Theorem 1.1 of the present paper in place of [5, Theorem 1.3]. Likewise [5, Theorem 1.2] can be proven as in [5] invoking Corollary 1.2 above in place of [5, Theorem 1.3]. However, we can actually obtain a stronger result, which we state here, in that we do not need the hypothesis made in [5, Theorem 1.2] that Ω is a Runge domain of holomorphy. (Note that [5, Theorem 1.3] involved a Runge domain of holomorphy but that this is not the case with Theorem 1.1 above.)

Theorem 1.4. *Let Ω be a bounded, connected open set in $\mathbb{C}^N$, let x_0 be a point of Ω, and let $\varepsilon > 0$. Then there exists a polynomially convex arc J in $\mathbb{C}^N$ such that $x_0 \in J \subset \Omega$ and $m(\Omega \setminus J) < \varepsilon$. Furthermore, J can be chosen so that $P(J) = C(J)$ and the set of polynomials zero-free on J is dense in $P(\overline{\Omega})$. The same statements hold with "arc" replaced by "simple closed curve" provided $N \geq 2$.*

The proof of Theorem 1.4 is essentially the same as that given for [5, Theorem 1.2] except for invoking Corollary 1.2 above in place of [5, Theorem 1.3] and replacing [5, Lemma 2.4] by the more general Lemma 2.5 below.

In the next section we present some preliminary results. Theorem 1.1 and Corollary 1.2 are proved in Section 3.

2. Preliminary Results

We begin with some topological lemmas concerning arcs.

Lemma 2.1. *Let λ be a closed set in $\mathbb{R}^N$, $N \geq 3$, of Lebesgue covering dimension at most 1, let a and a' be two points in λ, and let Ω be a connected open set of $\mathbb{R}^N$ that contains a and a'. Then there is an arc J from a to a' contained in Ω that intersects λ only in the end points a and a' of J and is such that the open arc $J \setminus \{a, a'\}$ is C^∞-smooth.*

Proof. The proof is essentially a repetition of the proof of the $n \geq 3$ case of [7, Theorem 1.2]. □

The above lemma becomes false with $N = 2$. There is, however, the following weaker result.

Lemma 2.2. *Let λ be an arc in $\mathbb{R}^2$, let a and a' be two points in λ, let $\lambda_{a,a'}$ be the subarc of λ from a to a', and let Ω be a neighborhood of $\lambda_{a,a'}$ in $\mathbb{R}^2$. Then there is an arc J from a to a' contained in Ω that intersects λ only in the end points a and a' of J and is such that the open arc $J \setminus \{a, a'\}$ is C^∞-smooth.*

Proof. The proof is similar to the proof of the $n = 2$ case of [7, Theorem 1.2] but requires some care, so we include the details. There is a conformal map $\varphi : \mathbb{U} \to \mathbb{C}^* \setminus \lambda$ of the open unit disc $\mathbb{U}$ onto the complement of λ in the Riemann sphere $\mathbb{C}^*$. By [2, Theorem 2.1] (which seems to be due to Marie Torhorst [9]), φ extends continuously to the closure of $\mathbb{U}$. Choose points p and p' in the boundary $\partial\mathbb{U}$ of $\mathbb{U}$ such that $\varphi(p) = a$ and $\varphi(p') = a'$ and such that on one of the open arcs α on $\partial\mathbb{U}$ determined by p and p' the function φ never takes either of the values a and a'. Then φ maps α onto the interior of the arc $\lambda_{a,a'}$. Let ℓ be an arc in $\mathbb{U} \cup \{p, p'\}$ with end points p and p', and set $J = \varphi(\ell)$. Then J is an arc from a to a' that intersects λ only in the points a and a'. By choosing ℓ to lie sufficiently near α and to be $\mathbb{C}^\infty$-smooth, we can insure that J is contained in Ω and that $J \setminus \{a, a'\}$ is C^∞-smooth. □

Lemma 2.3. *Let Ω be a connected open set in $\mathbb{R}^2$, and let $J_1, \ldots, J_n$ be finitely many disjoint arcs in Ω. Then $\Omega \setminus (J_1 \cup \cdots \cup J_n)$ is connected.*

Proof. We merely sketch the proof, leaving the details to the reader. By induction it suffices to consider the case when there is just one arc J. For that case, first show that Ω contains a connected neighborhood V of J such that the complement of V in $\mathbb{R}^2$ is connected. Since V is then homeomorphic to the plane, it is a standard fact that $V \setminus J$ is connected. Connectedness of $\Omega \setminus J$ follows. □

Lemma 2.4. *Let E be a compact set that is contained in an arc in $\mathbb{R}^N$. Then every connected neighborhood of E contains an arc J that contains E and has the additional property that each component of $J\setminus E$ is a C^∞-smooth open arc. In addition, J can be taken to be a simple closed curve rather than an arc provided $N \geq 2$.*

As in the statements of Theorem 1.1 and Corollary 1.2, we tacitly assume in the above lemma that the set E contains at least two points.

Proof. It is sufficient to construct the desired arc; the existence of the desired simple closed curve then follows immediately from [7, Theorem 1.2]. The case $N = 1$ is trivial. We first treat the case $N \geq 3$; the case $N = 2$ requires a more involved argument.

Let Ω be a connected neighborhood of E. Let $\sigma : [0, 1] \to \mathbb{R}^N$ be an arc through E, and assume without loss of generality that the end points of σ are in E. Set $L = \sigma^{-1}(E)$. The set $[0, 1] \setminus L$ is an at most countable union of disjoint open intervals $(a_1, b_1), (a_2, b_2), \ldots$. Note that the diameters $\mathrm{diam}\big(\sigma([a_j, b_j])\big)$ go to zero as $j \to \infty$ (if there are infinitely many intervals (a_j, b_j)). In particular, $\sigma([a_j, b_j])$ is contained in Ω for all but finitely many j. For each j such that $\sigma([a_j, b_j])$ is contained in Ω, choose a connected neighborhood Ω_j of $\sigma([a_j, b_j])$ contained in Ω and of diameter no more than twice the diameter of $\sigma([a_j, b_j])$. For j such that $\sigma([a_j, b_j])$ is not contained in Ω, set $\Omega_j = \Omega$. Note that then $\mathrm{diam}(\Omega_j) \to 0$ as $j \to \infty$.

By Lemma 2.1 there is an arc $\sigma_1 : [a_1, b_1] \to \Omega_1$ from $\sigma(a_1)$ to $\sigma(b_1)$ that is C^∞-smooth except possibly at its end points and that intersects E only in its end points $\sigma(a_1)$ and $\sigma(b_1)$. Continuing inductively, we can choose, for each $j = 2, 3, \ldots$, an arc $\sigma_j : [a_j, b_j] \to \Omega_j$ from $\sigma(a_j)$ to $\sigma(b_j)$ that is C^∞-smooth except possibly at its end points and that intersects $E \cup \sigma_1([a_1, b_1]) \cup \cdots \cup \sigma_{j-1}([a_{j-1}, b_{j-1}])$ only in its end points $\sigma(a_j)$ and $\sigma(b_j)$. Now defining $\tau : [0, 1] \to \Omega$ to coincide with σ on L and to coincide with σ_j on $[a_j, b_j]$ for each $j = 1, 2, \ldots$ yields the desired arc. (Continuity of τ is a consequence of the conditions that $\mathrm{diam}\big(\sigma_j([a_j, b_j])\big) \leq \mathrm{diam}(\Omega_j)$ and $\mathrm{diam}(\Omega_j) \to 0$ as $j \to \infty$.) This concludes the proof in the case $N \geq 3$.

We now consider the case $N = 2$, which we will establish in two steps. First we will obtain an arc through E that is contained in Ω and for which no smoothness is asserted, and subsequently we will obtain the arc whose existence is asserted in the statement of the lemma.

Let Ω, σ, and L be as before. Since L is a compact set contained in the (relatively) open set $\sigma^{-1}(\Omega)$ of $[0, 1]$, the set L is contained in a finite union of intervals that are open in $[0, 1]$ and are contained in

$\sigma^{-1}(\Omega)$. Consequently, we can choose points

$$0 = c_0 < d_0 < c_1 < d_1 < \cdots < c_n < d_n = 1$$

such that

$$\begin{aligned} E &\subset \sigma\Big([c_0, d_0) \cup (c_1, d_1) \cup \cdots \cup (c_{n-1}, d_{n-1}) \cup (c_n, d_n]\Big) \\ &\subset \sigma\Big([c_0, d_0] \cup [c_1, d_1] \cup \cdots \cup [c_{n-1}, d_{n-1}] \cup [c_n, d_n]\Big) \subset \Omega. \end{aligned}$$

By modifying σ near each of the points $d_0, \ldots, d_{n-1}$ and $c_1, \ldots, c_n$, we may assume that there are open Euclidean balls $B_{d_0}, \ldots, B_{d_{n-1}}$ and $B_{c_1}, \ldots, B_{c_n}$ centered at $\sigma(d_0), \ldots, \sigma(d_{n-1})$ and $\sigma(c_1), \ldots, \sigma(c_n)$, respectively, whose closures are disjoint and lie in Ω such that the intersection of $\sigma([0, 1])$ with each of these balls is a straight line segment. Choose, from $\Omega \setminus \sigma\big([c_0, d_0] \cup \cdots \cup [c_n, d_n]\big)$, points $q'_0, \ldots, q'_{n-1}$ and $p'_1, \ldots, p'_n$ in $B_{d_0}, \ldots, B_{d_{n-1}}$ and $B_{c_1}, \ldots, B_{c_n}$, respectively. The set $\Omega \setminus \sigma\big([c_0, d_0] \cup \cdots \cup [c_n, d_n]\big)$ is connected by Lemma 2.3, so there is an arc from q'_0 to p'_1 in $\Omega \setminus \sigma\big([c_0, d_0] \cup \cdots \cup [c_n, d_n]\big)$. By discarding initial and final segments of this arc, we can obtain an arc λ in $\Omega \setminus \big[\sigma\big([c_0, d_0] \cup \cdots \cup [c_n, d_n]\big) \cup B_{d_0} \cup B_{c_1}\big]$ whose end points $\tilde{q}$ and $\tilde{p}$ lie on the boundary of B_{d_0} and B_{c_1}, respectively. Let $L_{\tilde{q}}$ and $L_{\tilde{p}}$ be the straight line segments from $\sigma(d_0)$ to $\tilde{q}$ and from $\sigma(c_1)$ to $\tilde{p}$, respectively. Set $\ell_0 = L_{\tilde{q}} \cup \lambda \cup L_{\tilde{p}}$. Then ℓ_0 is an arc in Ω that intersects $\sigma\big([c_0, d_0] \cup \cdots \cup [c_n, d_n]\big)$ only in the end points $\sigma(d_0)$ and $\sigma(c_1)$ of ℓ_0.

Continuing inductively we can similarly choose, for each $j = 1, \ldots, n-1$, an arc ℓ_j in Ω from $\sigma(d_j)$ to $\sigma(c_{j+1})$ that, aside from its end points, is disjoint from $\sigma\big([c_0, d_0] \cup \cdots \cup [c_n, d_n]\big)$ and from each of the previously chosen arcs $\ell_1, \ldots, \ell_{j-1}$. Then $\sigma\big([c_0, d_0] \cup \cdots \cup [c_n, d_n]\big) \cup \ell_0 \cup \cdots \cup \ell_{n-1}$ is an arc in Ω that contains E.

The passage from the arc just obtained to the desired arc J is similar to the proof of the lemma in the case $N \geq 3$ but with Lemma 2.1 replaced by Lemma 2.2, so we will compress the details. Let $\tilde{\sigma} : [0, 1] \to \Omega$ be a parametrization of the arc just obtained satisfying $\tilde{\sigma}|L = \sigma|L$. With $[0, 1] \setminus L = (a_1, b_1) \cup (a_2, b_2) \cup \cdots$ as in the case $N \geq 3$, we choose, for each $j = 1, 2, \ldots$, a connected neighborhood Ω_j of $\tilde{\sigma}([a_j, b_j])$ contained in Ω in such a way that $\operatorname{diam}(\Omega_j) \to 0$ as $j \to \infty$. By Lemma 2.2 there is an arc $\sigma_1 : [a_1, b_1] \to \Omega_1$ from $\tilde{\sigma}(a_1)$ to $\tilde{\sigma}(b_1)$ that is C^∞-smooth except possibly at its end points and that intersects $\tilde{\sigma}([0, a_1] \cup [b_1, 1])$ only in its end points $\tilde{\sigma}(a_1)$ and $\tilde{\sigma}(b_1)$. Define $\tau_1 : [0, 1] \to \Omega$ to coincide with $\tilde{\sigma}$ on $[0, a_1] \cup [b_1, 1]$ and to coincide with σ_1 on $[a_1, b_1]$. Then τ_1 is an arc. Continuing inductively, we can obtain, for each $j = 2, 3, \ldots$, an arc $\sigma_j : [a_j, b_j] \to \Omega_j$ from $\tilde{\sigma}(a_j)$ to $\tilde{\sigma}(b_j)$ and an arc τ_j so that σ_j is C^∞-smooth except possibly at its end points and intersects $\tau_{j-1}([0, a_j] \cup [b_j, 1])$ only in its end points $\tilde{\sigma}(a_j)$ and $\tilde{\sigma}(b_j)$, and τ_j coincides with τ_{j-1} on $[0, a_j] \cup [b_j, 1]$ and coincides with σ_j on $[a_j, b_j]$. The sequence (τ_n) converges uniformly, and its limit is the desired arc. □

Next we present three results we will need concerning polynomial convexity. The first of these is an almost immediate consequence of [6, Lemma 3.2], and as mentioned in the introduction generalizes [5, Lemma 2.4].

Lemma 2.5. *Let Y be a compact set in $\mathbb{C}^N$, let x_0 be a point of Y, and let $\varepsilon > 0$. Let $\{p_j\}$ be a countable collection of polynomials on $\mathbb{C}^N$ such that $p_j(x_0) \neq 0$ for all j. Then there exists a totally disconnected, compact polynomially convex set E with $x_0 \in E \subset Y$ such that*
(i) *each p_j is zero-free on E*
(ii) *$m(Y \setminus E) < \varepsilon$.*

Proof. By making a complex affine change of coordinates, we may assume without loss of generality that $x_0 = 0$ and that Y is contained in the open unit ball B of $\mathbb{C}^N$. Then [6, Lemma 3.2] gives a totally disconnected, compact polynomially convex set K with $0 \in K \subset B$ such that
(i) each p_j is zero-free on K
(ii) $m(B \setminus K) < \varepsilon$.
Let $E = K \cap Y$. Then E is compact and totally disconnected. Also $x_0 = 0 \in E \subset Y$, each p_j is zero-free on E, and $m(Y \setminus E) \leq m(B \setminus K) < \varepsilon$. Because K is a totally disconnected, compact polynomially convex set, it follows from the Shilov idempotent theorem that $P(K) = C(K)$ (see [1, p. 48, Corollary 3] for instance), and hence, $P(E) = C(E)$, so E is polynomially convex. □

The following result, a special case of [7, Theorem 1.3], will play a key role in the proof of Theorem 1.1.

Theorem 2.6. *Let $Y \subset \mathbb{C}^N$ be a compact polynomially convex set, and let Γ be a rectifiable arc both of whose end points lie in Y but that is otherwise disjoint from Y. If a nonempty open subarc of Γ is contained in a purely one-dimensional analytic subvariety V of $\mathbb{C}^N$ but Γ is not entirely contained in V, then $Y \cup \Gamma$ is polynomially convex.*

The next result, which is [7, Theorem 1.7], will be used in the proof of Corollary 1.2.

Theorem 2.7. *Let λ be an arc in $\mathbb{C}^N$, and let E be a compact subset of λ that is polynomially convex. Suppose that $\lambda \setminus E$ is locally rectifiable.*

Then λ is polynomially convex. Furthermore, if $P(E) = C(E)$, then $P(\lambda) = C(\lambda)$.

Finally we will need the following technical lemma.

Lemma 2.8. *Let $\sigma : [a,b] \to \mathbb{C}^N$, $N \geq 2$, be a C^∞-smooth arc. Fix a positive integer n, an $\varepsilon > 0$, and an open interval U contained in $[a,b]$. Then there exists a C^∞-smooth arc $\tau : [a,b] \to \mathbb{C}^N$ and a purely one-dimensional analytic subvariety V of $\mathbb{C}^N$ such that*
(i) *some nonempty open subarc of $\tau([a,b])$ is contained in V but $\tau([a,b])$ is not entirely contained in V,*
(ii) $\|\tau - \sigma\|_{C^n} < \varepsilon$, *and*
(iii) *τ coincides with σ except in U.*

The proof of Lemma 2.8 will use the following real variable lemma whose proof we also include.

Lemma 2.9. *Let $[a,b] \subset \mathbb{R}$ be an interval whose interior contains 0, let n be a positive integer, and let $f : [a,b] \to \mathbb{R}^N$ be a C^∞-smooth map such that $f(0) = f'(0) = \cdots = f^{(n)}(0) = 0$. Fix $\varepsilon > 0$, and fix a neighborhood U of 0 in (a,b). Then there exists a C^∞-smooth map $g : [a,b] \to \mathbb{R}^N$ such that g coincides with f on a neighborhood of 0, such that $\|g\|_{C^n} < \varepsilon$, and such that the support of g is contained in U.*

Proof. It suffices to consider the case $N = 1$. Taylor's theorem shows that there is a constant C_1 (depending only on $\max\{-a, b\}$ and the particular formula one uses for the C^n-norm) such that every C^∞-smooth function u on $[a,b]$ satisfying $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$ also satisfies the inequality

$$\|u\|_{C^n} \leq C_1 \|u^{(n)}\|_{[a,b]}.$$

Choose a C^∞-smooth function $\psi : [a,b] \to \mathbb{R}$ such that $0 \leq \psi \leq 1$ everywhere, ψ is identically 1 on a neighborhood of 0, and the support of ψ is contained in U. Then it follows from Leibniz' formula that there exists a constant C_2 such that every C^∞-smooth function u on $[a,b]$ satisfies the inequality

$$\|\psi u\|_{C^n} \leq C_2 \|u\|_{C^n}.$$

Choose $\delta > 0$ small enough that

$$\|f^{(n)}\|_{[-\delta,\delta]} < \varepsilon / 2C_1C_2.$$

Choose a C^∞-smooth function $\phi : [a,b] \to \mathbb{R}$ such that $0 \leq \phi \leq 1$ everywhere, ϕ is identically 1 on $[-\delta, \delta]$, and the support of ϕ lies in a sufficiently small neighborhood of $[-\delta, \delta]$ that

$$\|\phi f^{(n)}\|_{[a,b]} \leq 2\|f^{(n)}\|_{[-\delta,\delta]}.$$

Let $h : [a, b] \to \mathbb{R}$ be the function satisfying $h^{(n)} = \phi f^{(n)}$ and $h(0) = h'(0) = \cdots = h^{(n-1)}(0) = 0$. Set $g = \psi h$. Then g satisfies all the conditions asserted in the statement of the lemma. □

Proof of Lemma 2.8. Without loss of generality assume that 0 is in U. Choose a polynomial map $F : \mathbb{C} \to \mathbb{C}^N$ such that for each component of F the Taylor coefficients at 0 match those of the corresponding component of σ up to order n but differ for order $n+1$. Set $V = F(\mathbb{C})$. Then V is a purely one-dimensional analytic subvariety of $\mathbb{C}^N$ (as a special case of Remmert's proper mapping theorem [3, Theorem N1]). Note that $\sigma(0)$ is in V, but there exists a point t_1 in (a, b) such that the point $\sigma(t_1)$ is not in V. Define $f : [a, b] \to \mathbb{C}^N$ by $f(t) = F(t) - \sigma(t)$. By the stability of smooth embeddings in the C^1-topology, there exists $\varepsilon' > 0$ such that every C^∞-smooth map $\tilde{\sigma} : [a, b] \to \mathbb{C}^N$ satisfying $\|\tilde{\sigma} - \sigma\|_{C^n} < \varepsilon'$ is a C^∞-smooth arc. By Lemma 2.9 there exists a C^∞-smooth map $g : [a, b] \to \mathbb{C}^N$ such that g coincides with f on a neighborhood of 0, such that $\|g\|_{C^n} < \min\{\varepsilon, \varepsilon'\}$, and such that the support of g is contained in U and does not contain the point t_1. Setting $\tau = \sigma + g$ yields the lemma. □

3. Proofs of Theorem 1.1 and Corollary 1.2

Corollary 1.2 is a special case of Theorem 1.1, but we give an independent proof since the corollary is much more easily established than the general theorem.

Proof of Corollary 1.2. Let Ω be an arbitrary connected neighborhood of E. Let J be the arc in Ω given by Lemma 2.4. Then Theorem 2.7 gives that J is polynomially convex and that $P(J) = C(J)$ if $P(E) = C(E)$. The statement about a simple closed curve then follows from Theorem 1.3. □

Proof of Theorem 1.1. The case $N = 1$ follows immediately from Lemma 2.4 since every arc in the complex plane is polynomially convex. From now on we assume that $N \geq 2$. We treat only the construction of the arc. The simple closed curve can be constructed similarly.

Let Ω be an arbitrary connected neighborhood of E. Lemma 2.4 yields the existence of an arc $\sigma_0 : [0, 1] \to \Omega$ such that $\sigma_0(\{0, 1\}) \subset E \subset \sigma_0([0, 1])$ and such that the restriction $\sigma_0|_{([0,1]\setminus\sigma_0^{-1}(E))}$ is a C^∞-embedding. The proof will be complete once we establish that there is an arc $\sigma : [0, 1] \to \Omega$ that has the properties just listed for σ_0 and has the additional property that setting $J = \sigma([0, 1])$ we have that $\widehat{J} = J \cup \widehat{E}$. We will obtain σ as a limit of a sequence of arcs $\sigma_n : [0, 1] \to$

Ω that will be constructed inductively. We will also simultaneously construct a sequence (K_n) of compact polynomially convex sets.

Let K_0 be a closed ball in $\mathbb{C}^N$ whose interior contains $\sigma_0([0,1])$. Choose a sequence (U_n) of neighborhoods of $\widehat{E}$ with $U_0 = \mathbb{C}^N$ and $\bigcap_{n=0}^{\infty} U_n = \widehat{E}$. Set $L = \sigma_0^{-1}(E)$. Roughly, the σ_n will be constructed by succesively changing σ_0 at most once on each member of a collection of disjoint open intervals contained in $[0,1] \setminus L$.

The sequence of arcs (σ_n) and the sequence of compact polynomially convex sets (K_n) will be chosen so that the following conditions hold for all $n = 1, 2, \ldots$.

(i) $\widehat{E} \subset \mathrm{Int}(K_n) \subset K_n \subset \mathrm{Int}(K_{n-1}) \cap U_n$.

(ii) K_n has C^∞-smooth boundary ∂K_n.

(iii) $\sigma_n|_L = \sigma_0|_L$.

(iv) The restriction of σ_n to $[0,1] \setminus L$ is a C^∞-immersion.

(v) $\|\sigma_n - \sigma_{n-1}\|_{[0,1]} < 1/2^n$.

(vi) $\|\sigma_n|_{([0,1]\setminus L)} - \sigma_{n-1}|_{([0,1]\setminus L)}\|_{C^n} < 1/2^n$.

(vii) σ_{n-1} is transverse to ∂K_n (i.e., at each point where $\sigma_{n-1}([0,1])$ and ∂K_n intersect, the real linear span of their tangent spaces is $\mathbb{C}^N$).

(viii) $\sigma_n([0,1]) \subset K_{n-1} \cup \sigma_{n-1}([0,1])$.

(ix) Condition (vii) implies that $\sigma_{n-1}^{-1}(\mathbb{C}^N \setminus K_n)$ is a finite union of open intervals $(a_1, b_1), \ldots, (a_u, b_u)$. The map σ_n coincides with σ_{n-1} everywhere on $[0,1]$ except on those intervals (a_j, b_j) such that $\sigma_{n-1}\big((a_j, b_j)\big)$ is contained entirely in K_{n-1}. Also $\sigma_n\big((a_1, b_1) \cup \cdots \cup (a_u, b_u)\big)$ is disjoint from K_n.

(x) For each component λ of $\sigma_n([0,1]) \setminus K_n$ there is a purely one-dimensional analytic subvariety of $\mathbb{C}^N$ that contains a nonempty open subarc of λ but does not contain all of λ.

Before constructing the sequences (σ_n) and (K_n) we show that their existence will yield the theorem. First we note a special property that the sequence (σ_n) has as a consequence of conditions (vii) and (ix). Given a point t_0 in $[0,1]$, if $\sigma_0(t_0)$ is not in $\widehat{E} = \bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} K_n = \bigcap_{n=0}^{\infty} \mathrm{Int}(K_n)$, then there is a smallest integer $s \geq 1$ such that $\sigma_0(t_0)$ is not in $\mathrm{Int}(K_s)$. In this case, there is an open interval $(\alpha, \beta) \subset [0,1]$ containing t_0 such that σ_{s-1} and σ_s may differ on (α, β), but $\sigma_0, \ldots, \sigma_{s-1}$ coincide on (α, β), and σ_n coincides with σ_s on (α, β) for all $n \geq s$. If instead $\sigma_0(t_0)$ is in $\widehat{E}$, then for each integer $n \geq 1$ there is a neighborhood of t_0 in $[0,1]$ on which σ_n coincides with σ_0.

Condition (v) implies that the sequence (σ_n) converges uniformly to a continuous map $\sigma : [0,1] \to \Omega$, and condition (vi) implies that $\sigma|_{([0,1]\setminus L)}$ is C^∞-smooth. Condition (iii) implies that $\sigma|L = \sigma_0|L$. In combination with condition (iv), the special property of the sequence (σ_n) observed in the preceding paragraph yields that $\sigma|_{([0,1]\setminus L)}$ is an immersion. Furthermore, σ is injective because given $t_1 \neq t_2$ in $[0,1]$ there exists an s such that $\sigma(t_1) = \sigma_s(t_1) \neq \sigma_s(t_2) = \sigma(t_2)$. Thus σ, or more precisely $J = \sigma([0,1])$, is an arc, and each component of $J \setminus E$ is a C^∞-smooth open arc.

Recall that $\sigma|L = \sigma_0|L$, and hence $J \supset \sigma_0(L) = E$. Consequently, $\widehat{J} \supset J \cup \widehat{E}$. To establish the reverse inclusion it suffices to show that $J \cup \widehat{E}$ is polynomially convex.

By conditions (ix) and (x), there are finitely many disjoint arcs $\lambda_1, \ldots, \lambda_u$ such that $K_n \cup \sigma_n([0,1]) = K_n \cup \lambda_1 \cup \cdots \cup \lambda_u$ where each λ_j intersects K_n precisely in its two end points and is such that there is a purely one-dimensional analytic subvariety of $\mathbb{C}^N$ that contains a nonempty open subarc of λ_j but does not contain all of λ_j. Consequently, repeated application of Theorem 2.6 shows that $K_n \cup \sigma_n([0,1])$ is polynomially convex. We will show that

$$\bigcap_{n=0}^{\infty} \Big[K_n \cup \sigma_n([0,1]) \Big] = J \cup \widehat{E} \tag{1}$$

thereby establishing the polynomial convexity of $J \cup \widehat{E}$. Clearly $\widehat{E}$ is contained in the left hand side of equation (1) by condition (i). For each point x_0 in J there is an s such that x_0 is in $\sigma_n([0,1])$ for all $n \geq s$. Since conditions (i) and (viii) show that the sequence of sets $\big(K_n \cup \sigma_n([0,1])\big)$ is decreasing, this yields that also J is contained in the left hand side of equation (1). Thus the left hand side of equation (1) contains the right hand side. For the reverse inclusion note that for a point x_0 in the left hand side of equation (1) that does not lie in $\widehat{E}$ there is an s such that x_0 does not lie in K_s; then x_0 is in $\sigma_n([0,1])$ for all $n \geq s$, and thus x_0 must be in J. This concludes the verification that the existence of the sequences (σ_n) and (K_n) will yield the theorem.

It remains to construct the sequences (σ_n) and (K_n). We already have σ_0 and K_0. We proceed by induction. Suppose for some $k \geq 0$ we have chosen $\sigma_0, \ldots, \sigma_k$ and $K_0, \ldots, K_k$ so that conditions (i)–(x) are satisfied for all $n = 1, \ldots, k$.

By well-known theorems in several complex variables regarding the existence of plurisubharmonic exhaustion functions and the equality of polynomial hulls and plurisubharmonic hulls (see for instance [8, Theorems II.5.11 and VI.1.18]) there exists a C^∞ strictly plurisubharmonic exhaustion function φ on $\mathbb{C}^N$ such that

$$\varphi(z) > 0 \text{ for } z \in \mathbb{C}^N \setminus (\mathrm{Int}(K_k) \cap U_{k+1})$$

and

$$\varphi(z) < 0 \text{ for } z \in E.$$

Set $K^r = \{z \in \mathbb{C}^N : \varphi(z) \leq r\}$. Set $M = \max_{z\in E} \varphi(z)$. Then for $M < r < 0$ the set K^r is a compact polynomially convex set such that $\widehat{E} \subset \mathrm{Int}(K^r) \subset K^r \subset \mathrm{Int}(K_k) \cap U_{k+1}$. Furthermore, since for these values of r the boundary of K^r is disjoint from E, we can choose, by Sard's theorem, a value r' satisfying $M < r' < 0$ such that $K^{r'}$ has C^∞-smooth boundary $\partial K^{r'}$ and σ_k is transverse to $\partial K^{r'}$. Set $K_{k+1} = K^{r'}$.

Because σ_k is transverse to ∂K_{k+1}, the set $\sigma_k^{-1}(\mathbb{C}^N \setminus K_{k+1})$ is a finite union of disjoint open intervals. Of those open intervals, let $(a_1, b_1), \ldots, (a_w, b_w)$ denote those whose image under σ_k is entirely contained in K_k. For each $j = 1, \ldots, w$, choose a point x_j in $\sigma_k\big((a_j, b_j)\big) \cap \mathrm{Int}(K_k)$. Then choose disjoint closed balls $\overline{B}_1, \ldots, \overline{B}_w$ centered at $x_1, \ldots, x_w$, respectively, with radii strictly less than $1/2^{k+1}$ and small enough that each $\overline{B}_j$ is contained in K_k and the intersection of each $\overline{B}_j$ with $K_{k+1} \cup \sigma_k([0, 1])$ is contained in $\sigma_k\big((a_j, b_j)\big)$. By applying Lemma 2.8 choose, for each $j = 1, \ldots, w$, a C^∞-smooth arc $\tau_j : [a_j, b_j] \to \Omega$ such that τ_j coincides with $\sigma_k|_{[a_j,b_j]}$ except on some interval that is mapped into $\overline{B}_j$ by both σ_k and τ_j, such that there is a purely one-dimensional analytic subvariety of $\mathbb{C}^N$ that contains a nonempty open subarc of $\tau_j([a_j, b_j])$ but does not contain all of $\tau_j([a_j, b_j])$, and such that $\|\tau_j - \sigma_k|_{[a_j,b_j]}\|_{C^{k+1}} < 1/2^{k+1}$. Finally, define $\sigma_{k+1} : [0, 1] \to \Omega$ to coincide with σ_k on $[0, 1] \setminus \big((a_1, b_1) \cup \cdots \cup (a_w, b_w)\big)$ and to coincide with τ_j on $[a_j, b_j]$ for each $j = 1, \ldots, w$. Then σ_{k+1} is an arc. Furthermore, conditions (i)–(x) hold for all $n = 1, \ldots, k + 1$. This completes the induction and the proof. □

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Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403

Email address: `aizzo@bgsu.edu`