APPROXIMATION BY AN ALGEBRA GENERATED BY HOLOMORPHIC AND CONJUGATE HOLOMORPHIC FUNCTIONS

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ABSTRACT. Using methods from the theory of uniform algebras, we give a simple proof of an approximation result of Sahutoğlu and Tikaradze with L^∞-pseudoconvex domains replaced by the open sets for which Gleason's problem is solvable.

1. THE RESULTS

In [ST19] Sönmez Sahutoğlu and Akaki Tikaradze proved, on what they referred to as L^∞-pseudoconvex domains, an approximation result that can be viewed as a several complex variables generalization of a weak form of an earlier approximation result in one complex variable due to Christopher Bishop[1] [Bis89]. They used their approximation result to give a generalization to several complex variables of a theorem of Sheldon Axler, Željko Čučkovič, and Nagisetti Rao regarding commuting Toeplitz operators [AČR00]. The main purpose of the present paper is to give a simple proof of the approximation result of Sahutoğlu and Tikaradze, under a different hypothesis on the underlying domain, using methods from the theory of uniform algebras.

We introduce here some notation and terminology we will use. Throughout the paper, Ω will be an open set in $\mathbb{C}^n$ or in the Riemann sphere. The boundary of Ω will be denoted by $b\Omega$. Following [ST19], given a holomorphic map $f : \Omega \to \mathbb{C}^m$ we will denote by $\Omega_{f,\lambda}$ the set of all nonisolated points of $f^{-1}(\lambda)$ and we set $\Omega_f = \bigcup_{\lambda \in \mathbb{C}^m} \Omega_{f,\lambda}$. For a compact space X, we denote by $C(X)$ the algebra of all continuous complex-valued functions on X. A *uniform algebra* on X is a supremum norm closed subalgebra of $C(X)$ that contains the constant functions and separates the points of X. In particular, a uniform algebra is a commutative Banach algebra. We will denote the maximal ideal space of a commutative Banach algebra A by $\mathfrak{M}_A$. Given $x \in A$ we will denote the Gelfand transform of x as usual by $\widehat{x}$. If A is a Banach algebra of continuous complex-valued functions on a subset of $\mathbb{C}^n$ and the complex coordinate functions $z_1, \ldots, z_n$ belong to A, we will let $\pi_A : \mathfrak{M}_A \to \mathbb{C}^n$ denote the map given

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[1] Christopher Bishop should not be confused with Errett Bishop after whom the antisymmetric decomposition, which will appear later in our paper, is named.

by $\pi_A(x) = \big(\widehat{z}_1(x), \ldots, \widehat{z}_n(x)\big)$. As usual $H^\infty(\Omega)$ will denote the algebra of bounded holomorphic functions on Ω equipped with the supremum norm. If A is an algebra of bounded continuous complex-valued functions on Ω and $f_1, \ldots, f_m$ are bounded continuous complex-valued functions on Ω, we will denote by $A[f_1, \ldots, f_m]$ the norm closed subalgebra of $L^\infty(\Omega)$ generated by A and the functions $f_1, \ldots, f_m$. This last notation, which is rather standard, differs from the notation in [ST19] in that in [ST19] the notation $A[f_1, \ldots, f_m]$ is used to denote the algebra generated by A and $f_1, \ldots, f_m$ without taking closure.

In the terminology of Sahutoğlu and Tikaradze, an L^∞-pseudoconvex domain is a pseudoconvex domain on which the $\overline{\partial}$-problem is solvable in L^∞. (See [ST19] for the precise definition.) The approximation theorem of Sahutoğlu and Tikaradze referred to above is the following.

Theorem 1 ([ST19], Theorem 1)**.** *Let Ω be a bounded L^∞-pseudoconvex domain in $\mathbb{C}^n$ and let $f_j \in H^\infty(\Omega)$ for $j = 1, \ldots, m$. Set $f = (f_1, \ldots, f_m)$. Suppose that $g \in C(\overline{\Omega})$ satisfies $g|_{b\Omega \cup \Omega_f} = 0$. Then g is in $H^\infty(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$.*

This theorem can be regarded as a partial extension to several variables of an approximation theorem of Christopher Bishop.

Theorem 2 ([Bis89], Theorem 1.2)**.** *Suppose that Ω is an open set in the Riemann sphere and that $f \in H^\infty(\Omega)$ is nonconstant on each component of Ω. Then $C(\overline{\Omega}) \subset H^\infty(\Omega)[\overline{f}]$.*

Sahutoğlu and Tikaradze's proof of Theorem 1 was inspired by Bishop's proof of Theorem 2, and like Bishop's proof, it is rather long and complicated. A simpler proof of Bishop's theorem was given by the second author of the present paper in [Izz93] using uniform algebra methods. Here we will use uniform algebra methods to give a simple proof of Theorem 1 with the hypothesis that Ω is an L^∞-pseudoconvex domain replaced by the hypothesis that Ω is open when regarded as a subset of the maximal ideal space $\mathfrak{M}_{H^\infty(\Omega)}$ of $H^\infty(\Omega)$. (We regard Ω as a subset of $\mathfrak{M}_{H^\infty(\Omega)}$ by identifying each point λ in Ω with the functional "evaluation at λ".) We state the result explicitly here.

Theorem 3. *Let Ω be a bounded open set in $\mathbb{C}^n$ such that Ω is open in $\mathfrak{M}_{H^\infty(\Omega)}$, and let $f_j \in H^\infty(\Omega)$ for $j = 1, \ldots, m$. Set $f = (f_1, \ldots, f_m)$. Suppose that $g \in C(\overline{\Omega})$ satisfies $g|_{b\Omega \cup \Omega_f} = 0$. Then g is in $H^\infty(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$.*

By exactly the same method we will also establish the analogous assertion for the algebra $A(\Omega)$ of continuous complex-valued functions on $\overline{\Omega}$ that are holomorphic on Ω.

Theorem 4. *Let Ω be a bounded open set in $\mathbb{C}^n$ such that Ω is open in $\mathfrak{M}_{A(\Omega)}$, and let $f_j \in A(\Omega)$ for $j = 1, \ldots, m$. Set $f = (f_1, \ldots, f_m)$. Suppose that $g \in C(\overline{\Omega})$ satisfies $g|_{b\Omega \cup \Omega_f} = 0$. Then g is in $A(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$.*

We will see also that a similar argument in combination with a result in the second author's paper [Izz03] yields yet another proof of Theorem 2.

The class of domains Ω for which Ω is open in $\mathfrak{M}_{H^\infty(\Omega)}$ (or $\mathfrak{M}_{A(\Omega)}$) is quite broad. To see this, note that for A a Banach algebra of continuous complex-valued functions on Ω containing the functions $z_1, \ldots, z_n$, the set Ω is open in $\mathfrak{M}_A$ whenever π_A is injective over Ω, since in that case Ω (regarded as a subset of $\mathfrak{M}_A$) coincides with $\pi_A^{-1}(\Omega)$. Furthermore this injectivity over Ω obviously holds whenever Gleason's problem is solvable for A, i.e., whenever, for every point $a = (a_1, \ldots, a_n) \in \Omega$, the functions $z_1 - a_1, \ldots, z_n - a_n$ generate the ideal of functions in A vanishing at a. Gleason's problem has been extensively studied and is known to be solvable for $H^\infty(\Omega)$ and $A(\Omega)$ on many classes of domains. (See for instance, [AS79] for the case of strongly pseudoconvex domains, or [Rud08] for the particular case of the ball.)

It is not evident what relationship there is between our condition that Ω be open in $\mathfrak{M}_{H^\infty(\Omega)}$ (or $\mathfrak{M}_{A(\Omega)}$) and the condition of Sahutoğlu and Tikaradze that Ω be an L^∞-pseudoconvex domain, i.e., it is not obvious whether either of these conditions implies the other. This issue may be addressed in a future paper.

Theorems 1, 3, and 4 can be reformulated using the notion of essential set. For a uniform algebra A on a compact space X, the *essential set* $\mathscr{E}$ for A is the smallest closed subset of X such that A contains every continuous complex-valued function on X that vanishes on $\mathscr{E}$. The existence of the essential set was proved by Herbert Bear [Bea59] (or see [Bro69, Section 2–8]). Theorem 4 asserts that under the given hypotheses on Ω and f, the essential set for $A(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$ is contained in $b\Omega \cup \overline{\Omega}_f$. The conclusion of Theorems 1 and 3 can be reformulated as the assertion that the essential set for $H^\infty(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$ regarded as a uniform algebra on its maximal ideal space is contained in $(\mathfrak{M}_{H^\infty(\Omega)} \setminus \Omega) \cup \overline{\Omega}_f$.

One reason for interest in the above theorems stems from an application to Toeplitz operators given by Sahutoğlu and Tikaradze. Let $L^2_a(\Omega)$ denote the Bergman space, i.e., the space of square integrable, holomorphic functions on Ω, and let $P : L^2(\Omega) \to L^2_a(\Omega)$ denote the Bergman projection, i.e., the orthogonal projection of $L^2(\Omega)$ onto $L^2_a(\Omega)$. For $\phi \in L^\infty(\Omega)$ the Toeplitz operator $T_\phi : L^2_a(\Omega) \to L^2_a(\Omega)$ is defined by the equation $T_\phi(f) = P(\phi f)$. The commuting Toeplitz operator problem is to characterize those functions $\phi, \psi \in L^\infty(\Omega)$ such that T_ϕ and T_ϕ commute. With the Hardy space in place of the Bergman space, the commuting Toeplitz operator problem was solved by Arlen Brown and Paul Halmos in [BH64]. On the Bergman space, the problem is still open even on the disk. There are, however, various partial solutions including the following result due to Axler, Čučković, and Rao [AČR00].

Theorem 5 ([AČR00]). *Let Ω be a domain in the complex plane, let ϕ be a nonconstant bounded holomorphic function on Ω, and let ψ is a bounded measurable function on Ω such that T_ϕ and T_ψ commute. Then ψ is holomorphic.*

Axler, Cučkovič, and Rao obtained this theorem as a consequence of Theorem 2 of Bishop. Sahutoğlu and Tikaradze used their partial extension of Bishop's theorem to several variables (Theorem 1 above), to give a generalization of the Axler-Cučkovič-Rao theorem to several variables.

Theorem 6 ([ST19], Corollary 2)**.** *Let Ω be a bounded L^∞-pseudoconvex domain in $\mathbb{C}^n$, let $g \in L^\infty(\Omega)$, and let $f_j \in H^\infty(\Omega)$ for all $j = 1, \ldots, m$. Suppose the Jacobian of the map $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ has rank n at some point $z \in \Omega$ and T_g commutes with T_{f_j} for all $1 \leq j \leq m$. Then g is holomorphic.*

As an intermediate step in the proof of Theorem 6, Sahutoğlu and Tikaradze used Theorem 1 to prove an L^p-approximation theorem.

Theorem 7 ([ST19], Corollary 1)**.** *Let Ω be a bounded L^∞-pseudoconvex domain in $\mathbb{C}^n$ and $f_j \in H^\infty(\Omega)$ for all $j = 1, \ldots, m$. Then the following are equivalent.*
(i) *$H^\infty(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$ is dense in $L^p(\Omega)$ for all $0 < p < \infty$.*
(ii) *$H^\infty(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$ is dense in $L^p(\Omega)$ for some $1 \leq p < \infty$.*
(iii) *the Jacobian of the map $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ has rank n at some point $z \in \Omega$.*

Repeating the proofs of Theorems 6 and 7 given in [ST19] with our Theorem 3 in place of Theorem 1 shows that Theorems 6 and 7 continue to hold with the hypothesis that Ω is an L^∞-domain replaced by the hypothesis that Ω is open in $\mathfrak{M}_{H^\infty(\Omega)}$. (See, however, Remark 10 at the end of our paper).

2. The Proofs

We will need the following elementary lemma whose proof we include for completeness.

Lemma 8. *Let Σ be a topological space and let A be a supremum normed Banach algebra of bounded continuous complex-valued functions on Σ that separates points and contains the constants. Let $f_1, \ldots, f_m$ be functions in A. Then the map $r : \mathfrak{M}_{A[\overline{f}_1,\ldots,\overline{f}_m]} \to \mathfrak{M}_A$ that sends each multiplicative linear functional on $A[\overline{f}_1, \ldots, \overline{f}_m]$ to its restriction to A is injective.*

Proof. By replacing the functions in A and the functions $\overline{f}_1, \ldots, \overline{f}_m$ by their continuous extensions to the closure of Σ in $\mathfrak{M}_A$, we may assume without loss of generality that Σ is compact and A and $A[\overline{f}_1, \ldots, \overline{f}_m]$ are uniform algebras on Σ. Now suppose φ_1 and φ_2 are two multiplicative linear functionals on $A[\overline{f}_1, \ldots, \overline{f}_m]$ whose restrictions to A coincide. Choose representing measures μ_1 and μ_2 on Σ for φ_1 and φ_2, respectively. Then for each $j = 1, \ldots, m$, we have

$$\varphi_1(\overline{f}_j) = \int \overline{f}_j \, d\mu_1 = \overline{\int f_j \, d\mu_1} = \overline{\varphi_1(f_j)} = \overline{\varphi_2(f_j)} = \overline{\int f_j \, d\mu_2} = \int \overline{f}_j \, d\mu_2 = \varphi_2(\overline{f}_j).$$

Consequently, $\varphi_1 = \varphi_2$. $\square$

Proof of Theorems 3 and 4. Set $A = H^\infty(\Omega)$, for the proof of Theorem 3, or $A = A(\Omega)$, for the proof of Theorem 4. Set $B = A[\overline{f}_1, \ldots, \overline{f}_m]$. Let $\widehat{B}$ denote the uniform algebra on $\mathfrak{M}_B$ whose members are the Gelfand transforms of the functions in B. By Lemma 8 we can regard $\mathfrak{M}_B$ as a subspace of $\mathfrak{M}_A$ by identifying each element of $\mathfrak{M}_B$ with its restriction to A. Since Ω is open in $\mathfrak{M}_A$ and is contained in the subspace $\mathfrak{M}_B$, the set Ω is open in $\mathfrak{M}_B$ as well.

We can regard g as defined and continuous on all of $\mathbb{C}^n$ by considering g to be identically zero on $\mathbb{C}^n \setminus \Omega$. Note that Ω, regarded as a subset of $\mathfrak{M}_B$, is closed in $\pi_B^{-1}(\Omega)$ (because it is the subset of $\pi_B^{-1}(\Omega)$ where the two continuous functions π_B and the identity function agree). Thus the closure of Ω in $\mathfrak{M}_B$ is contained in $\Omega \cup \pi_B^{-1}(\mathbb{C}^n \setminus \Omega)$. Since the function $g \circ \pi_B$ is identically zero on $\pi_B^{-1}(\mathbb{C}^n \setminus \Omega)$, it follows that there is a well-defined continuous function $\widetilde{g}$ on $\mathfrak{M}_B$ given by

$$\widetilde{g}(x) = \begin{cases} (g \circ \pi_B)(x) & \text{for } x \text{ in the closure of } \Omega \text{ in } \mathfrak{M}_B \\ 0 & \text{for } x \text{ in } \mathfrak{M}_B \setminus \Omega. \end{cases}$$

By applying the Bishop antisymmetric decomposition (see [Bro69, Theorem 2.7.5], [Gam84, Theorem II.13.1], or [Sto71, Theorem 12.1]), we will show that $\widetilde{g}$ is in $\widehat{B}$. It follows that g is in B.

Let E be a maximal set of antisymmetry for $\widehat{B}$. Since the real and imaginary parts of each of $f_1, \ldots, f_m$ lie in B, the set E must be contained in a common level set of the functions $\widehat{f}_1, \ldots, \widehat{f}_m$. Let $\lambda_1, \ldots, \lambda_m$ denote the respective constant values of $\widehat{f}_1, \ldots, \widehat{f}_m$ on E. By the definition of Ω_f, each point of the set $L_\lambda = \{z \in \Omega : \big(f_1(z), \ldots, f_m(z)\big) = (\lambda_1, \ldots, \lambda_m)\}$ that is not in Ω_f is an isolated point of L_λ. Because Ω is open in $\mathfrak{M}_B$, it follows that each point of L_λ that is not in Ω_f is also an isolated point of the set $\widetilde{L}_\lambda = \{z \in \mathfrak{M}_B : \big(\widehat{f}_1(z), \ldots, \widehat{f}_m(z)\big) = (\lambda_1, \ldots, \lambda_m)\}$. Since each maximal set of antisymmetry for a uniform algebra on its maximal ideal space is connected [Sto71, Remarks 12.7], it follows that E must be either a singleton set or else be contained in $(\mathfrak{M}_B \setminus \Omega) \cup \Omega_f$. Since $\widetilde{g}$ is identically zero on $(\mathfrak{M}_B \setminus \Omega) \cup \Omega_f$, we conclude that $\widetilde{g}|_E$ is in $\widehat{B}|_E$. Therefore, $\widetilde{g}$ is in $\widehat{B}$ by the Bishop antisymmetric decomposition. □

We now show how a similar argument yields a new proof of the theorem of Bishop discussed earlier.

Proof of Theorem 2. Set $B = H^\infty(\Omega)[\overline{f}]$. Let $\widehat{B}$ denote the uniform algebra on $\mathfrak{M}_B$ whose members are the Gelfand transforms of the functions in B. Regard $\mathfrak{M}_B$ as a subspace of $\mathfrak{M}_{H^\infty(\Omega)}$ via Lemma 8.

There is a continuous map $\pi_{H^\infty(\Omega)} \to \overline{\Omega}$ that is the identity on Ω and takes $\mathfrak{M}_{H^\infty(\Omega)} \setminus \Omega$ onto $b\Omega$. When $\Omega \subset \mathbb{C}$ is bounded, $\pi_{H^\infty(\Omega)}$ is just the Gelfand transform of z. In the general case the definition of $\pi_{H^\infty(\Omega)}$ is more complicated. See [Gam70]. In particular, Ω is open in $\mathfrak{M}_{H^\infty(\Omega)}$, and hence, in the subspace $\mathfrak{M}_B$. Let π be the restriction of $\pi_{H^\infty(\Omega)}$ to $\mathfrak{M}_B$.

Each maximal set of antisymmetry for $\widehat{B}$ must be contained in a level set of $\widehat{f}$, and by [Sto71, Remarks 12.7] must be connected. Since each level set of a nonconstant holomorphic function of one complex variable is discrete, and Ω is open in $\mathfrak{M}_B$, it follows that each maximal set of antisymmetry for $\widehat{B}$ is either a singleton set of else is contained in $\mathfrak{M}_{H^\infty(\Omega)} \setminus \Omega = \pi^{-1}(b\Omega)$. Invoking the Bishop antisymmetric decomposition, we conclude that the essential set for $H^\infty(\Omega)[\overline{f}]$ is contained in $\pi^{-1}(b\Omega)$. By [Izz03, Theorem 4.1], which we quote below for the reader's convenience, it follows at once that $C(\overline{\Omega}) \subset H^\infty(\Omega)[\overline{f}]$. ☐

Theorem 9 ([Izz03])**.** *Let Ω be an open set in the Riemann sphere, and let A be a uniformly closed algebra of bounded continuous complex-valued functions on Ω. If $\Omega \subset \mathbb{C}$ is bounded assume that $A \supset A(\Omega)$, and if Ω is unbounded assume that $A \supset H^\infty(\Omega)$. Let $\mathscr{E}$ denote the essential set of A regarded as a uniform algebra on $\mathfrak{M}_A$. Then $A \supset C(\overline{\Omega})$ if and only if $\mathscr{E} \subset \pi^{-1}(b\Omega)$.*

When Ω is bounded, the proof of this theorem is rather easy. The case of unbounded Ω is more difficult.

Remark 10. While the basic idea of the proof of Theorem 7 given in [ST19] is correct, there is an incorrect statement there in the proof of the implication (ii) implies (iii). (The algebra generated by the functions $z_1, \ldots, z_n$ is not dense in $H^\infty(B)$ for B an open ball in $\mathbb{C}^n$.) We therefore repeat the proof of this implication avoiding that error. Suppose that $H^\infty(\Omega)[\overline{f}_1, \ldots, \overline{f}_m]$ is dense in $L^p(\Omega)$ for some $1 \leq p < \infty$. Then applying [IL13, Theorem 4.2 or Lemma 4.3] yields that at some point $z \in \Omega$ the differentials of the functions in the set $H^\infty(\Omega) \cup \{\overline{f}_1, \ldots, \overline{f}_m\}$ span a $2n$-dimensional vector space (the complexified cotangent space to $\mathbb{C}^n$). Since the differential of every function in $H^\infty(\Omega)$ lies in the n-dimensional space spanned by $dz_1, \ldots, dz_n$, it follows that the Jacobian of $f = (f_1, \ldots, f_m)$ has rank n at z.

3. ACKNOWLEDGMENT

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REFERENCES

[AČR00] Sheldon Axler, Željko Čučković, and N. V. Rao, *Commutants of analytic Toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 1951–1953. MR 1694299

[AS79] P. Ahern and R. Schneider, *Holomorphic Lipschitz functions in pseudoconvex domains*, Amer. J. Math. **101** (1979), 543–565.

[Bea59] H. S Bear, *Complex function algebras*, Trans. Amer. Math. Soc. **90** (1959), 383–393.

[BH64] Arlen Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/64), 89–102. MR 160136

[Bis89] Christopher Bishop, *Approximating continuous functions by holomorphic and harmonic functions*, Trans. Amer. Math. Soc. **311** (1989), 781–811.

[Bro69] Andrew Browder, *Introduction to function algebras*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[Gam70] T. W. Gamelin, *Localization of the corona problem*, Pacific J. Math. **34** (1970), 73–81.

[Gam84] Theodore W. Gamelin, *Uniform algebras*, second edition ed., Chelsea, 1984.

[IL13] Alexander J. Izzo and Bo Li, *Generators for algebras dense in L^p-spaces*, Studia Math. **217** (2013), no. 3, 243–263. MR 3119758

[Izz93] Alexander J. Izzo, *Uniform approximation by holomorphic and harmonic functions*, J. London Math. Soc. **47** (1993), 129–141.

[Izz03] ———, *Algebras containing bounded holomorphic functions*, Indiana Univ. Math. J. **52** (2003), no. 5, 1305–1342.

[Rud08] Walter Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1980 edition. MR 2446682

[ST19] Sönmez Sahutoğlu and Akaki Tikaradze, *On a theorem of Bishop and commutants of Toeplitz operators in $\mathbb{C}^n$*, Rend. Circ. Mat. Palermo (2) **68** (2019), no. 2, 237–246. MR 4148742

[Sto71] Edgar Lee Stout, *The theory of uniform algebras*, Bogden & Quigley, Inc., Tarrytown-on-Hudson, N. Y., 1971. MR 0423083

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