# DIRAC SERIES FOR COMPLEX CLASSICAL LIE GROUPS: A MULTIPLICTY-ONE THEOREM

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ABSTRACT. This paper computes the Dirac cohomology $H_D(\pi)$ of irreducible unitary Harish-Chandra modules $\pi$ of complex classical groups viewed as real reductive groups. More precisely, unitary representations with nonzero Dirac cohomology are shown to be unitarily induced from unipotent representations. When nonzero, there is a unique, multiplicity free $K-$type in $\pi$ contributing to $H_D(\pi)$. This confirms conjectures formulated by the first named author and Pandžić in 2011.

## 1. INTRODUCTION

The Dirac operator was first introduced in the representation theory of real reductive groups by Parthasarathy [P1, P2] and Schmid in order to give geometric realization of the discrete series. A byproduct, the Dirac inequality, has proved very useful to provide necessary conditions for unitarity. In the case of real rank one groups, the work of [BSi] and [BB], shows that this necessary condition is also sufficient. The Dirac inequality plays a crucial role in the determination of representations with $(\mathfrak{g}, K)-$cohomology in the work of [E] and [VZ] for complex and real groups, subsequently expanded by [Sa] to find necessary and sufficient conditions for the unitarity of irreducible representations with regular integral infinitesimal character.

In order to find sharper estimates for the spectral gap in the case of locally symmetric spaces, Vogan in [V2] introduced the notion of **Dirac cohomology** for irreducible representations. He formulated a conjecture on its relationship with the infinitesimal character of the representation.

We recall the construction of Dirac operator and Dirac cohomology. Let $G$ be a connected real reductive Lie group. Fix a Cartan involution $\theta$, and write $K := G^\theta$ for the maximal compact subgroup. Denote by $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding decomposition of the complexification. Let $\langle\,,\rangle$ be an invariant nondegenerate form such that $\langle\,,\,\rangle\,|_{\mathfrak{p}_0}$ is positive definite, and $\langle\,,\,\rangle\,|_{\mathfrak{k}_0}$ is negative definite. Fix $Z_1, \ldots, Z_n$ an orthonormal basis of $\mathfrak{p}_0$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to $\langle\,,\,\rangle$. The **Dirac operator** $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is defined as

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i.$$

The operator $D$ does not depend on the choice of the orthonormal basis $Z_i$ and is $K$-invariant for the diagonal action of $K$ induced by the adjoint actions on both factors.

Define $\Delta : \mathfrak{k} \to U(\mathfrak{g}) \otimes C(\mathfrak{p})$ by $\Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$, where $\alpha : \mathfrak{k} \to C(\mathfrak{p})$ is the composition of $\mathrm{ad} : \mathfrak{k} \longrightarrow \mathfrak{so}(\mathfrak{p})$ with the embedding $\mathfrak{so}(\mathfrak{p}) \cong \wedge^2(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$. Write $\mathfrak{k}_\Delta := \alpha(\mathfrak{k})$, and denote by $\Omega_{\mathfrak{g}}$ (resp. $\Omega_{\mathfrak{k}}$) the Casimir operator of $\mathfrak{g}$ (resp. $\mathfrak{k}$). Let $\Omega_{\mathfrak{k}_\Delta}$ be the image of $\Omega_{\mathfrak{k}}$ under $\Delta$. Then ([P1])

$$(1) \qquad D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + (\|\rho_c\|^2 - \|\rho_{\mathfrak{g}}\|^2)1 \otimes 1,$$

where $\rho_{\mathfrak{g}}$ and $\rho_c$ are the corresponding half sums of positive roots of $\mathfrak{g}$ and $\mathfrak{k}$.

Let

$$\widetilde{K} := \{(k, s) \in K \times \mathrm{Spin}(\mathfrak{p}_0) \ : \ \mathrm{Ad}(k) = p(s)\},$$

where $p : \mathrm{Spin}(\mathfrak{p}_0) \to \mathrm{SO}(\mathfrak{p}_0)$ is the spin double covering map. If $\pi$ is a $(\mathfrak{g}, K)$-module, and if $S_G$ denotes a spin module for $C(\mathfrak{p})$, then $\pi \otimes S_G$ is a $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), \widetilde{K})$ module.

The action of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is the obvious one, and $\widetilde{K}$ acts on both factors; on $\pi$ through $K$ and on $S_G$ through the spin group $\mathrm{Spin}\,\mathfrak{p}_0$. The Dirac operator acts on $\pi \otimes S_G$. The Dirac cohomology of $\pi$ is defined as the $\widetilde{K}$-module

$$(2) \qquad H_D(\pi) = \mathrm{Ker}\, D/(\mathrm{Im}\, D \cap \mathrm{Ker} D).$$

The following foundational result on Dirac cohomology, conjectured by Vogan, was proven by Huang and Pandžić in 2002. Let $\mathfrak{h}$ be a $\theta-$stable Cartan subalgebra with Cartan decomposition $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{k}$.

**Theorem 1.1** ([HP1] Theorem 2.3). *Let $\pi$ be an irreducible $(\mathfrak{g}, K)$-module. Assume that the Dirac cohomology of $\pi$ is nonzero, and that it contains the $\widetilde{K}$-type with highest weight $\gamma \in \mathfrak{t}^* \subset \mathfrak{h}^*$. Then the infinitesimal character of $\pi$ is conjugate to $\gamma + \rho_c$ under $W(\mathfrak{g}, \mathfrak{h})$.*

1.1. **Dirac Series.** Denote by $\widehat{G}$ be the set of equivalence classes of irreducible unitary $(\mathfrak{g}, K)$-modules. If $\pi \in \widehat{G}$, then $\pi \otimes S_G$ acquires a natural inner product, and $D$ is self-adjoint. As a result, Dirac cohomology simplifies to

$$(3) \qquad H_D(\pi) = \mathrm{Ker}\, D = \mathrm{Ker}\, D^2.$$

For a unitary irreducible representation, (1) is a nonnegative scalar on any $\widetilde{K}-$type. If $\chi_\pi$ is the infinitesimal character of $\pi$, and $\tau$ is the highest weight of a $K-$type in $\pi \otimes S_G$, then

$$(4) \qquad \|\chi_\pi\|^2 \leq \|\tau + \rho_c\|^2$$

This is **Parthasarathy's Dirac operator inequality**. Moreover, by Theorem 3.5.2 of [HP2], the equality holds precisely when $\tau$ is the highest weight of a $\widetilde{K}$-type in $H_D(\pi)$ (see Section 2.3).

Let $\widehat{G}^d$ be the representations with nonzero Dirac cohomology. This subset forms an interesting part of $\widehat{G}$. For convenience, we call these representations **Dirac series** of $G$ (terminology suggested by J.-S. Huang).

When $G$ is a complex Lie group viewed as a real Lie group, a necessary condition for $\pi \in \widehat{G}^d$ is that twice the infinitesimal character $\lambda$ of $\pi$ must satisfying the regular integral

condition (12) given in Section 2.3. For this paper we adopt the following setting. We focus on the cases when the infinitesimal character is **regular half-integral** – to emphasize, $2\lambda$ satisfies (12) but $\lambda$ is not integral. This is because in the case of $\lambda$ regular integral, these are unitary representations with nontrivial $(\mathfrak{g}, K)-$cohomology, and the results in [E] and [VZ] imply that any representation in $\widehat{G}^d$ is unitarily induced from the trivial representation on a Levi component. This is not the case for half-integral regular parameter.

We begin by determining the representations with half-integral regular parameter which are unitary and not unitarily induced from any unitary representation on a proper Levi component. This can be read off from [B1] and [V1] for the classical groups, i.e. $GL(n, \mathbb{C})$, $SO(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$. We give a self contained derivation of the unitary dual at half-integral regular infinitesimal character for these groups, along with a brief discussion on the cases of genuine representations of the $Spin$ groups.

For $GL(n, \mathbb{C})$, these representations are just unitary characters. Yet this is not the case for the other classical groups. In [B1], a larger class of representations is identified which are called the *building blocks* of the unitary dual in the sense that

- they are unitary and are not unitarily induced from unitary representations on proper Levi components,
- any other unitary representation is obtained by unitary induction and continuous deformations from unitarily induced modules (complementary series)

They turn out to have the additional property that the annihilator in the universal enveloping algebra is maximal. We call these *cuspidal unipotent representations*. Following [BV], we consider a larger class of representations which we call *unipotent*. They have properties analogous to the representations studied in [BV] which are called *special unipotent* and have the properties conjectured by Arthur in relation to the residual spectrum of locally symmetric spaces.

A general discussion of the notion of *unipotent representation* is beyond the scope of this paper. We have included an explicit list for the classical groups and a partial discussion in Appendix A. It is a paraphrase of [B3] which identifies the representations as iterated $\Theta$ lifts from one dimensional representations.

The following conjecture on $\widehat{G}^d$ was formulated in [BP]:

**Conjecture 1.2** ([BP] Conjecture 1.1)**.** *Let $G$ be a connected complex simple Lie group and $\pi \in \widehat{G}$ whose infinitesimal character is regular and half-integral. Then $\pi \in \widehat{G}^d$ if and only if $\pi$ is parabolically induced from a unipotent representation with nonzero Dirac cohomology, tensored with a unitary character.*

Conjecture 1.2 generalizes to real reductive Lie groups, where unitary induction is replaced by the more general cohomological induction in a range where unitarity is preserved. In the complex case, Parthasarathy's Dirac inequality (4) implies that all $\pi \in \widehat{G}$ with regular integral infinitesimal character are unitarily induced from unitary characters of parabolic subgroups, and hence the conjecture follows immediately.

Here is the list of all nontrivial unipotent representations with half-integral regular infinitesimal characters for complex classical groups. In all cases the representations have maximal primitive ideal. The parameters are explicit, and fit in the parametrization in Appendix A. Note that the ones in Type $B$, $C$ and $D$ are not induced from unitary representations on proper Levi components.

**Type $A_n$:** The infinitesimal character satisfies

$$(5) \qquad 2\lambda = (b-1, b-3, \ldots, a, a-1, \ldots, -a+1, -a, \ldots, -b+3, -b+1),$$

where we assume $b > a$. The corresponding unipotent representation is spherical of the form

$$\pi_u = \mathrm{Ind}_{GL(a) \times GL(b)}^{GL(a+b)} (triv \otimes triv).$$

It is also the $\Theta-$lift of the trivial representation of $GL(2b+1)$ to $GL(2a+2b+1)$.

**Type $B_n$:** The infinitesimal character $\lambda$ satisfies

$$(6) \qquad 2\lambda = (2b-1, 2b-3, \ldots, 2a+3, 2a+1, 2a, 2a-1, \ldots, 2, 1).$$

with $b \geq a$. The nilpotent orbit has columns $(2b+1, 2a)$, and the representation is the $\Theta-$lift of the trivial representation of $Sp(2a)$ to $SO(2b+2a+1)$.

**Type $C_n$:** The infinitesimal character satisfies

$$(7) \qquad 2\lambda = (2n-1, 2n-3, \ldots, 3, 1).$$

and there are two representations, the components of the Segal-Shale-Weil representation. The nilpotent orbit has columns $(2n-1, 1)$ and the representations are the $\Theta-$lifts of the two characters of $O(1)$ to $Sp(2n)$.

**Type $D_n$:** The infinitesimal character satisfies

$$(8) \qquad 2\lambda = (2b-2, 2b, \ldots, 2a+2, 2a, 2a-1, 2a-2, \ldots, 1, 0)$$

with $b \geq a$. (When $b = a$, the parameter is $(2a-1, 2a-2, \ldots, 1, 0)$). There are two representations with maximal primitive ideal. The nilpotent orbit has columns $(2b, 2a-1, 1)$ and the representations are $\Theta-$lifts from the Segal-Shale-Weil representations which in turn are $\Theta-$lifts of the characters of $O(1)$. This is a case of two iterations of $\Theta-$lifts from 1-dimensional representations.

As already mentioned, the unitarily induced representations from the unipotent ones listed above are generalizations of the representations with nontrivial $(\mathfrak{g}, K)-$cohomology. As far as locally symmetric spaces and the work of [A], it is expected that they would provide new examples of local factors of automorphic forms.

We follow the same strategy in the case of the $Spin$ groups. Here are the parameters of unipotent representations with half-integral regular infinitesimal characters:

**$Spin(2n+1, \mathbb{C})$:** Apart from the infinitesimal characters in (6),

$$(9) \qquad 2\lambda = (2n-1, 2n-3, \ldots, 3, 1)/2.$$

**$Spin(2n, \mathbb{C})$:** Apart from the infinitesimal characters in (8), there is also

$$(10) \qquad 2\lambda = (2n-1, 2n-3, \ldots, 3, \pm 1)/2$$

Unlike the parameters in (6) and (8), these parameters correspond to *genuine* representations, i.e. they do not factor through $SO(2n+1, \mathbb{C})$ or $SO(2n, \mathbb{C})$. Moreover, they have maximal primitive ideal, and are unitarily induced from a unitary character of a Levi component of type $A_{n-1}$. Note that half-integral means $2\lambda$ is integral, not that the coordinates are half-integers. Consequently, just like the case of type $A$, one only needs to consider unitary characters for the genuine representations of Spin groups.

We are now ready to state the unitarity results in [V1] and [B1] for complex classical $G$:

**Theorem 1.3** (Theorem 3.1). *Let $G$ be a classical complex Lie group. Any $\pi \in \widehat{G}$ with regular, half-integral infinitesimal character is of the form*

$$\pi := \mathrm{Ind}_{MN}^{G}((\mathbb{C}_\xi \otimes \pi_u) \otimes \mathbf{1}),$$

*where $P = MN$ is a parabolic subgroup of $G$ with Levi factor $M$, and $\mathbb{C}_\mu$ is a unitary character on $M$. Moreover, $\pi_u$ is either the trivial representation, or a unipotent representation with infinitesimal character given in $(5) - (8)$.*

By the paragraph after Equation (5), $\pi_u$ is induced from the trivial representation in Type $A$. Using induction in stages, we will assume from now on that $\pi_u = triv$ for Type $A$.

A self-contained proof of Theorem 1.3 for all classical groups is in Sections 3 to 6. The case of Theorem 1.3 for Spin groups is also discussed in Section 4.5 and 6.5. When $\pi$ is not unitary, we will specify precisely on which $K-$types the Hermitian form is indefinite. This will be useful in proving the analogous theorem for exceptional groups of Type $E$.

Using this, we will prove the following:

**Theorem 1.4.** *Conjecture 1.2 holds for complex connected classical Lie groups and the Spin groups.*

1.2. **Spin-lowest $K-$type.** Following [D1], we are interested in studying **spin-lowest $K-$type** (spin-LKT) of an admissible $(\mathfrak{g}, K)-$module. See Definition 2.3 for the precise meaning of spin-lowest $K-$type in the setting of complex Lie groups. If $\pi \in \widehat{G}^d$, then the spin-lowest $K-$types are precisely those contributing to $H_D(\pi)$. More explicitly, let $\tau$ be the highest weight of the $\widetilde{K}-$type occurring $H_D(\pi)$. Then

$$[V_\mathfrak{k}(\tau) \ : \ H_D(\pi)] = \sum_{\eta \text{ spin}-\mathrm{LKT}} [V_\mathfrak{k}(\eta) : \pi] \cdot [V_\mathfrak{k}(\eta) \otimes S_G \ : \ V_\mathfrak{k}(\tau)],$$

where $V_\mathfrak{a}(\eta)$ is the irreducible, finite-dimensional $\mathfrak{a}-$module with highest weight $\eta$. In view of this, the following conjecture, formulated in [BP], makes $\widehat{G}^d$ and $H_D(\pi)$ precise.

**Conjecture 1.5** ([BP] Conjecture 4.1 and J.-S. Huang). *Let $G$ be a connected complex simple Lie group, and $\pi \in \widehat{G}^d$. Then $\pi$ has a unique spin-lowest $K-$type $V_\mathfrak{k}(\eta)$ which occurs with multiplicity one.*

Here is the second main result of this paper:

**Theorem 1.6.** *Conjecture 1.5 holds for complex connected classical Lie groups and the Spin groups.*

We believe that Theorems 1.4 and 1.6 should hold for all complex reductive groups. Indeed, based on the results in [DD, D2, DW], these theorems are shown to be true for exceptional groups of type $G_2$, $F_4$, $E_6$ and $E_7$. We give full details on the case of complex $E_8$ in a forthcoming work.

The manuscript is organized as follows. Section 2 includes some preliminary results on complex simple Lie groups, Dirac cohomology and spin-lowest $K-$types. Sections 3–6 state the classification of the unitary dual for complex classical Lie groups with half integral regular infinitesimal character (cf. [B1], [V1]) and gives complete proofs. Section 7 proves a stronger version of Conjecture 1.5 for unipotent representations, which is essential for the determination of $H_D(\pi)$ in Section 8. In Appendix A, we give an overview of unipotent representations for complex classical Lie groups. Finally, in Appendix B, we present some calculations on `atlas` ([ALTV], [At]) for the modules appearing in Sections 4–6, offering examples for the results in these sections.

## 2. Preliminaries

Let $G$ be a connected complex simple Lie group viewed as a real Lie group. Fix a maximal compact subgroup $K$ and a Borel subgroup $B$. Then $T := K \cap B$ is a maximal torus in $K$.

Denote by $\mathfrak{t}_0$ the Lie algebra of $T$. Then $\mathfrak{a}_0 := \sqrt{-1}\mathfrak{t}_0$ is a maximally split Cartan subalgebra of $\mathfrak{g}_0$. Let $A := \exp(\mathfrak{a}_0)$. Then $H = TA$ is a Cartan subgroup of $G$ with Lie algebra $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$.

The realization of the complexification of $\mathfrak{g}_0$ in (2.1.3) – (2.1.7) of [B1] gives

$$(11) \qquad \mathfrak{g} \cong \mathfrak{g}_0 \oplus \mathfrak{g}_0, \quad \mathfrak{h} \cong \mathfrak{h}_0 \oplus \mathfrak{h}_0, \quad \mathfrak{t} \cong \{(x, -x) : x \in \mathfrak{h}_0\}, \quad \mathfrak{a} \cong \{(x, x) : x \in \mathfrak{h}_0\}$$

(we drop the subscripts of the Lie algebras to denote their complexifications).

Let $\rho$ be the half sum of positive roots in $\Delta_G^+$. A choice of positive roots of $\mathfrak{g}$ is

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{\alpha \times 0\} \cup \{0 \times (-\alpha)\}_{\alpha \in \Delta_G^+}.$$

Denote by $W$ the Weyl group $W(\mathfrak{g}_0, \mathfrak{h}_0)$, which has identity element $e$ and longest element $w_0$. Then $W(\mathfrak{g}, \mathfrak{h}) \simeq W \times W$.

2.1. **Classification of irreducible modules.** The classification of irreducible $(\mathfrak{g}, K)$-modules for complex Lie groups was first obtained by Parthasarathy-Rao-Varadarajan [PRV] and Zhelobenko [Zh]. Let $(\lambda_L, \lambda_R) \in \mathfrak{h}_0^* \times \mathfrak{h}_0^*$ be such that $\lambda_L - \lambda_R$ is a weight of a finite dimensional holomorphic representation of $G$. Using (11), we can view $(\lambda_L, \lambda_R)$ as a real-linear functional on $\mathfrak{h}$ (we will also sometimes denote it as $\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix}$), and write $\mathbb{C}_{(\lambda_L, \lambda_R)}$ as the character of $H$ with differential $(\lambda_L, \lambda_R)$ (which exists) with

$$\mathbb{C}_{(\lambda_L, \lambda_R)}|_T = \mathbb{C}_\mu := \mathbb{C}_{\lambda_L - \lambda_R}, \quad \mathbb{C}_{(\lambda_L, \lambda_R)}|_A = \mathbb{C}_\nu := \mathbb{C}_{\lambda_L + \lambda_R}.$$

Put $X(\lambda_L, \lambda_R) := K$-finite part of $\mathrm{Ind}_B^G(\mathbb{C}_{(\lambda_L,\lambda_R)} \otimes \mathbf{1})$.

**Theorem 2.1.** ([PRV], [Zh]) *The $K-$type with extremal weight $\mu := \lambda_L - \lambda_R$ occurs with multiplicity one in $X(\lambda_L, \lambda_R)$. Let $J(\lambda_L, \lambda_R)$ be the unique subquotient of $X(\lambda_L, \lambda_R)$ containing this $K-$type.*

a) *Every irreducible admissible $(\mathfrak{g}, K)$-module is of the form $J(\lambda_L, \lambda_R)$.*
b) *Two such modules $J(\lambda_L, \lambda_R)$ and $J(\lambda'_L, \lambda'_R)$ are equivalent if and only if there exists $w \in W$ such that $w\lambda_L = \lambda'_L$ and $w\lambda_R = \lambda'_R$.*
c) *$J(\lambda_L, \lambda_R)$ admits a nondegenerate Hermitian form if and only if there exists $w \in W$ such that $w(\lambda_L - \lambda_R) = \lambda_L - \lambda_R, w(\lambda_L + \lambda_R) = -\overline{(\lambda_L + \lambda_R)}$.*

The $W \times W-$orbit of $(\lambda_L, \lambda_R)$ is the infinitesimal character of $J(\lambda_L, \lambda_R)$.

In general we normalize hermitian forms on irreducible modules to be positive on the lowest $K-$type. Occasionally we will say that the form is indefinite on a set of $K-$types, with the understanding that if one of them is a lowest $K-$type, then the form is normalized as stated above.

2.2. **PRV-component.** In this subsection, we summarize Corollaries 1 and 2 to Theorem 2.1 of [PRV] on the decomposition of the tensor product $V_{\mathfrak{k}}(\sigma_1) \otimes V_{\mathfrak{k}}(\sigma_2)$ for highest weights $\sigma_1$ and $\sigma_2$.

**Theorem 2.2.** ([PRV]) *The component $V_{\mathfrak{k}}(\{\sigma_1 + w_0\sigma_2\})$ occurs exactly once in $V_{\mathfrak{k}}(\sigma_1) \otimes V_{\mathfrak{k}}(\sigma_2)$, where $\{\sigma_1 + w_0\sigma_2\}$ is the unique dominant element to which $\sigma_1 + w_0\sigma_2$ is conjugate under the action of $W$. Moreover, any other component $V_{\mathfrak{k}}(\eta')$ occurring in $V_{\mathfrak{k}}(\sigma_1) \otimes V_{\mathfrak{k}}(\sigma_2)$ must be of the form*

$$\eta' = \{\sigma_1 + w_0\sigma_2\} + \sum_{i=1}^{l} n_i\alpha_i, \ \ where \ n_i \in \mathbb{N}.$$

*In particular,*

$$\|\{\sigma_1 + w_0\sigma_2\} + \rho\| < \|\eta' + \rho\|.$$

The factor $V_{\mathfrak{k}}(\{\sigma_1 + w_0\sigma_2\})$ is usually called the **PRV-component** of $V_{\mathfrak{k}}(\sigma_1) \otimes V_{\mathfrak{k}}(\sigma_2)$.

2.3. **Hermitian modules with Dirac cohomology.** Let $\pi$ be an irreducible $(\mathfrak{g}, K)-$module for a complex Lie group $G$. By Theorem 1.1 and (11), $\pi$ has Dirac cohomology if and only if its Zhelobenko parameter $(w_1\lambda_L, w_2\lambda_R)$ satisfies

(12)
$$\begin{cases} w_1\lambda_L - w_2\lambda_R = \tau + \rho \\ w_1\lambda_L + w_2\lambda_R = 0, \end{cases}$$

where $V_{\mathfrak{k}}(\tau)$ is a $\widetilde{K}-$type in $H_D(\pi)$. The second equation implies $\lambda_R = -w_2^{-1}w_1\lambda_L$. Since $\tau + \rho$ is regular integral, the first equation implies that $2w_1\lambda_L$ is regular integral.

Write $\lambda = w_1\lambda_L$. The module can be written as $\pi = J(\lambda, -s\lambda)$ with $2\lambda$ regular integral, and the first equation of (12) implies that the *only* $\widetilde{K}-$type that can appear in $H_D(\pi)$ is $V_{\mathfrak{k}}(2\lambda - \rho)$. Furthermore, if $J(\lambda, -s\lambda)$ is Hermitian (e.g. if $J(\lambda, -s\lambda)$ is unitary), it follows as in [BP] that $s$ is an involution.

Assume further that $\pi = J(\lambda, -s\lambda) \in \widehat{G}$, i.e. it is unitary. To relate the above arguments in terms of Parthasarathy's Dirac inequality, note that $V_{\mathfrak{k}}(\tau)$ is in $H_D(\pi)$ if and only if

$$2\lambda = \tau + \rho,$$

which is precisely when the equality holds in (4). Moreover, if the $K-$type $V_{\mathfrak{k}}(\eta)$ in $\pi$ contributes to $H_D(\pi)$, then by Theorem 2.2 it must come from the PRV component of

$$V_{\mathfrak{k}}(\eta) \otimes S_G = 2^{[\frac{l}{2}]} \, V_{\mathfrak{k}}(\eta) \otimes V_{\mathfrak{k}}(\rho),$$

where the equality comes from Lemma 2.2 of [BP]. This leads to the following definition given in [D1].

**Definition 2.3.** *The* **spin norm** *of the $K-$type $V_{\mathfrak{k}}(\eta)$ is defined as*

(13) $$\|\eta\|_{\mathrm{spin}} := \|\{\eta - \rho\} + \rho\|$$

*For any irreducible admissible $(\mathfrak{g}, K)-$module $\pi$, we define*

(14) $$\|\pi\|_{\mathrm{spin}} := \min \|\eta\|_{\mathrm{spin}},$$

*where $\eta$ runs over all the $K-$types occurring in $\pi$. A module $V_{\mathfrak{k}}(\eta)$ is called a* **spin-lowest** *$K-$type of $\pi$ if it occurs in $\pi$ and $\|\eta\|_{\mathrm{spin}} = \|\pi\|_{\mathrm{spin}}$.*

Using the terminology in Definition 2.3, the results of this section can be summarized as follows.

**Proposition 2.4.** *Let $\pi = J(\lambda, -s\lambda) \in \widehat{G}$ with $2\lambda$ regular integral, and $s \in W$ an involution. Then $\|\pi\|_{spin} \geq \|2\lambda\|$, and the equality holds if and only if $J(\lambda, -s\lambda) \in \widehat{G}^d$.*

*In such cases, $H_D(\pi)$ consists of a single $\widetilde{K}-$type $V_{\mathfrak{k}}(2\lambda - \rho)$ with multiplicity*

$$
\begin{aligned}
[V_{\mathfrak{k}}(2\lambda - \rho) : H_D(\pi)] &= \sum_{\eta \ spin-LKT} [V_{\mathfrak{k}}(\eta) : \pi] \cdot [V_{\mathfrak{k}}(\eta) \otimes S_G : V_{\mathfrak{k}}(2\lambda - \rho)] \\
&= 2^{[\frac{l}{2}]} \sum_{\eta \ spin-LKT} [V_{\mathfrak{k}}(\eta) : \pi] \cdot [V_{\mathfrak{k}}(\eta) \otimes V_{\mathfrak{k}}(\rho) : V_{\mathfrak{k}}(2\lambda - \rho)] \\
&= 2^{[\frac{l}{2}]} \sum_{\eta \ spin-LKT} [V_{\mathfrak{k}}(\eta) : \pi].
\end{aligned}
$$

Conjecture 1.5 can be rephrased in the following sharper form. This is the main result of the paper in the case of groups of classical type.

**Conjecture 2.5.** *Let $\pi = J(\lambda, -s\lambda) \in \widehat{G}$. Then*

$$[\pi \otimes V_{\mathfrak{k}}(\rho) \; : \; V_{\mathfrak{k}}(2\lambda - \rho)] := \sum_{\kappa} [V_{\mathfrak{k}}(\kappa) \; : \; \pi] \cdot [V_{\mathfrak{k}}(\kappa) \otimes V_{\mathfrak{k}}(\rho) \; : \; V_{\mathfrak{k}}(2\lambda - \rho)] = \begin{cases} 1 & \text{if } \pi \in \widehat{G}^d \\ 0 & \text{otherwise} \end{cases}.$$

*Consequently, if $\pi \in \widehat{G}^d$, then $H_D(\pi) = 2^{[\frac{l}{2}]} V_{\mathfrak{k}}(2\lambda - \rho)$ by Proposition 2.4.*

## 3. Unitary Dual

We use the notation and terminology in the previous section. We determine the unitary representations $J(\lambda, -s\lambda)$ with $2\lambda$ regular and integral; as already mentioned, $s$ must be an involution. The results were first proved in [B1] and [V1], and can be summarized as follows.

**Theorem 3.1** ([B1], [V1])**.** *Let $G$ be a classical complex Lie group. Any irreducible unitary representation $\pi := J(\lambda, -s\lambda)$ of $G$ with $2\lambda$ regular and integral must be of the form*

$$\pi := \operatorname{Ind}_{LU}^{G}((\mathbb{C}_\mu \otimes \pi_u) \otimes \mathbf{1}),$$

*where $P = LU$ is a parabolic subgroup of $G$ with Levi factor $L$, $\mathbb{C}_\mu$ is a unitary character of $L$, and $\pi_u$ is either the trivial representation, or one of the unipotent representations listed in (6) – (8) for Type B, C or D:*

**Type $B_n$:** *The spherical unipotent representations*

$$\pi_u = J \begin{pmatrix} -b+1/2, \ldots, -1/2; -a, \ldots, -1 \\ -b+1/2, \ldots, -1/2; -a, \ldots, -1 \end{pmatrix}, \quad 0 < a \le b \text{ integers and } a + b = n.$$

*It has $K-$spectrum*

$$V_{\mathfrak{k}}(\alpha_1, \alpha_1, \ldots, \alpha_a, \alpha_a, \underbrace{0, \ldots, 0}_{b-a}), \quad \alpha_1 \ge \cdots \ge \alpha_a \ge 0.$$

**Type $C_n$:** *The Oscillator representations*

$$\pi_u^{even} = J \begin{pmatrix} -n+1/2, & \ldots, & -1/2 \\ -n+1/2, & \ldots, & -1/2 \end{pmatrix} \text{ and } \pi_u^{odd} = J \begin{pmatrix} -n+1/2, & \ldots, & -1/2 \\ -n+1/2, & \ldots, & 1/2 \end{pmatrix},$$

*Their $K-$spectra are given by*

$$V_{\mathfrak{k}}(2k, 0, \ldots, 0) \quad and \quad V_{\mathfrak{k}}(2k+1, 0, \ldots, 0), \quad k \ge 0$$

**Type $D_n$:** *The unipotent representations*

$$\pi_u^{even} = J \begin{pmatrix} -a+1/2, & \ldots, & -3/2, & -1/2, & ; -b+1, & \ldots, & 0 \\ -a+1/2, & \ldots, & -3/2, & -1/2, & ; -b+1, & \ldots, & 0 \end{pmatrix} \text{ and}$$

$$\pi_u^{odd} = J \begin{pmatrix} -a+1/2, & \ldots, & -3/2, & -1/2, & ; -b+1, & \ldots, & 0 \\ -a+1/2, & \ldots, & -3/2, & 1/2, & ; -b+1, & \ldots, & 0 \end{pmatrix}$$

*with $0 < a \le b$ integers and $a + b = n$. Their $K-$spectra are*

$$V_{\mathfrak{k}}(\alpha_1, \ldots, \alpha_{2a}, \underbrace{0, \ldots, 0}_{b-a}), \quad \alpha_1 \ge \cdots \ge \alpha_{2a} \ge 0, \quad \sum_i \alpha_i \text{ is even/odd.}$$

3.1. **Bottom Layer K−types.** We use the standard realizations of the classical groups and Lie algebras. As in [B2], we will use the notion of relevant $K$−types to detect non-unitarity of $\pi$.

**Definition 3.2.** *The $K$−types $V_{\mathfrak{k}}(1,\ldots,1,0,\ldots,0,-1,\ldots,-1)$ with equal number of $1$ and $-1$ for type $A$, and $V_{\mathfrak{k}}(1,\ldots,1,0,\ldots,0)$ and $V_{\mathfrak{k}}(2,1,\ldots,1,0,\ldots,0)$ in types $B$, $C$, $D$ will be called* **cx-relevant**. *The ones with coordinates $\pm 1$ only, will be called* **fundamental cx-relevant**.

We will make heavy use of bottom layer $K$−types as detailed in [KnV]. The special case of complex groups is in Section 2.7 of [B1]. For the classical groups of Type $B$, $C$ or $D$, the results in coordinates are as follows. Write the lowest $K$−type of $J(\lambda, -s\lambda)$ as

$$\mu = (\ldots,\underbrace{r,\ldots,r}_{\mu_r},\ldots,\underbrace{1,\ldots,1}_{\mu_1},\underbrace{0,\ldots,0}_{\mu_0}) = (\ldots,r^{\mu_r},\ldots,1^{\mu_1},0^{\mu_0}).$$

Let

$$M_1 = \prod_{r \geq 1} GL(\mu_r) \times G(\mu_0) \qquad J_1 = \bigotimes_{r \geq 1} J_{GL(\mu_r)}(\lambda_L^r, \lambda_R^r) \otimes J_{G(\mu_0)}(\lambda_L^0, \lambda_R^0)$$

$$M_2 = \prod_{r \geq 2} GL(\mu_r) \times G(\mu_1 + \mu_0) \quad J_2 = \bigotimes_{r \geq 2} J_{GL(\mu_r)}(\lambda_L^r, \lambda_R^r) \otimes J_{G(\mu_1 + \mu_0)}(\lambda_L^1 \cup \lambda_L^0, \lambda_R^1 \cup \lambda_R^0)$$

be Levi components of real parabolic subalgebras containing the centralizer of $\mu$, and irreducible modules. Let

(15) $$I_1 := \mathrm{Ind}_{M_1}^G(J_1), \quad I_2 := \mathrm{Ind}_{M_2}^G(J_2)$$

be induced modules containing $J(\lambda, -s\lambda)$. We only specify the information on the Levi subgroup for parabolic induction when there is no danger of confusion. Bottom layer $K$−types are of the form $\mu_i = \mu + \mu_{M_i}$ where $\mu_{M_i}$ are $K \cap M_i$−types in $J_i$ so that $\mu_i$ is dominant. They possess the crucial property that the multiplicities and signatures of $\mu_{M_i}$ on the $J_i$ and $\mu_i$ in the induced modules in (15) and the lowest $K$−type factor $J$ coincide. By Section 2.7 of [B1], some of the bottom layer $K$−types for $I_1$ are obtained by adding $(1,\ldots,1,0,\ldots,0,-1,\ldots,-1)$ (equal number of 1 and −1) to the coordinates equal to $r \geq 1$ in $\mu$. In addition one can add $(1,\ldots,1,0,\ldots,0)$ to the coordinates of $\mu$ equal to 0; an even number in cases $C, D$. For $I_2$, there are extra bottom layer $K$−types obtained by replacing the coordinates $(1^{\mu_1}, 0^{\mu_0})$ with $(2^{\mu_2'}, 1^{\mu_1'}, 0^{\mu_0'})$ which also denote a $K \cap M_2$−type coming from $J_{G(\mu_1 + \mu_0)}$.

3.2. **Necessary Conditions for Unitarity.**

**Proposition 3.3.** *Assume that $\lambda$ is half-integral regular. The parameter $(\lambda_L^r, \lambda_R^r)$ in (15) for $r \geq 1$ consists of at most two strings,*

$$\begin{pmatrix} A, & \ldots, & \frac{r}{2}+1, & \frac{r}{2}, & \frac{r}{2}-1, & \ldots, & a \\ -a, & \ldots, & -\frac{r}{2}+1, & -\frac{r}{2}, & -\frac{r}{2}-1, & \ldots, & -A \end{pmatrix},$$

*and/or*

$$\begin{pmatrix} B, & \dots, & \frac{r+1}{2}, & \frac{r-1}{2}, & \dots, & b \\ -b, & \dots, & -\frac{r-1}{2}, & -\frac{r+1}{2}, & \dots, & -B \end{pmatrix}$$

*only* $(a + A = r,$ *and* $B + b = r)$.

*Proof.* The irreducible module $J_{GL(\mu_r)}(\lambda_L^r, \lambda_R^r)$ in (15) has $1-$dimensional lowest $K-$type $V_{\mathfrak{k} \cap \mathfrak{gl}(\mu_r)}(r, \dots, r)$. The condition that $2\lambda$ be regular integral implies that $J(\lambda_L^r, \lambda_R^r)$ is unitarily induced irreducible from a finite dimensional $J_e \times J_o$ of a Levi component $GL_e \times GL_o \subset GL(\mu_r)$, where the parameters of $J_e$ and $J_o$ come from the $\mathbb{Z}$ and $\mathbb{Z} + \frac{1}{2}$ coordinates of $J(\lambda_L^r, \lambda_R^r)$ respectively.

Note that by Theorem 2.1(c), and the assumption that $J(\lambda_L, \lambda_R)$ has an invariant Hermitian form, both $J_e$ and $J_o$ have invariant Hermitian forms. Using Casimir's inequality [V1, Lemma 12.6], unless $J_e$ and $J_o$ are unitary characters, otherwise $J(\lambda_L^r, \lambda_R^r)$ have indefinite form on $K-$types $V_{\mathfrak{k} \cap \mathfrak{gl}(\mu_r)}(r + 1, r, \dots, r, r - 1)$ and $V_{\mathfrak{k} \cap \mathfrak{gl}(\mu_r)}(r, \dots, r)$. Since these $K-$types are bottom layer in the induced modules (15), $J$ is unitary only if $J(\lambda_L^r, \lambda_R^r)$ is unitary and induced from unitary characters. So $\begin{pmatrix} \lambda_L^r \\ \lambda_R^r \end{pmatrix}$ must consist of at most two strings as in the statement of the Proposition. $\qquad \square$

**Remark 3.4.** *Since all Levi subgroups of $G = GL(n, \mathbb{C})$ consist only of $GL-$factors, one can apply the above Proposition for all $r \in \mathbb{Z}$ to conclude that Theorem 3.1 holds for Type A. Hence we focus on the classical groups of Type B, C and D from now on.*

**Corollary 3.5.** *Assume $\mu_1 \neq 0$. Then*

$$\begin{pmatrix} \lambda_L^1 \\ \lambda_R^1 \end{pmatrix} = \begin{cases} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} & \textit{in types } B, C \\ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \textit{ or } \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix} & \textit{in type } D. \end{cases}$$

*Proof.* The statement is a direct consequence of the fact that $2\lambda$ is assumed regular integral. $\qquad \square$

We consider $J_{G(\mu_1 + \mu_0)}(\lambda_L^1, \lambda_L^0, \lambda_R^1, \lambda_R^0)$ appearing in $J_2$ of (15). A consequence of Proposition 3.3 and Corollary 3.5 is that we can write the parameter as

(16)        $(\lambda_{rel}, -s_{rel}\lambda_{rel}) := (\lambda^1, \lambda^0, -\lambda^1, \lambda^0)$    with    $\lambda^1 = \underbrace{(1, \dots, 1)}_{\mu_1}$    $\mu_1 = 0, 1, 2.$

Specifically, $\lambda_{rel} = (\lambda^1, \lambda^0)$ and $s_{rel}$ is an involution so that $s_{rel}(\lambda^1, \lambda^0) = (-\lambda^1, \lambda^0)$. Sections 4–6 is devoted to proving the following:

**Theorem 3.6.** *Assume that the parameter is half-integral regular, and $\mu_r = 0$ for $r \geq 2$ so that $\lambda = \lambda_{rel}$. Then $J(\lambda, -s_{rel}\lambda)$ is unitary if and only if it is of the form given in Theorem 3.1; i.e. unipotent tensored with a unitary character. When it is not unitary, the form is indefinite on cx-relevant $K-$types.*

**Corollary 3.7.** *Let $J(\lambda, -s\lambda)$ be an irreducible module with half integral regular infinitesimal character. Then Theorem 3.1 holds.*

*Proof.* The Corollary (and therefore Theorem 3.1) follows immediately from properties of bottom layer $K$−types. Suppose $J(\lambda_{rel}, -s_{rel}\lambda_{rel})$ is not of the form given in Theorem 3.1. Then by Proposition 3.6 it must be non-unitary, which has indefinite form on cx-relevant $K$−types. Since all cx-relevant $K$−types are bottom layer in $I_2$, this implies that $J(\lambda, -s\lambda)$ is not unitary.

On the other hand, if $J(\lambda_{rel}, -s_{rel}\lambda_{rel})$ is of the form given in Theorem 3.1, then by induction in stages $I_2$ is of the form given by Theorem 3.1, with $J(\lambda, -s\lambda)$ being its lowest $K$−type subquotient. Since it is a subquotient of the unitary module $I_2$, $J$ is unitary. A sharper result holds – by Theorem 14.1 of [B1], $I_2 = J(\lambda, -s\lambda)$. □

3.3. **General Strategy.** By the corollary above, it suffices to prove Theorem 3.6. In particular, when the parameter is not as in Theorem 3.1, the form is indefinite on a cx-relevant $K$−type. These give rise to bottom layer $K$−type in the general case.

To treat the case $J((\lambda^1, \lambda^0), (-\lambda^1, \lambda^0))$ given in Theorem 3.6, the spherical case $J(\lambda^0, \lambda^0)$ plays an important role. Write $\lambda = \lambda^0$ from now on. We define a parabolic subgroup $P(\lambda)$ and a representation $\pi_{L(\lambda)}$ on its Levi component so that the induced module $I_{P(\lambda)} := \mathrm{Ind}_{P(\lambda)}^{G}(\pi_{L(\lambda)})$ is Hermitian, and the cx-relevant $K$−types occur with full multiplicity in the spherical subquotient $J(\lambda, \lambda)$. The induction step proceeds as follows. Deform $\lambda$ and the induced module $I_{P(\lambda)}$ to $\lambda + t\nu$ where $\nu$ is central for $L(\lambda)$, so that the norm of the parameter becomes larger, and the multiplicities of the cx-relevant $K$−types do not change for small $t$. Let $t_0 > 0$ be the nearest where the multiplicities change; $P(\lambda + t_0\nu)$ changes as well. If the condition in Theorem 3.1 are not satisfied, the induction hypothesis holds, so the form is *indefinite on cx-relevant $K$−types*, that is, the form has different signatures on the lowest $K$−type and at least one of the cx-relevant $K$−types, and the semi-continuity of the signature implies that the form was indefinite on cx-relevant $K$−types at $t = 0$. The exceptions are when $J(\lambda + t_0\nu, \lambda + t_0\nu)$ is unitary, or the deformation goes on to "$\infty$". In the first case we find a non-spherical factor in the deformed induced module with a pair of indefinite cx-relevant $K$−types. In the second case, the Casimir inequality implies that the form is indefinite on the trivial and adjoint $K$−types.

*We will henceforth concentrate on the cases when $\lambda$ is NOT regular integral. The cases when $\lambda$ is regular integral, are covered by [E]; the unipotent representations occurring are $\pi_u = Triv$.*

## 4. Proof of Theorem 3.6 – Type $B$

Let $G = SO(2m+1, \mathbb{C})$ and $K = SO(2m+1)$. The $K$−types have highest weights $\eta$ with coordinates integers only. Since $\rho = (m - 1/2, \ldots, 1/2)$, $2\lambda = \{\eta - \rho\} + \rho$, $2\lambda$ must have integer coordinates only; so $\lambda$ has integer and half-integer coordinates. Since we assume that $\lambda$ is regular half-integral but not integral, the integral system determined by $\lambda$ is type $C \times C$.

4.1. **Spherical Representations.** In the next few subsections, we will prove the following Proposition.

**Proposition 4.1.** *Let $\lambda$ be regular half-integral. The spherical irreducible module $J(\lambda, \lambda)$ is unitary if and only it is unipotent, i.e. the parameter is*

$$\lambda = \left(-K_0 + \frac{1}{2}, \ldots, -\frac{1}{2}; -N_0, \ldots, -1\right)$$

*with $N_0 \leq K_0$. This is a unipotent representation attached to the nilpotent orbit $[2^{2N_0} 1^{2K_0 - 2N_0 + 1}]$.*

*When not unitary, the form is indefinite on the set of cx-relevant $K-$types with highest weights*

$$CXB := \{(0, \ldots, 0), \quad (1, \ldots, 1, 0, \ldots, 0), \quad (2, 0, \ldots, 0)\}.$$

The unipotent representation in Proposition 4.1 is unitary because it can be realized via the dual pair correspondence as $\Theta(\mathrm{triv}_{Sp})$, from the pair $Sp(2N_0, \mathbb{C}) \times SO(2K_0 + 2N_0 + 1, \mathbb{C})$ in the stable range.

In order to prove the non-unitarity of other parameters, we use the strategy in Section 3.3. We construct an induced module $I_{P(\lambda)}$ having $J(\lambda, \lambda)$ as a quotient. Let $\lambda$ be half-integral and dominant for the standard positive system, i.e.

$$\lambda = (\ldots \lambda_i \geq \lambda_{i+1} \geq \cdots \geq 0), \quad 2\lambda_i \in \mathbb{N}.$$

If $\lambda$ is further assumed to be regular, then the above inequalities are strict. We construct a parabolic subgroup $P(\lambda) = L(\lambda)U(\lambda)$ and an induced module $I_{P(\lambda)}$ so that $J(\lambda, \lambda)$ is the spherical irreducible factor in $I_{P(\lambda)}$, and the multiplicities of the cx-relevant $K-$types are the same.

(i) If $1/2$ is a coordinate of $\lambda$, form the longest *string*

$$\kappa_0 := (-K_0 + 1/2, \ldots, -1/2)$$

such that all the half-integers starting from $1/2$ to $K_0 - 1/2$ are coordinates of $\lambda$, but $K_0 + 1/2$ is not. If the coordinate $1$ occurs, form the longest string

$$\sigma_0 := (-N_0, \ldots, -1)$$

where $N_0$ is the largest integer coordinate that occurs in $\lambda$, but $N_0 + 1$ does not. Add a factor to $L(\lambda)$ of type $G(K_0 + N_0) = SO(2K_0 + 2N_0 + 1)$ and the spherical irreducible representation with parameter

$$\begin{pmatrix} -K_0 + 1/2 & ,\ldots, & -1/2 & ; & -N_0 & ,\ldots, & -1 \\ -K_0 + 1/2 & ,\ldots, & -1/2 & ; & -N_0 & ,\ldots, & -1 \end{pmatrix}$$

If $1/2$ is not a coordinate, let $k_1 - 1/2 > 0$ be the smallest half-integer coordinate, and form the string $\kappa_1 = (k_1 - 1/2, \ldots, K_1 - 1/2)$ with increasing coordinates differing by 1 as before. Add a factor $GL(K_1 - k_1 + 1)$, and the 1-dimensional representation of $GL(K_1 - k_1 + 1)$ with parameter

$$\begin{pmatrix} k_1 - 1/2 & ,\ldots, & K_1 - 1/2 \\ k_1 - 1/2 & ,\ldots, & K_1 - 1/2 \end{pmatrix}$$

to $L(\lambda)$. Similarly if 1 does not occur as a coordinate, form $\sigma_1 = (n_1, \ldots, N_1)$, add a factor $GL(N_1 - n_1 + 1)$ to the Levi component $L(\lambda)$, and the 1-dimensional representation of $GL(N_1 - n_1 + 1)$ with parameter

$$\begin{pmatrix} n_1 & ,\ldots, & N_1 \\ n_1 & ,\ldots, & N_1 \end{pmatrix}$$

(ii) Remove the coordinates in Step (i) from $\lambda$, and repeat on the remainder until there are no half-integer coordinates left. Since the assumption was that at most one coordinate was equal to $1/2$, only $GL-$factors are created.

(iii) Repeat Steps (i) and (ii) on the integer coordinates until there are none left.

The process produces a parabolic subgroup, and an induced module on its Levi component. The Levi component is

$$(17) \qquad L(\lambda) := \prod_{i>0} GL(\sigma_j) \times \prod_{j>0} GL(\kappa_i) \times G(K_0 + N_0).$$

If $\lambda$ is assumed to be regular, its corresponding strings $\kappa_i$, $\sigma_j$ satisfy

$$(18) \qquad \begin{cases} k_i > 2 & \text{if } 1/2 \text{ is a coordinate,} \\ k_i \geq 2 & \text{otherwise} \end{cases} \quad \text{and} \quad \begin{cases} n_j > 2 & \text{if } 1 \text{ is a coordinate,} \\ n_j \geq 2 & \text{otherwise} \end{cases}$$

In the proof of Proposition 4.1 below, we begin with $J(\lambda, \lambda)$ where $\lambda$ is regular and half-integral. Then we deform some $GL$–strings $\kappa_i$, $\sigma_j$, $i, j > 0$ upward and analyze the new parameter $\lambda_{new}$ and its corresponding induced module $I(\lambda_{new})$. Here $\lambda_{new}$ is half-integral but is not necessarily regular (see Example 4.4 below). Nevertheless, by the above construction of $\kappa$ and $\sigma-$strings, it is easy to see that the more general parameters satisfy

$$(19) \qquad \begin{cases} k_{i+1} - K_i \geq 2, \text{ or} \\ k_i \leq k_{i+1} \leq K_{i+1} \leq K_i, \end{cases} \quad \text{and} \quad \begin{cases} n_{j+1} - N_j \geq 2, \text{ or} \\ n_i \leq n_{i+1} \leq N_{i+1} \leq N_i \end{cases} .$$

We say the strings $\kappa_i$, $\kappa_{i+1}$ (or $\sigma_j$, $\sigma_{j+1}$) **nested** if its parameters satisfy (19) for all $i, j \geq 0$. The parabolic subgroup is determined by the order of the factors, and the integer and half-integer strings are interchangeable.

The main property of the cx-relevant $K-$types is the following Lemma.

**Lemma 4.2.** *Let $\lambda$ be dominant whose coordinates are half-integers. Assume that the strings of $\lambda$ satisfy (18) and (19). The multiplicities of the cx-relevant $K-$types in $I_{P(\lambda)}$ coincide with those in $J(\lambda, \lambda)$.*

*Proof.* This kind of result can be found in [B2]. The main difference is that $(2, 0, \ldots, 0)$ is not petite/single petaled. The condition that the value of $\check{\alpha}$ for $\alpha$ a long root on the highest weight of the $K-$type be $\leq 3$ is satisfied except for the case of $(2, 0, \ldots, 0)$ and a long root. The crucial property needed is that $SL(2)-$intertwining operators be isomorphisms on these $K-$types. Condition (18) insures that this property is still valid for the larger class of $K-$types. We sketch the steps.

Recall that $\lambda$ was assumed dominant. Then $J(\lambda, \lambda)$ is the image of the long intertwining operator from $I_B(\lambda, \lambda)$ to $I_B(-\lambda, -\lambda)$. The module $I_{P(\lambda)}$ is a homomorphic image of $I_B(\lambda, \lambda)$. The long intertwining operator $A_{w_0}$ factors into

$$I_B(\lambda, \lambda) \longrightarrow I_{P(\lambda)} \longrightarrow I_B(-\lambda, -\lambda).$$

We only need to show that the intertwining operator on the right is an injection on the cx-relevant $K$−types. We need to "*flip*" the coordinates of the $\kappa_i$ and $\sigma_j$ into their negatives. This is done by embedding into a larger induced module where it is possible to factor the operator further into ones induced from $SL(2)'_\alpha$s. Condition (18) insures that they are isomorphisms on the restrictions of the cx-relevant $K$−types. This is also the reason that we have put $\kappa_0$ and $\sigma_0$ into the Levi component. $\qquad\square$

We finish this subsection by giving a necessary condition on the spherical parameter:

**Lemma 4.3.** *If $J(\lambda, \lambda)$ is unitary, then the string $\kappa_0 = (-K_0 + 1/2, \ldots, 1/2)$ must appear in $(\lambda, \lambda)$.*

*Proof.* The coordinates on the spherical part of $I_1$ in Equation (15) are all $\geq 1$. The Casimir inequality implies that the form is indefinite on the adjoint $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ $K$−type and the trivial $K$−type $V_{\mathfrak{k}}(0, \ldots, 0)$ . These give rise to bottom layer $K$−types of $I_1$, and hence the irreducible $J(\lambda, \lambda)$ is not unitary. $\qquad\square$

4.2. **Proof of Proposition 4.1 $-$ $\lambda = \kappa_0 \cup \sigma_0$.** If only $\sigma_0$ occurs in $\lambda$, then it is not unitary by Lemma 4.3. Furthermore, the case when $\lambda = \kappa_0 \cup \sigma_0$ with $K_0 \geq N_0$ is unitary. So assume

(20) $$\lambda = \kappa_0 \cup \sigma_0 \quad \text{satisfying} \quad N_0 > K_0 \geq 1.$$

Let

$$Ind(\lambda_t) := \operatorname{Ind}_{GL(\sigma_0) \times G(K_0)}^{G}((1 + t, \ldots, N_0 + t) \otimes \operatorname{triv}),$$

The signatures and multiplicities of the fundamental cx-relevant $K$−types of the form $V_{\mathfrak{k}}(1, \ldots, 1, 0, \ldots, 0)$ coincide on $Ind(\lambda_0)$ and $J(\lambda, \lambda)$. Indeed, $Ind(\lambda)$ is a homomorphic image of $\operatorname{Ind}_B^G(\lambda, \lambda)$, and the intertwining operator changing $(1, \ldots, N_0)$ to $(-N_0, \ldots, -1)$ involves only $(\check\alpha, w\lambda)$ which are integers $\geq 2$:

$$\begin{pmatrix} i \\ i \end{pmatrix} \mapsto \begin{pmatrix} -i \\ i \end{pmatrix}$$

The kernel of the intertwining operator has lowest $K$−type of highest weight $(2i)$ for $1 \leq i \leq N_0$. So the intertwining operator is an isomorphism on the cx-relevant $K$−types $V_{\mathfrak{k}}(1, \ldots, 1, 0, \ldots, 0)$ (but *not* necessarily for $V_{\mathfrak{k}}(2, 0 \ldots, 0)$). These values remain unchanged for all $Ind(\lambda_t)$ with $t \in [0, 1/2)$ because the multiplicities do not change. At $t = 1/2$,

$$\lambda_{1/2} = (3/2, 5/2, \ldots, N_0 + 1/2; -K_0 + 1/2, \ldots, -1/2)$$
$$= (-N_0 - 1/2, \ldots, -1/2) \cup (3/2, \ldots, K_0 - 1/2).$$

So the induced module $I_{P(\lambda_{1/2})}$ defined in Section 4.1 is given by

$$I_{P(\lambda_{1/2})} = \operatorname{Ind}_{GL(K_0 - 1) \times G(N_0)}^{G}((3/2, \ldots, K_0 - 1/2) \otimes \operatorname{triv}),$$

and differs from $Ind(\lambda_{1/2})$. More precisely, apart from $J(\lambda_{1/2}, \lambda_{1/2})$, $Ind(\lambda_{1/2})$ has a non-spherical irreducible factor whose parameter is given by

$$\begin{pmatrix} 1/2, \ldots, K_0 - 1/2; & 3/2, \ldots, K_0 + 1/2; & K_0 + 3/2, \ldots, N_0 + 1/2 \\ 3/2, \ldots, K_0 + 1/2; & 1/2, \ldots, K_0 - 1/2; & K_0 + 3/2, \ldots, N_0 + 1/2 \end{pmatrix}$$

This module has indefinite form on the $K-$types $V_{\mathfrak{k}}(\underbrace{1, \ldots, 1}_{2K_0}, 0, \ldots, 0)$ and

(21)
$$\begin{cases} V_{\mathfrak{k}}(\underbrace{1, \ldots, 1}_{2K_0}, 1) & \text{if } N_0 = K_0 + 1; \\ V_{\mathfrak{k}}(\underbrace{1, \ldots, 1}_{2K_0}, 1, 1, 0, \ldots, 0) & \text{otherwise} \end{cases}$$

Indeed, the second $K-$type is bottom layer for the parabolic subgroup with Levi component $GL(K_0) \times G(N_0 - K_0)$. The *spherical part* of the parameter $\begin{pmatrix} K_0 + 3/2, \ldots, N_0 + 1/2 \\ K_0 + 3/2, \ldots, N_0 + 1/2 \end{pmatrix}$ is a finite dimensional representation of $G(N_0 - K_0)$, so the form is indefinite on the trivial and adjoint $K-$types of $G(N_0 - K_0)$.

Consequently, by semicontinuity of signatures, $Ind(\lambda_0)$ and $J(\lambda, \lambda)$ also have indefinite form on the $K-$types given in (21).

4.3. **Proof of Proposition 4.1 – Other Strings.** Assume $\lambda$ contains strings other than $\kappa_0$ and $\sigma_0$. We do an induction upward on the length of the parameter, downward on the number of strings.

Assume there is a $\kappa_i = (k_i - 1/2, \ldots, K_i - 1/2)$ with $i > 0$ or $\sigma_j = (n_j, \ldots, N_j)$ with $j > 0$. Replace it by $(k_i - 1/2 + t, \ldots, K_i - 1/2 + t)$ (or $(n_j + t, \ldots, N_j + t)$), and denote the new parameter by $\lambda_t$. At $t = 0$, $I_{P(\lambda)} = I_{P(\lambda_0)}$, and the signatures of cx-relevant $K-$types do not change for $0 \leq t < 1/2$. At $t = 1/2$, if the induction hypothesis (condition for the form to be indefinite on the cx-relevant $K-$types) holds for $J(\lambda_{1/2}, \lambda_{1/2})$ we conclude that $J(\lambda, \lambda)$ is not unitary, with form indefinite on the cx-relevant $K-$types. It may happen that $I_{P(\lambda_{1/2})}$ is unchanged, and we can continue to deform $t$ upward. $I_{P(\lambda)}$ may be unchanged as $t \longrightarrow \infty$. In this case the form is indefinite on the adjoint $K-$type $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$. We call this an initial case. The other case is when the spherical module $J(\lambda_{1/2}, \lambda_{1/2})$ is unitary. This is the case $\sigma_0 \cup \kappa_0$ with $K_0 \geq N_0$. Note that it includes the case when the spherical module is the trivial representation.

In summary, these cases, which we call *initial cases* are

(a) There is a string $\kappa_i$ or $\sigma_j$ with $i, j > 0$ such that $P(\lambda_t)$ does not change as $t \to \infty$,
(b) The strings are

$$(-K_0 + 1/2, \ldots, -1/2; -N_0, \ldots, -1), \quad \text{with } K_0 < N_0$$

as in the previous section.
(c) The strings are $(-K_0 + 1/2, \ldots, -1/2; -N_0, \ldots, -1) \cup \xi$ satisfying

$$\xi = (K_0, \ldots, K_1) \text{ or } (N_0 + 1/2, \ldots, N_1 - 1/2),$$

so that the deformation of $\xi$ to $t = 1/2$ yields a unitary spherical module. This means that $K_1 \geq N_0$ in one case, $K_0 \geq N_1$ in the other case. See Example 4.4 for more details. In Case (a), as already mentioned, the Casimir inequality implies that the spherical irreducible module at $t = 1/2$ has indefinite form on the trivial and adjoint $K-$types $V_{\mathfrak{k}}(0, \ldots, 0)$ and $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$.

Case (b) was discussed in the previous section.

For Case (c), we give details for $\xi = (K_0)$. The other $\xi$ are similar. $I_{P(\lambda_{1/2})}$ has another irreducible factor with parameter containing

$$\begin{pmatrix} -K_0 + 1/2 & -K_0 - 1/2 & K_0 - 3/2 & \ldots & 1/2 \\ -K_0 - 1/2 & -K_0 + 1/2 & K_0 - 3/2 & \ldots & 1/2 \end{pmatrix}$$

with the rest of the spherical part formed of integer coordinates coming from $\sigma_0$.

The lowest $K-$type is $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ and $V_{\mathfrak{k}}(2, 0, \ldots, 0)$ is bottom layer. Since for such a parameter the form on the $GL(2)-$factor is indefinite on $(1, 1)$ and $(2, 0) = (1, 1) + (1, -1)$, semicontinuity of the signature implies the same for the parameter at $\lambda$.

The proof of Proposition 4.1 is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Example 4.4.** *Let* $\lambda = (-11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (3, 4) \cup (6)$. *Note that* $\kappa_0$ *is longer than* $\sigma_0$. *Deform all* $\sigma_i$ *into* $\kappa_i$ *for* $i > 0$:

$$\begin{aligned} \lambda =\ & (-11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (\mathbf{3}, \mathbf{4}) \cup (\mathbf{6}) \\ \longrightarrow\ & (-11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (7/2, 9/2) \cup (13/2) \\ =\ & (-13/2, -11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (7/2, 9/2) \end{aligned}$$

*Deform the new* $\kappa_i$ *for* $i > 0$ *and get*

$$\begin{aligned} & (-13/2, -11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (\mathbf{7/2}, \mathbf{9/2}) \\ \longrightarrow\ & (-13/2, -11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (13/2, 15/2) \\ =\ & (-15/2, -13/2, -11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (\mathbf{13/2}) \\ \longrightarrow\ & (-17/2, -15/2, -13/2, -11/2, -9/2, -7/2, -5/2, -3/2, -1/2; -1) \cup (9) \end{aligned}$$

*and we are in Case (c) above.*

4.4. **Non-spherical Case.** Now we study the case when $\mu_1 > 0$. Then the parameter $(\lambda_{rel}, -s\lambda_{rel})$ does not have a $\kappa_0$, or else the regularity condition is violated. Consider the spherical part of the parameter. It only contains $\kappa_i$ for $i > 0$ and $\sigma_j$ for $j \geq 0$. By Lemma 4.3 this spherical parameter yields indefinite form on $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ and $V_{\mathfrak{k}}(0, \ldots, 0)$, both are bottom layer in $J(\lambda_{rel}, -s\lambda_{rel})$. Therefore there cannot be any spherical parameter, and the only unitary case is $(\lambda_{rel}, -s\lambda_{rel}) = (1/2, -1/2)$.

4.5. **Spin Groups.** In this section, we give a brief idea on how our results can be extended to Spin groups $G = Spin(2n + 1, \mathbb{C})$. We only consider genuine representations of $G$, i.e. representations whose $K-$types have highest weights with coordinates of the form $\mathbb{N} + \frac{1}{2}$ only. As $\rho = (m - 1/2, \ldots, 1/2)$, so $2\lambda = \{\eta - \rho\} + \rho$ must have coordinates of the form $\mathbb{N} + \frac{1}{2}$ only. The integral system for $\lambda$ is type A.

We study the case when the lowest $K-$type of $J(\lambda_L, \lambda_R)$ is

$$Spin = V_{\mathfrak{k}}(\frac{1}{2}, \ldots, \frac{1}{2}).$$

The parameter is

$$
\begin{array}{lll}
\lambda_L = & (1/4, \ldots, 1/4) & + (\nu_1, \ldots, \nu_k, -\nu_k, \ldots, -\nu_1) \\
\lambda_R = & (-1/4, \ldots, -1/4) & + (\nu_1, \ldots, \nu_k, -\nu_k, \ldots, -\nu_1)
\end{array}
$$

(22)          or

$$
\begin{array}{lll}
\lambda_L = & (1/4, \ldots, 1/4) & + (\nu_1, \ldots, \nu_k, 0, -\nu_k, \ldots, -\nu_1) \\
\lambda_R = & (-1/4, \ldots, -1/4) & + (\nu_1, \ldots, \nu_k, 0, -\nu_k, \ldots, -\nu_1)
\end{array}
$$

The symmetry $\nu_i \longleftrightarrow -\nu_i$ follows from the assumption that the parameter must be Hermitian. Since $2\lambda_L = (\frac{1}{2} + 2\nu_1, \ldots, \frac{1}{2} - 2\nu_1)$ must be regular integral consisting of half-integers, it follows that

(23)                                   $2\nu_i \in \mathbb{Z}$ for all $i$,

satisfying $\nu_i \pm \nu_j \neq 0$, and $\nu_i \neq 0$.

Separate the $\nu_i$ into integers $\nu_a$ and half-integers $\nu_b$. The Hermitian property implies that $\nu_a$ must be conjugate to $-\nu_a$ by the symmetric group, and similarly for $\nu_b$.

There are two finite dimensional Hermitian representations $F_a$ and $F_b$ of Type A (with lowest $K-$types $V_{\mathfrak{u}}(\frac{1}{2}, \ldots, \frac{1}{2})$) so that

(24)                         $J(\lambda_L, \lambda_R) = \mathrm{Ind}_{GL \times GL}^{G}(F_a \otimes F_b).$

The restriction of $V_{\mathfrak{k}}(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ to $GL$ contains

$$V_{\mathfrak{u}}(\frac{3}{2}, \frac{1}{2}, \ldots, -\frac{1}{2}) = V_{\mathfrak{u}}(\frac{1}{2}, \ldots, \frac{1}{2}) \otimes V_{\mathfrak{u}}(1, 0, \ldots, 0, -1).$$

Therefore, as in Proposition 3.3, the Hermitian form of $J(\lambda_L, \lambda_R)$ on the $K-$types $V_{\mathfrak{k}}(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$ and $V_{\mathfrak{k}}(\frac{1}{2}, \ldots, \frac{1}{2})$ is indefinite unless $F_a, F_b$ are unitary characters. In the case when there is only $F_a$ or $F_b$ in (24), we obtain the genuine unipotent representation with infinitesimal character given in (9).

## 5. Proof of Theorem 3.6 – Type C

Let $G = Sp(2m, \mathbb{C})$ and $K = Sp(2m)$. The $K-$types have highest weights $\eta$ formed of integers only. Since $\rho = (m, \ldots, 1)$, $2\lambda = \{\eta - \rho\} + \rho$ must have positive integer coordinates only. So $\lambda$ must have integers and half integer coordinates only. Since $\lambda$ is regular half-integral but not integral, the integral system determined by $\lambda$ is type $B \times D$.

### 5.1. **Spherical Representations.**

**Proposition 5.1.** *Let $\lambda$ be regular half-integral. The spherical irreducible module $J(\lambda, \lambda)$ is unitary if and only it is unipotent, i.e. the parameter is*

$$\lambda = \left(-K_0 + \frac{1}{2}, \ldots, -\frac{1}{2}\right) \quad or \quad \lambda = (-N_0, \ldots, -1)$$

*The first representation is the spherical component of the Oscillator representation attached to the nilpotent orbit $[2^1 1^{2N_0-2}]$, and the second case is the trivial representation attached to $[1^{2N_0}]$.*

*When not unitary, the form is indefinite on the set of cx-relevant $K-$types with highest weights*

$$CXC := \{(0, \ldots, 0), \quad (1, 1, 0, \ldots, 0), \quad (2, 0, \ldots, 0)\}.$$

Unlike Types B or D, only (1,1,0,...,0), rather than (1,...,1, 0,...,0) suffices. The proof will be given in the next subsection. The unipotent representation is unitary because, when not the trivial module, it is the spherical component of the Oscillator representation.

As in the case of Type B, we construct a parabolic subgroup $P(\lambda) = L(\lambda)U(\lambda)$ and an induced module $I_{P(\lambda)}$ so that $J(\lambda, \lambda)$ is the spherical irreducible factor in $I_{P(\lambda)}$, and the multiplicities of the cx-relevant $K-$types coincide in the two modules. Write $\lambda$ dominant for the standard positive system, i.e.

$$\lambda = (\ldots \lambda_i \geq \lambda_{i+1} \geq \cdots \geq 0), \quad 2\lambda_i \in \mathbb{Z}.$$

Since the parameters we are going to study are obtained by deforming a regular parameter *upward*, we can further assume that all $\lambda_i$ are positive.

(i) If $1/2$ is a coordinate of $\lambda$, form the longest *string*

$$\kappa_0 = (-K_0 + 1/2, \ldots, -1/2)$$

such that all the half-integers starting from $1/2$ to $K_0 - 1/2$ are coordinates of $\lambda$, but $K_0 + 1/2$ is not. If the coordinate 1 occurs, form the longest string

$$\sigma_0 = (-N_0, \ldots, -1)$$

where $1, \ldots, N_0$ occur as coordinates in $\lambda$, but $N_0+1$ does not. Add a factor of $L(\lambda)$ of type $G(K_0 + N_0) = Sp(2K_0 + 2N_0)$ and the spherical irreducible representation with parameter

$$\begin{pmatrix} -K_0 + 1/2 & ,\ldots, & -1/2 & ; & -N_0, & \ldots, & -1 \\ -K_0 + 1/2 & ,\ldots, & -1/2 & ; & -N_0, & \ldots, & -1 \end{pmatrix}$$

If $1/2$ is not a coordinate, let $k_1 - 1/2 > 0$ be the smallest half-integer coordinate, and form the longest string $\kappa_1 = (k_1 - 1/2, \ldots, K_1 - 1/2)$ increasing by 1, as before. Add a factor $GL(K_1 - k_1 + 1)$, and the 1-dimensional representation with parameter

$$\begin{pmatrix} k_1 - 1/2 & ,\ldots, & K_1 - 1/2 \\ k_1 - 1/2 & ,\ldots, & K_1 - 1/2 \end{pmatrix}$$

to $M(\lambda)$. Similarly if 1 does not occur as a coordinate, form $\sigma_1 = (n_1, \ldots, N_1)$ and add a factor $GL(N_1 - n_1 + 1)$ to the Levi component $M(\lambda)$.

(ii) Remove the coordinates in Step (i) from $\lambda$, and repeat on the remainder until there are no half-integer coordinates left. Since the assumption was that at most one coordinate was equal to $1/2$, only $GL-$factors are created.

(iii) Repeat Steps (i) and (ii) on the integer coordinates until there are none left.

The process produces a parabolic subgroup, and an induced module on its Levi component. The Levi component is

$$(25) \qquad \prod_{i>0} GL(\sigma_j) \times \prod_{j>0} GL(\kappa_i) \times G(K_0 + N_0).$$

As in the case of Type B, we are interested in the cases when the *strings* satisfy the properties:

$$(26) \qquad \begin{cases} k_i > 2 & \text{if } 1/2 \text{ is a coordinate,} \\ k_i \geq 2 & \text{otherwise} \end{cases} \quad \text{and} \quad \begin{cases} n_j > 2 & \text{if } 1 \text{ is a coordinate,} \\ n_j \geq 2 & \text{otherwise} \end{cases}$$

along with the nested condition:

$$(27) \qquad \begin{cases} k_{i+1} - K_i \geq 2, \text{ or} \\ k_i \leq k_{i+1} \leq K_{i+1} \leq K_i, \end{cases} \quad \text{and} \quad \begin{cases} n_{j+1} - N_j \geq 2, \text{ or} \\ n_i \leq n_{i+1} \leq N_{i+1} \leq N_i \end{cases} .$$

The main property of the cx-relevant $K-$types is the following Lemma.

**Lemma 5.2.** *Let $\lambda$ be such that (26) and (27) are satisfied. The multiplicities of the cx-relevant $K-$types is the same in $I_{P(\lambda)}$ and $J(\lambda, \lambda)$.*

*Proof.* The proof follows the one for the analogous result in Type B. We have to show that certain $SL(2)_\alpha-$operators are isomorphisms. For the cx-relevant $K-$types this follows from conditions (26) and (27) and the fact that the coordinates of the highest weights of the $K-$types are $\leq 2$. $\qquad\square$

5.2. **Proof of Proposition 5.1** $- \lambda = \sigma_0 \cup \kappa_i$ **or** $\kappa_0 \cup \sigma_i$. If $\lambda$ contains only $\sigma_0 = (-N_0, \ldots, -1)$ or $\kappa_0 = (-K_0 + 1/2, \ldots, -1/2)$, the parameter is unitary. So consider $\lambda = \sigma_0 \cup \kappa_i$ or $\kappa_0 \cup \sigma_i$ for $i = 0$ or $1$, and the induced module

$$\text{Ind}_{GL(K_i) \times G(N_0)}^{G} (\kappa_i \otimes (-N_0, \ldots, -1)) \quad \text{or} \quad \text{Ind}_{GL(N_i) \times G(K_0)}^{G} (\sigma_i \otimes (-K_0 + 1/2, \ldots, -1/2)).$$

If $i = 1$, i.e. $k_1 \geq 3/2$ or $N_1 \geq 2$, then the above induced modules admit deformations where the multiplicities of all cx-relevant $K-$types coincide with that of $J(\lambda, \lambda)$ for $0 \leq t < 1/2$. If $i = 0$, the deformations still preserve multiplicities of the cx-relevant $K-$types of the form $V_{\mathfrak{k}}(1, \ldots, 1, 0, \ldots, 0)$. There are two cases:

(a) Suppose If $k_i - N_0 > 1$ or $n_i - K_0 \geq 1$ (so that $i = 1$), or equivalently one has $|n - k| \geq 3/2$ for all $n \in \sigma_i$ and $k \in \kappa_j$, the deformations on $\kappa_1$ or $\sigma_1$ does not produce new $P(\lambda)$ for all $t \geq 0$. So by Casimir inequality the form is indefinite on the trivial and the adjoint $K-$type $V_{\mathfrak{k}}(2, 0, \ldots, 0)$.

(b) Otherwise, At $t = 1/2$, the spherical parameter acquires a new $\sigma_1$ or $\kappa_1$. As in Type B, we can apply induction hypothesis and reduce to the *initial cases* when the spherical parameter at $t = 1/2$ is either the trivial representation, or the spherical Oscillator

representation. These are

$$(N_0 + 1/2, \ldots, N_1 + 1/2) \cup (-N_0, \ldots, -1) \quad \text{or} \quad (K_0, \ldots, K_1) \cup (-K_0 + 1/2, \ldots, -1/2)$$

The argument for type B applies. At $t = 1/2$ there is another factor

$$\begin{pmatrix} K_0 + 1/2 & -K_0 + 1/2 & K_1 + 1/2 & \ldots & K_0 + 3/2 & K_0 - 3/2 & \ldots & 1/2 \\ K_0 - 1/2 & -K_0 - 1/2 & K_1 + 1/2 & \ldots & K_0 + 3/2 & K_0 - 3/2 & \ldots & 1/2 \end{pmatrix}$$

(28)     respectively

$$\begin{pmatrix} N_0 + 1 & -N_0 & N_1 + 1 & \ldots & N_0 + 2 & N_0 - 1 & \ldots & 1 \\ N_0 - 1 & -N_0 - 1 & N_1 + 1 & \ldots & N_0 + 2 & N_0 - 1 & \ldots & 1 \end{pmatrix}$$

The $K$−types $V_{\mathfrak{k}}(2, 0, \ldots, 0)$ and $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ are bottom layer for the parameter in (28), and the form is indefinite. In this case one can in fact show that at $t = 0$ the form is indefinite on $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ and $V_{\mathfrak{k}}(0, \ldots, 0)$. The reason is that one can deform the string $\kappa_1$ or $\sigma_1$ all the way to a place where the module is unitarily induced irreducible, and $V_{\mathfrak{k}}(2, 0, \ldots, 0)$ occurs with full multiplicity in the spherical irreducible module. So its sign must be the same as that of $V_{\mathfrak{k}}(0, \ldots, 0)$. Therefore, $J(\lambda, \lambda)$ has indefinite forms on $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ and $V_{\mathfrak{k}}(0, \ldots, 0)$.

**Remark 5.3.** *More generally, if $\lambda = \sigma_i \cup \kappa_j$ satisfies $k_j \leq N_i + 1 \leq K_j$ or $n_i \leq K_j \leq N_i$, i.e. there are $n \in \sigma_i$ and $k \in \kappa_j$ such that $|n - k| = 1/2$, then one can deform both strings $\sigma_i$, $\kappa_j$ downwards simultaneously*

$$\sigma_i \cup \kappa_j \mapsto \sigma_i \cup \kappa_j - (t, \ldots, t),$$

*until it reaches Case (b) above. Then one can conclude that $J(\lambda, \lambda)$ has indefinite forms on $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ and $V_{\mathfrak{k}}(0, \ldots, 0)$.*

5.3. **Proof of Proposition 5.1 − Other Strings.** We do an induction, downward on the number of strings, upward on the length of the parameter, as in type B. The claim is that if there is a string $\kappa_1$ or $\sigma_1$, the spherical module cannot be unitary.

For $i > 0$, let $\xi = (k_i - 1/2, \ldots, K_i - 1/2)$ or $(n_i, \ldots, N_i)$ be a string. Deform upward $\xi_t = (k_i - 1/2 + t, \ldots, K_i - 1/2 + t)$ or $(n_i + t, \ldots, N_i + t)$. The signatures and multiplicities of the all cx-relevant $K$−types do not change for $0 \leq t < 1/2$. At $t = 1/2$, one of several cases may occur:

(a) There is no $\xi$, that is, $\lambda = \kappa_0 \cup \sigma_0$. We have dealt with this in the previous section.
(b) $P(\lambda_{1/2}) = P(\lambda_0)$. Continue deforming upwards. If no change occurs as $t \to \infty$ (this includes Case (a) in Section 5.2), the form is indefinite on $V_{\mathfrak{k}}(0, \ldots, 0)$ and the adjoint $K$−type $V_{\mathfrak{k}}(2, 0, \ldots, 0)$.
(c) $P(\lambda_{1/2}) \neq P(\lambda_0)$. Then we are in the setting of Remark 5.3, and the form is indefinite on $V_{\mathfrak{k}}(0, \ldots, 0)$ and $V_{\mathfrak{k}}(1, 1, \ldots, 0)$.

The cases when indefiniteness is first detected on the $K$−type $V_{\mathfrak{k}}(2, 0, \ldots, 0)$ rather than $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ is when the entries of two different strings in $\lambda$ differ by at least 1. For example, this holds for the strings $\lambda = (21/2, 23/2) \cup (8, 9) \cup (7/2, 9/2, 11/2)$.

5.4. **Non-spherical Case.** Consider the case $\mu_1 = 1$ and the parameter contains $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$.
As before, there cannot be a $\kappa_0$ present. The fundamental cx-relevant $K-$types for the spherical parameter produce bottom layer $K-$types. We are reduced to the cases when these bottom layer $K-$types do not detect non-unitarity. By the last paragraph in the previous section, this is the case when there is a $\kappa_i, \sigma_j$ with $i, j > 0$ in the spherical parameter deforming to $\infty$. The case when there is only $\kappa_1 = (3/2, \ldots, K_1 - 1/2)$ in the spherical parameter gives a unitary representation. We are reduced to the case when there is another string $\kappa_i \geq 5/2$ and/or $n_j \geq 2$ deforming to $\infty$. The $K-$types

$$V_{\mathfrak{k}}(1, 0, \ldots, 0) \quad V_{\mathfrak{k}}(2, 1, 0, \ldots, 0)$$

occur with the same multiplicities in the unitarily induced module from $GL(1) \times G(\mu_0)$ with $J(\lambda^0, \lambda^0)$ on the $G(\mu_0)-$factor, and in $J(\lambda, -s\lambda)$. The form is indefinite on these $K-$types, since they restrict to $K \cap M-$types for which the form on $J(\lambda^0, \lambda^0)$ is indefinite.

## 6. Proof of Theorem 3.6 – Type $D$

Let $G = SO(2m, \mathbb{C})$ and $K = SO(2m)$. The $K-$types have highest weight with integer coordinates only. Since $\rho = (m - 1, \ldots, 1, 0)$, it follows that $2\lambda = \{\eta - \rho\} + \rho$ has integer coordinates only. So $2\lambda$ is regular integral it has integer coordinates only. Since $\lambda$ is not assumed integral, its coordinates are integers and half integers, and the integral system is of type $D \times D$.

### 6.1. **Spherical Representations.**

**Proposition 6.1.** *Let $\lambda$ be regular half-integral. The spherical irreducible module $J(\lambda, \lambda)$ is unitary if and only it is unipotent, i.e.*

$$\lambda = \left( -K_0 + \frac{1}{2}, \ldots, -\frac{1}{2}; -N_0 + 1, \ldots, -1, 0 \right) \quad satisfying \quad N_0 \geq K_0.$$

*When $K_0 > 0$, the representation is attached to the nilpotent orbit $[3^1 2^{2K_0} 1^{2N_0 - 2K_0 - 1}]$.*

*When $K_0 = 0$, the nilpotent orbit is the trivial one.*

*When not unitary, the form is indefinite on the set of cx-relevant $K-$types with highest weights*

$$CXD := \{(0, \ldots, 0), \quad (1, \ldots, 1, 0, \ldots, 0), \quad (2, 0, \ldots, 0)\}.$$

The proof will take up most of the next few subsections. The unipotent representations are unitary because they can be realized via the dual pair correspondence in the stable range, as $\Theta(\text{triv}_{Sp})$, with the pair $Sp(2K_0, \mathbb{C}) \times SO(2K_0 + 2N_0, \mathbb{C})$ and one of the components of the the Oscillator representation on the $Sp-$factor.

As in Type B and C, we construct a parabolic subgroup $P(\lambda) = L(\lambda)U(\lambda)$ and an induced module $I_{P(\lambda)}$ for each $\lambda$ dominant for the standard positive system, i.e.

$$\lambda = (\ldots \lambda_i \geq \lambda_{i+1}, \cdots \geq \lambda_{m-1} \geq |\lambda_m| \geq 0), \quad 2\lambda_i \in \mathbb{Z}.$$

(i) If $1/2$ is a coordinate of $\lambda$, form the longest *string*

$$\kappa_0 = (-K_0 + 1/2, \ldots, -1/2)$$

such that all the half-integers staring from $1/2$ to $K_0 - 1/2$ are coordinates of $\lambda$, but $K_0 + 1/2$ is not. If the coordinate $0$ occurs, form the longest string

$$\sigma_0 = (-N_0 + 1, \ldots, -1, 0)$$

where $N_0 - 1$ is the largest integer coordinate that occurs in $\lambda$, but $N_0$ does not. Add a factor of type $G(K_0 + N_0) = SO(2K_0 + 2N_0)$ to $L(\lambda)$, and the spherical irreducible representation with parameter

$$\begin{pmatrix} -K_0 + 1/2 & , \ldots, & -1/2 & ; & -N_0 + 1, & \ldots, & -1 & 0 \\ -K_0 + 1/2 & , \ldots, & -1/2 & ; & -N_0 + 1, & \ldots, & -1 & 0 \end{pmatrix}.$$

If $1/2$ is not a coordinate, let $k_1 - 1/2 > 0$ be the smallest half-integer coordinate, and form the longest string $\kappa_1 = (k_1 - 1/2, \ldots, K_1 - 1/2)$ going up by one as before. Add a factor $GL(K_1 - k_1 + 1)$, to $L(\lambda)$, and the 1-dimensional representation with parameter

$$\begin{pmatrix} k_1 - 1/2 & , \ldots, & K_1 - 1/2 \\ k_1 - 1/2 & , \ldots, & K_1 - 1/2 \end{pmatrix}.$$

Similarly if $0$ does not occur as a coordinate, form $\sigma_1 = (n_1, \ldots, N_1)$ and add a factor $GL(N_1 - n_1 + 1)$ to the Levi component $L(\lambda)$.

(ii) Remove the coordinates in Step (i) from $\lambda$, and repeat on the remainder of half integer coordinates until there are no half-integer coordinates left. Similarly for the integer coordinates. Since the regularity assumption implies that at most one coordinate can be equal to $1/2$, and at most one coordinate equal to $0$, only $GL-$factors are created.

The process produces a parabolic subgroup, and an irreducible module on its Levi component. The Levi component is

$$(29) \qquad \prod_{i>0} GL(\sigma_j) \times \prod_{j>0} GL(\kappa_i) \times G(K_0 + N_0).$$

The parameters $\lambda$ we are going to study satisfy:

$$(30) \qquad \begin{cases} k_i > 2 & \text{if } 1/2 \text{ is a coordinate,} \\ k_i \geq 2 & \text{otherwise} \end{cases} \quad \text{and} \quad \begin{cases} n_j > 1 & \text{if } 0 \text{ is a coordinate,} \\ n_j \geq 1 & \text{otherwise} \end{cases}$$

and the nested condition

$$(31) \qquad \begin{cases} k_{i+1} - K_i \geq 2, \text{ or} \\ k_i \leq k_{i+1} \leq K_{i+1} \leq K_i, \end{cases} \quad \text{and} \quad \begin{cases} n_{j+1} - N_j \geq 2, \text{ or} \\ n_i \leq n_{i+1} \leq N_{i+1} \leq N_i \end{cases}.$$

The main property of the cx-relevant $K-$types is the following Lemma.

**Lemma 6.2.** *Assume that the strings of $\lambda$ satisfy (30) and (31). The multiplicities of the cx-relevant $K-$types in $I_{P(\lambda)}$ coincide with those in $J(\lambda, \lambda)$.*

*Proof.* The proof follows the analogous result for Type B. In this case all cx-relevant $K-$types are petite/single petaled. This is because $(\check{\alpha}, \lambda) \leq 3$ for all roots.          □

As in Type B, we have a necessary condition on the spherical parameter:

**Lemma 6.3.** *If $J(\lambda, \lambda)$ is unitary, then the string $\sigma_0 = (-N_0 + 1, \ldots, 1, 0)$ must appear in $(\lambda, \lambda)$.*

*Proof.* The coordinates on the spherical part of $I_1$ in Equation (15) are all $\geq 1/2$. As in Lemma 4.3, the irreducible representation $J(\lambda, \lambda)$ has indefinite form on the adjoint $K-$type $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ and the trivial $K-$type.          □

6.2. **Proof of Proposition 6.1** $- \lambda = \kappa_0 \cup \sigma_0$. The case when $N_0 \geq K_0$ is unitary. So assume

$$(32) \qquad \lambda = \kappa_0 \cup \sigma_0 \quad \text{satisfying} \quad K_0 > N_0.$$

By Lemma 6.3, we assume $N_0 > 0$, and let

$$Ind(\lambda_t) := \text{Ind}_{GL(\kappa_0) \times G(N_0)}^{G} \left( (1/2 + t, \ldots, K_0 - 1/2 + t) \otimes (-N_0 + 1, \ldots, -1, 0) \right).$$

The multiplicities of all cx-relevant $K-$types in $Ind(\lambda_t)$ and $J(\lambda, \lambda)$ still coincide for small $t$. This is as before: $Ind(\lambda)$ is a homomorphic image of $\text{Ind}_B^G(\lambda, \lambda)$, and the intertwining

operator changing $(1/2, \ldots, K_0 - 1/2)$ to $(-K_0 + 1/2, \ldots, -1/2)$ involves only $(\check{\alpha}, w\lambda)$ which are half-integers or $\geq 2$:

$$\begin{pmatrix} 1/2, 0 \\ 1/2, 0 \end{pmatrix} \mapsto \begin{pmatrix} 0, -1/2 \\ 1/2, 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1/2, 3/2 \\ 1/2, 3/2 \end{pmatrix} \mapsto \begin{pmatrix} -3/2, -1/2 \\ 1/2, 3/2 \end{pmatrix}$$

depending whether $K_0$ is even or odd. In the first case, the $SL(2)-$intertwining operator is an isomorphism, in the other case the kernel of the intertwining operator has lowest $K-$type $(2, 2)$. So the intertwining operator is an isomorphism on the cx-relevant $K-$types.

The signatures (and multiplicities) of the fundamental cx-relevant $K-$types of $Ind(\lambda_t)$ do not change for $0 \leq t < 1/2$. At $t = 1/2$, the parameter is

$$\lambda_{1/2} = (1, \ldots, K_0; -N_0 + 1, \ldots, -1, 0) = (-K_0, \ldots, -1, 0) \cup (1, \ldots, N_0 - 1).$$

As in the case in Type B, $J(\lambda_{1/2}, \lambda_{1/2})$ and $Ind(\lambda_{1/2})$ are different on the level of fundamental $K-$types, and $Ind(\lambda_{1/2})$ has another factor with parameter

(33)
$$\begin{pmatrix} -N_0 + 1 & \ldots & N_0 & ; & -K_0 & \ldots & -N_0 - 1 \\ -N_0 & \ldots & N_0 + 1 & ; & -K_0 & \ldots & -N_0 - 1 \end{pmatrix}$$

and lowest $K-$type $\mu_0 = (\underbrace{1, \ldots, 1}_{2N_0}, 0, \ldots 0)$.

If $K_0 - N_0$ is odd, the factor is not Hermitian, and there are two of them which are Hermitian dual to each other; one of the coordinates $-K_0, \ldots, -N_0 - 1$ is changed to its negative in both $\lambda_L$ and $\lambda_R$. When $K_0 - N_0 > 0$ is even, the signature is indefinite on $\mu_0$ and $\mu_1 = (\underbrace{1, \ldots, 1}_{2N_0 + 2}, 0, \ldots 0)$.

In both cases, $Ind(\lambda_{1/2})$, and hence $Ind(\lambda)$ and $J(\lambda, \lambda)$, has indefinite signature on the fundamental cx-relevant $K-$types.

### 6.3. Proof of Proposition 6.1 $-$ Other Strings. 
We follow the reasoning for type B. We do a downward induction on the length of $\lambda$, and the number of strings. The case when there are no strings other than $\kappa_0, \sigma_0$, was dealt with in the previous section. As in Type B, there are three *initial cases*:

(a) There is a string $\kappa_i$ or $\sigma_j$ with $i, j > 0$ such that $P(\lambda_t)$ does not change as $t \to \infty$,
(b) The strings are

$$(-K_0 + 1/2, \ldots, -1/2; -N_0 + 1, \ldots, 1, 0), \quad \text{with } K_0 > N_0$$

as in the previous section.
(c) The strings are $\lambda = \kappa_0 \cup \sigma_0 \cup \xi$, where

$$\xi = (K_0, \ldots, K_1 - 1) \quad \text{or} \quad (N_0 - 1/2, \ldots, N_1 - 3/2),$$

so that the deformation of $\xi$ to $t = 1/2$ yields a unitary spherical module. This means that $N_0 \geq K_1$ in one case, $N_1 \geq K_0$ in the other case.

As in Type B, Case (a) and (b) yield indefinite signatures on the trivial and adjoint $K-$type $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$. And Case (c) yields indefinite form on $V_{\mathfrak{k}}(1, 1, 0, \ldots, 0)$ and $V_{\mathfrak{k}}(2, 0, \ldots, 0)$.

6.4. **Non-spherical case.** If $\mu_1 > 0$, the parameter contains

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.$$

Suppose $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ occurs. If the parameter has no spherical part, there is nothing to be done; the parameter is unitary. If the parameter has a spherical part, there cannot be a $\sigma_0$ or else the regularity of the parameter is violated. Lemma 6.3 implies that the Hermitian form is indefinite on the trivial and adjoint $K-$types. Both are bottom layer if the lowest $K-$type has coordinates greater than one.

The proof of the claim is reduced to the case when the non-spherical parameter is exactly $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$, and the spherical parameter contains a $\sigma_0$.

The only case when the bottom layer $K-$type does not detect non-unitarity is in Case (c) in Section 6.3, which occurs when there is no $\kappa_0$ (due to regularity of $\lambda$), and a string $\kappa_i$ ($i > 0$) in the spherical parameter such that it is deformed to $\xi = (N_0 - 1/2, \ldots, N_1 - 3/2)$. The case when the spherical part is exactly $\sigma_0 \cup \kappa_1$ with $\kappa_1 = (3/2, \ldots, K_1 - 1/2)$ and $N_0 \geq K_1$ is unitary. Otherwise, we have $\kappa_1 = (3/2, \ldots, K_1 - 1/2)$ and $N_0 < K_1$ which is not unitary on the level of bottom layer $K-$types by Case (b) above, or there is a string $\kappa_i$ in the spherical parameter satisfying $k_i - 1/2 \geq 5/2$. The fact that $k_i - 1/2 \geq 5/2$ implies that the $K-$types

$$V_{\mathfrak{k}}(2, 1, 0, \ldots, 0) \quad \text{and} \quad V_{\mathfrak{k}}(1, 1, 1, 0, \ldots, 0)$$

occur with the same multiplicity in $J(\lambda, -s\lambda)$ and in the unitarily induced module from the spherical part. Since their restrictions to the Levi component contain $K-$types with indefinite form, the conclusion follows.

6.5. **Spin Groups.** As in Section 4.5, we study genuine representations of $G = Spin(2n, \mathbb{C})$ in this section. The $K-$types have highest weights with coordinates in $\mathbb{N} \cup +\frac{1}{2}$ only, except the last coordinate can be $-\frac{1}{2}$. As already mentioned, $\rho = (m - 1, \ldots, 1, 0)$, so $2\lambda = \{\eta - \rho\} + \rho$ must have coordinates of the form $\mathbb{N} + \frac{1}{2}$ only (the last coordinate can be $-\frac{1}{2}$). The integral system for $\lambda$ is type A.

We consider the case wen the lowest $K-$type of $J(\lambda_L, \lambda_R)$ is $Spin^{\pm} = V_{\mathfrak{k}}(\frac{1}{2}, \ldots \frac{1}{2}, \pm\frac{1}{2})$. Using the same arguments as in Section 4.5, all such irreducible modules must be of the form

(34) $$J(\lambda_L, \lambda_R) = Ind_{GL \times GL}^{G}(F_a \otimes F_b).$$

Unless $F_a, F_b$ are one dimensional, the form is indefinite on the lowest $K-$type $V_{\mathfrak{k}}(\frac{1}{2}, \ldots, \frac{1}{2})$ and 'adjoint' $K-$type $V_{\mathfrak{k}}(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \mp\frac{1}{2})$. In the case when there is only $F_a$ in (34), and the $GL$ corresponds to either one of the two subroot system of $D_n$, one obtains the genuine unipotent representations with infinitesimal character given in (10).

## 7. A POSITIVITY RESULT

In this section, we sharpen the results in Section 5.4-5.6 in [BP]. We investigate the PRV-components of $\pi_u \otimes V_{\mathfrak{k}}(\rho)$ when $\pi_u$ is a unipotent representation with half-integral regular infinitesimal character for a classical group.

By [BP, Section 5.4-5.6], all $\pi_u \in \widehat{G}^d$ for Type $B_n$, while for Type $C_n$ and $D_n$ $\pi_u^{even/odd} \in \widehat{G}^d$ if and only if $n$ is even/odd. Moreover, the spin-lowest $K$−type is unique for all such $\pi_u$'s (this will be verified in Proposition 7.1 below).

Since the $K$−types of $\pi_u$ are multiplicity free, Theorem 1.6 holds for all $\pi_u \in \widehat{G}^d$. In order to prove Theorems 1.4 and 1.6 for general $\pi \in \widehat{G}^d$, we need the following refinement of the results in [BP]:

**Proposition 7.1.** *Let $G$ be a connected complex classical simple Lie group and $\pi_u = J(\lambda, -s\lambda)$ be a unipotent representation given in Theorem 1.3. If $\pi_u \in \widehat{G}^d$, then there is a unique $K$−type $V_{\mathfrak{k}}(\eta)$ in $\pi_u$ such that $\delta := \{\eta - \rho\} = 2\lambda - \rho$ realizes the minimum of $\{\eta' - \rho\}$ over the $K$−spectrum of $\pi_u$. Furthermore,*

$$
\text{(35)} \qquad \pi_u \otimes V_{\mathfrak{k}}(\rho) = V_{\mathfrak{k}}(\delta) \oplus \bigoplus_{\delta' \neq \delta} m_{\delta'} V_{\mathfrak{k}}(\delta'),
$$

*where $m_{\delta'}$ are positive integers and*

$$
\text{(36)} \qquad \delta' = \delta + \sum_{i=1}^{l} m_i \alpha_i, \ \ satisfying \ m_i \in \mathbb{Z}_{\geq 0}.
$$

*If $\pi_u \notin \widehat{G}^d$, then all $\widetilde{K}$−types of $\pi_u \otimes V_{\mathfrak{k}}(\rho)$ have extremal weights of the form (36) for $\delta$ with norm strictly greater than $\|2\lambda - \rho\|$.*

*Proof.* The statement is obvious when $\pi_u = triv$ is the trivial representation. So we assume $\pi_u$ is not trivial from now on. Let $\eta'$ be any $K$−type of $\pi_u$ other than a spin-lowest $K$−type $\eta$. Put $\delta' := \{\eta' - \rho\}$. In view of Theorem 2.2, it suffices to prove that (36) holds for $\delta$ and $\delta'$.

**Type $B_n$:** Let $V_{\mathfrak{k}}(\eta') = V_{\mathfrak{k}}(\alpha_1, \alpha_1, \ldots, \alpha_a, \alpha_a, \underbrace{0, \ldots, 0}_{b-a})$ be a $K$−type in $\pi_u$. Since $\rho = (n - 1/2, n - 3/2, \ldots, 1/2)$, the PRV-component $\delta'$ is, up to the action of $W(B_n)$,

$$
\text{(37)} \qquad x\delta' = (n - 2a - 1/2, n - 2a - 3/2, \ldots, 1/2, B_1, \ldots, B_{2a})
$$

The minimum is attained when all $B_i = 1/2$, and this can only be achieved from

$$
\eta = (n - 1, n - 1, n - 3, \ldots, n - 2a - 1, n - 2a - 1, 0, \ldots, 0).
$$

It follows that

$$
\text{(38)} \qquad \delta = (n - 2a - 1, \ldots, 1/2, 1/2, \ldots, 1/2).
$$

Any other $K$−type must give rise to a $\delta'$ with at least $B_1 \geq 3/2$, and $B_i \geq 1/2$. The difference $x\delta' - \delta$, from (37) and (38), is a sum of short positive roots; on each nonzero

coordinate it is $B_i - 1/2$ times the corresponding short root. The difference $x\delta' - \delta'$, as in (37), is clearly a sum of positive roots since the two are conjugate, and $\delta'$ is dominant.

**Type C$_\mathbf{n}$:** Here $V_{\mathfrak{k}}(\eta') = V_{\mathfrak{k}}(2k, 0, \ldots, 0)$ or $V_{\mathfrak{k}}(2k+1, 0, \ldots, 0)$ and $\rho = (n, n-1, \ldots, 1)$. The PRV-component is, up to $W(C_n)$,

$$\delta' = (n-1, n-2, \ldots, 1, |n-2k|) \quad \text{or} \quad (n-1, \ldots, |n-2k-1|). \tag{39}$$

The minimum is attained at $k = \frac{n}{2}$ if $n$ is even, $k = \frac{n\pm 1}{2}$ if $n$ is odd. Thus

$$\delta = (n-1, n-2, \ldots, 1, 0) \quad \text{or} \quad (n-1, n-2, \ldots, 1, \mathbf{1}). \tag{40}$$

The argument for Type B applies to derive the conclusion in the statement of the Proposition.

Also, since $\delta + \rho$ is equal to $2\lambda = (2n-1, \ldots, 3, 1)$ if and only if $\delta = (n-1, n-2, \ldots, 1, 0)$, it also follows that $H_D(\pi^{even}) \neq 0$ and $H_D(\pi^{odd}) = 0$ if $n$ is even, and the reverse is true if $n$ is odd.

**Type D$_\mathbf{n}$:** We only consider $b > a > 0$ and omit the easier case when $b = a$. Here

$$V_{\mathfrak{k}}(\eta') = V_{\mathfrak{k}}(\alpha_1, \ldots, \alpha_{2a}, \underbrace{0, \ldots, 0}_{b-a}),$$

where $\sum_i \alpha_i$ is even/odd if $\pi_u^{even/odd}$ is being considered, and $\rho = (n-1, \ldots, 1, 0)$. Then the PRV-component, up to the action of $W(D_n)$, is

$$\delta' = (n-2a-1, \ldots, 1, 0, |n-1-\alpha_1|, \ldots, |n-2a-\alpha_{2a}|) \tag{41}$$

Even though $W(D_n)$ only allows an even number of sign changes, in the case $b > a$ there is a coordinate equal to 0, so we can change all coordinates to $\geq 0$. As in type C,

$$\delta = (n-2a-1, \ldots, 1, 0, \ldots, 0) \quad \text{or} \quad (n-2a-1, \ldots, 1, \mathbf{1}, 0, \ldots, 0),$$

and $H_D(\pi_u^{even}) \neq 0$ if and only if $\delta$ take the first value. We omit further details which are as in Types B and C. $\qquad\square$

The above proposition demonstrates a strong positivity result on the $\widetilde{K}$-types appearing in the tensor product decomposition of $\pi_u \otimes V_{\mathfrak{k}}(\rho)$ for unipotent representations $\pi_u$. In fact, similar calculations have been carried out for other irreducible unitary representations, and so far there are no counter-examples to the following conjecture, which sharpens Conjecture 1.5 in view of Proposition 2.4:

**Conjecture 7.2.** *Proposition 7.1 holds for any $\pi \in \widehat{G}^d$.*

## 8. Proof of Theorems 1.4 and 1.6

We prove Theorems 1.4 and 1.6 by sharpening the results in Section 2.2 of [BP]. To conform to the notation in that section, write $\pi_{\mathfrak{m}} = J(\lambda_{\mathfrak{m}}, -s\lambda_{\mathfrak{m}})$ for a unitary representation such that the center of $M$ acts trivially. In particular, when $\pi_{\mathfrak{m}}$ is 1-dimensional, it is the trivial representation. This case occurs in all classical types, and is the only case for type

$A$ and Spin groups. We assume that $\lambda_{\mathfrak{m}}$ is regular integral dominant for a positive system $\Delta_M$, and $\lambda$ is regular half-integral. The relations

(42)
$$
\begin{aligned}
\lambda_{\mathfrak{m}} + s\lambda_{\mathfrak{m}} &= \mu_{\mathfrak{m}}, & 2\lambda_{\mathfrak{m}} &= \mu_{\mathfrak{m}} + \nu_{\mathfrak{m}}, \\
\lambda_{\mathfrak{m}} - s\lambda_{\mathfrak{m}} &= \nu_{\mathfrak{m}}, & 2s\lambda_{\mathfrak{m}} &= \mu_{\mathfrak{m}} - \nu_{\mathfrak{m}}, \\
\lambda &= \xi/2 + \lambda_{\mathfrak{m}}, & \mu &= \xi + \mu_{\mathfrak{m}}, \\
s\lambda &= \xi/2 + s\lambda_{\mathfrak{m}}, & \nu &= \nu_{\mathfrak{m}}.
\end{aligned}
$$

hold, with $s \in W_M \subset W$. The unitary character $\xi$ can be assumed dominant for a choice of $\Delta(\mathfrak{n})$. We denote $\Delta = \Delta_M \cup \Delta(\mathfrak{n})$. However $\lambda$ may not be dominant for $\Delta$, so let $\Delta'$ be the positive system for which $\lambda$ is dominant. Since $\lambda$ is dominant for $\Delta_M$,

$$
\Delta_M \subset \Delta' \cap \Delta.
$$

For $\pi_{\mathfrak{m}}$, we assume in addition that

(i) $\pi_{\mathfrak{m}}$ is unitary,
(ii) $\lambda$ is regular half-integral,
(iii) $\pi_{\mathfrak{m}} \otimes V_{\mathfrak{k}\cap\mathfrak{m}}(\rho_{\mathfrak{m}})$ contains only $\widetilde{K \cap M}-$types of the form

$$
\delta'_M = \delta_M + \sum_{\gamma \in \Delta_M} m_\gamma \gamma, \quad m_\gamma \in \mathbb{N}, \quad \text{with} \quad \delta_M = 2\lambda_{\mathfrak{m}} - \rho_{\mathfrak{m}}
$$

By Proposition 7.1, this covers all $\pi_u$ in Theorem 3.1 with $H_D(\pi_u) \neq 0$ for classical types, and the case of $\pi_u = triv$ for Spin groups.

By Proposition 2.4, the only $\widetilde{K}$-type that can appear in the Dirac cohomology of $\pi$ must have extremal weight $\tau' := 2\lambda - \rho'$, where $2\rho'$ is the sum of all positive roots in $\Delta'$. By abuse of notations, we write $V_{\mathfrak{k}}(\tau')$ as the $\widetilde{K}$-type with extremal weight $\tau'$. The relation

(43)
$$
\begin{aligned}
\tau' =&\, 2\lambda - \rho' = \xi + \mu_{\mathfrak{m}} + \nu_{\mathfrak{m}} - \rho' = \xi + 2\lambda_{\mathfrak{m}} - \rho' = \xi + \delta_M + \rho_{\mathfrak{m}} - \rho' = \\
&\, \xi + \delta_M - w_{\mathfrak{m}}\rho + \rho_{\mathfrak{n}} - \rho' = \delta_M + (\xi + \rho_{\mathfrak{n}}) - (w_{\mathfrak{m}}\rho + \rho'),
\end{aligned}
$$

because $w_{\mathfrak{m}}\rho = -\rho_{\mathfrak{m}} + \rho_{\mathfrak{n}}$. Furthermore,

(44)
$$
w_{\mathfrak{m}}\rho + \rho' = \sum_{\beta \in \Delta' \cap \Delta(\mathfrak{n})} \beta
$$

Continuing with the proof of [BP, Theorem 2.4] in Section 2.2,

(45)
$$
\begin{aligned}
\left[\pi \otimes V_{\mathfrak{k}}(\rho) : V_{\mathfrak{k}}(\tau')\right] &= [\pi : V_{\mathfrak{k}}(\tau') \otimes V_{\mathfrak{k}}(\rho)] \\
&= [\pi_{\mathfrak{m}} \otimes \mathbb{C}_\xi : V_{\mathfrak{k}}(\tau')|_M \otimes V_{\mathfrak{k}}(\rho)|_M] \\
&= [\pi_{\mathfrak{m}} \otimes \mathbb{C}_\xi \otimes V_{\mathfrak{k}}(\rho)|_M : V_{\mathfrak{k}}(\tau')|_M] \\
&= [\pi_{\mathfrak{m}} \otimes \mathbb{C}_\xi \otimes (V_{\mathfrak{k}\cap\mathfrak{m}}(\rho_{\mathfrak{m}}) \otimes \mathbb{C}_{\rho_{\mathfrak{n}}} \otimes \textstyle\bigwedge^{\bullet}\mathfrak{n}^*) : V_{\mathfrak{k}}(\tau')|_M] \\
&= [\pi_{\mathfrak{m}} \otimes V_{\mathfrak{k}\cap\mathfrak{m}}(\rho_{\mathfrak{m}}) \otimes \mathbb{C}_{\xi+\rho_{\mathfrak{n}}} \otimes \textstyle\bigwedge^{\bullet}\mathfrak{n}^* : V_{\mathfrak{k}}(\tau')|_M].
\end{aligned}
$$

The penultimate step above uses [BP, Lemma 2.3], and that $\bigwedge^\bullet \mathfrak{n}^*$ consists of weights of the form $-\sum_{\alpha \in S} \alpha$, where $S$ is a subset of the roots in $\Delta(\mathfrak{n})$.

**Proposition 8.1.** *Let $\pi = \mathrm{Ind}_M^G(\mathbb{C}_\xi \otimes \pi_\mathfrak{m})$ be an irreducible, unitary representation with $\pi_\mathfrak{m}$ satisfying (i)-(iii). Then*

$$(46) \qquad [\pi_\mathfrak{m} \otimes V_{\mathfrak{k} \cap \mathfrak{m}}(\rho_\mathfrak{m}) \otimes \mathbb{C}_{\xi + \rho_\mathfrak{n}} \otimes \bigwedge{}^\bullet \mathfrak{n}^* : V_\mathfrak{k}(\tau')|_M] = [\pi_\mathfrak{m} \otimes V_{\mathfrak{k} \cap \mathfrak{m}}(\rho_\mathfrak{m}) : V_{\mathfrak{k} \cap \mathfrak{m}}(\delta_M)].$$

*(Recall that $H_D(\pi_\mathfrak{m})$ is either zero or a multiple of $V_{\mathfrak{k} \cap \mathfrak{m}}(\delta_M)$.)*

*Proof.* We use (iii); the fact that $\pi_\mathfrak{m} \otimes V_{\mathfrak{k} \cap \mathfrak{m}}(\rho_\mathfrak{m})$ is a sum of $\widetilde{K \cap M}$−types of the form

$$\delta'_M = \delta_M + \sum_{\gamma \in \Delta_M} m_\gamma \gamma.$$

Tensoring with $\mathbb{C}_{\xi + \rho_\mathfrak{n}} \otimes \bigwedge^\bullet \mathfrak{n}^*$, the $\widetilde{K \cap M}$-types that appear must have highest weights of the form

$$\delta'_M + \xi + \rho_\mathfrak{n} - \sum_{\alpha \in S} \alpha$$

for some $S \subseteq \Delta(\mathfrak{n})$.

Combining the arguments above, any $\widetilde{K \cap M}$-type appearing on the left module in (46) must have highest weights of the form
(47)
$$\delta'_M + \xi + \rho_\mathfrak{n} - \sum_{\alpha \in S} \alpha$$

$$= \left( \delta_M + \sum_{\gamma \in \Delta_M, \, m_\gamma \geq 0} m_\gamma \gamma \right) + \xi + \rho_\mathfrak{n} - \left( \sum_{\alpha \in S \cap \Delta'} \alpha + \sum_{\beta \in S \cap (-\Delta')} \beta \right)$$

$$= \left( \delta_M + \sum_{\gamma \in \Delta_M, \, m_\gamma \geq 0} m_\gamma \gamma \right) + \xi + \rho_\mathfrak{n} - \left( \sum_{\alpha \in \Delta(\mathfrak{n}) \cap \Delta'} \alpha - \sum_{\beta' \in (\Delta(\mathfrak{n}) \backslash S) \cap \Delta'} \beta' + \sum_{\beta \in S \cap (-\Delta')} \beta \right)$$

$$= \tau' + \sum_{\gamma \in \Delta_M, \, m_\gamma \geq 0} m_\gamma \gamma + \sum_{\beta' \in S_1} \beta' - \sum_{\beta \in S_2} \beta,$$

where $S_1 := (\Delta(\mathfrak{n}) \setminus S) \cap \Delta'$ and $S_2 := S \cap (-\Delta')$.

Consider the squared norm of the weight in (47):
(48)
$$\left\| \tau' + \sum_{\gamma \in \Delta_M, \, m_\gamma \geq 0} m_\gamma \gamma + \sum_{\beta' \in S_1} \beta' - \sum_{\beta \in S_2} \beta \right\|^2 = \|\tau'\|^2 +$$

$$2 \left\langle \tau', \sum_{\gamma \in \Delta_M, \, m_\gamma \geq 0} m_\gamma \gamma + \sum_{\beta' \in S_1} \beta' - \sum_{\beta \in S_2} \beta \right\rangle + \left\| \sum_{\gamma \in \Delta_M, \, m_\gamma \geq 0} m_\gamma \gamma + \sum_{\beta' \in S_1} \beta' - \sum_{\beta \in S_2} \beta \right\|^2$$

By construction, $\tau'$ is a dominant weight in $\Delta'$. On the other hand, we have seen from above that

$$\gamma \in \Delta_M \subset \Delta'; \quad \beta' \in \Delta'; \quad -\beta \in \Delta'.$$

Thus $\langle \tau', \gamma \rangle$, $\langle \tau', \beta' \rangle$, $\langle \tau', -\beta \rangle$ are all non-negative. Therefore,

$$\left\| \tau' + \sum_{\gamma \in \Delta_M, \ m_\gamma \geq 0} m_\gamma \gamma + \sum_{\beta' \in S_1} \beta' - \sum_{\beta \in S_2} \beta \right\|^2 \geq \|\tau'\|^2.$$

Equality occurs exactly when $\delta'_M = \delta_M$, and $S_1$, $S_2$ are both empty. The latter condition further implies that $S = \widetilde{\Delta(\mathfrak{n})} \cap \Delta'$.

Since $V_\mathfrak{k}(\tau')|_M$ has $\widetilde{K \cap M}$−types of norm less than or equal to $\tau'$, the left module in (46) contains $V_{\mathfrak{k} \cap \mathfrak{m}}(\tau')$ with multiplicity equal to $[\pi_\mathfrak{m} \otimes V_{\mathfrak{k} \cap \mathfrak{m}}(\rho_\mathfrak{m}) : V_{\mathfrak{k} \cap \mathfrak{m}}(\delta_M)]$. $\qquad\square$

We now present the proof of Theorem 1.4 and Theorem 1.6 for all $\pi = \mathrm{Ind}_M^G(\mathbb{C}_\xi \otimes \pi_u)$ in Theorem 3.1. The same argument holds for Spin groups with $\pi_u = triv$. It suffices to prove

$$(49) \qquad [\pi \otimes V_\mathfrak{k}(\rho) : V_\mathfrak{k}(2\lambda - \rho)] = \begin{cases} 1 & \text{if } \pi_u \in \widehat{M}^d \\ 0 & \text{if } \pi_u \notin \widehat{M}^d \end{cases}.$$

The special case when $M = G$ and $\pi = \pi_u$ is the content of Section 7.

By applying $\pi_\mathfrak{m} = \pi_u \in \widehat{M}^d$ to (45) and (46),

$$\left[ \pi \otimes V_\mathfrak{k}(\rho) : V_\mathfrak{k}(\tau') \right] = \left[ \pi_u \otimes V_{\mathfrak{k} \cap \mathfrak{m}}(\rho_\mathfrak{m}) : V_{\mathfrak{k} \cap \mathfrak{m}}(\delta_M) \right].$$

When $\pi_u \in \widehat{M}^d$, the proof in Proposition 7.1 implies that $\pi_u$ has a unique spin-lowest $K$−type and hence the right hand side of the above equation is equal to 1.

The case $H_D(\pi_u) = 0$ occurs in Types $C$ and $D$ only. By the proof of Proposition 8.1, in particular Equation (47), the $\widetilde{K \cap M}$-types appearing in the left module on the last line of (45) has highest weights

$$(50) \qquad \tau' + \sum_{\gamma \in \Delta_M, \ m_\gamma \geq 0} m_\gamma \gamma + \sum_{\beta' \in S_1} \beta' - \sum_{\beta \in S_2} \beta + \mathbf{e_i},$$

where $\mathbf{e_i}$ is the unit vector corresponding to the bolded $\mathbf{1}$ in the proof of Proposition 7.1. Consider the sum of coordinates of the expression in (50): since all the roots are of the form $2\mathbf{e_i}$ and/or $\mathbf{e_i} \pm \mathbf{e_j}$ in Type $C$ and $D$, the sum of coordinates in (50) must be of opposite parity with that of $\tau'$. Therefore, the multiplicity $[\pi \otimes V_\mathfrak{k}(\rho) : V_\mathfrak{k}(\tau')]$ in (45) is zero.

Hence (49) holds, and this completes the proofs of Theorems 1.4 and 1.6. $\qquad\square$

## APPENDIX A. THE NOTION OF UNIPOTENT REPRESENTATION

James Arthur made conjectures in the 1980's which state (roughly) that automorphic representations occurring in the residual spectrum of a locally symmetric space associated to a number field $F$, should be associated to $^\vee G-$equivalence classes of homomorphisms

$$\Phi : \mathcal{W}_F \times SL(2) \longrightarrow {}^\vee G$$

where $\mathcal{W}_F$ is the Weil group. There are additional conditions such as the image not contained in any proper Levi component, and $\Phi(\mathcal{W}_F)$ be bounded. We refer to [A] for a very detailed analysis. For $F$ a local field, one expects such representations to be the building blocks of the unitary dual. The homomorphism $\Phi \mid_{\mathbb{C}^\times}$ determines a semisimple orbit and, in the case of $F = \mathbb{C}$ (which is the case in this paper) should correspond to unitary induction. The infinitesimal character conjectured by Arthur is

$$d\Phi \left( 1, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right).$$

When $\Phi \mid_{\mathcal{W}_F} = Triv$, the infinitesimal character is $^\vee h/2$ where $\{^\vee e, \ ^\vee h, \ ^\vee f\}$ is a Lie triple associated to $\Phi(SL(2))$. In the general case, the data for $\Phi$ correspond to a $^\vee G-$orbit, semisimple times unipotent.

In [BV], for the above reason, the special case $\Phi|_{\mathbb{C}^\times} = Triv$ is studied. These correspond to unipotent conjugacy classes. A set of representations $\pi$ associated to $\Phi$ are assumed to satisfy

- $\mathrm{Ann}(\pi) \subset U(\mathfrak{g})$ is maximal subject to the prescribed infinitesimal character.

These representations are called *special unipotent Arthur packets* associated to the nilpotent orbit in $^\vee\mathfrak{g}$ determined by $\Phi$. The main result is that these packets satisfy the properties conjectured by Arthur.

The *building blocks* of the unitary dual is conjectured to be the packets associated to $\Phi$ satisfying $\Phi \mid_{\mathcal{W}_F} = Triv$ and such that the orbit of $^\vee e$ does not meet any proper Levi component. It is clear that this cannot be the case; the best known example is $G = Sp(2n, \mathbb{C})$ and the Segal-Shale-Weil (also called oscillator) representation. It is unitary, not unitarily induced from any representation on a proper Levi component, and its infinitesimal character is not of the form $^\vee h/2$.

For $GL(n, \mathbb{C})$, the unitary dual is determined in [V1], and for the other classical groups in [B1]. The building blocks for $GL(n, \mathbb{C})$ are 1-dimensional unitary representations of Levi components. For the other groups, a set of building blocks is identified explicitly in [B1]. They can be characterized as irreducible representations which are

- unitary with half-integral infinitesimal character,
- their annihilator in the universal algebra is maximal for the given infinitesimal character.

They have properties analogous to the Arthur packets of special unipotent representations. A minimal set of building blocks requires that the representations not be unitarliy induced irreducible from proper Levi components. In [B1] the class of unipotent representations is extended to include some unitarily induced representations from proper Levi factors (and

even some representations in complementary series which fall under the category of special unipotent). This is in line with the parameters introduced by Arthur where the image of $\Phi$ meet a proper Levi component. A parametrization in terms of the homomorphism $\Phi$ is given in [BV, Chapter 11]; the infinitesimal character is modified according to certain elements in the centralizer of the Lie triple.

A different parametrization, motivated by the *orbit philosophy* is in [B3]. It is in terms of nilpotent orbits $\mathcal{O} \subset \mathfrak{g}$. It is shown there that they can be obtained by iterating $\Theta-$lifts and tensoring with unitary characters starting with a 1-dimensional representation on $O(n, \mathbb{C})$ or the trivial of $Sp(2n, \mathbb{C})$.

Another definition of unipotent representations is given and studied in [LMM]. It is our understanding that the representations listed below match those in [LMM].

The packets associated to $^\vee h/2$ are called *special unipotent*. For the more general infinitesimal characters, they are called *unipotent*. To be completely clear what we mean by unipotent representation, the list of infinitesimal characters is in the next section.

A.1. **Parameters of Unipotent Representations.** We rely on [BV] and [B3]. For each $\mathcal{O} \subset \mathfrak{g}$ we will give an infinitesimal character $(\lambda_\mathcal{O}, \lambda_\mathcal{O})$, and a set of $(\lambda_\mathcal{O}, w\lambda_\mathcal{O})$ such that $\{L(\lambda_\mathcal{O}, w\lambda_\mathcal{O})\}$ are the unipotent representations with asymptotic support $\mathcal{O}$. In all cases $\lambda_\mathcal{O}$ and $-\lambda_\mathcal{O}$ are in the same $W-$orbit.

**Main Properties of $\lambda_\mathcal{O}$.** Suppose $\Pi$ is an irreducible representation with infinitesimal character $(\lambda_\mathcal{O}, \lambda_\mathcal{O})$. Then $\lambda_\mathcal{O}$ and $\Pi$ must satisfy:

(1) $\mathrm{Ann}(\Pi) \subset U(\mathfrak{g})$ is the maximal primitive ideal $\mathcal{I}_{\lambda_\mathcal{O}}$ with infinitesimal character $(\lambda_\mathcal{O}, \lambda_\mathcal{O})$,

(2) $|\{\Pi : \mathrm{Ann}(\Pi) = \mathcal{I}_{\lambda_\mathcal{O}}\}| = | \widehat{A(\mathcal{O})} |$, where $A(\mathcal{O})$ is the component group of the centralizer of an $e \in \mathcal{O}$,

(3) $\Pi$ is unitary.

We call such representations unipotent. The list of $\lambda_\mathcal{O}$ is given below. The choices satisfying (3) rely on the determination of the unitary dual for classical groups in [B1]. The parameter will always have integer and half-integer coordinates, and the corresponding system of integral co-roots is maximal.

**Definition A.1.** *A special orbit $\mathcal{O}$ (in the sense of Lusztig) is called **stably trivial** if Lusztig's quotient $\overline{A}(\mathcal{O})$ equals the full component group $A(\mathcal{O})$.*

For a definition and discussion of $\overline{A}(\mathcal{O})$, see [L], chapter 13.

The set of unipotent representations as defined above contains the building blocks of the unitary dual. They are attached to $\mathcal{O}$ which are not induced (in the sense of Lusztig-Spaltenstein) from any proper Levi component. For $\mathcal{O}$ special (in the sense of Lusztig) and not induced from a nilpotent orbit on a proper Levi component, $\lambda_\mathcal{O} = h(^\vee\mathcal{O})/2$ where $^\vee\mathcal{O}$ is the Barbasch-Spaltenstein-Vogan dual of $\mathcal{O}$. For other special $\mathcal{O}$ which are induced from proper Levi components, condition (2) may not be satisfied if they are not stably trivial. See the example below. The component group $A(\mathcal{O})$ depends on the isogeny class

of $G$. To make a definition that includes all cases, one would have to take the isogeny class into account. We leave this for future considerations. It is our understanding that a definition of unipotent closely related to the one above is considered in [LMM] addresses this problem.

The partitions in the next examples denote rows.

**Example A.2.**
• $\mathcal{O} = (2222) \subset \mathfrak{sp}(8)$ *is stably trivial,* $A(\mathcal{O}) = \overline{A(\mathcal{O})} \cong \mathbb{Z}_2$, $\lambda_{\mathcal{O}} = (2, 1, 1, 0)$. *In this case* $^\vee\mathcal{O}$ *corresponds to the partition* $(531)$, *and* $\lambda_{\mathcal{O}} = h(^\vee\mathcal{O})/2$.
• $\mathcal{O} = (222) \subset \mathfrak{sp}(6)$ *has dual orbit* $^\vee\mathcal{O}$ *corresponding to* $(331)$ *but is not stably trivial;* $A(\mathcal{O}) \cong \mathbb{Z}_2$, *while* $\overline{A(\mathcal{O})} \cong 1$. *In this case* $h(^\vee\mathcal{O})/2 = (1, 1, 0)$, *and for this infinitesimal character, conditions (1) and (3) are satisfied, but (2) is not satisfied. The choice of infinitesimal character in this case will be* $\lambda_{\mathcal{O}} = (3/2, 1/2, 1/2)$. *There are two parameters,*

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} 3/2 & 1/2 & 1/2 \\ 3/2 & 1/2 & 1/2 \end{pmatrix} \qquad and \qquad \begin{pmatrix} 3/2 & 1/2 & 1/2 \\ 1/2 & 3/2 & -1/2 \end{pmatrix}$$

*Note that* $(1, 1, 0)$ *is in the root lattice and drops down to the adjoint group,* $(3/2, 1/2, 1/2)$ *while is not, so genuine for* $Sp(2n, \mathbb{C})$.
• $\mathcal{O} = (211)$ *in* $\mathfrak{sp}(4, \mathbb{C})$ *is not special in the sense of Lusztig. The parameter is* $\lambda_{\mathcal{O}} = (3/2, 1/2)$ *and the representations are the two components of the oscillator representation:*

$$\begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} 3/2 & 1/2 \\ 3/2 & 1/2 \end{pmatrix} \qquad and \qquad \begin{pmatrix} 3/2 & 1/2 \\ 3/2 & -1/2 \end{pmatrix}$$

A.2. **Type A.** The group $G$ is $GL(n)$. Nilpotent orbits are determined by their Jordan canonical form. An orbit is given by a partition, i.e. a sequence of numbers in decreasing order $\mathcal{O} \longleftrightarrow (n_1, \ldots, n_k)$ that add up to $n$. Let $(m_1, \ldots, m_l)$ be the dual partition. The component group of $\mathcal{O}$ is trivial. The infinitesimal character is

$$\lambda_{\mathcal{O}} = \left( \frac{m_1 - 1}{2}, \ldots, -\frac{m_1 - 1}{2}, \ldots, \frac{m_l - 1}{2}, \ldots, -\frac{m_l - 1}{2} \right).$$

The orbit is induced from the trivial orbit on the Levi component $\mathfrak{m}$ of a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ with $\mathfrak{m} = \mathfrak{gl}(m_1) \times \cdots \times \mathfrak{gl}(m_l)$. The corresponding unipotent representation is spherical and induced irreducible from the trivial representation on the same Levi component. All orbits are *special* and *stably trivial.*

A.3. **Type B.** We describe the case $SO(2m + 1)$. For $O(2m + 1)$ there are twice the parameters, the parameters for $SO$ are tensored with *sgn.*

A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). Then $\mathcal{O}$ is parametrized by a partition $\mathcal{O} \longleftrightarrow (n_1, \ldots, n_k)$ of $2m + 1$ such that every even entry occurs an even number of times. Let $(m'_0, \ldots, m'_{2p'})$ be the transpose partition (add an $m'_{2p'} = 0$ if necessary, in order to have an odd number of terms). If $\mathcal{O}$ is represented by a tableau, these are the sizes of the columns in decreasing order. If there are any $m'_{2j} = m'_{2j+1}$, then pair them together and remove them from the partition. Then

relabel and pair up the remaining columns $(m_0)(m_1, m_2)\ldots(m_{2p-1}m_{2p})$. The members of each pair have the same parity and $m_0$ is odd. $\lambda_{\mathcal{O}}$ is given by the coordinates

$$(m_0) \longleftrightarrow (\frac{m_0 - 2}{2}, \ldots, \frac{1}{2}),$$

(51)
$$(m'_{2j} = m'_{2j+1}) \longleftrightarrow (\frac{m'_{2j} - 1}{2}, \ldots, -\frac{m'_{2j} - 1}{2})$$

$$(m_{2i-1}m_{2i}) \longleftrightarrow (\frac{m_{2i-1}}{2}, \ldots, -\frac{m_{2i} - 2}{2}).$$

In case $m'_{2j} = m'_{2j+1}$, $\mathcal{O}$ is induced from an orbit

$$\mathcal{O}_\mathfrak{m} \subset \mathfrak{m} = \mathfrak{so}(*) \times \mathfrak{gl}(\frac{m'_{2j} + m'_{2j+1}}{2})$$

where $\mathfrak{m}$ is the Levi component of a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$. $\mathcal{O}_\mathfrak{m}$ is the trivial nilpotent on the $\mathfrak{gl}-$factor. The component groups satisfy $A_G(\mathcal{O}) \cong A_M(\mathcal{O}_\mathfrak{m})$. Each unipotent representation is unitarily induced from a unipotent representation attached to $\mathcal{O}_\mathfrak{m}$.

Similarly if some $m_{2i-1} = m_{2i}$, then $\mathcal{O}$ is induced from a

$$\mathcal{O}_\mathfrak{m} \subset \mathfrak{m} \cong \mathfrak{so}(*) \times \mathfrak{gl}(\frac{m_{2i-1} + m_{2i}}{2}), \quad (0) \text{ on the } \mathfrak{gl} - \text{factor.}$$

Here $A_G(\mathcal{O}) \ncong A_M(\mathcal{O}_\mathfrak{m})$, but each unipotent representation is (not necessarily unitarily) induced irreducible from a representation on the Levi component $\mathfrak{m}$, unipotent on $\mathfrak{so}(*)$, and a character on the $\mathfrak{gl}$-factor.

The *stably trivial* orbits are the ones such that every odd sized part appears an even number of times, except for the largest size. An orbit is called triangular if it has partition

$$\mathcal{O} \longleftrightarrow (2m + 1, 2m - 1, 2m - 1, \ldots, 3, 3, 1, 1).$$

We give the explicit Langlands parameters of the unipotent representations. There are $\mid A_G(\mathcal{O})\mid$ distinct representations. Let

$$(\underbrace{k, \ldots, k}_{r_k}, \ldots, \underbrace{1, \ldots 1}_{r_1})$$

be the rows of the Jordan form of the nilpotent orbit. The numbers $r_{2i}$ are even. The reductive part of the centralizer (when $G$ is the orthogonal group) of the nilpotent element is a product of $O(r_{2i+1})$, and $Sp(r_{2j})$.

The columns are paired as in (51). The pairs $(m'_{2j} = m'_{2j+1})$ contribute to the spherical part of the parameter,

(52)
$$(m'_{2j} = m'_{2j+1}) \longleftrightarrow \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} \frac{m'_{2j}-1}{2} & , & \cdots & , & -\frac{m'_{2j}-1}{2} \\ \frac{m'_{2j}-1}{2} & , & \cdots & , & -\frac{m'_{2j}-1}{2} \end{pmatrix}.$$

The singleton $(m_0)$ contributes to the spherical part,

(53)
$$(m_0) \longleftrightarrow \begin{pmatrix} \frac{m_0-2}{2} & , & \cdots & , & \frac{1}{2} \\ \frac{m_0-2}{2} & , & \cdots & , & \frac{1}{2} \end{pmatrix}.$$

Let $(\eta_1, \ldots, \eta_p)$ with $\eta_i = \pm 1$, one for each $(m_{2i-1}, m_{2i})$. An $\eta_i = 1$ contributes to the spherical part of the parameter, with coordinates as in (52) and (53). An $\eta_i = -1$ contributes

$$
(54) \qquad \begin{pmatrix} \frac{m_{2i-1}}{2} & , & \cdots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}}{2} & , & \cdots & , & -\frac{m_{2i}-2}{2} \\ \frac{m_{2i-1}}{2} & , & \cdots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}-2}{2} & , & \cdots & , & -\frac{m_{2i}}{2} \end{pmatrix}.
$$

If $m_{2p} = 0$, $\eta_p = 1$ only for $SO$.

**Explanation.**

(1) Odd sized rows contribute a $\mathbb{Z}_2$ to $A(\mathcal{O})$, even sized rows a 1.
(2) When there are no $m'_{2j} = m'_{2j+1}$, every row size occurs. The inequalities

$$
\ldots (m_{2i-1} \geq m_{2i}) > (m_{2i+1} \geq m_{2i+2}) \ldots
$$

imply that there are $m_{2i} - m_{2i+1}$ rows of size $2i + 1$. Each pair $(m_{2i-1} \geq m_{2i})$ contributes exactly 2 parameters corresponding to the $\mathbb{Z}_2$ in $A(\mathcal{O})$.
(3) The pairs $(m'_{2j} = m'_{2j+1})$ lengthen the sizes of the rows without changing their parity. The component group does not change, they do not affect the number of parameters.

As already mentioned, when $G = O(2m+1, \mathbb{C})$ the unipotent representations are obtained from those of $SO(2m, \mathbb{C})$ by lifting them to $O(2m, \mathbb{C})$, and also tensoring with $sgn$.

In case $m_{2i-1} = m_{2i}$ even, there is another choice of parameter:

$$
(55) \qquad (m_{2i-1} = m_{2i}) \longleftrightarrow (\frac{m_{2i-1}-1}{2}, \ldots, -\frac{m_{2i}-1}{2}).
$$

The representations are unitarily induced irreducible from representations of the same type on Levi components $GL(2m_{2i-1}) \times SO(2n + 1 - 2m_{2i-1})$. The number of parameters no longer matches $|A(\mathcal{O})|$, but special unipotent representations are included.

A.4. **Type C.** A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $\mathcal{O} \longleftrightarrow (n_1, \ldots, n_k)$ of $2n$ such that every odd part occurs an even number of times. Let $(c'_0, \ldots, c'_{2p'})$ be the dual partition (add a $c'_{2p'} = 0$ if necessary in order to have an odd number of terms). As in type B, these are the sizes of the columns of the tableau corresponding to $\mathcal{O}$. If there are any $c'_{2j-1} = c'_{2j}$ pair them up and set aside. Then relabel and pair up the remaining columns $(c_0 c_1) \ldots (c_{2p-2} c_{2p-1})(c_{2p})$. The members of each pair have the same parity. The last one, $c_{2p}$, is always even. Then form a parameter

$$
(56) \qquad (c'_{2j-1} = c'_{2j}) \longleftrightarrow (\frac{c_{2j}-1}{2}, \ldots, -\frac{c_{2j}-1}{2}),
$$

$$
(57) \qquad (c_{2i} c_{2i+1}) \longleftrightarrow (\frac{c_{2i}}{2}, \ldots, -\frac{c_{2i+1}-2}{2}),
$$

$$
(58) \qquad c_{2p} \longleftrightarrow (\frac{c_{2p}}{2}, \ldots, 1).
$$

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B.

The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.

An orbit is called triangular if it corresponds to the partition $(2m, 2m, \ldots, 4, 4, 2, 2)$.

We give a parametrization of the unipotent representations in terms of their Langlands parameters. There are $\mid A_G(\mathcal{O}) \mid$ representations.

Let

$$(\underbrace{k, \ldots, k}_{r_k}, \ldots, \underbrace{1, \ldots, 1}_{r_1})$$

be the rows of the Jordan form of the nilpotent orbit. The numbers $r_{2i+1}$ are even. The reductive part of the centralizer of the nilpotent element is a product of $Sp(r_{2i+1})$, and $O(r_{2j})$.

The elements $(c'_{2j-1} = c'_{2j})$ and $c_{2p}$ contribute to the spherical part of the parameter as in (52) and (53). Let $(\eta_1, \ldots, \eta_p)$ be such that $\eta_i = \pm 1$, one for each $(c_{2i}, c_{2i+1})$. An $\eta_i = 1$ contributes to the spherical part, according to the infinitesimal character. An $\eta_i = -1$ contributes

$$(59) \qquad \begin{pmatrix} \frac{c_{2i}}{2} & , & \cdots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}}{2} & \cdots & , & -\frac{c_{2i+1}-2}{2} \\ \frac{c_{2i}}{2} & , & \cdots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}-2}{2} & \cdots & , & -\frac{c_{2i+1}}{2} \end{pmatrix}.$$

The explanation is similar to type $B$.

In case $c_{2i} = c_{2i+1}$ odd, there is another choice of parameter:

$$(60) \qquad (c_{2i} = c_{2i+1}) \longleftrightarrow (\frac{c_{2i-1}-1}{2}, \ldots, -\frac{c_{2i}-1}{2}).$$

The representations are unitarily induced irreducible from representations of the same type on Levi components
$GL(2c_{2i}+1) \times Sp(2n - 2c_{2i})$. The number of parameters no longer matches $|A(\mathcal{O})|$, but special unipotent representations are included.

A.5. **Type D.** We treat the case $G = SO(2m)$. A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $\mathcal{O} \longleftrightarrow (n_1, \ldots, n_k)$ of $2m$ such that every even part occurs an even number of times. Let $(m'_0, \ldots, m'_{2p'-1})$ be the dual partition (add a $m'_{2p'-1} = 0$ if necessary), the sizes of the columns of the tableau corresponding to $\mathcal{O}$. If there are any $m'_{2j} = m'_{2j+1}$ pair them up and remove from the partition. Then pair up the remaining columns $(m_0, m_{2p-1})(m_1, m_2) \ldots (m_{2p-3}, m_{2p-2})$. The members of each pair have the same parity

and $m_0, m_{2p-1}$ are both even. The infinitesimal character is

$$(m'_{2j} = m'_{2j+1}) \longleftrightarrow (\frac{m'_{2j} - 1}{2} \dots, -\frac{m'_{2j} - 1}{2})$$

(61)

$$(m_0 m_{2p-1}) \longleftrightarrow (\frac{m_0 - 2}{2}, \dots, -\frac{m_{2p-1}}{2}),$$

$$(m_{2i-1} m_{2i}) \longleftrightarrow (\frac{m_{2i-1}}{2} \dots, -\frac{m_{2i} - 2}{2})$$

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. An exception occurs for $G = SO(2m)$ when the partition is formed of pairs $(m'_{2j} = m'_{2j+1})$ only. In this case there are two nilpotent orbits corresponding to the partition. There are also two nonconjugate Levi components of the form $\mathfrak{gl}(m'_0) \times \mathfrak{gl}(m'_2) \times \dots \mathfrak{gl}(m'_{2p'-2})$ of parabolic subalgebras. There are two unipotent representations each induced irreducible from the trivial representation on the corresponding Levi component.

The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.

A nilpotent orbit is triangular if it corresponds to the partition $(2m-1, 2m-1, \dots, 3, 3, 1, 1)$.

The parametrization of the unipotent representations follows from types B,C, with the pairs $(m'_{2j} = m'_{2j+1})$ and $(m_0, m_{2p-1})$ contributing to the spherical part of the parameter only. Similarly for $(m_{2i-1}, m_{2i})$ with $\epsilon_i = 1$ spherical only, while $\epsilon_i = -1$ contributes analogous to (54) and (59).

The explanation parallels that for types $B, C$.

When $G = O(2m, \mathbb{C})$ the unipotent representations are obtained from those of $SO(2m, \mathbb{C})$ by lifting them to $O(2m, \mathbb{C})$, and also tensoring with $sgn$. In the case when all $m'_{2j} = m'_{2j+1}$ the representations associated to the two nilpotent orbits have the same lift, and it is invariant under tensoring with $sgn$. Otherwise tensoring with $sgn$ gives inequivalent unipotent representations.

As in types B,C, when $m_{2i-1} = m_{2i}$ is even, there is another choice of infinitesimal character:

(62)            $$(m_{2i-1} = m_{2i}) \longleftrightarrow (\frac{m_{2i-1} - 1}{2}, \dots, -\frac{m_{2i} - 1}{2}).$$

The representations are unitarily induced irreducible from representations of the same type on Levi components $GL(2m_{2i}) \times SO(2n - 2m_{2i-1})$. The number of parameters no longer matches $|A(\mathcal{O})|$, but special unipotent representations are included.

## Appendix B. Some Atlas Calculations

In this section, we illustrate some of the results on signatures on cx-relevant $K-$types considered in Sections 4–6 using the software `atlas` [ALTV, At]. The calculations are carried out using the function `print_sig_irr_long`, which is available at

http://klein.mit.edu/~dav/atlassem/.

**B.1. Section 4.2, Equation** (21). Let $G = SO(7, \mathbb{C})$, and $\lambda = (-1/2; -2, -1)$. The `atlas` is

```
atlas> set G = complexification(SO(7))
atlas> set all = all_parameters_gamma(G,[4,2,1,4,2,1]/2)
atlas> all[0]
Value: final parameter(x=47,lambda=[5,3,1,5,3,1]/2,nu=[4,2,1,4,2,1]/2)
```

The signature of some of the $K-$types are given by:

```
atlas> print_sig_irr_long(all[0],KGB(G,0),15)
sig  x  lambda                       hw                        dim
s    0  [  1,  1,  1, -1, -1, -1 ]/2  [ -2, -1,  0,  2,  1,  0 ]  1
s    0  [  1,  1,  1,  1,  1, -1 ]/2  [ -2, -1,  0,  3,  2,  0 ]  21
1    0  [ 1, 1, 1, 1, 1, 1 ]/2        [ -2, -1,  0,  3,  2,  1 ]  35
```

The $K-$types of $J(\lambda, -s\lambda)$ are in the column labelled `hw`. More precisely, by adding the $i^{th}$-coordinate and the $(i + \mathrm{rank}(G))^{th}$-coordinate of the vector in the `hw` column, one can get the highest weight of a $K-$type in usual coordinates. For example, $[-2, -1, 0, 3, 2, 0]$ corresponds to the highest weight $(-2+3, -1+2, 0+0) = (1, 1, 0)$ in the usual coordinates.

The `sig` column represents the signature of the Hermitian form of $J(\lambda_{rel}, -s\lambda_{rel})$. The form is definite if and only if the entries of the `sig` column are all scalars or all scalar multiples of `s`. In particular, the above output shows that the form is indefinite on the $K-$types $V_{\mathfrak{k}}(1, 1, 0)$ and $V_{\mathfrak{k}}(1, 1, 1)$, which matches Equation (21).

**B.2. Section 4.3, Case (c).** Let $G = SO(9, \mathbb{C})$ and $\lambda = (-5/2, -3/2, -1/2) \cup (2)$. We are in the setting of Case (c). Its $K-$type signatures are given by

```
sig x  lambda                       hw                    dim
1   0  [ 1,  1,  1,  1,-1,-1,-1,-1]/2 [-3,-2,-1,  0,  3,  2,  1,  0] 1
1   0  [ 1,  1,  1,  1,  1,  1,-1,-1]/2 [-3,-2,-1,  0,  4,  3,  1,  0] 36
s   0  [ 3,  1,  1,  1,  1,-1,-1,-1]/2 [-2,-2,-1,  0,  4,  2,  1,  0] 44
1   0  [ 3,  3,  1,  1,  1,  1,-1,-1]/2 [-2,-1,-1,  0,  4,  3,  1,  0] 495
s   0  [ 3,  1,  1,  1,  3,  1,-1,-1]/2 [-2,-2,-1,  0,  5,  3,  1,  0] 910
```

In this case, the $K-$types $V_{\mathfrak{k}}(1, 1, 0, 0)$ and $V_{\mathfrak{k}}(2, 0, 0, 0)$ have different signatures.

**B.3. Section 5.4, non-spherical Type C.** Let $G = Sp(8, \mathbb{C})$ and parameter $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \cup (-2, -1) \cup (3/2)$. The `atlas` code for this parameter is

```
atlas> set G = Sp(8,C)
atlas> set all = all_parameters_gamma(G,[4,3,2,1,4,3,2,1]/2)
atlas> LKT(all[1])
Value: (KGB element #0,[ 1, 0, 0, 0, 0, 0, 0, 0 ]/1)
```

The signatures of the $K-$types are:

```
sig  x  lambda                 hw                          dim
1    0  [ 1,0,0,0,0,0,0,0 ]/1  [ -3,-3,-2,-1, 4, 3, 2, 1 ]  8
s    0  [ 1,1,1,0,0,0,0,0 ]/1  [ -3,-2,-1,-1, 4, 3, 2, 1 ]  48
```

The $K-$types $V_{\mathfrak{k}}(1,0,0,0)$ and $V_{\mathfrak{k}}(1,1,1,0)$ have different signatures.

B.4. **Section 6.2, Equation (33).** This is an example where the Hermitian form is indefinite on a single $K-$type. Let $G = SO(6,\mathbb{C})$ and the parameter be given by $(-3/2, -1/2; 0)$. Then the signatures are given by:

```
sig   x  lambda                hw                       dim
1     0  [ 0, 0, 0, 0, 0, 0 ]/1  [ -2, -1,  0,  2,  1,  0 ]  1
1+s   0  [ 1, 1, 0, 0, 0, 0 ]/1  [ -1,  0,  0,  2,  1,  0 ]  15
1     0  [ 1, 0, 0, 1, 0, 0 ]/1  [ -1, -1,  0,  3,  1,  0 ]  20
s     0  [ 1, 1, 1, 1, 0, 0 ]/1  [ -1,  0,  1,  3,  1,  0 ]  45
```

The $K-$type $V_{\mathfrak{k}}(1,1,0)$ has indefinite signature as in Equation (33) with an odd number of spherical coordinates.

B.5. **Section 6.4, non-spherical Type D.** Let $G = SO(10,\mathbb{C})$. Let $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \cup (-2,-1,0) \cup$ (5/2) be the parameter, where the spherical part satisfies Case (c) of Section 6.3. Then the signatures of the $K-$types are given by:

```
sig   x  lambda                       hw                       dim
s     0  [1,0,0,0,0,0,0,0,0,0 ]/1  [-3,-3,-2,-1,0,4,3,2,1,0 ]  10
s     0  [1,1,1,0,0,0,0,0,0,0 ]/1  [-3,-2,-1,-1,0,4,3,2,1,0 ]  120
1+2s  0  [1,1,0,0,0,1,0,0,0,0 ]/1  [-3,-2,-2,-1,0,5,3,2,1,0 ]  320
1+s   0  [2,0,0,0,0,1,0,0,0,0 ]/1  [-2,-3,-2,-1,0,5,3,2,1,0 ]  210
s     0  [1,1,1,1,0,1,0,0,0,0 ]/1  [-3,-2,-1,0,0,5,3,2,1,0 ]  1728
```

The $K-$types $V_{\mathfrak{k}}(1,1,1,0,0)$ and $V_{\mathfrak{k}}(2,1,0,0,0)$ have opposite signatures. Moreover, this is the only place where the signatures are different on the level of cx-relevant $K-$types.

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