



Complete radiation boundary conditions for the Helmholtz equation II: domains with corners

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Received: 15 March 2022 / Revised: 7 March 2023 / Accepted: 5 April 2023 /

Published online: 22 April 2023

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Abstract

This paper continues Part I (Hagstrom and Kim in Numer Math 141(4):917–966, 2019) of the investigation on the complete radiation boundary condition (CRBC) in waveguides. In this paper, we propose corner compatibility conditions for CRBC applied to the Helmholtz equation posed in $\mathbb{R}^2$. Since CRBC is developed as a high-order absorbing boundary condition approximating the radiation condition by using rational functions via the cross-sectional Fourier analysis, it is well-studied and its accurate performance is validated on a straight/planar fictitious boundary in waveguides. However in the presence of corners on artificial absorbing boundaries such as boundaries of rectangular domains, a special treatment for corner conditions is required. We design and validate the accurate CRBC with the corner compatibility conditions on rectangular domains. We also analyze the existence and uniqueness of solutions to the Helmholtz equation coupled with CRBC with the corner compatibility conditions. Finally, numerical experiments illustrating the accuracy of CRBC will be presented.

Mathematics Subject Classification 65N30 · 65N12 · 74J20 · 76Q05

The research of the first author was supported in part by NSF Grant DMS-2012296. The research of the second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF-2018R1D1A1B07047416) funded by the Ministry of Education, Science and Technology. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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1 Introduction

This paper is concerned with an analysis of the complete radiation boundary condition (CRBC) applied to the time-harmonic wave propagation problem in a domain with corners. The CRBC is an efficient high order absorbing boundary condition that can be easily used for truncating unbounded domains of wave propagation problems to a finite computational domain so that wave phenomena can be well described in the computational domain without any noticeable pollution from fictitious reflections.

We consider the Helmholtz equation

$$\Delta u + k^2 u = f \text{ in } \mathbb{R}^2 \quad (1.1)$$

with the Sommerfeld radiation condition at infinity, where k is a positive wavenumber and f is a wave source with compact support. The numerical solution of the model problem (1.1) requires a domain truncation technique for which artificial reflections can be made as small as desired. To this end many different types of absorbing boundary conditions, also known as transparent or non-reflecting boundary conditions, have been developed and analyzed such as Dirichlet-to-Neumann (DtN) conditions [19, 26, 28, 33], approximate far-field expansions [8, 34], pole conditions [24, 25, 41] and perfectly matched layers (PML) [9, 12, 13, 32].

Rational approximation approaches [3, 15, 20, 38] to the non-local pseudodifferential operator related with the radiation condition have also been employed to construct efficient absorbing boundary conditions. Among others CRBC is devised in [21, 30] as a rational approximation to the exact radiation condition pertaining to the square root function. The rational approximation of CRBC can be rephrased as a system of differential equations of certain auxiliary functions with respect to tangential derivatives on artificial boundaries and so it is well-suited for a high-order absorbing boundary condition on a straight/planar boundary. Thus CRBC is successfully used on cross-sectional boundaries for computation of scattering waves in waveguides, and it is also found that the optimal minimization of reflection errors and efficient handling of both propagating and evanescent modes including grazing modes, make CRBC an attractive technique. In addition, it is shown in [31] that its performance can be improved by hybridization with PML.

On the other hand, when the wave propagation takes place in the open space, artificial boundaries for absorbing boundary conditions are required to surround wave sources or scatterers. As a simplest computational domain with such boundaries, we can take a rectangular region, and in this case it is demanded to provide certain conditions at corners for two systems of the boundary differential equations employed for CRBC from each side. These must be constructed so that the resulting problem coupled with CRBC incorporating the corner compatibility conditions has a unique solution and produces approximate radiating solutions of high accuracy. As a goal of this paper, we design and validate such compatibility conditions at corners for CRBC. The study on corner conditions for absorbing boundary conditions based on approximations of the pseudodifferential operators of the radiation condition has been developed for a long time such as [6, 14, 20, 22, 43] in time-domain computations and [38, 40] in frequency domain problems. However, we will provide an improved

analysis including the existence and uniqueness of solutions to the Helmholtz equation supplemented with CRBC satisfying the corner compatibility conditions. To this end, we carry out a spectral analysis based on [37] to understand the distribution of eigenvalues associated with CRBC and the completeness of eigenvectors, which is essential for verifying the existence and uniqueness of solutions. It is worth noting that there have been approaches to avoid corners of fictitious boundaries by using regular boundaries [4, 27].

This paper is organized as follows. In Sect. 2 we review CRBC for absorbing boundary conditions in the half space. The initial CRBC is introduced by certain relations involving the normal derivatives of auxiliary functions defined on a neighborhood of the boundary and then it is transformed to a system of differential equations of the auxiliary functions with respect to the tangential derivatives on the boundary. Section 3 is devoted to a study of the eigenvalue problem with CRBC as the boundary condition in the unit interval. Here we study the asymptotic behavior of eigenvalues and completeness of eigenvectors. In Sect. 4 the model problem on a square domain in $\mathbb{R}^2$ is introduced together with CRBC for two radiating directions orthogonal to the fictitious boundaries. The CRBC is defined in terms of auxiliary functions defined on a neighborhood of the fictitious boundaries and this will motivate the derivation of the corner conditions of CRBC imposed only on the artificial boundaries. The existence of solutions to the problem truncated by CRBC is also established by using the spectral analysis presented in Sect. 3. We propose the corner compatibility conditions for CRBC in Sect. 5, and the model problem coupled with CRBC satisfying the corner conditions is reformulated in a variational form in Sect. 6, which will be utilized for obtaining finite element approximations. In Sect. 7 we complete the proof of the existence of unique solutions to the model problem. Finally, numerical experiments demonstrating the efficiency of the CRBC will be presented in Sect. 8.

We remark that our spectral analysis also applies in three space dimensions and thus motivates and validates the construction of edge and corner conditions in the three-dimensional case. However we focus here on the two-dimensional case for ease of presentation.

2 Complete radiation boundary conditions in the half space

We consider the Helmholtz equation in the free space $\mathbb{R}^2$ of $x = (x_1, x_2)$

$$\Delta u + k^2 u = f \text{ in } \mathbb{R}^2, \quad (2.1)$$

where k is a positive wavenumber and f is a compactly supported source function in $L^2(\mathbb{R}^2)$ vanishing for $x_1 > -\delta$ with $\delta > 0$. Let $\mathbb{R}_-^2$ be the left half space of $x_1 < 0$ and let Γ denote the boundary of $\mathbb{R}_-^2$. The limiting absorption principle (see e.g. [1, 44]) shows that the problem (2.1) with the Sommerfeld radiation condition has a unique solution. By taking the Fourier transform $\hat{u}$ defined by

$$\hat{u}(x_1, \xi) = \int_{\mathbb{R}} u(x_1, x_2) e^{-ix_2 \xi} dx_2$$

with respect to x_2 for appropriately regular functions u , the radiating solution to the problem (2.1) can be written as

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{u}_0(\xi) e^{ix_1 \sqrt{k^2 - \xi^2} + ix_2 \xi} d\xi \quad \text{for } x_1 > 0, \quad (2.2)$$

where $\hat{u}_0$ is the Fourier transform of $u|_{\Gamma}$ and the branch of the square root function is chosen such that $\sqrt{k^2 - \xi^2} = i\sqrt{\xi^2 - k^2}$ for $|\xi| > k$. The integral representation (2.2) for the radiating solution u can be seen as a superposition of propagating modes for $|\xi| \leq k$ and evanescent modes for $|\xi| > k$. The precise meaning of the integral (2.2) in terms of weighted Sobolev spaces can be found in [10]. The radiating solution u satisfies the Dirichlet-to-Neumann (DtN) condition on Γ ,

$$\partial_1 u = T(u) \text{ on } \Gamma,$$

where ∂_j represents the derivative with respect to x_j and T is the DtN map defined by

$$T(\phi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} ik \sqrt{1 - \frac{\xi^2}{k^2}} \hat{\phi}(\xi) e^{ix_2 \xi} d\xi. \quad (2.3)$$

A study on local absorbing boundary conditions by approximating the square root function involved in the DtN map was initiated by the seminal works [16, 17], and since then various absorbing boundary conditions have been developed, for instance, Padé-type approximations [4, 5, 45] and optimized rational approximations [15, 21, 30].

2.1 Formulation of complete radiation boundary conditions

Now we present a brief description of CRBC approximating the radiation condition based on the DtN map T in the half space. Following the idea in [21, 30], for non-negative integers n_p, n_e with $P := n_p + n_e$, we consider the auxiliary variables ϕ_j defined in $x_1 > -\delta$ solving the Helmholtz equation and the recurrence relations,

$$\phi_0 = u, \quad (2.4)$$

$$(\partial_1 + a_j)\phi_j = (-\partial_1 + \tilde{a}_j)\phi_{j+1} \quad \text{for } j = 0, 1, \dots, P-1 \quad (2.5)$$

with the terminal condition $\partial_1 \phi_P = 0$ on Γ . Here the parameters $a_j, \tilde{a}_j$ are given by

$$a_j = -ic_j k, \quad \tilde{a}_j = -i\tilde{c}_j k \quad \text{for } j = 0, \dots, n_p - 1 \quad (2.6)$$

with $0 < c_j, \tilde{c}_j \leq 1$ for propagating modes related to $|\xi| \leq k$ and

$$a_j = c_j k, \quad \tilde{a}_j = \tilde{c}_j k \quad \text{for } j = n_p, \dots, n_p + n_e - 1 \quad (2.7)$$

with $c_j, \tilde{c}_j > 0$ for evanescent modes with $|\xi| > k$. The essential idea is that for some specific values of ξ the functions $\hat{\phi}_j(\xi)$ associated with the radiating solution will vanish for $j = j_0 < P$ in which case the terminal condition is exact. Thus we have a mechanism for interpolating the exact radiation condition.

The recurrence relations (2.4)–(2.5) with the terminal condition can be reformulated to an equivalent form suitable for numerical computations cooperating with the finite element method. To do this we remove the normal derivatives from the recursive formulas by first multiplying the recursive formulas by the operator ∂_1 and second eliminating the second derivatives using the Helmholtz equation with simple algebraic manipulations as in [21, 30]. Denoting $\Phi = (\phi_0, \dots, \phi_P)^\top$, the resulting practical CRBC is given by

$$-\partial_1 u \mathbf{e}_0 = -\mathbb{L} \partial_2 \Phi + (-k^2 \mathbb{L} + \mathbb{M}) \Phi \quad \text{on } \Gamma, \quad (2.8)$$

where $\mathbf{e}_j$ for $j = 0, \dots, P$ is the j th vector of the standard basis in $\mathbb{C}^{P+1}$. In addition, $\mathbb{L}$ and $\mathbb{M}$ are the $(P+1) \times (P+1)$ tridiagonal symmetric (not Hermitian) matrices defined by

$$\mathbb{L}_{j,j-1} = \frac{1}{a_{j-1} + \tilde{a}_{j-1}}, \quad \mathbb{L}_{j,j} = \frac{1}{a_{j-1} + \tilde{a}_{j-1}} + \frac{1}{a_j + \tilde{a}_j}, \quad \mathbb{L}_{j,j+1} = \frac{1}{a_j + \tilde{a}_j}$$

and

$$\mathbb{M}_{j,j-1} = \frac{-a_{j-1}^2}{a_{j-1} + \tilde{a}_{j-1}}, \quad \mathbb{M}_{j,j} = \frac{a_{j-1}\tilde{a}_{j-1}}{a_{j-1} + \tilde{a}_{j-1}} + \frac{a_j\tilde{a}_j}{a_j + \tilde{a}_j}, \quad \mathbb{M}_{j,j+1} = \frac{-\tilde{a}_j^2}{a_j + \tilde{a}_j}$$

for $j = 0, \dots, P$ with the convention ignoring a_ℓ and $\tilde{a}_\ell$ with ℓ outside of the index range of the parameters, $0 \leq \ell \leq P-1$.

2.2 Error analysis

We remark that CRBC has been investigated for the waveguide problem in [21, 30], for which the radiating solution is an infinite series consisting of discrete modes as opposed to the Fourier integral (2.2) of the half space problem. Noting that the Fourier transform $\hat{\phi}_j$ of the j -th auxiliary variable ϕ_j with respect to x_2 can be written as

$$\hat{\phi}_j(x_1, \xi) = A_j(\xi)e^{ix_1\sqrt{k^2-\xi^2}} + B_j(\xi)e^{-ix_1\sqrt{k^2-\xi^2}}$$

for $x_1 > -\delta$, it follows from the Fourier transform of the recurrence relation (2.5) and the terminal condition that the reflection coefficient is given by

$$\mathcal{Z}(\xi) := \frac{B_0(\xi)}{A_0(\xi)} = \prod_{j=0}^{P-1} \frac{(a_j + i\sqrt{k^2 - \xi^2})(\tilde{a}_j + i\sqrt{k^2 - \xi^2})}{(a_j - i\sqrt{k^2 - \xi^2})(\tilde{a}_j - i\sqrt{k^2 - \xi^2})}.$$

Therefore the solution to the Helmholtz equation satisfying CRBC is written as

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{1}{1 + \mathcal{Z}(\xi)} e^{ix_1 \sqrt{k^2 - \xi^2}} + \frac{\mathcal{Z}(\xi)}{1 + \mathcal{Z}(\xi)} e^{-ix_1 \sqrt{k^2 - \xi^2}} \right) \hat{u}_0(\xi) e^{ix_2 \xi} d\xi$$

for $x_1 > 0$, which allows us to have the DtN map for CRBC,

$$T_{\text{CRBC}}(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}} ik \sqrt{1 - \frac{\xi^2}{k^2}} \left(\frac{1 - \mathcal{Z}(\xi)}{1 + \mathcal{Z}(\xi)} \right) \hat{\phi}(\xi) e^{ix_2 \xi} d\xi. \quad (2.9)$$

Here $\mathcal{Z}$ can be factored as $\mathcal{Z} = \mathcal{H}_p \mathcal{H}_e$ depending on the types of parameters, where

$$\mathcal{H}_p = \prod_{j=0}^{n_p-1} \frac{(c_j - \sqrt{1 - \xi^2/k^2})(\tilde{c}_j - \sqrt{1 - \xi^2/k^2})}{(c_j + \sqrt{1 - \xi^2/k^2})(\tilde{c}_j + \sqrt{1 - \xi^2/k^2})},$$

$$\mathcal{H}_e = \prod_{j=n_p}^{n_p+n_e-1} \frac{(c_j - \sqrt{\xi^2/k^2 - 1})(\tilde{c}_j - \sqrt{\xi^2/k^2 - 1})}{(c_j + \sqrt{\xi^2/k^2 - 1})(\tilde{c}_j + \sqrt{\xi^2/k^2 - 1})}.$$

satisfy $|\mathcal{H}_p| < 1$, $|\mathcal{H}_e| = 1$ for $|\xi| < k$ and $|\mathcal{H}_p| = 1$, $|\mathcal{H}_e| < 1$ for $|\xi| > k$. Hence we can see that the purely imaginary parameters $a_j, \tilde{a}_j$ for $0 \leq j \leq n_p - 1$ are responsible for attenuating the reflections of propagating modes whereas the real parameters $a_j, \tilde{a}_j$ for $n_p \leq j \leq n_p + n_e - 1$ play a role of reducing the errors from slowly decaying evanescent modes. Denoting $\mathcal{P}(\mu) = \prod_{j=0}^{P-1} (a_j/k + i\mu)(\tilde{a}_j/k + i\mu)$, then CRBC is related to a (P, P) -type rational approximation to $\sqrt{1+x}$ of the form

$$f_P(x) = \sqrt{1+x} \frac{\mathcal{P}(-\sqrt{1+x}) - \mathcal{P}(\sqrt{1+x})}{\mathcal{P}(-\sqrt{1+x}) + \mathcal{P}(\sqrt{1+x})}. \quad (2.10)$$

Noting that the radiating solution u generated by a source δ away from the absorbing boundary Γ satisfies

$$\hat{u}_0(\xi) = e^{i\delta \sqrt{k^2 - \xi^2}} \hat{u}(-\delta, \xi),$$

the error of the Fourier transform of $T_{\text{CRBC}}(u|_{\Gamma})$ for the radiating solution u is given by

$$\mathcal{E} := i\sqrt{k^2 - \xi^2} \varrho \hat{u}(-\delta, \xi),$$

where $\varrho := \varrho_1 \varrho_2$ with

$$\varrho_1 := \frac{2\mathcal{Z}(\xi)}{1 + \mathcal{Z}(\xi)} \quad \text{and} \quad \varrho_2 := e^{i\delta \sqrt{k^2 - \xi^2}}. \quad (2.11)$$

Here ϱ_1 is interpreted as the relative error of the (P, P) -type rational function representation (2.10) of CRBC approximating the square root function and ϱ_2 reveals the exponentially small amplitude of evanescent modes on Γ .

2.3 Parameter selection

It is crucial for the accuracy of the absorbing boundary condition to reduce the maximal value of ϱ . One approach is to take a sufficiently small $\epsilon > 0$ and a sufficiently large $M > 1$ such that the Fourier integral for grazing modes satisfying $|k - |\xi|| < \epsilon k$ and evanescent modes for $|\xi| > Mk$ are ignorable, and then to minimize ϱ_1 on the set $G := \{|k - |\xi|| \geq \epsilon k \text{ and } |\xi| \leq Mk\}$. Since $|\mathcal{Z}| \leq 1$ for $\xi \in \mathbb{R}$ and $\mathcal{Z} \rightarrow 1$ as $|\xi|$ approaches infinity, it holds that

$$\max_{|k-|\xi|| \geq \epsilon} |\varrho| \leq C \max_{\xi \in G} \{\max_{|\mathcal{Z}| \leq 1} |\varrho_1|, e^{-k\delta\sqrt{M^2-1}}\}$$

for some positive constant C independent of CRBC parameters. From here on, the constants C and c represent generic positive numbers which have different values at different places but do not depend on CRBC parameters and functions to be estimated. We can nearly minimize the maximal value of ϱ_1 by minimizing $\mathcal{H}_p$ in $[0, (1 - \epsilon)k]$ and $\mathcal{H}_e$ in $[(1 + \epsilon)k, Mk]$. Denoting $z = \sqrt{1 - \xi^2/k^2}$, the min-max problem of $\mathcal{H}_p$ can be rephrased as finding the parameters c_j and $\tilde{c}_j$ for $j = 0, \dots, n_p - 1$ solving

$$\rho_p := \min_{c_j, \tilde{c}_j \in [\gamma, 1]} \max_{z \in [\gamma, 1]} \prod_{j=0}^{n_p-1} \frac{(c_j - z)(\tilde{c}_j - z)}{(c_j + z)(\tilde{c}_j + z)} \quad (2.12)$$

with $\gamma = \sqrt{\epsilon(2 - \epsilon)}$. It is related with the third Zolotarev problem about the least deviation from zero, see e.g., [36, 46]. As shown in [15, 30], it can be solved analytically by using elliptic functions [2]: $c_j = s_{2j}$ and $\tilde{c}_j = s_{2j+1}$ for $j = 0, \dots, n_p - 1$ with

$$s_j = \operatorname{dn} \left(\left(1 - \frac{2j+1}{4n_p} \right) K(\tilde{\gamma}), \tilde{\gamma} \right) \text{ for } j = 0, \dots, 2n_p - 1, \quad (2.13)$$

where dn is the delta amplitude of the Jacobi elliptic functions and $K(\tilde{\gamma})$ represents the complete elliptic integral with $\tilde{\gamma} = \sqrt{1 - \gamma^2}$, and it turns out that the minimal value ρ_p decays exponentially with respect to the number of the imaginary parameters, n_p ,

$$\rho_p \leq C e^{-cn_p / \ln(1/\gamma)}. \quad (2.14)$$

The min-max problem (2.12) can also be solved numerically by using the Remez exchange algorithm [39] based on the equioscillation theorem.

By solving the min-max problem for $d_j = c_j/\sqrt{M^2 - 1}$ and $\tilde{d}_j = \tilde{c}_j/\sqrt{M^2 - 1}$,

$$\rho_e := \min_{d_j, \tilde{d}_j \in [\gamma, 1]} \max_{z \in [\gamma, 1]} \prod_{j=n_p}^{n_p+n_e-1} \frac{(d_j - z)(\tilde{d}_j - z)}{(d_j + z)(\tilde{d}_j + z)} \quad (2.15)$$

with $\gamma = \sqrt{\epsilon(2 + \epsilon)}/\sqrt{M^2 - 1}$, the analogous result for $\mathcal{H}_e$ holds as well.

For the actual implementation of CRBC, it is of importance to determine appropriate ϵ and M for efficient performance of CRBC, however it is not clear to understand how to determine desired ϵ and M . For general data we develop worst-case estimates below which guarantee accuracy.

Representing u using the fundamental solution of the Helmholtz equation we have

$$u(x) = -\frac{i}{4} \int_{y_1 < -\delta} H_0^1(k|x - y|) f(y) dy,$$

which motivates the estimation of ϵ and M for the Hankel function,

$$H_0^1(k|x - y|) = \frac{1}{\pi} \int_{\mathbb{R}} e^{i\xi(x_2 - y_2)} \frac{e^{-\sqrt{\xi^2 - k^2}|x_1 - y_1|}}{\sqrt{k^2 - \xi^2}} d\xi. \quad (2.16)$$

Theorem 2.1 *There exists a constant C independent of P and k such that for any $\tau > 0$ and separation $\delta > 0$ from the support of f and the radiation boundary $x_1 = 0$ there exist P CRBC parameters,*

$$P \leq C \left(\ln \frac{1}{\tau} + \ln \frac{1}{k\delta} \right)^2, \quad (2.17)$$

such that the error u_R satisfies

$$|u_R(x)| \leq \tau \|f\|_{L^1}. \quad (2.18)$$

If $k\delta \geq 1$ the second term in the estimate is absent.

Proof The error due to the artificial boundary can be represented by the perturbation to the fundamental solution $H(x, y) = \frac{-i}{4} H_0^1(k|x - y|)$ to the Helmholtz equation. More precisely, we begin by noticing that the fundamental solution G to the Helmholtz equation satisfying the CRBC on Γ can be written as

$$G(x, y) = H(x, y) + H_R(x, y),$$

where H_R to be determined represents a reflection term due to the artificial boundary. First of all, from (2.16) we see that the Fourier transform of H is given by, for $y_1 <$

$$x_1 < 0,$$

$$\hat{H}((x_1, \xi), y) = \left(\frac{-ie^{-i\xi y_2}}{2} \frac{e^{\sqrt{\xi^2 - k^2} y_1}}{\sqrt{k^2 - \xi^2}} \right) e^{-\sqrt{\xi^2 - k^2} x_1} = A(\xi, y) e^{-\sqrt{\xi^2 - k^2} x_1}.$$

By applying the Helmholtz operator in x to G it can be shown that

$$-(\Delta_x + k^2)H_R(x, y) = 0$$

so that the Fourier transform of H_R can be written in the form

$$\hat{H}_R((x_1, \xi), y) = B(\xi, y) e^{\sqrt{\xi^2 - k^2} x_1} \text{ for } x_1 < 0.$$

Then, by applying the CRBC on Γ to G we get $B(\xi, y) = \mathcal{Z}(\xi)A(\xi, y)$, from which it follows

$$H_R(x, y) = -\frac{i}{4\pi} \int_{\mathbb{R}} e^{i\xi(x_2 - y_2)} \mathcal{Z}(\xi) \frac{e^{\sqrt{\xi^2 - k^2}(x_1 + y_1)}}{\sqrt{k^2 - \xi^2}} d\xi. \quad (2.19)$$

Therefore the error function is written as

$$u_R(x) = \int_{y_1 < -\delta} H_R(x, y) f(y) dy. \quad (2.20)$$

We estimate H_R using the symmetry with respect to $\xi = 0$ and the inequalities $x_1 + y_1 \leq -\delta$, $|\mathcal{Z}(\xi)| \leq 1$:

$$\begin{aligned} |H_R(x, y)| &\leq \frac{1}{2\pi} (I_1 + I_2 + I_3 + I_4) \\ I_1 &= \int_0^{1-\epsilon} \frac{|\mathcal{Z}(\eta k)|}{\sqrt{1 - \eta^2}} d\eta, \\ I_2 &= \int_{1-\epsilon}^{1+\epsilon} \frac{|\mathcal{Z}(\eta k)|}{|\sqrt{1 - \eta^2}|} d\eta, \\ I_3 &= \int_{1+\epsilon}^M \frac{|\mathcal{Z}(\eta k)| e^{-\sqrt{\eta^2 - 1} k \delta}}{\sqrt{1 - \eta^2}} d\eta, \\ I_4 &= \int_M^\infty \frac{e^{-\sqrt{\eta^2 - 1} k \delta}}{\sqrt{1 - \eta^2}} d\eta. \end{aligned}$$

For I_1 , we use (2.14) with $\gamma = O(\sqrt{\epsilon})$ and

$$\int_0^{1-\epsilon} \frac{1}{\sqrt{1 - \eta^2}} d\eta \leq \frac{\pi}{2}$$

we get

$$|I_1| \leq C e^{cn_p / \ln \epsilon^{-1}}. \quad (2.21)$$

For I_2 , due to the fact that $|\mathcal{Z}| \leq 1$ and $\sqrt{1+\eta} > 1$ for $|1-\eta| < \epsilon$,

$$|I_2| \leq \int_{1-\epsilon}^{1+\epsilon} \frac{1}{|\sqrt{1-\eta}|} d\eta \leq 4\sqrt{\epsilon} \quad (2.22)$$

For I_3 , we use a bound for (2.15) analogous to (2.14) with $\gamma = O(\sqrt{\epsilon}/M)$ and with n_p replaced by n_e to see

$$|\mathcal{Z}| \leq C e^{-cn_e / (\ln \epsilon^{-1} + \ln M)}.$$

Since by a change of variables $\sqrt{\eta^2 - 1} = \zeta$

$$\int_{1+\epsilon}^M \frac{e^{-\sqrt{\eta^2-1}k\delta}}{\sqrt{\eta^2-1}} d\eta = \int_{\sqrt{\epsilon(2+\epsilon)}}^{\sqrt{M^2-1}} \frac{e^{-k\delta\zeta}}{\sqrt{\zeta^2+1}} d\zeta \leq \int_0^M e^{-k\delta\zeta} d\zeta \leq \frac{1}{k\delta}, \quad (2.23)$$

we have

$$|I_3| \leq \frac{C}{k\delta} e^{-cn_e / (\ln \epsilon^{-1} + \ln M)}. \quad (2.24)$$

Lastly, for I_4 using the same change of variable as in (2.23) we have

$$|I_4| \leq \int_{\sqrt{M^2-1}}^{\infty} e^{-k\delta\zeta} d\zeta = \frac{1}{k\delta} e^{-\sqrt{M^2-1}k\delta} \quad (2.25)$$

Combining (2.21)–(2.25), we deduce

$$|H_R(x, y)| \leq C \left(e^{-cn_p / \ln \epsilon^{-1}} + \sqrt{\epsilon} + \frac{1}{k\delta} e^{-cn_e / (\ln \epsilon^{-1} + \ln M)} + \frac{1}{k\delta} e^{-cMk\delta} \right). \quad (2.26)$$

To meet some error tolerance τ we must choose:

$$\epsilon = O(\tau^2), \quad (2.27)$$

$$n_p = O\left(\ln \frac{1}{\tau} \cdot \ln \frac{1}{\epsilon}\right) = O\left(\ln^2 \frac{1}{\tau}\right), \quad (2.28)$$

$$M = O\left(\frac{\ln \frac{1}{\tau} + \ln \frac{1}{k\delta}}{k\delta}\right), \quad (2.29)$$

$$n_e = O\left((\ln \frac{1}{\tau} + \ln \frac{1}{k\delta})(\ln \frac{1}{\epsilon} + \ln M)\right) = O\left((\ln \frac{1}{\tau} + \ln \frac{1}{k\delta})^2\right) \quad (2.30)$$

where we have assumed $k\delta < 1$. For larger values of $k\delta$ the terms involving $k\delta$ in the estimates are absent. Using (2.27)–(2.30), (2.26) and (2.20) we establish the bound (2.17) on P . $\square$

As noticed in the proof of the above theorem, we can choose parameters by following the procedure to make the maximal value of the reflection less than τ relative to $\|f\|_{L^1}$ for given τ

- i. set $\epsilon = \sqrt{\tau}$ for a frequency range of grazing modes,
- ii. determine n_p and choose $2n_p$ imaginary parameters such that $|\rho_p| < \tau$ by solving (2.12),
- iii. determine M such that $(k\delta)^{-1}e^{-k\delta\sqrt{M^2-1}} < \tau$ for the upper bound of evanescent modes,
- iv. determine n_e and choose $2n_e$ real parameters such that $|\rho_e| < \tau$ by solving (2.15).

Although this result for the half space is not applicable directly to domains with corners, it can be used as a useful guide to select parameters of CRBC with high accuracy. In doing so, τ represents an error bound of reflected waves in the maximal norm relative to the source term in the half space but it does not mean that it can make actual relative L^2 -errors of approximate solutions in domains with corners less than τ . However we can expect smaller relative L^2 -errors by taking smaller τ since smaller τ gives CRBC of higher order accuracy.

On the other hand, as will be seen in the numerical experiments in Sect. 8, for special data one can achieve much better results, that is obtain the desired accuracy for smaller values of P , with different choices. Developing a solution-adaptive strategy for choosing the parameters would be of interest, but we argue that the general procedure described above leads to choices of P which are far more efficient than the use of simple approximate conditions if one takes into account the fact that the artificial boundary can be located very close to the sources.

We remark that Theorem 2.1 also holds in three space dimensions. The estimate is slightly worse than what is proven in [23] for CRBC in the time domain. Roughly speaking, purely glancing waves do not propagate to the boundary in finite time. We note that a discussion of optimal rational approximants for the Helmholtz equation is given in [15] with glancing modes excluded. The implementation there uses optimal grids - although the authors choose to call this a PML, in fact the method is more closely related to CRBC. For applications of the usual PML method in the half space, which can also provide the exponential convergence of high-frequency propagating modes, it is shown in [11] that PML also suffers from slow convergence of grazing modes only reciprocally proportional to PML strength and PML width whereas [35] avoids the grazing modes in solutions by taking imaginary wavenumber in the model problem.

3 Eigenvalue problem with CRBC on the interval $D_1 = (0, 1)$

In this section, we analyze an eigenvalue problem with the CRBC in the unit open interval $D_1 \subset \mathbb{R}$. The eigenvalue problem to be investigated is supplemented with the homogeneous Dirichlet condition at $x = 0$ and CRBC at $x = 1$. The result in this section will be used for the existence and uniqueness of solutions to the Helmholtz equation posed in the square domain $(0, 1)^2 \subset \mathbb{R}^2$ supplemented with CRBC on boundaries later.

3.1 Eigenvalues of CRBC

For a given set of parameters a_j and $\tilde{a}_j$ we consider the eigenvalue problem to find $\lambda \in \mathbb{C}$ and non-zero solutions $u \in L^2(D_1)$ satisfying

$$\begin{aligned} \frac{d^2}{dx^2}u + \lambda^2 u &= 0 \text{ in } D_1, \\ u(0) &= 0 \text{ and CRBC at } x = 1. \end{aligned} \quad (3.1)$$

The above eigenvalue problem (3.1) can be written in a system of differential equations as follows: find $\tilde{\Phi} = (\tilde{\varphi}_0, \dots, \tilde{\varphi}_P)^\top \in (L^2(D_1))^{P+1}$ such that

$$\frac{d^2}{dx^2}\tilde{\Phi} + \lambda^2 \tilde{\Phi} = 0 \text{ in } D_1, \quad (3.2)$$

subject to the conditions

$$\tilde{\varphi}_0(0) = 0, \quad (3.3)$$

$$\left(\frac{d}{dx} + a_j\right)\tilde{\varphi}_j = \left(-\frac{d}{dx} + \tilde{a}_j\right)\tilde{\varphi}_{j+1} \text{ in } D_1 \text{ for } j = 0, \dots, P-1, \quad (3.4)$$

$$\frac{d\tilde{\varphi}_P}{dx}(1) = 0, \quad (3.5)$$

where the 0-th component $\tilde{\varphi}_0$ of $\tilde{\Phi}$ is an eigenvector to the problem (3.1).

By taking the negative real axis branch cut for the square root we assume that

$$-\pi/2 < \arg(\lambda) \leq \pi/2 \quad (3.6)$$

and hence $\Re(\lambda) \geq 0$. Also, we note that $a_j, \tilde{a}_j \neq i\lambda$ for $j = 0, \dots, P-1$ due to the conditions (2.6)–(2.7) for $a_j, \tilde{a}_j$ and (3.6).

Lemma 3.1 *The eigenparameters λ are neither real nor purely imaginary. In addition, the eigenvalues λ^2 satisfy $\Im(\lambda^2) < 0$.*

Proof If $\lambda = 0$, then it is easy to show that non-zero $\tilde{\Phi}$ can not fulfill (3.4)–(3.5) since all $\tilde{\varphi}_j$ are polynomials of degree 1. Therefore, $\lambda \neq 0$ and the components $\tilde{\varphi}_j$ of the eigenvectors are of the form

$$\tilde{\varphi}_j = A_j e^{i\lambda x} + B_j e^{-i\lambda x}. \quad (3.7)$$

and (3.4) gives

$$(a_j + i\lambda)A_j = (\tilde{a}_j - i\lambda)A_{j+1} \text{ and } (a_j - i\lambda)B_j = (\tilde{a}_j + i\lambda)B_{j+1}. \quad (3.8)$$

We first claim that if λ^2 is an eigenvalue, then $a_j, \tilde{a}_j \neq -i\lambda$ for $j = 0, \dots, P-1$. Indeed, suppose that $a_\ell = -i\lambda$ for some ℓ as the other case $\tilde{a}_\ell = -i\lambda$ for some ℓ can

be treated similarly. From (3.8), it can be shown that that $A_j = 0$ for $j = \ell + 1, \dots, P$. Since $A_P = 0$, the terminal condition (3.5) implies $B_P = 0$. Thus the second equation of (3.8) gives $B_j = 0$ for $j = 0, \dots, P$. Due to (3.3), we have $A_0 = 0$ and hence it follows from the first equation of (3.8) that $A_j = 0$ for $j = 0, \dots, P$, which means that $\tilde{\Phi} = 0$ and λ^2 cannot be an eigenvalue. Once we normalize $\tilde{\varphi}_0$ such that $A_0 = 1$, $B_0 = -1$ by (3.3), a simple computation from (3.8) and (3.5) leads to

$$e^{2i(\lambda - \frac{\pi}{2})} = \prod_{j=0}^{P-1} \frac{(a_j - i\lambda)(\tilde{a}_j - i\lambda)}{(a_j + i\lambda)(\tilde{a}_j + i\lambda)}. \quad (3.9)$$

Now, suppose that λ is a positive real number. Due to the conditions (2.6)–(2.7) for the parameters a_j and $\tilde{a}_j$, the magnitude of the right-hand side of (3.9) is less than 1 whereas the left-hand side has magnitude 1, which implies that λ^2 with $\lambda > 0$ cannot be an eigenvalue. The similar argument can be used to verify that there is no eigenvalue with purely imaginary λ such that $\Im(\lambda) > 0$.

Since λ is neither real nor purely imaginary, $\Im(\lambda^2)$ is non-zero. Finally, in order to prove that $\Im(\lambda^2) < 0$, we suppose that $\Im(\lambda^2) > 0$ and so $\Im(\lambda) > 0$. Then the left-hand side of (3.9) has magnitude less than 1 while that of the right-hand side is larger than 1, which contradicts and the proof is completed. $\square$

Since eigenvalues are related with an eigenvalue problem of a holomorphic Fredholm operator valued function (see (3.17) below), the eigenvalues are discrete and all eigenvalues have finite algebraic multiplicity (see [37, Theorem 1.3.1, Corollary 3.1.3]), which allows to order eigenvalues in magnitude with λ_n^2 being the n -th eigenvalue.

Lemma 3.2 *For $|n| \gg 1$, the eigenvalue λ_n^2 has the asymptotic formula*

$$\lambda_n^2 = \frac{(2n+1)^2\pi^2}{4} + 2 \sum_{j=0}^{P-1} (a_j + \tilde{a}_j) + O(n^{-1}). \quad (3.10)$$

Proof To motivate this, note that if $|\lambda_n| \gg 1$ then the right-hand side of (3.9) is approximately 1. Therefore

$$\lambda_n = \frac{(2n+1)\pi}{2} + o(1).$$

Then we have

$$\begin{aligned} \frac{i\lambda_n - a_j}{i\lambda_n + a_j} &= \frac{1 + 2ia_j/((2n+1)\pi)}{1 - 2ia_j/((2n+1)\pi)} + o(n^{-1}) \\ &= 1 + \frac{4ia_j}{(2n+1)\pi} + o(n^{-1}). \end{aligned}$$

Substituting it and the analogous formula for $\tilde{a}_j$ into (3.9), we find

$$e^{2i(\lambda_n - \frac{\pi}{2})} = 1 + \frac{4i}{(2n+1)\pi} \sum_{j=0}^{P-1} (a_j + \tilde{a}_j) + o(n^{-1}).$$

Taking the logarithm gives

$$\lambda_n = \frac{(2n+1)\pi}{2} + \frac{2}{(2n+1)\pi} \sum_{j=0}^{P-1} (a_j + \tilde{a}_j) + o(n^{-1})$$

and (3.10) follows. $\square$

3.2 Eigenvector expansion

Now, we shall establish that the set of eigenvectors is complete in $(L^2(D_1))^{P+1}$, which allows the eigenvector expansion for functions in $(L^2(D_1))^{P+1}$. To this end, we look at the system of equations of the auxiliary variables with the modified recursions serving as boundary conditions: to find $\lambda^2 \in \mathbb{C}$ and non-trivial solutions $\Phi = (\tilde{\varphi}_0, \dots, \tilde{\varphi}_P)^\top$ in $(L^2(D_1))^{P+1}$ satisfying (3.2) with the boundary conditions (3.3), (3.5) and with (3.4) replaced by

$$\left(\frac{d}{dx} + a_j\right) \tilde{\varphi}_j = \left(-\frac{d}{dx} + \tilde{a}_j\right) \tilde{\varphi}_{j+1} \quad \text{at } x = 0, 1 \quad \text{for } j = 0, \dots, P-1. \quad (3.11)$$

The only difference between the two eigenvalue problems is that the recursive formulas of (3.4) hold in the whole domain D_1 whereas the modified recursions of the conditions (3.11) are imposed only at the two boundary points $x = 0$ and 1 .

Obviously, eigenvalues of the problem (3.2) with the conditions (3.3)–(3.5) are eigenvalues of the expanded eigenvalue problem with (3.11) instead of (3.4). We note that the expanded eigenvalue problem has additional eigenvalues. However, eigenvectors associated with the additional eigenvalues do not contribute to eigenvector expansions for functions satisfying (3.4). Indeed, let λ^2 be an eigenvalue of the expanded eigenvalue problem for an eigenvector Φ such that

$$\delta_j = \frac{d}{dx} \tilde{\varphi}_{j+1} - \tilde{a}_j \tilde{\varphi}_{j+1} + \frac{d}{dx} \tilde{\varphi}_j + a_j \tilde{\varphi}_j \neq 0.$$

Then it solves the problem

$$\frac{d^2}{dx^2} \delta_j + \lambda^2 \delta_j = 0 \quad \text{and} \quad \delta_j(0) = \delta_j(1) = 0.$$

This is satisfied for $\lambda^2 = n^2\pi^2$, $\delta_j = C_j \sin(n\pi x)$. Here we claim that for each n the eigenspace associated with λ^2 is of P -dimension. To prove it, we see that

$$\tilde{\varphi}_j = A_j \sin(n\pi x) + B_j \cos(n\pi x) \quad (3.12)$$

for some constants A_j and B_j , and the boundary conditions (3.3), (3.5) and (3.11) are satisfied if

$$B_0 = A_P = 0, \quad (3.13)$$

$$n\pi A_j + a_j B_j = -n\pi A_{j+1} + \tilde{a}_j B_{j+1}, \quad j = 0, \dots, P-1. \quad (3.14)$$

These represent $P+2$ linearly independent linear equations in $2P+2$ unknowns and so the null space has dimension P . If the full extended system has a complete set of eigenvectors then we can expand any vector function Φ for which all $\delta_j = 0$. These will only involve the eigenvectors corresponding to the original problem (see Theorem 3.7).

From now on, we shall prove the completeness of eigenvectors of the expanded eigenvalue problem (3.2) with (3.3), (3.5) and (3.11). To do this, we use the spectral theory in [37] investigating a general class of non-self-adjoint boundary eigenvalue problems in $L^2(D_1)$, in particular, of the form

$$\frac{d}{dx}\mathcal{Y} = \lambda\mathcal{A}\mathcal{Y} \text{ in } D_1 \quad (3.15)$$

with the boundary condition given by

$$\tilde{W}^{(0)}(\lambda)\mathcal{Y}(0) + \tilde{W}^{(1)}(\lambda)\mathcal{Y}(1) = 0, \quad (3.16)$$

where $\mathcal{Y} \in (L^2(D_1))^N$ and $\mathcal{A}$ is an $N \times N$ invertible matrix for $N \in \mathbb{N}$. Also, $\tilde{W}^{(j)}(\lambda)$ are $N \times N$ matrices. The theory deals with boundary eigenvalue problems for which eigenparameters are involved non-linearly in boundary conditions and gives sufficient conditions for the norm convergence of the eigenvector expansions. The eigenvalue problem (3.15)–(3.16) is related to the holomorphic Fredholm operator-valued function of $\lambda \in \mathbb{C}$ from $(L^2(D_1))^N$ to $(L^2(D_1))^N \times \mathbb{C}^N$ defined by

$$T(\lambda)\mathcal{Y} = \begin{pmatrix} \frac{d}{dx}\mathcal{Y} - \lambda\mathcal{A}\mathcal{Y} \\ \tilde{W}^{(0)}(\lambda)\mathcal{Y}(0) + \tilde{W}^{(1)}(\lambda)\mathcal{Y}(1) \end{pmatrix} := \begin{pmatrix} T^D(\lambda)\mathcal{Y} \\ T^R(\lambda)\mathcal{Y} \end{pmatrix}, \quad (3.17)$$

and $(\mu, \mathcal{Y})$ is an eigenpair of the problem if and only if $T(\mu)\mathcal{Y} = 0$.

The eigenvector expansion under consideration is defined as an infinite sum of certain eigenvectors and associated vectors of (3.17), (precisely they are called the canonical system of eigenvectors and associated vectors), where eigenvectors and associated vectors for an eigenvalue μ are defined as a set $\{\mathcal{Y}^0, \mathcal{Y}^1, \dots, \mathcal{Y}^m\}$ satisfying $T(\lambda)\mathcal{Y}$ has a zero at $\lambda = \mu$ of multiplicity $\geq m+1$ with a root function $\mathcal{Y} = \mathcal{Y}^0 + (\lambda - \mu)\mathcal{Y}^1 + \dots + (\lambda - \mu)^m\mathcal{Y}^m$ (see [37, Definition 1.6.1]).

The desired series expansion in terms of eigenvectors and associated vectors of (3.17) is guaranteed if the boundary eigenvalue problem (3.15) with the boundary condition (3.16) satisfies the so-called *Birkhoff regular condition*, whose definition is given as follows. For an eigenvalue λ_ν of $\mathcal{A}$ for $\nu = 1, 2, \dots, N$, let φ_ν denote $\arg(\lambda_\nu)$. For $\lambda \in \mathbb{C} \setminus \{0\}$ we set

$$\gamma_\nu(\lambda) = \begin{cases} 0 & \text{if } \Re(\lambda e^{i\varphi_\nu}) < 0, \\ 1 & \text{if } \Re(\lambda e^{i\varphi_\nu}) > 0, \\ 0 & \text{if } \Re(\lambda e^{i\varphi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\varphi_\nu}) > 0, \\ 1 & \text{if } \Re(\lambda e^{i\varphi_\nu}) = 0 \text{ and } \Im(\lambda e^{i\varphi_\nu}) < 0. \end{cases}$$

Finally, we define the $N \times N$ diagonal matrix

$$\Delta(\lambda) = \text{diag}(\gamma_1(\lambda), \dots, \gamma_N(\lambda)).$$

Definition 3.3 The boundary eigenvalue problem (3.15)–(3.16) is said to be *Birkhoff regular* if there exist $N \times N$ matrices $W^{(j)}$ independent of λ satisfying

$$|\tilde{W}^{(j)}(\lambda) - W^{(j)}| = O(\lambda^{-1}) \text{ as } \lambda \rightarrow \infty. \quad (3.18)$$

and

$$W^{(0)}(I_N - \Delta(\lambda)) + W^{(1)}\Delta(\lambda)$$

is invertible for $\lambda \in \mathbb{C} \setminus \{0\}$, where I_N represents the $N \times N$ identity matrix.

The definition of the Birkhoff regular condition is a special case of the general definition given in [37, Definition 4.1.2]. The following theorem quoted from [37] provides a sufficient condition for the existence of a series expansion of functions in $(L^2(D_1))^N$ in terms of eigenvectors and associated vectors.

Theorem 3.4 ([37, Theorem 5.3.2]) *If the boundary eigenvalue problem (3.15) with the boundary condition (3.16) is Birkhoff regular, then every function $\mathcal{F} \in (L^2(D_1))^N$ has a series expansion*

$$\mathcal{F} = \sum_{n=0}^{\infty} \sum_{q=0}^{m_n-1} c_n^q \mathcal{Y}_n^q,$$

where c_n^q are complex coefficients and $\{\mathcal{Y}_n^q\}_{q=0}^{m_n-1}$ is the canonical system of eigenvectors and associated vectors for the eigenvalue λ_n of (3.15), (3.16).

To follow the theory in [37], we convert the second order eigenvalue problem to a first order boundary eigenvalue problem by introducing $\tilde{\psi}_j$ such that

$$\frac{d}{dx} \tilde{\varphi}_j = -\lambda \tilde{\psi}_j \quad \text{and} \quad \frac{d}{dx} \tilde{\psi}_j = \lambda \tilde{\varphi}_j.$$

Then we seek eigenvalues $\lambda \in \mathbb{C}$ such that there is a non-zero function

$$\mathcal{Y} = (\tilde{\varphi}_0, \tilde{\varphi}_1, \dots, \tilde{\varphi}_P, \tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_P)^\top \in (L^2(D_1))^{2P+2}$$

satisfying (3.15) and the boundary condition (3.16) with $N = 2P + 2$, where $\mathcal{A}$ is a $(2P + 2) \times (2P + 2)$ matrix defined by

$$\mathcal{A} = \begin{bmatrix} 0 & -I_{P+1} \\ I_{P+1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes I_{P+1}, \quad (3.19)$$

and $\tilde{W}^{(0)}$ and $\tilde{W}^{(1)}$ are $(2P + 2) \times (2P + 2)$ matrices given by

$$\tilde{W}^{(0)}(\lambda) = \begin{bmatrix} \tilde{V}^{(0)} & -\tilde{J}^{(0)} \\ 0_{P+1} & 0_{P+1} \end{bmatrix} \quad \text{and} \quad \tilde{W}^{(1)} = \begin{bmatrix} 0_{P+1} & 0_{P+1} \\ \tilde{V}^{(1)} & -\tilde{J}^{(1)} \end{bmatrix} \quad (3.20)$$

with $(P + 1) \times (P + 1)$ matrices

$$\begin{aligned} \tilde{V}^{(0)} &= \begin{bmatrix} a_0 & -\tilde{a}_0 & & & \\ & a_1 & -\tilde{a}_1 & & \\ & & \ddots & \ddots & \\ & & & a_{P-1} & -\tilde{a}_{P-1} \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, & \tilde{J}^{(0)} &= \begin{bmatrix} \lambda & \lambda & & & \\ & \lambda & \lambda & & \\ & & \ddots & \ddots & \\ & & & \lambda & \lambda \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \\ \tilde{V}^{(1)} &= \begin{bmatrix} a_0 & -\tilde{a}_0 & & & \\ & a_1 & -\tilde{a}_1 & & \\ & & \ddots & \ddots & \\ & & & a_{P-1} & -\tilde{a}_{P-1} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, & \tilde{J}^{(1)} &= \begin{bmatrix} \lambda & \lambda & & & \\ & \lambda & \lambda & & \\ & & \ddots & \ddots & \\ & & & \lambda & \lambda \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \end{aligned}$$

Here 0_{P+1} denotes the $(P + 1) \times (P + 1)$ zero matrix. By the change of basis using the transition matrix

$$\mathcal{S} = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \otimes I_{P+1},$$

$\mathcal{A}$ can be diagonalized and by abuse of notation we continue to write $\mathcal{A}$ for $\mathcal{S}^{-1}\mathcal{A}\mathcal{S}$. Similarly, we continue to use $\tilde{W}^{(0)}$ and $\tilde{W}^{(1)}$ for $2\mathcal{S}^{-1}\tilde{W}^{(0)}\mathcal{S}$ and $2\mathcal{S}^{-1}\tilde{W}^{(1)}\mathcal{S}$, respectively. Thus, we can assume that $\mathcal{A}$ is a diagonal matrix

$$\mathcal{A} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \otimes I_{P+1} \quad (3.21)$$

and $\tilde{W}^{(0)}$ and $\tilde{W}^{(1)}$ are given by

$$\begin{aligned}\tilde{W}^{(0)}(\lambda) &= \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \otimes \tilde{V}^{(0)} - \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \otimes \tilde{J}^{(0)}, \\ \tilde{W}^{(1)}(\lambda) &= \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \otimes \tilde{V}^{(1)} - \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \otimes \tilde{J}^{(1)}.\end{aligned}\quad (3.22)$$

Before we go further to study the Birkhoff regularity of the eigenvalue problem, we investigate the algebraic multiplicity m_n of eigenvalues of the problem (3.21) and (3.22) by analyzing the determinant of the characteristic matrix function (see [37, Section 1.11, Section 3.2]). To do this, we shall recall the characteristic matrix function of (3.17) in a decomposition of $T(\lambda)$: let

$$Y(x, \lambda) = \begin{bmatrix} e^{i\lambda x} & 0 \\ 0 & e^{-i\lambda x} \end{bmatrix} \otimes I_{P+1},$$

which is a generator of solutions of the form $\mathcal{Y} = Y(x, \lambda)c_0$ for $c_0 \in \mathbb{C}^{2P+2}$ to the homogeneous equation $T^D(\lambda)\mathcal{Y} = 0$ and is called the fundamental matrix function of (3.17). We define $Z(\lambda) : \mathbb{C}^{2P+2} \rightarrow (H^1(0, 1))^{2P+2}$ by $Z(\lambda) = Y(x, \lambda)c_0$ for $c_0 \in \mathbb{C}^{2P+2}$. We can also find the right inverse of $T^D(\lambda)$ for the zero initial condition, which is denoted by $U(\lambda)$, for $g \in (L^2(0, 1))^{2P+2}$

$$U(\lambda)g = Y(x, \lambda) \int_0^x Y(t, \lambda)^{-1} g(t) dt.$$

The maps defined up to now come together in the following short exact sequence.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}^{2P+2} & \xrightarrow{Z(\lambda)} & (H^1(0, 1))^{2P+2} & \xrightarrow{T^D(\lambda)} & (L^2(0, 1))^{2P+2} \longrightarrow 0 \\ & & & \searrow M(\lambda) & \downarrow T^R(\lambda) & & \\ & & & & \mathbb{C}^{2P+2} & & \end{array}$$

$\overset{U(\lambda)}{\curvearrowright}$

In particular, since $T^D(\lambda)$ has the right inverse $U(\lambda)$, the exact sequence is splitting, i.e., $(H^1(0, 1))^{2P+2} \simeq \mathbb{C}^{2P+2} \oplus (L^2(0, 1))^{2P+2}$, and the map $(Z(\lambda), U(\lambda)) : \mathbb{C}^{2P+2} \times (L^2(0, 1))^{2P+2} \rightarrow (H^1(0, 1))^{2P+2}$ defined by $(Z(\lambda), U(\lambda))(c_0, g) = Z(\lambda)c_0 + U(\lambda)g$ is invertible. By introducing

$$M(\lambda)c_0 = T^R(\lambda)Z(\lambda)c_0, \quad (3.23)$$

which is called the characteristic matrix function of (3.17), we can decompose $T(\lambda)$ into

$$T(\lambda) = \begin{pmatrix} T^D(\lambda) \\ T^R(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & \text{id}_{(L^2(0,1))^{2P+2}} \\ \text{id}_{\mathbb{C}^{2P+2}} & T^R(\lambda)U(\lambda) \end{pmatrix} \begin{pmatrix} M(\lambda) & 0 \\ 0 & \text{id}_{(L^2(a,b))^{2P+2}} \end{pmatrix} (Z(\lambda), U(\lambda))^{-1}.$$

Since the first and last matrix functions in the decomposition are invertible, the spectrum of $T(\lambda)$ is determined by the characteristic matrix function $M(\lambda)$. From the definition (3.23) of the characteristic matrix function, we see that

$$M(\lambda) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where

$$\begin{aligned} M_{11} &= -i(\tilde{V}^{(0)} + i\tilde{J}^{(0)}) + ie^{i\lambda}(\tilde{V}^{(1)} + i\tilde{J}^{(1)}), \\ M_{12} &= -i(\tilde{V}^{(0)} - i\tilde{J}^{(0)}) + e^{-i\lambda}(\tilde{V}^{(1)} - i\tilde{J}^{(1)}), \\ M_{21} &= -i(\tilde{V}^{(0)} + i\tilde{J}^{(0)}) + e^{i\lambda}(\tilde{V}^{(1)} + i\tilde{J}^{(1)}), \\ M_{22} &= -i(\tilde{V}^{(0)} - i\tilde{J}^{(0)}) - ie^{-i\lambda}(\tilde{V}^{(1)} - i\tilde{J}^{(1)}). \end{aligned}$$

Denoting

$$\alpha_j^\pm = a_j \pm i\lambda, \text{ and } \tilde{\alpha}_j^\pm = \tilde{a}_j \pm i\lambda, \quad (3.24)$$

$$a = 1 + ie^{i\lambda}, \quad b = -i + e^{-i\lambda}, \quad c = i + e^{i\lambda} \text{ and } d = 1 - ie^{-i\lambda}, \quad (3.25)$$

$M(\lambda)$ can be written as

$$M(\lambda) = \begin{bmatrix} a \begin{bmatrix} \alpha_0^+ & -\tilde{\alpha}_0^- & & & \\ & \alpha_1^+ & -\tilde{\alpha}_1^- & & \\ & & \ddots & \ddots & \\ & & & \alpha_{P-1}^+ & -\tilde{\alpha}_{P-1}^- \\ 1/a & 0 & \cdots & 0 & -e^{i\lambda}/a \end{bmatrix} & b \begin{bmatrix} \alpha_0^- & -\tilde{\alpha}_0^+ & & & \\ & \alpha_1^- & -\tilde{\alpha}_1^+ & & \\ & & \ddots & \ddots & \\ & & & \alpha_{P-1}^- & -\tilde{\alpha}_{P-1}^+ \\ -i/b & 0 & \cdots & 0 & -ie^{-i\lambda}/b \end{bmatrix} \\ c \begin{bmatrix} \alpha_0^+ & -\tilde{\alpha}_0^- & & & \\ & \alpha_1^+ & -\tilde{\alpha}_1^- & & \\ & & \ddots & \ddots & \\ & & & \alpha_{P-1}^+ & -\tilde{\alpha}_{P-1}^- \\ i/c & 0 & \cdots & 0 & ie^{i\lambda}/c \end{bmatrix} & d \begin{bmatrix} \alpha_0^- & -\tilde{\alpha}_0^+ & & & \\ & \alpha_1^- & -\tilde{\alpha}_1^+ & & \\ & & \ddots & \ddots & \\ & & & \alpha_{P-1}^- & -\tilde{\alpha}_{P-1}^+ \\ 1/d & 0 & \cdots & 0 & -e^{-i\lambda}/d \end{bmatrix} \end{bmatrix}. \quad (3.26)$$

Lemma 3.5 *The determinant of $M(\lambda)$ is*

$$-2(-2i)^P (e^{-i\lambda} - e^{i\lambda})^P \left(e^{-i\lambda} \prod_{j=0}^{P-1} (a_j - i\lambda)(\tilde{a}_j - i\lambda) + e^{i\lambda} \prod_{j=0}^{P-1} (a_j + i\lambda)(\tilde{a}_j + i\lambda) \right). \quad (3.4)$$

Thus an eigenvalue λ with $\Im(\lambda) < 0$ has algebraic multiplicity 1 whereas an eigenvalue λ with $\lambda = n\pi$ has algebraic multiplicity P .

Proof Since eigenvalues λ with $\Im(\lambda) < 0$ are solutions to the equation (3.9), the determinant must have the factor

$$w^0 := \underbrace{e^{-i\lambda} \prod_{j=0}^{P-1} (a_j - i\lambda)(\tilde{a}_j - i\lambda)}_{:=w^-} + \underbrace{e^{i\lambda} \prod_{j=0}^{P-1} (a_j + i\lambda)(\tilde{a}_j + i\lambda)}_{:=w^+}.$$

Using (3.24), w^- and w^+ are simply written as

$$w^- = e^{-i\lambda} \prod_{j=0}^{P-1} \alpha_j^- \tilde{\alpha}_j^- \quad \text{and} \quad w^+ = e^{i\lambda} \prod_{j=0}^{P-1} \alpha_j^+ \tilde{\alpha}_j^+.$$

Taking the number of linear factors $\alpha_j^\pm$ and $\tilde{\alpha}_j^\pm$ with respect to λ into account, we note that $\det(M(\lambda))$ is of the form

$$\det(M(\lambda)) := z(e^{i\lambda})w^0 = z(e^{i\lambda})w^- + z(e^{i\lambda})w^+,$$

where $z(e^{i\lambda})$ is a function of $e^{i\lambda}$, and it does not include any terms with mixed signs such as $\alpha_j^+ \alpha_\ell^-$, $\alpha_j^- \alpha_\ell^+$, $\alpha_j^+ \tilde{\alpha}_\ell^-$ and $\alpha_j^- \tilde{\alpha}_\ell^+$. We will find the common factor $z(e^{i\lambda})$ by examining w^- and w^+ in computing $\det(M(\lambda))$.

We begin by denoting the multipliers of $w^\pm$ in $\det M(\lambda)$ by $z^\pm(e^{i\lambda})$, respectively, and we will show that

$$z^+(e^{i\lambda}) = z^-(e^{i\lambda}) = -2(-2i)^P (e^{-i\lambda} - e^{i\lambda})^P.$$

In order to examine the term $z^+(e^{i\lambda})w^+$ of $\det(M(\lambda))$, we note that the first P rows of M_{11} , M_{12} , M_{21} and M_{22} have the common factors a , b , c and d defined by (3.25), respectively. Since w^+ is determined by how many α_j^+ are selected in M_{11} and how many remaining α_j^+ are selected in M_{21} (or equivalently how many $\tilde{\alpha}_j^+$ are selected in M_{12} and M_{22}), it suffices to find the determinant of the matrix $\tilde{M}$ obtained by replacing α_j^- , $\tilde{\alpha}_j^-$ with 0 and α_j^+ , $\tilde{\alpha}_j^+$ with 1,

$$\tilde{M}(\lambda) = \begin{bmatrix} \begin{bmatrix} a & 0 \\ a & 0 \\ & \ddots \\ & a & 0 \end{bmatrix} & \begin{bmatrix} 0 & -b \\ 0 & -b \\ & \ddots \\ 0 & -b \end{bmatrix} \\ \begin{bmatrix} c & 0 \\ c & 0 \\ & \ddots \\ c & 0 \end{bmatrix} & \begin{bmatrix} 0 & -d \\ 0 & -d \\ & \ddots \\ 0 & -d \end{bmatrix} \end{bmatrix} \begin{matrix} \leftarrow (P+1)\text{th row} \\ \leftarrow (2P+2)\text{th row} \end{matrix} \quad (3.28)$$

$\uparrow$
 $(P+1)\text{th column}$

Then we add the $(P + 1)$ -th row multiplied by i to the $(2P + 2)$ -th row to eliminate the $(2P + 2, P + 1)$ entry $ie^{i\lambda}$ and use the cofactor expansion along the $(P + 1)$ -th column, which includes only one non-zero entry $-e^{i\lambda}$ at $(P + 1, P + 1)$. We conduct the cofactor expansion along the $(P + 1)$ -th column and then do it once again along the $(P + 1)$ -th column (which also has only one non-zero entry 2 at $(P + 1, 2P + 1)$) in computing the $(P + 1, P + 1)$ minor to see that $\det(\tilde{M})$ is given by

$$\det(\tilde{M}) = (-1)^{P+1} 2e^{i\lambda} \det(\tilde{M}_1),$$

where $\tilde{M}_1$ is a $2P \times 2P$ matrix of the form

$$\tilde{M}_1 = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \otimes I_P.$$

Using the fact that the determinant of $\tilde{M}_1$ is $(bc - ad)^P = (2i)^P (e^{-i\lambda} - e^{i\lambda})^P$ yields that

$$\begin{aligned} z^+(e^{i\lambda})w^+ &= \det(\tilde{M}(\lambda)) \prod_{j=0}^{P-1} \alpha_j^+ \tilde{\alpha}_j^+ \\ &= -2(-2i)^P (e^{-i\lambda} - e^{i\lambda})^P e^{i\lambda} \prod_{j=0}^{P-1} \alpha_j^+ \tilde{\alpha}_j^+. \end{aligned}$$

Similarly, one can show that

$$z^-(e^{i\lambda})w^- = -2(-2i)^P (e^{-i\lambda} - e^{i\lambda})^P e^{-i\lambda} \prod_{j=0}^{P-1} \alpha_j^- \tilde{\alpha}_j^-,$$

and so it is shown that (3.4) is the determinant of M . Thus, it turns out that eigenvalues λ are the zeros of $e^{2i\lambda} = 1$ or $w^0 = 0$ (the equation (3.9)).

Now, we are left to show that the zeros of $e^{2i\lambda} = 1$ and $w^0 = 0$ are all simple. It is easy to see that the zeros of $e^{2i\lambda} = 1$ are simple and hence their algebraic multiplicity is P due to the P -th power of $(e^{-i\lambda} - e^{i\lambda})^P$. For the case of (3.9), we shall show that the derivative of the function

$$F(\lambda) = e^{2i\lambda} \prod_{j=0}^{P-1} \frac{(a_j + i\lambda)(\tilde{a}_j + i\lambda)}{(a_j - i\lambda)(\tilde{a}_j - i\lambda)} + 1$$

does not have a zero in the fourth quadrant of $\mathbb{C}$. To do this we find the derivative of the function

$$F(\lambda) = e^{2i\lambda} \prod_{j=0}^{2P-1} \frac{b_j + i\lambda}{b_j - i\lambda} + 1$$

with $b_j = a_j$ and $b_{P+j} = \tilde{a}_j$ for $j = 0, \dots, P-1$:

$$F'(\lambda) = 2ie^{2i\lambda} \prod_{j=0}^{2P-1} \frac{b_j + i\lambda}{b_j - i\lambda} + e^{2i\lambda} \sum_{\ell=0}^{2P-1} \frac{2ib_\ell}{(b_\ell - i\lambda)^2} \prod_{j \neq \ell} \frac{b_j + i\lambda}{b_j - i\lambda}.$$

If λ is a zero of $F(\lambda) = 0$, then we have

$$F'(\lambda) = -2i - \sum_{\ell=0}^{2P-1} \frac{2ib_\ell}{(b_\ell - i\lambda)^2} \frac{b_\ell - i\lambda}{b_\ell + i\lambda} = -2i \left(1 + \sum_{\ell=0}^{2P-1} \frac{b_\ell}{b_\ell^2 + \lambda^2} \right).$$

Therefore, it suffices to show that $1 + \mathfrak{f}(\lambda) + \mathfrak{g}(\lambda) = 0$ has no zero λ with $-\pi/2 < \arg(\lambda) < 0$, where

$$\begin{aligned} \mathfrak{f}(\lambda) &= \sum_{j=0}^{n_p-1} \frac{-ic_j k}{\lambda^2 - c_j^2 k^2} + \frac{-i\tilde{c}_j k}{\lambda^2 - \tilde{c}_j^2 k^2}, \\ \mathfrak{g}(\lambda) &= \sum_{j=n_p}^{n_p+n_e-1} \frac{c_j k}{\lambda^2 + c_j^2 k^2} + \frac{\tilde{c}_j k}{\lambda^2 + \tilde{c}_j^2 k^2}. \end{aligned}$$

Now we observe that linear fractional transformations of the form

$$\eta(z) = \frac{-ic}{z - c^2} \quad \text{and} \quad \zeta(z) = \frac{\sigma}{z + \sigma^2}$$

with $c > 0$ and $\sigma > 0$ have the following mapping properties

$$\begin{aligned} \eta, \zeta : \{z \in \mathbb{C} : -\pi/2 < \arg(z) < 0\} &\rightarrow \{z \in \mathbb{C} : \Re(z) > 0\} \\ \eta, \zeta : \{z \in \mathbb{C} : -\pi < \arg(z) \leq -\pi/2\} &\rightarrow \{z \in \mathbb{C} : \Im(z) > 0\}. \end{aligned}$$

Then the required result follows from the fact that $\Re(\mathfrak{f}(\lambda)) > 0$, $\Re(\mathfrak{g}(\lambda)) > 0$ for $-\pi/4 < \arg(\lambda) < 0$ and $\Im(\mathfrak{f}(\lambda)) > 0$, $\Im(\mathfrak{g}(\lambda)) > 0$ for $-\pi/2 < \arg(\lambda) \leq -\pi/4$, which completes the proof. $\square$

We further transform the boundary condition (3.16) by multiplying it by an invertible diagonal matrix $I_2 \otimes \Lambda$ with $\Lambda = \text{diag}(\omega^{-1}, \dots, \omega^{-1}, 1)$, which results in the equivalent boundary condition of the form (3.16) with

$$\tilde{W}^{(0)}(\lambda) = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \otimes \Lambda \tilde{V}^{(0)} - \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \otimes J^{(0)}, \quad (3.4)$$

$$\tilde{W}^{(1)}(\lambda) = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \otimes \Lambda \tilde{V}^{(1)} - \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \otimes J^{(1)} \quad (3.5)$$

with abuse of notations for $\tilde{W}^{(0)}$ and $\tilde{W}^{(1)}$, where $J^{(j)}$ are the matrices obtained from $\tilde{J}^{(j)}$ by replacing λ with 1. It is clear that $\tilde{W}^{(0)}(\lambda)$ and $\tilde{W}^{(1)}(\lambda)$ are asymptotically

constant for large λ , i.e. there exist $(2P+2) \times (2P+2)$ matrices $W^{(0)}$ and $W^{(1)}$ such that

$$\tilde{W}^{(0)}(\lambda) = W^{(0)} + O(\lambda^{-1}) \quad \text{and} \quad \tilde{W}^{(1)}(\lambda) = W^{(1)} + O(\lambda^{-1}),$$

where

$$\begin{aligned} W^{(0)} &= \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \otimes E_{P,0} - \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \otimes J^{(0)}, \\ W^{(1)} &= - \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \otimes J^{(1)}. \end{aligned} \quad (3.6)$$

Here $E_{\ell,j}$ is a $(P+1) \times (P+1)$ matrix whose only non-zero element is one at the (ℓ, j) component with $0 \leq \ell, j \leq P$.

On the other hand, we recall that $\mathcal{A}$ has the eigenvalues i and $-i$ of multiplicity $P+1$ and see that if $\gamma_\nu(\lambda) = 1$ for the first $P+1$ eigenvalues i counting their multiplicity for $\lambda \in \mathbb{C} \setminus \{0\}$, then $\gamma_\nu(\lambda) = 0$ for the remaining $P+1$ eigenvalues, $-i$, and vice versa. It implies that

$$\Delta(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{P+1} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_{P+1}.$$

Consequently, according to Definition 3.3 it follows that the boundary eigenvalue problem (3.15) with the boundary condition (3.16) (and hence (3.2) with (3.3), (3.5) and (3.11)) is Birkhoff regular, once it is established that

$$W_0 := W^{(0)} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{P+1} \right) + W^{(1)} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_{P+1} \right) \quad (3.7)$$

and

$$W_1 := W^{(0)} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_{P+1} \right) + W^{(1)} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_{P+1} \right) \quad (3.8)$$

are invertible.

Lemma 3.6 *The matrices W_0 and W_1 defined as above are invertible.*

Proof A simple computation using (3.6), (3.7) and (3.8) shows that

$$W_0 = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix} \otimes E_{P,0} - \begin{bmatrix} -i & 0 \\ 1 & 0 \end{bmatrix} \otimes J^{(0)} - \begin{bmatrix} 0 & i \\ 0 & 1 \end{bmatrix} \otimes J^{(1)}.$$

Writing the matrix W_0 in 2×2 block form gives

$$W_0 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

with

$$W_{11} = E_{P,0} + iJ^{(0)}, \quad W_{12} = -iJ^{(1)}$$

$$W_{21} = iE_{P,0} - J^{(0)}, \quad W_{22} = -J^{(1)},$$

and so we have

$$iW_{11} = W_{21} \quad \text{and} \quad iW_{12} = -W_{22}.$$

Here we note that

$$\det(W_{11}) = (-i)^P \quad \text{and} \quad \det(W_{22}) = (-1)^{P+1}.$$

Multiplying W_0 by an invertible matrix

$$\begin{bmatrix} I_{P+1} & 0 \\ -iI_{P+1} & I_{P+1} \end{bmatrix}$$

gives

$$\begin{bmatrix} I_{P+1} & 0 \\ -iI_{P+1} & I_{P+1} \end{bmatrix} W_0 = \begin{bmatrix} W_{1,1} & W_{1,2} \\ 0 & 2W_{2,2} \end{bmatrix},$$

from which it then follows that $\det(W_0) = -2(2i)^P \neq 0$.

A similar computation leads to $\det(W_1) = -2(-2i)^P \neq 0$. Therefore, both W_0 and W_1 are invertible. $\square$

The eigenvector expansion of $f \in L^2(D_1)$ can be established from Theorem 3.4.

Theorem 3.7 *Let $\mathcal{A}$ and $\tilde{W}^{(j)}$ be given as in (3.19) and (3.20), respectively, for the boundary eigenvalue problem (3.15)–(3.16). Then every function $f \in L^2(D_1)$ has a series expansion $f = \sum_{n=0}^{\infty} c_n \tilde{\varphi}_{0,n}$ where $\tilde{\varphi}_{0,n}$ is the zero-th component of an eigenvector $\mathcal{Y}_n := \mathcal{Y}_n^0 = (\tilde{\varphi}_{0,n}, \dots, \tilde{\varphi}_{P,n}, \tilde{\psi}_{0,n}, \dots, \tilde{\psi}_{P,n})^\top$ for the eigenvalue λ_n with $\Im(\lambda_n) < 0$ of (3.15)–(3.16), satisfying (3.4).*

Proof For f smooth and compactly supported in $(0, 1 - \delta)$ we define f_j by $f_0 = f$ and

$$f_{j+1}(x) = -f_j(x) - (a_j + \tilde{a}_j) e^{\tilde{a}_j x} \int_{1-\delta}^x f_j(t) e^{-\tilde{a}_j t} dt \quad (3.9)$$

for $j = 0, \dots, P-1$. Indeed, the sequence of the functions f_j is constructed in such a way that they solve the recurrence relations

$$-\frac{d}{dx} f_{j+1} + \tilde{a}_j f_{j+1} = \frac{d}{dx} f_j + a_j f_j \quad \text{for } j = 0, \dots, P-1$$

and vanish for $1 - \delta < x < 1$.

We define $\mathcal{F} = (f_0, \dots, f_{P-1}, 0, \dots, 0)^\top \in (L^2(D_1))^{2P+2}$. By Theorem 3.4 there exist constants c_n^q such that

$$\mathcal{F} = \sum_{n=0}^{\infty} \sum_{q=0}^{m_n-1} c_n^q \mathcal{Y}_n^q. \quad (3.10)$$

Due to Lemma 3.5 the geometric multiplicity is equal to the algebraic multiplicity, that is, $m_n = 1$ for $\Im(\lambda) < 0$ and $m_n = P$ for $\lambda = n\pi$. Furthermore, it can be shown that the series (3.10) involves only eigenvectors $\mathcal{Y} = (\tilde{\varphi}_0, \dots, \tilde{\varphi}_P, \tilde{\psi}_0, \dots, \tilde{\psi}_P)^\top$ for eigenvalues λ with $\Im(\lambda) < 0$, by examining

$$\delta_j(\mathcal{Y}) := \frac{d}{dx} \tilde{\varphi}_{j+1} - \tilde{a}_j \tilde{\varphi}_{j+1} + \frac{d}{dx} \tilde{\varphi}_j + a_j \tilde{\varphi}_j.$$

Indeed, it is enough to show that $\hat{\mathcal{Y}}_n := \sum_{q=0}^{P-1} c_n^q \mathcal{Y}_n^q$ in the series (3.10), where $\mathcal{Y}_n^q$, $q = 0, \dots, P-1$ are the P linearly independent eigenvectors corresponding to $\lambda = n\pi$, must vanish. If so, $c_n^q = 0$ since $\mathcal{Y}_n^q$ are linearly independent. We recall that the components $\tilde{\varphi}_j^q$ of $\mathcal{Y}_n^q$ are of the form (3.12) with coefficients $A_0^q, \dots, A_P^q, B_0^q, \dots, B_P^q$ satisfying (3.13)–(3.14). By simple computation using (3.14), we get

$$\delta_j(\mathcal{Y}_n^q) = (a_j A_j^q - \tilde{a}_j A_{j+1}^q - n\pi B_j^q - n\pi B_{j+1}^q) \sin(n\pi x) := C_j^q \sin(n\pi x)$$

and so

$$\delta_j(\hat{\mathcal{Y}}_n) = \delta_j\left(\sum_{q=0}^{P-1} c_n^q \mathcal{Y}_n^q\right) = \left(\sum_{q=0}^{P-1} c_n^q C_j^q\right) \sin(n\pi x).$$

Since $\delta_j(\mathcal{F}) = 0$ and $\delta_j(\mathcal{Y}) = 0$ for eigenvectors $\mathcal{Y}$ associated with eigenvalues λ with $\Im(\lambda) < 0$, we have

$$0 = \delta_j(\mathcal{F}) = \sum_{\lambda=n\pi} \delta_j(\hat{\mathcal{Y}}_n) = \sum_{\lambda=n\pi} \left(\sum_{q=0}^{P-1} c_n^q C_j^q\right) \sin(n\pi x).$$

Thus it follows that $\sum_{q=0}^{P-1} c_n^q C_j^q = 0$ due to the linear independence of the sine functions, which means $\delta_j(\hat{\mathcal{Y}}_n) = 0$. If $\hat{\mathcal{Y}}_n$ was non-zero, then $\hat{\mathcal{Y}}_n$ would be an eigenvector for $\lambda = n\pi$ since it is the linear combination of eigenvectors for $\lambda = n\pi$, and so we would conclude that $n\pi$ is an eigenvalue with the eigenvector satisfying (3.4) saying $\delta_j(\hat{\mathcal{Y}}_n) = 0$. This possibility is excluded by Lemma 3.1. We thus conclude that the eigenvectors corresponding to the eigenvalue $n\pi$ do not contribute to the expansion of f . The final result follows from the density of smooth compactly supported functions in $L^2(D_1)$. $\square$

Remark 3.8 The analysis of the eigenvalue problem in this section allows us to have a series expansion of functions in $L^2(D_1)$ in terms of eigenvectors satisfying CRBC at one boundary. It is also possible with minor modifications to extend it to the eigenvalue problem equipped with CRBC at both boundaries of the unit interval.

4 Model problems in a domain with corners

In this section, we consider an application of CRBC to a time-harmonic wave propagation problem posed in a domain with corners. Let $\Omega = (0, 1)^2$ be the square domain in $\mathbb{R}^2$ with north, west, south and east boundaries denoted by Γ_N , Γ_W , Γ_S and Γ_E , respectively. In order to focus on a corner condition of CRBC at one corner, CRBC is imposed on Γ_N and Γ_E while a homogeneous Dirichlet boundary condition is given on Γ_S and Γ_W , so that we will analyze a corner condition for CRBC at the north-east corner denoted by NE. In the model problem, we assume that wave sources are supported away from the boundaries $\Gamma_N \cup \Gamma_E$. We consider the problem to find u in $H^1(\Omega)$ satisfying

$$\begin{cases} \Delta u + k^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_W \cup \Gamma_S, \\ \text{CRBC} & \text{on } \Gamma_E \cup \Gamma_N. \end{cases} \quad (4.1)$$

for a source function f compactly supported in $(0, 1 - \delta)^2 \subset \Omega$ with $0 < \delta < 1$.

We notice that solutions to the above problem satisfy the Helmholtz equation on a neighborhood of $\Gamma_E \cup \Gamma_N$, say $\Omega_\delta = \Omega \setminus (0, 1 - \delta]^2$. The CRBC on Γ_E is defined by a sequence of auxiliary variables ϕ_j^E satisfying

$$\begin{cases} \Delta \phi_j^E + k^2 \phi_j^E = 0 & \text{in } (1 - \delta, 1) \times (0, 1), \text{ for } j = 0, \dots, P, \\ \phi_0^E = u & \text{in } (1 - \delta, 1) \times (0, 1), \\ (\partial_1 + a_j) \phi_j^E = (-\partial_1 + \tilde{a}_j) \phi_{j+1}^E & \text{in } (1 - \delta, 1) \times (0, 1), \text{ for } j = 0, \dots, P - 1, \\ \partial_1 \phi_P^E = 0 & \text{on } \Gamma_E. \end{cases} \quad (4.2)$$

Similarly, the CRBC on Γ_N is defined by the auxiliary variables ϕ_ℓ^N satisfying

$$\begin{cases} \Delta \phi_\ell^N + k^2 \phi_\ell^N = 0 & \text{in } (0, 1) \times (1 - \delta, 1), \text{ for } \ell = 0, \dots, P, \\ \phi_0^N = u & \text{in } (0, 1) \times (1 - \delta, 1), \\ (\partial_2 + a_\ell) \phi_\ell^N = (-\partial_2 + \tilde{a}_\ell) \phi_{\ell+1}^N & \text{in } (0, 1) \times (1 - \delta, 1), \text{ for } \ell = 0, \dots, P - 1, \\ \partial_2 \phi_P^N = 0 & \text{on } \Gamma_N. \end{cases} \quad (4.3)$$

Now, we establish the existence of the solution u to the problem (4.1) and find a series expansion of u in terms of eigenvectors of (3.2) with the boundary conditions (3.3)–(3.5). This series representation will give a motivation for finding a corner compatibility condition of a practical CRBC formulation suitable for discretization methods such as the finite element method for numerical computation.

Since $L^2(\Omega) = L^2(D_1) \otimes L^2(D_1)$, by Theorem 3.7 we can assume that $f \in L^2(\Omega)$ with compact support can be written as

$$f(x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} c_{n_1, n_2} \tilde{\varphi}_{0, n_1}(x_1) \tilde{\varphi}_{0, n_2}(x_2), \quad (4.4)$$

where the series includes only eigenvectors for eigenvalues with $\Im(\lambda_{n_j}) < 0$. Here f is considered as the $(0, 0)$ -component of the function $\mathcal{F} = (f_{j,\ell})_{j,\ell=0}^P$ with $f_{j,\ell} = f$ for $(j, \ell) = (0, 0)$ and $f_{j,\ell}$ defined analogously to (3.9) with respect to ℓ with the initial function $f_{j,0}$ otherwise, which can be expanded as

$$\mathcal{F}(x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} c_{n_1, n_2} \mathcal{Y}_{n_1}(x_1) \otimes \mathcal{Y}_{n_2}(x_2).$$

Theorem 4.1 *There exists a solution u in $H^1(\Omega)$ satisfying $\Delta u \in L^2(\Omega)$ of the form*

$$u(x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} u_{n_1, n_2} \tilde{\varphi}_{0, n_1}(x_1) \tilde{\varphi}_{0, n_2}(x_2) \quad (4.5)$$

with $u_{n_1, n_2} = c_{n_1, n_2} (k^2 - \lambda_{n_1}^2 - \lambda_{n_2}^2)^{-1}$ to the problem (4.1).

Proof It suffices to show that (4.5) is well-defined and it solves the problem (4.1). To this end, we let

$$\begin{aligned} U(x_1, x_2) &= \sum_{n_1, n_2=0}^{\infty} u_{n_1, n_2} \mathcal{Y}_{n_1}(x_1) \otimes \mathcal{Y}_{n_2}(x_2) \\ &= \sum_{n_1, n_2=0}^{\infty} a_{n_1, n_2} \left(c_{n_1, n_2} \mathcal{Y}_{n_1}(x_1) \otimes \mathcal{Y}_{n_2}(x_2) \right) \end{aligned} \quad (4.6)$$

with $a_{n_1, n_2} = (k^2 - \lambda_{n_1}^2 - \lambda_{n_2}^2)^{-1}$. We first show that

(i) a_{n_1, n_2} is of bounded bivariation, i.e.,

$$\sum_{n_1, n_2=0}^{\infty} |a_{n_1, n_2} - a_{n_1+1, n_2} - a_{n_1, n_2+1} + a_{n_1+1, n_2+1}| < \infty, \quad (4.7)$$

(ii) $a_{n_1, 0}$ of bounded variation, i.e., $\sum_{n_1=0}^{\infty} |a_{n_1, 0} - a_{n_1+1, 0}| < \infty$,

(iii) a_{0, n_2} of bounded variation, i.e., $\sum_{n_2=0}^{\infty} |a_{0, n_2} - a_{0, n_2+1}| < \infty$,

by using the asymptotic behavior (3.10) of eigenvalues. Indeed, due to the fact that

$$\begin{aligned} &|a_{n_1, n_2} - a_{n_1+1, n_2} - a_{n_1, n_2+1} + a_{n_1+1, n_2+1}| \\ &= \left| \frac{(\lambda_{n_1}^2 - \lambda_{n_1+1}^2)(\lambda_{n_2}^2 - \lambda_{n_2+1}^2)(2k^2 - \lambda_{n_1}^2 - \lambda_{n_1+1}^2 - \lambda_{n_2}^2 - \lambda_{n_2+1}^2)}{(k^2 - \lambda_{n_1}^2 - \lambda_{n_2}^2)(k^2 - \lambda_{n_1+1}^2 - \lambda_{n_2}^2)(k^2 - \lambda_{n_1}^2 - \lambda_{n_2+1}^2)(k^2 - \lambda_{n_1+1}^2 - \lambda_{n_2+1}^2)} \right| \\ &= \frac{O(n_1)O(n_2)O(n_1^2 + n_2^2)}{O((n_1^2 + n_2^2)^4)} = O\left(\frac{1}{(n_1^2 + n_2^2)^2}\right), \end{aligned} \quad (4.8)$$

$$\begin{aligned}
 |a_{n_1,0} - a_{n_1+1,0}| &= \left| \frac{\lambda_{n_1}^2 - \lambda_{n_1+1}^2}{(k^2 - \lambda_{n_1}^2 - \lambda_0^2)(k^2 - \lambda_{n_1+1}^2 - \lambda_0^2)} \right| \\
 &= \frac{O(n_1)}{O(n_1^4)} = O\left(\frac{1}{n_1^3}\right)
 \end{aligned} \tag{4.9}$$

and the analogous asymptotic behavior for $|a_{0,n_2} - a_{0,n_2+1}|$, (i), (ii) and (iii) holds.

Now, since two series

$$\begin{aligned}
 \mathcal{F}_{\infty,N}(x_1, x_2) &= \sum_{n_1}^{\infty} \sum_{n_2=0}^N c_{n_1,n_2} \mathcal{Y}_{n_1}(x_1) \otimes \mathcal{Y}_{n_2}(x_2) \\
 \mathcal{F}_{M,\infty}(x_1, x_2) &= \sum_{n_2}^{\infty} \sum_{n_1=0}^M c_{n_1,n_2} \mathcal{Y}_{n_1}(x_1) \otimes \mathcal{Y}_{n_2}(x_2)
 \end{aligned}$$

converges for each $M, N \in \mathbb{N}$ as a projection onto $(L^2(D_1))^{2P+2} \otimes \text{span}\{\mathcal{Y}_{n_2}\}_{n_2=0}^N$ and $\text{span}\{\mathcal{Y}_{n_1}\}_{n_1=0}^M \otimes (L^2(D_1))^{2P+2}$, respectively, and a_{n_1,n_2} satisfies (i), (ii) and (iii), we can apply Dedekind's test (see e.g., [18]) to conclude that the series (4.6) converges in $(L^2(\Omega))^{4(P+1)^2}$. Therefore (4.5) is well-defined as the $(0, 0)$ -component of U and satisfies the Dirichlet boundary condition on Γ_S and Γ_W .

Furthermore, one can show that $\lambda_{n_j} a_{n_1,n_2}$ is of bounded bivariation, in case of $n_j = n_1$, by using (4.8) and $\lambda_{n_1} - \lambda_{n_1+1} = O(1)$

$$\begin{aligned}
 &|\lambda_{n_1} a_{n_1,n_2} - \lambda_{n_1+1} a_{n_1+1,n_2} - \lambda_{n_1} a_{n_1,n_2+1} + \lambda_{n_1} a_{n_1+1,n_2+1}| \\
 &\leq |\lambda_{n_1}| |a_{n_1,n_2} - a_{n_1+1,n_2} - a_{n_1,n_2+1} + a_{n_1+1,n_2+1}| + C |a_{n_1+1,n_2} - a_{n_1+1,n_2+1}| \\
 &= O\left(\frac{n_1}{(n_1^2 + n_2^2)^2}\right) + O\left(\frac{n_2}{(n_1^2 + n_2^2)^2}\right) = O\left(\frac{1}{(n_1^2 + n_2^2)^{3/2}}\right),
 \end{aligned}$$

and $\lambda_{n_1} a_{n_1,0}$ is of bounded variation ($\lambda_{n_2} a_{0,n_2}$ as well)

$$\begin{aligned}
 |\lambda_{n_1} a_{n_1,0} - \lambda_{n_1+1} a_{n_1+1,0}| &\leq |\lambda_{n_1}| |a_{n_1,0} - a_{n_1+1,0}| + C |a_{n_1+1,0}| \\
 &= O\left(\frac{1}{n_1^2}\right) + O\left(\frac{1}{n_1^2}\right) = O\left(\frac{1}{n_1^2}\right),
 \end{aligned}$$

and so Dedekind's test again verifies that

$$v_j(x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} -\lambda_{n_j} u_{n_1,n_2} \tilde{\psi}_{0,n_j}(x_j) \tilde{\varphi}_{0,n_\ell}(x_\ell) \quad j \neq \ell$$

converge in $L^2(\Omega)$ for $j, \ell = 1, 2$. Noting that $v_j = \partial_j u$ in the weak sense, we see that u is in $H^1(\Omega)$.

For the partial sum $U_{M,N} = \sum_{n_1=0}^M \sum_{n_2=0}^N u_{n_1,n_2} \mathcal{Y}_{n_1}(x_1) \mathcal{Y}_{n_2}(x_2)$ and the analogous one $\mathcal{F}_{M,N}$ for $\mathcal{F}$, it holds that $(\Delta + k^2)U_{M,N} = \mathcal{F}_{M,N}$. Thus, for any smooth function $\mathcal{G}$ with compact support

$$\begin{aligned} (U, (\Delta + k^2)\mathcal{G})_{\Omega} &= \lim_{M,N \rightarrow \infty} (U_{M,N}, (\Delta + k^2)\mathcal{G})_{\Omega} \\ &= \lim_{M,N \rightarrow \infty} (\mathcal{F}_{M,N}, \mathcal{G})_{\Omega} = (\mathcal{F}, \mathcal{G})_{\Omega}, \end{aligned}$$

which implies that U is the solution to the Helmholtz equation in the weak sense, and $\Delta U \in (L^2(\Omega))^{4P+4}$. In particular the $(0, 0)$ -component reads the Helmholtz equation of the problem (4.1).

Due to the fact that $f_{j,\ell}$ vanishes in Ω_{δ} , the auxiliary variables ϕ_j^E and ϕ_{ℓ}^N , $(j, 0)$ - and $(0, \ell)$ -components of U , respectively, defined by

$$\begin{aligned} \phi_j^E(x_1, x_2) &= \sum_{n_1, n_2=0}^{\infty} u_{n_1, n_2} \tilde{\varphi}_{j, n_1}(x_1) \tilde{\varphi}_{0, n_2}(x_2), \\ \phi_{\ell}^N(x_1, x_2) &= \sum_{n_1, n_2=0}^{\infty} u_{n_1, n_2} \tilde{\varphi}_{0, n_1}(x_1) \tilde{\varphi}_{\ell, n_2}(x_2) \end{aligned}$$

solve the Helmholtz equation in $(1 - \delta, 1) \times (0, 1)$ and $(0, 1) \times (1 - \delta, 1)$, and satisfy (4.2) and (4.3), respectively. Therefore they define CRBC on Γ_E and Γ_N , which completes the proof. $\square$

Remark 4.2 If u is a solution to the problem (4.1), then there exist doubly indexed auxiliary variables, (j, ℓ) -components of U ,

$$\phi_{j,\ell}(x_1, x_2) = \sum_{n_1, n_2=0}^{\infty} u_{n_1, n_2} \tilde{\varphi}_{j, n_1}(x_1) \tilde{\varphi}_{\ell, n_2}(x_2)$$

satisfying

$$\begin{aligned} (\Delta + k^2)\phi_{j,\ell} &= 0 \text{ in } \Omega_{\delta}, \\ (\partial_1 + a_j)\phi_{j,\ell} &= (-\partial_1 + \tilde{a}_j)\phi_{j+1,\ell} \text{ in } \Omega, \\ (\partial_2 + a_{\ell})\phi_{j,\ell} &= (-\partial_2 + \tilde{a}_{\ell})\phi_{j,\ell+1} \text{ in } \Omega \end{aligned} \tag{4.10}$$

with $\phi_{j,0} = \phi_j^E$, $\phi_{0,\ell} = \phi_{\ell}^N$, and $\partial_1 \phi_{P,\ell}(1, \cdot) = 0$ on Γ_E , $\partial_2 \phi_{j,P}(1, \cdot) = 0$ on Γ_N for $j, \ell = 0, \dots, P$. These relations lead to a corner condition to be imposed at NE in deriving practical CRBC. We discuss this corner compatibility condition in more detail in the following section.

5 Practical CRBC with the corner compatibility condition

In this section we derive a practical CRBC on Γ_E and Γ_N from CRBC analyzed in the preceding section and define a corner compatibility condition. Let Φ^E and

Φ^N represent vector functions whose components are the auxiliary variables for each absorbing boundary,

$$\Phi^E = \sum_{j=0}^P \phi_j^E \mathbf{e}_j \quad \text{and} \quad \Phi^N = \sum_{\ell=0}^P \phi_\ell^N \mathbf{e}_\ell.$$

As studied for the half space problem in Sect. 2, CRBC on each absorbing boundary can be written as

$$(\partial_1 u) \mathbf{e}_0 = (\mathbb{L} \partial_2^2 \Phi^E + (k^2 \mathbb{L} - \mathbb{M}) \Phi^E) \text{ on } \Gamma_E \quad (5.1)$$

and

$$(\partial_2 u) \mathbf{e}_0 = (\mathbb{L} \partial_1^2 \Phi^N + (k^2 \mathbb{L} - \mathbb{M}) \Phi^N) \text{ on } \Gamma_N. \quad (5.2)$$

In order to complete the systems of differential equations (5.1) and (5.2), boundary conditions for the auxiliary variables ϕ_j^E and ϕ_ℓ^N at two end points of Γ_E and Γ_N , respectively, are required. Clearly, at one end of Γ_E and Γ_N , they satisfy

$$\phi_j^E(1, 0) = 0 \quad \text{and} \quad \phi_\ell^N(0, 1) = 0. \quad (5.3)$$

Motivated by Remark 4.2, the condition at the other end, which is the required corner condition at NE, is derived as follows. From the relations (4.10) in Remark 4.2, one can show that auxiliary variables ϕ_j^E and ϕ_ℓ^N satisfy

$$\begin{aligned} (\partial_2 \phi_j^E) \mathbf{e}_0 &= (\partial_2 \phi_{j,0}) \mathbf{e}_0 = -\mathbb{L} \partial_2^2 \Phi_j^N - \mathbb{M} \Phi_j^N, \\ (\partial_1 \phi_\ell^N) \mathbf{e}_0 &= (\partial_1 \phi_{0,\ell}) \mathbf{e}_0 = -\mathbb{L} \partial_1^2 \Phi_\ell^E - \mathbb{M} \Phi_\ell^E \end{aligned} \quad (5.4)$$

at NE, where $\Phi_j^N = \sum_{\ell=0}^P \phi_{j,\ell} \mathbf{e}_\ell$ and $\Phi_\ell^E = \sum_{j=0}^P \phi_{j,\ell} \mathbf{e}_j$ for $j, \ell = 0, \dots, P$. All derivative terms in the right hand side of the boundary conditions (5.4) will be eliminated by using the Helmholtz equation as follows. Denoting

$$\Phi^{NE} = \sum_{j,\ell=0}^P \phi_{j,\ell} \mathbf{e}_j \otimes \mathbf{e}_\ell,$$

it is clear that

$$\Phi^{NE} = \sum_{j,\ell=0}^P \phi_{j,\ell} \mathbf{e}_j \otimes \mathbf{e}_\ell = \sum_{j=0}^P \mathbf{e}_j \otimes \Phi_j^N = \sum_{\ell=0}^P \Phi_\ell^E \otimes \mathbf{e}_\ell. \quad (5.5)$$

Thus, it follows from (5.4) and (5.5) that

$$\begin{aligned} \sum_{j=0}^P (\mathbb{L} \mathbf{e}_j \otimes \partial_2 \phi_j^E \mathbf{e}_0) &= - \sum_{j=0}^P (\mathbb{L} \mathbf{e}_j \otimes \mathbb{L} \partial_2^2 \Phi_j^N + \mathbb{L} \mathbf{e}_j \otimes \mathbb{M} \Phi_j^N) \\ &= -((\mathbb{L} \otimes \mathbb{L}) \partial_2^2 \Phi^{NE} + (\mathbb{L} \otimes \mathbb{M}) \Phi^{NE}) \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} \sum_{\ell=0}^P (\partial_1 \phi_\ell^N \mathbf{e}_0 \otimes \mathbb{L} \mathbf{e}_\ell) &= - \sum_{\ell=0}^P (\mathbb{L} \partial_1^2 \Phi_\ell^E \otimes \mathbb{L} \mathbf{e}_\ell + \mathbb{M} \Phi_\ell^E \otimes \mathbb{L} \mathbf{e}_\ell) \\ &= -((\mathbb{L} \otimes \mathbb{L}) \partial_1^2 \Phi^{NE} + (\mathbb{M} \otimes \mathbb{L}) \Phi^{NE}). \end{aligned} \quad (5.7)$$

Adding (5.6) and (5.7) followed by using the Helmholtz equation removes all derivative terms in the right hand sides, which results in

$$\mathbb{L} \partial_2 \Phi_0^E \otimes \mathbf{e}_0 + \mathbf{e}_0 \otimes \mathbb{L} \partial_1 \Phi_0^N = (k^2 \mathbb{L} \otimes \mathbb{L} - \mathbb{L} \otimes \mathbb{M} - \mathbb{M} \otimes \mathbb{L}) \Phi^{NE} \quad (5.8)$$

at NE.

Therefore, the model problem (4.1) can be written as the problem to find u defined in Ω , Φ^E on Γ_E , Φ^N on Γ_N and $\Phi^{NE} \in \mathbb{C}^{(P+1)^2}$ satisfying

$$\Delta u + k^2 u = f \text{ in } \Omega, \quad (5.9)$$

$$u = 0 \text{ on } \Gamma_S \cup \Gamma_W, \quad (5.10)$$

$$\begin{aligned} (\partial_1 u) \mathbf{e}_0 &= (\mathbb{L} \partial_2^2 \Phi^E + (k^2 \mathbb{L} - \mathbb{M}) \Phi^E) \text{ on } \Gamma_E, \\ (\partial_2 u) \mathbf{e}_0 &= (\mathbb{L} \partial_1^2 \Phi^N + (k^2 \mathbb{L} - \mathbb{M}) \Phi^N) \text{ on } \Gamma_N, \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} &(\mathbb{L} \partial_2 \Phi^E) \otimes \mathbf{e}_0 + \mathbf{e}_0 \otimes (\mathbb{L} \partial_1 \Phi^N) \\ &= (k^2 \mathbb{L} \otimes \mathbb{L} - \mathbb{L} \otimes \mathbb{M} - \mathbb{M} \otimes \mathbb{L}) \Phi^{NE} \text{ at NE}, \end{aligned} \quad (5.12)$$

$$\Phi^E(1, 0) = 0 \text{ and } \Phi^N(0, 1) = 0. \quad (5.13)$$

6 Variational reformulation

For a variational formulation of the problem (5.9)–(5.13), we introduce Sobolev spaces

$$\begin{aligned} H_{SW}^1(\Omega) &= \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_S \cup \Gamma_W\}, \\ H_{SE}^1(\Gamma_E) &= \{v \in H^1(\Gamma_E) : v(1, 0) = 0\}, \\ H_{NW}^1(\Gamma_N) &= \{v \in H^1(\Gamma_N) : v(0, 1) = 0\} \end{aligned}$$

and

$$\begin{aligned} V = \{(\phi, \Phi^E, \Phi^N, \Phi^{NE}) \in H_{SW}^1(\Omega) \times (H_{SE}^1(\Gamma_E))^{P+1} \times (H_{NW}^1(\Gamma_N))^{P+1} \times \mathbb{C}^{(P+1)^2} : \\ \phi = \phi_0^E \text{ on } \Gamma_E, \quad \phi = \phi_0^N \text{ on } \Gamma_N \text{ and} \\ \phi_j^E = \phi_{j,0}^{NE}, \quad \phi_\ell^N = \phi_{0,\ell}^{NE} \text{ at NE for } j, \ell = 0, \dots, P\}. \end{aligned}$$

Applying the L^2 -inner product of (5.9) with test functions $\xi \in H_{SW}^1(\Omega)$ and integrating it by parts lead to

$$(\nabla u, \nabla \xi)_\Omega - k^2(u, \xi)_\Omega - (\partial_1 u, \xi)_{\Gamma_E} - (\partial_2 u, \xi)_{\Gamma_N} = (f, \xi)_\Omega. \quad (6.1)$$

where $(\cdot, \cdot)_\mathcal{D}$ is the L^2 -inner product on a domain $\mathcal{D}$. For the boundary integrals on Γ_E and Γ_N in (6.1), we take test functions

$$\begin{aligned} \Psi^E &= (\psi_0^E, \dots, \psi_P^E)^\top \in (H_{SE}^1(\Gamma_E))^{P+1}, \\ \Psi^N &= (\psi_0^N, \dots, \psi_P^N)^\top \in (H_{NW}^1(\Gamma_N))^{P+1} \end{aligned}$$

with $\psi_0^E = \xi$ on Γ_E and $\psi_0^N = \xi$ on Γ_N and apply L^2 -inner products of (5.11) with Ψ^E and Ψ^N to obtain that

$$\begin{aligned} -(\partial_1 u, \xi)_{\Gamma_E} &= (\mathbb{L} \partial_2 \Phi^E, \partial_2 \Psi^E)_{\Gamma_E} \\ &\quad + ((-k^2 \mathbb{L} + \mathbb{M}) \Phi^E, \Psi^E)_{\Gamma_E} - (\mathbb{L} \partial_2 \Phi^E(\text{NE}), \Psi^E(\text{NE}))_{\mathbb{C}^{P+1}}, \\ -(\partial_2 u, \xi)_{\Gamma_N} &= (\mathbb{L} \partial_1 \Phi^N, \partial_1 \Psi^N)_{\Gamma_N} \\ &\quad + ((-k^2 \mathbb{L} + \mathbb{M}) \Phi^N, \Psi^N)_{\Gamma_N} - (\mathbb{L} \partial_1 \Phi^N(\text{NE}), \Psi^N(\text{NE}))_{\mathbb{C}^{P+1}}, \end{aligned} \quad (6.2)$$

where $(\cdot, \cdot)_{\mathbb{C}^N}$ is the standard inner product for $\mathbb{C}^N$. In order to deal with the inner product of the vectors composed of the corner values at NE in (6.2), we denote test vectors in $\mathbb{C}^{(P+1)^2}$ by

$$\Psi^{NE} = \sum_{j,\ell=0}^P \psi_{j,\ell}^{NE} \mathbf{e}_j \otimes \mathbf{e}_\ell$$

with $\psi_{j,0}^{NE} = \psi_j^E(\text{NE})$ and $\psi_{0,\ell}^{NE} = \psi_\ell^N(\text{NE})$. Then noting that $\psi_0^E = \Psi^E$, we observe that

$$\begin{aligned} \left(\mathbb{L} \partial_2 \Phi^E(\text{NE}), \Psi^E(\text{NE}) \right)_{\mathbb{C}^{P+1}} &= \left(\mathbb{L} \partial_2 \Phi^E(\text{NE}) \otimes \mathbf{e}_0, \Psi_0^E(\text{NE}) \otimes \mathbf{e}_0 \right)_{\mathbb{C}^{(P+1)^2}} \\ &= \left(\mathbb{L} \partial_2 \Phi^E(\text{NE}) \otimes \mathbf{e}_0, \Psi^{NE} \right)_{\mathbb{C}^{(P+1)^2}}. \end{aligned} \quad (6.3)$$

By (6.3) and the analogous equation for $\mathbb{L}\partial_1\Phi^N(\text{NE})$, the corner condition (5.12) produces

$$\begin{aligned} & (\mathbb{L}\partial_2\Phi^E(\text{NE}), \Psi^E(\text{NE}))_{\mathbb{C}^{(P+1)^2}} + (\mathbb{L}\partial_1\Phi^N(\text{NE}), \Psi^N(\text{NE}))_{\mathbb{C}^{(P+1)^2}} \\ &= ((k^2\mathbb{L} \otimes \mathbb{L} - \mathbb{L} \otimes \mathbb{M} - \mathbb{M} \otimes \mathbb{L})\Phi^{NE}, \Psi^{NE})_{\mathbb{C}^{(P+1)^2}}. \end{aligned} \quad (6.4)$$

Finally, combining (6.1), (6.2) and (6.4) gives a variational problem to find $(u, \Phi^E, \Phi^N, \Phi^{NE}) \in V$ such that

$$A((u, \Phi^E, \Phi^N, \Phi^{NE}), (\xi, \Psi^E, \Psi^N, \Psi^{NE})) = (f, \xi)_\Omega \quad (6.5)$$

for all $(\xi, \Psi^E, \Psi^N, \Psi^{NE}) \in V$, where

$$\begin{aligned} & A((u, \Phi^E, \Phi^N, \Phi^{NE}), (\xi, \Psi^E, \Psi^N, \Psi^{NE})) \\ &= A_I(u, \xi) + B_E(\Phi^E, \Psi^E) + B_N(\Phi^N, \Psi^N) + (\mathcal{R}\Phi^{NE}, \Psi^{NE})_{\mathbb{C}^{(P+1)^2}} \end{aligned}$$

with

$$\begin{aligned} A_I(u, \xi) &= (\nabla u, \nabla \xi)_\Omega - k^2(u, \xi)_\Omega, \\ B_E(\Phi^E, \Psi^E) &= (\mathbb{L}\partial_2\Phi^E, \partial_2\Psi^E)_{\Gamma_E} + ((-k^2\mathbb{L} + \mathbb{M})\Phi^E, \Psi^E)_{\Gamma_E}, \\ B_N(\Phi^N, \Psi^N) &= (\mathbb{L}\partial_1\Phi^N, \partial_1\Psi^N)_{\Gamma_N} + ((-k^2\mathbb{L} + \mathbb{M})\Phi^N, \Psi^N)_{\Gamma_N}, \\ \mathcal{R} &= (-k^2\mathbb{L} \otimes \mathbb{L} + \mathbb{L} \otimes \mathbb{M} + \mathbb{M} \otimes \mathbb{L}). \end{aligned}$$

7 Existence of unique solutions to problem (6.5)

We first show that the problem (6.5) has a unique solution in V . To do this, we begin with some properties of matrices defining the corner compatibility condition. Let $\mathbb{L}_z$ and $\mathbb{M}_z$ be the $P \times P$ submatrices of $\mathbb{L}$ and $\mathbb{M}$ obtained by removing the 0-th row and 0-th column of $\mathbb{L}$ and $\mathbb{M}$, respectively. Also, $\mathcal{R}_z$ denotes the $P^2 \times P^2$ submatrix of $\mathcal{R}$ associated with elements $e_j \otimes e_\ell$ for $j, \ell = 1, \dots, P$,

$$\mathcal{R}_z = -k^2\mathbb{L}_z \otimes \mathbb{L}_z + \mathbb{L}_z \otimes \mathbb{M}_z + \mathbb{M}_z \otimes \mathbb{L}_z.$$

Their properties are listed in the following lemma.

Lemma 7.1 *The matrices $\mathbb{L}_z$, $\mathbb{M}_z$ and $\mathcal{R}_z$ have the following properties:*

- (i) *The matrices $\mathbb{L}_z$ and $\mathbb{M}_z$ are invertible.*
- (ii) *If λ is an eigenvalue of $\mathbb{L}_z^{-1}\mathbb{M}_z$, then $\Im(\lambda) < 0$.*
- (iii) *The matrix $\mathcal{R}_z$ is invertible.*

Proof To prove (i), assume that $\mathbb{L}_z \Theta = 0$ for $\Theta = (\theta_1, \dots, \theta_P)^\top \in \mathbb{C}^P$. Then, it is easy to show that

$$(L_z \Theta, \Theta)_{\mathbb{C}^P} = \sum_{j=0}^{P-1} \frac{1}{a_j + \tilde{a}_j} |\theta_j + \theta_{j+1}|^2 = 0$$

with $\theta_0 = 0$. Since both real and imaginary parts need to be zero, that is,

$$\begin{aligned} \Re((\mathbb{L}_z \Theta, \Theta)_{\mathbb{C}^P}) &= \sum_{j=n_p}^{P-1} \frac{1}{kc_j + k\tilde{c}_j} |\theta_j + \theta_{j+1}|^2 = 0, \\ \Im((\mathbb{L}_z \Theta, \Theta)_{\mathbb{C}^P}) &= \sum_{j=0}^{n_p-1} \frac{1}{kc_j + k\tilde{c}_j} |\theta_j + \theta_{j+1}|^2 = 0, \end{aligned}$$

from the assumptions (2.6) and (2.7) for a_j and $\tilde{a}_j$ it follows that $\theta_j = 0$ for $j = 1, \dots, P$, which establishes the invertibility of $\mathbb{L}_z$. As for $\mathbb{M}_z$, we use backward Gaussian elimination to show that $\det \mathbb{M}_z = \prod_{j=0}^{P-1} a_j \tilde{a}_j / (a_j + \tilde{a}_j) \neq 0$, which completes the proof of (i).

For (ii), consider the linear problem to find $\Theta \in \mathbb{C}^P$ satisfying

$$(-\lambda \mathbb{L}_z + \mathbb{M}_z) \Theta = E \quad (7.1)$$

for $E = (E_1, \dots, E_P)^\top \in \mathbb{C}^P$. It is enough to show that if $\Im(\lambda) \geq 0$ then (7.1) is uniquely solvable. First of all, we notice that (7.1) has a unique solution for $\lambda = 0$ by (ii). Now we assume that $\lambda \neq 0$ and we introduce

$$Q_{j,\ell} = \begin{cases} \prod_{m=j}^{\ell} \frac{a_m + i\sqrt{\lambda}}{\tilde{a}_m - i\sqrt{\lambda}} & \text{for } j \leq \ell, \\ 1 & \text{for } j > \ell, \end{cases} \quad R_{j,\ell} = \begin{cases} \prod_{m=j}^{\ell} \frac{\tilde{a}_m + i\sqrt{\lambda}}{a_m - i\sqrt{\lambda}} & \text{for } j \leq \ell, \\ 1 & \text{for } j > \ell, \end{cases}$$

and $Z_{j,\ell} = Q_{j,\ell} R_{j,\ell}$. Then the same computation as that used in [21, Lemma 8.5] and [30, Lemma 5.2] with a minor modification shows that if $1 + Z_{0,P-1} \neq 0$, then the problem (7.1) has a unique solution of the form

$$\Theta_\ell = \sum_{j=1}^P S_{\ell,j} E_j,$$

where

$$S_{\ell,j} = \begin{cases} \frac{(1 - Z_{0,\ell-1}) R_{\ell,j-1} (1 + Z_{j,P-1})}{-2i\lambda(1 + Z_{0,P-1})} & \text{for } \ell \leq j, \\ \frac{(1 - Z_{0,j-1}) Q_{j,\ell-1} (1 + Z_{\ell,P-1})}{-2i\lambda(1 + Z_{0,P-1})} & \text{for } \ell > j. \end{cases}$$

Therefore, we have only to show that if $\Im(\lambda) \geq 0$ and $\lambda \neq 0$ then $1 + Z_{0,p-1} \neq 0$. In fact, this result is obtained by observing that

$$\left| \frac{-ic_j k + i\sqrt{\lambda}}{-ic_j k - i\sqrt{\lambda}} \right| \begin{cases} < 1 & \text{for } \Im(\lambda) > 0, \\ < 1 & \text{for } \lambda > 0, \\ = 1 & \text{for } \lambda < 0, \end{cases} \quad \text{and} \quad \left| \frac{c_j k + i\sqrt{\lambda}}{c_j k - i\sqrt{\lambda}} \right| \begin{cases} < 1 & \text{for } \Im(\lambda) > 0, \\ = 1 & \text{for } \lambda > 0, \\ < 1 & \text{for } \lambda < 0, \end{cases}$$

for $c_j > 0$, from which together with the analogous inequalities for $\tilde{c}_j > 0$ it follows that $1 + Z_{0,p-1} \neq 0$ and the proof of (ii) is completed.

To prove (iii), it suffices to show that zero is not an eigenvalue of the following matrix

$$(\mathbb{L}_z^{-1} \otimes \mathbb{L}_z^{-1}) \mathcal{R}_z = \left(-\frac{k^2}{2} I + \mathbb{L}_z^{-1} \mathbb{M}_z \right) \otimes I + I \otimes \left(-\frac{k^2}{2} I + \mathbb{L}_z^{-1} \mathbb{M}_z \right). \quad (7.2)$$

Here, we recall that every eigenvalue of the Kronecker sum (7.2) arises as a sum of eigenvalues of $-\frac{k^2}{2} I + \mathbb{L}_z^{-1} \mathbb{M}_z$. Since all eigenvalues of $-\frac{k^2}{2} I + \mathbb{L}_z^{-1} \mathbb{M}_z$ have negative imaginary parts due to the result of (ii), it follows that $(\mathbb{L}_z^{-1} \otimes \mathbb{L}_z^{-1}) \mathcal{R}_z$ does not have a zero eigenvalue, which completes the proof of (iii). $\square$

Theorem 7.2 *The problem (6.5) has a unique solution $(u, \Phi^E, \Phi^N, \Phi^{NE}) \in V$.*

Proof We know that there exists a solution to the problem (4.1) by Theorem 4.1. Thus, it suffices to show that if

$$A((u, \Phi^E, \Phi^N, \Phi^{NE}), (\xi, \Psi^E, \Psi^N, \Psi^{NE})) = 0 \text{ for all } (\xi, \Psi^E, \Psi^N, \Psi^{NE}) \in V, \quad (7.3)$$

then $(u, \Phi^E, \Phi^N, \Phi^{NE}) = 0$ in V .

[1] To show that $u = 0$ in Ω :

Let $\{u_n\}_{n=1}^\infty$ be a sequence of compactly supported functions converging to u in $L^2(\Omega)$. By the existence of a solution discussed in Theorem 4.1, there exists a solution $(\xi_n, \Psi_n^E, \Psi_n^N, \Psi_n^{NE}) \in V$ to the problem (6.5) with $f = \bar{u}_n$, where $\bar{\cdot}$ represents the complex conjugate. Noting that $A(\alpha, \beta) = A(\bar{\beta}, \bar{\alpha})$ for $\alpha, \beta \in V$, we can show that

$$(u, u_n)_\Omega = A((u, \Phi^E, \Phi^N, \Phi^{NE}), (\xi_n, \Psi_n^E, \Psi_n^N, \Psi_n^{NE})) = 0.$$

Since u_n converges to u in $L^2(\Omega)$, it follows that $u = 0$ in Ω .

[2] To show that $\Phi^E = 0$ on Γ_E and $\Phi^N = 0$ on Γ_N under the conditions $\phi_0^E = 0$ on Γ_E and $\phi_0^N = 0$ on Γ_N :

As the equations for the auxiliary variables Φ^N on Γ_N are identical with those for the auxiliary variables Φ^E on Γ_E , it is enough to show that $\Phi^E = 0$ on Γ_E . By taking test functions $(\xi, \Psi^E, \Psi^N, \Psi^{NE})$ in V such that $\Psi^N = 0$, $\Psi^E \in (C_0^\infty(\Gamma_E))^{P+1}$ and $\Psi^{NE} = 0$ in (7.3), it can be shown that the auxiliary variables Φ^E satisfy

$$-\mathbb{L} \partial_2^2 \Phi^E + (-k^2 \mathbb{L} + \mathbb{M}) \Phi^E = 0 \text{ on } \Gamma_E \quad (7.4)$$

with $\Phi^E(0) = 0$ (identifying Γ_E with the interval $(0, 1)$). Now, we claim that ϕ_j^E for $j = 1, \dots, P$ can be written as

$$\phi_j^E = \sum_{q=0}^{j-1} P_{q,j-1} A_q \sin(\tilde{\alpha}_q x_2) \quad (7.5)$$

for some constants A_q . Here $\tilde{\alpha}_q^2 := k^2 + \tilde{a}_q^2$ and

$$P_{q,j-1} = \prod_{m=q+1}^{j-1} \frac{a_m + \tilde{a}_q}{\tilde{a}_m - \tilde{a}_q}$$

for $q = 0, \dots, j-2$ with $P_{j-1,j-1} = 1$. To prove it, we proceed by an induction argument on j . Since $\phi_0^E = 0$ on Γ_E , it follows from the 0-th equation of (7.4) that we have an equation for ϕ_1^E ,

$$\frac{1}{a_0 + \tilde{a}_0} \partial_2^2 \phi_1^E + \left(\frac{k^2 + \tilde{a}_0^2}{a_0 + \tilde{a}_0} \right) \phi_1^E = 0 \text{ on } \Gamma_E$$

with $\phi_1^E(0) = 0$, which leads that

$$\phi_1^E(x_2) = P_{0,0} A_0 \sin(\tilde{\alpha}_0 x_2)$$

for some constant A_0 . Assume that up to the j -th ($j < P$) auxiliary variables are given by the formula (7.5). The j -th equation of (7.4) shows that ϕ_{j+1}^E satisfies the equation

$$\begin{aligned} \frac{1}{a_j + \tilde{a}_j} \partial_2^2 \phi_{j+1}^E + \left(\frac{k^2 + \tilde{a}_j^2}{a_j + \tilde{a}_j} \right) \phi_{j+1}^E &= \frac{-1}{a_{j-1} + \tilde{a}_{j-1}} \partial_2^2 \phi_{j-1}^E + \left(\frac{-k^2 - a_{j-1}^2}{a_{j-1} + \tilde{a}_{j-1}} \right) \phi_{j-1}^E \\ &+ \left(\frac{-1}{a_{j-1} + \tilde{a}_{j-1}} + \frac{-1}{a_j + \tilde{a}_j} \right) \partial_2^2 \phi_j^E + \left(\frac{-k^2 + a_{j-1} \tilde{a}_{j-1}}{a_{j-1} + \tilde{a}_{j-1}} + \frac{-k^2 + a_j \tilde{a}_j}{a_j + \tilde{a}_j} \right) \phi_j^E \end{aligned} \quad (7.6)$$

with $\phi_{j+1}^E(0) = 0$, which can be written simply as

$$\partial_2^2 \phi_{j+1}^E + \tilde{\alpha}_j^2 \phi_{j+1}^E = \sum_{q=0}^{j-1} (a_j + \tilde{a}_q)(\tilde{a}_j + \tilde{a}_q) P_{q,j-1} A_q \sin(\tilde{\alpha}_q x_2).$$

for $q = 0, \dots, j-1$. By solving the second order ordinary differential equation we see that ϕ_{j+1}^E is of the form

$$\begin{aligned}\phi_{j+1}^E(x_2) &= A_j \sin(\tilde{\alpha}_j x_2) + \sum_{q=0}^{j-1} \frac{a_j + \tilde{a}_q}{\tilde{a}_j - \tilde{a}_q} P_{q,j-1} A_q \sin(\tilde{\alpha}_q x_2) \\ &= A_j \sin(\tilde{\alpha}_j x_2) + \sum_{q=0}^{j-1} P_{q,j} A_q \sin(\tilde{\alpha}_q x_2) = \sum_{q=0}^j P_{q,j} A_q \sin(\tilde{\alpha}_q x_2)\end{aligned}$$

for some constant A_j , which verifies (7.5).

Now, since the last equation of the system (7.4) reveals that

$$\partial_2^2 \phi_{P-1}^E + (k^2 + a_{P-1}^2) \phi_{P-1}^E + \partial_2^2 \phi_P^E + (k^2 - a_{P-1} \tilde{a}_{P-1}) \phi_P^E = 0, \quad (7.7)$$

substituting (7.5) for $j = P - 1$ and $j = P$ into (7.7) gives

$$-(a_{P-1} + \tilde{a}_{P-1}) \sum_{q=0}^{P-1} \tilde{a}_q P_{q,P-1} A_q \sin(\tilde{\alpha}_q x_2) = 0 \text{ on } \Gamma_E.$$

Finally, since $\tilde{a}_q P_{q,P-1} \neq 0$ and $\sin(\tilde{\alpha}_q x_2)$ for $q = 0, \dots, P-1$ are linearly independent, we can conclude that $A_q = 0$ for $q = 0, \dots, P-1$, which implies that $\Phi^E = 0$ on Γ_E .

[3] To show that $\Phi^{NE} = 0$ in $\mathbb{C}^{(P+1)^2}$ under the conditions that $\phi_{j,0}^{NE} = \phi_{0,\ell}^{NE} = 0$ for $j, \ell = 0, \dots, P$:

Since $u = 0$, $\Phi^E = 0$ and $\Phi^N = 0$, the problem (7.3) is reduced to the linear problem for the corner values,

$$\mathcal{R}_z \left(\sum_{j,\ell=1,P} \phi_{j,\ell}^{NE} \mathbf{e}_j \otimes \mathbf{e}_\ell \right) = 0.$$

Since $\mathcal{R}_z$ is invertible by Lemma 7.1, it follows that $\phi_{j,\ell}^{NE} = 0$ for $j, \ell = 1, \dots, P$, which completes the proof of uniqueness of solutions. $\square$

8 Numerical experiments

This section provides numerical examples illustrating the performance of the CRBC for time-harmonic wave propagation problems in $\mathbb{R}^2$. In the first example, we consider a source problem in the square domain $\Omega = (0, 1)^2$ with the CRBC on $\Gamma_E \cup \Gamma_N$ and the homogeneous Dirichlet boundary condition on $\Gamma_W \cup \Gamma_S$. The source function f is prescribed in a way that the analytical solution is given by

$$u^{ex}(r, \theta) = \chi(r) \sum_{n=1}^4 \frac{1}{(2n)^2} H_{2n}^1(kr) \sin(2n\theta)$$

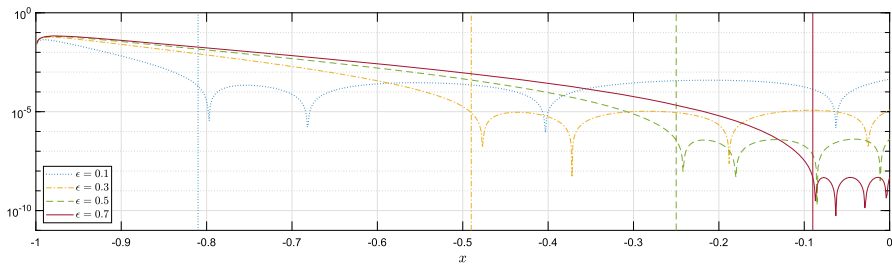


Fig. 1 The plots of the error, $\sqrt{1+x}\varrho_1$, of the rational function representation (2.10) of CRBC with $n_p = 2$ as an approximation to $\sqrt{1+x}$ of $x = -\xi^2/k^2$ for $\xi \in [0, k]$. The CRBC parameters are chosen such that the error is minimized in the region of $\xi \in [0, (1-\epsilon)k]$. The vertical lines represent the bounds of x corresponding to $\xi = (1-\epsilon)k$

in polar coordinates, where $k = 4$ and χ is a smooth cut-off function vanishing for $0 < r < 0.25$ and being one for $r > 0.9$.

For the first test, instead of following the parameter selection procedure in Sect. 2.3, we consider the performance of CRBC as a function of n_p and n_e once ϵ and M are set for the bounds of spectral ranges of propagative and evanescent modes. Assuming that the separation between the wave source and the absorbing boundary is $\delta = 0.1$, we set M for the upper bound of evanescent modes to be handled by CRBC by making (2.25) less than τ for a tolerance τ for maximal reflection errors of CRBC, for instance, $\tau = 10^{-4}$. The parameters a_j and $\tilde{a}_j$ for the CRBC on $\Gamma_E \cup \Gamma_N$ are chosen analytically by the formula (2.13) in Sect. 2.3 to test the performance of CRBC with various values of $\epsilon = 0.01, 0.1, 0.3, 0.5, 0.7$ and 0.9 and so CRBC are tuned for attenuating reflections uniformly for the frequency range of $\xi, \pm[0, (1-\epsilon)k]$ and $\pm[(1+\epsilon)k, Mk]$. See Fig. 1 for the plots of the error, $\sqrt{1+x}\varrho_1$, of the rational function representation (2.10) of CRBC with $n_p = 2$ as an approximation to $\sqrt{1+x}$ of $x = -\xi^2/k^2$ for $\xi \in [0, k]$ for $\epsilon = 0.1, 0.3, 0.5$ and 0.7 .

The finite element computations are conducted with mesh size $h = 1/400$ and with bilinear elements using the `deal.II` finite element library [7]. From the resulting relative L^2 -errors reported in Fig. 2, we observe that the errors decrease until finite element errors are dominant as n_p increases. The black dash-dot horizontal lines represent the relative L^2 -error $\approx 2.33 \times 10^{-4}$ of the finite element approximation to the Helmholtz equation with the exact Dirichlet boundary condition on $\partial\Omega$, and the green dash horizontal lines do the relative L^2 -error of the L^2 -projection $\approx 1.02 \times 10^{-4}$ of the analytical solution onto the finite element space. It is also seen that the errors of CRBC with $n_e = 0$ and 1 can reach the level of finite element errors with increasing n_p but slowly, which implies that CRBC is required to have at least 2 auxiliary variables corresponding to real parameters in order to handle evanescent modes efficiently in this particular example. We note that ϵ is positively correlated with τ as seen in (2.27) but since CRBC requires more parameters to reduce reflection errors in the larger spectral ranges (the case of smaller ϵ) as indicated in (2.28), (2.30), relative L^2 -errors are not necessarily monotonically decreasing with decreasing ϵ for fixed n_p and n_e .

Next we examine CRBC when parameters are selected via the procedure in Sect. 2.3. But we can see that ϵ is not necessarily too small as observed in the preceding tests

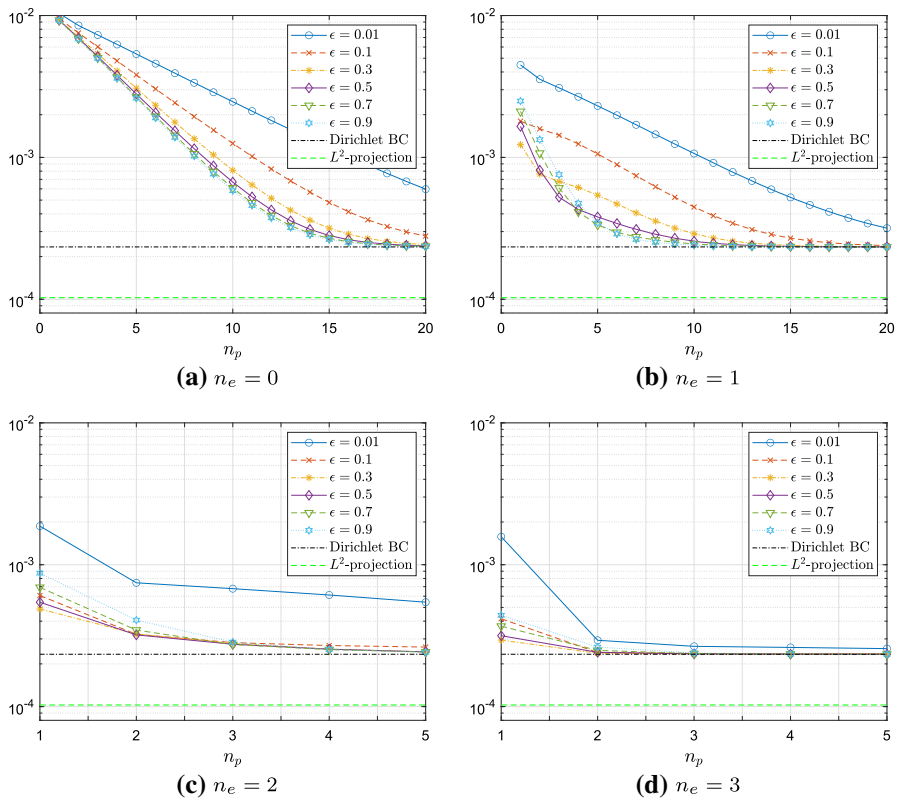


Fig. 2 Relative L^2 -errors as a function of n_p for various ϵ and $\tau = 10^{-4}$ (for determining M) with CRBC imposed on $\Gamma_E \cup \Gamma_N$

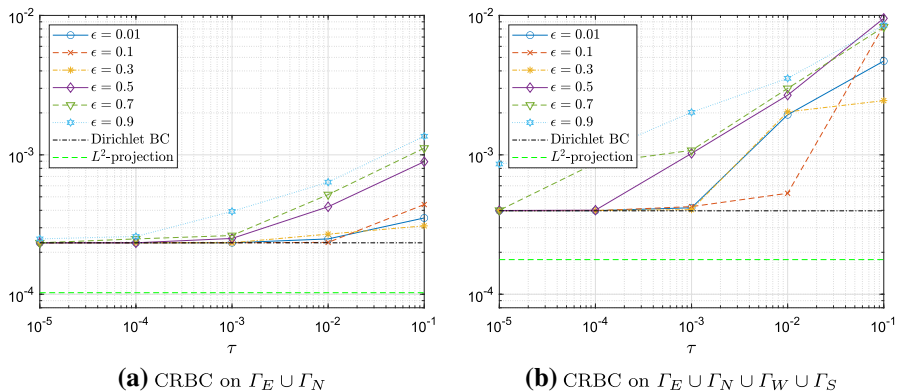
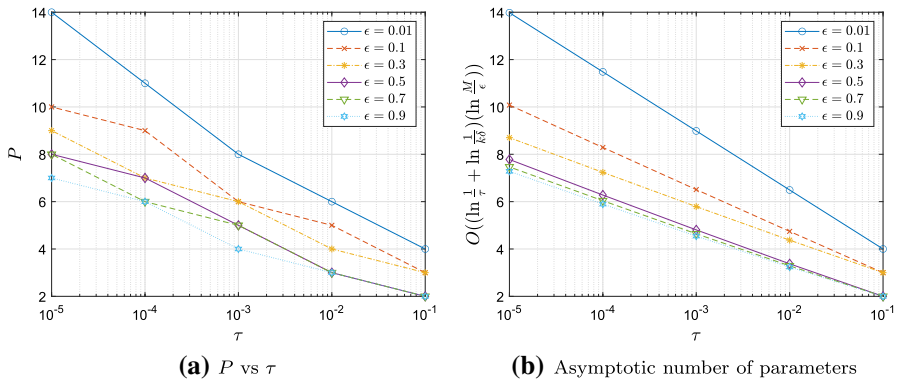
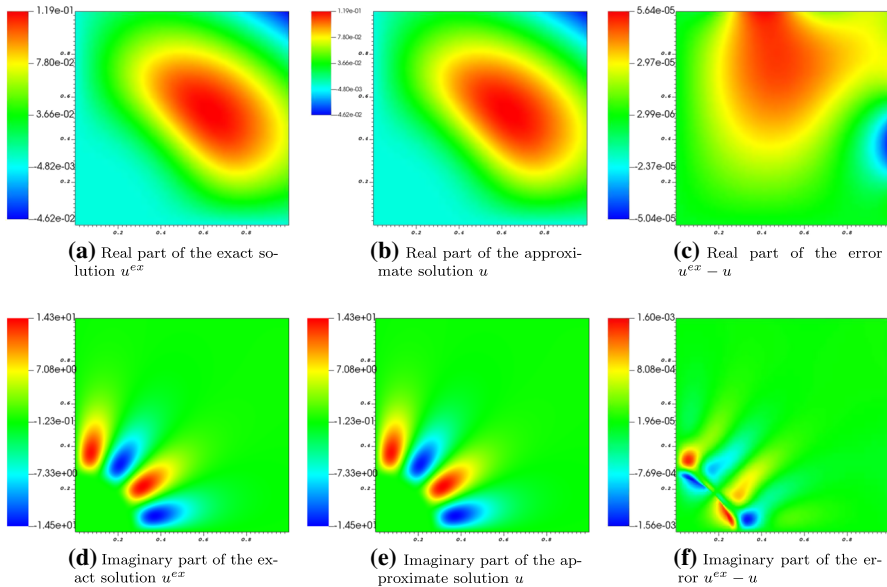


Fig. 3 Relative L^2 -errors with respect to τ for each ϵ

Fig. 4 CRBC order P Table 1 (n_p, n_e) for each ϵ and τ

τ	ϵ					
	0.01	0.1	0.3	0.5	0.7	0.9
1e-01	(2,2)	(1,2)	(1,2)	(1,1)	(1,1)	(1,1)
1e-02	(2,4)	(2,3)	(1,3)	(1,2)	(1,2)	(1,2)
1e-03	(3,5)	(2,4)	(2,4)	(1,4)	(1,4)	(1,3)
1e-04	(4,7)	(3,6)	(2,5)	(2,5)	(1,5)	(1,5)
1e-05	(5,9)	(3,7)	(2,7)	(2,6)	(2,6)	(1,6)

Fig. 5 Snapshots of the finite element solution satisfying CRBC on $\Gamma_E \cup \Gamma_N$ and its error $u^{ex} - u$ with $(n_p, n_e) = (2, 4)$, $\tau = 10^{-3}$ (for determining M) and $\epsilon = 0.1$

and so we consider ϵ as an independent parameter rather than one depending on τ of order $O(\tau^2)$. Assuming this and expressing the terms containing ϵ explicitly in (2.28) and (2.30), we get the number of parameters P of order

$$O\left(\left(\ln \frac{1}{\tau} + \ln \frac{1}{k\delta}\right) \cdot \left(\ln \frac{1}{\epsilon} + \ln\left(\frac{1}{k\delta} + \ln\left(\ln \frac{1}{\tau} + \ln \frac{1}{k\delta}\right)\right)\right)\right) \quad (8.1)$$

from the parameter selection procedure from ii. to iv. in Sect. 2.3. Here τ is introduced as a tolerance of maximal reflection errors of CRBC rather than a tolerance of relative L^2 -errors of approximate solutions. The parameter τ only reflects the error of the approximate boundary condition but does not include any discretization error resulting from the finite element method and so relative L^2 -errors of approximate solutions are not necessarily smaller than τ . However it is expected that the relative L^2 -errors decrease monotonically with respect to τ in that the more accurate CRBC is achieved by using the smaller τ . Figure 3a shows the performance of CRBC with respect to $\tau = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ and 10^{-5} . Table 1 reports (n_p, n_e) and Fig. 4 exhibits order P used for computations and the asymptotic behavior (8.1). It looks that approximate solutions obtained by CRBC with smaller $\epsilon \leq 0.3$ converge faster than those with larger $\epsilon \geq 0.5$ with respect to τ since more parameters are used for smaller ϵ . We see that approximate solutions of the relative L^2 -error $\approx 2.3 \times 10^{-4}$ can be achieved using only 5 or 6 auxiliary variables, for example, $(n_p, n_e) = (2, 3)$ with $\tau = 10^{-2}$ and $\epsilon = 0.1$, and $(n_p, n_e) = (2, 4)$ with $\tau = 10^{-3}$ and $\epsilon = 0.1, 0.3$. The snapshots of the real and imaginary parts of the finite element approximate solution satisfying CRBC of $(n_p, n_e) = (2, 4)$ with $\epsilon = 0.1$ are presented in Fig. 5, that exhibits that their errors are concentrated at peaks of solutions rather than the artificial boundary.

The second example is a time-harmonic wave propagation problem with $k = 4$ in the whole domain $\mathbb{R}^2$. The computational domain is restricted to $\Omega = (-0.5, 0.5)^2$ and CRBC is imposed on the four artificial boundaries. The external source function f is determined by the analytical solution defined by

$$u^{ex}(r, \theta) = \chi(r) \sum_{n=0}^6 \frac{1}{(n+1)^2} H_n^1(kr) e^{in\theta}$$

in polar coordinates. In this example, the cut-off function χ is supported on $r > 0.3$ and is defined by one for $r > 0.4$ so that the separation from the source and the fictitious boundaries is $\delta = 0.1$. Figure 6 shows the convergence of relative L^2 -errors with respect to n_p for each n_e and ϵ , which have the similar behavior to those in the problem with CRBC on $\Gamma_E \cup \Gamma_N$ of the first example. Approximate solutions satisfying CRBC can be as close to the exact solution as the finite element solution satisfying the exact Dirichlet boundary condition on $\partial\Omega$, whose relative L^2 -error is approximately 3.96×10^{-4} . It also reveals that a sufficient number of real parameters related to evanescent modes are required for fast convergence, for example, $n_e \geq 3$ for $\epsilon = 0.3, 0.5$ and $n_e \geq 4$ for $\epsilon = 0.1$. By using the same pairs of (n_p, n_e) in Table 1 the convergence of relative L^2 -errors is presented in Fig. 3b. It shows that approximate solutions of errors $\approx 3.96 \times 10^{-4}$ can be obtained by using 6 auxiliary

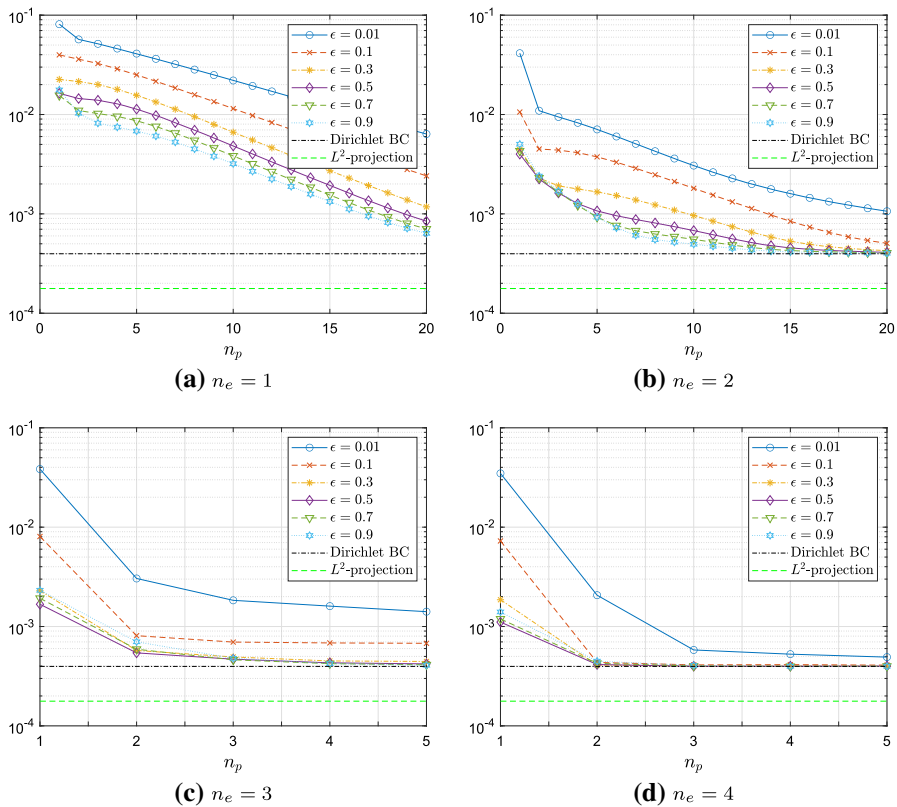


Fig. 6 Relative L^2 -errors as a function of n_p for various ϵ and $\tau = 10^{-4}$ (for determining M) with CRBC imposed on the four sides

variables when $\tau = 10^{-3}$ and $\epsilon = 0.1, 0.3$. Finally, the snapshots of an approximate solution satisfying CRBC of $(n_p, n_e) = (2, 4)$ with $\epsilon = 0.1$ on the four absorbing boundaries are given in Fig. 7.

These experiments illustrate the importance of the real parameters when k is not large. For larger values of k and larger separations δ the propagating modes become more prominent. As an example to show this we consider a time-harmonic wave propagation problem with $k = 20$ in the whole domain $\mathbb{R}^2$. The computational domain is restricted to $\Omega = (-0.5, 0.5)^2$ and CRBC is imposed on the four artificial boundaries $\Gamma_E \cup \Gamma_W \cup \Gamma_N \cup \Gamma_S$. The external source function f is determined by the analytical solution defined by

$$u^{ex}(r, \theta) = \chi(r) \sum_{n=0}^4 \frac{1}{(n+1)^2} H_n^1(kr) e^{in\theta}$$

in polar coordinates. In this example, the cut-off function χ is supported on $r > 0.1$ and is defined by one for $r > 0.3$. Figure 8 shows the convergence of relative L^2 -errors.

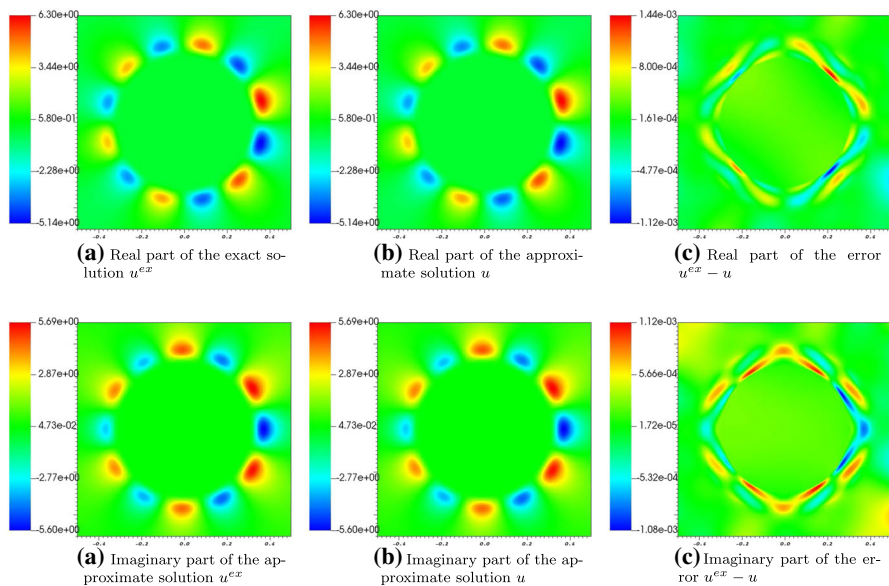


Fig. 7 Snapshots of the finite element solution satisfying CRBC on $\Gamma_E \cup \Gamma_N \cup \Gamma_W \cup \Gamma_S$ and its error $u^{ex} - u$ with $(n_p, n_e) = (2, 4)$, $\tau = 10^{-3}$ (for determining M) and $\epsilon = 0.1$

Interestingly, for large wavenumber $k = 20$, CRBC with sufficient parameters (its error $\approx 4.67 \times 10^{-4}$) can give better approximations than the Dirichlet condition (its error $\approx 1.06 \times 10^{-3}$ represented by the black dash-dot lines in the plots). In addition, unlike the previous case it appears that CRBC can be an effective absorbing boundary condition for sufficiently large n_p without an effort to absorb evanescent modes, though for finer error tolerances we expect they would be needed. This observation can be drawn from Fig. 9 presenting relative L^2 -errors vs. finite element mesh size h with $h = 1/400, 1/800$ and $1/1600$ as well. It is observed that the relative L^2 -errors of approximate solutions satisfying CRBC, for $n_p \geq 4$ if $n_e \geq 0$, or for $n_p \geq 3$ if $n_e \geq 1$, decrease at the optimal rate, which indicates the errors from the boundary condition do not deteriorate the finite element solutions seriously. Finally, the snapshots of an approximate solution satisfying CRBC with $(n_p, n_e) = (3, 0)$ and $\epsilon = 0.5$ on the four absorbing boundaries are given in Fig. 10.

As the last example, we use CRBC as an absorbing boundary condition for finding a scattered field u^{sc} arising from an incident plane wave $u^{in}(x) = e^{ikd \cdot x}$ of wavenumber $k = 20$ hitting a sound soft disc D_R of radius $R = 0.2$ centered at the origin $(0, 0)$. Here $d = (\cos \phi, \sin \phi)$ is the propagation direction of the plane wave and $x = (x_1, x_2)$. Noting that the plane wave can be written as a series

$$u^{in}(r, \theta) = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in(\theta-\phi)}$$

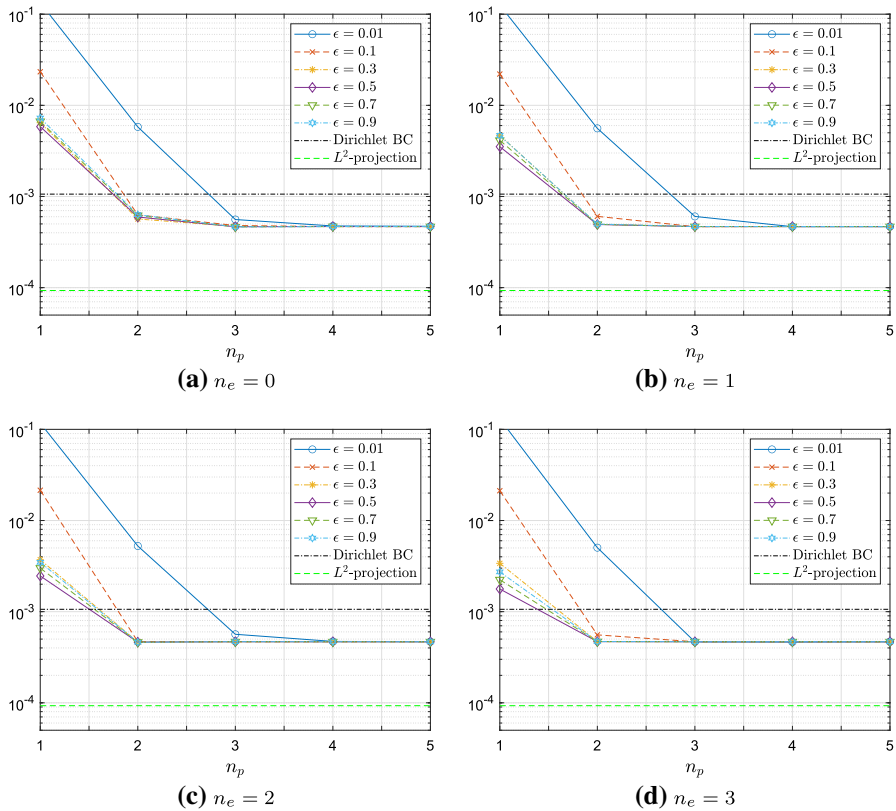


Fig. 8 Relative L^2 -errors as a function of n_p for various ϵ

in polar coordinates (r, θ) , from the Dirichlet condition $u^{in} + u^{sc} = 0$ on $|x| = R$ it can be shown that the scattered field u^{sc} has the series representation

$$u^{sc}(r, \theta) = \sum_{n=-\infty}^{\infty} A_n H_n^1(kr) e^{in\theta} \text{ for } r > 0.2 = R$$

with $A_n = -i^n J_n(kR) e^{-in\phi} / H_n^1(kR)$. For numerical computation of the scattered fields, we take the domain $\Omega = (-0.6, 0.6)^2 \setminus \overline{D_R}$ and impose CRBC of order $(n_p, n_e) = (2, 2)$ on the four exterior boundaries of Ω . The domain is decomposed into quadrilaterals with finite element mesh size $h \approx 0.0023$. With $\tau = 10^{-4}$ to determine M and with $\epsilon = 0.3$ that gives a good performance in the previous examples, we get $\rho_p \approx 6.21 \times 10^{-6}$ and $\rho_e \approx 5.49 \times 10^{-6}$. When $\phi = 0$, the resulting finite element solution satisfying CRBC has a relative L^2 -error $\approx 3.57 \times 10^{-4}$. Here the error is calculated with the analytical scattered field of a finitely truncated series of index n from -30 to 30 . It shows that we can obtain an approximate scattered field accurate within to 0.036% in L^2 -norm with only 4 auxiliary variables. This error is between the relative L^2 -projection error $\approx 6.35 \times 10^{-5}$ and the relative L^2 -error

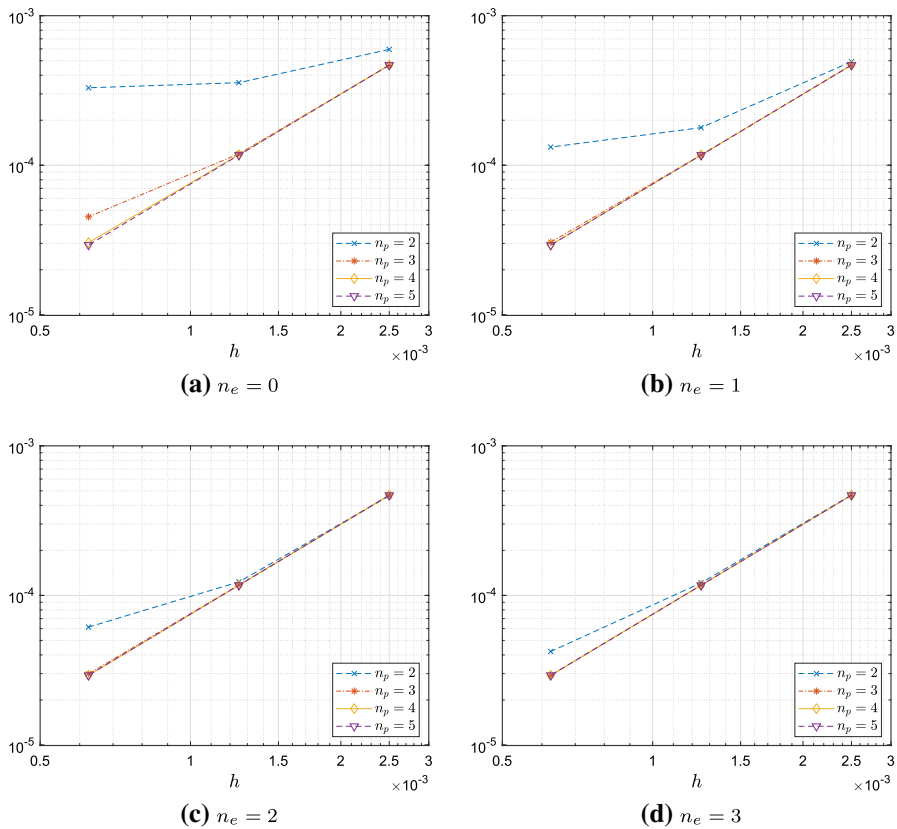


Fig. 9 Relative L^2 -errors vs. h with $h = 1/400, 1/800$ and $1/1600$ as various pairs of order (n_p, n_e) with $\epsilon = 0.5$ and $\tau = 10^{-4}$ (for determining M)

$\approx 9.05 \times 10^{-4}$ of the solution satisfying the exact Dirichlet condition on the boundary of the physical domain as seen in the previous examples. Thus the CRBC solution is more accurate than the solution obtained using exact Dirichlet data. Errors of solutions satisfying CRBC of other orders (n_p, n_e) are also reported in Table 2 and it can be shown that the accuracy is quite good even for $n_p = 1$ (all errors of the test cases are less than 0.7%) and the cases of $n_p \geq 2$ produce sufficiently accurate approximate solutions. In particular, CRBC of order $(n_p, n_e) = (3, 0)$ gives the best result with the smallest degrees of freedom (smallest auxiliary variables) among all these experiments. We also test the performance of CRBC of order $(n_p, n_2) = (2, 2)$ with the same parameters $\tau = 10^{-4}$ and $\epsilon = 0.3$, when the incident plane wave propagates in the direction $\mathbf{d} = (\cos \phi, \sin \phi)$ with various incident angles ϕ , and the relative L^2 -errors are reported in Table 3. It shows that the propagation direction does not have any influence on the performance of CRBC (Figs. 11, 12, 13). The snapshots of the approximate scattered fields and their errors are presented in Fig. 12 for $\phi = 0$ and in Fig. 13 for $\phi = \pi/3$. At last, we compare the performance of CRBC with that of PML, one of the well-known absorbing boundary techniques. To do this, we introduce

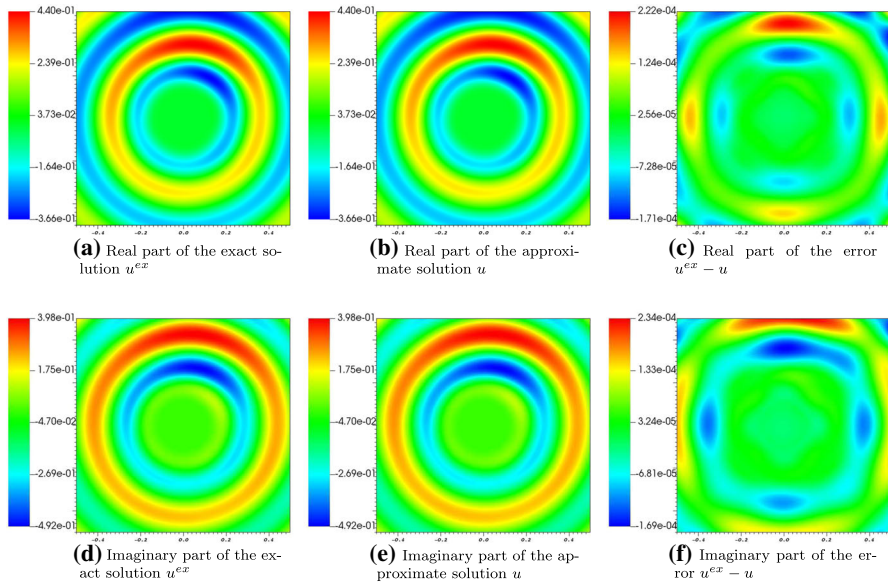


Fig. 10 Snapshots of the finite element solution satisfying CRBC on $\Gamma_E \cup \Gamma_N \cup \Gamma_W \cup \Gamma_S$ and its error $u^{ex} - u$ with $(n_p, n_e) = (3, 0)$, $\tau = 10^{-4}$ (for determining M), $\epsilon = 0.5$ and $h = 1/400$

Table 2 Relative L^2 -errors of finite element solutions to the scattering problem satisfying CRBC of various order (n_p, n_e)

n_e	n_p		
	1 ($\rho_p \approx 3.52 \times 10^{-3}$)	2 ($\rho_p \approx 6.21 \times 10^{-6}$)	3 ($\rho_p \approx 1.09 \times 10^{-8}$)
0	6.92×10^{-3}	6.80×10^{-4}	3.58×10^{-4}
1 ($\rho_e \approx 3.31 \times 10^{-3}$)	4.26×10^{-3}	3.87×10^{-4}	3.60×10^{-4}
2 ($\rho_e \approx 5.49 \times 10^{-6}$)	3.42×10^{-3}	3.57×10^{-4}	3.58×10^{-4}

Table 3 Relative L^2 -errors of finite element approximate solutions satisfying CRBC of order $(n_p, n_e) = (2, 2)$ for the incident angle ϕ

ϕ	$\pi/4$	$\pi/6$	$\pi/8$	$\pi/10$	$\pi/12$
rel. L^2 -errors	3.41×10^{-4}	3.44×10^{-4}	3.48×10^{-4}	3.50×10^{-4}	3.35×10^{-4}

a Cartesian PML defined by the quadratic stretching function

$$\tilde{x}(x) = \begin{cases} x + i \frac{3\sigma}{kW^3} \int_T^x (t-T)^2 dt & \text{if } x > T, \\ x & \text{if } 0 \leq x \leq T, \end{cases}$$

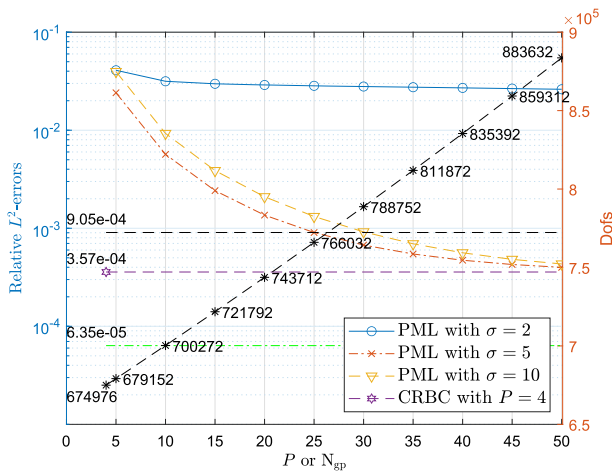


Fig. 11 Comparing relative L^2 -errors of solutions for CRBC with those for PML. The black *-markers represent the total degrees of freedom of the finite element problems including those of auxiliary variables for CRBC and those in the artificial layer for PML. The horizontal black dash line and green dash-dot line represent the errors of the solution satisfying the exact Dirichlet condition on the exterior boundary of the physical domain and the relative L^2 -projection error, respectively

satisfying $\tilde{x}(-x) = -\tilde{x}(x)$, where $T = 0.6$ and the positive constants W and σ represent the PML width and strength, respectively, with $\tilde{y}(y)$ being defined analogously, that is, the absorbing layer of PML is the region of $(T + W)^2 \setminus \Omega \subset \mathbb{R}^2$. The PML is truncated with the homogeneous Dirichlet condition on the exterior boundary of PML, $x = \pm(T + W)$ or $y = \pm(T + W)$. We take $h = 0.3/2^7 \approx 0.0023$ and set the PML width $W = hN_{gp}$, where N_{gp} stands for the number of grid points along the x - and y -axes and it corresponds to the number of auxiliary variables $P = n_p + n_e$ of CRBC.

With the PML strength $\sigma = 2, 5$ and 10 , the resulting relative L^2 -errors of finite element approximate solutions satisfying PML are exhibited in Fig. 11. It shows that $\sigma = 2$ is too small to absorb propagating waves going into the layer and $\sigma = 10$ is so large that the medium property in PML is highly anisotropic, which results in polluting the finite element approximations with a uniform mesh [29, 42]. The PML with in-between $\sigma = 5$ gives the best performance among these, however even the PML with $\sigma = 5$, $N_{gp} = 50$ (degrees of freedom = 883,632) gives the approximate solution with the relative L^2 -error $\approx 3.99 \times 10^{-4}$, which is still larger than the relative L^2 -error $\approx 3.57 \times 10^{-4}$ of the approximate solution satisfying CRBC using only 4 auxiliary variables, $P = 4$, (degrees of freedom = 674,976). As the solve using Dirichlet boundary conditions required 658,432 degree of freedom, the additional degrees of freedom with CRBC are 16,544 and for the PML of comparable accuracy the additional degrees of freedom are 225,200, which are 13 times as many degree of freedom as CRBC. These experiments demonstrate the efficiency and accuracy of CRBC as an absorbing boundary condition for the scattering problem.

In the end, we note that CRBC requires the somewhat elaborate variational problem (6.5) in finite element implementations and it may need more efforts for extension to

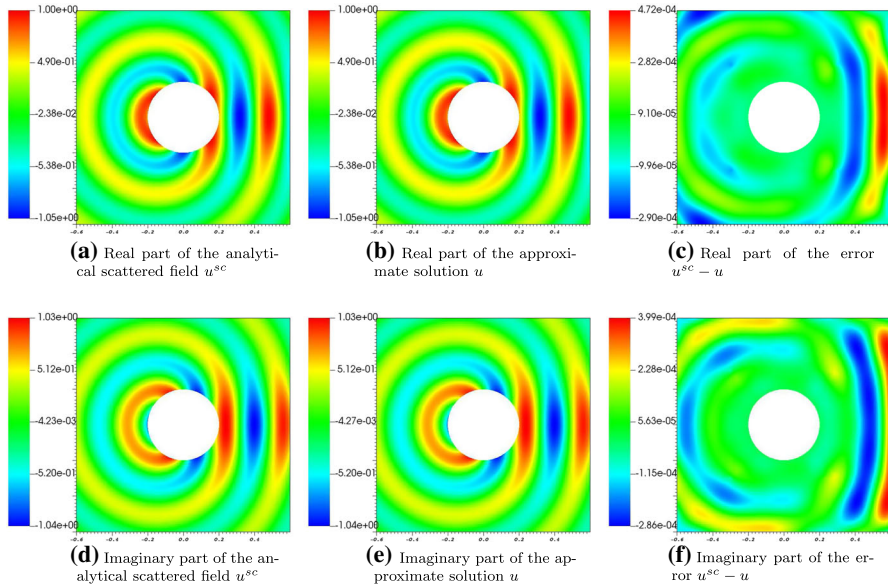


Fig. 12 Snapshots of the finite element solution satisfying CRBC on $\Gamma_E \cup \Gamma_N \cup \Gamma_W \cup \Gamma_S$ and its error $u^{sc} - u$ with $(n_p, n_e) = (2, 2)$, $\tau = 10^{-4}$ (for determining M), $\epsilon = 0.3$ and $h \approx 0.0023$ for the incident angle $\phi = 0$

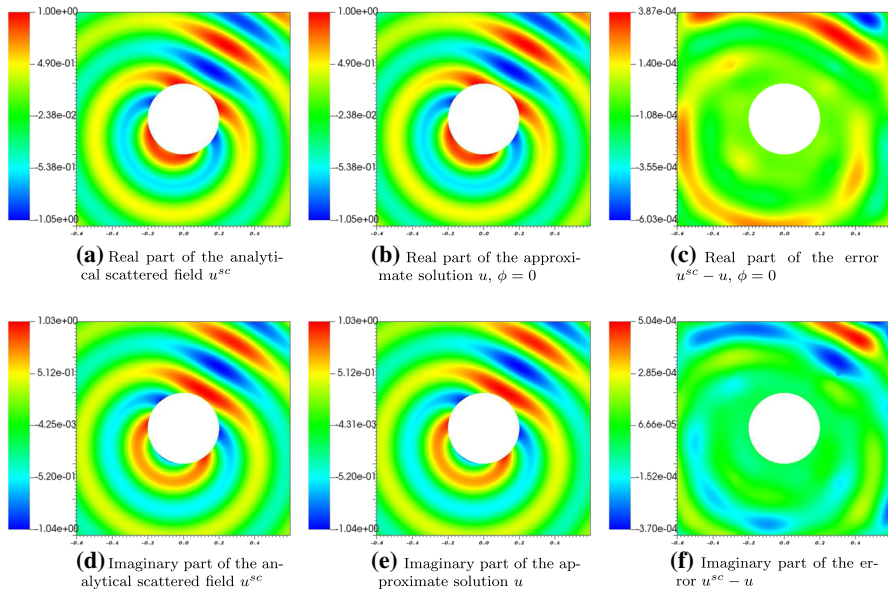


Fig. 13 Snapshots of the finite element solution satisfying CRBC on $\Gamma_E \cup \Gamma_N \cup \Gamma_W \cup \Gamma_S$ and its error $u^{sc} - u$ with $(n_p, n_e) = (2, 2)$, $\tau = 10^{-4}$ (for determining M), $\epsilon = 0.3$ and $h \approx 0.0023$ for the incident angle $\phi = \pi/3$

scattering problem in $\mathbb{R}^3$ as opposed to PML that requires only changing coefficient matrices. However, CRBC can gain the higher efficiency and accuracy in return for the extra effort. In addition, CRBC parameters can be chosen to provide any desired accuracy with minimal cost based on the prescriptions listed above, while PML may require experimentation to determine the layer width and damping profile.

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