

THE BAD LOCUS IN THE MODULI OF SUPER RIEMANN SURFACES WITH RAMOND PUNCTURES

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ABSTRACT. The bad locus in the moduli of super Riemann surfaces with Ramond punctures parametrizes those super Riemann surfaces that have more than the expected number of independent closed holomorphic 1-forms. There is a super period map that depends on certain discrete choices. For each such choice, the period map blows up along a divisor that contains the bad locus. Our main result is that away from the bad locus, at least one of these period maps remains finite. In other words, we identify the bad locus as the intersection of the blowup divisors. The proof abstracts the situation into a question in linear algebra, which we then solve. We also give some bounds on the dimension of the bad locus.

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1. INTRODUCTION

The super vector space Ω of closed holomorphic differentials on a general super Riemann surface of genus g with $2r$ Ramond punctures has dimension $g|r$. The super period map as defined by Witten in [W15] sends this space to a $g|r$ dimensional maximal isotropic subspace $P(\Omega)$ of period space, which is a $2g|2r$ -dimensional super vector space with a supersymplectic pairing. Unfortunately there is a bad locus B in the moduli space parametrizing super Riemann surfaces¹ where the odd dimension of Ω is strictly greater than r , so the period map fails to be injective there. This bad locus B was studied by Witten in [W15], mostly in the case $r = 1$. The goal of this note is to study the bad locus for higher r .

When $r = 1$, the bad locus has codimension 1 and it is the locus where Witten's super period map blows up. We will see in Theorem 8 that in general the bad locus (or at least one of its components) has codimension r . The period map depends on some discrete choices: a symplectic basis of the integral homology, as in the bosonic case, plus an orientation ψ of each of the Ramond punctures. Each choice of ψ determines a period matrix that blows up along a certain divisor Y''_σ . Here $\sigma = \sigma(\psi)$ is a sign choice, determined by ψ but retaining less information than ψ : there are 2^{2r} choices of ψ , but only 2^r choices for σ . The blowup divisor Y''_σ depends only on σ . Our main result, Theorem 2, is that the intersection Y of these divisors equals the bad locus B . The Y''_σ therefore give a more-or-less explicit set of equations for the bad locus. (This story has a variant that depends additionally on the choice of a pairing ν of the $2r$ punctures. It follows from our result that this intersection of the Y''_σ is independent of the choice of ν .)

We review the setup and state the main result in section 2. Our strategy for the proof is to abstract the situation and translate it to a question in linear algebra, Conjecture 3. A special case of this conjecture, sufficient for our needs here, is proved in Appendix A. The full conjecture was proved in [D22]. The reduction of the main theorem to this linear algebra result is explained in section 5.

¹The supermoduli space of super Riemann surfaces with Ramond punctures is a Deligne-Mumford superstack ([CV19], [OV22], [BR21], [MZ19]) and so locally has an étale cover by superschemes, and a compatible system of universal curves. All our statements pertain to the stack, and can be easily translated into statements over the étale covers.

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2. PERIODS OF SUPER RIEMANN SURFACES WITH RAMOND PUNCTURES

A genus g super Riemann surface X with $2r$ Ramond punctures determines a curve C of the same genus g with a divisor $D = \sum_{i=1}^{2r} x_i$ consisting of $2r$ distinct labeled points and a twisted spin structure, i.e. a line bundle L satisfying

$$(1) \quad L^{\otimes 2} = K_C(D).$$

Riemann-Roch gives

$$(2) \quad \chi(L) = h^0(L) - h^0(L(-D)) = r.$$

For $r > 0$ an easy general position argument shows that generically $h^0(L) = r$. This means that the set of super Riemann surfaces with $h^0(L) = r$ can be identified with an open subset of the moduli space

$$(3) \quad M := \{(C, D, L) \mid C \text{ a curve of genus } g, D \in C^{2r}, L \in \text{Pic}^{g+r-1}(C) \text{ satisfies (1)}\}.$$

Following Witten [W15], we refer to the complement

$$B := \{(C, D, L) \mid h^0(L) \geq r + 1\} \subset M.$$

of this open subset as the “bad locus”.

The super vector space Ω of closed 1-forms on X has dimension $g|h^0(L)$. (Our sign convention is that when X is split, the forms pulled back from the reduced curve C are considered even, e.g. for local coordinates (z, θ) on X , dz is even and $d\theta$ is odd. Both this and the opposite convention occur in the literature.) In the split case, the even closed forms are pulled back from $H^0(C, K_C)$ while the odd closed forms are exact, of the form $da(z)\theta$ where $a = a(z)$ is a section of the twisted spin bundle L , cf. [W15], section 5.1, or our appendix B.2. So generically Ω is of dimension $g|r$.

We recall Witten’s superperiod map

$$P : \Omega \rightarrow \Lambda = \Lambda_0 \oplus \Lambda_- \cong \mathbb{C}^{2g|2r}.$$

This depends on some discrete choices. The $2g$ even periods, just as in the classical theory, depend on the choice of a symplectic basis $(a_1, \dots, a_g; b_1, \dots, b_g)$ of $H_1(X, \mathbb{Z})$. Since the residue gives a natural trivialization of the fiber of $K_C(D)$ at each puncture $x_i \in D$, equation (1) specifies a trivialization of the square $(L|x_i)^2$ of the fiber of L at x_i . By an *orientation* ψ_i of the puncture we mean a square root of this, i.e. a trivialization of $L|x_i$ that squares to the above trivialization of $K_C(D)$. The $2r$ odd periods depend on a choice of orientation at each of the $2r$ punctures.

These discrete choices determine identifications

$$\Lambda_0 = H^1(C, \mathbb{C})$$

and

$$\Pi\Lambda_1 = H^0(L/L(-D)) = \bigoplus_{i=1}^{2r} L_{x_i}.$$

The even part of the superperiod map is then restriction from X to C followed by integration on the symplectic basis $(a_1, \dots, a_g; b_1, \dots, b_g)$. The odd part is just the restriction from C to D :

$$H^0(C, L) \rightarrow H^0(D, L) = H^0(L/L(-D)),$$

cf. Appendix B.2.

One bad thing about the “bad” locus B is that this superperiod map P is not injective there: On $X \in B$ there are (odd) closed 1-forms with all periods equal to zero, as can be seen from the long exact sequence

$$(4) \quad 0 \rightarrow H^0(L(-D)) \rightarrow H^0(L) \xrightarrow{P_1} H^0(L/L(-D)) \rightarrow H^1(L(-D)) \rightarrow \dots$$

where $h^0(C, L(-D)) \geq 1$.

The target of the superperiod map is period superspace $\Lambda \cong \mathbb{C}^{2g|2r}$. This comes equipped with a super symplectic pairing Q (symplectic on $\mathbb{C}^{2g}$, orthogonal on $\mathbb{C}^{0|2r}$) which we may think of as a super version of the usual intersection pairing on $H^1(X, \mathbb{R})$. In the obvious notation,

$$Q = \sum_{k=1}^g a_k \wedge b_k - \sum_{i=1}^{2r} (\psi_i)^2,$$

where we recall that the orientations ψ_i form a basis for Λ_1 . The period map can be described by a $(g|r) \times (2g|2r)$ matrix $\mathcal{P}$. In order to obtain a $(g|r) \times (g|r)$ period matrix, we need to split period space as a sum of two complementary Q-Lagrangian subspaces, $\Lambda = \Lambda_A \oplus \Lambda_B$. For the even part we use the span of $(a_1, \dots, a_g)$ for $\Lambda_{A,0}$ and of $(b_1, \dots, b_g)$ for $\Lambda_{B,0}$. For the odd part we fix a pairing ν of the $2r$ punctures, e.g. we could pair x_{2i-1} with x_{2i} . We then take $\Lambda_{A,1}$ as the span of $x_{2i-1} + \sqrt{-1}x_{2i}$ and $\Lambda_{B,1}$ as the span of $x_{2i-1} - \sqrt{-1}x_{2i}$. The period map then decomposes: $P = (P_A, P_B)$, with corresponding decomposition of the matrix, $\mathcal{P} = (\mathcal{P}_A, \mathcal{P}_B)$. The $(g|r) \times (g|r)$ period matrix is then $\mathcal{P}_A^{-1} \mathcal{P}_B$. This is defined only where $\mathcal{P}_A$ is invertible. It depends on:

- the data $X = (C, D, L)$ underlying the super Riemann surface with labeled Ramond punctures,
- the symplectic basis $(a_1, \dots, a_g; b_1, \dots, b_g)$ of $H_1(X, \mathbb{Z})$, and
- the orientations ψ_i .

Sometimes we may want to replace the specific pairing $(x_{2i-1} \text{ with } x_{2i})$ by an arbitrary pairing ν and to emphasize the dependence on this choice. A pairing ν is specified by relabeling the $2r$ points as:

$$x_{ij}, \quad i = 1, \dots, r, \quad j \in \pm := \{+, -\}.$$

By gluing the points (x_{i+}, x_{i-}) in each pair we obtain a nodal curve $\overline{C}$ (of arithmetic genus $g + r$, with r nodes). Specifying the pairing of the points is equivalent to specifying the normalization map $\nu : C \rightarrow \overline{C}$.

Having specified the pairing ν , a *sign choice* σ is a specification of a spin bundle, aka theta characteristic, on the nodal curve $\overline{C}$, i.e., a line bundle L_σ on the nodal curve $\overline{C}$ satisfying

$$(5) \quad L_\sigma^{\otimes 2} = K_{\overline{C}}.$$

In particular, $L := \nu^* L_\sigma$ is a twisted spin structure on C satisfying (1). Equivalently, given L , σ specifies for each $i = 1, \dots, r$ an isomorphism

$$(6) \quad \sigma_i : L_{x_{i+}} \rightarrow L_{x_{i-}}$$

whose square:

$$\sigma_i^{\otimes 2} : K_C(D)|_{x_{i+}} = L_{x_{i+}}^{\otimes 2} \rightarrow L_{x_{i-}}^{\otimes 2} = K_C(D)|_{x_{i-}}$$

is identified via residues with multiplication by -1 ,

$$\sigma_i^{\otimes 2} = -1 : \mathbb{C} \rightarrow \mathbb{C}.$$

(L_σ clearly determines a σ as in (6). Conversely, given σ as in (6), we recover L_σ as the subsheaf of L consisting of sections s of L satisfying

$$(7) \quad s(x_{i-}) = -\sqrt{-1}\sigma_i s(x_{i+})$$

for all $1 \leq i \leq r$. In particular, $h^0(\overline{C}, L_\sigma) > 0$ if L has a *global* section satisfying the gluing equations (7) for all $1 \leq i \leq r$.)

Recall that an orientation ψ is a trivialization of $L|_D$ compatible via residues with (1). In our current notation this amounts to trivializations ψ_{ij} of each $L_{x_{ij}}$. Such a ψ combined with formula (6) therefore determines a $\sigma = \sigma(\psi)$. Changing ψ at both x_{i+} and x_{i-} leaves σ unchanged. On a given X there are 2^{2r} ψ 's but only 2^r σ 's.

Recall that the odd parts

$$P_1 : \Omega_1 \rightarrow \Lambda_1, \quad P_{A,1} : \Omega_1 \rightarrow \Lambda_{A,1}$$

of the superperiod map depend on the orientation ψ . The kernels, however, depend only on $\sigma = \sigma(\psi)$. Indeed, if we identify Ω_1 with $H^0(C, L)$, we find the identifications :

$$(8) \quad \text{Ker}(P_1) = H^0(L(-D)) \subset \text{Ker}(P_{A,1}) = H^0(\overline{C}, L_\sigma) \subset H^0(C, L).$$

We will write $\mathcal{P}^{\nu, \sigma}$ when we want to emphasize that the matrix $\mathcal{P}$ representing the period map depends on a choice of pairing ν and sign σ .

3. MODULI SPACES AND THE MAIN THEOREM

We are interested in describing the locus in moduli space where the period matrix is not defined, i.e., the locus where $\mathcal{P}_A$ is singular for all possible signs σ and for some or all pairings ν . To do this, we need to consider some covers of the moduli space M of (3). Let

$$M' := \{\nu : C \rightarrow \overline{C}, D, L\}$$

be the (bosonic) moduli space parametrizing the data of a curve C with labeled divisor D , a pairing ν on the points of D , and a twisted spin bundle L satisfying (1). There is a forgetful map

$$\pi : M' \rightarrow M.$$

There is a further 2^r -sheeted cover

$$\pi' : M'' \rightarrow M'$$

where

$$M'' := \{\nu : C \rightarrow \overline{C}, D, L_\sigma\}$$

parametrizes curves C with labeled divisor D , a pairing of the points of D , and a choice of a spin bundle L_σ satisfying (5).

Let $\mathcal{P}$ denote (the component of degree $g + r - 1$ of) the relative Picard of the universal curve over the Deligne-Mumford compactification of the moduli space M_{g+r} of curves of genus $g + r$, so

$$\mathcal{P} := \{C', L' \mid C' \text{ a stable curve of genus } g + r, L' \in \text{Pic}^{g+r-1}(C')\}.$$

In $\mathcal{P}$ there is a natural divisor $\mathcal{P}_e$ parametrizing pairs $\{C', L'\}$ where L' is an effective line bundle: $h^0(L') > 0$. There is a natural map of M'' to $\mathcal{P}$ sending $\{\nu : C \rightarrow \overline{C}, D, L_\sigma\}$ to $\{\overline{C}, L_\sigma\}$. We let Y'' denote the inverse image of the divisor $\mathcal{P}_e$:

$$Y'' = Y''_\sigma := \{(\nu : C \rightarrow \overline{C}, D, L_\sigma) \mid L_\sigma^{\otimes 2} = K_{\overline{C}}, h^0(L_\sigma) \geq 1\} \subset M''.$$

This is a divisor in M'' . If $(C, D, L, \nu, \sigma) \in Y''$ then there exists a global section of L satisfying (7) for all $1 \leq i \leq r$, i.e., there exists a global section s whose odd A-periods all vanish. So $\mathcal{P}_{A,1}^{\nu, \sigma}$ has a non-trivial kernel and the matrix $\mathcal{P}_{A,1}^{\nu, \sigma}$ is singular.

As shown in Appendix A of [W15], the period matrix (viewed as a map on M'') blows up (has a pole) along this divisor. This divisor Y''_σ does not depend on a choice of sign σ ; the subscript σ simply reminds us that it is a divisor in M'' , whose fiber over a point of M' is indexed by the σ 's.) The reason that σ occurs here rather than the full orientation ψ was explained at the end of the

previous section: Y'' parametrizes points where the (odd part of the) superperiod map fails to be injective. According to (8), this depends only on σ rather than ψ .

Descending to M' , we have the closed locus

$$Y' := \{(\nu : C \rightarrow \overline{C}, D, L) \mid L^{\otimes 2} = K_C(D), h^0(L_\sigma) \geq 1 \forall \sigma\} = \{y \in M' \mid \pi'^{-1}(y) \subset Y''\} \subset M'$$

parametrizing super Riemann surfaces with a given pairing of their $2r$ Ramond punctures D such that the period matrix blows up for *every* sign choice σ , i.e., $(C, D, L, \nu) \in Y'$ if $\mathcal{P}_A^{\nu, \sigma}$ is singular for all σ . So the complement of Y' is the open subset where the period matrix, for at least one sign choice σ , makes sense. Somewhat informally we write $Y' = \cap_\sigma Y''_\sigma$. (Informal because $Y'' \subset M''$ while $Y' \subset M'$.)

Descending further to M , we have the closed locus

$$Y := \{(C, D, L) \mid (\nu : C \rightarrow \overline{C}, D, L) \in Y' \forall \nu\} = \{y \in M \mid \pi^{-1}(y) \subset Y'\} \subset M,$$

where $(C, D, L) \in Y$ if $\mathcal{P}_A^{\nu, \sigma}$ is singular for all pairings ν and signs σ , and also the a priori larger locus

$$\tilde{Y} := \{(C, D, L) \mid (\nu : C \rightarrow \overline{C}, D, L) \in Y' \text{ for some } \nu\} = \pi(Y') \subset M.$$

The distinction is that (C, D, L) is in $\tilde{Y}$ if $(C, D, L, \nu) \in Y'$ for some pairing ν , while in order to be in Y this must hold for all ν .

Lemma 1. $B \subseteq Y \subseteq \tilde{Y}$

Proof. If $(C, D, L) \in B$, then $h^0(L(-D)) \geq 1$ and so there exists a global section of L satisfying the gluing equation (7) for all $1 \leq i \leq r$ and for all pairings ν and signs σ . $\square$

Our main result can now be stated as follows:

Theorem 2. $B = Y = \tilde{Y}$.

4. LINEAR ALGEBRA

Consider a set of $2r$ abstract points

$$p_{ij}, i = 1, \dots, r, j \in \pm := \{+1, -1\}.$$

By a section, or r -section, we mean a function

$$\sigma : \{1, \dots, r\} \rightarrow \pm,$$

or equivalently the set of r points

$$p_{i, \sigma(i)}, (i = 1, \dots, r).$$

Our study of Ramond punctures leads to the following:

Conjecture 3. Let V be a vector space of dimension $\geq r$ containing r pairs of points $p_{ij}, i = 1, \dots, r, j \in \pm$ that span V . The following conditions (*), (**) are equivalent:

- (*) For each of the 2^r sections σ , the r points $p_{i, \sigma(i)}, (i = 1, \dots, r)$ are linearly dependent.
- (**) There is a subset $I \subset \{1, \dots, r\}$ of some cardinality $k < r$ such that the $2k$ points $p_{ij}, i \in I, j \in \pm$ span a subspace of dimension $< k$.

In section A of the appendix we prove

Theorem 4. The conjecture holds when V is r dimensional.

We will need only this case. The full Conjecture has been proved in [D22].

5. FROM RAMOND PUNCTURES TO LINEAR ALGEBRA

In this section we reduce Theorem 2 to the linear algebra result, Theorem 4.

We begin by proving that condition (*) in Conjecture 3 and Theorem 4 characterizes the locus $\tilde{Y}$. Let V be the vector space

$$V := (H^0(L)/H^0(L-D))^*$$

of linear functions on $H^0(L)$ that vanish on $H^0(L-D)$. (Equivalently, it is the dual of the image of the odd superperiod map P_1 .) By Riemann-Roch, its dimension is r . We are interested mostly in the case that $h^0(L) = r$, so $H^0(L-D) = 0$ and $V = H^0(L)^*$. For each $i \in \{1, \dots, r\}$ and $j \in \pm$, fix an orientation ψ_{ij} , i.e. a trivialization of the fiber $L|_{x_{ij}}$ of L at x_{ij} , compatible with the residue trivialization of $L^{\otimes 2} = K_C(D)$. Evaluation at x_{ij} then becomes a vector $q_{ij} \in V$. We let $p_{ij} := q_{i+} + j\sqrt{-1}q_{i-}$, or explicitly:

$$p_{i+} := q_{i+} + \sqrt{-1}q_{i-}, \quad p_{i-} := q_{i+} - \sqrt{-1}q_{i-}.$$

The collection of $2r$ vectors q_{ij} clearly spans V , and therefore so do the p_{ij} . A sign choice σ picks, for each i , a vector $p_{i\sigma(i)}$ which is one of p_{i+}, p_{i-} . The condition for being in Y''_σ is that the r points $p_{i\sigma(i)}$, $i = 1, \dots, r$ should be contained in a hyperplane in V . Condition (*) of Theorem 4 therefore characterizes $\tilde{Y}$.

Next we describe a certain locus W in moduli space which we will show is equal to both B (Lemma 7) and $\tilde{Y}$ (Lemma 6), and thereby we prove Theorem 2. For every subset

$$I \subset \{1, \dots, r\},$$

the divisor

$$D_I := \sum_{i \in I} (x_{i+} + x_{i-})$$

which is of degree $2k$ where $k := \#I$, determines the locus:

$$W_I := \{(C, D, L) | h^0(L(-D_I)) \geq r + 1 - k\} \subset M.$$

For example, the bad locus B equals $W_\emptyset$. Now let

$$W := \cup_I W_I.$$

Note 5. *These loci obey a duality. Let $k := \#I$ and let I' denote the complement of I . We have $h^0(L(-D_I)) = \chi(L(-D_I)) + h^1(L(-D_I)) = r - 2k + h^0(K \otimes L^{-1}(D - D_{I'})) = r - 2k + h^0(L - D_{I'})$ so $W_{I'} = W_I$.*

Lemma 6. *Condition (**) of Conjecture 3 and Theorem 4 characterizes W . In particular, $W = \tilde{Y}$.*

Proof. Let $k := \#I$. The condition that the $2k$ points p_{ij} , $i \in I$, $j \in \pm$ are contained in a subspace of dimension $k - 1$ implies that we are in W_I . The converse fails in general, because $h^0(L)$ can be $> r$ if we are at a point of B . However, if we are at a point of $\tilde{Y}$ that is not in $B = W_\emptyset$, then $h^0(L) = r$, $H^0(L-D) = 0$ and $V = H^0(L)^*$, and then the condition that the $2k$ points p_{ij} , $i \in I$, $j \in \pm$ are contained in a subspace of dimension $k - 1$ is equivalent to being in W_I . So Condition (**) of Theorem 4 characterizes W . $\square$

The proof of Theorem 2 is now reduced to the following result:

Lemma 7. *We have the inclusion and equality:*

$$B = W_\emptyset = \cup_I W_I = W \subset Y.$$

In particular, $B = Y = \tilde{Y}$.

Proof. A point of W_I is given by (C, D, L) such that $h^0(L(-D_I)) \geq r + 1 - \#I$. For elements of $H^0(L(-D_I))$, the conditions for being in some Y''_σ are clearly satisfied automatically at points of I since such an element vanishes along the divisor D_I , so we only need to impose $r - \#I$ conditions at the remaining points in $D - D_I$, ending with $h^0(L_\sigma) \geq 1$. Thus $W = \cup_I W_I \subset Y$.

On the other hand, each W_I is contained in B . Indeed, say we have a point of W_I that is not in B . Then $h^0(L) = r$ and the divisor D_I imposes at most $k - 1$ conditions on sections of L , where we assume $1 \leq k := \#I \leq r - 1$. But the equality $W_I = W_{I'}$ implies that the divisor $D_{I'}$ imposes at most $r - k - 1$ conditions on sections of L . So $h^0(L - D) \geq r - (k - 1) - (r - k - 1) = 2$, contradicting (2). $\square$

6. DIMENSION COUNTS

Among general line bundles L of degree $g - 1 + r$ on a curve C of genus g , for $0 \leq r < g$, the locus where $h^0(L) > r$ has codimension $r + 1$ by Brill-Noether theory [BN, ACGH]. One might therefore naively expect that the bad locus B should have codimension $r + 1$ in M . The residue theorem implies that this expectation is wrong: it is off by at least 1. In this section we show that there are no further corrections:

Theorem 8.

- In the unpunctured case $r = 0$, the moduli space $M = M^+ \cup M^-$ is reducible. The bad locus B consists of all of M^- plus the irreducible “vanishing thetanull” divisor in M^+ .
- For $1 \leq r < g$, every component of B has codimension $\leq r$ and there is a component of codimension exactly r .
- For $r \geq g$ the bad locus is empty.

Proof. The case $r = 0$ is well known: M^- is the moduli of odd spin bundles L , where $h^0(L)$ is odd and in particular $h^0(L) \geq 1$. M^+ is the moduli of even spin bundles L , for which $h^0(L) = 0$ generically and $h^0(L)$ is even and ≥ 2 along the irreducible divisor of vanishing thetanulls [A71, M71].

For $r \geq g$, the degree of a twisted spin bundle L is $\geq 2g - 1$, so $h^1(L) = 0$ and $h^0(L) = r$, so the bad locus is empty.

Assume $1 \leq r < g$. To see that the codimension is at most r (and thus off by 1 from the Brill-Noether expectation), it is convenient to switch to the $2r$ -sheeted cover

$$\tilde{M} := \{(C, D, L, p) \mid (C, D, L) \in M \text{ and } p \in D\}$$

and to $\tilde{B}$, the inverse image there of B . Consider the map

$$f : \tilde{M} \rightarrow \text{Pic}^{g-r}(\mathcal{C}/M),$$

sending

$$(C, D, L, p) \mapsto N := L(p - D).$$

Here $\mathcal{C}$ is the universal curve over M . In the Picard, the effective locus

$$G_{g-r}^0 := \{N \mid h^0(N) > 0\}$$

has codimension r . We claim that

$$\tilde{B} = f^{-1}(G_{g-r}^0),$$

which implies that the codimension is at most r .

Indeed in one direction,

$$L \in \tilde{B} \iff h^0(L) > r \iff h^0(L(-D)) > 0 \implies h^0(N) > 0.$$

But conversely, a non-zero section $s \in H^0(N)$ gives a meromorphic 1-form

$$s^2 \in H^0(K(-(D-p))(p))$$

with a possible pole only at p . By the residue theorem, it actually cannot have a pole there, so s must be a non-zero section in $H^0(N-p) = H^0(L-D)$.

To conclude, we need to exhibit a component of codimension r . Let A be a generic odd theta characteristic:

$$A^{\otimes 2} = K, h^0(A) = 1.$$

Let $a \in \text{Sym}^{g-1}(C)$ be the unique divisor of a non-zero section of A , and let $d \in \text{Sym}^r(C)$ be the sum of any r of these points. (Recall we are assuming $r \leq g-1$.) Let $p \in d$ be one of these points. Generically such p is not in $a-d$, which we will assume. Take $D := 2d, L := A(d)$. We claim that $f(\tilde{M})$ is transversal to G_{g-r}^0 at $f(C, D, L)$, so the component of $\tilde{B}$ through (C, D, L) has codimension r .

This is seen by comparing the tangent spaces to $f(\tilde{M})$ and G_{g-r}^0 at

$$f(C, D, L) = N := L(p-D) = A(d+p-2d) = A(-(d-p)).$$

The tangent there to G_{g-r}^0 is spanned by the $g-r$ tangent lines to the Abel-Jacobi image of C in its Jacobian at the points of $a-d+p$, or projectively by the corresponding $g-r$ points of the canonical image of C in canonical space. The tangent to $f(\tilde{M})$ is likewise spanned by the $2r$ tangent lines to the Abel-Jacobi (or: canonical) image of C at the points of $D = 2d$. (Since our D is non-reduced, this really means the r osculating planes to C at the points of d .) This follows from the relation

$$N^{\otimes 2} = K(-(D-p)+p),$$

which implies that the differential of f at each of these points is $\pm \frac{1}{2}$ times the corresponding differential of Abel-Jacobi, so the spans are the same.

The transversality is therefore equivalent to saying that the divisor

$$2d + (a-d+p) = a+d+p$$

is not contained in any canonical divisor. From our assumption that $h^0(A) = 1$ it follows that the unique canonical divisor containing a equals $2a$. This does not contain $a+d+p$ because of our assumption that $p \notin a-d$.

□

Appendices

A. PROOF OF THE LINEAR ALGEBRA RESULT

We rephrase Theorem 4 as follows:

Theorem 9. *Let $\{P_1, \dots, P_r\}$, $P_i \neq 0$ be a set of nonzero subspaces of the r dimensional vector space $V = (H^0(L)/H^0(L-D))^*$ such that $\dim P_i \leq 2$. For each P_i , fix two spanning vectors $\{p_i, p'_i\}$ (with one necessarily redundant if $\dim P_i = 1$) and define*

$$V_r = \{\{v_1, \dots, v_r\} \mid v_i \in \{p_i, p'_i\}\}.$$

*If **no** element of V_r spans V , then there exists a subset $\{P_{i_1}, \dots, P_{i_k}\}$ for $k \leq r$ such that*

$$\dim \text{span}\{P_{i_1}, \dots, P_{i_k}\} < k.$$

Let $\rho = \max\{\dim(\text{span } v) \mid \forall v \in V_r\}$ and let $V_r^\rho = \{v \in V_r \mid \dim \text{span } v = \rho\}$. For each $v \in V_r$ let the set $v' \in V_r$ be defined by the rule that if $v_i \in v$ then $v'_i \in v'$. Let

$$v_{-j} = \{v_1, \dots, \hat{v}_j, \dots, v_r\}, \quad v_{+i'} = \{v_1, \dots, v_r, v'_i\}.$$

For $A \subset (v \cup v') \cap W$, the square brackets $[A]$ will denote the indices of the elements in A . For example, if $A = \{v_1, v'_1, v_2, v_3\}$, then $[A]$ is the multi-set $\{1, 1, 2, 3\}$ where the two copies of 1 correspond to v_1 and v'_1 . We say that $i \in [A]$ is a *double* if $v_i \in A$ and $v'_i \in A$.

Definition 10. Let $v \in V_r^\rho$, let $W = \text{span } v$, and let $W' = \text{span } v'$. We say that $w \subset (v \cup v') \cap W$ is a spanning set if it satisfies the following conditions: (1) $\text{span } w = W$, (2) $\text{card } w = \rho$, and (3) $[w]$ contains no doubles.

Note that a spanning set w is just a certain choice of basis for W . We let $\max(v)$ denote the set of all spanning sets in $(v \cup v') \cap W$ and let $w^c \subset v$ denote the complement of w .

Lemma 11. Let $v \in V_r^\rho$ and let $w \in \max(v)$. If $i \in [w^c]$, then $P_i \subset \text{span } w$.

Proof. Suppose $P_i \not\subset \text{span } w$. Then $v'_i \notin \text{span } w$, or $v_i \notin \text{span } w$. WLOG, let $v'_i \notin \text{span } w$ and note that, under this assumption, we have that $\dim \text{span } w_{+i'} = \rho + 1$. Recall that we assumed that the maximal dimension of any element in V_r^ρ is equal to ρ . Since $w_{+i'} \in V_r^\rho$, the equality $\dim \text{span } w_{+i} = \rho + 1$ contradicts the maximality of ρ and thus, $P_i \subset \text{span } w$. $\square$

Definition 12. Let $v \in V_r^\rho$ and let $P_i \subset W = \text{span } v$. We say that a nonzero subset $\iota(P_i) \subset v$ is a minimal set for P_i if P_i is contained in $\text{span } \iota(P_i)$ but there exists no proper subset of $\iota(P_i)$ whose span contains P_i .

If we fix $w \in \max(v)$, then there is a unique minimal set $\iota(P_i) \subset w$ for $P_i = \text{span}\{v_i, v'_i\}$ consisting of the unique set of generators in w for v_i and v'_i , e.g, if

$$v'_i = a_1 v_{i_1} + \dots + a_\rho v_{i_\rho},$$

is the linear combination for v'_i in terms of the basis $w = \{v_{i_1}, \dots, v_{i_\rho}\}$, then $v_{i_1}, \dots, v_{i_\rho} \in \iota(P_i)$, and similarly for v_i .

Lemma 13. Fix $w \in \max(v)$ and let $i \in [w^c]$. Let $\iota(P_i) \subset W$ be minimal for P_i , then for each $j \in [\iota(P_i)]$, we have that $P_j \subset W$. Furthermore, if $\iota(P_j)$ is a minimal set for P_j and $l \in [\iota(P_j)]$, then $P_l \subset W$. More generally, any n -th descendant of P_i is contained in W .

Proof. By definition, if $v_j \in \iota(P_i)$, then $P_i \not\subset \text{span } \iota(P_i)_{-j}$ and so either v_i or v'_i is not in $\text{span } \iota(P_i)_{-j}$. WLOG, let $v'_i \notin \text{span } \iota(P_i)_{-j}$ and note that this assumption implies that $v'_i \notin \text{span } w_{-j}$. Since $v'_i \notin \text{span } w_{-j}$, the set $w_{-j, +i'}$ is a spanning set and since $j \in [w_{-j, +i'}^c]$ we find that $P_j \subset \text{span } w_{-j, +i'} \subset W$ by Lemma 11. For the same $w \in \max(v)$ as above, let $\iota(P_j) \subset w$ be the minimal set for P_j and let $l \in [\iota(P_j)]$. We can use the same proof to prove that $P_l \subset W$ if we can find a minimal set B for P_j such that $l \in [B]$ and $j \notin [B]$. Since $P_j \subset \text{span } w_{-j, +i'}$, there exists a unique $B \subset w_{-j, +i'}$ minimal for P_j . For $l \neq j, i'$, note that $l \in [\iota(P_j)]$ if and only if $l \in [B]$. since we already know that P_j and P_i are contained in W , we may assume that $l \neq j, i'$. Now since $j \notin [B]$, we can use the same proof as above to conclude that $P_l \subset W$. $\square$

Fix $w \in \max(v)$ and define

$$[\Upsilon_n] = \{j \mid v_j \in [\iota(P_i)], \text{ such that } \iota(P_i) \subset w \text{ and } i \in \Upsilon_{n-1}\} \cup \Upsilon_{n-1},$$

where

$$[\Upsilon_0] = \{j \mid v_j \in [\iota(P_i)], \text{ such that } \iota(P_i) \subset w \text{ and } i \in [w^c],\}$$

is the “initial condition.” Note that we meant what we wrote when we defined $[\Upsilon_0]$, i.e., we do *not* want $\Upsilon_0 = [w^c]$. Clearly, the sets $\Upsilon_1, \Upsilon_2, \dots$ will eventually converge to some subset Υ of w . That is, there exists some $n_0 \gg 0$ such that $\Upsilon_{n_0} = \Upsilon_{n_0+i_0}, \forall i_0 \geq 0$ and so $\Upsilon = \Upsilon_{n_0}$.

Lemma 14. *If $j \in [\Upsilon]$, then $P_j \subset \text{span } \Upsilon$.*

Proof. If $j \in [\Upsilon]$, then $j \in [\Upsilon_n]$ for some n . By construction, $\iota(P_j) \subset \Upsilon_{n+1}$ and so $P_j \subset \text{span } \Upsilon_{n+1}$. In particular, $P_j \subset \text{span } \Upsilon$ since $\Upsilon_{n+1} \subset \Upsilon$. □

The proof of Theorem 9 is now trivial:

Proof of Theorem 9. Let

$$[\Upsilon] = \{j_1, \dots, j_h\}$$

and note that $\dim \text{span } \Upsilon \leq h$ and that $\text{span } \Upsilon$ contains at least h planes, i.e. $P_{j_1}, \dots, P_{j_h} \subset \text{span } \Upsilon$ by Lemma 14. Recall that by construction $j_1, \dots, j_h \in [w]$, but $P_i \subset \text{span } \Upsilon$ for all $i \in [w^c]$ since we defined $\bigcup_{i \in w^c} \iota(P_i) = \Upsilon_0$. Since $\text{card } w^c \geq 1$, the set $\{P_{j_1}, \dots, P_{j_h}\} \cup \bigcup_{i \in w^c} P_i$ satisfies the conclusion of Theorem 9. □

B. ODD PERIODS

In [W15], the odd periods are defined to be restrictions of elements of Ω to the components of the Ramond divisor, see Section B.2. Throughout the paper, we computed the odd periods as values of elements in $H^0(C, L)$ along the points of D . In this section of the appendix we prove (Lemma 17) that these values agree with the values of the odd periods as defined by Witten. Throughout this section we consider only *split* super Riemann surfaces, e.g. ones defined over a point or more generally over a (bosonic) base.

B.1. $\Omega_1 = H^0(C, L)$. We begin by recalling Witten’s argument that the odd elements of Ω can be identified with global sections of L . Let Ber'_X denote the subsheaf of $\text{Ber}_X(R)$ whose sections have a θ -independent residue along R . One can check that

$$(9) \quad \text{Ber}'_X = \Omega_C^1 \oplus \Pi L.$$

In contrast to an ordinary curve, the dualizing sheaf Ber_X on the supercurve X is not equal to the cotangent bundle Ω_X^1 : this is of course obvious given that Ber_X has rank $0|1$ while Ω_X^1 has rank $1|1$. However, there is a canonical isomorphism $\alpha : H^0(X, \text{Ber}'_X) \cong \Omega$ which in local superconformal coordinates (z, θ) sends $s = (g(z) + \theta f(z))[dz|d\theta]$ to

$$(10) \quad \alpha(s) = g(z)d\theta + (f(z) + \theta g'(z))dz.$$

Composing α^{-1} with (9) we find that $\Omega = H^0(C, \Omega_C^1) \oplus \Pi H^0(C, L)$ (we will write $=$ whenever an isomorphism is canonical), and so $\dim \Omega = \dim H^0(X, \text{Ber}'_X) = g|h^0(L)$.

B.2. Odd periods. One can check that around each $R_i \in R$, there exist superconformal coordinates (z, θ) such that R_i is defined by the equation $z = 0$. In these coordinates a general element of Ω is of the form $f(z, \theta)dz + g(z, \theta)d\theta$ and its restriction to R_i (to compute: set $z = 0$ and apply the condition $ds = 0$) can be shown to be equal to

$$g(0) d\theta,$$

where the constant $g(0) \in \mathbb{C}$ is unique up to a superconformal change of coordinates $\theta \mapsto -\theta$. Let $\sigma = \{\sigma_i \in \{+, -\}\}$ denote a choice of θ or $-\theta$ for each $R_i \in R$.

Definition 15. For each $R_i \in R$, the odd period $\mathfrak{o}_i^\sigma(s)$ of $s \in \Omega$ with respect σ is the constant $\sigma_i \cdot g(0)$.

Lemma 16. The odd (resp. even) periods of even (resp. odd) elements in Ω are equal to zero.

Proof. If s is an even element in Ω , then its restriction to R_i must also be even; however $g_0 d\theta$ is odd for all non-zero g_0 . Thus $g_0 = 0$.

Now let $s \in \Omega_1$ be odd and let $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ be A- and B-cycles on C . The even (A- and B-) periods of s are the integrals

$$\int_{a_i} \tau^*(s), \quad \int_{b_i} \tau^*(s)$$

where $\tau : C \subset X$ is the canonical inclusion. In local superconformal coordinates (z, θ) , the odd closed one form s can be written as $s = f(z)\theta dz + g(z)d\theta$. The pullback $\tau^*(s)$ is equal to 0 since $\tau^*(\theta) = 0$ and thus both of the above integrals are equal to zero. $\square$

We will now show that odd periods can be computed as residues along D . We need to explain what we mean by a residue of an element in Ω . Using the identification $\Omega = H^0(X, \text{Ber}'_X)$ and the splitting (9) we can write every element s in Ω as a sum $s_0 + s_1$ of a holomorphic differential s_0 on C and a global section s_1 of the twisted spin structure L . For each $p_i \in D$, we define

$$\text{res}_i(s) = \text{res}_{p_i}(s_0) + \text{res}_{p_i}(s_1) = \text{res}_{p_i}(s_1)$$

where $\text{res}_{p_i}(s_0) = 0$ because s_0 is holomorphic near p_i . Since s_1 is not a differential, we need to explain what we mean by $\text{res}_{p_i}(s_1)$: Locally near each $p_i \in D$, there exists coordinates $U(z)$ such that $\Omega_U^1 \cong \mathcal{O}_U$ and such that p_i is defined by $z = 0$. Let dz be a generator for Ω_U^1 . We then have two possible generators for $L|_U$, namely $\theta := \sqrt{dz/z}$ and $-\theta := -\sqrt{dz/z}$. For a fixed generator θ , there is a canonical isomorphism $L|_U = \Omega_U^1(D) = \Omega_U^1(p_i)$ sending θ to dz/z . Let $\sigma = \{\sigma_i \in \{+, -\}\}_{i=1}^{2r}$ denote the sign of the generator for each $L|_{U_i}$. Then each $s \in \Omega_1$ is locally of the form $\sigma_i \cdot g(z)\theta$ and is sent to $\sigma_i \cdot \frac{g(z)}{z} dz$ in $\Omega_U^1(p_i)$: And,

$$(11) \quad \text{res}_{p_i}(s_1) = \sigma_i \cdot \text{res}_{p_i} \left(\frac{g(z)}{z} dz \right).$$

Lemma 17. Let σ be fixed and let $s \in \Omega$. Then for each point $p_i \in D$, we have

$$(1/2\pi i) \mathfrak{o}_{p_i}(s) = \text{res}_{p_i}(s).$$

Proof. The lemma is obvious for even elements Ω so let s be an odd element and let $U(z, \sigma \cdot \theta)$ be superconformal coordinates near R_i so that R_i is defined by $z = 0$. Then there exist unique g such that

$$s|_U = \sigma_i \cdot (g(z)d\theta + g'(z)\theta dz)$$

and $\alpha^{-1}(s) = (\sigma_i \cdot g(z))[dz|d\theta]$. The odd periods are now easy to read off: namely, $\mathfrak{o}_i^\sigma(s) = \sigma_i \cdot g(0)$.

The section $\alpha^{-1}(s)$ is identified with the section $\sigma_i \cdot g(z)\theta$ of $L|_U$ by (9) where $\theta = \sqrt{\frac{dz}{z}}$. The section $\sigma_i \cdot g(z)\theta$ is in turn identified with the section $\sigma_i \cdot \frac{g(z)}{z} dz$ of $\Omega_U^1(p_i)$ under the identification $L|_U = \Omega_U^1(p_i)$. It is now easy to see that

$$2\pi i \text{res}_{z=0} \left(\frac{g(z)}{z} dz \right) = \sigma_i \cdot g(0) = \mathfrak{o}_i^\sigma(s)$$

$\square$

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