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Anosov groups: local mixing, counting and equidistribution

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Let G be a connected semisimple real algebraic group, and $\Gamma < G$ a Zariski dense Anosov subgroup with respect to a minimal parabolic subgroup. We describe the asymptotic behavior of matrix coefficients $\langle (\exp tv).f_1, f_2\rangle$ in $L^2(\Gamma\backslash G)$ as $t \to \infty$ for any $f_1, f_2 \in C_c(\Gamma\backslash G)$ and any vector v in the interior of the limit cone of Γ. These asymptotics involve higher-rank analogues of Burger–Roblin measures, which are introduced in this paper. As an application, for any affine symmetric subgroup H of G, we obtain a bisector counting result for Γ–orbits with respect to the corresponding generalized Cartan decomposition of G. Moreover, we obtain analogues of the results of Duke, Rudnick and Sarnak as well as Eskin and McMullen for counting discrete Γ–orbits in affine symmetric spaces $H\backslash G$.

22E40, 37A17, 37A25, 37A44

1 Introduction

Let G be a connected semisimple real algebraic group. We fix a Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$, where K is a maximal compact subgroup and $\exp \mathfrak{a}^+$ is a positive Weyl chamber of a maximal real split torus of G. Let $\Gamma < G$ be a Zariski dense discrete subgroup. Consider a matrix coefficient of $L^2(\Gamma\backslash G)$ given by

$$\langle \exp(tu) f_1, f_2 \rangle = \int_{\Gamma\backslash G} f_1(x \exp(tu)) f_2(x)\, dx, \tag{1.1}$$

where $u \in \mathfrak{a}^+ - \{0\}$ and dx denotes the G–invariant measure on $\Gamma\backslash G$. Understanding its asymptotic behavior as $t \to \infty$ is of basic importance in the study of dynamics of flows in $\Gamma\backslash G$, and has many applications, including to equidistribution and counting problems. A classical result due to Howe and Moore [20] implies that

$$\lim_{t\to\infty} \langle \exp(tu) f_1, f_2 \rangle = \frac{1}{\mathrm{Vol}(\Gamma\backslash G)} \int f_1\, dx \int f_2\, dx. \tag{1.2}$$

In particular, if Γ has infinite covolume in G, then

$$\lim_{t\to\infty} \langle \exp(tu) f_1, f_2 \rangle = 0. \tag{1.3}$$

This leads us to the following local mixing type question: for a given unit vector $u \in \mathfrak{a}^+$, do there exist a normalizing function $\Psi_{\Gamma,u}\colon (0,\infty) \to (0,\infty)$ and locally finite Borel measures μ_u, μ_u^* on $\Gamma\backslash G$ such that for any $f_1, f_2 \in C_c(\Gamma\backslash G)$,[1]

$$\lim_{t\to\infty} \Psi_{\Gamma,u}(t) \langle \exp(tu) f_1, f_2 \rangle = \mu_u(f_1)\mu_u^*(f_2)\,? \tag{1.4}$$

When G has rank one, this was completely answered by Roblin [42] and Winter [51] for geometrically finite subgroups and by Oh and Pan [31] for coabelian subgroups of convex cocompact subgroups.

When G has rank at least two, the location of the vector u relative to the *limit cone* of Γ turns out to play an important role. The limit cone of Γ, which we denote by $\mathcal{L}_\Gamma$, is defined as the smallest closed cone in $\mathfrak{a}^+$ containing the Jordan projection of Γ. Benoist [1] showed that $\mathcal{L}_\Gamma$ is convex and has nonempty interior. Indeed, it is not hard to show that if $u \notin \mathcal{L}_\Gamma$, then for any $f_1, f_2 \in C_c(\Gamma\backslash G)$,

$$\langle \exp(tu) f_1, f_2 \rangle = 0 \quad \text{for all } t \text{ large enough;}$$

see Proposition 2.19.

[1] For a topological space X, the notation $C_c(X)$ means the space of all continuous functions on X with compact support, and if M is a compact group acting continuously on X, the notation $C_c(X)^M$ means the subspace of $C_c(X)$ consisting of M–invariant functions.

The main goal of this paper is to prove the local mixing result, giving a positive answer to question (1.4) for all directions u in the interior of $\mathcal{L}_\Gamma$, and to describe applications to counting and equidistribution results associated to a symmetric subgroup H of G for a large class of discrete subgroups, called Anosov subgroups. Anosov subgroups of G are defined with respect to any parabolic subgroup. In this paper, we focus on Zariski dense Anosov groups with respect to a minimal parabolic subgroup P of G. Let $\mathcal{F} := G/P$ be the Furstenberg boundary. We denote by $\mathcal{F}^{(2)}$ the unique open G–orbit in $\mathcal{F}\times\mathcal{F}$ via the diagonal action. A Zariski dense discrete subgroup $\Gamma < G$ is called *Anosov* with respect to P if there exists a finitely generated word hyperbolic group Σ such that $\Gamma = \Phi(\Sigma)$, where $\Phi\colon \Sigma \to G$ is a P–Anosov representation, ie Φ induces a continuous equivariant map ζ from the Gromov boundary $\partial\Sigma$ to $\mathcal{F}$ such that for all $x \neq y \in \partial\Sigma$, $(\zeta(x), \zeta(y))$ belongs to $\mathcal{F}^{(2)}$. The definition of a P–Anosov representation for a general discrete subgroup requires a certain contraction property, which is automatic for Zariski dense subgroups; see Guichard and Wienhard [17, Theorem 1.5]. The notion of Anosov representations was first introduced by Labourie for surface groups [24], and then extended by Guichard and Wienhard [17] to general word hyperbolic groups.

If G has rank one, the class of Zariski dense Anosov groups coincides with the class of Zariski dense convex cocompact subgroups of G; see [17, Theorem 5.15]. Guichard and Wienhard [17, Theorem 1.2] showed that P–Anosov representations form an open subset of the space $\mathrm{Hom}(\Sigma, G)$; this abundance of Anosov subgroups contrasts with the fact that there are only countably many lattices in any simple algebraic group not locally isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. The class of Anosov subgroups includes subgroups of a real-split simple algebraic group which arise as the Zariski dense image of a Hitchin representation (see Hitchin [19]) of a surface subgroup studied by Labourie [24] and Fock and Goncharov [12], as well as Schottky groups (cf Lemma 7.2). We refer to Kapovich, Leeb and Porti [21], Guéritaud, Guichard, Kassel and Wienhard [16] and Bochi, Potrie and Sambarino [3] for other equivalent definitions of Anosov subgroups, and to Kassel [22] and Wienhard [50] for excellent survey articles.

In the whole paper, by an Anosov subgroup we mean a Zariski dense Anosov subgroup with respect to a minimal parabolic subgroup. In the rest of the introduction, we let $\Gamma < G$ be an Anosov subgroup.

We recall the definition of the growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R}\cup\{-\infty\}$ given by Quint [36]: for any vector $u \in \mathfrak{a}^+$,

$$\psi_\Gamma(u) := \|u\| \inf_{\substack{\text{open cones } \mathcal{C}\subset\mathfrak{a}^+ \\ u\in\mathcal{C}}} \tau_{\mathcal{C}}, \tag{1.5}$$

where $\tau_{\mathcal{C}}$ is the abscissa of convergence of the series $\sum_{\gamma\in\Gamma,\,\mu(\gamma)\in\mathcal{C}} e^{-t\|\mu(\gamma)\|}$. Here $\mu\colon G\to\mathfrak{a}^+$ is the Cartan projection and $\|\cdot\|$ is the norm on $\mathfrak{a}$ induced from a left-invariant Riemannian metric on G/K. Observe that in the rank one case, ψ_Γ is simply the critical exponent of Γ. Quint [36] showed that ψ_Γ is a concave and upper semicontinuous function, which is positive on $\operatorname{int}\mathcal{L}_\Gamma$; here $\operatorname{int}\mathcal{L}_\Gamma$ denotes the interior of $\mathcal{L}_\Gamma$. If $2\rho\in\mathfrak{a}^*$ denotes the sum of all positive roots with respect to the choice of $\mathfrak{a}^+$, then $\psi_\Gamma\le 2\rho$. When Γ is a lattice, it follows from Gorodnik and Oh [13] that $\psi_\Gamma=2\rho$. On the other hand, when Γ is of infinite covolume in a simple Lie group of rank at least 2, Quint [39] deduced from Oh [30] that $\psi_\Gamma\le 2(\rho-\eta_G)$, where $2\eta_G$ is the sum of the maximal strongly orthogonal subset of the root system of G.

Local mixing Let N^+ and N^- denote the maximal expanding and contracting horospherical subgroups, respectively, associated with $\mathfrak{a}^+$ (see (2.1) and (2.2)), and M the centralizer of $\exp\mathfrak{a}$ in K. For each $u\in\operatorname{int}\mathcal{L}_\Gamma$, Quint [37] constructed a higher-rank analogue of the Patterson–Sullivan density supported on the limit set Λ_Γ, which is the minimal Γ–invariant subset of $\mathcal{F}$. Using this, we define the $N^\pm M$–invariant Burger–Roblin measures m_u^{BR} and $m_u^{\mathrm{BR}_*}$, respectively, on $\Gamma\backslash G$ (see (3.8) and (3.10)), which can be considered as the higher-rank generalizations of the Burger–Roblin measures in the rank one case; see Burger [5], Roblin [42] and Oh and Shah [32]. We denote by i the opposition involution of $\mathfrak{a}$ (Definition 3.1), and set $r:=\operatorname{rank}(G)=\dim\mathfrak{a}\ge 1$.

Theorem 1.6 *For any $u\in\operatorname{int}\mathcal{L}_\Gamma$, there exists a $\kappa_u>0$ such that for all $f_1, f_2\in C_c(\Gamma\backslash G)^M$,*

$$\lim_{t\to+\infty} t^{(r-1)/2}e^{t(2\rho-\psi_\Gamma)(u)}\int_{\Gamma\backslash G} f_1(x\exp(tu))\,f_2(x)\,dx=\kappa_u\cdot m_{\mathrm{i}(u)}^{\mathrm{BR}}(f_1)\,m_u^{\mathrm{BR}_*}(f_2).$$

This theorem is not expected to hold for $u\in\partial\mathcal{L}_\Gamma$ in view of [9, Theorem 1.1]. See Theorem 7.12 for a more refined version of this theorem; in fact, it is this refined version which is needed in the application to counting problems as stated in Theorems 1.8 and 1.10.

Equidistribution of maximal horospheres We also obtain the following equidistribution result for translates of maximal horospheres:

Theorem 1.7 *For any $u\in\operatorname{int}\mathcal{L}_\Gamma$, $f\in C_c(\Gamma\backslash G)^M$, $\phi\in C_c(N^+)$ and $x=[g]\in\Gamma\backslash G$,*

$$\lim_{t\to\infty} t^{(r-1)/2}e^{t(2\rho-\psi_\Gamma)(u)}\int_{N^+} f(xn\exp(tu))\phi(n)\,dn=\kappa_u\cdot m_{\mathrm{i}(u)}^{\mathrm{BR}}(f)\,\mu_{gN^+,u}^{\mathrm{PS}}(\phi),$$

where dn and $\mu_{gN^+,u}^{\mathrm{PS}}$ are, respectively, the Lebesgue and Patterson–Sullivan measures on gN^+, as defined in (4.7) *and* (4.1).

Bisector counting for a generalized Cartan decomposition Let H be a symmetric subgroup of G, ie H is the identity component of the set of fixed points for an involution σ of G. Up to a conjugation, we may assume that σ commutes with the Cartan involution θ, which fixes K. We then have a generalized Cartan decomposition $G = H\mathcal{W}(\exp \mathfrak{b}^+)K$, where $\mathfrak{b}^+ \subset \mathfrak{a}^+$ and $\mathcal{W}$ is a subgroup of the Weyl group; see Section 8 for details. Set $r_0 := \operatorname{rank} H\backslash G = \dim \mathfrak{b}$. Note that $1 \le r_0 \le r$.

Theorem 1.8 *For any $v \in \mathfrak{b}^+ \cap \operatorname{int} \mathcal{L}_\Gamma$, there exist $c > 0$ and a norm $|\cdot|$ on $\mathfrak{b}$ such that for any right $H \cap M$–invariant bounded subset $\Omega_H \subset H$ with $\mu^{\mathrm{PS}}_{H,v}(\partial\Omega_H) = 0$ and any left M–invariant bounded subset $\Omega_K \subset K$ with $\mu^{\mathrm{PS},*}_{K,\mathrm{i}(v)}(\partial\Omega_K^{-1}) = 0$, we have*

$$\lim_{T\to\infty} \frac{\#(\Gamma \cap \Omega_H(\exp \mathfrak{b}_T^+)\,\Omega_K)}{e^{\psi_\Gamma(v)T} \cdot T^{(r_0-r)/2}} = c\mu^{\mathrm{PS}}_{H,v}(\Omega_H)\mu^{\mathrm{PS},*}_{K,\mathrm{i}(v)}(\Omega_K^{-1}),$$

where $\mathfrak{b}_T^+ = \{w \in \mathfrak{b}^+ : |w| \le T\}$ and $\mu^{\mathrm{PS}}_{H,v}$ and $\mu^{\mathrm{PS},*}_{K,\mathrm{i}(v)}$ are measures on H and K defined in Definition 8.7 and Lemma 9.20, respectively.

When Γ is a cocompact lattice in a rank one Lie group and H is compact, this goes back to Margulis' thesis from 1970; see [27] for an English translation published in 2004. In the case when Γ is a geometrically finite subgroup of a rank one Lie group, this was shown in Roblin [42] for H compact, and in Oh and Shah [32] and Mohammadi and Oh [29] for general symmetric subgroups. For lattices of higher-rank Lie groups, the bisector counting problem was studied in Gorodnik and Oh [13] and Gorodnik, Oh and Shah [14; 15]. For nonlattices of higher-rank Lie groups, it was studied for H compact, by Quint [40] and Thirion [48] for Schottky groups[2] and by Sambarino [45] for Anosov subgroups; see also Carvajales [7]. Hence the main novelty of this paper lies in our treatment of noncompact symmetric subgroups H in a general higher-rank case. It is interesting to note the presence of the decaying polynomial term $T^{(r_0-r)/2}$ when $\mathfrak{a} \ne \mathfrak{b}$, as the results in loc. cit. have all purely exponential terms. We mention that a related counting result was obtained for $\mathrm{SO}(p, q-1)\backslash \mathrm{SO}(p, q)$ in a recent paper of Carvajales [6]; in this case, $\mathfrak{b}$ lies in the wall of $\mathfrak{a}$ and hence Theorem 1.8 does not apply, and the asymptotic is again purely exponential.

By the concavity and upper semicontinuity of ψ_Γ, there exists a unique unit vector $u_\Gamma \in \mathfrak{a}^+$ (called the maximal growth direction) such that

$$\psi_\Gamma(u_\Gamma) = \max_{v\in\mathfrak{a}^+,\, \|v\|=1} \psi_\Gamma(v).$$

[2]Thirion's work applies to the so-called ping-pong groups, which are slightly more general than Schottky groups.

It is known that $u_\Gamma \in \operatorname{int} \mathcal{L}_\Gamma$ [40; 45]. When $u_\Gamma \in \mathfrak{b}^+$, the norm $|\cdot|$ in Theorem 1.8 associated to u_Γ may simply be taken as the Euclidean norm $\|\cdot\|$ as above, ie the one obtained from the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}$ induced by the Killing form. For a general vector $v \in \mathfrak{b}^+ \cap \operatorname{int} \mathcal{L}_\Gamma$, one may take any norm that arises from an inner product for which v and $(\nabla \psi_\Gamma(v))^\perp = \{w \in \mathfrak{b} : \langle \nabla \psi_\Gamma(v), w \rangle = 0\}$ are orthogonal.

Example When $\mathfrak{a} = \mathfrak{b}$, we automatically have $u_\Gamma \in \mathfrak{b}^+ \cap \operatorname{int} \mathcal{L}_\Gamma$; so Theorem 1.8 applies. For groups G of rank one, this is always the case for any symmetric subgroup H. In general, this case arises as follows: let ι be any involution of G that commutes with the Cartan involution θ and fixes $\mathfrak{a}$ pointwise. Then defining $\sigma := \iota \circ \theta$, we have $\sigma|_{\mathfrak{a}} = -1$, and hence $\mathfrak{a} = \mathfrak{b}$. For example, for any element $m \in K$ of order two which commutes with $\exp \mathfrak{a}$, $\iota(g) := mgm$ satisfies the above conditions. More specifically, the pair $G = \mathrm{PGL}_n(\mathbb{R})$ and $H = \mathrm{PO}(p, n-p)$ may be realized this way by taking $m = \operatorname{diag}(\mathrm{Id}_p, -\mathrm{Id}_{n-p})$.

Counting in affine symmetric spaces Around 1993, Duke, Rudnick and Sarnak [10] and Eskin and McMullen [11] showed the following; see also Benoist and Oh [2], Gorodnik and Oh [13] and Gorodnik, Oh and Shah [14; 15].

Theorem 1.9 (Duke–Rudnick–Sarnak, Eskin–McMullen) *Let $\Gamma < G$ be a lattice such that $\mathsf{v}_0\Gamma \subset H\backslash G$ is closed for $\mathsf{v}_0 = [H]$. Suppose that $(H \cap \Gamma)\backslash H$ has finite volume. We have, as $T \to \infty$,*

$$\lim_{T\to\infty} \frac{\#(\mathsf{v}_0\Gamma \cap \mathsf{v}_0(\exp \mathfrak{b}_T^+)K)}{\mathsf{m}(\mathsf{v}_0(\exp \mathfrak{b}_T^+)K)} = \frac{\mathrm{Vol}((\Gamma \cap H)\backslash H)}{\mathrm{Vol}(\Gamma\backslash G)},$$

where $\mathfrak{b}_T^+ = \{w \in \mathfrak{b}^+ : \|w\| \le T\}$ and m is a suitably normalized G–invariant measure on $H\backslash G$.

In order to state an analogue of Theorem 1.9 for Γ Anosov, we introduce the following condition on an H–orbit: a closed orbit $[e]H \subset \Gamma\backslash G$ is said to be *uniformly proper* if there exists a right K–invariant neighborhood $\mathcal{O}$ of $[e]$ in $\Gamma\backslash G$ such that

$$\{[h] \in (\Gamma \cap H)\backslash H : [h] \exp(\mathfrak{b}^+ \cap \mathcal{L}_\Gamma) \cap \mathcal{O} \neq \varnothing\}$$

is bounded.[3] We show in Lemma 9.35 that $[e]H$ is uniformly proper whenever $r = r_0$ and the limit set Λ_Γ is contained in the open set $HP/P \subset \mathcal{F}$; cf [46, Proposition 7.1.8].

[3] As $[e]H$ is closed, the inclusion $(H \cap \Gamma)\backslash H \to \Gamma\backslash G$ is a proper map, and hence the set $S_b := \{[h] \in (\Gamma \cap H)\backslash H : [h] \exp b \cap \mathcal{O} \neq \varnothing\}$ is bounded for each $b \in \mathfrak{b}^+ \cap \mathcal{L}_\Gamma$. Now the uniform properness of $[e]H$ means that the union $\bigcup_{b \in \mathfrak{b}^+ \cap \mathcal{L}_\Gamma} S_b$ is also bounded.

When $\mathsf{v}_0\Gamma$ is closed, the measure $\mu^{\mathrm{PS}}_{H,v}$ in Theorem 1.8 induces a locally finite Borel measure on $(\Gamma\cap H)\backslash H$, whose total measure will be called the skinning constant $\mathrm{sk}_{\Gamma,v}(H)$ of H with respect to Γ and v; this constant is positive if and only if $\Lambda_\Gamma \cap HP/P \neq \varnothing$; see (8.8).

Theorem 1.10 *Let $\Gamma < G$ be an Anosov subgroup such that $\mathsf{v}_0\Gamma \subset H\backslash G$ is closed for $\mathsf{v}_0 = [H]$. Suppose that $u_\Gamma \in \mathfrak{b}^+$ and $[e]H$ is uniformly proper. Then $\mathrm{sk}_{\Gamma,u_\Gamma}(H) < \infty$ and there exists a constant $c > 0$ such that*

$$\lim_{T\to\infty} \frac{\#(\mathsf{v}_0\Gamma \cap \mathsf{v}_0(\exp \mathfrak{b}_T^+)K)}{e^{\delta_\Gamma T}\cdot T^{(r_0-r)/2}} = c\,\mathrm{sk}_{\Gamma,u_\Gamma}(H),$$

where $\mathfrak{b}_T^+ = \{w \in \mathfrak{b}^+ : \|w\| \le T\}$ and $\delta_\Gamma = \psi_\Gamma(u_\Gamma)$.

See Theorem 9.28 for a more refined version, where u_Γ is replaced by a more general $v \in \mathfrak{b}^+ \cap \operatorname{int}\mathcal{L}_\Gamma$.

Remark 1.11 (1) When G has rank one, this is proved in Oh and Shah [32] and Mohammadi and Oh [29] for any geometrically finite group Γ under the finite skinning constant hypothesis, based on the fact that the Bowen–Margulis–Sullivan measure is finite. In the higher-rank case, the corresponding Bowen–Margulis–Sullivan measure is infinite; see Sambarino [45, Proposition 3.5] and also Lee and Oh [25, Corollary 4.9]. For this reason, the finite skinning constant hypothesis seems insufficient for our approach to work.

(2) We mention a recent work of Carvajales [7, Theorem B] for a counting result on the orbit $\mathsf{v}_0\Gamma$ where $G = \mathrm{PSL}_n(\mathbb{R})$ and $H = \mathrm{SO}(p, n-p)$ under the assumption that $\Gamma \cap H$ is finite. In this case, $\mathfrak{a} = \mathfrak{b}$ and the uniform properness of $[e]H$ easily follows, and hence Theorem 1.10 applies.

On the proofs The following mixing result for the Bowen–Margulis–Sullivan measures was proved by Thirion [48] for Schottky groups and by Sambarino [45] for Anosov groups which arise from representations of the fundamental group of a closed negatively curved Riemannian manifold, using thermodynamic formalism. The general case was recently proved by Chow and Sarkar [8], based on the fundamental work of Bridgeman, Canary, Labourie and Sambarino [4]:

Theorem 1.12 *For any $u \in \operatorname{int}\mathcal{L}_\Gamma$ and any $f_1,\ f_2 \in C_c(\Gamma\backslash G)^M$,*

$$\lim_{t\to\infty} t^{(r-1)/2} \int_{\Gamma\backslash G} f_1(x)\, f_2(x\exp(tu))\, dm_u^{\mathrm{BMS}}(x) = \kappa_u \cdot m_u^{\mathrm{BMS}}(f_1)\, m_u^{\mathrm{BMS}}(f_2),$$

where m_u^{BMS} is the BMS–measure associated to u; see (3.5) *and* (7.11).

This theorem is known only for Zariski dense Anosov subgroups with respect to a minimal parabolic subgroup, and this is the main reason we restrict our attention to this class of groups in this paper.

Remark 1.13 In fact, Chow and Sarkar proved in [8] that a version of this theorem holds for general $f_1, f_2 \in C_c(\Gamma\backslash G)$ which are not necessarily M–invariant, provided the right-hand side is replaced by the sum $\kappa_u \sum_Y m_u^{\mathrm{BMS}}|_Y(f_1)\, m_u^{\mathrm{BMS}}|_Y(f_2)$, where the sum is taken over all P°–minimal subsets Y of $\Gamma\backslash G$. Given this result, the M–invariance condition in Theorems 1.6, 1.7 and 1.8 should not be necessary with an appropriate modification of the main terms.

Using the product structures of the Haar measure dx and $dm_u^{\mathrm{BMS}}(x)$, one can deduce mixing for one measure from that of the other via the study of transversal intersections. This observation is originally due to Roblin [42] in the case of the unit tangent bundle of a rank one locally symmetric manifold, and has been extended and utilized in Mohammadi and Oh [29] and Oh and Shah [32] to the frame bundle. This study leads us to generalize the definition of the family of Burger–Roblin measures m_u^{BR} and $m_u^{\mathrm{BR}_*}$ for $u \in \operatorname{int}\mathcal{L}_\Gamma$, which turn out to control the asymptotic behavior of matrix coefficients as in Theorem 1.6 in a quite similar way as in the rank one case.

As in Duke, Rudnick and Sarnak [10] and Eskin and McMullen [11] (also as in [32] and [29]), passing from Theorem 1.6 to Theorems 1.8 and 1.10 requires the following equidistribution statement for translates of H–orbits; more precisely, we need Proposition 8.11. The idea of using mixing in the equidistribution and counting problem goes back to Margulis [27]:

Proposition 1.14 *For any $v \in \mathfrak{b}^+ \cap \operatorname{int}\mathcal{L}_\Gamma$, there exists $\kappa_v > 0$ such that for any $f \in C_c(\Gamma\backslash G)^M$ and $\phi \in C_c(H)^{H\cap M}$,*

$$\lim_{t\to\infty} t^{(r-1)/2} e^{(2\rho-\Theta)(tv)} \int_H f([h]\exp(tv))\phi(h)\, dh = \kappa_v \cdot \mu_{H,v}^{\mathrm{PS}}(\phi) m_{\mathrm{i}(v)}^{\mathrm{BR}}(f),$$

where dh denotes the Haar measure on H, and $\Theta \in \mathfrak{a}^*$ is given by $\Theta(w) = \langle \nabla\psi_\Gamma(v), w\rangle$ as in (8.3).

When Γ is a lattice or when G is of rank one, the equidistribution for the translates

$$\int_H f([h]\exp b_i)\phi(h)\, dh \tag{1.15}$$

is known for a general sequence $b_i \to \infty$ in $\mathfrak{b}^+$; see [11, Theorem 1.2] and [32].

In our setting of a general Anosov subgroup, Proposition 1.14 (or Proposition 8.11) applies only for those sequences b_i from the same fixed direction. Such lack of uniformity in (1.15) makes it difficult to obtain the main terms of the counting functions in Theorems 1.8 and 1.10 directly from Proposition 1.14. A similar issue as this was treated in [45]. This difficulty in our setting is overcome by introducing a norm $|\cdot|$ on $\mathfrak{a}$ induced by an inner product with respect to which v and $\ker\Theta$ are orthogonal to each other, where Θ is a linear form on $\mathfrak{a}$ tangent to the growth indicator function at v. With respect to such a norm, the set of elements in the cone $\mathcal{L}_\Gamma \cap \mathfrak{b}^+$ of norm at most T can be expressed as the union $\bigcup_{w\in\ker\Theta} R_T(w)$, where $R_T(w)$ is an *interval* of the form $\big[t_w, \frac{1}{2}(-|w|^2 + \sqrt{|w|^4 + 4T^2})\big]$ for some $t_w \geq 0$. By analyzing an appropriate integral over the interval $R_T(w)$ (Lemma 9.4), we then get the desired main term as in Proposition 9.10.

Organization We start by reviewing some basic notions in Section 2, including higher-rank analogues of Patterson–Sullivan measures as defined by Quint [37]. Section 3 introduces generalized BMS–measures; in particular, higher-rank versions of Burger–Roblin measures are defined. The product structure of these measures is discussed in Section 4. We then deduce equidistribution of translates of PS measures on horospheres from local mixing in Section 5. This is then used in Section 6 to show mixing for the Haar measure and equidistribution of translates of Lebesgue measures on maximal horospheres. Properties of the main types of discrete subgroups we study are discussed in Section 7. The remainder of the paper is mainly devoted to proving the claimed counting statements. As a first step towards this, we prove equidistribution of translates of orbits of symmetric subgroups in $\Gamma\backslash G$ in Section 8. These equidistribution statements are combined with the strong wavefront property in Section 9 to give the various counting results.

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2 (Γ, ψ)–Patterson–Sullivan measures

Let G be a connected, semisimple real algebraic group, and $\Gamma < G$ a Zariski dense discrete subgroup. In this section, we review the notion of (Γ, ψ)–Patterson–Sullivan

measures associated to a certain class of linear forms ψ on $\mathfrak{a}$, as constructed by Quint in [37]. We present these measures as analogously as possible to the Patterson–Sullivan measures on the limit set of Γ in the rank one case.

We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Fixing a left G–invariant and right K–invariant Riemannian metric on G induces a Weyl-group invariant inner product and corresponding norm on $\mathfrak{a}$, which we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The choice of this Riemannian metric induces a G–invariant metric $d(\cdot, \cdot)$ on G/K. The identity coset $[e]$ in G/K is denoted by o.

Let $A := \exp \mathfrak{a}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we set

$$N = N^-$$

to be the maximal contracting horospherical subgroup for A: for an element a in the interior of A^+,

$$N^- = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}. \tag{2.1}$$

Note that $\log(N)$ is the sum of all positive root subspaces for our choice of $\mathfrak{a}^+$. Similarly, we will also need to consider the maximal expanding horospherical subgroup

$$N^+ := \{g \in G : a^n g a^{-n} \to e \text{ as } n \to +\infty\}. \tag{2.2}$$

We set

$$P^+ = MAN^+ \quad \text{and} \quad P = MAN;$$

they are minimal parabolic subgroups of G. The quotient

$$\mathcal{F} = G/P$$

is known as the Furstenberg boundary of G, and is isomorphic to K/M.

Definition 2.3 (Busemann function) The Iwasawa cocycle $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$, $\exp \sigma(g, \xi)$ is the A–component of gk in the KAN decomposition, where $\xi = [k] \in K/M$:

$$gk \in K \exp(\sigma(g, \xi)) N.$$

The Busemann function $\beta : \mathcal{F} \times G/K \times G/K \to \mathfrak{a}$ is now defined as follows: for $\xi \in \mathcal{F}$ and $[g], [h] \in G/K$,

$$\beta_\xi([g], [h]) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi).$$

Observe that the Busemann function is continuous in all three variables. To ease notation, we will write $\beta_\xi(g,h)=\beta_\xi([g],[h])$. The following identities will be used throughout the article:

$$\begin{aligned}\beta_\xi(g,h)+\beta_\xi(h,q)&=\beta_\xi(g,q),\\ \beta_{g\xi}(gh,gq)&=\beta_\xi(h,q),\\ \beta_\xi(e,g)&=-\sigma(g^{-1},\xi).\end{aligned}\tag{2.4}$$

Geometrically, if $\xi=[k]\in K/M$, then for any unit vector $u\in\mathfrak{a}^+$,

$$\langle\beta_\xi(g,h),u\rangle=\lim_{t\to+\infty} d([g],\xi_t)-d([h],\xi_t),$$

where $\xi_t=k\exp(tu)o\in G/K$.

Definition 2.5 (conformal measures and densities) Given $\psi\in\mathfrak{a}^*$ and a closed subgroup $\Delta<G$, a Borel probability measure ν_ψ on $\mathcal{F}$ is called a (Δ,ψ)–conformal measure if, for any $\gamma\in\Delta$ and $\xi\in\mathcal{F}$,

$$\frac{d\gamma_*\nu_\psi}{d\nu_\psi}(\xi)=e^{\psi(\beta_\xi(e,\gamma))},\tag{2.6}$$

where $\gamma_*\nu_\psi(Q)=\nu_\psi(\gamma^{-1}Q)$ for any Borel subset $Q\subset\mathcal{F}$.

Definition 2.7 (Lebesgue measure) Let m_o denote the K–invariant probability measure on $\mathcal{F}$, and ρ denote the half sum of all positive roots with respect to $\mathfrak{a}^+$. Then, using the decomposition of the Haar measure in the KAN coordinates, one can check the following (cf [41, Proposition 3.3]): if m_o is a $(G,2\rho)$–conformal measure, ie for any $g\in G$ and $\xi\in\mathcal{F}$,

$$\frac{dg_*m_o}{dm_o}(\xi)=e^{2\rho(\beta_\xi(e,g))}.\tag{2.8}$$

Limit set and limit cone

Definition 2.9 (limit set) The limit set Λ_Γ of Γ is defined to be the set of all points $x\in\mathcal{F}$ such that the Dirac measure δ_x is a limit point (in the space of Borel probability measures on $\mathcal{F}$) of $\{\gamma_*m_o:\gamma\in\Gamma\}$.

We refer to [25, Lemma 2.13] for an alternative definition. Benoist showed that Λ_Γ is the minimal Γ–invariant closed subset of $\mathcal{F}$. Moreover, Λ_Γ is Zariski dense in $\mathcal{F}$ [1, Section 3.6].

An element of G is called elliptic if it is contained in a compact subgroup, and hyperbolic if it is conjugate to an element of A^+. Any $g \in G$ can be written as the commuting product

$$g = g_h g_e g_u, \tag{2.10}$$

where g_h is hyperbolic, g_e is elliptic and g_u is unipotent. An element $g \in G$ is called *loxodromic* if g_h is conjugate to an element of $\operatorname{int} A^+$.

Lemma 2.11 *For any open subset $U \subset \mathcal{F}$ with $U \cap \Lambda_\Gamma \neq \varnothing$, $U \cap \Lambda_\Gamma$ is not contained in any smooth submanifold of $\mathcal{F}$ of smaller dimension.*

Proof This is proved in [51] when G has rank one, and our proof is similar. Suppose that there exists an open subset $U \subset \mathcal{F}$ such that $U \cap \Lambda_\Gamma \neq \varnothing$ is contained in a smooth submanifold S of $\mathcal{F}$ of smaller dimension. Since Γ is Zariski dense, the set of attracting fixed points of loxodromic elements of Γ is dense in Λ_Γ. Hence there exists a loxodromic element $\gamma \in \Gamma$ whose attracting fixed point is contained in $U \cap \Lambda_\Gamma$. We may assume without loss of generality that $\gamma = am$, where $m \in M$ and $a \in \operatorname{int} A^+$, and hence $[e] = P \in \mathcal{F}$ is the attracting fixed point of γ. Since $\mathfrak{n}^+ := \operatorname{Lie} N^+$ is nilpotent, the map $\mathfrak{n}^+ \to G/P$ given by $x \mapsto \exp(x)[e]$ is algebraic and its image $N^+[e]$ is Zariski open and dense in G/P for $[e] = P$. Therefore $N^+[e] \cap \Lambda_\Gamma$ is Zariski dense in $N^+[e]$. On the other hand, since U is a neighborhood of $[e]$, we may assume without loss of generality that $U \subset N^+[e]$ by replacing U by a smaller open neighborhood of $[e]$ if necessary. We now claim that the hypothesis that $U \cap \Lambda_\Gamma \subset S$ implies that $N^+[e] \cap \Lambda_\Gamma$ cannot be Zariski dense in $N^+[e]$, which then yields a desired contradiction.

Choose a basis of $\mathfrak{n}^+$ consisting of eigenvectors of Ad_a, and for $x \in \mathfrak{n}^+$, we write $x = (x_1, \dots, x_d)$ for the coordinates with respect to this basis. It follows that there exist $0 < c_i < 1$, $i = 1, \dots, d$ such that

$$\operatorname{Ad}_a x = (c_1 x_1, \dots, c_d x_d). \tag{2.12}$$

Choose $\ell \in \mathbb{N}$ such that $c_1 > \max_{1 \le i \le d} c_i^{\ell+1}$. By the implicit function theorem, after shrinking U and rearranging the indices if necessary, we may assume that $U \cap S = \{[\exp(x)] \in U : x_1 = f(x_2, \dots, x_d)\}$ for some smooth function f. Let p be the Taylor polynomial of f of degree ℓ. Then, by shrinking U further, there exists $C > 0$ such that for all $x \in U$,

$$|f(x_2, \dots, x_d) - p(x_2, \dots, x_d)| \le C \|(x_2, \dots, x_d)\|^{\ell+1}. \tag{2.13}$$

(Here $\|\cdot\|$ denotes the Euclidean norm on $\mathfrak{n}^+$.) Since the action of Ad_a on the polynomial ring $\mathbb{R}[\mathfrak{n}^+]$ is diagonalizable, we can write

$$x_1 - p(x_2, \dots, x_d) = \sum_{i=1}^{k} p_i(x), \tag{2.14}$$

where $p_i \in \mathbb{R}[\mathfrak{n}^+]$ are nonzero polynomials such that $p_i(\mathrm{Ad}_a\, x) = \beta_i \cdot p_i(x)$, where $1 > \beta_1 > \cdots > \beta_k > 0$. Note that $\beta_1 \geq c_1$, due to the presence of x_1 in (2.14). Since $U \cap \Lambda_\Gamma \subset S$, combining (2.13) and (2.14), we conclude

$$\left| \sum_{i=1}^{k} p_i(x) \right| \leq C \|(x_2, \dots, x_d)\|^{\ell+1} \quad \text{whenever } [\exp x] \in U \cap \Lambda_\Gamma. \tag{2.15}$$

Now let $x \in \mathfrak{n}^+$ be such that $[\exp x] \in \Lambda_\Gamma$. Since $[e] \in U \cap \Lambda_\Gamma$ and $\mathrm{Ad}_{\gamma^n} x \to 0$ as $n \to \infty$, we have $[\exp \mathrm{Ad}_{\gamma^n} x] \in U \cap \Lambda_\Gamma$ for all sufficiently large n. Applying (2.12) and (2.15), we obtain

$$\left| \sum_{i=1}^{k} p_i(\mathrm{Ad}_{\gamma^n} x) \right| = \left| \sum_{i=1}^{k} \beta_i^n \cdot p_i(\mathrm{Ad}_{m^n} x) \right| \leq C \big(\max_{1 \leq i \leq d} c_i^{\ell+1} \big)^n \|\mathrm{Ad}_{m^n} x\|^{\ell+1}.$$

Therefore, by dividing by β^n,

$$\left| p_1(\mathrm{Ad}_{m^n} x) + \sum_{i=2}^{k} \left(\frac{\beta_i}{\beta_1} \right)^n p_i(\mathrm{Ad}_{m^n} x) \right| \leq C \left(\frac{\max_{1 \leq i \leq d} c_i^{\ell+1}}{\beta_1} \right)^n \|\mathrm{Ad}_{m^n} x\|^{\ell+1}.$$

Since M is a compact subgroup, it follows that $m^{n_i} \to e$ for some sequence $n_i \to \infty$. Taking the limit along this subsequence yields $p_1(x) = 0$. This shows that

$$\{x \in \mathfrak{n}^+ : [\exp x] \in \Lambda_\Gamma\} \subset \{x \in \mathfrak{n}^+ : p_1(x) = 0\},$$

implying that $N^+[e] \cap \Lambda_\Gamma$ is not Zariski dense in $N^+[e]$. This finishes the proof. □

We remark that a weaker version of this lemma was proved for $G = \mathrm{SL}_n(\mathbb{C})$ by Cantat; see [18, Section 6].

Definition 2.16 (Cartan projection) The Cartan projection $\mu : G \to \mathfrak{a}^+$ is defined as follows: for each $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that

$$g \in K \exp(\mu(g)) K.$$

The Jordan projection of g is defined as $\lambda(g) \in \mathfrak{a}^+$, where $\exp \lambda(g)$ is the element of A^+ conjugate to g_h, where g_h is as in (2.10).

Definition 2.17 (limit cone) The *limit cone* $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ of Γ is defined as the smallest closed cone containing the *Jordan projection* of Γ.

Quint showed the following.

Theorem 2.18 [36, Theorem IV.2.2] *The growth indicator function ψ_Γ, defined in* (1.5), *is concave, upper semicontinuous, and satisfies*

$$\mathcal{L}_\Gamma = \{u \in \mathfrak{a}^+ : \psi_\Gamma(u) > -\infty\}.$$

Moreover, $\psi_\Gamma(u)$ is nonnegative on $\mathcal{L}_\Gamma$ and positive on $\operatorname{int}\mathcal{L}_\Gamma$.

Proposition 2.19 *Let m be a locally finite Borel measure on $\Gamma\backslash G$ and $\mathcal{C} \subset \mathfrak{a}^+$ a closed cone with $\mathcal{L}_\Gamma \subset \operatorname{int}\mathcal{C}$. For any $f_1, f_2 \in C_c(\Gamma\backslash G)$, there exists $t_0 > 0$ such that for all $v \in \mathfrak{a}^+ - \mathcal{C}$ with $\|v\| \geq t_0$,*

$$\int_{\Gamma\backslash G} f_1(x\exp(v))\, f_2(x)\, d\mathsf{m}(x) = 0.$$

In particular, if $u \in \mathfrak{a}^+ - \mathcal{L}_\Gamma$, then $\int_{\Gamma\backslash G} f_1(x\exp(tu))\, f_2(x)\, d\mathsf{m}(x) = 0$ for all $t \gg 1$ large enough.

Proof It suffices to check that for any given compact subset $L \subset G$ such that $L = L^{-1}$, we have

$$L\exp(-v) \cap \Gamma L = \varnothing$$

for all sufficiently large $v \in \mathfrak{a}^+ - \mathcal{C}$.

Suppose that there exist a compact subset $L \subset G$, sequences $\ell_n, \ell_n' \in L$, $\gamma_n \in \Gamma$, and $v_n \in \mathfrak{a} - \mathcal{C}$ with $\|v_n\| = t_n \to \infty$ such that

$$\ell_n \exp(-v_n) = \gamma_n \ell_n'.$$

We may assume that v_n/t_n converges to some unit vector $v \in (\mathfrak{a}^+ - \operatorname{int}\mathcal{C})$; hence $v \notin \mathcal{L}_\Gamma$. By [1, Lemma 4.6], there exists a compact subset $M = M(L)$ of $\mathfrak{a}$ such that for all $g \in G$,

$$\mu(LgL) \subset \mu(g) + M.$$

Note that $\mathcal{L}_\Gamma$ is equal to the asymptotic cone of $\mu(\Gamma)$ by [1, Theorem 1.2]. Since $v \notin \mathcal{L}_\Gamma$, we can find an open cone $\mathcal{D}$ containing v such that $\mathcal{D} \cap \mu(\Gamma)$ is finite. Then

$$\mu(\gamma_n^{-1}) = \mu(\ell_n' \exp(v_n)\ell_n^{-1}) \in \mu(\exp(v_n)) + M.$$

As $\mathcal{D}$ is open, there exists n_0 such that $(1/t_n)(v_n + M) \subset \mathcal{D}$ for all $n \geq n_0$. Since $\mu(\exp(v_n)) + M = t_n(v_n/t_n + (1/t_n)M)$, we conclude $\mu(\gamma_n^{-1}) \in \mathcal{D}$ for all $n \geq n_0$. This yields a contradiction.

The second claim follows from the first one as we can find a closed cone $\mathcal{C}$ such that $u \notin \mathcal{C}$ and $\mathcal{L}_\Gamma \subset \operatorname{int}\mathcal{C}$. □

Set

$$D_\Gamma := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma \text{ on } \mathfrak{a}^+\},$$

which is a nonempty set [38, Section 4.1]. An element $\psi \in D_\Gamma$ is said to be tangent to ψ_Γ at $u \in \mathfrak{a}$ if $\psi(u) = \psi_\Gamma(u)$. The following collection of linear forms is of particular importance:

$$D_\Gamma^\star := \{\psi \in D_\Gamma : \psi \text{ is tangent to } \psi_\Gamma \text{ at some } u \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+\}. \tag{2.20}$$

Definition 2.21 (Patterson–Sullivan measures) For $\psi \in \mathfrak{a}^*$, a (Γ, ψ)–conformal measure supported on Λ_Γ will be called a (Γ, ψ)–PS measure.

Generalizing the work of Patterson [34] and Sullivan [47], Quint [37] constructed a (Γ, ψ)–PS measure for every $\psi \in D_\Gamma^\star$.

Maximal growth direction Since ψ_Γ is concave, upper semicontinuous, and the unit norm ball in $\mathfrak{a}$ is strictly convex, there exists a unique unit vector $u_\Gamma \in \mathcal{L}_\Gamma$ (called the maximal growth direction) such that

$$\delta_\Gamma := \max_{u \in \mathfrak{a}^+, \|u\|=1} \psi_\Gamma(u) = \psi_\Gamma(u_\Gamma). \tag{2.22}$$

Note that u_Γ must be stabilized by the *opposition involution*; see Definition 3.1.

Example 2.23 If $G = \mathrm{PSL}_3(\mathbb{R})$, then $u_\Gamma = \operatorname{diag}(1/\sqrt{2}, 0, -1/\sqrt{2})$ for any Zariski dense subgroup Γ.

Uniqueness of tangent forms.

The claims (2) and (3) of the following lemma follow from [38, Section 4.1]; see also [43, Lemma 4.8].

Lemma 2.24 (1) *For any $u \in \operatorname{int} \mathcal{L}_\Gamma$, there exists a linear form $\psi_u \in D_\Gamma^\star$ tangent to ψ_Γ at u.*

(2) *For any $u \in \operatorname{int} \mathcal{L}_\Gamma$ at which ψ_Γ is differentiable, there exists a unique linear form $\psi_u \in D_\Gamma^\star$ tangent to ψ_Γ at u, and it is given by*

$$\psi_u(\cdot) = \langle (\nabla \psi_\Gamma)(u), \cdot \rangle = D_u \psi_\Gamma(\cdot).$$

(3) *If $u_\Gamma \in \operatorname{int} \mathcal{L}_\Gamma$ and ψ_Γ is differentiable at u_Γ, then ψ_{u_Γ} is given by*

$$\psi_{u_\Gamma}(\cdot) = \delta_\Gamma \langle u_\Gamma, \cdot \rangle.$$

Proof Let $P \subset \mathfrak{a}$ be an affine hyperplane such that $P \cap \mathfrak{a}^+$ is an $(r-1)$–simplex and $P \cap \mathcal{L}_\Gamma$ is a bounded convex subset of $P \simeq \mathbb{R}^{r-1}$. Since $P \cap \mathcal{L}_\Gamma$ is convex and $\psi_\Gamma : \mathfrak{a} \to \mathbb{R}$ is concave, the set

$$S := \{(x, y) \in (P \cap \mathcal{L}_\Gamma) \times \mathbb{R} : 0 \le y \le \psi_\Gamma(x)\}$$

is convex. Since $\mathbb{R}(P \cap \operatorname{int} \mathcal{L}_\Gamma) \supset \operatorname{int} \mathcal{L}_\Gamma$, it suffices to prove (1) for $u \in P \cap \operatorname{int} \mathcal{L}_\Gamma$. Since $(u, \psi_\Gamma(u)) \in \partial S$, the supporting hyperplane theorem implies that there exists a hyperplane $C \subset P \times \mathbb{R}$ passing through $(u, \psi_\Gamma(u))$ such that the interior of S is contained in a connected component of $P \times \mathbb{R} - C$. As $u \in \operatorname{int} \mathcal{L}_\Gamma$, such a hyperplane C must be the graph of a function. We may therefore write $C = \{(x, \varphi(x)) \in P \times \mathbb{R}\}$ for some affine map $\varphi : P \to \mathbb{R}$ satisfying $\varphi(x) \ge \psi_\Gamma(x)$ for all $x \in P \cap \mathcal{L}_\Gamma$. Consider the unique linear form in $\mathfrak{a}^*$ which extends φ, which we also denote by φ by abuse of notation. Since $\varphi(x) \ge \psi_\Gamma(x)$ for all $x \in P \cap \mathcal{L}_\Gamma$ and $\mathcal{L}_\Gamma$ has nonempty interior, it follows that $\varphi \ge \psi_\Gamma$. Since $\varphi(u) = \psi_\Gamma(u)$, this proves (1).

To prove (2), define $\psi_u(\cdot) := \langle (\nabla \psi_\Gamma)(u), \cdot \rangle$. By differentiating $\psi_\Gamma(tu) = t\psi_\Gamma(u)$ with respect to t, we get by the chain rule that

$$\langle \nabla \psi_\Gamma(tu), u \rangle = \psi_\Gamma(u). \tag{2.25}$$

Hence $\psi_u(u) = \psi_\Gamma(u)$ by plugging $t = 1$. Let $\mathfrak{b}$ be a vector space such that $\mathfrak{a} = \mathbb{R}u \oplus \mathfrak{b}$, and let $v \in \mathfrak{b}$ be arbitrary. Consider the closed interval $I = \{s \in \mathbb{R} : u + sv \in \mathcal{L}_\Gamma\}$ and let $f(s) := \psi_\Gamma(u + sv)$. Note that $f : I \to \mathbb{R}$ is concave, differentiable at $s = 0$, and $f'(0) = \langle \nabla \psi_\Gamma(u), v \rangle$. Hence, using (2.25),

$$f(s) \le \psi_\Gamma(u) + s\langle \nabla \psi_\Gamma(u), v \rangle = \langle \nabla \psi_\Gamma(u), u + sv \rangle = \psi_u(u + sv).$$

As $v \in \mathfrak{b}$ is arbitrary, this implies $\psi_u \ge \psi_\Gamma$. Hence $\psi_u \in D_\Gamma^\star$.

To show the uniqueness, suppose that $\psi \in D_\Gamma^\star$ is tangent to ψ_Γ at u. Let $v \in \mathfrak{b}$ be arbitrary. Define $g : I \to \mathbb{R}$ by $g(s) := \psi(u + sv)$. Then $g \ge f$ and $g(0) = f(0)$. Since f is a concave function on an interval I and differentiable at $0 \in S$, it follows that $g(s) = f(0) + sf'(0)$. Since $f(0) = \psi_\Gamma(u)$ and $f'(0) = \langle \nabla \psi_\Gamma(u), v \rangle$, it follows that

$$\psi(u + sv) = \psi_\Gamma(u) + s\langle \nabla \psi_\Gamma(u), v \rangle = \langle \nabla \psi_\Gamma(u), u + sv \rangle.$$

This proves the uniqueness.

Next we claim that $\nabla \psi_\Gamma(u_\Gamma) = cu_\Gamma$ for some $c \ne 0$. Consider a curve $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{S}^{r-1} \cap \mathfrak{a}^+$ such that $\alpha(0) = u_\Gamma$. By definition of u_Γ, $s \mapsto \psi_\Gamma(\alpha(s))$ achieves its

maximum at $s = 0$. Hence, its derivative at $s = 0$ vanishes, and $\langle \nabla \psi_\Gamma(u_\Gamma), \alpha'(0) \rangle = 0$. Since $\alpha'(0) \in \mathrm{T}\mathbb{S}^{r-1}$ can be arbitrary, $\nabla \psi_\Gamma(u_\Gamma)$ is parallel to u_Γ. Combining this with (2.25), the claim follows. Since $\psi_\Gamma(u_\Gamma) = \delta_\Gamma$, we have $c = \delta_\Gamma$. This completes the proof of the lemma. □

3 Generalized BMS–measures

Using the notation introduced in Section 2, given a pair of Γ–conformal measures on $\mathcal{F}$, we now define an MA–invariant locally finite Borel measure on $\Gamma\backslash G$, which we call a generalized BMS–measure. Haar measures, BR–measures, and BMS–measures are all constructed in this way.

Definition 3.1 (opposition involution) Denote by $w_0 \in K$ a representative of the unique element of the Weyl group $N_K(A)/M$ such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The opposition involution $\mathrm{i}\colon \mathfrak{a} \to \mathfrak{a}$ is defined by

$$\mathrm{i}(u) = -\mathrm{Ad}_{w_0}(u).$$

Note that for all $g \in G$, we have

$$\lambda(g^{-1}) = \mathrm{i}(\lambda(g)), \quad \mu(g^{-1}) = \mathrm{i}(\mu(g)), \quad \mathrm{i}(\mathfrak{a}^+) = \mathfrak{a}^+, \quad \psi_\Gamma \circ \mathrm{i} = \psi_\Gamma.$$

In particular, i preserves $\operatorname{int} \mathcal{L}_\Gamma$.

Note that for all rank one groups, i is the identity map.

Example When $G = \mathrm{PSL}_d(\mathbb{R})$, with the Riemannian metric given by the inner product $\langle X, Y \rangle = \mathrm{tr}(XY^t)$, we have

$$\begin{aligned} \mathfrak{a} &= \{\mathrm{diag}(t_1, \dots, t_d) : t_1 + \cdots + t_d = 0\}, \\ \mathfrak{a}^+ &= \{\mathrm{diag}(t_1, \dots, t_d) \in \mathfrak{a} : t_1 \geq \cdots \geq t_d\}, \end{aligned} \tag{3.2}$$

and $\langle \cdot, \cdot \rangle \colon \mathfrak{a} \to \mathbb{R}$ is given by $\langle X, Y \rangle = \mathrm{tr}(XY)$. The opposition involution is given by

$$\mathrm{i}(\mathrm{diag}(t_1, \dots, t_d)) = \mathrm{diag}(-t_d, \dots, -t_1).$$

For each $g \in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.$$

Observe that $(gm)^{\pm} = g^{\pm}$ for all $g \in G$, $m \in M$; we may thus also view the above as maps from G/M to $\mathcal{F}$. Hence, for the identity element $e \in G$, $e^+ = [P]$, $e^- = [w_0 P]$ and $g^{\pm} = g(e^{\pm})$ for any $g \in G$. Let $\mathcal{F}^{(2)}$ denote the unique open G–orbit in $\mathcal{F} \times \mathcal{F}$:

$$\mathcal{F}^{(2)} = G.(e^+, e^-) = \{(gP, gw_0P) \in G/P \times G/P : g \in G\}.$$

Example If $G = \mathrm{PGL}_d(\mathbb{R})$, $\mathcal{F}$ may be identified with the space of complete flags $\{\{V_1 \subset \cdots \subset V_{d-1}\} : \dim V_i = i\}$ in $\mathbb{R}^d$; $\mathcal{F}^{(2)}$ is then identified with the set of pairs of flags $(\{V_1 \subset \cdots \subset V_{d-1}\}, \{W_1 \subset \cdots \subset W_{d-1}\})$ in general position, ie $V_i \oplus W_{d-i} = \mathbb{R}^d$ for all $1 \le i \le d-1$.

Definition 3.3 (Hopf parametrization) The homeomorphism $G/M \to \mathcal{F}^{(2)} \times \mathfrak{a}$ given by $gM \mapsto (g^+, g^-, b = \beta_{g^-}(e, g))$ is called the Hopf parametrization of G/M.

Example Using the linear fractional transformation action of $G = \mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^2 \cup \widehat{\mathbb{R}}$, we have $P^- = \mathrm{Stab}(\infty)$ and $P^+ = \mathrm{Stab}(0)$, where $P = P^-$ and P^+ are the upper and lower triangular subgroups of G, respectively. Hence

$$g^+ = gP = g(\infty) \quad \text{and} \quad g^- = gw_0P = g(0) \in \partial\mathbb{H}^2 = G/P,$$

where $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

We will make use of the following identities, which are all straightforward:

$$\textbf{(3.4)} \qquad \begin{aligned} \beta_{g^-}(e, g) &= -\sigma(g^{-1}, kw_0P) = -\mathrm{i}(\log a) && \text{if } g = kan \in KAN^+, \\ \beta_{g^+}(e, g) &= \ -\sigma(g^{-1}, kP) \ = \log a && \text{if } g = kah \in KAN^-. \end{aligned}$$

In particular, for $a \in A$,

$$\beta_{e^+}(e, a) + \mathrm{i}(\beta_{e^-}(e, a)) = 0.$$

The generalized BMS–measure m_{ν_1,ν_2} Fix a pair of linear forms $\psi_1, \psi_2 \in \mathfrak{a}^*$. Let $\nu_1 = \nu_{\psi_1}$ and $\nu_2 = \nu_{\psi_2}$ be, respectively, (Γ, ψ_1)– and (Γ, ψ_2)–conformal measures on $\mathcal{F}$. Using the Hopf parametrization, define a locally finite Borel measure $\widetilde{m}_{\nu_1,\nu_2}$ on G/M as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times \mathfrak{a}$,

$$\textbf{(3.5)} \qquad d\widetilde{m}_{\nu_1,\nu_2}(g) = e^{\psi_1(\beta_{g^+}(e,g)) + \psi_2(\beta_{g^-}(e,g))}\, d\nu_1(g^+)\, d\nu_2(g^-)\, db,$$

where $db = d\ell(b)$ is the Lebesgue measure on $\mathfrak{a}$.

Lemma 3.6 *The measure $\widetilde{m}_{\nu_1,\nu_2}$ is left Γ–invariant and right A–quasi-invariant: for all $a \in A$,*

$$a_*\widetilde{m}_{\nu_1,\nu_2} = e^{(-\psi_1 + \psi_2 \circ \mathrm{i})(\log a)}\, \widetilde{m}_{\nu_1,\nu_2}.$$

Proof Let $\gamma \in \Gamma$ and $g \in G$ be arbitrary. Note

$$\beta_{\gamma g^{\pm}}(e, \gamma g) = \beta_{\gamma g^{\pm}}(e, \gamma) + \beta_{\gamma g^{\pm}}(\gamma, \gamma g) = \beta_{\gamma g^{\pm}}(e, \gamma) + \beta_{g^{\pm}}(e, g).$$

Recall the conformality of the measures ν_1 and ν_2:

$$d\nu_1(\gamma g^+) = e^{\psi_1(\beta_{g^+}(e,\gamma^{-1}))} d\nu_1(g^+) \quad \text{and} \quad d\nu_2(\gamma g^-) = e^{\psi_2(\beta_{g^-}(e,\gamma^{-1}))} d\nu_2(g^-).$$

Combining these, we have

$$\begin{aligned} &d\tilde{m}_{\nu_1,\nu_2}(\gamma g) \\ &\quad = e^{\psi_1(\beta_{\gamma g^+}(e,\gamma g)) + \psi_2(\beta_{\gamma g^-}(e,\gamma g))}\, d\nu_1(\gamma g^+)\, d\nu_2(\gamma g^-)\, d\ell(b + \beta_{\gamma g^-}(e,\gamma)) \\ &\quad = e^{\psi_1(\beta_{g^+}(e,g)) + \psi_2(\beta_{g^-}(e,g))}\, d\nu_1(g^+)\, d\nu_2(g^-)\, d\ell(b) \\ &\quad = d\tilde{m}_{\nu_1,\nu_2}(g). \end{aligned}$$

Let $a \in A$. By the identities (2.4) and (3.4),

$$\begin{aligned} \beta_{g^{\pm}}(e, ga) = \beta_{g^{\pm}}(e, g) + \beta_{g^{\pm}}(g, ga) &= \beta_{g^{\pm}}(e, g) + \beta_{e^{\pm}}(e, a), \\ ga^{\pm} &= g^{\pm}, \\ \beta_{e^+}(e, a) = -\mathrm{i}\beta_{e^-}(e, a) &= \log a. \end{aligned}$$

Combining with definition (3.5), we have

$$\begin{aligned} &d\tilde{m}_{\nu_1,\nu_2}(ga) \\ &\quad = e^{\psi_1(\beta_{g^+}(e,ga)) + \psi_2(\beta_{g^-}(e,ga))}\, d\nu_1(g^+)\, d\nu_2(g^-)\, d\ell(b + \beta_{e^-}(e,a)) \\ &\quad = e^{\psi_1(\beta_{g^+}(e,g) + \beta_{e^+}(e,a)) + \psi_2(\beta_{g^-}(e,g) + \beta_{e^-}(e,a))}\, d\nu_1(g^+)\, d\nu_2(g^-)\, d\ell(b) \\ &\quad = e^{(\psi_1 - \psi_2 \circ \mathrm{i})(\log a)} d\tilde{m}_{\nu_1,\nu_2}(g). \end{aligned}$$

This proves the claim. □

The measure $\tilde{m}_{\nu_1,\nu_2}$ gives rise to a left Γ–invariant and right M–invariant measure on G, by integrating along the fibers of $G \to G/M$ with respect to the Haar measure on M. By abuse of notation, we will also denote this measure by $\tilde{m}_{\nu_1,\nu_2}$. We denote by m_{ν_1,ν_2} the measure on $\Gamma\backslash G$ induced by $\tilde{m}_{\nu_1,\nu_2}$, and call it the generalized BMS–measure associated to the pair (ν_1, ν_2).

BMS–measure $m^{\mathrm{BMS}}_{\nu_\psi,\nu_{\psi\circ\mathrm{i}}}$ Let $\psi \in \mathfrak{a}^*$ and let ν_ψ and $\nu_{\psi\circ\mathrm{i}}$ be (Γ, ψ)– and $(\Gamma, \psi\circ\mathrm{i})$–PS measures, respectively. We set

$$m^{\mathrm{BMS}}_{\nu_\psi,\nu_{\psi\circ\mathrm{i}}} := m_{\nu_\psi,\nu_{\psi\circ\mathrm{i}}} \tag{3.7}$$

and call it the Bowen–Margulis–Sullivan measure associated to $(\nu_\psi, \nu_{\psi\circ \mathrm{i}})$. Then by Lemma 3.6, $m^{\mathrm{BMS}}_{\nu_\psi,\nu_{\psi\circ \mathrm{i}}}$ is an A–invariant measure, whose support is given by

$$\mathrm{supp}(m^{\mathrm{BMS}}_{\nu_\psi,\nu_{\psi\circ \mathrm{i}}}) = \{x \in \Gamma\backslash G : x^{\pm} \in \Lambda_\Gamma\};$$

since Λ_Γ is Γ–invariant, the condition $x^{\pm} \in \Lambda_\Gamma$ is a well-defined condition.

BR–measures $m^{\mathrm{BR}}_{\nu_\psi}$ We set

$$m^{\mathrm{BR}}_{\nu_\psi} = m_{m_o,\nu_\psi}, \tag{3.8}$$

and call it the Burger–Roblin measure associated to ν_ψ. The support of $m^{\mathrm{BR}}_{\nu_\psi}$ is given by

$$\mathrm{supp}(m^{\mathrm{BR}}_{\nu_\psi}) = \{x \in \Gamma\backslash G : x^{-} \in \Lambda_\Gamma\}.$$

Lemma 3.9 *The Burger–Roblin measure $m^{\mathrm{BR}}_{\nu_\psi}$ is right N^+–invariant.*

Proof Let $g \in G$ and $n \in N^+$. By the identities (2.4) and (3.4), we have $gn^- = g^-$, $\beta_{n^-}(e,n) = 0$ and

$$\beta_{gn^-}(e,gn) = \beta_{gn^-}(e,g) + \beta_{gn^-}(g,gn) = \beta_{gn^-}(e,g) + \beta_{n^-}(e,n) = \beta_{g^-}(e,g).$$

On the other hand, by the conformality (2.6),

$$\begin{aligned} dm_o(gn^+) &= e^{2\rho(\beta_{gn^+}(gng^{-1},e))} dm_o(g^+) \\ &= e^{2\rho(\beta_{gn^+}(gng^{-1},gn)+\beta_{gn^+}(gn,e))} dm_o(g^+) \\ &= e^{2\rho(\beta_{g^+}(e,g)-\beta_{gn^+}(e,gn))} dm_o(g^+). \end{aligned}$$

Combining these, we have

$$\begin{aligned} d\tilde{m}_{m_o,\nu_\psi}(gn) &= e^{2\rho(\beta_{gn^+}(e,gn))+\psi(\beta_{g^-}(e,gn))}\, dm_o(gn^+) d\nu_\psi(g^-) d\ell(b) \\ &= e^{2\rho(\beta_{g^+}(e,g))+\psi(\beta_{g^-}(e,g))}\, dm_o(g^+) d\nu_\psi(g^-) d\ell(b) \\ &= d\tilde{m}_{m_o,\nu_\psi}(g). \end{aligned}$$

□

Similarly, but with a different parametrization $g = (g^+, g^-, b = \beta_{g^+}(e,g))$, we define the N^-–invariant locally finite measure

$$d\tilde{m}^{\mathrm{BR}_*}_{\nu_\psi}(g) = e^{\psi(\beta_{g^+}(e,g))+2\rho(\beta_{g^-}(e,g))}\, d\nu_\psi(g^+)\, dm_o(g^-)\, db. \tag{3.10}$$

Haar measure m^{Haar} Recall that the K–invariant probability measure m_o is a conformal density for the linear form 2ρ. We denote by $dx = dm^{\mathrm{Haar}}(x)$ the generalized BMS–measure associated to the pair (m_o, m_o):

$$dx = dm^{\mathrm{Haar}} := dm_{m_o,m_o}. \tag{3.11}$$

Since m_o is a $(G, 2\rho)$–conformal measure, dm^{Haar} is G–invariant; the proofs of Lemmas 3.6 and 3.9 show that m_o is invariant under AM and $N^{\pm}$. As these subgroups generate G, the G–invariance follows.

4 Disintegration of BMS– and BR–measures along N–orbits

In this section, we fix a linear form $\psi \in D_\Gamma^\star$, a (Γ, ψ)–PS measure ν_ψ and a $(\Gamma, \psi\circ\mathrm{i})$–PS measure $\nu_{\psi\circ\mathrm{i}}$ on $\mathcal{F}$. To simplify the notation, we write

$$\nu := \nu_\psi, \quad \nu_{\mathrm{i}} := \nu_{\psi\circ\mathrm{i}}, \quad \widetilde{m}^{\mathrm{BMS}} := \widetilde{m}^{\mathrm{BMS}}_{\nu_\psi,\nu_{\psi\circ\mathrm{i}}}, \quad \widetilde{m}^{\mathrm{BR}} := \widetilde{m}^{\mathrm{BR}}_{\nu_{\psi\circ\mathrm{i}}}, \quad \widetilde{m}^{\mathrm{BR}*} := \widetilde{m}^{\mathrm{BR}*}_{\nu_\psi}.$$

4.1 PS measures on $gN^{\pm}$

We start by defining measures on $N^{\pm}$. Firstly, for $g \in G$, define $\mu^{\mathrm{PS}}_{gN^+} := \mu^{\mathrm{PS}}_{gN^+,\nu}$ and $\mu^{\mathrm{PS}}_{gN^-} := \mu^{\mathrm{PS}}_{gN^-,\nu_{\mathrm{i}}}$ on $N^{\pm}$, for $n \in N^+$ and $h \in N^-$, by the formulas

$$\begin{aligned} d\mu^{\mathrm{PS}}_{gN^+}(n) &:= e^{\psi(\beta_{(gn)^+}(e,gn))}\, d\nu((gn)^+), \\ d\mu^{\mathrm{PS}}_{gN^-}(h) &:= e^{\psi\circ\mathrm{i}(\beta_{(gh)^-}(e,gh))}\, d\nu_{\mathrm{i}}((gh)^-). \end{aligned} \tag{4.1}$$

These are left Γ–invariant; for any $\gamma \in \Gamma$ and $g \in G$,

$$\mu^{\mathrm{PS}}_{\gamma gN^\pm} = \mu^{\mathrm{PS}}_{gN^\pm}.$$

When $xN^{\pm}$ is closed in $\Gamma\backslash G$ for $x = [g] \in \Gamma\backslash G$, $\mu^{\mathrm{PS}}_{gN^\pm}$ induces a locally finite Borel measure on $\mathrm{Stab}_{N^\pm}(x)\backslash N^\pm \simeq xN^\pm$, which we will denote by $d\mu^{\mathrm{PS}}_{xN^\pm}$.

Recalling that A normalizes $N^{\pm}$, we will use the following lemma:

Lemma 4.2 *For any $g \in G$, $a \in A$ and $n_0, n \in N^+$, we have*

$$d\mu^{\mathrm{PS}}_{gN^+}(an_0na^{-1}) = e^{-\psi(\log a)} d\mu^{\mathrm{PS}}_{gan_0N^+}(n).$$

Proof By (2.4) and (3.4), we have $(gan_0na^{-1})^+ = gan_0n^+$, and

$$\begin{aligned}\beta_{gan_0n^+}(e, gan_0na^{-1}) &= \beta_{gan_0n^+}(e, gan_0n) + \beta_{gan_0n^+}(gan_0n, gan_0na^{-1})\\ &= \beta_{gan_0n^+}(e, gan_0n) + \beta_{e^+}(e, a^{-1}).\end{aligned}$$

Also note that $\beta_{e^+}(e, a^{-1}) = -\log a$. Consequently,

$$\begin{aligned}d\mu^{\mathrm{PS}}_{gN^+}(an_0na^{-1}) &= e^{\psi(\beta_{gan_0n^+}(e, gan_0na^{-1}))}\, d\nu(gan_0n^+)\\ &= e^{-\psi(\log a)} e^{\psi(\beta_{gan_0n^+}(e, gan_0n))}\, d\nu(gan_0n^+)\\ &= e^{-\psi(\log a)}\, d\mu^{\mathrm{PS}}_{gan_0N^+}(n).\end{aligned}$$

□

The measures $\mu^{\mathrm{PS}}_{gN^\pm}$ allow us to decompose the BMS–measure as follows: the product map $N^+ \times P^- \to G$ is a diffeomorphism onto a Zariski open neighborhood of e.

4.2 Product structure of BMS–measures

Given $g \in G$, the BMS–measure $\widetilde{m}^{\mathrm{BMS}}$ can be disintegrated in gN^+P^- as follows.

Lemma 4.3 *For $g \in G$, $f \in C_c(gN^+P^-)$ and $nham \in N^+N^-AM$,*

$$\widetilde{m}^{\mathrm{BMS}}(f) = \int_{N^+} \left(\int_{N^-AM} f(gnham)\, dm\, da\, d\mu^{\mathrm{PS}}_{gnN^-}(h) \right) d\mu^{\mathrm{PS}}_{gN^+}(n).$$

Proof By the identities (2.4) and (3.4), we have $gnha^- = gnh^-$, $gnha^+ = gn^+$ and

$$\begin{aligned}\beta_{gnh^\pm}(e, gnha) &= \beta_{gnh^\pm}(e, gnh) + \beta_{gnh^\pm}(gnh, gnha)\\ &= \beta_{gnh^\pm}(e, gnh) + \beta_{e^\pm}(e, a).\end{aligned}$$

Note that $\beta_{e^-}(e, a) = -\mathrm{i}\log a$ and $\beta_{e^+}(e, a) = \log a$. Hence

$$\begin{aligned}&d\widetilde{m}^{\mathrm{BMS}}(gnha)\\ &\quad= e^{\psi(\mathrm{i}(\beta_{gnh^-}(e, gnha) + \beta_{gnh^+}(e, gnha)))}\, d\nu_{\mathrm{i}}(gnh^-)\, d\nu(gn^+)\, d\ell(\beta_{gnh^-}(e, gnha))\\ &\quad= e^{\psi(\mathrm{i}\beta_{gnh^-}(e, gnh) - \log a + \beta_{gnh^+}(e, gnh) + \log a)}\\ &\qquad\qquad \times d\nu_{\mathrm{i}}(gnh^-)\, d\nu(gn^+)\, d\ell(\beta_{gnh^-}(e, gnh) + \log a)\\ &\quad= da\, d\mu^{\mathrm{PS}}_{gnN^-}(h)\, d\mu^{\mathrm{PS}}_{gN^+}(n).\end{aligned}$$

Hence for $nham \in N^+N^-AM$,

$$d\widetilde{m}^{\mathrm{BMS}}(gnham) = dm\, d\widetilde{m}^{\mathrm{BMS}}(gnha) = dm\, da\, d\mu^{\mathrm{PS}}_{gnN^-}(h) d\mu^{\mathrm{PS}}_{gN^+}(n),$$

proving the claim. □

In a similar manner, one can decompose the BMS–measure according to gP^-N^+:

Lemma 4.4 *For $g \in G$, $f \in C_c(gP^-N^+)$, and $hamn \in N^-AMN^+$,*

$$\tilde{m}^{\mathrm{BMS}}(f) = \int_{N^-AM} \left(\int_{N^+} f(ghamn)\, d\mu^{\mathrm{PS}}_{ghamN^+}(n) \right) e^{-\psi(\log a)}\, dm\, da\, d\mu^{\mathrm{PS}}_{gN^-}(h).$$

Proof For each $m \in M$, consider the change of variable $n_0 = mnm^{-1}$. Then for $hamn \in N^-AMN^+$, we have

$$\begin{aligned} d\tilde{m}^{\mathrm{BMS}}(ghamn) &= d\tilde{m}^{\mathrm{BMS}}(ghan_0m) \\ &= dm\, d\tilde{m}^{\mathrm{BMS}}(ghan_0). \end{aligned}$$

By the identity (3.4), we have $\beta_{gh^-}(e, ghan_0) = \beta_{gh^-}(e, gh) - \mathrm{i}\log a$ and

$$\begin{aligned} &\mathrm{i}(\beta_{gh^-}(e, ghan_0)) + \beta_{ghan_0^+}(e, ghan_0) \\ &\qquad = \mathrm{i}(\beta_{gh^-}(e, gh) - \mathrm{i}\log a) + \beta_{ghan_0^+}(e, ghan_0). \end{aligned}$$

It follows that

$$\begin{aligned} &d\tilde{m}^{\mathrm{BMS}}(ghan_0) \\ &\quad = e^{\psi(\beta_{ghan_0^+}(e, ghan_0))}\, dv(ghan_0^+)\, e^{-\psi(\log a)}\, da\, e^{\psi\circ\mathrm{i}(\beta_{gh^-}(e,gh))}\, dv_{\mathrm{i}}(gh^-) \\ &\quad = d\mu^{\mathrm{PS}}_{ghamN^+}(n)\, e^{-\psi(\log a)}\, da\, d\mu^{\mathrm{PS}}_{gN^-}(h). \end{aligned}$$

Hence

$$d\tilde{m}^{\mathrm{BMS}}(ghamn) = d\mu^{\mathrm{PS}}_{ghamN^+}(n)\, e^{-\psi(\log a)}\, dm\, da\, d\mu^{\mathrm{PS}}_{gN^-}(h),$$

finishing the proof. □

Define, for $ham \in N^-AM$,

$$d\mu^{\mathrm{PS}}_{gP^-}(ham) = e^{-\psi(\log a)}\, dm\, da\, d\mu^{\mathrm{PS}}_{gN^-}(h).$$

This also allows us to succinctly rewrite the decomposition in Lemma 4.4 as follows: for any $f \in C_c(gP^-N^+)$,

$$\tilde{m}^{\mathrm{BMS}}(f) = \int_{P^-} \int_{N^+} f(gpn)\, d\mu^{\mathrm{PS}}_{gpN^+}(n)\, d\mu^{\mathrm{PS}}_{gP^-}(p). \tag{4.5}$$

Lebesgue measures on $gN^{\pm}$ For $g \in G$, we note that the Haar measure on $gN^{\pm}$ can be given as follows: for $n \in N^{\pm}$,

$$d\mu^{\mathrm{Leb}}_{gN^-}(n) = e^{2\rho(\beta_{(gn)^-}(e,gn))} dm_o((gn)^-), \tag{4.6}$$

$$d\mu^{\mathrm{Leb}}_{gN^+}(n) = e^{2\rho(\beta_{(gn)^+}(e,gn))} dm_o((gn)^+). \tag{4.7}$$

Using (2.6), it can be checked that these are N^- and N^+ invariant measures respectively. Moreover, $d\mu^{\mathrm{Leb}}_{gN^{\pm}}$ does not depend on $g \in G$, so we simply write dn.

4.3 Decomposition of m^{BR}

Similarly to Lemma 4.4, the BR– and BR$_*$–measures can be decomposed in terms of the gP^-N^+ decomposition of G:

For all $f \in C_c(gP^-N^+)$,

$$\begin{aligned} \tilde{m}^{\mathrm{BR}}(f) &= \int_{P^-}\int_{N^+} f(gpn)\, dn\, d\mu^{\mathrm{PS}}_{gP^-}(p), \\ \tilde{m}^{\mathrm{BR}_*}(f) &= \int_{P^-}\int_{N^+} f(ghamn)\, d\mu^{\mathrm{PS}}_{ghamN^+}(n) e^{-2\rho(\log a)}\, dm\, da\, dh. \end{aligned} \tag{4.8}$$

We also have the following description of the BR–measures $\tilde{m}^{\mathrm{BR}}_{\nu_\psi}$ and $\tilde{m}^{\mathrm{BR}_*}_{\nu_\psi}$ on G/M.

Lemma 4.9 *For $g = k\exp(b)n \in KAN^+$ and $[g] = gM$, we have*

$$d\tilde{m}^{\mathrm{BR}}_{\nu_\psi}([g]) = e^{-(\psi\circ \mathrm{i})(b)}\, dn\, d\ell(b)\, d\tilde{\nu}_\psi(k),$$

where $\tilde{\nu}_\psi$ is given by

$$d\tilde{\nu}_\psi(k) := d\nu_\psi(k^-)\, dm. \tag{4.10}$$

For $g = k\exp(b)n \in KAN^-$ and $[g] = gM$, we have

$$d\tilde{m}^{\mathrm{BR}_*}_{\nu_\psi}([g]) = e^{\psi(b)}\, dn\, d\ell(b)\, d\tilde{\nu}_\psi(k),$$

where $\tilde{\nu}_\psi$ is given by

$$d\tilde{\nu}_\psi(k) := d\nu_\psi(k^+)\, dm.$$

Proof For $g = k\exp(b)n \in KAN^+$, we have $\beta_g{}^-(e,g) = -\mathrm{i}(b)$. Since m_o is a $(G, 2\rho)$–conformal measure, we have

$$\begin{aligned} d\tilde{m}^{\mathrm{BR}}_{\nu_\psi}([g]) &= e^{2\rho(\beta_{k\exp(b)n^+}(e,k\exp(b)n))} e^{-(\psi\circ\mathrm{i})(b)} dm_o(k\exp(b)n^+) db\, d\nu_\psi(k^-)\, dm \\ &= e^{-(\psi\circ\mathrm{i})(b)}\, dn\, db\, d\tilde{\nu}_\psi(k). \end{aligned}$$

The second statement can be proved similarly. □

5 BMS–mixing and translates of PS–measures

In this section, we fix

(1) an element $u \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+$,

(2) a linear form $\psi \in D_\Gamma^\star$ tangent to ψ_Γ at u,

(3) a (Γ, ψ)–PS measure $\nu = \nu_\psi$ on $\mathcal{F}$, and

(4) a $(\Gamma, \psi \circ \mathrm{i})$–PS measure $\nu_{\mathrm{i}} = \nu_{\psi \circ \mathrm{i}}$ on $\mathcal{F}$.

As before, we set

$$m^{\mathrm{BMS}} = m^{\mathrm{BMS}}_{\nu,\nu_{\mathrm{i}}} \quad \text{and} \quad \mu^{\mathrm{PS}}_{gN^+} = \mu^{\mathrm{PS}}_{gN^+,\nu}.$$

For all $t \geq 0$ and $v \in \ker \psi$, define

$$a(t, v) := \exp(tu + \sqrt{t}v) \in A.$$

Definition 5.1 We say that m^{BMS} satisfies the *local mixing property* if there exist functions $\Psi\colon (0,\infty) \to (0,\infty)$ and $J\colon \ker \psi \to (0,\infty)$ such that:

(1) For all $v \in \ker \psi$ and $f_1,\ f_2 \in C_c(\Gamma\backslash G)^M$,

$$\lim_{t\to\infty} \Psi(t) \int_{\Gamma\backslash G} f_1\big(x\, a(t,v)\big) f_2(x)\, dm^{\mathrm{BMS}}(x) = J(v)\, m^{\mathrm{BMS}}(f_1)\, m^{\mathrm{BMS}}(f_2). \tag{5.2}$$

(2) There exists a $C = C(f_1, f_2) > 0$ such that for all $(t, v) \in (0,\infty) \times \ker \psi$ with $a(t, v) \in A^+$,

$$\left| \Psi(t) \int_{\Gamma\backslash G} f_1(x\, a(t,v))\, f_2(x)\, dm^{\mathrm{BMS}}(x) \right| < C.$$

The main goal of this section is to establish the following result.

Proposition 5.3 *Suppose that m^{BMS} satisfies the local mixing property for the pair (Ψ, J). Then for any $x = [g] \in \Gamma\backslash G$, $v \in \ker \psi$, $f \in C_c(\Gamma\backslash G)^M$ and $\phi \in C_c(N^+)$,*

$$\lim_{t\to\infty} \Psi(t) \int_{N^+} f(xn\, a(t,v))\phi(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n) = J(v)\, m^{\mathrm{BMS}}(f)\, \mu^{\mathrm{PS}}_{gN^+}(\phi), \tag{5.4}$$

and there exists a $C' = C'(f, \phi) > 0$ such that

$$\left| \Psi(t) \int_{N^+} f(xn\, a(t,v))\phi(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n) \right| < C'$$

for all $(t, v) \in (0,\infty) \times \ker \psi$ with $a(t, v) \in A^+$.

For $\varepsilon > 0$, let G_ε denote the open ball of radius ε around e in G. For a subgroup $S < G$, we define $S_\varepsilon := S \cap G_\varepsilon$. The choices $S = P^\pm$, $N^\pm$ and A are the only subgroups we will require. We will carry out a thickening argument using PS measures as in [33]; the following lemma is needed.

Lemma 5.5 *For any $g \in G$,*

$$\nu(gN^+(e^+)) > 0 \quad \textit{and} \quad \nu_{\mathrm{i}}(gN^-(e^-)) > 0.$$

Proof The Zariski density of Λ_Γ in $\mathcal{F}$ is proved in [1]. This also follows from Lemma 2.11. Since each $gN^\pm(e^\pm)$ is a Zariski open subset of $\mathcal{F}$ and the support of ν is equal to Λ_Γ, the conclusion follows. □

We will also need the following result on a continuity property of the PS measures [32, Proposition 2.15]:

Lemma 5.6 *For any fixed $\rho \in C_c(N^\pm)$ and $g \in G$, the map $N^\mp \to \mathbb{R}$ given by $h \mapsto \mu^{\mathrm{PS}}_{ghN^\pm}(\rho)$ is continuous.*

Proof We will only prove the case when $\rho \in C_c(N^+)$; the other case can be proved similarly. Define a function $\widetilde{\rho}_g \colon N^- \times G/P \to \mathbb{R}$ by

$$\widetilde{\rho}_g(h,\xi) := \begin{cases} \rho(n) e^{\psi(\beta_{ghn^+}(e,ghn))} & \text{if } \xi = ghn^+ \text{ for some } n \in N^+, \\ 0 & \text{otherwise.} \end{cases}$$

Since $N^+ \cap P = \{e\}$, $\widetilde{\rho}_g$ is well-defined. By the continuity of the Busemann function, $\widetilde{\rho}_g$ is continuous in $h \in N^-$. This gives

$$\mu^{\mathrm{PS}}_{ghN^+}(\rho) = \int_{N^+} \rho(n) e^{\psi(\beta_{ghn^+}(e,ghn))}\, d\nu(ghn^+) = \int_{G/P} \widetilde{\rho}_g(h,\xi)\, d\nu(\xi).$$

Hence $|\mu^{\mathrm{PS}}_{gh_1N^+}(\rho) - \mu^{\mathrm{PS}}_{gh_2N^+}(\rho)| \le \max_{\xi \in G/P} |\widetilde{\rho}_g(h_1,\xi) - \widetilde{\rho}_g(h_2,\xi)|$. The continuity of $\widetilde{\rho}_g$ then implies the claimed statement. □

A function on $N^\pm$ is said to be *radial* if it is invariant under conjugation by elements of M, ie $f(mnm^{-1}) = f(n)$ for all $m \in M$, and $n \in N^\pm$.

Corollary 5.7 *Given $\varepsilon > 0$ and $g \in G$, there exist $R > 1$ and a nonnegative radial function $\rho_{g,\varepsilon} \in C_c(N^-_R)$ such that for all $n \in N^+_\varepsilon$,*

$$\mu^{\mathrm{PS}}_{gnN^-}(\rho_{g,\varepsilon}) > 0.$$

Proof For each $j \in \mathbb{N}$, let $\phi_j \in C_c(N^-_{j+1})$ be a nonnegative radial function such that $\phi_j|_{N^-_j} = 1$. By Lemma 5.5, for each $n \in N^+$ there exists some $j_n \in \mathbb{N}$ such that $\mu^{\mathrm{PS}}_{gnN^-}(N^-_{j_n}) > 0$. By Lemma 5.6, for each $n \in N^+$ there exists $r_n > 0$ such that

$$\mu^{\mathrm{PS}}_{gn_0N^-}(N^-_{j_n}) > 0 \quad \text{for all } n_0 \in \mathcal{B}(n) := \{n_0 \in N^+ : \mathrm{dist}(n, n_0) < r_n\}.$$

Using the relative compactness of N^+_ε, we choose $n_1, \dots, n_k \in N^+$ such that $N^+_\varepsilon \subset \bigcup_{i=1}^k \mathcal{B}(n_i)$. Choosing $R := \max(j_{n_1}, \dots, j_{n_k}) + 1$ and $\rho_{g,\varepsilon} := \phi_{R-1}$ completes the proof. □

Proof of Proposition 5.3 Fixing $v \in \ker(\psi)$, for simplicity, we write $a_t = a(t, v)$. Let $x = [g]$ and $\varepsilon_0 > 0$ be such that $\phi \in C_c(N^+_{\varepsilon_0})$. By Corollary 5.7, there exist $R > 0$ and a nonnegative $\rho_{g,\varepsilon_0} \in C_c(N^-_R)$ such that

$$\mu^{\mathrm{PS}}_{gnN^-}(\rho_{g,\varepsilon_0}) > 0 \quad \text{for all } n \in N^+_{\varepsilon_0}.$$

Given an arbitrary $\varepsilon > 0$, choose a nonnegative function $q_\varepsilon \in C_c(A_\varepsilon)$ satisfying $\int_A q_\varepsilon(a)\, da = 1$. Then

$$\begin{aligned}
\textbf{(5.8)} \quad & \int_{N^+} f(xna_t)\phi(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n) \\
&= \int_{N^+} f(xna_t)\phi(n) \\
&\qquad \times \left(\frac{1}{\mu^{\mathrm{PS}}_{gnN^-}(\rho_{g,\varepsilon_0})} \int_{N^-A} \rho_{g,\varepsilon_0}(h) q_\varepsilon(a)\, da\, d\mu^{\mathrm{PS}}_{gnN^-}(h) \right) d\mu^{\mathrm{PS}}_{gN^+}(n) \\
&= \int_{N^+} \left(\int_{N^-A} f(xna_t) \frac{\phi(n)\rho_{g,\varepsilon_0}(h) q_\varepsilon(a)}{\mu^{\mathrm{PS}}_{gnN^-}(\rho_{g,\varepsilon_0})}\, da\, d\mu^{\mathrm{PS}}_{gnN^-}(h) \right) d\mu^{\mathrm{PS}}_{gN^+}(n).
\end{aligned}$$

We now define a right M–invariant function $\widetilde{\Phi}_\varepsilon \in C_c(gN^+_{\varepsilon_0}N^-_R A_\varepsilon M) \subset C_c(G)$ by

$$\widetilde{\Phi}_\varepsilon(g_0) := \begin{cases} \dfrac{\phi(n)\rho_{g,\varepsilon_0}(h) q_\varepsilon(a)}{\mu^{\mathrm{PS}}_{gnN^-}(\rho_{g,\varepsilon_0})} & \text{if } g_0 = gnham, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the continuity of $\widetilde{\Phi}_\varepsilon$ is a consequence of Lemma 5.6. Also observe that $\widetilde{\Phi}_\varepsilon$ depends on our choice of representative for $x = [g]$.

We now assume without loss of generality that $f \geq 0$ and define, for all $\varepsilon > 0$, functions $f^\pm_\varepsilon$ as follows: for all $z \in \Gamma\backslash G$,

$$f^+_\varepsilon(z) := \sup_{b \in N^+_\varepsilon N^-_\varepsilon A_\varepsilon} f(zb) \quad \text{and} \quad f^-_\varepsilon(z) := \inf_{b \in N^+_\varepsilon N^-_\varepsilon A_\varepsilon} f(zb).$$

Since $u \in \operatorname{int}\mathfrak{a}^+$, for every $\varepsilon > 0$ there exists $t_0(R,\varepsilon) > 0$ such that

$$a_t^{-1} N_R^- a_t \subset N_\varepsilon^- \quad \text{for all } t \geq t_0(R,\varepsilon).$$

Then, as $\operatorname{supp}(\widetilde{\Phi}_\varepsilon) \subset g N_{\varepsilon_0}^+ N_R^- A_\varepsilon M$, we have

$$f(xna_t)\widetilde{\Phi}_\varepsilon(gnha) \leq f_{2\varepsilon}^+(xnhaa_t)\widetilde{\Phi}_\varepsilon(gnha) \tag{5.9}$$

for all $nha \in N^+N^-A$ and $t \geq t_0(R,\varepsilon)$. Similarly,

$$f_{2\varepsilon}^-(xnhaa_t)\widetilde{\Phi}_\varepsilon(gnha) \leq f(xna_t)\widetilde{\Phi}_\varepsilon(gnha).$$

We now use $f_{2\varepsilon}^+$ to give an upper bound on the limit we are interested in; $f_{2\varepsilon}^-$ is used in an analogous way to provide a lower bound. Entering the definition of Φ_ε and the above inequality (5.9) into (5.8) gives

$$\begin{aligned}
&\limsup_{t\to\infty} \Psi(t) \int_{N^+} f(xna_t)\phi(n)\, d\mu_{gN^+}^{\mathrm{PS}}(n) \\
&\quad \leq \limsup_{t\to\infty} \Psi(t) \\
&\qquad \times \int_{N^+} \Big(\int_{N^- AM} f_{2\varepsilon}^+(xnhaa_t)\widetilde{\Phi}_\varepsilon(gnha)\, dm\, da\, d\mu_{gnN^-}^{\mathrm{PS}}(h) \Big) d\mu_{gN^+}^{\mathrm{PS}}(n) \\
&\quad = \limsup_{t\to\infty} \Psi(t) \int_G f_{2\varepsilon}^+([g_0]a_t)\widetilde{\Phi}_\varepsilon(g_0)\, d\widetilde{m}^{\mathrm{BMS}}(g_0) \\
&\quad = \limsup_{t\to\infty} \Psi(t) \int_{\Gamma\backslash G} f_{2\varepsilon}^+([g_0]a_t)\Phi_\varepsilon([g_0])\, dm^{\mathrm{BMS}}([g_0]),
\end{aligned}$$

where

$$\Phi_\varepsilon([g_0]) := \sum_{\gamma\in\Gamma} \Phi_\varepsilon(\gamma g_0),$$

and Lemma 4.3 was used in the second-to-last line of the above calculation. By the standing assumption (5.2), we then have

$$\begin{aligned}
\limsup_{t\to\infty} \Psi(t) \int_{N^-} f(xna_t)\phi(n)\, d\mu_{gN^+}^{\mathrm{PS}}(n) &\leq J(v) m^{\mathrm{BMS}}(f_{2\varepsilon}^+) m^{\mathrm{BMS}}(\Phi_\varepsilon) \\
&= J(v) m^{\mathrm{BMS}}(f_{2\varepsilon}^+) \widetilde{m}^{\mathrm{BMS}}(\widetilde{\Phi}_\varepsilon).
\end{aligned}$$

Using Lemma 4.3 and the M–invariance of $\widetilde{\Phi}_\varepsilon$, we have

$$\begin{aligned}
\widetilde{m}^{\mathrm{BMS}}(\widetilde{\Phi}_\varepsilon) &= \int_{N^+} \Big(\int_{N^- A} \widetilde{\Phi}_\varepsilon(gnha)\, da\, d\mu_{gnN^-}^{\mathrm{PS}}(h) \Big) d\mu_{gN^+}^{\mathrm{PS}}(n) \\
&= \int_{N^+} \frac{\phi(n)}{\mu_{gnN^-}^{\mathrm{PS}}(\rho_{g,\varepsilon_0})} \Big(\int_{N^- A} \rho_{g,\varepsilon_0}(h) q_\varepsilon(a)\, da\, d\mu_{gnN^-}^{\mathrm{PS}}(h) \Big) d\mu_{gN^+}^{\mathrm{PS}}(n) \\
&= \mu_{gN^+}^{\mathrm{PS}}(\phi).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, taking $\varepsilon \to 0$ gives

$$\limsup_{t\to\infty} \Psi(t) \int_{N^-} f(xna_t)\phi(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n) \le J(v)\, m^{\mathrm{BMS}}(f)\, \mu^{\mathrm{PS}}_{gN^+}(\phi).$$

The lower bound given by replacing $f^+_{2\varepsilon}$ with $f^-_{2\varepsilon}$ in the above calculations proves the first statement.

For the second claim of the proposition, observe that if $tu + \sqrt{t}v \in \mathfrak{a}^+$, then

$$f(xna_t)\widetilde{\Phi}_\varepsilon(gnha) \le f^+_{R+\varepsilon}(xnhaa_t)\widetilde{\Phi}_\varepsilon(gnha),$$

as in (5.9). Hence

$$\Psi(t) \int_{N^+} f(xna_t)\phi(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n) \le C(f^+_{R+\varepsilon}, \Phi_\varepsilon).$$

Choosing $C'(f,\phi) := C(f^+_{R+\varepsilon}, \Phi_\varepsilon)$ finishes the proof. □

6 Translates of Lebesgue measures and Haar mixing

We continue with the setup of Section 5: recall that we have fixed $u \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+$, a linear form $\psi \in D^\star_\Gamma$ such that $\psi(u) = \psi_\Gamma(u)$, a (Γ,ψ)–PS measure $\nu = \nu_\psi$ and a $(\Gamma, \psi\circ \mathrm{i})$–PS measure $\nu_{\mathrm{i}} = \nu_{\psi\circ\mathrm{i}}$ on $\mathcal{F}$. We set

$$m^{\mathrm{BMS}} = m^{\mathrm{BMS}}_{\nu,\nu_{\mathrm{i}}}, \quad m^{\mathrm{BR}} = m^{\mathrm{BR}}_{\nu_{\mathrm{i}}}, \quad m^{\mathrm{BR}_*} = m^{\mathrm{BR}_*}_{\nu} \quad \text{and} \quad \mu^{\mathrm{PS}}_{gN^+} = \mu^{\mathrm{PS}}_{gN^+,\nu}.$$

The main goal in this section is to prove a local mixing statement for the Haar measure on $\Gamma\backslash G$. In order to do this, we first convert equidistribution of translates of $\mu^{\mathrm{PS}}_{gN^+}$ (Proposition 5.3) into equidistribution of translates of the Lebesgue measure on xN^+:

Proposition 6.1 *Suppose that m^{BMS} satisfies the local mixing property for the pair (Ψ, J). Then for any $x = [g] \in \Gamma\backslash G$, $v \in \ker\psi$, $f \in C_c(\Gamma\backslash G)^M$ and $\phi \in C_c(N^+)$,*

$$\lim_{t\to\infty} \Psi(t) e^{(2\rho-\psi)(tu+\sqrt{t}v)} \int_{N^+} f(xn\,a(t,v))\phi(n)\, dn = J(v)\, m^{\mathrm{BR}}(f)\, \mu^{\mathrm{PS}}_{gN^+}(\phi),$$

and there exists $C'' = C''(f,\phi) > 0$ such that

$$\left| \Psi(t) e^{(2\rho-\psi)(tu+\sqrt{t}v)} \int_{N^+} f(xn\,a(t,v))\phi(n)\, dn \right| < C''$$

for all $(t,v) \in (0,\infty)\times\ker\psi$ with $a(t,v) \in A^+$.

Proof For $\varepsilon_0 > 0$, set $\mathcal{B}_{\varepsilon_0} = N^-_{\varepsilon_0} A_{\varepsilon_0} M N^+_{\varepsilon_0}$. Note that $MN^+_{\varepsilon_0} = N^+_{\varepsilon_0} M$ by the choice of the invariant metric on G. Given $x_0 \in \Gamma\backslash G$, let $\varepsilon_0(x_0)$ denote the maximum number r such that the map $G \to \Gamma\backslash G$ given by $h \mapsto x_0 h$ for $h \in G$ is injective on $\mathcal{B}_r$. Note that $\psi(u) = \psi_\Gamma(u) = \psi_\Gamma(\mathrm{i}(u))$. Fixing $v \in \ker \psi$ we set, for all $t \in \mathbb{R}$,

$$a_t := a(t, v).$$

By using a partition of unity if necessary, it suffices to prove that for any $x_0 \in \Gamma\backslash G$ and $\varepsilon_0 = \varepsilon_0(x_0)$, the claims of the proposition hold for any nonnegative $f \in C(x_0\mathcal{B}_{\varepsilon_0})^M$, nonnegative $\phi \in C(N^+_{\varepsilon_0})$ and $x = [g] \in x_0\mathcal{B}_{\varepsilon_0}$; that is,

$$\textbf{(6.2)} \qquad \lim_{t\to\infty} \Psi(t) e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f(xna_t)\phi(n)\, dn = J(v)\, m^{\mathrm{BR}}(f)\, \mu^{\mathrm{PS}}_{gN^+}(\phi),$$

and for $\log a_t \in \mathfrak{a}^+$, we have

$$\Psi(t) e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f(xna_t)\phi(n)\, dn < C'' \quad \text{for some } C'' = C''(f, \phi).$$

Moreover, we may assume that f is given for all $g \in G$ as

$$f([g]) = \sum_{\gamma\in\Gamma} \widetilde{f}(\gamma g)$$

for some nonnegative $\widetilde{f} \in C_c(g_0\mathcal{B}_{\varepsilon_0}) \subset C_c(G)$. Note that for $x = [g] \in [g_0]\mathcal{B}_{\varepsilon_0}$,

$$\textbf{(6.3)} \qquad \int_{N^+} f([g]na_t)\phi(n)\, dn = \sum_{\gamma\in\Gamma} \int_{N^+} \widetilde{f}(\gamma g n a_t)\phi(n)\, dn.$$

Note that $\widetilde{f}(\gamma g n a_t) = 0$ unless $\gamma g n a_t \in g_0\mathcal{B}_{\varepsilon_0}$. Taken together with the fact that $\mathrm{supp}(\phi) \subset N^+_{\varepsilon_0}$, it follows that the summands in (6.3) are nonzero for only finitely many elements $\gamma \in \Gamma \cap g_0\mathcal{B}_{\varepsilon_0} a_t^{-1} N^+_{\varepsilon_0} g^{-1}$.

Suppose $\gamma g N^+_{\varepsilon_0} a_t \cap g_0\mathcal{B}_{\varepsilon_0} \neq \varnothing$. Then

$$\gamma g a_t \in g_0 N^-_{\varepsilon_0} A_{\varepsilon_0} M N^+,$$

and there are unique elements $p_{t,\gamma} \in N^-_{\varepsilon_0} A_{\varepsilon_0} M$ and $n_{t,\gamma} \in N^+$ such that

$$\gamma g a_t = g_0 p_{t,\gamma} n_{t,\gamma} \in g_0 P^-_{\varepsilon_0} N^+.$$

Let $\Gamma_{t,v}$ denote the subset $\Gamma \cap g_0(N^-_{\varepsilon_0} A_{\varepsilon_0} M N^+) a_t^{-1} g^{-1}$. Note that although $\Gamma_{t,v}$ may possibly be infinite, only finitely many of the terms in the sums we consider

will be nonzero. This gives

$$\begin{aligned}\int_{N^+} f([g]na_t)\phi(n)\,dn &= \sum_{\gamma\in\Gamma}\int_{N^+}\widetilde{f}(\gamma g n a_t)\phi(n)\,dn\\ &= \sum_{\gamma\in\Gamma_{t,v}}\int_{N^+}\widetilde{f}(\gamma g a_t(a_t^{-1}na_t))\phi(n)\,dn\\ &= e^{-2\rho(\log a_t)}\sum_{\gamma\in\Gamma_{t,v}}\int_{N^+}\widetilde{f}(\gamma g a_t n)\phi(a_t n a_t^{-1})\,dn\\ &= e^{-2\rho(\log a_t)}\sum_{\gamma\in\Gamma_{t,v}}\int_{N^+}\widetilde{f}(g_0 p_{t,\gamma} n_{t,\gamma} n)\phi(a_t n a_t^{-1})\,dn\\ &= e^{-2\rho(\log a_t)}\sum_{\gamma\in\Gamma_{t,v}}\int_{N^+}\widetilde{f}(g_0 p_{t,\gamma} n)\phi(a_t\, n_{t,\gamma}^{-1} n\, a_t^{-1})\,dn.\end{aligned}$$

Since $\operatorname{supp}(\widetilde{f})\subset g_0\mathcal{B}_{\varepsilon_0}$, we have

$$\begin{aligned}&\sum_{\gamma\in\Gamma_{t,v}}\int_{N^+}\widetilde{f}(g_0 p_{t,\gamma} n)\phi(a_t\, n_{t,\gamma}^{-1} n\, a_t^{-1})\,dn\\ &\qquad\le \sum_{\gamma\in\Gamma_{t,v}}\Big(\sup_{n\in N^+_{\varepsilon_0}}\phi(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}(a_t n a_t^{-1}))\Big)\cdot\int_{N^+}\widetilde{f}(g_0 p_{t,\gamma} n)\,dn\end{aligned}$$

and

$$\begin{aligned}&\sum_{\gamma\in\Gamma_{t,v}}\int_{N^+}\widetilde{f}(g_0 p_{t,\gamma} n)\phi(a_t\, n_{t,\gamma}^{-1} n\, a_t^{-1})\,dn\\ &\qquad\ge \sum_{\gamma\in\Gamma_{t,v}}\Big(\inf_{n\in N^+_{\varepsilon_0}}\phi(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}(a_t n a_t^{-1}))\Big)\cdot\int_{N^+}\widetilde{f}(g_0 p_{t,\gamma} n)\,dn.\end{aligned}$$

Since u belongs to $\operatorname{int}\mathcal{L}_\Gamma$, there exist $t_0(v)>0$ and $\alpha>0$ such that

$$a_t N_r^+ a_t^{-1}\subset N^+_{re^{-\alpha t}}\quad\text{for all } r>0 \text{ and } t>t_0(v).$$

Therefore, for all $n\in N^+_{\varepsilon_0}$ and $t>t_0(v)$, we have

$$\textbf{(6.4)}\qquad \phi^-_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1} a_t^{-1})\le\phi(a_t\, n_{t,\gamma}^{-1} a_t^{-1}(a_t n a_t^{-1}))\le\phi^+_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}),$$

where, for all $n\in N^+$ and $\varepsilon>0$,

$$\phi^+_\varepsilon(n):=\sup_{b\in N^+_\varepsilon}\phi(nb)\quad\text{and}\quad\phi^-_\varepsilon(n):=\inf_{b\in N^+_\varepsilon}\phi(nb).$$

We now have the chain of inequalities (for $t > t_0(v)$)

$$\begin{aligned}\textbf{(6.5)}\quad \sum_{\gamma\in\Gamma_{t,v}} \phi^-_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}) \int_{N^+_{\varepsilon_0}} \widetilde{f}(g_0 p_{t,\gamma} n)\, dn \\ \le e^{2\rho(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\, dn \\ \le \sum_{\gamma\in\Gamma_{t,v}} \phi^+_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}) \int_{N^+_{\varepsilon_0}} \widetilde{f}(g_0 p_{t,\gamma} n)\, dn.\end{aligned}$$

By Lemmas 5.5 and 5.6, there exist $R > 0$ and a radial function $\rho \in C_c(N^+_R)$ such that $\rho(n) \ge 0$ for all $n \in N^+$, and $\mu^{\mathrm{PS}}_{g_0 pN^+}(\rho) > 0$ for all $p \in N^-_{\varepsilon_0} A_{\varepsilon_0} M$. Define $\widetilde{F} \in C_c(g_0 N^-_{\varepsilon_0} A_{\varepsilon_0} M N^+_R)$ by

$$\textbf{(6.6)}\quad \widetilde{F}(g) := \begin{cases} \dfrac{\rho(n)}{\mu^{\mathrm{PS}}_{g_0 pN^+}(\rho)} \displaystyle\int_{N^+_{\varepsilon_0}} \widetilde{f}(g_0 pv)\, dv & \text{if } g = g_0 pn \in g_0 N^-_{\varepsilon_0} A_{\varepsilon_0} M N^+_R, \\ 0 & \text{otherwise.}\end{cases}$$

Since ρ is radial, $\widetilde{F}$ is right M–invariant. The key property of $\widetilde{F}$ we will use is the following: for all $p \in P^-_{\varepsilon_0}$,

$$\int_{N^+} \widetilde{F}(g_0 pn)\, d\mu^{\mathrm{PS}}_{g_0 pN^+}(n) = \int_{N^+_R} \widetilde{F}(g_0 pn)\, d\mu^{\mathrm{PS}}_{g_0 pN^+}(n) = \int_{N^+_{\varepsilon_0}} \widetilde{f}(g_0 pn)\, dn.$$

Returning to (6.5), we now give an upper bound for $\int_N f([g]na_t)\phi(n)\, dn$; the lower bound can be dealt with in a similar fashion. We observe

$$\begin{aligned} e^{2\rho(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\, dn \\ \le \sum_{\gamma\in\Gamma_{t,v}} \phi^+_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}) \int_{N^+_{\varepsilon_0}} \widetilde{f}(g_0 p_{t,\gamma} n)\, dn \\ = \sum_{\gamma\in\Gamma_{t,v}} \phi^+_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}) \int_{N^+_R} \widetilde{F}(g_0 p_{t,\gamma} n)\, d\mu^{\mathrm{PS}}_{g_0 p_{t,\gamma} N^+}(n) \\ = \sum_{\gamma\in\Gamma_{t,v}} \int_{N^+_R} \widetilde{F}(g_0 p_{t,\gamma} n)\phi^+_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1})\, d\mu^{\mathrm{PS}}_{g_0 p_{t,\gamma} N^+}(n).\end{aligned}$$

Similarly as before, we have, for all $t > t_0(v)$ and $n \in N^+_R$,

$$\begin{aligned}\textbf{(6.7)}\quad \phi^+_{\varepsilon_0 e^{-\alpha t}}(a_t n_{t,\gamma}^{-1} a_t^{-1}) &= \phi^+_{\varepsilon_0 e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1} n(n)^{-1} a_t^{-1}) \\ &\le \phi^+_{(R+\varepsilon_0) e^{-\alpha t}}(a_t n_{t,\gamma}^{-1} n\, a_t^{-1}).\end{aligned}$$

Hence (6.5) is bounded above by

$$\leq \sum_{\gamma\in\Gamma_{t,v}} \int_{N_R^+} \widetilde{F}(g_0 p_{t,\gamma} n)\phi^+_{(R+\varepsilon_0)e^{-\alpha t}}(a_t\, n_{t,\gamma}^{-1} n\, a_t^{-1})\, d\mu^{\mathrm{PS}}_{g_0 p_{t,\gamma} N^+}(n)$$

$$= \sum_{\gamma\in\Gamma_{t,v}} \int_{N^+} \widetilde{F}(g_0 p_{t,\gamma} n_{t,\gamma} a_t^{-1} n a_t)\phi^+_{(R+\varepsilon_0)e^{-\alpha t}}(n)\, d\mu^{\mathrm{PS}}_{g_0 p_{t,\gamma} N^+}(n_{t,\gamma} a_t^{-1} n a_t).$$

By Lemma 4.2,

$$d\mu^{\mathrm{PS}}_{g_0 p_{t,\gamma} N^+}(n_{t,\gamma} a_t^{-1} n a_t) = e^{-\psi(\log a_t^{-1})} d\mu^{\mathrm{PS}}_{g_0 p_{t,\gamma} n_{t,\gamma} a_t^{-1} N^+}(n).$$

Since $g_0 p_{t,\gamma} n_{t,\gamma} a_t^{-1} = \gamma g$, it follows that for all $t > t_0(v)$,

$$e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g] n a_t)\phi(n)\, dn$$

$$\leq \sum_{\gamma\in\Gamma_{t,v}} \int_{N^+} \widetilde{F}(\gamma g n a_t)\phi^+_{(R+\varepsilon_0)e^{-\alpha t}}(n)\, d\mu^{\mathrm{PS}}_{\gamma g N^+}(n)$$

$$\leq \int_{N^+} \Big(\sum_{\gamma\in\Gamma} \widetilde{F}(\gamma g n a_t) \Big) \phi^+_{(R+\varepsilon_0)e^{-\alpha t}}(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n).$$

Define a function F on $\Gamma\backslash G$ by

$$F([g]) := \sum_{\gamma\in\Gamma} \widetilde{F}(\gamma g).$$

Then for any $\varepsilon > 0$ and for all $t > t_0(v)$ such that $(R+\varepsilon_0)e^{-\alpha t} \leq \varepsilon$,

$$\Psi(t) \int_{N^+} F([g] n a_t)\phi_\varepsilon^-(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n) \leq \Psi(t) e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g] n a_t)\phi(n)\, dn$$

$$\leq \Psi(t) \int_{N^+} F([g] n a_t)\phi_\varepsilon^+(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n).$$

Since F is right M–invariant, by Proposition 5.3, letting $\varepsilon \to 0$ gives

$$\lim_{t\to\infty} \Psi(t) e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g] n a_t)\phi(n)\, dn = J(v) m^{\mathrm{BMS}}(F)\, \mu^{\mathrm{PS}}_{gN^+}(\phi).$$

From the definition of F, together with Lemma 4.4 in the form (4.5), and (4.8), we have

$$m^{\mathrm{BMS}}(F) = \widetilde{m}^{\mathrm{BMS}}(\widetilde{F}) = \int_{P^-} \Big(\int_{N^+} \widetilde{F}(g_0 p n)\, d\mu^{\mathrm{PS},}_{g_0 p N^+}(n) \Big) d\mu^{\mathrm{PS}}_{g_0 P^-}(p)$$

$$= \int_{P^-} \Big(\int_{N^+} \widetilde{f}(g_0 p n)\, dn \Big) d\mu^{\mathrm{PS}}_{g_0 P^-}(p)$$

$$= \widetilde{m}^{\mathrm{BR}}(\widetilde{f}) = m^{\mathrm{BR}}(f).$$

This finishes the proof of the first statement. For the second statement, note that the following inequalities corresponding to (6.4), and (6.7) hold with the weaker assumption $tu + \sqrt{t}v \in \mathfrak{a}^+$, rather than $t > t_0(v)$: for all $n \in N^+_{\varepsilon_0}$,

$$\phi(a_t n^{-1}_{t,\gamma} a_t^{-1}(a_t n a_t^{-1})) \le \phi^+_{\varepsilon_0}(a_t n^{-1}_{t,\gamma} a_t^{-1}),$$

and for all $n \in N^+_R$,

$$\phi^+_{\varepsilon_0}(a_t\, n^{-1}_{t,\gamma}\, a_t^{-1}) = \phi^+_{\varepsilon_0}(a_t\, n^{-1}_{t,\gamma} n(n)^{-1}\, a_t^{-1}) \le \phi^+_{R+\varepsilon_0}(a_t\, n^{-1}_{t,\gamma} n\, a_t^{-1}).$$

Now proceeding similarly as in the proof of the first statement, we have

$$\int_{N^+} f([g]na_t)\phi(n)\,dn \le e^{(\psi-2\rho)(\log a_t)} \int_{N^+} \Big(\sum_{\gamma\in\Gamma} \widetilde{F}(\gamma g n a_t) \Big) \phi^+_{R+\varepsilon_0}(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n),$$

and hence

$$\begin{aligned} \Psi(t) e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\,dn & \\ &\le \Psi(t) \int_{N^+} F([g]na_t)\phi^+_{R+\varepsilon_0}(n)\, d\mu^{\mathrm{PS}}_{gN^+}(n) \\ &\le C'(F, \phi^+_{R+\varepsilon_0}) \end{aligned}$$

provided $\log a_t \in \mathfrak{a}^+$. By setting $C''(f,\phi) := C'(F, \phi^+_{R+\varepsilon_0})$, this finishes the proof of the proposition. □

With the help of Proposition 5.3, we are now ready to prove:

Proposition 6.8 *Suppose that m^{BMS} satisfies the local mixing property for the pair (Ψ, J). Then for any $f_1, f_2 \in C_c(\Gamma\backslash G)^M$ and $v \in \ker\psi$, we have*

$$\lim_{t\to\infty} \Psi(t) e^{(2\rho-\psi)(tu+\sqrt{t}v)} \int_{\Gamma\backslash G} f_1(xa(t,v))\, f_2(x)\, dx = J(v)\, m^{\mathrm{BR}}(f_1)\, m^{\mathrm{BR}*}(f_2),$$

and there exists a $C_0 = C_0(f_1, f_2) > 0$ such that if $a(t,v) \in \mathfrak{a}^+$,

$$\left| \Psi(t) e^{(2\rho-\psi)(tu+\sqrt{t}v)} \int_{\Gamma\backslash G} f_1(xa(t,v))\, f_2(x)\, dx \right| < C_0.$$

Proof The hypotheses above coincide with those of Propositions 5.3 and 6.1; this allows us to apply Proposition 6.1 in the following argument.

By compactness, we can find $\varepsilon_0 > 0$ and $x_i \in \Gamma\backslash G$ for $i = 1, \dots, \ell$ such that the map $G \to \Gamma\backslash G$ given by $g \to x_i g$ is injective on $R_{\varepsilon_0} = N^-_{\varepsilon_0} A_{\varepsilon_0} N^+_{\varepsilon_0} M$, and $\bigcup_{i=1}^{\ell} x_i R_{\varepsilon_0/2}$ contains both $\operatorname{supp} f_1$ and $\operatorname{supp} f_2$. As before, set $a_t = \exp(tu + \sqrt{t}v)$. We use

continuous partitions of unity to write f_1 and f_2 as finite sums $f_1 = \sum_{i=1}^{\ell} f_{1,i}$ and $f_2 = \sum_{j=1}^{\ell} f_{2,j}$ with $\operatorname{supp} f_{1,i} \subset x_i R_{\varepsilon_0/2}$ and $\operatorname{supp} f_{2,j} \subset x_j R_{\varepsilon_0/2}$. Writing $p = ham \in N^- AM$ and using the decomposition of the Haar measure on G given by

$$d(hamn) = e^{-2\rho(\log a)}\, dn\, dm\, da\, dh,$$

cf [23, Proposition 8.45], we have

$$\begin{aligned}\textbf{(6.9)}\quad &\int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\, dx \\ &= \sum_{i,j} \int_{R_{\varepsilon_0}} f_{1,i}(x_i pna_t) f_{2,j}(x_i pn) e^{-2\rho(\log a)}\, dn\, dm\, da\, dh \\ &= \sum_{i,j} \int_{N^-_{\varepsilon_0} A_{\varepsilon_0} M} \Big(\int_{N^+_{\varepsilon_0}} f_{1,i}(x_i pna_t) f_{2,j}(x_i pn)\, dn \Big) e^{-2\rho(\log a)}\, dm\, da\, dh.\end{aligned}$$

Applying Proposition 6.1, it follows (cf also (4.8)) that

$$\begin{aligned}&\lim_{t\to\infty} \Psi(t) e^{(2\rho-\psi)(\log a_t)} \int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\, dx \\ &\quad = J(v) \sum_i m^{\mathrm{BR}}(f_{1,i}) \sum_j \int_{N^-_{\varepsilon_0} A_{\varepsilon_0} M} \mu^{\mathrm{PS}}_{x_i pN^+}(f_{2,j}(x_i p\,\cdot)) e^{-2\rho(\log a)}\, dm\, da\, dh \\ &\quad = J(v) \sum_i m^{\mathrm{BR}}(f_{1,i}) \sum_j m^{\mathrm{BR}_*}(f_{2,j}) \\ &\quad = J(v) m^{\mathrm{BR}}(f_1)\, m^{\mathrm{BR}_*}(f_2),\end{aligned}$$

where the second-last equality is valid by (4.8). This justifies the first statement. For the second statement, note that if $tu + \sqrt{t}v \in \mathfrak{a}^+$, (6.9) together with Proposition 6.1 gives

$$\left| \Psi(t) e^{(2\rho-\psi)(\log a_t)} \int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\, dx \right| \le C_0(f_1, f_2),$$

where

$$C_0(f_1, f_2) := \sum_{i,j} C''(f_{1,i}, f_{2,j}) \int_{N^-_{\varepsilon_0} A_{\varepsilon_0} M} e^{-2\rho(\log(a(t,v))}\, dm\, da\, dh.$$

This completes the proof. □

We make the following observation, which will be used in the proof of Theorem 7.9.

Corollary 6.10 *Fix $u \in \mathcal{L}_\Gamma \cap \operatorname{int}\mathfrak{a}^+$, and let $\psi \in D_\Gamma^\star$ be tangent to ψ_Γ at u. For $k = 1, 2$, let ν_k and $\overline{\nu}_k$ be (Γ, ψ)– and $(\Gamma, \psi \circ \mathrm{i})$–PS measures on Λ_Γ, respectively. Setting $m_k^{\mathrm{BMS}} := m_{\nu_k, \overline{\nu}_k}^{\mathrm{BMS}}$, suppose that there exist functions $\Psi_k : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all $f_1, f_2 \in C_c(\Gamma\backslash G)^M$,*

$$\lim_{t\to\infty} \Psi_k(t) \int_{\Gamma\backslash G} f_1(x \exp(tu))\, f_2(x)\, dm_k^{\mathrm{BMS}}(x) = m_k^{\mathrm{BMS}}(f_1) m_k^{\mathrm{BMS}}(f_2).$$

Then $\nu_1 = \nu_2$ and $\overline{\nu}_2 = \overline{\nu}_2$.

Proof By an argument similar to the proof of Proposition 6.8,

$$\lim_{t\to\infty} \Psi_k(t) e^{t(2\rho - \psi_\Gamma)(u)} \int_{\Gamma\backslash G} f_1(x \exp(tu))\, f_2(x)\, dx = m_{\overline{\nu}_k}^{\mathrm{BR}}(f_1)\, m_{\nu_k}^{\mathrm{BR}*}(f_2).$$

Fix $f_1 \in C_c(\Gamma\backslash G)^M$ with $m_{\overline{\nu}_k}^{\mathrm{BR}}(f_1) > 0$ for each $k = 1, 2$. Considering $f_2 \in C_c(\Gamma\backslash G)^M$ with $m_{\nu_k}^{\mathrm{BR}*}(f_2) > 0$, it follows from the hypothesis that $c_0 := \lim_{t\to\infty} \Psi_1(t)/\Psi_2(t) > 0$. Set $c_1 := c_0 \cdot m_{\overline{\nu}_2}^{\mathrm{BR}}(f_1)/m_{\overline{\nu}_1}^{\mathrm{BR}}(f_1)$. Then for any $f_2 \in C_c(\Gamma\backslash G)^M$, we have

$$m_{\nu_1}^{\mathrm{BR}*}(f_2) = c_1 \cdot m_{\nu_2}^{\mathrm{BR}*}(f_2). \tag{6.11}$$

Recall from Lemma 3.6 that for all $a \in A$, $a_* m_{\nu_k}^{\mathrm{BR}*} = e^{(-\psi + 2\rho)(\log a)}\, m_{\nu_k}^{\mathrm{BR}*}$. We claim that $\nu_1 = \nu_2$. Let $g_0 \in G$ be arbitrary. Fix $\varepsilon > 0$ and consider $F \in C(G/P)$ supported in $(g_0 N_\varepsilon^+)^-$. Choose $q_1 \in C_c(A)$, and a radial function $q_2 \in C_c(N)$ such that $\int_A q_1\, da = \int_{N^-} q_2\, dn = 1$. Define $\widetilde{f}_2 \in C_c(G)$ by

$$\widetilde{f}_2(g) := \begin{cases} F(g_0 h^+) q_1(a) q_2(n) e^{-\psi(\log a)} & \text{if } g = g_0 h m a n \in g_0 N^+ P^-, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\widetilde{f}_2$ is M–invariant, as q_2 is radial. Defining $f_2 \in C_c(\Gamma\backslash G)$ by

$$f_2([g]) := \sum_{\gamma\in\Gamma} \widetilde{f}_2(\gamma g),$$

a direct computation shows that

$$\begin{aligned} m_{\nu_k}^{\mathrm{BR}*}(f_2) &= \int_{N^+} \int_{P^-} \widetilde{f}_2(g_0 h m a n)\, e^{\psi(\log a)}\, dm\, da\, dn\, d\nu_k(g_0 h^+) \\ &= \int_{N^+} F(g_0 h^+) \Big(\int_{P^-} q_1(a) q_2(n)\, dm\, da\, dn \Big)\, d\nu_k(g_0 h^+) \\ &= \nu_k(F). \end{aligned}$$

Hence for any $g_0^- \in G/P$, it holds that $\nu_1(F) = c_1 \cdot \nu_2(F)$ for all $F \in C(G/P)$ supported in $(g_0 \mathcal{O}_\varepsilon)^-$. By using a partition of unity, we get $\nu_1(F) = c_1 \cdot \nu_2(F)$ for all $F \in C(G/P)$. Since $|\nu_1| = |\nu_2| = 1$, we have $c_1 = 1$ and hence $\nu_1 = \nu_2$. Repeating the same argument for $\mathrm{i}(v)$, we also get $\overline{\nu}_1 = \overline{\nu}_2$ (this implies $c_0 = 1$). □

7 Anosov groups

7.1 Anosov subgroups

Let Σ be a finitely generated word hyperbolic group and let $\partial\Sigma$ denote the Gromov boundary of Σ. We call a Zariski dense discrete subgroup $\Gamma < G$ *Anosov* with respect to P if it arises as the image of a P–Anosov representation of Σ. A representation $\Phi\colon \Sigma \to G$ is P–Anosov if Φ induces a continuous equivariant map $\zeta\colon \partial\Sigma \to \mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y \in \partial\Sigma$, following Labourie [24] and Guichard and Wienhard [17].

Let $\tau_d\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$ be the d–dimensional irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$. For any torsion-free uniform lattice Σ in $\mathrm{PSL}_2(\mathbb{R})$, the connected component of $\tau_d|_\Sigma$ in the space $\mathrm{Hom}(\Sigma, \mathrm{PSL}_d(\mathbb{R}))$ is called the Hitchin component. Representations $\Sigma \to \mathrm{PSL}_d(\mathbb{R})$ in the Hitchin component are known to be P–Anosov [24]. In fact, Hitchin components are defined for representations of Σ into any split real simple Lie group G, and all representations $\Sigma \to G$ in the Hitchin component are known to be P–Anosov [12; 17].

We mention that if $\rho_i\colon \Sigma \to G_i$ are P_i–Anosov, where P_i is a minimal parabolic subgroup of G_i, then $\rho_1 \times \rho_2\colon \Sigma \to G_1 \times G_2$ is $P_1 \times P_2$–Anosov whenever its image is Zariski dense. Indeed, if $\zeta_i\colon \partial\Sigma \to G_i/P_i$ denotes the limit map of ρ_i, the map $\zeta(x) = (\zeta_1(x), \zeta_2(x))$ provides the desired limit map for $\rho_1 \times \rho_2$; hence $\{(\rho_1(g), \rho_2(g)) : g \in \Sigma\}$ is an Anosov subgroup of $G_1 \times G_2$.

One subclass of Anosov groups consists of Schottky groups, which generalize the Schottky subgroups of rank one Lie groups.

For a loxodromic element $g \in G$, we denote by $y_g \in \mathcal{F}$ the unique attracting fixed point of g. Let $\gamma_1, \dots, \gamma_p$ be loxodromic elements of G, where $p \geq 2$. For each $1 \leq i \leq p$, set $\xi_i^{+1} = y_{\gamma_i}$ and $\xi_i^{-1} = y_{\gamma_i^{-1}}$.

Definition 7.1 The subgroup Γ generated by $\{\gamma_1, \dots, \gamma_p\}$ is called Schottky if there exist open subsets $b_i^{\pm}, B_i^{\pm} \subset \mathcal{F}$ for $1 \leq i \leq p$, and a $0 < \varepsilon < 1$ such that:

(1) For all $i \neq j$ and $\omega, \varpi \in \{-1, 1\}$, we have that $b_i^{\omega} \subset B_j^{\varpi}$.

(2) $\overline{b_i^{\omega}} \times \overline{b_j^{\varpi}} \subset \mathcal{F}^{(2)}$ whenever $i \neq j$ or $\omega \neq \varpi$.

(3) For all i and $\omega \in \{-1, 1\}$, we have that $\xi^{\omega} \in \operatorname{int} b_i^{\omega}$, $\gamma_i^{\omega} B_i^{\omega} \subset b_i^{\omega} \subset B_i^{\omega}$ and the restriction of γ_i to B_i^{ω} is ε–Lipschitz.

(4) The intersection $\bigcap_{1 \leq i \leq p,\, \omega \in \{1,-1\}} B_i^{\omega}$ is nonempty.

This is the same definition as given in [40, Section 4.2], except for the extra condition (2), which we added to ensure the following lemma:

Lemma 7.2 *Any Zariski dense Schottky subgroup $\Gamma < G$ is Anosov.*

Proof The Gromov boundary $\partial\Gamma$ can be identified with the set of infinite words of the form $\underline{a} = (a_0 a_1 a_2 \cdots)$, where $a_i \in \{\gamma_1^{\pm 1}, \dots, \gamma_p^{\pm 1}\}$ and $a_i \neq a_{i+1}^{-1}$. Fix an element

$$\xi_0 \in \bigcap_{1 \le i \le p,\, \omega \in \{1,-1\}} B_i^{\omega},$$

which exists by (3). Under the above definition of a Schottky group, the proof of [38, Proposition 3.3] gives that the map $\underline{a} \mapsto \lim_{n\to\infty}(a_0 a_1 \cdots a_n)\xi_0$ induces a Γ–equivariant homeomorphism $\zeta\colon \partial\Gamma \to \Lambda_\Gamma$; see also [40, Proposition 4.5]. Let $\xi \neq \eta \in \Lambda_\Gamma$, and write them as

$$\xi = \lim_{n\to\infty}(a_0 a_1 \cdots a_n)\xi_0 \quad \text{and} \quad \eta = \lim_{n\to\infty}(a_0' a_1' \cdots a_n')\xi_0$$

for some infinite words $\underline{a}$ and $\underline{a}'$. Let k be the smallest integer such that $a_k \neq a_k'$, and set $\gamma := a_0 \cdots a_{k-1} \in \Gamma$. It follows from (1) and (2) that $\gamma^{-1}\xi \in \overline{b_i^{\omega}}$ and $\gamma^{-1}\eta \in \overline{b_j^{\varpi}}$, where $i \neq j$ or $\omega \neq \varpi$. Hence $(\gamma^{-1}\xi, \gamma^{-1}\eta) \in \mathcal{F}^{(2)}$ by (2); consequently, we get that $(\xi, \eta) \in \mathcal{F}^{(2)}$. This shows that Γ is Anosov. □

Schottky groups are found everywhere, in the following sense:

Lemma 7.3 *Any Zariski dense discrete subgroup Γ contains a Zariski dense Schottky subgroup.*

Proof This follows from the proof of a more general theorem [1, Proposition 4.3]. We give a sketch of the proof for the sake of completeness. Since the set of loxodromic elements of Γ is Zariski dense, we may choose a loxodromic element $\gamma_1 \in \Gamma$. There exists a proper Zariski closed subset $F_{\gamma_1} \subset G$ which contains all Zariski connected and Zariski closed proper subgroups of G containing γ_1 [49, Proposition 4.4]. We may choose a second loxodromic element $\gamma_2 \in \Gamma - F_{\gamma_1}$ such that $\{(y_{\gamma_2^{\pm 1}}, y_{\gamma_1}), (y_{\gamma_2^{\pm 1}}, y_{\gamma_1^{-1}})\} \subset \mathcal{F}^{(2)}$. Moreover, we can assume that γ_2^k generates a Zariski connected subgroup, and hence that $\gamma_2^k \notin F_{\gamma_1}$ for any $k \in \mathbb{N}$. Let $\varphi_i \in G$ be such that $\gamma_i \in \varphi_i(\operatorname{int} A^+)\varphi_i^{-1}$. Then $y_{\gamma_i^{\pm 1}} = \varphi_i e^{\pm}$. For $\varepsilon > 0$, let $b_i^{\pm 1}(\varepsilon)$ be the ε–neighborhood of $\varphi_i e^{\pm}$ and set $B_i^{\pm 1}(\varepsilon) = \varphi_i N_{\varepsilon^{-1}}^{\pm} e^{\pm}$, where $N_{\varepsilon^{-1}}^{\pm}$ denotes the ε^{-1}–neighborhood of e in $N^{\pm}$. For any $\varepsilon_1, \varepsilon_2 > 0$, we get that $\gamma_i^{\pm k} B_i^{\pm 1}(\varepsilon_2) \subset b_i^{\pm 1}(\varepsilon_1)$ for all sufficiently large k. It follows that we can find k, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that γ_1^k and γ_2^k satisfy the conditions in Definition 7.1 with $b_i^{\pm 1} := b_i^{\pm 1}(\varepsilon_1)$ and $B_i^{\pm 1} := B_i^{\pm 1}(\varepsilon_2)$. □

7.2 BMS–mixing

The following theorem was first proved by Sambarino [45], when Σ is the fundamental group of a compact negatively curved manifold, using the work of Thirion [48]. Using the reparametrization theorem of [4] for a general case, Chow and Sarkar [8] proved the following theorem; their proof uses a different symbolic coding than [45] and works for functions which are not necessarily M–invariant.

Theorem 7.4 *Let $u \in \operatorname{int} \mathcal{L}_\Gamma$. Let $\psi \in D^\star_\Gamma$ be tangent to ψ_Γ at u, and let ν and ν_{i} be respectively (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$–PS measures on Λ_Γ. Set $m^{\mathrm{BMS}} := m^{\mathrm{BMS}}_{\nu,\nu_{\mathrm{i}}}$. Then there exists $\kappa_u > 0$ such that for any $v \in \ker \psi$ and any $f_1, f_2 \in C_c(\Gamma\backslash G)^M$,*

$$\lim_{t\to\infty} t^{(r-1)/2} \int_{\Gamma\backslash G} f_1(x) f_2(x \exp(tu + \sqrt{t}v))\, dm^{\mathrm{BMS}}(x) = \kappa_u e^{-I(v)/2} m^{\mathrm{BMS}}(f_1) m^{\mathrm{BMS}}(f_2),$$

where $I : \ker(\psi) \to \mathbb{R}$ is given by

$$I(v) := c \cdot \frac{\|v\|_*^2 \|u\|_*^2 - \langle v, u\rangle_*^2}{\|u\|_*^2} \tag{7.5}$$

for some inner product $\langle \cdot, \cdot \rangle_*$ and some $c > 0$. Moreover, the left-hand side is uniformly bounded over all $(t, v) \in (0, \infty) \times \ker \psi$ with $tu + \sqrt{t}v \in \mathfrak{a}^+$.

Although the second statement of Theorem 7.4 is not stated in [45, Theorem 3.8], its proof uses the same technique as [48, Theorem 1.1], where the corresponding statement can be found.

Remark 7.6 Theorem 7.4 is the main reason for the assumption that Γ is a Anosov subgroup. In fact, all our results stated in the introduction hold whenever Γ satisfies Theorems 7.4 and 7.7.

In the rest of the paper, let Γ be an Anosov subgroup of G. The following theorem was proved by Sambarino [43] for a special case and by Potrie and Sambarino [35, Propositions 4.6 and 4.11] in general.

Theorem 7.7 (1) $\mathcal{L}_\Gamma - \{0\} \subset \operatorname{int} \mathfrak{a}^+$.

(2) *ψ_Γ is strictly concave and analytic on $\{v \in \operatorname{int} \mathcal{L}_\Gamma : \|v\| = 1\}$.*

By Lemma 2.24 and Theorem 7.7, we get the following corollary:

Corollary 7.8 *For each unit vector $u \in \operatorname{int} \mathcal{L}_\Gamma$, there exists a unique $\psi_u \in D_\Gamma^\star$ tangent to ψ_Γ at u, which is given by*

$$\psi_u(\cdot) = \langle \nabla \psi_\Gamma(u), \cdot \rangle.$$

Theorem 7.9 *For any $\psi \in D_\Gamma^\star$, there exists a unique (Γ, ψ)–PS measure on Λ_Γ.*

Proof When Γ is the fundamental group of a closed negatively curved Riemannian manifold, this follows by combining [44, Theorem 3.1] and [44, Proposition 7.8]. The general case follows from Theorem 7.4 by Corollary 6.10. □

Note that this theorem implies the Γ–ergodicity of every (Γ, ψ)–PS measure.

Remark 7.10 The Γ–ergodicity of a (Γ, ψ)–PS measure for $\psi \in D_\Gamma^\star$ also follows from [7, Theorem A.2] together with the reparametrization theorem [8, Theorem 4.10]. By Sullivan's well-known argument, it implies the uniqueness of (Γ, ψ)–PS measure.

For each $u \in \operatorname{int} \mathcal{L}_\Gamma$, let ν_u denote the unique (Γ, ψ_u)–PS measure on Λ_Γ. We now set

$$m_u^{\mathrm{BMS}} := m_{\nu_u, \nu_{\mathrm{i}(u)}}^{\mathrm{BMS}}, \quad m_{\mathrm{i}(u)}^{\mathrm{BR}} := m_{\nu_{\mathrm{i}(u)}}^{\mathrm{BR}}, \quad m_u^{\mathrm{BR}*} := m_{\nu_u}^{\mathrm{BR}*}. \tag{7.11}$$

We mention that all three measures are infinite measures when G has rank at least two; cf [25, Corollary 4.9]. We also refer to [25] and [26], where ergodic properties of these measures are discussed.

We deduce the following from Propositions 5.3 and 6.8, and Theorem 7.4.

Theorem 7.12 *Let $u \in \operatorname{int} \mathcal{L}_\Gamma$.*

(1) *For any $f_1, f_2 \in C_c(\Gamma\backslash G)^M$ and $v \in \ker \psi_u$,*

$$\lim_{t\to+\infty} t^{(r-1)/2} e^{(2\rho-\psi_u)(tu+\sqrt{t}v)} \int_{\Gamma\backslash G} f_1(x \exp(tu+\sqrt{t}v)) f_2(x)\, dx = \kappa_u\, e^{-I(v)/2} m_{\mathrm{i}(u)}^{\mathrm{BR}}(f_1) m_u^{\mathrm{BR}*}(f_2).$$

(2) *For any $f \in C_c(\Gamma\backslash G)^M$, $\phi \in C_c(N^+)$, $x = [g] \in \Gamma\backslash G$, and $v \in \ker \psi_u$,*

$$\lim_{t\to\infty} t^{(r-1)/2} e^{(2\rho-\psi_u)(tu+\sqrt{t}v)} \int_{N^+} f(xn \exp(tu+\sqrt{t}v))\phi(n)\, dn = \kappa_u\, e^{-I(v)/2} m_{\mathrm{i}(u)}^{\mathrm{BR}}(f)\, \mu_{gN^+,\nu_u}^{\mathrm{PS}}(\phi).$$

Moreover, the left-hand sides of the above equalities are uniformly bounded for all $(t, v) \in (0,\infty) \times \ker \psi_u$ with $tu + \sqrt{t}v \in \mathfrak{a}^+$.

Recalling that $\psi_u(u) = \psi_\Gamma(u)$, the special case of Theorem 7.12 when $v = 0$ now implies Theorems 1.6 and 1.7.

8 Equidistribution of translates of $\Gamma\backslash\Gamma H$

Symmetric subgroups of G

Let $H < G$ be a symmetric subgroup; that is to say, H is the identity component of the set of fixed points of an involution σ of G. We start by reviewing some general structure theory regarding symmetric subgroups; see [46, Chapter 6] for more details on this. The involution σ induces a Lie algebra involution on $\mathfrak{g}$, which (using a slight abuse of notation) we also denote by σ. There exists a Cartan involution of G that commutes with σ; without loss of generality, we may assume that θ from Section 2 commutes with σ. These involutions give rise to the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ into the $+1$ and -1 eigenspace decompositions of θ and σ, respectively. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ such that $\mathfrak{b} := \mathfrak{a} \cap \mathfrak{q}$ is a maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{q}$. Denote the dimension of $\mathfrak{a}$ by r and the dimension of $\mathfrak{b}$ by r_0.

Let $\Sigma_\sigma \subset \mathfrak{b}^*$ be the root system of $\mathfrak{b}$, ie

$$\Sigma_\sigma = \{\lambda \in \mathfrak{b}^* - \{0\} : \exists X \in \mathfrak{g} - \{0\} \text{ such that } \mathrm{ad}_Y X = \lambda(Y)X \text{ for all } Y \in \mathfrak{b}\}.$$

From now on, we fix a closed positive Weyl chamber $\mathfrak{b}^+ \subset \mathfrak{b}$ for Σ_σ, which has been chosen compatibly with $\mathfrak{a}^+$ as follows [14, Section 3]: denoting the positive roots of $\mathfrak{a}$ by Σ^+, we assume that there exists a collection of positive roots Σ_σ^+ of $\mathfrak{b}$ such that the elements of Σ_σ^+ are all obtained by restricting elements of Σ^+ to $\mathfrak{b}$, hence $\mathfrak{b}^+ \subset \mathfrak{a}^+$. We will write $B = \exp(\mathfrak{b})$ and $B^+ = \exp(\mathfrak{b}^+)$.

Let $\mathcal{W}_\sigma := \mathrm{N}_K(\mathfrak{b})/Z_K(\mathfrak{b})$ and $\mathcal{W}_{\sigma,\theta} := \mathrm{N}_{K\cap H}(\mathfrak{b})/Z_{K\cap H}(\mathfrak{b})$. There then exists a finite set of representatives $\mathcal{W} \subset \mathrm{N}_K(\mathfrak{a}) \cap \mathrm{N}_K(\mathfrak{b})$ for $\mathcal{W}_{\sigma,\theta}\backslash\mathcal{W}_\sigma$, and we have the generalized Cartan decomposition

$$G = H \exp(\mathfrak{b}) K = H\mathcal{W} \exp(\mathfrak{b}^+) K, \tag{8.1}$$

in the sense that for any $g \in G$, there exist unique elements $b \in B^+$ and $\omega \in \mathcal{W}$ such that

$$g \in H\omega b K.$$

Directions in $\mathfrak{b}^+ \cap \operatorname{int} \mathcal{L}_\Gamma$

Let Γ be an Anosov subgroup of G. In the rest of this section, we assume that

$$\mathfrak{b}^+ \cap \operatorname{int}(\mathcal{L}_\Gamma) \neq \varnothing.$$

Since $\mathcal{L}_\Gamma \subset \operatorname{int}(\mathfrak{a}^+)$ by Theorem 7.7, it follows that $\mathfrak{b}^+ \cap \operatorname{int}(\mathfrak{a}^+) \neq \varnothing$.

We now fix a unit vector (with respect to the norm $\|\cdot\|$ on $\mathfrak{a}$)

$$v \in \mathfrak{b}^+ \cap \operatorname{int} \mathcal{L}_\Gamma$$

and set

$$\delta := \psi_\Gamma(v) > 0. \tag{8.2}$$

By Corollary 7.8, the linear form $\Theta \in \mathfrak{a}^*$ defined as

$$\Theta(w) = \langle \nabla \psi_\Gamma(v), w \rangle \tag{8.3}$$

gives the unique linear form in $D_\Gamma^\star$ such that $\Theta(v) = \psi_\Gamma(v) = \delta$.

Lemma 8.4 *We have* $\mathcal{L}_\Gamma \cap \ker \Theta = \{0\}$.

Proof We use the fact that ψ_Γ is strictly concave (Theorem 7.7). Since $\Theta \geq \psi_\Gamma$ on $\mathfrak{a}$, and $\Theta(v) = \psi_\Gamma(v)$, it follows that $\Theta(w) > \psi_\Gamma(w)$ for all vectors $w \in \mathfrak{a}^+ - \mathbb{R}v$. Since $\psi_\Gamma \geq 0$ on $\mathcal{L}_\Gamma$, we have $\Theta > 0$ on $\mathcal{L}_\Gamma - \{0\}$, ie $\mathcal{L}_\Gamma \cap \ker \Theta = \{0\}$. □

We use the following notation: for $t > 0$ and $w \in \ker \Theta$,

$$a(t, w) := \exp(tv + \sqrt{t}w) \quad \text{and} \quad \underline{a}(t, w) := tv + \sqrt{t}w.$$

We set $m^{\mathrm{BR}} = m^{\mathrm{BR}}_{\nu_{\mathrm{i}(v)}}$ and $m^{\mathrm{BR}*} = m^{\mathrm{BR}_*}_{\nu_v}$.

Patterson–Sullivan measures on H

Let $P = MAN$ be the minimal parabolic subgroup. Since $\mathfrak{b}^+ \cap \operatorname{int}(\mathfrak{a}^+) \neq \varnothing$, it follows that $M = Z_K(\mathfrak{b})$, and the unipotent subgroup whose Lie algebra is the sum of positive root spaces corresponding to Σ_σ coincides with N.

Lemma 8.5 *We have* $H \cap N = \{e\}$.

Proof Fix $\alpha \in \Sigma_\sigma$. Since $v \in \mathfrak{b}^+ \cap \operatorname{int} \mathfrak{a}^+$, we have $\alpha(v) > 0$. Letting $X \in \mathfrak{n}$ be such that $[v, X] = \alpha(v)X$, we have $\sigma(X) = \alpha(v)^{-1}\sigma([v, X]) = \alpha(v)^{-1}[\sigma(v), \sigma(X)] = \alpha(v)^{-1}[-v, \sigma(X)]$, ie $[v, \sigma(X)] = -\alpha(v)\sigma(X)$. Therefore $\sigma(X) \in \mathfrak{n}^+$. Since σ fixes H pointwise and swaps N and N^+, we get $H \cap N = \{e\}$. □

By [28, Theorem 3(iii)],

$$H \cap P = (H \cap M)(H \cap A)(H \cap N) = (H \cap M)(H \cap A). \tag{8.6}$$

Together with the fact $H \cap B = H \cap N = \{e\}$, it then follows that $H \cap MBN = H \cap M$.

Definition 8.7 Define a measure $\mu_H^{\mathrm{PS}} = \mu_{H,v}^{\mathrm{PS}}$ on H as follows: for $\phi \in C_c(H)$, let

$$\mu_H^{\mathrm{PS}}(\phi) = \int_{h_0 \in H/(H\cap P)} \int_{p \in H\cap P} \phi(h_0 p) e^{\Theta(\beta_{h_0^+}(e, h_0 p))}\, dp\, d\nu(h_0^+),$$

where dp is a right Haar measure on $H \cap P$; for $h_0 \in H/(H \cap P)$, h_0^+ is well-defined and independent of the choice of representative. The measure defined above is $\Gamma \cap H$–invariant: for any $\gamma \in \Gamma \cap H$, $\gamma_* \mu_H^{\mathrm{PS}} = \mu_H^{\mathrm{PS}}$. Therefore, if $\Gamma\backslash\Gamma H$ is closed in $\Gamma\backslash G$, $d\mu_{H,v}^{\mathrm{PS}}$ induces a locally finite Borel measure on $\Gamma\backslash\Gamma H \simeq (\Gamma \cap H)\backslash H$, which we denote by $\mu_{[e]H}^{\mathrm{PS}} = \mu_{[e]H,v}^{\mathrm{PS}}$.

Note that $\operatorname{supp} \mu_{[e]H}^{\mathrm{PS}} = \{[h] \in \Gamma\backslash\Gamma H : h^+ \in \Lambda_\Gamma\}$, and hence

$$|\mu_{[e]H}^{\mathrm{PS}}| = 0 \quad \text{if and only if} \quad \Lambda_\Gamma \cap HP/P = \varnothing. \tag{8.8}$$

For a subset $S \subset G$ and $\varepsilon > 0$, set $S_\varepsilon := \{s \in S : d(e, s) \le \varepsilon\}$. Let $M' \subset M$ be a Borel section for the map $m \mapsto (H \cap M)m$, and $P' = M'BN$ the subset of the minimal parabolic subgroup $P = MAN$. The map $H \times P' \to G$ given by $(h, p') \mapsto hp'$ is injective, which is an open map in a neighborhood of e; see [46, Proposition 7.1.8(ii)] and [11, Proposition 4.3]. For $\varepsilon > 0$, let $\rho_\varepsilon \in C((NB)_\varepsilon)$ be a nonnegative function such that

$$\int_{NB} \rho_\varepsilon(nb)\, dn\, db = 1$$

and $\rho_\varepsilon(mnbm^{-1}) = \rho_\varepsilon(nb)$ for all $m \in M$ and $nb \in NB$. Fixing $\phi \in C_c(H)^{H\cap M}$ and $\varepsilon > 0$ smaller than the injectivity radius of $\operatorname{supp}(\phi)$, define $\widetilde{\Phi}_\varepsilon \in C_c(G)$ by

$$\widetilde{\Phi}_\varepsilon(g) := \begin{cases} \phi(h)\rho_\varepsilon(nb) & \text{if } g = hm'nb \in HP', \\ 0 & \text{otherwise.} \end{cases} \tag{8.9}$$

Observe that $\widetilde{\Phi}_\varepsilon$ is right M–invariant. Define now $\Phi_\varepsilon \in C_c(\Gamma\backslash G)$ by

$$\Phi_\varepsilon([g]) = \sum_{\gamma\in\Gamma} \widetilde{\Phi}_\varepsilon(\gamma g).$$

Since $\widetilde{\Phi}_\varepsilon$ is right M–invariant, so is Φ_ε.

Lemma 8.10 *For $\phi \in C_c(H)^{H\cap M}$ and Φ_ε as above, we have*

$$m^{\mathrm{BR}_*}(\Phi_\varepsilon) = \mu_H^{\mathrm{PS}}(\phi)(1 + O(\varepsilon)).$$

Proof Note that $\operatorname{supp}(\widetilde{\Phi}_\varepsilon) \subset HM(NB)_\varepsilon$. The proof of the lemma now relies on the following three observations.

Firstly, for each $hm \in HM'$ and $n \in N$,

$$d(hmn) = e^{2\rho(\beta_{(hmn)^-}(e,hmn))} dm_o(hmn^-) = dn$$

is a Lebesgue measure on hmN.

Secondly, for $g = hm'nb \in HP'$, the decomposition

$$\beta_{g^+}(e,g) = \beta_{h^+}(e,h) + \beta_{e^+}(e,nb) = \beta_{h^+}(e,h) + \mathrm{i}(\log b)$$

induces an isomorphism $A \cong (A \cap H) \times B$. This implies that

$$d(\beta_{g^+}(e,g)) = d(\beta_{h^+}(e,h))\, d(\beta_{e^+}(e,p)) = d(\beta_{h^+}(e,h))\, d(b),$$

and for all $hm' \in HM'$ and $nb \in (NB)_\varepsilon$,

$$e^{\Theta(\beta_{h^+}(e,hm'nb))} = e^{\Theta(\beta_{h^+}(e,h))}(1 + O(\varepsilon))$$

by continuity of the Busemann function.

Finally, we also have

$$\begin{aligned}\beta_{(hm'nb)^-}(e,hm'nb) &= \beta_{(hm'n)^-}(e,hm'n) + \beta_{(hm'n)^-}(hm'n,hm'nb)\\ &= \beta_{(hm'n)^-}(e,hm'n) + \beta_{e^-}(e,b)\\ &= \beta_{(hm'n)^-}(e,hm'n) - \mathrm{i}(\log b).\end{aligned}$$

Consequently, $e^{2\rho(\beta_{(hm'nb)^-}(e,hm'nb))} = e^{2\rho(\beta_{(hm'n)^-}(e,hm'n))}(1 + O(\varepsilon))$. Using the definition of $m^{\mathrm{BR}*}(\Phi_\varepsilon)$ and the second and third observations above gives

$$\begin{aligned}m^{\mathrm{BR}*}(\Phi_\varepsilon) &= \int_{H\times M'\times (NB)_\varepsilon} \phi(h)\rho_\varepsilon(nb) e^{\Theta(\beta_{h^+}(e,hm'nb))+2\rho(\beta_{(hm'n)^-}(e,hm'nb))}\\ &\qquad \times dm_o((hm'n)^-)\, d(\beta_{h^+}(e,hm'nb))\, d\nu(h^+)\\ &= (1+O(\varepsilon)) \int_{H\times M'\times (NB)_\varepsilon} \phi(h) e^{\Theta(\beta_{h^+}(e,h))} \rho_\varepsilon(nb) e^{2\rho(\beta_{(hm'n)^-}(e,hm'n))}\\ &\qquad \times dm_o((hm'n)^-)\, db\, d(\beta_{h^+}(e,h))\, d\nu(h^+).\end{aligned}$$

We now choose a section $H_0 \subset H$ for the map $h \mapsto h(H \cap P)$, and write $h = h_0 m a_h \in H_0(M \cap H)(A \cap H) = H_0(H \cap P)$. Using the first observation above, we then have

$$\begin{aligned}m^{\mathrm{BR}*}(\Phi_\varepsilon) &= (1 + O(\varepsilon)) \int_{H_0(H\cap P)} \phi(h_0 m a_h) e^{\Theta(\beta_{h_0^+}(e,h))} \left(\int_{(NB)_\varepsilon} \rho_\varepsilon(nb) dn\, db \right)\\ &\qquad \times d(a_h)\, d\nu(h_0^+)\\ &= (1 + O(\varepsilon)) \int_{h_0 \in H/(H\cap P)} \int_{p\in H\cap P} \phi(h_0 p) e^{\Theta(\beta_{h_0^+}(e,h_0 p))}\, dp\, d\nu(h_0^+)\\ &= (1 + O(\varepsilon))\mu_H^{\mathrm{PS}}(\phi). \qquad \square\end{aligned}$$

Equidistribution of translates of $\Gamma\backslash\Gamma H$

Proposition 8.11 *For any $f \in C_c(\Gamma\backslash G)^M$, $\phi \in C_c(H)^{H\cap M}$ and $w \in \mathfrak{b} \cap \ker\Theta$,*

$$\lim_{t\to\infty} t^{\frac{1}{2}(r-1)} e^{(2\rho-\Theta)(\underline{a}(t,w))} \int_H f([h]a(t,w))\phi(h)\,dh = \kappa_v e^{-\frac{1}{2}I(w)} m^{\mathrm{BR}}(f)\mu_H^{\mathrm{PS}}(\phi),$$

and there exists $C' = C'(f,\phi) > 0$ such that for all (t,w) with $\underline{a}(t,w) \in \mathfrak{b}^+$,

$$\left| t^{\frac{1}{2}(r-1)} e^{(2\rho-\Theta)(\underline{a}(t,w))} \int_H f([h]a(t,w))\phi(h)\,dh \right| < C'. \tag{8.12}$$

Proof For $\varepsilon > 0$, let $R_\varepsilon := N_\varepsilon A_\varepsilon N_\varepsilon^+ M$ and define $f_\varepsilon^\pm \in C_c(\Gamma\backslash G)$ by

$$f_\varepsilon^+(y) = \sup_{g\in R_\varepsilon} f(yg) \quad \text{and} \quad f_\varepsilon^-(y) = \inf_{g\in R_\varepsilon} f(yg). \tag{8.13}$$

Since R_ε is right M–invariant, it follows that $f_\varepsilon^\pm \in C_c(\Gamma\backslash G)^M$. Let $C_0 \subset H$ denote the support of ϕ; we may assume that C_0 injects to its image under the map $G \mapsto \Gamma\backslash G$. Choosing $\rho_\varepsilon \in C_c((NB)_\varepsilon)$ and defining Φ_ε as above, we let $d\lambda(m')$ denote the density on M' of total mass one such that

$$d(hm'nb) = dh\,d\lambda(m')\,dn\,db \tag{8.14}$$

is a Haar measure on G, where $h \in H$, $m' \in M'$, $n \in N$ and $b \in B$.

We then obtain

$$\begin{aligned}\int_H f([h]a(t,w))\phi(h)\,dh &= \int_{C_0} f([h]a(t,w))\phi(h)\bigg(\int_{(NB)_\varepsilon} \rho_\varepsilon(nb)\,dn\,db\bigg)\,dh \\ &= \int_{C_0(NB)_\varepsilon} f([h]a(t,w))\Phi_\varepsilon([h]nb)\,dh\,dn\,db.\end{aligned}$$

Using the M–invariance of f and the definitions of Φ_ε and $d\lambda$ gives

$$f([h]a(t,w))\Phi_\varepsilon([h]nb) = \int_{M'} f([h]m'a(t,w))\Phi_\varepsilon([h]m'nb)\,d\lambda(m'),$$

and so

$$\begin{aligned}&\int_H f([h]a(t,w))\phi(h)\,dh \\ &\qquad = \int_{C_0M'(NB)_\varepsilon} f([h]m'a(t,w))\Phi_\varepsilon([h]m'nb)\,dh\,d\lambda(m')\,dn\,db.\end{aligned}$$

Since $v \in \operatorname{int}\mathfrak{b}^+$, for all (t,w) such that $\underline{a}(t,w) \in \mathfrak{b}^+$ and for all $nb \in (NB)_\varepsilon$, we have

$$\begin{aligned}f(xhm'a(t,w)) &= f(xhm'nba(t,w)\cdot(a(t,w)^{-1}(nb)^{-1}a(t,w))) \\ &\le f_\varepsilon^+(xhm'nba(t,w)).\end{aligned} \tag{8.15}$$

Consequently,

$$\begin{aligned}\int_{C_0} f([h]a(t,w))\phi(h)\,dh &\\ &\le \int_{C_0 M'(NB)_\varepsilon} f_\varepsilon^+([h]m'nba(t,w))\Phi_\varepsilon([h]m'nb)\,dh\,d\lambda(m')\,dn\,db\\ &= \int_{\Gamma\backslash G} f_\varepsilon^+(ya(t,w))\Phi_\varepsilon(y)\,dy.\end{aligned}$$

A similar computation shows that

$$\int_{C_0} f([h]a(t,w))\phi(h)\,dh \ge \int_{\Gamma\backslash G} f_\varepsilon^-(ya(t,w))\Phi_\varepsilon(y)\,dy.$$

On the other hand, we have

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{1}{2}(r-1)} e^{(2\rho-\Theta)(\underline{a}(t,w))} \int_{\Gamma\backslash G} f_\varepsilon^\pm(ya(t,w))\Phi_\varepsilon(y)\,dy &\\ = \kappa_v e^{-\frac{1}{2}I(w)} m^{\mathrm{BR}}(f_\varepsilon^\pm) m^{\mathrm{BR}_*}(\Phi_\varepsilon)&\\ = \kappa_v e^{-\frac{1}{2}I(w)} m^{\mathrm{BR}}(f_\varepsilon^\pm)\mu_H^{\mathrm{PS}}(\phi)(1+O(\varepsilon)).&\end{aligned}$$

Taking $\varepsilon \to 0$ in the last equality proves the first statement. The second statement is clear with the choice of $C' = C(f_\varepsilon^+, \Phi_\varepsilon)$, finishing the proof. □

9 Counting in affine symmetric spaces

Let Γ be an Anosov subgroup of G, and let H be a symmetric subgroup of G. We continue to use the notation v, ν, δ, r, r_0, Θ, $G = H\mathcal{W}\exp(\mathfrak{b}^+)K$, etc from Section 8; hence $v \in \mathfrak{b}^+ \cap \operatorname{int}\mathcal{L}_\Gamma$ is a unit vector (with respect to the norm $\|\cdot\|$ on $\mathfrak{a}$). We denote by $|\cdot|$ the norm on $\mathfrak{a}$ induced by an inner product $(\cdot,\cdot)$ with respect to which v and $\ker\Theta$ are orthogonal to each other and for which $|v| = 1$.

In the following we fix a convex cone $\mathcal{C} \subset \mathfrak{b}^+ \cap (\operatorname{int}(\mathfrak{a}^+) \cup \{0\})$ such that

$$v \in \operatorname{int}_{\mathfrak{b}}\mathcal{C} \quad \text{and} \quad \bar{\mathcal{C}} \cap \ker\Theta = \{0\}, \tag{9.1}$$

where $\operatorname{int}_{\mathfrak{b}}\mathcal{C}$ means the interior of $\mathcal{C}$ in the relative topology of $\mathfrak{b}$. Note that there are convex cones which contain $\mathcal{L}_\Gamma \cap \mathfrak{b}^+$ and satisfy (9.1) by Theorem 7.7(1) and Lemma 8.4.

Remark 9.2 If $v = u_\Gamma \in \mathfrak{b}^+$, then by Lemma 2.24, $\Theta(w) = \delta_\Gamma\langle u_\Gamma, w\rangle$, hence the cone $\mathfrak{b}^+ \cap (\operatorname{int}(\mathfrak{a}^+) \cup \{0\})$ satisfies the conditions placed on $\mathcal{C}$ above.

By the condition (9.1), we have

$$\mathcal{C} \subset \{tv + \sqrt{t}w : t \geq 0, w \in \ker\Theta\}. \tag{9.3}$$

Write

$$\mathcal{C}_T := \{w \in \mathcal{C} : |w| < T\} \quad \text{for } T > 1.$$

For $w \in \ker\Theta$, set

$$R_T(w) := \{t \in \mathbb{R} : tv + \sqrt{t}w \in \mathcal{C}_T\}.$$

For all sufficiently large T, $R_T(w)$ is an interval of the form

$$R_T(w) = \big[t_w, \tfrac{1}{2}\big(-|w|^2 + \sqrt{|w|^4 + 4T^2}\big)\big].$$

Lemma 9.4 (1) *There exists a $c > 0$ such that for all $w \in \mathfrak{b} \cap \ker\Theta$ and $T > 1$,*

$$e^{-\delta T} T^{\frac{1}{2}(r-r_0)} \int_{R_T(w)} t^{\frac{1}{2}(r_0-r)} e^{\delta t}\, dt \leq \delta^{-1} e^{-c\delta|w|^2}.$$

(2) *We have*

$$\lim_{T\to\infty} e^{-\delta T} T^{\frac{1}{2}(r-r_0)} \int_{R_T(w)} t^{\frac{1}{2}(r_0-r)} e^{\delta t}\, dt = \delta^{-1} e^{-\frac{1}{2}\delta|w|^2}. \tag{9.5}$$

Proof Note that $(v, x) > 0$ for any nonzero vector $x \in \mathcal{C}$. Since $\mathcal{C}$ is a convex cone with $\overline{\mathcal{C}} \cap \ker\Theta = \{0\}$, it follows that there exists $0 < \theta_0 < \frac{\pi}{2}$ such that the angle between any vector in $\mathcal{C}$ and v is at most θ_0. Now, as v is perpendicular to $\ker\Theta$ with respect to $(\cdot,\cdot)$, we have that for any t such that $\underline{a}(t, w) \in \mathcal{C}$,

$$\frac{\sqrt{t}|w|}{t|v|} \leq \tan\theta_0, \quad \text{or, equivalently,} \quad |w|^2 \leq \tan^2\theta_0 \cdot t.$$

In particular, for $t \in R_T(w)$, we have

$$T^2 \geq t^2 + t|w|^2 \geq \left(\frac{1}{\tan^2\theta_0} + \frac{1}{\tan^4\theta_0}\right)|w|^4.$$

This gives the upper bound

$$\frac{-|w|^2 + \sqrt{|w|^4 + 4T^2}}{2} - T = -\frac{|w|^2}{2} + \frac{|w|^4}{2\big(\sqrt{|w|^4 + 4T^2} + 2T\big)} \leq -c|w|^2,$$

with

$$c := \frac{1}{2}\left(1 - \left(1 + 4\left(\frac{1}{\tan^2\theta_0} + \frac{1}{\tan^4\theta_0}\right)\right)^{-\frac{1}{2}}\right) > 0. \tag{9.6}$$

Now, by changing variables,

$$\int_{t_w}^{\frac{1}{2}\left(-|w|^2+\sqrt{|w|^4+4T^2}\right)} t^{\frac{1}{2}(r_0-r)} e^{\delta t}\, dt = \int_{t_w-T}^{\frac{1}{2}\left(-|w|^2+\sqrt{|w|^4+4T^2}\right)-T} (t+T)^{\frac{1}{2}(r_0-r)} e^{\delta(t+T)}\, dt,$$

and hence

$$\begin{aligned} e^{-\delta T} T^{\frac{1}{2}(r-r_0)} \int_{t_w}^{\frac{1}{2}\left(-|w|^2+\sqrt{|w|^4+4T^2}\right)} t^{\frac{1}{2}(r_0-r)} e^{\delta t}\, dt &= \int_{t_w-T}^{\frac{1}{2}\left(-|w|^2+\sqrt{|w|^4+4T^2}\right)-T} \left(\frac{t}{T}+1\right)^{\frac{1}{2}(r_0-r)} e^{\delta t}\, dt \\ &\le \int_{-\infty}^{-c|w|^2} e^{\delta t}\, dt = \frac{1}{\delta} e^{-c\delta|w|^2}, \end{aligned} \tag{9.7}$$

which proves (1).

The second claim (2) follows as well, because by the dominated convergence theorem, (9.7) converges to

$$\int_{-\infty}^{-\frac{1}{2}|w|^2} e^{\delta t}\, dt = \frac{1}{\delta} e^{-\frac{1}{2}\delta|w|^2}. \qquad \square$$

We fix a left $(H \cap M)$–invariant function $\tau_H \in C_c(H)$ with its support injecting to $\Gamma\backslash G$, and a right M–invariant function $\tau_K \in C(K)$. Define a function $Z_T : G \to \mathbb{R}$ by

$$Z_T(g) := \begin{cases} \tau_H(h)\mathbf{1}_{\mathcal{C}_T}(\log b)\tau_K(k) & \text{if } g = hbk \in H\exp(\mathcal{C})K, \\ 0 & \text{if } g \notin H\exp(\mathcal{C})K. \end{cases}$$

Since $\mathcal{C} \subset \operatorname{int}(\mathfrak{a}^+) \cup \{0\}$, hM and Mk are uniquely determined and hence Z_T is well-defined.

Definition 9.8 (bisector counting function) Define $F_T = F_{T,\tau_H,\tau_K} : \Gamma\backslash G \to \mathbb{R}$ for $T > 0$ by

$$F_T([g]) = \sum_{\gamma\in\Gamma} Z_T(\gamma g). \tag{9.9}$$

For $\Phi \in C_c(\Gamma\backslash G)$ and a left M–invariant Borel function f on K, we define the following M–invariant function on $\Gamma\backslash G$: for $x \in \Gamma\backslash G$,

$$\Phi * f(x) := \int_{k\in K} \Phi(xk) f(k)\, dk.$$

In the definition (9.11) below, the integral over the trivial subspace $\{0\}$ should be interpreted as $\int_0 f(w)dw = f(0)$. In particular, if $\mathfrak{b} \cap \ker\Theta = \{0\}$, then $c_v = s_v\kappa_v/\delta$.

Proposition 9.10 *Let $\Phi \in C_c(\Gamma\backslash G)$. As $T \to \infty$, we have*

$$\langle F_T, \Phi\rangle \sim c_v\, e^{\delta T} T^{\frac{1}{2}(r_0-r)} \mu_H^{\mathrm{PS}}(\tau_H)\, m^{\mathrm{BR}}(\Phi * \tau_K),$$

where c_v is given by

$$c_v := \frac{s_v\kappa_v}{\delta} \int_{\mathfrak{b}\cap\ker\Theta} e^{-\frac{1}{2}(I(w)+\delta|w|^2)}\, dw. \tag{9.11}$$

Here κ_v and $I(w)$ are as in Theorem 7.4 and in (7.5) respectively, and $s_v = 1/|\det S_v|$, where $S_v\colon \mathfrak{a} \to \mathfrak{a}$ is any linear map such that $S_v|_{\ker\Theta} = \mathrm{Id}$ and $S_v v$ is a unit vector orthogonal to $\ker\Theta$ with respect to the inner product on $\mathfrak{a}$ induced by the Killing form.

Proof In view of the decomposition (8.1), we will need the following formula for the Haar measure dg on G; for all $\phi \in C_c(G)$,

$$\int_G \phi(g)\, dg = \sum_{w\in\mathcal{W}} \int_H \int_K \Big(\int_{\mathfrak{b}^+} \phi(hw(\exp b)k)\xi(b)\, db \Big) dk\, dh, \tag{9.12}$$

where $\xi\colon \mathfrak{b} \to \mathbb{R}$ is given by

$$\xi(b) = \prod_{\alpha\in\Sigma_\sigma^+} (\sinh\alpha(b))^{\ell_\alpha^+} (\cosh\alpha(b))^{\ell_\alpha^-}. \tag{9.13}$$

Here $\ell_\alpha^\pm := \dim(\mathfrak{g}_\alpha^\pm)$, where each $\mathfrak{g}_\alpha^\pm$ is the ±1 eigenspace of the root space $\mathfrak{g}_\alpha$ with respect to the involution $\theta\sigma$; cf [46] and [14, page 18].

Substituting $b = \underline{a}(t,w)$ for $t \geq 0$ and $w \in \mathfrak{b}\cap\ker\Theta$ gives $db = s_v t^{\frac{1}{2}(r_0-1)}\, dt\, dw$. Now,

$$\langle F_T, \Phi\rangle = \int_K \tau_K(k) \int_{b\in\mathcal{C}_T} \Big(\int_{[e]H} \Phi([h](\exp b)k)\tau_H(h)\, d[h] \Big) \xi(b)\, db\, dk. \tag{9.14}$$

Hence

$$\begin{aligned}
&\langle F_T, \Phi\rangle \\
&\quad = \int_{\mathfrak{b}\cap\ker\Theta} \int_{t\in R_T(w)} t^{\frac{1}{2}(r_0-1)} \xi(\underline{a}(t,w)) \Big(\int_{[e]H} (\Phi * \tau_K)([h]a(t,w))\tau_H(h)\, d[h] \Big) \\
&\qquad\qquad \times s_v\, dt\, dw \\
&\quad = s_v e^{\delta T} T^{\frac{1}{2}(r_0-r)} \int_{\mathfrak{b}\cap\ker\Theta} p_T(w)\, dw,
\end{aligned}$$

where we define $p_T(w)$ to be

$$\textbf{(9.15)}\quad e^{-\delta T} T^{\frac{1}{2}(r-r_0)} \times \int_{R_T(w)} t^{\frac{1}{2}(r_0-1)} \xi(\underline{a}(t,w)) \bigg(\int_{[e]H} (\Phi * \tau_K)([h]a(t,w)) \tau_H(h)\, d[h] \bigg) dt.$$

We next look for an integrable function on $\mathfrak{b} \cap \ker \Theta$ that bounds the family of functions $p_T(w)$ from above, in order to apply the dominated convergence theorem. By (9.13), there exists a constant $c_1 > 0$ such that for all (t,w) with $\underline{a}(t,w) \in \mathcal{C}$,

$$\textbf{(9.16)}\qquad e^{-2\rho(\underline{a}(t,w))} \xi(\underline{a}(t,w)) \le c_1.$$

By Proposition 8.11, we may assume that

$$\left| t^{\frac{1}{2}(r-1)} e^{2\rho(\underline{a}(t,w))-\delta t} \int_{[e]H} (\Phi * \tau_K)([h]a(t,w)) \tau_H(h)\, d[h] \right| \le c_1$$

as well. Hence

$$p_T(w) \le c_1^2 \psi(w),$$

where $\psi(w) = \delta^{-1} e^{-c\delta|w|^2}$ is as given in Lemma 9.4. Since ψ is integrable over $\mathfrak{b} \cap \ker \Theta$, we may apply Proposition 8.11 and the dominated convergence theorem to deduce that

$$\begin{aligned} &\int_K \tau_K(k) \int_{b \in \mathcal{C}_T} \bigg(\int_{[e]H} \Phi([h](\exp b)k) \tau_H(h)\, d[h] \bigg) \xi(b)\, db\, dk \\ &\quad \sim s_v \kappa_v e^{\delta T} T^{\frac{1}{2}(r_0-r)} \cdot \mu_H^{\mathrm{PS}}(\tau_H) \int_{w \in \mathfrak{b} \cap \ker \Theta} \delta^{-1} e^{-\frac{1}{2}\delta|w|^2} e^{-\frac{1}{2}I(w)} m^{\mathrm{BR}}(\Phi * \tau_K)\, dw \\ &\quad = c_v e^{\delta T} T^{\frac{1}{2}(r_0-r)} \mu_H^{\mathrm{PS}}(\tau_H)\, m^{\mathrm{BR}}(\Phi * \tau_K), \end{aligned}$$

with c_v as given in the statement of the proposition. □

We now fix a left $(H \cap M)$–invariant compact subset $\Omega_H \subset H$ and a right M–invariant compact subset $\Omega_K \subset K$. Let $\varepsilon > 0$ be a number smaller than the injectivity radius at $[e]$ in $\Gamma\backslash G$. There exists a symmetric neighborhood $\mathcal{O}_\varepsilon$ of e in G such that for all $T \ge 1$,

$$\textbf{(9.17)}\qquad \begin{aligned} \Omega_{H,\varepsilon}^- \exp(\mathcal{C}_{T,\varepsilon}^-) \Omega_{K,\varepsilon}^- &\subset \bigcap_{g \in \mathcal{O}_\varepsilon} \Omega_H \exp(\mathcal{C}_T) \Omega_K g, \\ \Omega_H \exp(\mathcal{C}_T) \Omega_K \mathcal{O}_\varepsilon &\subset \Omega_{H,\varepsilon}^+ \exp(\mathcal{C}_{T,\varepsilon}^+) \Omega_{K,\varepsilon}^+, \end{aligned}$$

where $\mathcal{C}_{T,\varepsilon}^+ = \mathcal{C}_T + \mathfrak{b}_\varepsilon$, $\mathcal{C}_{T,\varepsilon}^- = \cap_{b \in \mathfrak{b}_\varepsilon} (\mathcal{C}_T + b)$, $\Omega_{K,\varepsilon}^+ = \Omega_K K_\varepsilon$, $\Omega_{K,\varepsilon}^- = \bigcap_{k \in K_\varepsilon} \Omega_K k$ and $\Omega_{H,\varepsilon}^\pm$ are defined similarly; see [14; 15]. We will additionally fix convex cones $\mathcal{C}^\flat, \mathcal{C}^\sharp \subset \mathfrak{b}^+ \cap (\mathrm{int}(\mathfrak{a}^+) \cup \{0\})$ such that $\mathcal{C}^\flat \subset \mathrm{int}_{\mathfrak{b}}\, \mathcal{C}$, $\mathcal{C} \subset \mathrm{int}_{\mathfrak{b}}\, \mathcal{C}^\sharp$, and which satisfy the

condition (9.1). Note that $\mathcal{C}^+_{T,\varepsilon}$ is no longer contained in $\mathcal{C}$, but there exists a $T_0 > 0$ (independent of $0 < \varepsilon < 1$) such that

$$\mathcal{C}^+_{T,\varepsilon} - \mathcal{C}^+_{T_0,\varepsilon} \subset \mathcal{C}^\sharp_{T+\varepsilon} \quad \text{and} \quad \mathcal{C}^\flat_{T-\varepsilon} - \mathcal{C}^\flat_{T_0} \subset \mathcal{C}^-_{T,\varepsilon}. \tag{9.18}$$

Let $\phi_\varepsilon \in C_c(G)$ be a nonnegative function such that $\int_G \phi_\varepsilon(g)\, dg = 1$ and $\operatorname{supp}(\phi_\varepsilon) \subset \mathcal{O}_\varepsilon$. Define $\Phi_\varepsilon \colon \Gamma\backslash G \to \mathbb{R}$ by

$$\Phi_\varepsilon([g]) = \sum_{\gamma\in\Gamma} \phi_\varepsilon(\gamma g). \tag{9.19}$$

Lemma 9.20 *Let $\tau \in C(K)$ be left M–invariant. Then*

$$\lim_{\varepsilon\to 0} m^{\mathrm{BR}}(\Phi_\varepsilon * \tau) = \int_K \tau(k^{-1}) \mu^{\mathrm{PS},*}_{K,\mathrm{i}(\upsilon)}(k),$$

where $\mu^{\mathrm{PS},*}_{K,\mathrm{i}(\upsilon)} := \widetilde{\nu}_{\Theta\circ\mathrm{i}}$ is given in (4.10).

Proof Set $\widetilde{\nu} := \widetilde{\nu}_{\Theta\circ\mathrm{i}}$. We use Lemma 4.9 and write

$$\begin{aligned} m^{\mathrm{BR}}(\Phi_\varepsilon * \tau) &= \int_K \int_G \phi_\varepsilon(gk)\tau(k)\, d\widetilde{m}^{\mathrm{BR}}(g)\, dk \\ &= \int_K \int_G \phi_\varepsilon(k' \exp(q) nk)\tau(k)\, e^{-\Theta(q)}\, dn\, dq\, d\widetilde{\nu}(k')\, dk. \end{aligned}$$

Substituting $g = \exp(q)nk \in AN^+K$, the density of the Haar measure is given by

$$dg = e^{-2\rho(q)} dn\, dq\, dk.$$

For $g \in G$, let $\kappa(g)$ denote the K–component of g, and let a_g denote the logarithm of the A–component of g in the decomposition $G = AN^+K$. Then

$$\begin{aligned} m^{\mathrm{BR}}(\Phi_\varepsilon * \tau) &= \int_K \int_G \phi_\varepsilon(k'g)\tau(\kappa(g))\, e^{(2\rho-\Theta)(a_g)}\, dg\, d\widetilde{\nu}(k') \\ &= \int_K \int_G \phi_\varepsilon(g)\tau(\kappa(k^{-1}g))\, e^{(2\rho-\Theta)(a_{k^{-1}g})}\, dg\, d\widetilde{\nu}(k). \end{aligned}$$

By shrinking $\mathcal{O}_\varepsilon$ if necessary, we can assume that for all $k \in K$,

$$\kappa(k^{-1}\mathcal{O}_\varepsilon) \subset k^{-1} K_\varepsilon.$$

By the uniform continuity of τ, there exist positive $\eta = \eta_\varepsilon \to 0$ as $\varepsilon \to 0$ such that for all $g \in \mathcal{O}_\varepsilon$ and $k \in K$,

$$\tau(k^{-1}) - \eta \le \tau(\kappa(k^{-1}g)) \le \tau(k^{-1}) + \eta.$$

It follows from the fact that the multiplication map $A\times N^+\times K\to G$ is a diffeomorphism that for some $C>1$, we have that for all $g\in\mathcal{O}_\varepsilon$ and $k\in K$,

$$1-C\varepsilon\le e^{(2\rho-\Theta)(a_{k^{-1}g})}\le 1+C\varepsilon.$$

Since $\int_G \phi_\varepsilon\,dg=1$, we get

$$(1-C\varepsilon)\int_K(\tau(k^{-1})-\eta)\,d\widetilde{\nu}(k)\le m^{\mathrm{BR}}(\Phi_\varepsilon*\tau)\le(1+C\varepsilon)\int_K(\tau(k^{-1})+\eta)\,d\widetilde{\nu}(k).$$

The claim now follows from letting $\varepsilon\to 0$. □

Corollary 9.21 *Let $\mathcal{C}\subset\mathfrak{b}^+\cap(\operatorname{int}\mathfrak{a}^+\cup\{0\})$ be a convex cone satisfying* (9.1). *If $\mu^{\mathrm{PS},*}_{K,\mathrm{i}(v)}(\partial\Omega_K^{-1})=\mu^{\mathrm{PS}}_H(\partial\Omega_H)=0$, then*

$$\lim_{T\to\infty}\frac{\#(\Gamma\cap\Omega_H\exp(\mathcal{C}_T)\Omega_K)}{e^{\delta T}T^{(r_0-r)/2}}=c_v\mu^{\mathrm{PS}}_H(\Omega_H)\mu^{\mathrm{PS},*}_{K,\mathrm{i}(v)}(\Omega_K^{-1}).$$

If $v=u_\Gamma\in\mathfrak{b}^+$, then we may take $\mathcal{C}$ to be $\mathfrak{b}^+$.

Proof Write $\widetilde{\nu}:=\mu^{\mathrm{PS},*}_{K,\mathrm{i}(v)}$. For $g\in H(B^+\cap\operatorname{int}A^+)K$, let $g=h_g(\exp b_g)k_g$ denote the HB^+K decomposition of $g\in G$; note that $h_g(H\cap M)$, b_g and Mk_g are uniquely defined.

Since $\Omega_H(H\cap M)=\Omega_H$ and $M\Omega_K=\Omega_K$, we may define $X_T\colon G\to\mathbb{R}$ by

$$X_T(g)=\begin{cases}\mathbf{1}_{\Omega_H}(h_g)\mathbf{1}_{\mathcal{C}_T}(b_g)\mathbf{1}_{\Omega_K}(k_g) & \text{for } g\in H\exp\mathcal{C}^\sharp K,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\phi_{K,\varepsilon}\in C(K_\varepsilon)^M$ and $\phi_{H,\varepsilon}\in C(H_\varepsilon)^{H\cap M}$ be nonnegative functions with integral equal to one. Set

$$\tau^\pm_{K,\varepsilon}:=\mathbf{1}_{\Omega^\pm_{K,2\varepsilon}}*\phi_{K,\varepsilon},\qquad \tau^\pm_{H,\varepsilon}:=\mathbf{1}_{\Omega^\pm_{H,2\varepsilon}}*\phi_{H,\varepsilon},$$
$$F^+_{T+\varepsilon}:=F_{\mathcal{C}^\sharp_{T+\varepsilon},\tau^\pm_{H,3\varepsilon},\tau^\pm_{K,3\varepsilon}},\quad F^-_{T-\varepsilon}:=F_{\mathcal{C}^\flat_{T-\varepsilon},\tau^-_{H,3\varepsilon},\tau^-_{K,3\varepsilon}}.$$

By definition,

$$\mathbf{1}_{\Omega^+_{K,\varepsilon}}\le\tau^+_{K,\varepsilon}\le\mathbf{1}_{\Omega^+_{K,3\varepsilon}}\quad\text{and}\quad\mathbf{1}_{\Omega^-_{K,3\varepsilon}}\le\tau^-_{K,\varepsilon}\le\mathbf{1}_{\Omega^-_{K,\varepsilon}}.\tag{9.22}$$

By (9.17), (9.18) and (9.22) there exists a uniform constant $C>0$ such that for all $g\in\mathcal{O}_\varepsilon$,

$$F^-_{T-\varepsilon}([g])-C\le\sum_{\gamma\in\Gamma}X_T(\gamma)\le F^+_{T+\varepsilon}([g])+C.$$

Integrating this against Φ_ε as given in (9.19), we get

$$\langle F^-_{T-\varepsilon}, \Phi_\varepsilon\rangle - C \le \sum_{\gamma\in\Gamma} X_T(\gamma) \le \langle F^+_{T+\varepsilon}, \Phi_\varepsilon\rangle + C.$$

For simplicity, we set $x_T := \sum_{\gamma\in\Gamma} X_T(\gamma) = \#\Gamma \cap \Omega_H \exp(\mathcal{C}_T)\Omega_K$. Hence by Proposition 9.10, fixing $\varepsilon_0 > 0$, for all $0 < \varepsilon < \varepsilon_0$ we have

$$\begin{aligned}\textbf{(9.23)}\qquad \limsup_{T\to\infty} \frac{x_T}{e^{\delta T} T^{(r_0-r)/2}} &\le c_v \mu_H^{\mathrm{PS}}(\tau^+_{H,3\varepsilon}) m^{\mathrm{BR}}(\Phi_\varepsilon * \tau^+_{K,3\varepsilon})\\ &\le c_v \mu_H^{\mathrm{PS}}(\tau^+_{H,3\varepsilon}) m^{\mathrm{BR}}(\Phi_\varepsilon * \tau^+_{K,9\varepsilon_0}).\end{aligned}$$

On the other hand, since $\widetilde{\nu}(\partial\Omega_K^{-1}) = \mu_H^{\mathrm{PS}}(\partial\Omega_H) = 0$, we get from (9.22) that

$$\textbf{(9.24)}\quad \lim_{\varepsilon\to 0} \mu_H^{\mathrm{PS}}(\tau^\pm_{H,3\varepsilon}) = \mu_H^{\mathrm{PS}}(\Omega_H) \quad\text{and}\quad \lim_{\varepsilon\to 0} \int_K \tau^\pm_{K,\varepsilon}(k^{-1})\, d\widetilde{\nu}(k) = \widetilde{\nu}(\Omega_K^{-1}).$$

Now letting $\varepsilon \to 0$ first using Lemma 9.20 and (9.24), and then letting $\varepsilon_0 \to 0$, we get

$$\limsup_{T\to\infty} \frac{x_T}{e^{\delta T} T^{(r_0-r)/2}} \le c_v \mu_H^{\mathrm{PS}}(\Omega_H)\widetilde{\nu}(\Omega_K^{-1}).$$

Similarly, we can show that

$$\liminf_{T\to\infty} \frac{x_T}{e^{\delta T} T^{(r_0-r)/2}} \ge c_v \mu_H^{\mathrm{PS}}(\Omega_H)\widetilde{\nu}(\Omega_K^{-1}),$$

which proves the corollary. □

Remark 9.25 This corollary implies that the asymptotic of $\#(\Gamma \cap \Omega_H \exp(\mathcal{C}_T)\Omega_K)$ is independent of $\mathcal{C}$.

Proof of Theorem 1.8 The theorem now follows directly from applying Corollary 9.21 and the following observation to a convex cone $\mathcal{C}$ such that $\mathfrak{b} \cap \mathcal{L}_\Gamma \subset \mathrm{int}_{\mathfrak{b}}\,\mathcal{C}$; by Lemma 8.4, such a cone always exists. □

Lemma 9.26 *Suppose that* $\mathfrak{b}^+ \cap \mathcal{L}_\Gamma \subset \mathrm{int}_{\mathfrak{b}}\,\mathcal{C}$. *Then*

$$\#(\Gamma \cap \Omega_H \exp(\mathfrak{b}^+ - \mathcal{C})\Omega_K) < \infty.$$

Proof We can find a smaller closed convex cone $\mathcal{C}' \subset \mathrm{int}_{\mathfrak{b}}\,\mathcal{C}$ such that $\mathfrak{b}^+ \cap \mathcal{L}_\Gamma \subset \mathrm{int}_{\mathfrak{b}}\,\mathcal{C}'$. Set $\mathcal{Q} := \mathfrak{b}^+ - \mathcal{C}$, $\mathcal{Q}' := \mathfrak{b}^+ - \mathcal{C}'$, and $\Omega'_H := \Omega_H H_1$, where H_1 means the unit neighborhood of e in H. We can find a bi-K–invariant neighborhood $\mathcal{O} \subset G$ of e such that for all $g \in \mathcal{O}$,

$$\Omega_H \exp(\mathcal{Q}_T) K \subset \Omega'_H \exp(\mathcal{Q}'_{T+1}) K g^{-1}.$$

Define $G_T([g]) = \sum_{\gamma\in\Gamma} \mathbf{1}_{\Omega'_H \exp(\mathcal{Q}'_T)K}(\gamma g)$. Now,

$$\#(\Gamma \cap \Omega_H \exp(\mathfrak{b}_T^+ - \mathcal{C})\Omega_K) = \sum_{\gamma\in\Gamma} \mathbf{1}_{\Omega_H \exp(\mathcal{Q}_T)K}(\gamma) \le \langle G_{T+1}, \Phi\rangle,$$

where Φ is a nonnegative K–invariant continuous function supported in $[e]\mathcal{O}$ and $\int_{\Gamma\backslash G} \Phi\, dg = 1$. Now note that

$$\langle G_T, \Phi\rangle = \int_{b\in\mathcal{Q}'_T} \Big(\int_{[e]\Omega'_H} \Phi([h](\exp b))\, d[h] \Big) \xi(b)\, db.$$

Recalling Ω'_H is $H\cap M$–invariant, let $0 \le \psi \le 1$ be an element of $C_c(H)^{H\cap M}$ which is one on Ω'_H. For $\varepsilon_0 > 0$ smaller than the injectivity radius of $\operatorname{supp}(\psi)$, let $\Psi = \psi \otimes \rho_{\varepsilon_0}$ be given as (8.9). Let $\Phi^+_{\varepsilon_0}(y) = \sup_{g\in N_{\varepsilon_0} A_{\varepsilon_0} N^+_{\varepsilon_0} M} \Phi(yg)$ be as in (8.13). Since the closure of $\mathcal{Q}'$ is disjoint from $\mathfrak{b}^+ \cap \mathcal{L}_\Gamma$, by Proposition 2.19 there exists a $T_0 > 0$ such that $\langle (\exp b)\Phi^+_{\varepsilon_0}, \Psi\rangle = 0$ for all $b \in \mathcal{Q}' - \mathcal{Q}'_{T_0}$. Then by the same argument as in the proof of Proposition 8.11, for all $\widetilde{b} \in \mathcal{Q}' - \mathcal{Q}'_{T_0}$ we get that

$$\begin{aligned}\int_{[e]H} \Phi([h](\exp \widetilde{b}))\psi[h]\, dh &= \int_{[e]H(NB)_{\varepsilon_0}} \Phi([h](\exp \widetilde{b}))\Psi([h]n\widetilde{b}) dn\, d\widetilde{b}\, dh \\ &\le \langle (\exp \widetilde{b})\Phi^+_{\varepsilon_0}, \Psi\rangle = 0.\end{aligned}$$

Hence

$$\textbf{(9.27)} \qquad \begin{aligned}\int_{\mathcal{Q}'} \int_{[e]H} \Phi([h](\exp b))\psi[h] dh \xi(b)\, db &= \int_{\mathcal{Q}'_{T_0}} \langle (\exp b)\Phi^+_{\varepsilon_0}, \Psi\rangle \xi(b)\, db \\ &\le \|\Phi^+_{\varepsilon_0}\|_2 \|\Psi\|_2 \mathrm{Vol}(\mathcal{Q}'_{T_0}).\end{aligned}$$

Hence for all $T > T_0$,

$$\#(\Gamma \cap \Omega_H \exp(\mathfrak{b}_T^+ - \mathcal{C})\Omega_K) \le \langle G_{T+1}, \Phi\rangle \le \|\Phi^+_{\varepsilon_0}\|_2 \|\Psi\|_2 \mathrm{Vol}(\mathcal{Q}'_{T_0}).$$

This implies the claim in view of Corollary 9.21. □

For $\omega \in \mathcal{W}$, set $\Gamma^\omega = \omega^{-1}\Gamma\omega$, $H^\omega = \omega^{-1}H\omega$ and $\Omega_{H^\omega} = \omega^{-1}\Omega_H\omega \subset H^\omega$. Then

$$\#(\Gamma \cap \Omega_H \omega \exp(\mathcal{C}_T)\Omega_K) = \#(\Gamma^\omega \cap \Omega_{H^\omega} \exp(\mathcal{C}_T)\Omega_K \omega).$$

Since $\omega \in K$, it follows that $\psi_\Gamma = \psi_{\Gamma^\omega}$, and that the involution which stabilizes H^ω commutes with θ. Hence

$$\delta = \psi_{\Gamma^\omega}(v) = \max_{b\in\mathfrak{b},\, \|b\|=1} \psi_{\Gamma^\omega}(b).$$

By applying Corollary 9.21 to Γ^ω and H^ω for each $\omega \in \mathcal{W}$, we can also deduce the asymptotic of $\#\big(\Gamma \cap \Omega_H \mathcal{W} \exp(\mathcal{C}_T)\Omega_K\big)$.

Theorem 1.10 follows from the following: we set $\mathrm{sk}_{\Gamma,v}(H) = |\mu^{\mathrm{PS}}_{[e]H,v}|$.

Theorem 9.28 *Suppose that $\mathsf{v}_0\Gamma \subset H\backslash G$ is discrete for $\mathsf{v}_0 = [H]$ and that $[e]H$ is uniformly proper. Then $\mathrm{sk}_{\Gamma,v}(H) < \infty$, and there exists a $c_v > 0$ such that*

$$\lim_{T\to\infty} \frac{\#(\mathsf{v}_0\Gamma \cap \mathsf{v}_0 \exp(\mathfrak{b}^+ \cap \mathcal{L}_\Gamma)_T K)}{e^{\psi_\Gamma(v)T}\, T^{(r_0-r)/2}} = c_v\, \mathrm{sk}_{\Gamma,v}(H). \tag{9.29}$$

Moreover, for $v = u_\Gamma$, we have

$$\lim_{T\to\infty} \frac{\#(\mathsf{v}_0\Gamma \cap \mathsf{v}_0(\exp \mathfrak{b}^+_T)K)}{e^{\psi_\Gamma(u_\Gamma)T}\, T^{(r_0-r)/2}} = c_{u_\Gamma}\, \mathrm{sk}_{\Gamma,u_\Gamma}(H),$$

where $\mathfrak{b}^+_T = \{w \in \mathfrak{b}^+ : \|w\| \le T\}$.

Proof Set $\mathcal{C} := \mathfrak{b}^+ \cap \mathcal{L}_\Gamma$. By Theorem 7.7(1) and Lemma 8.4, $\mathcal{C}$ satisfies the conditions in (9.1).

By hypothesis, there exists a K–invariant open neighborhood $\mathcal{O} \subset \Gamma\backslash G$ of $[e]$ such that $Y_0 := \{[e]h \in \Gamma\backslash\Gamma H : [e]h \exp \mathcal{C} \cap \mathcal{O} \neq \varnothing\}$ is bounded. Set

$$F_T([g]) := \sum_{\gamma \in (\Gamma\cap H)\backslash\Gamma} \mathbf{1}_{\mathsf{v}_0 \exp \mathcal{C}_T K}(\mathsf{v}_0\gamma g).$$

Let $\mathcal{O}_\varepsilon \subset G$ and Φ_ε be as in (9.19). We may assume that Φ_ε is K–invariant as our functions F_T are K–invariant, in deducing that

$$\langle F_{T-\varepsilon}, \Phi_\varepsilon\rangle \le F_T([e]) \le \langle F_{T+\varepsilon}, \Phi_\varepsilon\rangle. \tag{9.30}$$

Observe that

$$\langle F_{T\pm\varepsilon}, \Phi_\varepsilon\rangle = \int_K \int_{b\in\mathcal{C}_{T\pm\varepsilon}} \Big(\int_{[e]H} \Phi_\varepsilon([h](\exp bk))\, d[h] \Big) \xi(b)\, db\, dk.$$

For $S > 0$, let $\tau_S \in C_c([e]H)$ be a function satisfying $0 \le \tau_S \le 1$ and $\tau_S = 1$ on the S–neighborhood of Y_0. By the definition of Y_0, it follows that

$$\langle F_{T\pm\varepsilon}, \Phi_\varepsilon\rangle = \int_K \int_{b\in\mathcal{C}_{T\pm\varepsilon}} \Big(\int_{[e]H} \Phi_\varepsilon([h](\exp bk))\tau_S([h])\, d[h] \Big) \xi(b)\, db\, dk.$$

This integral is the same as the one in (9.14), as τ_S is compactly supported. Hence

$$\lim_{T\to\infty} \frac{\langle F_{T\pm\varepsilon}, \Phi_\varepsilon\rangle}{e^{\delta(T\pm\varepsilon)}\, T^{(r_0-r)/2}} = c_v \mu^{\mathrm{PS}}_{[e]H}(\tau_S) m^{\mathrm{BR}}(\Phi_\varepsilon).$$

By sending $\varepsilon \to 0$ and applying Lemma 9.20, we get from (9.30) that

$$\lim_{T\to\infty} \frac{F_T([e])}{e^{\delta T}\, T^{(r_0-r)/2}} = c_v \mu^{\mathrm{PS}}_{[e]H}(\tau_S).$$

It follows that $\mu^{\mathrm{PS}}_{[e]H}(\tau_S)$ is a constant function of $S > 0$, and hence

$$|\mu^{\mathrm{PS}}_{[e]H}| = \mu^{\mathrm{PS}}_{[e]H}(\tau_S) < \infty.$$

This proves the first claim. The second claim follows from the first one in view of the following lemma, by taking $\mathcal{C} = \mathfrak{b}^+ \cap \mathcal{L}_\Gamma$. □

Lemma 9.31 *Let $\mathcal{C} \subset \mathfrak{b}^+$ be a convex cone with $v = u_\Gamma \in \mathrm{int}_{\mathfrak{b}}\, \mathcal{C}$. Set $Q := \mathfrak{b}^+ - \mathcal{C}$. Then there exist $0 < \delta' < \delta_\Gamma = \psi_\Gamma(u_\Gamma)$ and $C > 0$ such that for all $T \geq 1$,*

$$\#(\Gamma \cap \Omega_H \exp(\mathcal{Q}_T)\Omega_K) \leq Ce^{\delta' T}.$$

Proof Choose a closed convex cone $\mathcal{C}' \subset \mathrm{int}_{\mathfrak{b}}\, \mathcal{C}$ so that $u_\Gamma \in \mathrm{int}_{\mathfrak{b}}\, \mathcal{C}'$. Set $\mathcal{Q}' := \mathfrak{b}^+ - \mathcal{C}'$ and $\Omega'_H := \Omega_H H_1$, where H_1 means the unit neighborhood of e in H. We can find a bi-K–invariant neighborhood $\mathcal{O} \subset G$ of e such that for all $g \in \mathcal{O}$,

$$\Omega_H \exp(\mathcal{Q}_T) K \subset \Omega'_H \exp(\mathcal{Q}'_{T+1}) K g^{-1}.$$

Define $G_T([g]) = \sum_{\gamma\in\Gamma} \mathbf{1}_{\Omega'_H \exp(\mathcal{Q}'_T)K}(\gamma g)$. Now

$$\#(\Gamma \cap \Omega_H \exp(\mathcal{Q}_T)\Omega_K) = \sum_{\gamma\in\Gamma} \mathbf{1}_{\Omega_H \exp(\mathcal{Q}_T)K}(\gamma) \leq \langle G_{T+1}, \Phi\rangle, \tag{9.32}$$

where Φ is a nonnegative K–invariant continuous function supported in $[e]\mathcal{O}$ and such that $\int_{\Gamma\backslash G} \Phi\, dg = 1$. Now note that

$$\langle G_T, \Phi\rangle = \int_{b\in\mathcal{Q}'_T} \Big(\int_{[e]\Omega'_H} \Phi([h](\exp b))\, d[h] \Big) \xi(b)\, db.$$

If we write $b = tu_\Gamma + \sqrt{t}w \in \mathcal{Q}'_T$ with $w \in \ker\Theta$, then $t^2 + t\|w\|^2 \leq T^2$ and $\|w\|^2 \geq \tan^2\theta_0 \cdot t$ for some $0 < \theta_0 < \frac{\pi}{2}$ depending on the distance between u_Γ and $\mathcal{Q}'$; cf the proof of Lemma 9.4. Hence if $b = tu_\Gamma + \sqrt{t}w \in \mathcal{Q}'_T$, then

$$0 \leq t \leq T \cdot \cos\theta_0. \tag{9.33}$$

Fix a nonnegative function $\phi \in C_c(H)$ which is 1 on Ω'_H. Since (8.12) gives that

$$\Big| t^{\frac{1}{2}(r-1)} e^{(2\rho-\Theta)(\underline{a}(t,w))} \int_H \Phi([h]a(t,w))\phi(h)\, dh \Big| < C',$$

it follows that for all $t \geq 1$,

$$\Big| \int_H \Phi([h]a(t,w))\phi(h)\, dh \Big| < C' e^{(\Theta-2\rho)(\underline{a}(t,w))}. \tag{9.34}$$

Using (9.16), (9.33) and $\Theta(u_\Gamma) = \delta_\Gamma$, we deduce that

$$\int_{b\in\mathcal{Q}'_T}\Big(\int_H \Phi([h]a(t,w))\phi(h)\,dh\Big)\xi(b)\,db \ll \int_{b\in\mathcal{Q}'_T} e^{\delta_\Gamma t}\,db \le e^{(\delta_\Gamma\cos\theta_0)T}\operatorname{Vol}(\mathfrak{b}_T).$$

As $\operatorname{Vol}(\mathfrak{b}_T) = O(T^{r_0})$, for any δ' satisfying $\delta_\Gamma\cos\theta_0 < \delta' < \delta_\Gamma$ we get

$$\int_{b\in\mathcal{Q}'_T}\Big(\int_H \Phi([h]a(t,w))\phi(h)\,dh\Big)\xi(b)\,db \ll e^{\delta' T}.$$

This proves the lemma by (9.32). □

Finally we give examples satisfying the hypothesis of Theorem 9.28.

Lemma 9.35 *Suppose that $\mathfrak{a} = \mathfrak{b}$ and $\Lambda_\Gamma \subset HP/P$.*

(1) *The orbit $[e]H$ is uniformly proper.*

(2) *The support of $\mu^{\mathrm{PS}}_{[e]H}$ is a compact subset of $(H\cap\Gamma)\backslash H$ and $|\mu^{\mathrm{PS}}_{[e]H}| > 0$.*

Proof The condition $\mathfrak{a} = \mathfrak{b}$ means $H\cap A = \{e\}$, and hence $H\cap P$ is compact by (8.6). Since Λ_Γ is a compact subset of $HP/P \simeq H/(H\cap P)$, it follows that $\Lambda_\Gamma \subset H_0P/P$ for some compact subset $H_0 \subset H$.

To show (1), let $\mathcal{C}$ be a closed cone contained in $\operatorname{int}\mathfrak{a}^+\cup\{0\}$. It suffices to show that for any given compact subset $Z \subset \Gamma\backslash G$,

$$\{[e]h \in \Gamma\backslash\Gamma H : [e]h\exp\mathcal{C}\cap Z \neq \varnothing\}$$

is bounded. Suppose not; then there exist $h_i \in H$ with $[e]h_i \to \infty$ in $\Gamma\backslash\Gamma H$, $\gamma_i \in \Gamma$ and $c_i \to \infty$ in $\exp\mathcal{C}$ such that $\gamma_i h_i c_i$ converges to some $g \in G$. Since $[e]H \subset \Gamma\backslash G$ is a closed subset, it follows that $c_i \to \infty$. We may write $h_i c_i = \gamma_i^{-1} g g_i$, where $g_i \to e$. Set $o = [K] \in G/K$. By passing to a subsequence, we may assume that $\gamma_i^{-1} g g_i o$ converges to some $\xi \in \mathcal{F}$ in the sense of [25, Definition 2.7]; note here that $\mu(\gamma_i^{-1} g g_i) \to \infty$ regularly in $\mathfrak{a}^+$ as $\mathcal{L}_\Gamma \subset \operatorname{int}\mathfrak{a}^+$. Since gg_i is bounded, it follows that $\xi \in \Lambda_\Gamma$ by [25, Lemma 2.12]. Since $h_i c_i \eta = \gamma_i^{-1} g g_i \eta \to \xi$ for all $\eta \in \mathcal{F}$ except for points on a proper submanifold of $\mathcal{F}$, we may choose $\eta = ne^+ \in N^+e^+$ so that $\lim_{i\to\infty} h_i c_i n e^+ = \xi$. Writing $\xi = h_0 e^+$ for some $h_0 \in H_0$, we have $\lim_{i\to\infty} h_i c_i n c_i^{-1} P = h_0 P$. Since $c_i \in \infty$ in $\exp\mathcal{C}$ and $\mathcal{C} \subset \operatorname{int}\mathfrak{a}^+\cup\{0\}$, we have $\lim_{i\to\infty} c_i n c_i^{-1} = e$. As HP is open, we may write $c_i n c_i^{-1} = h'_i p_i \in HP$ with $h'_i \in H$ and $p_i \in P$ both tending to e. It follows that $h_i h'_i P \to h_0 P$ as $i \to \infty$. Since

$H \cap P$ is compact and the sequence h_i' is bounded, it follows that the sequence h_i is bounded, yielding a contradiction. This proves claim (1).

Since $\mathrm{supp}(\mu^{\mathrm{PS}}_{[e]H}) = \{[e]h \in [e]H : he^+ \in \Lambda_\Gamma\} \subset \{[e]h_0 : h_0 \in H_0(H \cap P)\}$, the first part of claim (2) follows, and the second part follows from (8.8). □

We remark that this lemma holds when H is replaced by N^+, by the same proof. Moreover, by replacing Proposition 8.11 by Theorem 7.12 and considering the Iwasawa decomposition $G = N(\exp \mathfrak{a})K$, the proofs of Theorems 1.8 and 9.28 apply for $H = N^+$ and $\mathfrak{b}^+ = \mathfrak{a}^+$.

Example 9.36 Let $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ and $H = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Then $\mathfrak{a} = \mathfrak{b}$ and G/P can be identified with $\mathbb{S}^2 \times \mathbb{S}^2$ and HP/P can be identified with $D \times D$, where D is the open northern hemisphere of $\mathbb{S}^2$. Let Σ be a nonelementary finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. For $i = 1, 2$, consider a convex cocompact representation $\pi_i : \Sigma \to \mathrm{PSL}_2(\mathbb{C})$ such that $\Lambda_{\pi_i(\Sigma)} \subset D$ and $\{(\pi_1(s), \pi_2(s)) : s \in \Sigma\}$ is Zariski dense in G. Then it is easy to see that $\Gamma := \{(\pi_1(s), \pi_2(s)) : s \in \Sigma\}$ is an Anosov subgroup with $\Lambda_\Gamma \subset D \times D$. Hence Γ satisfies the hypothesis of Lemma 9.35.

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