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# THE MULTINOMIAL TILING MODEL

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Given a graph $\mathcal{G}$ and collection of subgraphs $T$ (called tiles), we consider covering $\mathcal{G}$ with copies of tiles in $T$ so that each vertex $v \in \mathcal{G}$ is covered with a predetermined multiplicity. The *multinomial tiling model* is a natural probability measure on such configurations (it is the uniform measure on standard tilings of the corresponding "blow-up" of $\mathcal{G}$).

In the limit of large multiplicities, we compute the asymptotic growth rate of the number of multinomial tilings. We show that the individual tile densities tend to a Gaussian field defined by an associated discrete Laplacian. We also find an exact discrete Coulomb gas limit when we vary the multiplicities.

For tilings of $\mathbb{Z}^d$ with translates of a single tile and a small density of defects, we study a crystallization phenomenon when the defect density tends to zero, and give examples of naturally occurring quasicrystals in this framework.

**1. Introduction.** The study of random tilings is a cornerstone area of combinatorics, probability, and statistical mechanics. In its simplest form, the random tiling model is the study of the set of tilings of a region (for example a subset of the plane) with translated copies of a finite collection of shapes, called prototiles. However even the simplest cases can lead to hard problems. For example the mere existence of a tiling of a region in $\mathbb{R}^2$ with a prescribed set of polyominos is an NP-complete problem [19], even if the prototiles consist in just the $3 \times 1$ and $1 \times 3$ rectangles [2]. Enumerating tilings is of course even harder.

However in the few cases where we *can* analyze random tilings, like random domino tilings (tilings with $2 \times 1$ and $1 \times 2$ rectangles) or lozenge tilings (tilings with $60^\circ$ rhombi), we find very rich behavior, with beautiful enumerative properties [12, 25, 9, 10, 21], phase transitions [16], limit shapes [15], conformal invariance [13], and Gaussian scaling limits [14]. Beyond these and other dimer models there are almost no other cases we can analyze in detail. There are other cases where enumeration is sometimes possible, like the 6-vertex model [20], but for these models very little is known about correlations, although they are sometimes predicted in physics to be Gaussian in the scaling limit and/or conformally invariant —such models were in fact the inspiration for conformal field theory.

We study here a variant of the random tiling problem: the *multinomial tiling problem*, which is tractable in the sense that we can give generating functions for enumerations, which in turn yield, in the limit of large multiplicity, asymptotic expressions for growth rates and Gaussian behavior for random tilings. This setting is quite general and works for tilings in arbitrary graphs, not just plane regions. Furthermore we find all of the phenomena discussed above: phase transitions, limit shapes, crystallization phenomena, and conformal invariance (which we study in a subsequent paper [17]).

It comes as an additional surprise that in certain situations our random tilings form *quasicrystals*. Quasicrystals are non-periodic crystalline structures; they were first found in nature by Schechtman et al [23]. Their physical and mathematical framework is still discussed,

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but examples of quasiperiodic tilings were first found by Berger [3] and familiar examples like Penrose tilings are now well understood [7]; they are sets of tiles which tile the plane but only in nonperiodic fashion, and with the "repetitivity" property that each local pattern of tiles recurs within a finite distance of any point. Our quasicrystals are a type of random tiling with the property that the statistical correlations between tile densities are quasiperiodic: the correlations are superpositions of plane waves with irrationally-related periods (see Section 6.2.3). For other examples of random quasicrystals, see for example [11, 8].

Let $\mathcal{G} = (V, E)$ be a finite graph and let $T = \{t_1, \dots, t_k\}$ be a collection of subsets of $V$, called *tiles*. (For the first part of the paper it is convenient to ignore the edges and think of $(\mathcal{G}, T)$ as a hypergraph, since the edges $E$ play no role; however later the structure of $\mathcal{G}$ as a graph will be relevant as well.)

Let $\mathbf{N} = \{N_v\}_{v\in\mathcal{G}}$ be nonnegative integers associated to vertices of $\mathcal{G}$. Define a new graph $\mathcal{G}_\mathbf{N}$, the "$\mathbf{N}$-fold blowup" of $\mathcal{G}$, to be the graph obtained by replacing each vertex $v$ of $\mathcal{G}$ with $N_v$ vertices, and each edge $uv$ with the complete bipartite graph $K_{N_u,N_v}$. Now each tile $t \in T$ can be lifted to a subset of $\mathcal{G}_\mathbf{N}$ in many ways: if $t$ has vertices $v_1, \dots, v_j$ then it has $N_{v_1} \cdots N_{v_j}$-many lifts.

We consider *tilings* of $\mathcal{G}_\mathbf{N}$ with lifts of copies of tiles in $T$. A tiling is a partition of the vertices of $\mathcal{G}_\mathbf{N}$ into disjoint sets each of which is a lift of a single tile of $T$. Let $\Omega(\mathbf{N})$ be the set of all tilings; we call these $\mathbf{N}$*-fold tilings*.

Let $w : T \to \mathbb{R}_{>0}$ be a positive real weight assigned to each tile. An $\mathbf{N}$-fold tiling $m$ is assigned a weight $w(m) := \prod_{t\in T} w_t^{m(t)}$ where $m(t)$ is the number of copies of $t$ used. The *partition function* for $\mathbf{N}$-fold tilings is defined to be

$$Z(w, \mathbf{N}) = \sum_{m\in\Omega(\mathbf{N})} w(m). \tag{1}$$

We let $\mu = \mu(\mathbf{N}, w)$ be the natural probability measure on $\Omega(\mathbf{N})$, giving an $\mathbf{N}$-fold tiling a probability proportional to its weight.

We note that $\mu$ is *not* the same as the uniform measure on tilings of $\mathcal{G}$ covering each vertex $N_v$ times; each such "multiple tiling" of $\mathcal{G}$ can be typically lifted to a tiling of $\mathcal{G}_\mathbf{N}$ in many ways.

1.1. *Results.* We compute a generating function for $Z(w, \mathbf{N})$ (Theorem 2.1), and the asymptotic growth rate of $Z(w, \mathbf{N})$ as $\min_{v\in\mathcal{G}} N_v \to \infty$, see (7). This computation involves solving a nonlinear system of equations (6); however the solution is realized as the minimum of a convex function (Theorem 3.1).

In Theorem 5.1 we show that in the $\mathbf{N} \to \infty$ limit the tile occupation fractions tend to a *Gaussian field* governed by a discrete Laplacian operator $\Delta$ on $\mathcal{G}$, the *tiling laplacian*.

For transitive graphs, when we vary the multiplicities, we obtain a *Coulomb gas*: defects in multiplicity interact via Coulombic potentials arising from $\Delta$ (see Section 5.4).

Under certain conditions on transitive graphs, our random multinomial tilings also undergo a crystallization phenomenon, where the correlations between distant tiles no longer decay; the tiling freezes into a periodic or quasiperiodic state. This occurs on $\mathbb{Z}^2$, for example, tiled with translates of the $L$-triomino and a small density of singleton monomers (Section 6.2.1). As the density of monomers tends to zero the correlation length of the system tends to infinity, and the system *freezes*. There is a spontaneous symmetry breaking, since there are three distinct crystalline states (corresponding to the three distinct—up to translation—periodic tilings of the plane with $L$ triominos).

For certain other polyominos we get similar freezing phenomena, and others we don't; the behavior depends on the presence and type of zeros of the underlying *characteristic* *polynomial* $p(z, u)$ on the unit torus $\mathbb{T}^2 \subset \mathbb{C}^2$. If the zeros on $\mathbb{T}^2$ are isolated and simple, we show that the resulting tiling will be a quasicrystal (Section 6.2.3). However there is a plethora of nongeneric behavior for the roots of $p$ as the tile type varies, yielding a similarly wide variety of behaviors for random tilings (Section 6.3).

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## 2. Combinatorics.

It is convenient to generalize our definition of tile, to allow the vertices of a tile to have multiplicity larger than one. The vertices of a tile $t$ then form a *multiset* of vertices of $\mathcal{G}$, that is, a subset in which each vertex $v$ has a nonnegative integer multiplicity $t_v$. We identify a tile with its multiset. A lift of a tile $t$ to $\mathcal{G}_{\mathbf{N}}$ corresponds to a choice, for each $v \in \mathcal{G}$, of $t_v$ distinct vertices of $\mathcal{G}_{\mathbf{N}}$ lying above $v$.

### 2.1. *Generating function.*

We associate a variable $x_v$ to each vertex $v \in \mathcal{G}$. To each tile $t \in T$ is associated the monomial $x_t = \prod_{v \in \mathcal{G}} \frac{x_v^{t_v}}{t_v!}$. (Here the factor $t_v!$ accounts for the indistinguishability of the vertices of the same type in a lift of $t$.) Let $P = P(x_1, \dots, x_V)$ be the polynomial $P = \sum_{t \in T} w_t x_t$. We call $P$ the *tiling polynomial*. The function $F(X_1, \dots, X_V) := \log P(e^{X_1}, \dots, e^{X_V})$ is called the *free energy* (see Section 3.2 below).

THEOREM 2.1. Let $x^{\mathbf{N}} = \prod_v x_v^{N_v}$ and $\mathbf{N}! = \prod_v N_v!$. Then

$$Z(w) := \sum_{\mathbf{N} \geq 0} Z(w, \mathbf{N}) \frac{x^{\mathbf{N}}}{\mathbf{N}!} = \exp(P)$$

where the sum is over all vectors of nonnegative multiplicities and $Z(w, \mathbf{N})$ denotes the partition function for $\mathbf{N}$-fold tilings from (1). If we fix the total number $K$ of tiles, then the corresponding generating function is

$$Z_K(w) := \sum_{\mathbf{N} \geq 0} Z(w, \mathbf{N}, K) \frac{x^{\mathbf{N}}}{\mathbf{N}!} = \frac{P^K}{K!}$$

where $Z(w, \mathbf{N}, K) = \sum_{m \in \Omega(\mathbf{N}, K)} w(m)$ and $\Omega(\mathbf{N}, K)$ now denotes the set of all tilings with $K$ tiles in total.

PROOF. Suppose we use tile $t$ with multiplicity $K_t$. Label the abstract copies of tile $t$ with labels $\ell \in \{1, \dots, K_t\}$. To place those tiles in $\mathcal{G}_{\mathbf{N}}$, at each vertex $v$ of $t$, we must choose $K_t$ subsets of size $t_v$ (one of each label $\ell$) out of the $N_v$ vertices of $\mathcal{G}_{\mathbf{N}}$ lying over $v$. Taking into account all tiles, this is a multinomial coefficient at vertex $v$: it is $\frac{N_v!}{\prod_t (t_v!)^{K_t}}$ total choices. We take the product of these over all vertices, and then need to divide by $\prod_t K_t!$, the set of choices of initial labellings. In total, the number of tilings with tile multiplicities $K_t$ and vertex multiplicities $\mathbf{N}$ is

$$\frac{\prod_v N_v!}{\prod_t K_t! \prod_{v,t} (t_v!)^{K_t}}.$$

Multiplying by $\prod_t (w_t \prod x_v^{t_v})^{K_t}$, the tile weights (and factors of $x$), dividing by $\mathbf{N}!$, this is

$$\prod_t \frac{(w_t \prod_v \frac{x_v^{t_v}}{t_v!})^{K_t}}{K_t!}$$

and summing over the $K_t$s gives the result. $\square$

2.2. *Feasible multiplicities.* For a given graph $\mathcal{G} = (V, E)$ and tiling set $T$, not all multiplicities $\mathbf{N} \in (\mathbb{Z}_{\geq 0})^V$ are feasible. The set of feasible multiplicities $\mathcal{M}_\mathbb{Z} = \mathcal{M}_\mathbb{Z}(T, \mathcal{G})$ is (just by definition) the set of nonnegative integer linear combinations of the vectors $v_t = \sum_{v \in V} t_v e_v$, where $\{e_v\}_{v \in V}$ are the standard basis vectors for $\mathbb{Z}^V$.

In other words we have $\mathcal{M}_\mathbb{Z} = D((\mathbb{Z}_{\geq 0})^T)$ where $D : \mathbb{R}^T \to \mathbb{R}^V$ is the linear map defined by $D(e_t) = \sum_{v \in V} t_v e_v$. In the standard basis the matrix of $D$ (which we also denote $D$) is called the *incidence matrix* of the tiling problem: $D = (D_{v,t})$ where $D_{v,t} = t_v$, the multiplicity of $v$ in $t$. Feasible multiplicities $\mathcal{M}_\mathbb{Z}$ are certain integer points in a real polytopal cone $\mathcal{M}_\mathbb{R} \subset \mathbb{R}^V$ defined by $\mathcal{M}_\mathbb{R} = D((\mathbb{R}_+)^T)$.

Typically not all integer points in $\mathcal{M}_\mathbb{R}$ are in $\mathcal{M}_\mathbb{Z}$. For example if all tiles have size $\delta$ then necessarily $\mathbf{N}$ sums to a multiple of $\delta$. More generally if $\phi$ is a homomorphism from $\mathbb{Z}^V$ to some abelian group, with the property that $\phi(D(e_t)) = 0$ for all tiles $t$ then we must have $\phi(\mathbf{N}) = 0$ as well in order for $\mathbf{N}$ to be feasible. In the language of tilings such a $\phi$ is called a “coloring" condition.

As a typical example of a coloring condition, suppose we wish to tile $\mathbb{Z}^2$ or a subgraph of it with translates of bars of length 3: translates of $\{(0,0),(1,0),(2,0)\}$ and $\{(0,0),(0,1),(0,2)\}$. Let $\phi : \mathbb{Z}^{\mathbb{Z}^2} \to \mathbb{R}$ be a homomorphism defined on basis vectors by $\phi(e_{(x,y)}) = \cos(2\pi x/3)$. Note that $\phi$ applied to the translate of any tile is zero:

$$\phi(e_{(x,y)} + e_{(x+1,y)} + e_{(x+2,y)}) = 0 = \phi(e_{(x,y)} + e_{(x,y+1)} + e_{(x,y+2)}).$$

We conclude that $\phi(\mathbf{N}) = 0$ for any feasible multiplicity. The same argument with $\phi'(e_{(x,y)}) = \cos(2\pi y/3)$ gives another linear constraint on $\mathbf{N}$. In fact there is a four-dimensional space of such constraint functions $\phi$, defined by their values on $e_{(x,y)}$ for

$$(x,y) \in \{(0,0),(1,0),(0,1),(1,1)\}.$$

2.3. *Homology.* The incidence map $D : \mathbb{R}^T \to \mathbb{R}^V$ is generally neither surjective nor injective. Letting $D^*$ be its transpose with respect to the standard basis, we have $\mathbb{R}^V \cong \mathrm{Im}(D) \oplus \ker(D^*)$ and $\mathbb{R}^T \cong \mathrm{Im}(D^*) \oplus \ker(D)$. These are orthogonal decompositions with respect to the standard inner products. The map $D$ is an isomorphism from $\mathrm{Im}(D^*)$ to $\mathrm{Im}(D)$, and likewise $D^*$ is an isomorphism from $\mathrm{Im}(D)$ to $\mathrm{Im}(D^*)$

We define $H_1(T,\mathbb{R}) := \mathbb{R}^V / \mathrm{Im}(D) \cong \ker(D^*)$. Over the integers we define $H_1(T,\mathbb{Z}) := \mathbb{Z}^V / \mathrm{Im}(D)$ to be the cokernel of the map $D$. Colorings $\phi$ are then elements of $H_1$, that is, they are functions $\phi$ on vertices which sum to 0 for each tile: $(D^*\phi)(t) = \sum_v t_v \phi(v) = 0$.

If all tiles have the same size $\delta$, then $H_1(T,\mathbb{Z})$ contains a copy of $\mathbb{Z}/\delta\mathbb{Z}$; the corresponding coloring functions are constant functions $f : V \to \mathbb{Z}/\delta\mathbb{Z}$.

For the above example with bars of length 3, consider tilings of an $n \times n$ grid, $n \geq 3$. Then $H_1(T,\mathbb{R}) \equiv \mathbb{R}^4$: an element of $H_1(T,\mathbb{R})$ is determined by its values on the lower left $2 \times 2$ square in the grid, which can be arbitrary reals.

The existence of nontrivial integer constraints has an effect on the long-range behavior of random tilings, see Section 6 below.

2.4. *Laplacian.* The *tiling laplacian* $\Delta : \mathbb{R}^V \to \mathbb{R}^V$ is the operator $\Delta = DCD^*$, where $C$ is the diagonal matrix of tile weights $w_t$. It has matrix $\Delta = (\Delta_{u,v})_{u,v \in V}$ with

$$\Delta_{u,v} = \sum_{t \in T} w_t t_u t_v. \tag{2}$$

Equivalently, for $f : V \to \mathbb{R}$ we have

$$(\Delta f)(v) = \sum_u (\sum_t w_t t_u t_v) f(u).$$

The Laplacian controls the covariances between tile densities, see Section 5.2 below.

2.5. *Gauge equivalence.* Tile weight functions $w, w'$ on $T$ are said to be *gauge equivalent* if there is a positive function $f : V \to \mathbb{R}_+$ such that for all $t \in T$, $w'_t = w_t \prod_{u \in t} f(u)$. We call $f$ a *gauge transformation*.

Note that if $w'$ is gauge equivalent to $w$, then the gauge transformation $f : \mathcal{G} \to \mathbb{R}_+$ from $w$ to $w'$ may not be unique: the set of functions $f$ satisfying $\prod_{v \in t} f(v) = 1$ for all $t$ is by definition the kernel of $D^*$, written multiplicatively (that is, $\log f \in H_1(T, \mathbb{R})$).

LEMMA 2.1. *For fixed multiplicities* $\mathbf{N}$*, gauge equivalent weight functions give the same probability measure on multinomial tilings.*

PROOF. Suppose $w'$ is gauge equivalent to $w$, that is $w'_t = w_t \prod_v f(v)^{t_v}$. An $\mathbf{N}$-fold tiling $m$ for weights $w'$ has weight

$$\begin{aligned}\prod_t (w'_t)^{m(t)} &= \prod_t \left( w_t^{m(t)} \prod_v f(v)^{t_v m(t)} \right) \\ &= \left( \prod_t w_t^{m(t)} \right) \prod_v f(v)^{\sum_t t_v m(t)} \\ &= \left( \prod_t w_t^{m(t)} \right) \prod_v f(v)^{N_v}.\end{aligned}$$

In particular its weight for $w'$ is equal to its weight for $w$ multiplied by a constant independent of $m$. □

**3. Asymptotics.** In this section, we compute the asymptotic growth of $Z(w, \mathbf{N})$ as $\mathbf{N} \to \infty$. In many cases, this leads to an essentially unique closed expression.

3.1. *Fixing the number of tiles.* Given the multiplicities $\mathbf{N}$, it is convenient to also fix the total number of tiles $K$: otherwise we have to sum over possibly many $K$ values, and larger $K$ values typically dominate entropically, resulting in a distribution skewed towards having only the smallest tiles.

If all tiles have the same size $\delta$, then the total number of tiles $K$ is fixed and determined by the multiplicities $\mathbf{N}$: we have $K = \sum_v N_v / \delta$. More generally, we proceed as follows. We adjoin a new "dummy" vertex $v_0$ to $\mathcal{G}$, connected to all other vertices. Let $\tilde{\mathcal{G}} = \mathcal{G} \cup \{v_0\}$ be this new graph. We add to each tile a number of copies of the dummy vertex $v_0$ so that all tiles now have the same size $\delta$. Let $x_0$ be a variable associated to the new vertex $v_0$, and let $P_0$ be the new tiling polynomial; it is a homogenization of $P$, replacing a monomial $z$ by $\frac{x_0^m}{m!} z$, where $m + \deg(z) = \delta$. The number $\delta$ is the degree of $P_0$, and the size of every tile.

Let $N_0$ be an arbitrary multiplicity at $v_0$, and $M = \sum_v N_v$ be the total multiplicity of the other vertices (not including $v_0$). For tileability we need $M + N_0$ to be a multiple of $\delta$:

$$M + N_0 = K\delta. \tag{3}$$

Note then that given the remaining multiplicities, the choice of $N_0$ is linearly related to the number of tiles $K$.

We assume for the rest of the paper, unless explicitly stated, that **all tiles have the same size** $\delta$. Notationally we can then use $\mathcal{G}$ instead of $\tilde{\mathcal{G}}$ and $P$ instead of $P_0$.

3.2. *Critical gauge and growth rate.* In this section we define the *critical gauge* and use it to compute the growth rate of $Z$. The critical gauge also allows us to compute tile probabilities and, eventually, the full correlation field between tiles (see Section 5 below).

For each $v \in V$ let $\alpha_v \in \mathbb{R}_+$ be fixed. Let $\vec{\alpha} = (\alpha_v)_{v\in V}$. We suppose $\vec{\alpha} \in \mathcal{M}_{\mathbb{R}}(\mathcal{G})$, that is, $\vec{\alpha}$ is in the real cone of feasible multiplicities. Take $N_v \to \infty$ simultaneously for each $v$, in such a way that each $\mathbf{N}$ is feasible: $\mathbf{N} \in \mathcal{M}_{\mathbb{Z}}(\mathcal{G})$, and $\frac{N_v}{K} \to \alpha_v$. The quantity $\alpha_v$ is the (asymptotic) fraction of tiles covering $v$, and

$$\sum_v \alpha_v = \delta. \tag{4}$$

From the second part of Theorem 2.1, we have

$$\frac{K!}{\mathbf{N}!} Z(w, \mathbf{N}) = [\mathbf{x}^{\mathbf{N}}] P^K,$$

where the notation on the right-hand side denotes the coefficient of $\mathbf{x}^{\mathbf{N}}$ in $P^K$.

We can obtain an expression for the growth of this coefficient as $\mathbf{N} \to \infty$ as above, using a standard Legendre transform argument.

THEOREM 3.1. For any $\vec{\alpha} \in \mathcal{M}_{\mathbb{R}}(\mathcal{G})$ and weight function $w$ there is a unique gauge-equivalent weight function $w'$ with the property that for all $v$ the sum of weights of tiles containing vertex $v$ (counted with multiplicity) is $\alpha_v$, that is

$$\sum_t w'_t t_v = \alpha_v. \tag{5}$$

The corresponding gauge transformations $f : V \to \mathbb{R}_{>0}$ are exactly those which solve the "criticality equations"

$$\frac{x_v (P)_{x_v}}{P} = \alpha_v \tag{6}$$

with $x_v = f(v)$. The growth rate of $\frac{K!}{\mathbf{N}!} Z(w, \mathbf{N})$ is

$$\sigma(w, \vec{\alpha}) := \lim_{K\to\infty} \frac{1}{K} \log \frac{K!}{\mathbf{N}!} Z(w, \mathbf{N}) = \log P(\mathbf{x}) - \sum_v \alpha_v \log \mathbf{x}_v \tag{7}$$

for any solution $\mathbf{x}$ to (6).

We call $\sigma(w, \vec{\alpha})$ the *exponential growth rate* of the multinomial tiling model. We call $w'$ of this theorem the *critical weight function*, or weight function in the *critical gauge*.

As discussed in the proof below, positive solutions to (6) always exist but are not in general unique. However two positive solutions differ only by a gauge equivalence in $H_1(T, \mathbb{R})$, and as a consequence give rise to the same weight function $w'$ and growth rate $\sigma$. See Section 3.4 below for an example with nonuniqueness.

PROOF. We assign positive real values (to be determined) to the variables $x_v$ and let $w'_t = w_t \prod_{v\in t} x_v$. Define $X_v = \log x_v$ and $F(X_1, \dots, X_V) := \log P(e^{X_1}, \dots, e^{X_V})$. Then $F$ is a smooth function of the $X_i$'s. Its gradient is

$$\nabla F = \left(\frac{x_1 P_{x_1}}{P}, \dots, \frac{x_V P_{x_V}}{P}\right).$$

However $x_v P_{x_v} = \sum_t w_t t_v \prod_u x_u^{t_u} = \sum_t t_v w'_t$, so $F$ has gradient lying in $\mathrm{Im}(D)$:

$$\nabla F = \frac{1}{P} D\left(\sum_t w'_t e_t\right).$$

Moreover we claim that $F$ is convex. If we interpret $P(x_1,\dots,x_V)$ (after scaling so that $P(1)=1$) as the probability generating function for a $\mathbb{R}^V$-valued random variable $Y$, then the Hessian matrix $H_F$ of $F$, that is $H_F = (\frac{\partial^2 \log P}{\partial X_u \partial X_v})_{u,v\in V}$, is the covariance matrix of $Y$, hence positive semidefinite.

Recalling that $\mathbb{R}^V$ has an orthogonal decomposition $\mathbb{R}^V = \mathrm{Im}(D)\oplus\ker(D^*)$, and noting that the support of $Y$ is contained in $\mathrm{Im}(D)$ and spans $\mathrm{Im}(D)$, we see that $F$ is strictly convex on $\mathrm{Im}(D)$, as these are directions where the variance is positive, and $F$ is constant on directions in $\ker(D^*)$, that is, those orthogonal to $\mathrm{Im}(D)$.

Let $S(\vec\alpha)$ be the Legendre dual of $F$: for $\vec\alpha\in\mathcal{M}_{\mathbb{R}}\subset\mathrm{Im}(D)$, we define

$$S(\alpha_1,\dots,\alpha_V) = \max_{X_1,\dots,X_V}\left\{-\log P(e^{X_1},\dots,e^{X_V}) + \alpha_1 X_1 + \cdots + \alpha_V X_V\right\}. \tag{8}$$

Then $S$ is strictly convex on $\mathcal{M}_{\mathbb{R}}$ and defined on all of $\mathcal{M}_{\mathbb{R}}\cap\{\sum_v \alpha_v = \delta\}$. From (8) we have

$$\alpha_v = \frac{\partial}{\partial X_v}\log P(e^{X_1},\dots,e^{X_V})$$

so that the criticality equations are satisfied with $\mathbf{x}_v = e^{X_v}$. Comparing with (7) we see that $-S(\vec\alpha) = \sigma(w,\vec\alpha)$ is the growth rate function.

The maximizing $X_v$ are unique up to a global additive constant and up to translations in $\ker D^*$; these translations correspond precisely to gauge transformations not changing the tile weights. The additive constant allows us to scale all weights so that $P(1)=1$. After this scaling (6) says precisely that the sum of weights of tiles containing $v$ (counted with multiplicity) is $\alpha_v$. ☐

Since $\sigma$ is concave, a solution to (6) can be found from the maximization of the concave function in (8).

The Legendre duality allows us not only to compute the growth rate. For each $\vec\alpha$ there is a probability distribution on tiles whose probability distribution function is $\tilde P(x_1,\dots,x_V) := \frac{P(e^{X_1}x_1,\dots,e^{X_V}x_V)}{P(e^{X_1},\dots,e^{X_V})}$, with the property that monomials of $\tilde P^K$ concentrate near $\mathbf{N}$, that is, the expectation of the vertex densities is $\mathbf{N}/K$. For this distribution tile $t$ has probability proportional to $w'_t$, and the expectation of $Y$ is $\vec\alpha$. Thus we have

COROLLARY 3.2. For the critical gauge $w'$, tile probabilities are proportional to tile weights, that is, the expected number of tiles of type $t$ is $Kw'_t$, where $K$ is the total number of tiles.

One consequence of this corollary is that there is, for any choice of tile probabilities (satisfying the necessary condition of summing to $\alpha_v$ at vertex $v$ for each $v$), a choice of tile weights $w_t$, unique up to gauge, for which the multinomial tiling model (in the large-$K$ limit) has those tile probabilities.

3.3. *Example.* Consider tilings of $\mathcal{G}_1 = \{1,2,3,4,5\}\subset\mathbb{Z}$ with tiles consisting of single vertices and pairs of adjacent vertices: the tiles are $T=\{1,2,3,4,5,12,23,34,45\}$. We add a dummy vertex $v_0$ to all singleton tiles and let $\mathcal{G}=\mathcal{G}_1\cup\{v_0\}$. Suppose $N_i = N$ for $i\neq v_0$, and all tile weights are $1$. Then

$$P = x_0(x_1+x_2+x_3+x_4+x_5) + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5.$$

We have $N_0 + 5N = 2K$. Let $\alpha = N/K$ and $\alpha_0 = N_0/K$ (note $\alpha_0 + 5\alpha = 2 = \delta$).

The feasible range of $\alpha$ is $\alpha\in[\frac15,\frac13]$: when $\alpha = 1/5$, $K=5N$ and we need to use only singleton tiles, and when $\alpha=1/3$, $K=3N$ and we need to use the maximum proportion of

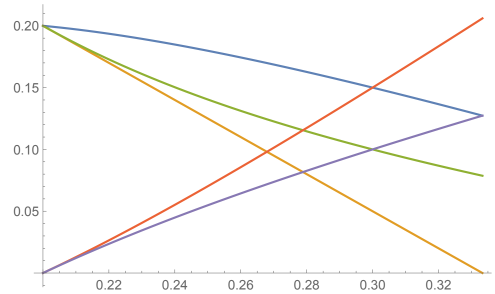


FIG 1. *Tile probabilities for* $\alpha \in [1/5, 1/3]$. $x_0x_1$ *in blue,* $x_0x_2$ *in orange,* $x_0x_3$ *green,* $x_1x_2$ *red,* $x_2x_3$ *purple.*

long tiles (which is two long tiles for every singleton tile); moreover the singleton tiles must be $x_1, x_3$ or $x_5$.

Solving the criticality equations (6) we find the tile probabilities

$$x_0x_1 = \frac{1}{4}(1+3\alpha - \sqrt{1-10\alpha+41\alpha^2})$$
$$x_0x_2 = \frac{1}{2}(1-3\alpha)$$
$$x_0x_3 = \frac{1}{2}(1-7\alpha + \sqrt{1-10\alpha+41\alpha^2})$$
$$x_1x_2 = \frac{1}{4}(-1+\alpha + \sqrt{1-10\alpha+41\alpha^2})$$
$$x_2x_3 = \frac{1}{4}(-1+9\alpha - \sqrt{1-10\alpha+41\alpha^2})$$

and the remaining probabilities are given by symmetry.

Tile probabilities are plotted in Figure 1.

3.4. *Example.* Here is an example with nontrivial homology. Consider tilings of a cycle of length 4: $\{1,2,3,4\}$ with dimers $\{12,23,34,41\}$. Then $P = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = (x_1+x_3)(x_2+x_4)$. The space $\mathrm{Im}(D) \subset \mathbb{R}^V$ is the orthocomplement of the vector $(1,-1,1,-1)$, so $H_1(T,\mathbb{R})$ has rank 1 and is generated by this vector. The feasible $\vec{\alpha}$ are those which satisfy $\sum_v \alpha_v = 2$ and are in $\mathrm{Im}(D)$, that is, satisfy the conditions $\alpha_1 + \alpha_3 = 1 = \alpha_2 + \alpha_4$. The criticality equations are

$$\frac{x_1x_2 + x_1x_4}{P} = \alpha_1, \frac{x_1x_2 + x_2x_3}{P} = \alpha_2, \frac{x_2x_3 + x_3x_4}{P} = \alpha_3, \frac{x_1x_4 + x_3x_4}{P} = \alpha_4,$$

which reduce to

$$\frac{x_1}{x_1+x_3} = \alpha_1, \quad \frac{x_2}{x_2+x_4} = \alpha_2.$$

Solutions are not unique: given any solution we can multiply $x_1, x_3$ by a constant $t$ and divide $x_2, x_4$ by $t$ to get another solution. We have

$$F(X_1,\dots,X_4) = \log((e^{X_1}+e^{X_3})(e^{X_2}+e^{X_4})).$$

This leads to the growth rate

$$\begin{aligned}\sigma(\vec{\alpha}) &= -\alpha_1\log\alpha_1 - \alpha_2\log\alpha_2 - \alpha_3\log\alpha_3 - \alpha_4\log\alpha_4\\ &= -\alpha_1\log\alpha_1 - \alpha_2\log\alpha_2 - (1-\alpha_1)\log(1-\alpha_1) - (1-\alpha_2)\log(1-\alpha_2)\\ &= h(\alpha_1)+h(\alpha_2)\end{aligned}$$

where $h(p)$ is the Shannon entropy $h(p) = -p\log p - (1-p)\log(1-p)$.

**4. Dimers.** A special case of the multinomial tiling model is the *multinomial dimer model*, where tiles are simply all pairs of adjacent vertices (also known as “dimers"). A 1-dimer tiling is then a perfect matching, also known as *dimer cover* of $\mathcal{G}$.

4.1. *Bipartite graphs.* For the dimer model, when $\mathcal{G}$ is bipartite, $V = B\cup W$, there is an equivalent but perhaps more efficient method of computing $Z(w,\mathbf{N})$. From (5) we need to look for a gauge function $x : V \to \mathbb{R}_{>0}$ such that

$$\alpha_{\mathrm{b}} = \sum_{\mathrm{w}\sim\mathrm{b}} w_{\mathrm{wb}}x_{\mathrm{w}}x_{\mathrm{b}} \tag{9}$$

and likewise for white vertices. However we can use (9) to define $x_{\mathrm{b}}$:

$$x_{\mathrm{b}} = \frac{\alpha_{\mathrm{b}}}{\sum_{\mathrm{w}\sim\mathrm{b}} w_{\mathrm{wb}}x_{\mathrm{w}}}.$$

Then we only have equations involving the remaining half of the variables: those at the white vertices $x_{\mathrm{w}}$. These equations are

$$\alpha_{\mathrm{w}} = \sum_{\mathrm{b}\sim\mathrm{w}} \alpha_{\mathrm{b}}\frac{w_{\mathrm{wb}}x_{\mathrm{w}}}{\sum_{\mathrm{w}\sim\mathrm{b}} w_{\mathrm{wb}}x_{\mathrm{w}}}. \tag{10}$$

In the standard case where $N_v \equiv N$, all the $\alpha_{\mathrm{w}}, \alpha_{\mathrm{b}}$ are equal and the saddle point equations correspond to the property that the *sum of (critical) edge weights at each vertex is* 1. This is just a restatement of Corollary 3.2 in this setting, since the sum of edge probabilities at each vertex is 1 for a random dimer cover.

4.2. *Aztec Diamond Example.* The *Aztec diamond* of order $n$ is a diamond-shaped subregion of $\mathbb{Z}^2$ of horizontal diameter $2n-1$; see Figure 2, left panel for the $n=4$ Aztec diamond. It is known to have $2^{n(n+1)/2}$ single-dimer covers (see [9] and [10]). Consider $\mathbf{N}$-fold dimer covers with $N_v \equiv N$ and $w_t \equiv 1$. The critical edge weights sum to 1 at each vertex, and are defined by this property and the property of being gauge equivalent to $w$, that is, around each square face the critical weights $a,b,c,d$ satisfy $ac = bd$. Surprisingly, the critical weights in this example are rational. The critical weights for $n=4$ are shown on the left, and for general $n$ (scaled by $n(n-1)$) on the right in Figure 2.

We can work out the growth rate $\sigma$ in this case as follows. Since $P=1$, and $\alpha_v = \frac{1}{k(k+1)}$ is a constant, we have

$$\sigma = -\frac{1}{k(k+1)}\sum_v \log x_v = -\frac{1}{k(k+1)}\log\prod x_v.$$

This product is the weight of any single dimer cover. The “all horizontal" dimer cover has dimers of weight $\frac{1}{k(k+1)}$ times: $k^2$ for the top row, $k(k-1)$ and $(k-1)k$ for the next row, and generally $k(k-i+1),(k-1)(k-i+2),\dots,(k-i+1)k$ for the $i$ row, for $i$ from 1

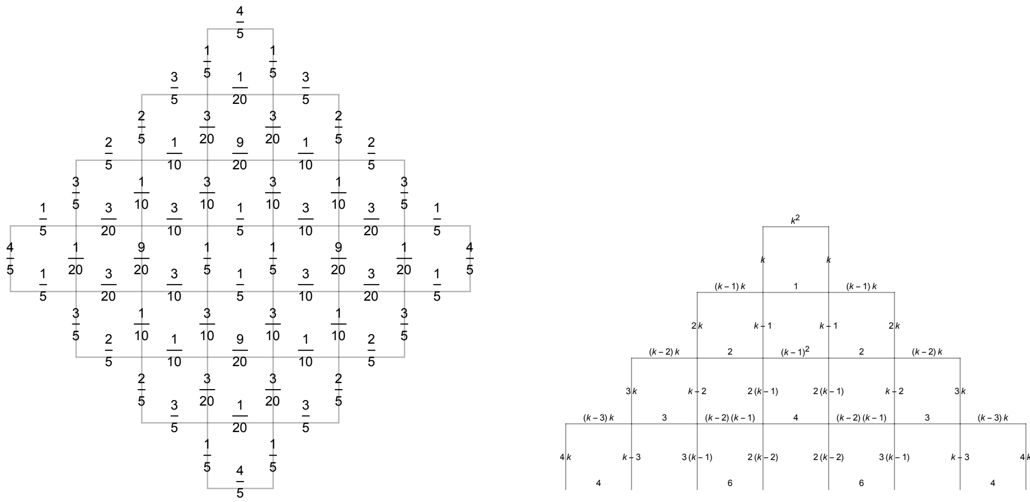

FIG 2. *Aztec diamond of order $k = 4$ (left) in critical gauge. On right, critical gauge edge weights for general $k$, multiplied by $k(k+1)$. The weight on an edge is a quadratic function of its $x, y$ coordinates and its parity.*

to $k$, then repeating for the bottom half of the diamond. The total product of edge weights of the dimer cover is

$$\frac{(k^k \cdot (k-1)^{k-1} \cdots 2^2 \cdot 1)^4}{(k(k+1))^{k(k+1)}}$$

This yields for the exponential growth rate the remarkable value $\sigma = 1 + O(\frac{\log k}{k^2})$.

Associated to a multinomial dimer cover with constant $N_v \equiv N$ of a subgraph of $\mathbb{Z}^2$ is a *height function* $h$ on the dual graph. The height function is defined to be zero on a fixed face, and the change in height across an edge wb (when crossing the edge so that the white vertex is on the left) is $-N/4$ plus the number of dimers on that edge. In [4], see also [5], the authors prove a limit shape phenomenon for single dimer covers: the (rescaled by $n$) height function for a random dimer cover of $AD(n)$ converges with probability one as $n \to \infty$ to a nonrandom piecewise analytic function on the rescaled diamond $|x| + |y| \leq 1$. For the $N$-fold dimer cover discussed above we get a similar, but analytic, limit shape. In the $N \to \infty$ limit it is just the function $h(x, y) = x^2 - y^2$. This follows from a short calculation using the above edge probabilities . The limit shape phenomenon for multinomial tilings is discussed more generally in [18].

4.3. *Path example.* For fixed $n > 0$ take $\mathcal{G}$ to be the (bipartite) $n \times n$ honeycomb graph of Figure 3, with edge weights 1. There are $\binom{2n}{n}$ single dimer covers: dimer covers correspond bijectively to monotone lattice paths from $(0, 0)$ to $(n, n)$. The bijection is obtained by taking a dimer cover and shrinking all horizontal edges of $\mathcal{G}$ to points.

Let us consider $\mathbf{N}$-dimer covers of $\mathcal{G}_n$, where $N_v \equiv N$ and $w_t = 1$. Index the white vertices $x_{i,j}$ as in the figure. The criticality equations (10) are

$$(11) \qquad \frac{x_{i,j}}{x_{i,j} + x_{i-1,j} + x_{i,j-1}} + \frac{x_{i,j}}{x_{i,j} + x_{i+1,j} + x_{i+1,j-1}} + \frac{x_{i,j}}{x_{i,j} + x_{i,j+1} + x_{i-1,j+1}} = 1$$

and boundary conditions $x_{i,j} = 0$ for $i < 0$ or $j < 0$ or $(i, j) = (n, n)$.

In the limit $n \to \infty$, there is a solution to (11) given by

$$x_{i,j} = (i + j)! \binom{i+j}{i}.$$

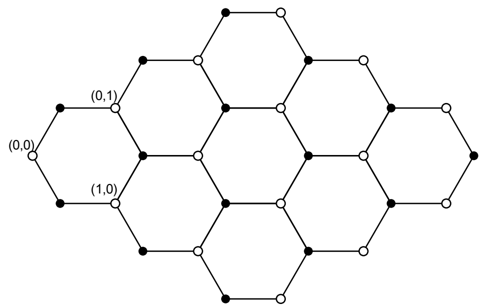


FIG 3. $3 \times 3$ *honeycomb graph.*

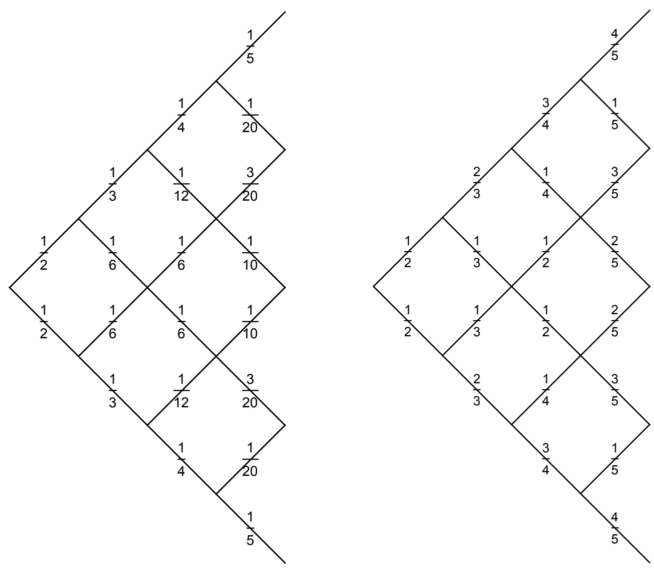

FIG 4. *Edge probabilities (left). Corresponding random walk probabilities (right).*

However we don't know if this solution is the only one up to gauge (since the graph is infinite, unicity does not necessarily hold).

The edge probabilities for this solution are (for horizontal, NE, SE edges respectively out of a white vertex $(i, j)$)

$$\frac{i+j}{i+j+1}, \quad \frac{j+1}{(i+j+1)(i+j+2)}, \quad \frac{i+1}{(i+j+1)(i+j+2)}$$

for the (horizontal, resp. NE, resp. SE) edge at $(i, j)$.

These edge probabilities give a unit flow on $\mathbb{N} \times \mathbb{N}$ from $(0, 0)$ to $\infty$, see Figure 4 left panel; the value on an edge represents the flow from left to right along that edge. The value on an edge is the probability that a certain monotone random walk uses that edge. The transition probabilities of this random walk are shown on the right in Figure 4; this random walk is the *Polya urn*[1].

[1]An urn starts out with one red and one green ball. A ball is selected at random and replaced along with another ball of the same color. This process is then repeated many times. The resulting distribution of the number of red balls after $k$ steps is uniform on $[1, k]$.

4.4. *Example in higher dimensions.* For a higher dimensional multinomial dimer example, consider the infinite subgraph of $\mathbb{Z}^3$ in the slab $0 \le x+y+z \le n$ where $n$ is even, and $w \equiv 1$. To get a finite graph we can quotient by a cofinite sublattice of $\{x+y+z=0\}$. Such a graph has a higher proportion of white vertices than black vertices (assuming the origin is white); the density ratio is $(n+1)/n$. Take $N_{\mathrm{w}}$ constant, and $N_{\mathrm{b}}$ a different constant with $N_{\mathrm{w}}/N_{\mathrm{b}} = n/(n+1)$. Then a critical gauge is given up to scale by: for $\mathrm{w} = (x,y,z)$, $x_{\mathrm{wb}} = n-x-y-z$ if $b = \mathrm{w}+e_i$ and $x_{\mathrm{wb}} = x+y+z$ if $\mathrm{b} = \mathrm{w}-e_i$ (here $e_1,e_2,e_3$ are the standard basis vectors).

There are analogous examples in all dimensions $d \ge 1$.

## 5. Multiplicities and fluctuations.

5.1. *Changing multiplicities.* We compute the change in growth rate $\sigma$ (from equation (7)) under a small change in the multiplicities $\alpha_v \to \alpha_v + d\alpha_v$. This will be used below to compute tile covariances. Since $\sum_v \alpha_v = \delta$, the sum of changes is necessarily zero: $\sum_v d\alpha_v = 0$.

Recall the incidence matrix $D = (D_{v,t})_{v\in V, t\in T}$, defined by $D_{v,t} = t_v$. Differentiating (5) we find for each vertex $u$:

$$\sum_t w_t t_u x_t \Big(\sum_v t_v \frac{dx_v}{x_v}\Big) = d\alpha_u,$$

or

$$\sum_{v,t} D_{u,t} w_t x_t D_{t,v} \frac{dx_v}{x_v} = d\alpha_u.$$

Thus

$$\sum_v \Delta_{u,v} \frac{dx_v}{x_v} = d\alpha_u, \tag{12}$$

where $\Delta = DCD^*$ is the tiling laplacian for the critical weights. Equation (12) says that as a function of $v$, $\frac{dx_v}{x_v}$ is harmonic with respect to the laplacian $\Delta$ at all vertices $u$ for which $d\alpha_u = 0$.

Equation (12) will have a solution if and only if $d\vec{\alpha}$ is in the image of $\Delta$, which is the same as $\mathrm{Im}(D)$, since the laplacian is invertible on $\mathrm{Im}(D)$, mapping it to itself (and $\Delta$ is zero on $\ker D^*$). The solution is unique up to an element of $\ker \Delta = \ker D^* = H_1(T,\mathbb{R})$; as discussed in Section 2.5 these correspond to gauge transformations not changing the tile weights.

5.2. *Covariance of tile densities.* Let $X_t$ be the random variable counting the number of occurrences of tile $t$ in an $\mathbf{N}$-fold tiling. We wish to compute the covariance $\mathrm{Cov}(X_t, X_{t'}) = \mathbb{E}[X_t X_{t'}] - \mathbb{E}[X_t]\mathbb{E}[X_{t'}]$ for two tiles $t,t'$.

Note that $X_t$ is itself a sum of $\{0,1\}$-valued random variables, $X_t = \sum_{i=1}^{M_t} X_t^i$, where the sum runs over all $M_t = \prod_v \binom{N_v}{t_v}$ possible lifts of the tile $t$ to $\mathcal{G}_{\mathbf{N}}$. It suffices to compute the covariance $\mathrm{Cov}(X_t^i, X_{t'}^j)$.

5.2.1. *The case $t \ne t'$.* Assume first that $t \ne t'$. By the symmetry of $\mathcal{G}_{\mathbf{N}}$, if $t$ and $t'$ are disjoint this is independent of $i$ and $j$: $\mathrm{Cov}(X_t^i, X_{t'}^j) = \mathrm{Cov}(X_t^1, X_{t'}^1)$. (If $t,t'$ overlap, see below.) We have

$$\begin{aligned}\mathbb{E}[X_t^1 X_{t'}^1] &= \Pr(X_t^1 = 1, X_{t'}^1 = 1)\\ &= \Pr(X_{t'}^1 = 1 | X_t^1 = 1)\Pr(X_t^1 = 1)\\ &= \Pr{}^*(X_{t'}^1 = 1)\Pr(X_t^1 = 1),\end{aligned}$$

where the star denotes the probability measure on the graph $\mathcal{G}_{\mathbf{N}^*}$ where we have reduced the multiplicities of vertices $v$ by $t_v$. We thus have

$$\mathbb{E}[X_t X_{t'}] = M_t M_{t'} \mathbb{E}[X_t^1 X_{t'}^1] = \mathbb{E}[X_t] \frac{M_{t'}}{M_{t'}^*} \mathbb{E}^*[X_{t'}] = \mathbb{E}[X_t]\mathbb{E}^*[X_{t'}],$$

since $M_{t'} = M_{t'}^*$ when $t, t'$ are disjoint.

When $t$ and $t'$ overlap, the number of disjoint lifts of $t$ and $t'$ is

$$M_{t,t'} := \prod_v \binom{N_v}{t_v, t'_v} = \prod_v \binom{N_v}{t_v}\binom{N_v - t_v}{t'_v} = M_t \prod_v \binom{N_v - t_v}{t'_v}.$$

Thus

(13)
$$\mathbb{E}[X_t X_{t'}] = M_{t,t'} \mathbb{E}[X_t^1 X_{t'}^1] = M_{t,t'} \mathbb{E}[X_t^1]\mathbb{E}^*[X_{t'}^1] = \mathbb{E}[X_t] \frac{M_{t,t'}}{M_t} \mathbb{E}^*[X_{t'}^1] = \mathbb{E}[X_t]\mathbb{E}^*[X_{t'}],$$

since the number of lifts of $t'$ in $\mathcal{G}_{\mathbf{N}^*}$ is exactly

$$\prod_v \binom{N_v - t_v}{t'_v} = \frac{M_{t,t'}}{M_t}.$$

Thus in either case, if $t \neq t'$,

$$\mathrm{Cov}(X_t, X_{t'}) = \mathbb{E}[X_t](\mathbb{E}^*[X_{t'}] - \mathbb{E}[X_{t'}]).$$

5.2.2. *The case $t = t'$.* Finally, if $t = t'$ are the same tile, then we have a slightly different computation. We have $X_t = X_t^1 + \cdots + X_t^{M_t}$ so

$$\mathbb{E}[X_t^2] = M_t \mathbb{E}[X_t^1] + \sum_{i \neq j} \mathbb{E}[X_t^i X_t^j].$$

Here in the sum we only get a contribution from disjoint lifts, so if $X_t^1$ and $X_t^2$ are disjoint, this is

$$\mathbb{E}[X_t^2] = \mathbb{E}[X_t] + M_{t,t} \mathbb{E}[X_t^1 X_t^2].$$

Thus from the computation in (13), with $t = t'$

$$\mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2 = \mathbb{E}[X_t] + \mathbb{E}[X_t](\mathbb{E}^*[X_t] - \mathbb{E}[X_t]).$$

5.2.3. *Computation of $\mathbb{E}^*$.* We can now compute $\mathbb{E}^*[X_{t'}]$ from the methods of section 5.1. We need to change $\alpha_v = \frac{N_v}{K}$ to $\frac{N_v - t_v}{K-1}$. Thus

$$d\alpha_v = \frac{\alpha_v K - t_v}{K-1} - \alpha_v = \frac{\alpha_v - t_v}{K-1}.$$

Recalling $\mathbb{E}[X_{t'}] = K w_{t'} x_{t'}$, we have (using (12), and ignoring lower order terms)

$$\begin{aligned}
\mathbb{E}^*[X_{t'}] - \mathbb{E}[X_{t'}] &= (K-1) w_{t'} (x_{t'} + dx_{t'}) - K w_{t'} x_{t'} \\
&= (K-1) w_{t'} x_{t'} (1 + \sum_u t'_u \frac{dx_u}{x_u}) - K w_{t'} x_{t'} \\
&= w_{t'} x_{t'} (-1 + (K-1) \sum_u t'_u \frac{dx_u}{x_u}) \\
&= w_{t'} x_{t'} (-1 + (K-1) \sum_{u,v} D_{t',u} \Delta^{-1}_{u,v} d\alpha_v)
\end{aligned}$$

$$= w_{t'} x_{t'} (-1 + \sum_{u,v} D_{t',u} \Delta^{-1}_{u,v} (\alpha_v - D_{v,t}))$$

$$= w_{t'} x_{t'} (-1 - (D^* \Delta^{-1} D)_{t',t} + \sum_{u,v} D_{t',u} \Delta^{-1}_{u,v} \alpha_v). \tag{14}$$

Now note that

$$\sum_v \Delta_{u,v} = \sum_{v,t} D_{u,t} w_t x_t D_{t,v} = \delta \sum_t D_{u,t} w_t x_t = \delta \alpha_u,$$

so $\Delta^{-1}_{u,v} \alpha_v = \frac{1}{\delta} 1_u$ and thus $\sum_{u,v} D_{t',u} \Delta^{-1}_{u,v} \alpha_v = 1$. The last sum in (14) cancels the $-1$ and we have

$$\mathbb{E}^*[X_{t'}] - \mathbb{E}[X_{t'}] = -w_{t'} x_{t'} \mathbb{K}_{t',t},$$

where $\mathbb{K} = D^* \Delta^{-1} D$, and so for $t \neq t'$ at critical gauge

$$\mathrm{Cov}(X_t, X_{t'}) = -K w_t w_t' \mathbb{K}_{t',t}.$$

Likewise when $t = t'$,

$$\mathrm{Var}(X_t) = K w_t (1 - w_t \mathbb{K}_{t,t}).$$

Combining these two cases we have

$$\mathrm{Cov}(X_t, X_{t'}) = K w_t (I - C\mathbb{K})_{t',t} \tag{15}$$

where $I$ is the identity matrix. Note that $C\mathbb{K} = (C\mathbb{K})^2$ is a projection matrix from $\mathbb{R}^T$ onto the subspace $C\,\mathrm{Im}(D^*)$, with kernel $\ker(D)$. Thus $I - C\mathbb{K}$ is the complementary projection.

5.3. *Gaussian fluctuations.* For the multinomial tiling model with multiplicities $\mathbf{N}$, as above let $X_t$ be the random variable representing the multiplicity of tile $t$. We consider a limit $K \to \infty$ as in Section 3. Scaling $P$ to 1, we can consider $P$ to be the probability generating function (pgf) of a single tile. Then $P^K$ is the pgf of placing $K$ i.i.d. tiles. The tile multiplicities under this process have a multinomial distribution, exactly analogous to rolling a $|T|$-sided biased die $K$ times. Each tile multiplicity is a binomial $B(K, w_t)$ random variable. These multiplicities tend in the limit of large $K$ to Gaussian random variables which are independent except for the constraint that their sum is $K$. When we impose the constraints on the multiplicities $N_v$ this is an additional linear constraint (which implies the first). The resulting random variable is also a multidimensional Gaussian: a multidimensional Gaussian conditioned to lie in an affine subspace is again a multidimensional Gaussian.

A multidimensional Gaussian is determined by its covariance matrix. The covariance can be obtained from the original covariance matrix and the constraint matrix: just take the positive definite quadratic form defining the Gaussian and restrict it to the affine space to get a new positive definite quadratic form, whose inverse restricted to that space is the covariance. In our case we have already computed the covariance matrix above in (15).

THEOREM 5.1. In the limit of large $K$ the joint distribution of the $X_t$ tends to a multidimensional Gaussian with mean $\mathbb{E}[\vec{X}] = K\vec{w}$ (where $\vec{w}$ is the vector of critical tile weights) and covariance matrix $\mathrm{Cov}(X_s, X_t) = K w_s (I - C\mathbb{K})_{t,s}$.

Examples are explored in Section 6.

5.4. *Coulomb gas.* In this section we assume for simplicity that $\mathcal{G}$ is regular, $T$ and $w$ are symmetric under a transitive group of automorphisms of $\mathcal{G}$, and $N_v \equiv N$. We also assume $t_v \in \{0,1\}$ for all tiles and vertices. We have $K = nN/\delta$, and $\alpha_v = \delta/n$ for all $v$. Under these conditions the critical vertex weights are $x_v \equiv x$ where $x$ is a constant.

Suppose that we take a small perturbation of the multiplicities $N_v = N + q_v$ where $\frac{q_v}{N} = o(1)$. We call $q_v$ the *$d$-charge* at $v$. It can be positive or negative; we'll take the sum of $d$-charges to be zero. Each $\alpha_v$ is then of the form $\alpha_v = \delta/n + d\alpha_v$ where $d\alpha_v = \frac{\delta q_v}{nN}$.

How does $\sigma$ change under this perturbation? To first order from (7) we have

$$d\sigma(\vec{\alpha}) = \sum_v \frac{P_{x_v}}{P} dx_v - \sum_v \alpha_v \frac{dx_v}{x_v} - \sum_v \log x_v d\alpha_v = -\sum_v \log x_v d\alpha_v = 0$$

since $\log x_v$ is constant. To second order we have

$$\sigma(\vec{\alpha} + d\vec{\alpha}) - \sigma(\vec{\alpha}) =$$

$$\sum_v \left(\frac{P_{x_v x_v}}{P} - \frac{P_{x_v}^2}{P^2}\right)\frac{dx_v^2}{2} + \sum_{u\neq v}\left(\frac{P_{x_u x_v}}{P} - \frac{P_{x_u}P_{x_v}}{P^2}\right)dx_u dx_v - \sum_v \frac{d\alpha_v dx_v}{x_v} + \sum_v \frac{\alpha_v}{x_v^2}\frac{dx_v^2}{2}.$$

Substituting $x_v = x, P_{x_v} = \alpha_v/x, P = 1, P_{x_v x_v} = 0$ and $\alpha_v = \delta/n$ gives

$$= \sum_v \left(\frac{\delta}{n} - \frac{\delta^2}{n^2}\right)\frac{dx_v^2}{2x^2} + \sum_{u\neq v}\left(\sum_t w_t t_u t_v - \frac{\delta^2}{n^2}\right)\frac{dx_u}{x}\frac{dx_v}{x} - \sum_v d\alpha_v \frac{dx_v}{x}$$

which we can write as

$$= \frac{d\vec{x}^t}{x}\left(\frac{1}{2}\Delta - \frac{\delta^2}{2n^2}J\right)\frac{d\vec{x}}{x} - d\vec{\alpha}^t \cdot \frac{d\vec{x}}{x}$$

where $J$ is the all-1's matrix. But

$$\frac{d\vec{x}^t}{x} J \frac{d\vec{x}}{x} = d\vec{\alpha}\Delta^{-1}J\Delta^{-1}d\vec{\alpha} = 0$$

(since $\Delta^{-1}$ has constant row and column sums, $\Delta^{-1}J\Delta^{-1}$ is a multiple of $J$). Using (12) we are left with

THEOREM 5.2. Under the above conditions on $\mathcal{G}, \mathbf{N}, T, w$ the second derivative of $\sigma(\vec{\alpha})$ in direction $d\vec{\alpha}$ (of sum 0) is

$$d^2\sigma = -\frac{1}{2} d\vec{\alpha}\Delta^{-1}d\vec{\alpha}.$$

If we fix the $d$-charges but allow them to move from vertex to vertex, considering $\sigma$ as a function of position of these charges, we see that $d$-charges interact with a potential defined by $\Delta^{-1}$. This is the *Coulomb potential* associated with $\Delta$.

5.4.1. *Example: dimer case.* Consider for example the case of the multinomial dimer model on a bipartite graph $\mathcal{G}$. Define the *charge* $\tilde{q}_v$ of a vertex of d-charge $q_v$ to be $\tilde{q}_v = d(v)q_v$ where $d(v) = 1$ for a white vertex and $d(v) = -1$ for a black vertex. Then note that $\tilde{\Delta}_{u,v} = d(u)\Delta_{u,v}d(v)$ is the *standard graph laplacian*.

Suppose we fix the charges $\tilde{q}_v$, but not their locations. We can interpret $\sigma$ as a function of charge positions as a potential energy which causes like charges to repel and opposite charges to attract. That is, $\sigma$ is larger when like charges are farther apart and opposite charges are closer. The *force* of repulsion/attraction is naturally given by the gradient of the Green's function $\tilde{\Delta}^{-1}$ (that is, the gradient of the potential). For $\mathcal{G} = \mathbb{Z}^2$, for example this is a "$1/r$" force for distant particles, where $r$ is the vector between them. For $\mathcal{G} = \mathbb{Z}^3$ it is a "$1/r^2$" force for distant particles. These conclusions are consistent with standard electrostatics in $\mathbb{R}^2$ and $\mathbb{R}^3$, when the charges are far apart (compared to the lattice spacing). In $2d$ these results agree with corresponding results obtained for the single dimer model by Ciucu [6].

**6. Crystallization.** In this section we restrict our graphs $\mathcal{G}$ to be $\mathbb{Z}^d$ or a quotient graph $\mathbb{Z}^d/n\mathbb{Z}^d$ (eventually for $n$ large). As we shall see, the behavior of a random tiling can depend on $n$ in such a way that the limit $n \to \infty$ might not exist.

For fixed finite $\mathcal{G}$ the Gaussian approximation (Theorem 5.1) only holds in the large-$K$ limit, so for a sequence of growing graphs in order to have a Gaussian limit we should assume that $K$ is growing much faster than the size of $\mathcal{G}$; in other words limits should be taken first in $K$ then in $|\mathcal{G}|$.

We consider tilings with tiles $T$ which are translates of one or more "prototiles". In the simplest case we have only two prototiles $\mathrm{t}, \mathrm{t}_0$, where $\mathrm{t}_0$ is a *single vertex*. Then a $\mathbf{N}$-fold tiling of $\mathcal{G}$ with $T$ is a tiling of $\mathcal{G}_{\mathbf{N}}$ with lifts of translates of $\mathrm{t}$ which has a number of *holes* (which are locations which are covered by lifts of translates of $\mathrm{t}_0$). We are interested in what happens when the fractional density of holes goes to zero. Depending on the shape of $\mathrm{t}$, the system will *crystallize*.

Let us explain what we mean by this crystallization. While there are several different definitions of what it means to be a crystal (see e.g. [22]), we define here a crystal as a structure with long-range order. By long range order for our random tilings we mean that tile correlations do not decay with distance: (some) pairs of distant (in $\mathcal{G}$) tiles have scaled multiplicities whose scaled covariances (covariances divided by $\sqrt{K}$) are bounded below by a positive constant independently of their distance in $\mathcal{G}$. See Theorem 6.1 below for a statement about the concrete form of the covariances. Depending on the tile shapes, correlation patterns of highly correlated tiles may be either periodic or aperiodic. If aperiodic, we call the system a quasicrystal.

Before giving a general result, we work out some explicit examples, which are of interest in their own right, and which illustrate the general situation. First in dimension one we consider the case of triominos with no holes (Section 6.1.1) and then with a positive fraction of holes (Section 6.1.2). In Section 6.2.1 we consider the $L$ triomino in $\mathbb{Z}^2$. Further examples in $\mathbb{Z}^2$ with more interesting behavior are studied in Sections 6.2.3 and 6.3.

6.1. *Examples in one dimension.*

6.1.1. *Example: bars of length* 3. Consider tilings of the cycle $\mathcal{G} = \mathbb{Z}/n\mathbb{Z}$ with translates of the triomino $\mathrm{t} = \{0, 1, 2\}$, that is $T = \{t_0, \ldots, t_{n-1}\}$ with $t_i = \{i, i+1, i+2\}$ with cyclic indices. We choose constant multiplicities $N_v \equiv N$. Then the total number of tiles is $K = Nn/3$ and the fraction of tiles per vertex is $\alpha = 3/n$. Since the graph is regular, for the critical gauge we have $x_v \equiv x$, and $P = nx^3 = 1$, so the weight per tile is $w_t = x^3 = 1/n$. The tile laplacian $\Delta$ satisfies, for a function $f \in \mathbb{R}^V$,

$$(\Delta f)(j) = \frac{1}{n}(3f(j) + 2f(j-1) + 2f(j+1) + f(j+2) + f(j-2))$$

with cyclic indices.

It is convenient to use the Fourier transform to invert $\Delta$. For $z$ an $n$th root of $1$ let $U_z$ be the subspace of $\mathbb{C}^V$ consisting of $z$-periodic functions

$$U_z = \{f \in \mathbb{C}^V \mid f(x+1) = zf(x)\}.$$

Here $U_z$ is the eigenspace for translation by 1 on $\mathcal{G}$ with eigenvalue $z$. The operator $\Delta$ preserves each $U_z$ and its action on $U_z$ is multiplication by

$$\lambda_z = \frac{1}{n}(1+z+z^2)(1+z^{-1}+z^{-2}).$$

The matrix $D$ can likewise be written in a Fourier basis, if we identify a tile $t_i$ with the location of its left endpoint $i$. The action of $D$ on $U_z$ is then multiplication by $1+z+z^2$, and that of $D^*$ is multiplication by $1+1/z+1/z^2$. We see that $\mathbb{K} = D^*\Delta^{-1}D$ is simply multiplication by $n$ on subspaces $U_z$ for which $z^2+z+1 \neq 0$, and 0 on subspaces $U_z$ for which $z^2+z+1=0$.

Suppose that $n$ is not a multiple of 3. Then $\mathbb{K}$ is a scalar, equal to multiplication by $n$. The covariance matrix is identically zero: $\mathrm{Cov}(X_t, X_{t'}) = 0$. This is not surprising since the tile multiplicities $X_i$ are not random: we necessarily have $X_i = N/3$ for all $i$.

Suppose that $n$ is a multiple of 3. Then $\frac{1}{n}\mathbb{K}$ is a projection matrix $\mathbb{P}_3$ onto the span of the subspaces $U_z$ where $z^2+z+1 \neq 0$. The covariance matrix is $\mathrm{Cov}(X_t, X_t') = \frac{K}{n}(I-\mathbb{P}_3)$, where $I-\mathbb{P}_3$ is the projection onto the (two-dimensional) span of the $U_z$ where $z^2+z+1 = 0$. That is, in the standard basis on $\mathbb{R}^V$,

$$\mathrm{Cov} = \frac{K}{n^2}\begin{pmatrix} 2 & -1 & -1 & 2 & \\ -1 & 2 & -1 & -1 & \\ -1 & -1 & 2 & -1 & \\ 2 & -1 & -1 & 2 & \\ & & & & \ddots \end{pmatrix}. \tag{16}$$

Pairs of tiles $t, t'$ at distance a multiple of 3 are perfectly correlated: we have a crystal. This is also not surprising since the multiplicity at $j$ determines $X_j$ as a function of $X_{j-1}, X_{j-2}$: we have $X_{j-2}+X_{j-1}+X_j = N$, which implies $X_j = X_{j+3}$ for all $j$.

6.1.2. *Example: bars of length* 3 *and singletons.* Let us consider a variant of the above model, where we also allow singleton tiles, with weight 1. As discussed in Section 3.1 we include a dummy vertex $v_0$ in $\mathcal{G}$, with multiplicity 2 for all singleton tiles; we can then use the multiplicity $N_0$ of the dummy vertex to control the number of singleton tiles. We have $N_0 + nN = 3K$, and $\alpha_0 = N_0/K$ and $\alpha = N/K$ satisfy $\alpha_0 + n\alpha = 3$. Also

$$P_0 = \frac{x_0^2}{2}\sum x_i + \sum x_i x_{i+1} x_{i+2} = \frac{n}{2}x_0^2 x + n x^3$$

where we replaced $x_i$ with $x$ by circular symmetry. A short calculation shows that the critical weights $w_0, w$ are $w_0 = \frac{\alpha_0}{n}$ and $w = \frac{2-\alpha_0}{2n}$.

In this case the laplacian can be partially diagonalized. We write

$$\mathbb{R}^{\mathcal{G}} = \left(\bigoplus_{z^n=1} U_z\right) \oplus U_0$$

where $U_z$ is as above and $U_0 \cong \mathbb{R}$ is the space of functions on $v_0$. The laplacian $\Delta$ preserves each $U_z$ for $z \neq 1, 0$, and acts by multiplication by

$$\lambda_z = w_0 + w(1+z+z^2)(1+z^{-1}+z^{-2})$$

on these $U_z$. The laplacian does not preserve $U_1$ but does preserve the sum $U_1 \oplus U_0$. On the space $U_1 \oplus U_0$, with basis given by $e_1 = (1, 1, \ldots, 1, 0)$ and $e_0 = (0, 0, \ldots, 0, 1)$, the laplacian acts as the matrix

$$\begin{pmatrix} w_0 + 9w & 2w_0 \\ 2nw_0 & 4nw_0 \end{pmatrix}. \tag{17}$$

More generally, for a tile of size $A$, the matrix would be

$$\begin{pmatrix} w_0 + A^2 w & (A-1)w_0 \\ (A-1)nw_0 & (A-1)^2 n w_0 \end{pmatrix}$$

where $n = |\mathcal{G}|$; we'll use this below. Likewise the kernel $\mathbb{K} = D^*\Delta^{-1}D$ has a similar decomposition into the subspaces $U_z$ and $U_1 \oplus U_0$. On $U_z, z \neq 1, 0$ it acts as multiplication by

$$\frac{(1+z+z^2)(1+z^{-1}+z^{-2})}{w_0 + w(1+z+z^2)(1+z^{-1}+z^{-2})}.$$

On $U_1 \oplus U_0$, if we take a single triomino $X_t$ it corresponds to the vector $\frac{3}{\sqrt{n}}e_1 + 0e_0$. Thus the contribution to the covariance is $\frac{1}{wn}$, which is $\frac{9}{n}$ times the $1,1$ entry in the inverse of (17).

Now using Theorem 5.1,

$$\text{Cov}(X_s, X_t) = Kw\delta_{s=t} - \frac{Kw}{n} - \frac{Kw^2}{n}\sum_{\substack{z^n=1\\ z\neq 1}} \frac{z^{s-t}(1+z+z^2)(1+z^{-1}+z^{-2})}{w_0 + w(1+z+z^2)(1+z^{-1}+z^{-2})}. \tag{18}$$

Here the second term $-\frac{Kw}{n}$ is the component of $-Kw^2(D^*\Delta^{-1}D)_{s,t}$ on the subspace $U_1$, that is $-Kw^2(\frac{1}{nw})$.

The covariance (18) only depends on $s-t$; without loss of generality assume $t=0$. Now (18) simplifies to

$$\text{Cov}(X_0, X_s) = \frac{Kw_0}{n}\sum_{\substack{z^n=1\\ z\neq 1}} \frac{z^s}{w_0/w + (1+z+z^2)(1+z^{-1}+z^{-2})}. \tag{19}$$

As long as $w_0/w \ll 1$, to leading order this sum is localized on the region where $z$ is close to one of the primitive cube roots of 1. If $\frac{1}{n^2} \ll \frac{w_0}{w} \ll 1$ and $n$ is large, we can approximate the sum with an integral. Writing $z = e^{i(2\pi/3 + 2\pi j/n)}$, and $\varepsilon = w_0/w$, and letting $u = 2\pi j/n$, the contribution near $e^{2\pi i/3}$ is, up to a multiplicative $1+o(1)$ factor,

$$\frac{Kw_0 \exp(\frac{2\pi i s}{3})}{2\pi}\int_{-\infty}^{\infty}\frac{e^{isu}du}{\varepsilon + 3u^2} = \frac{Kw_0\exp(\frac{2\pi i s}{3})}{2\sqrt{3\varepsilon}}e^{-|s|\sqrt{\varepsilon/3}}.$$

The integral near the other primitive cube root of 1 contributes the complex conjugate, and we get

$$\text{Cov}(X_0, X_s) \sim \frac{Kw_0\cos(\frac{2\pi s}{3})}{\sqrt{3\varepsilon}}e^{-|s|\sqrt{\varepsilon/3}}. \tag{20}$$

We see the onset of the 3-periodic correlations between the $X_s$ as $\varepsilon \to 0$.

Note that the variances and covariances here are much larger than in the $w_0 = 0$ case: here they are of order

$$\frac{Kw_0}{\sqrt{\varepsilon}} = K\sqrt{ww_0} = \frac{K\sqrt{\alpha_0(2-\alpha_0)}}{2n},$$

which is (for constant $\alpha_0$) a factor of $n$ larger than that for the pure triomino case above.

For smaller $\varepsilon$, write $\varepsilon = \beta/n^2$; in this case expanding near the primitive cube roots of 1 gives

$$\begin{aligned}\text{Cov}(X_0, X_s) &\sim \frac{K\beta\cos(\frac{2\pi s}{3})}{n^2}\sum_{j\in\mathbb{Z}}\frac{1}{\beta + 12\pi^2 j^2}\\ &= \frac{K\sqrt{\beta}\cos(\frac{2\pi s}{3})}{2n^2\sqrt{3}}\coth(\frac{\sqrt{\beta}}{2\sqrt{3}}) = \frac{K\cos(\frac{2\pi s}{3})}{n^2}(1 + \frac{\beta}{36} + O(\beta^2)),\end{aligned}$$

where for the last line we used the Mittag-Leffler expansion for the hyperbolic cotangent function

$$\pi\coth(\pi z)=\frac{1}{z}+2\sum_{n=1}^{\infty}\frac{z}{z^2+n^2}.$$

6.1.3. *Quasiperiodic example in dimension* 1. On $\mathbb{Z}/n\mathbb{Z}$ let $\mathrm{t}$ be the tile with characteristic polynomial $p(z)=2/z+1+2z$. It has two roots of modulus 1 which are not roots of unity. We tile with $\mathrm{t}$ and $\mathrm{t}_0$ the singleton, and as before let $\varepsilon=w_0/w$. The covariance function is given by the analogue of (19) where the denominator is replaced by $w_0/w+|p(z)|^2$. For $1/n^2\ll w_0/w\ll 1$ we get

$$\mathrm{Cov}(X_0,X_s)\sim\frac{Kw_0}{2\pi i}\int_{S^1}\frac{z^s}{\varepsilon+p^2}\frac{dz}{z}.$$

The integral can be localized near the two roots $e^{\pm i\theta_0}$ of $p$. Letting $z=e^{i(\theta_0+x)}$ we get

$$\mathrm{Cov}(X_0,X_s)\sim 2Kw_0\,\mathrm{Re}\left[\frac{e^{is\theta_0}}{2\pi}\int_{\mathbb{R}}\frac{e^{isx}dx}{\varepsilon+2p'(\theta_0)^2x^2}\right]=\frac{Kw_0\cos(s\theta_0)e^{-|s|\sqrt{\varepsilon a}}}{\sqrt{\varepsilon a}}$$

where $a=2p'(\theta_0)^2=32\sin^2(\theta_0)$.

For a similar example without multiplicity, we can take $p(z)=1+z+z^3+z^5+z^6$.

6.2. *Examples in higher dimensions.*

6.2.1. *Example: L triomino.* For a $2d$ example, consider tilings of the torus $\mathbb{Z}^2/n\mathbb{Z}^2$ with translates of the triomino $\mathrm{t}=\{(0,0),(1,0),(0,1)\}$, and constant multiplicities $\mathbf{N}\equiv N$. Let $\Gamma$ be the sublattice of $\mathbb{Z}^2$ generated by $(1,1)$ and $(3,0)$. Translates of $\mathrm{t}$ by $\Gamma$ tile $\mathbb{Z}^2$. Translates of this tiling by the three cosets of $\Gamma$ in $\mathbb{Z}^2$ give three distinct periodic tilings.

We have $K=Nn^2/3$, $\alpha=3/n^2$. The graph is regular, so for the critical gauge we have $x_v\equiv x$, and $w_t=1/n^2$.

For $z,u$ two $n$th roots of 1, define $U_{z,u}$ to be the subspace of $\mathbb{C}^V\cong\mathbb{C}^{n^2}$ consisting of $(z,u)$-periodic functions, that is, functions $f:\mathbb{Z}^2/n\mathbb{Z}^2\to\mathbb{C}$ such that $f(i+1,j)=zf(i,j)$ and $f(i,j+1)=uf(i,j)$.

The action of $\Delta$ on $U_{z,u}$ is multiplication by

$$\lambda_{z,w}=\frac{1}{n^2}(1+z+u)(1+z^{-1}+u^{-1}).$$

If we index tiles according to the location of their lower left vertex, then the matrix $D$ corresponds to multiplication by $1+z+u$, and $D^*$ by $1+z^{-1}+u^{-1}$. Then $\mathbb{K}=D^*\Delta^{-1}D$ is multiplication by $n^2$ of spaces $U_{z,u}$ for which $1+z+u\neq 0$, and zero on spaces $U_{z,u}$ where $1+z+u=0$.

If $n$ is not a multiple of 3, then there are no subspaces $U_{z,u}$ where $1+z+u=0$, and so $\mathbb{K}=n^2I$ is a scalar multiple of the identity, and Cov is the zero matrix: all tile multiplicities are determined and constant. If $n$ is a multiple of 3, then $\mathrm{Cov}=\frac{K}{n^2}(I-\mathbb{P})$ where $I-\mathbb{P}$ is the projection onto the span of $U_{\omega,\omega^2}\oplus U_{\omega^2,\omega}$ (here $\omega=e^{2\pi i/3}$). The covariance between tiles is $2K/n^4$ if they lie in the same translate of $\Gamma$ and $-K/n^4$ if they do not. Again this is a *perfect crystal*; the tiles differing by translates in $\Gamma$ are perfectly correlated.

Let us now allow a small fraction of singleton tiles: $N_0+n^2N=3K$, and $\alpha_0=N_0/K$. As before $w_0=\frac{\alpha_0}{2n^2}$ and $w=\frac{2-\alpha_0}{2n^2}$. The laplacian $\Delta$ still preserves each $U_{z,u}$ for $(z,u)\neq(1,1)$, and on $U_{z,u}$ acts by multiplication by

$$\lambda_{z,w}=w_0+w(1+z+u)(1+z^{-1}+u^{-1}).$$

On $U_{1,1} \oplus U_0$ it acts as the matrix

$$\begin{pmatrix} w_0 + 9w & 2w_0 \\ 2n^2 w_0 & 4n^2 w_0 \end{pmatrix}.$$

We have

$$\begin{aligned}\mathrm{Cov}(X_{(0,0)}, X_{(s,t)}) &= Kw\delta_{s=t=0} - \frac{Kw}{n^2} - \frac{Kw^2}{n^2} \sum_{\substack{z^n=1=u^n \\ (z,u)\neq(1,1)}} \frac{z^s u^t (1+z+u)(1+z^{-1}+u^{-1})}{w_0 + w(1+z+u)(1+z^{-1}+u^{-1})} \\ &= \frac{Kw_0}{n^2} \sum_{\substack{z^n=1=u^n \\ (z,u)\neq(1,1)}} \frac{z^s u^t}{w_0/w + (1+z+u)(1+z^{-1}+u^{-1})}.\end{aligned}$$

Now assume $w_0/w \ll 1$ and is fixed as $n \to \infty$. The sum is then localized to the region near $(z,u) = (\omega, \omega^2)$ or $(\omega^2, \omega)$. Near the first point write $z = e^{i(2\pi/3+x)}$ and $u = e^{i(4\pi/3+y)}$, and take $\varepsilon = w_0/w$. The contribution near this point is

$$\sim \frac{Kw_0 \exp(\frac{2\pi i(s+2t)}{3})}{(2\pi)^2} \iint_{\mathbb{R}^2} \frac{e^{i(sx+ty)}dx\,dy}{\varepsilon + x^2 - xy + y^2}.$$

The integral here for $(s,t) \neq (0,0)$ is a Bessel-K function (see the appendix Section 8),

$$= \frac{Kw_0 \exp(\frac{2\pi i(s+2t)}{3})}{\pi\sqrt{3}} B(\frac{2}{\sqrt{3}}\sqrt{\varepsilon(s^2+st+t^2)}).$$

Summing over both roots, and taking $\varepsilon$ small we have

$$\mathrm{Cov}(X_{(0,0)}, X_{(s,t)}) = \frac{Kw_0 \cos(\frac{2\pi}{3}(s+2t))}{\pi\sqrt{3}} \left(\log\frac{1}{\varepsilon} - \log\frac{2(s^2+st+t^2)}{3} - 2\gamma_E + O(\varepsilon)\right)$$

where $\gamma_E$ is the Euler gamma.

For the variance, that is, when $(s,t) = (0,0)$,

$$\mathrm{Var}(X_{0,0}) \sim \frac{Kw_0}{(2\pi i)^2} \iint \frac{dzdu}{(1+u)(z-r_1)(z-r_2)}$$

where $r_1, r_2$ are the roots of the denominator which is a quadratic polynomial in $z$. Let $r_1$ be the root inside the unit circle; then using residues this is

$$= \frac{Kw_0}{2\pi i} \int_{S^1} \frac{du}{(1+u)(r_1-r_2)}$$

$$= \frac{Kw_0}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{3 + 6\varepsilon + \varepsilon^2 + (4+4\varepsilon)\cos\theta + 2\cos(2\theta)}}$$

and splitting the integral into parts near $\theta = 2\pi/3, 4\pi/3$ and the remainder, we arrive at

$$= \frac{Kw_0}{\pi\sqrt{3}}(\log\frac{9}{\varepsilon})(1+o(1)).$$

See Figure 5 for a plot of the covariances for small $\varepsilon$.

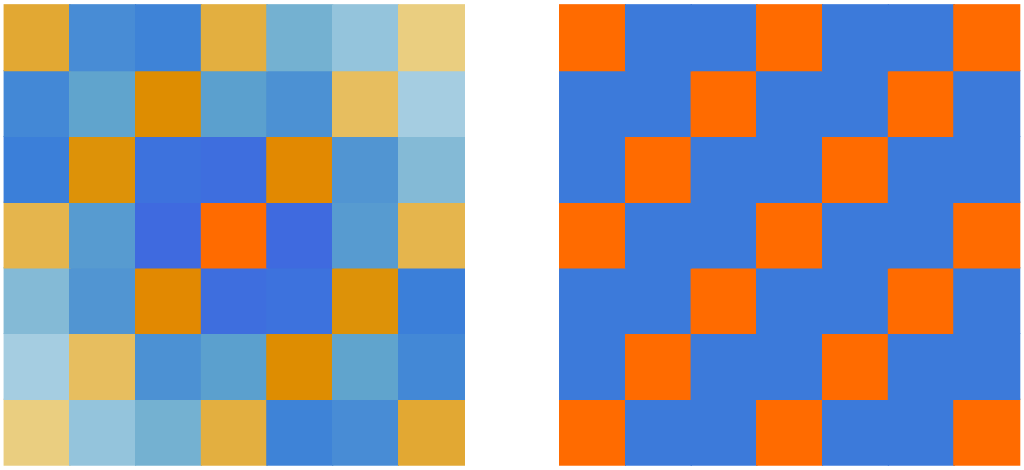

FIG 5. *Covariances for the L polyomino for* $\varepsilon = 0.001$ *(left) and in the limit* $\varepsilon = 0$ *(right). Here values are scaled to the range from* $1$ *(orange) to* $-1$ *(dark blue) with* $0$ *being white.*

6.2.2. *Examples in* $\mathbb{Z}^2$ *with simple zeros..* The above example of the $L$ triomino can be generalized. Consider the case of tiling with a polyomino $\mathrm{t}$ in $\mathbb{Z}^2$ and a small density of singletons $\mathrm{t}_0$. The characteristic polynomial $p(z,u)$ of a polyomino $\mathrm{t}$ is defined as

$$p(z,u) = \sum_{(i,j)\in \mathrm{t}} z^i u^j.$$

As in the previous section we have

$$\begin{aligned}\mathrm{Cov}(X_{0,0}, X_{s,t}) &= \frac{Kw_0}{n^2} \sum_{\substack{z^n=1=w^n\\(z,u)\neq(1,1)}} \frac{z^s u^t}{w_0/w + p(z,u)p(1/z,1/u)} \\ &\sim \frac{Kw_0}{(2\pi)^2} \iint_{(S^1)^2} \frac{z^s u^t}{\varepsilon + p(z,u)p(1/z,1/u)} \frac{dz}{iz}\frac{du}{iu}. \qquad (21)\end{aligned}$$

Suppose now that $p(z,u)$ has a finite number of roots $\{(z_i,u_i)\}_{i=1,\dots,k}$ on $\mathbb{T}^2 = S^1 \times S^1$. A root $(z,u)$ is *simple* if $\frac{zP_z}{uP_u} \notin \mathbb{R}$; this means the zero set of $p$ intersects the unit torus transversely at that point. We suppose for the moment that all roots $(z_i,u_i)$ are simple.

For $w_0/w$ a small constant the integral (21) can be localized near each root. Near a simple root $(z_0,u_0)$ the integral is approximated by

$$\frac{Kw_0 z_0^s u_0^t}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{i(sx+ty)}\,dx\,dy}{\varepsilon + ax^2 + bxy + cy^2}$$

where the quadratic form in the denominator is

$$ax^2 + bxy + cy^2 = |z_0 P_z(z_0,u_0)x + u_0 P_u(z_0,u_0)y|^2.$$

Since the root is simple this quadratic form is positive definite, and the integral is defined and finite (for $(s,t)\neq(0,0)$). This integral is a (linear image of a) Bessel-K function (see section 8). Summing over all roots, the covariance is a superposition of these Bessel-type functions.

We also need the $s=t=0$ case, that is, the variance, which we cannot get using Bessel functions. We have

$$\mathrm{Var}(X_{0,0}) = \frac{Kw_0}{(2\pi)^2} \iint \frac{1}{\varepsilon + p(z,u)p(1/z,1/u)} \frac{dz}{iz}\frac{du}{iu}.$$

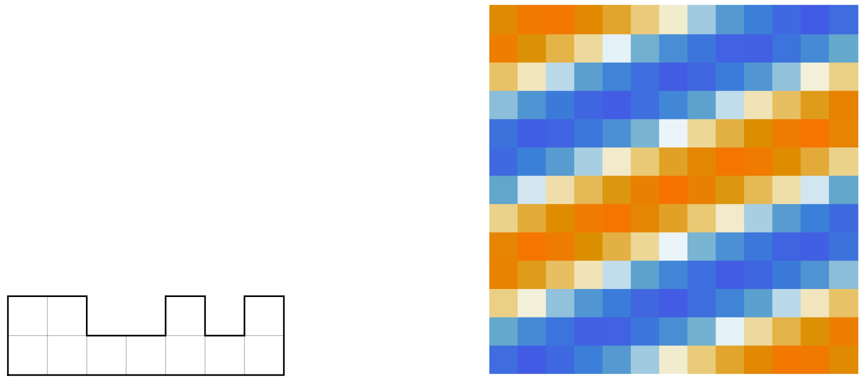

FIG 6. *The key polyomino and its (quasiperiodic) covariance function in the limit $\varepsilon \to 0$.*

Localize to a ball $B$ of radius $M\sqrt{\varepsilon}$ around a simple root $(z_0, u_0)$ (where $M$ is a large constant). Let $z = z_0 e^{is\sqrt{\varepsilon}}$ and $u = u_0 e^{it\sqrt{\varepsilon}}$. The contribution from this ball is

$$\frac{Kw_0}{(2\pi)^2}\int_B \frac{ds\,dt}{1+as^2+bst+ct^2+O(\varepsilon^{1/2})} = \frac{Kw_0}{2\pi}\left(\frac{1}{\sqrt{4ac-b^2}}\log\frac{1}{\varepsilon}+O(1)\right).$$

This is then summed over all roots.

THEOREM 6.1. Suppose $\mathrm{t}$ is a polyomino whose characteristic polynomial has only simple roots $\{(z_j,u_j)\}_{j=1,\dots,k}$ on $\mathbb{T}^2$, and consider tilings of $\mathbb{Z}^2$ with translates of $\mathrm{t}$ and the singleton $\mathrm{t}_0$. Fix a small $\varepsilon = w_0/w$, then the covariance function is, up to a $1+o_\varepsilon(1)$ factor,

$$\mathrm{Cov}(X_{0,0},X_{s,t}) = Kw_0\sum_{j=1}^{k}\frac{z_j^s u_j^t}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{e^{i(sx+ty)}\,dx\,dy}{\varepsilon+|z_jP_z(z_j,u_j)x+u_jP_u(z_j,u_j)y|^2}.$$

6.2.3. *Quasiperiodic example in dimension* 2. This is an example with simple roots which are not roots of unity. Consider the “key" polyomino (Figure 6) with

$$p(z,u) = 1+z+z^2+z^3+z^4+z^5+z^6+u(1+z+z^4+z^6).$$

It has 2 simple roots on $\mathbb{T}^2$, $(z_0,u_0),(\bar z_0,\bar u_0)$, and $z_0,u_0$ have arguments $\theta,\phi$ which are not rationally related to each other, that is, there are no integers $m_1,m_2$ such that $z_0^{m_1}u_0^{m_2}=1$ except $m_1=0=m_2$ (this requires a short Galois theory argument, see the appendix, Section 9).

This implies that the resulting covariance function $\mathrm{Cov}(X_{0,0},X_{s,t})$ is a quasiperiodic function of $(s,t)$ in the $\varepsilon\to 0$ limit: specifically, to leading order

$$\mathrm{Cov}(X_{0,0},X_{s,t}) = Kw_0C(\log\frac{1}{\varepsilon})\cos(s\theta+t\phi)\Big(1+o_\varepsilon(1)\Big) \tag{22}$$

for a constant $C$. See Figure 6 for a numerical plot.

A simpler quasiperiodic example, if we allow multiplicities, is the tile with $p(z,w) = 2+3z+4u$. It also has 2 simple roots (with arguments corresponding to the angles of the 2, 3, 4-triangle). The covariance formula is again of the form (22) where $\theta,\phi$ are these angles.

6.3. *Other examples.* Not all polyominos have roots on $\mathbb{T}^2$. For example the tile of Figure 8 has the property that its characteristic polynomial has no roots on $\mathbb{T}^2$.

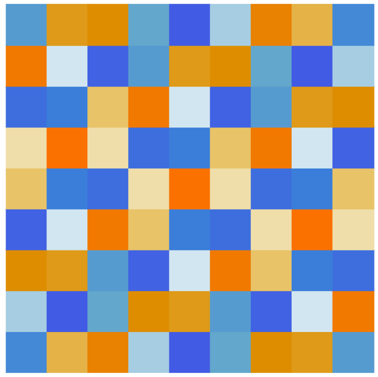

FIG 7. *Quasiperiodic covariance function (when $\varepsilon = 0$) for the $2,3,4$-weighted $L$ polyomino.*

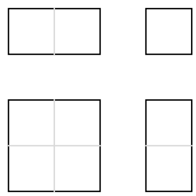

FIG 8. *A (nonconnected) polyomino with no crystal structure.*

As a consequence its covariance function $\mathrm{Cov}(X_{0,0}, X_{s,t})$ decays exponentially in $|s|+|t|$ even for $w_0 = 0$. This example is however somewhat special since its characteristic polynomial $p(z,u) = (1+z+z^3)(1+u+u^3)$ is a product of two $1d$ polynomials. A genuinely 2d example which does not factor is not easy to find. Here is one:

$$p(z,u) = 1+z+z^2+z^3+z^7+z^9+z^{12}+z^{13}+z^{17}+u.$$

For polyominos with multiplicity, an easy example is the one with $p(z,u) = 3+z+u$.

The third class of polyominos in $\mathbb{Z}^2$ has characteristic polynomials with roots on $\mathbb{T}^2$ which are either *not simple* or *not isolated*. And generally for $d$-dimensional polyominos with $d > 2$ the roots on $\mathbb{T}^d$ are not typically isolated. It is harder to formulate a general theory encompassing all these cases. We will simply illustrate with a few examples.

6.3.1. *The square polyomino.* We consider the square polyomino with $p(z,u) = (1+z)(1+u)$. We evaluate the integral (21). We first perform a contour integral over $u$. Assume $t \geq 0$.

$$\frac{Kw_0}{(2\pi i)^2}\int \frac{z^s u^t}{\varepsilon + (1+z)(1+u)(1+\frac{1}{z})(1+\frac{1}{u})}\frac{du}{u}\frac{dz}{z} = \frac{Kw_0}{(2\pi i)^2}\int \frac{z^s u^t}{\varepsilon z u + p^2}\,du\,dz$$

$$= \frac{Kw_0}{(2\pi i)^2}\int \frac{z^s u^t}{(1+z)^2(u-r_1)(u-r_2)}du\,dz.$$

Roots $r_1, r_2$ of the denominator are real with product 1; choose $|r_1| < 1 < |r_2|$. We get

$$= \frac{Kw_0}{2\pi}\int_0^{2\pi} \frac{z^s r_1^t}{(1+z)^2(r_1-r_2)}d\theta = \frac{Kw_0}{2\pi\sqrt{\varepsilon}}\int \frac{\cos(\theta s) r_1^t}{\sqrt{8+\varepsilon+8\cos\theta}}d\theta.$$

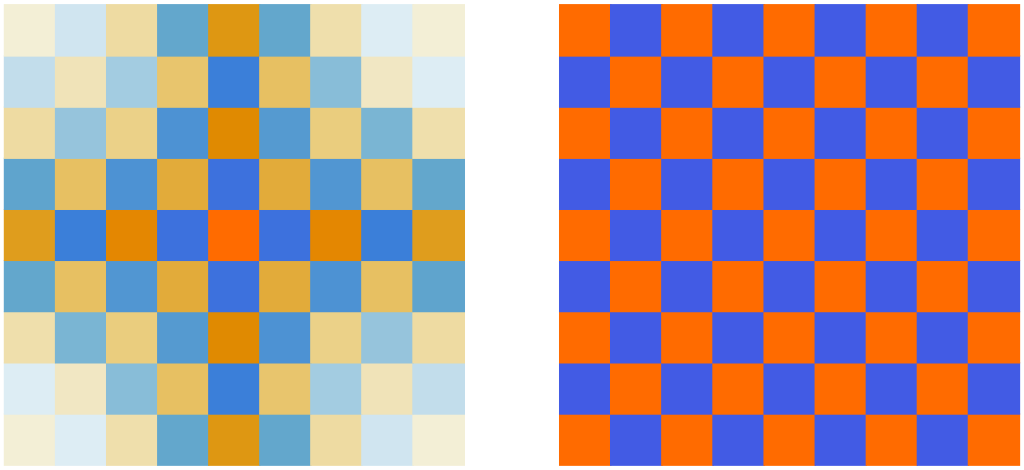

FIG 9. *Covariances for the square polyomino, when $\varepsilon = .001$ (left), and in the limit $\varepsilon \to 0$ (right).*

This is an elliptic function. For small $\varepsilon$ the integral concentrates near $\theta = \pi$, and is to leading order, for constant $s,t$,

$$Kw_0 \frac{(-1)^{s+t}}{2\pi\sqrt{\varepsilon}} \left( \log \frac{16}{\sqrt{\varepsilon}} - 2h_s - 2h_t + O(\varepsilon) \right)$$

where $h_m = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m-1}$.

We thus have

$$\mathrm{Cov}(X_{0,0}, X_{s,t}) = Kw_0 \frac{(-1)^{s+t}}{2\pi\sqrt{\varepsilon}} \left( \log \frac{16}{\sqrt{\varepsilon}} - 2h_{|s|} - 2h_{|t|} + O(\varepsilon) \right).$$

6.3.2. *The $+$ polyomino.* We consider the polyomino with $p(z,u) = 1 + z + 1/z + u + 1/u$. We need to compute ($Kw_0/n$ times) the Fourier series of $\frac{1}{\varepsilon + |p|^2}$. Let $z = e^{i\theta}$ and $u = e^{i\phi}$. The polynomial $p = 1 + 2\cos\theta + 2\cos\phi$ vanishes on a whole curve on $\mathbb{T}^2$, where $\theta$ runs over the range $\theta \in [\pi/3, 5\pi/3]$. For small $\varepsilon$ the integral concentrates near this curve. Let $(\theta_0, \phi_0)$ be a point on the curve. For $\theta$ fixed, the denominator has the form $\varepsilon + a(\theta)(\phi - \phi_0)^2 + O(\phi - \phi_0)^3$ where $a(\theta) = 3 - 4\cos\theta - 4\cos^2\theta$.

The contribution for this slice is (with $x = \phi - \phi_0$, and to leading order)

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{dx}{\varepsilon + a(\theta)x^2} = \frac{1}{2\sqrt{\varepsilon a(\theta)}}.$$

We thus have to leading order (for fixed $(s,t)$ as $\varepsilon \to 0$)

$$\mathrm{Cov}(X_{0,0}, X_{s,t}) = \frac{Kw_0}{2\pi\sqrt{\varepsilon}} \int_{\pi/3}^{5\pi/3} \frac{\cos(s\theta + t\phi)d\theta}{\sqrt{3 - 4\cos\theta - 4\cos^2\theta}}$$

where $\cos\phi + \cos\theta = 1/2$.

Multiplied by $\sqrt{\varepsilon}$ these covariances still tend to zero as $|s| + |t| \to \infty$, although the decay is slow, of order $1/(s^2 + t^2)^{1/4}$. See Figure 10 for a numerical plot.

**7. Further directions.** Because tilings are so diverse, there are many directions for further research. Here are some ideas.

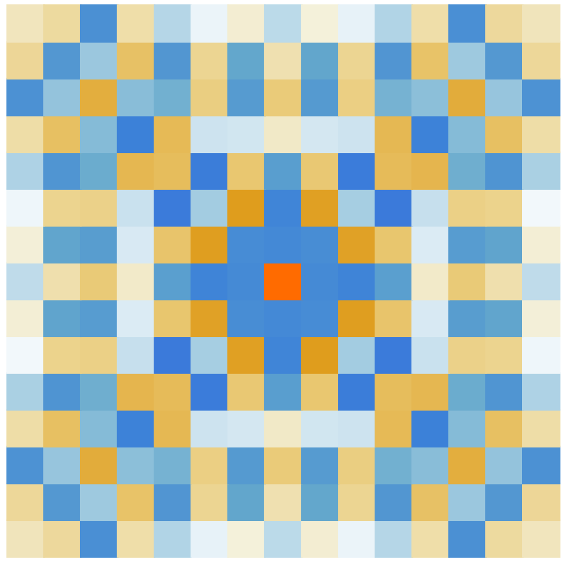

FIG 10. *Covariances for the "plus" polyomino, in the limit $\varepsilon \to 0$.*

1. How is the covariance function for a 3D polyomino different? Typically the characteristic polynomial intersects the unit 3-dimensional torus $\mathbb{T}^3$ in a 1-dimensional set. Is there an analogue of Theorem 6.1?
2. Find interesting examples with multiple tiles in $\mathbb{Z}^2$.
3. For tilings of a planar domain with copies of the $L$ polyomino and a small density of monomers, understand the influence of the boundary on the tiling. Can one create situations where there is coexistence of the multiple phases? That is, for the three possible sublattices, one might find a subregion where one is prevalent, abutting on another subregions where another sublattice is prevalent.
4. What behavior do we expect for the Coulomb gas for a general tiling problem? What about the $L$ triomino?
5. Wang tiles (squares with colored edges, tiled so that adjacent tiles share the same color) can be used to emulate any Turing machine. What is the multinomial-tiling analog of such a computation?
6. What is the natural multinomial analogue of the random partition model? What is the limit shape of a random such partition in that model (in the appropriate limit)?

**8. Appendix: The Bessel-K function.** For material in this section see [1]. The Bessel-K function, or modified Bessel function of the second kind, $B(s)$, can be defined for $s \in \mathbb{R} \setminus \{0\}$ by the integral

$$B(s) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \frac{e^{isx}\,dx\,dy}{1 + x^2 + y^2}.$$

For $z \in \mathbb{C}$, $B(|z|)$ is the Green's function for the massive laplacian, that is, it satisfies

$$(I - \Delta)B(|z|) = \delta_0$$

where $\delta_0$ is the point measure.

The function $B(s)$ has a logarithmic singularity at the origin, with expansion

$$B(s) = \log\frac{1}{s} + \log 2 - \gamma_E + O(s^2 \log s).$$

An integral of the form

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{e^{i(sx+ty)}\,dx\,dy}{\varepsilon+ax^2+bxy+cy^2}$$

where the quadratic form $ax^2+bxy+cy^2$ is positive definite, can be converted into a Bessel-K integral with a linear change of coordinates, yielding

$$=\frac{1}{\sqrt{c-\frac{b^2}{4a}}}B\left(\sqrt{\frac{\varepsilon(cs^2-bst+at^2)}{ac-b^2/4}}\right).$$

**9. Appendix: Irrationality of arguments.** Let $(z,u)$ be a root on $\mathbb{T}^2$ of

$$p(Z,W)=1+Z+Z^2+Z^3+Z^4+Z^5+Z^6+U(1+Z+Z^4+Z^6).$$

We prove here that there are no integers $(m_1,m_2)\neq(0,0)$ for which $z^{m_1}u^{m_2}=1$.

Note that $u=s(z):=-\frac{1+z+\cdots+z^6}{1+z+z^4+z^6}$, so

$$-\frac{1+z+z^4+z^6}{1+z+\cdots+z^6}=1/u=-\frac{1+z^{-1}+\cdots+z^{-6}}{1+z^{-1}+z^{-4}+z^{-6}}$$

from which we find that $z$ satisfies the irreducible (over $\mathbb{Q}$) polynomial

$$q(z)=z^{10}+2z^9+3z^8+4z^7+5z^6+3z^5+5z^4+4z^3+3z^2+2z+1=0.$$

We can use this along with $p(z,u)=0$ to eliminate $z$ and get a polynomial equation for $u$, which is the irreducible polynomial

$$8u^{10}+56u^9+150u^8+145u^7-83u^6-255u^5-83u^4+145u^3+150u^2+56u+8=0.$$

Since $u,1/u$ are not algebraic integers, neither is any power of $u$, so we can't have $z^{m_1}u^{m_2}=1$ for integers $m_1,m_2$ unless $m_2=0$. However $z$ is not itself a root of 1. This completes the proof.

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