



RESEARCH ARTICLE

Automorphy lifting with adequate image

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Abstract

Let F be a CM number field. We generalise existing automorphy lifting theorems for regular residually irreducible p -adic Galois representations over F by relaxing the big image assumption on the residual representation.

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1. Introduction

This paper closely builds on [ACC⁺18], which proves modularity lifting theorems for regular n -dimensional Galois representations over a CM number field F without any self-duality condition. In this paper, we generalise the main results of [ACC⁺18] to relax the big image assumption on the residual representation from ‘enormous image’ to ‘adequate image’. Following [Tho12], we define ‘adequate image’:

Definition 1.1. Let k be a finite field of characteristic p , such that $p \nmid n$, and let $G \subset \mathrm{GL}_n(k)$ be a subgroup which acts absolutely irreducibly on $V = k^n$. We suppose that k is large enough to contain all eigenvalues of all elements of G . If $g \in G$ and $\alpha \in k$ is an eigenvalue g , we write $e_{g,\alpha} : V \rightarrow V$ for the g -equivariant projection to the generalised α -eigenspace. We say that G is **adequate** if the following conditions are satisfied:

1. $H^0(G, \mathrm{ad}^0 V) = 0$.
2. $H^1(G, k) = 0$.

3. $H^1(G, \text{ad}^0 V) = 0$.
4. For every irreducible $k[G]$ -submodule $W \subset \text{ad}^0 V$, there exists an element $g \in G$ with an eigenvalue α , such that $\text{tr } e_{g,\alpha} W \neq 0$.

Our main theorems are as follows:

Theorem 1.2. *Let F be an imaginary CM or totally real field, let $c \in \text{Aut}(F)$ be complex conjugation and let p be a prime. Suppose given a continuous representation $\rho : G_F \rightarrow \text{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

1. ρ is unramified almost everywhere.
2. For each place $v \mid p$ of F , the representation $\rho|_{G_{F_v}}$ is crystalline. The prime p is unramified in F .
3. $\overline{\rho}$ is absolutely irreducible and decomposed generic. The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is adequate.
4. There exists $\sigma \in G_F - G_{F(\zeta_p)}$, such that $\overline{\rho}(\sigma)$ is a scalar. We have $p > n^2$.
5. There exists a cuspidal automorphic representation π of $\text{GL}_n(\mathbf{A}_F)$ satisfying the following conditions:
 - (a) π is regular algebraic of weight λ , this weight satisfying

$$\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} < p - 2n$$

for all τ .

- (b) There exists an isomorphism $\iota : \overline{\mathbf{Q}}_p \rightarrow \mathbf{C}$, such that $\overline{\rho} \cong \overline{r_\iota(\pi)}$, and the Hodge-Tate weights of ρ satisfy the formula for each $\tau : F \hookrightarrow \overline{\mathbf{Q}}_p$:

$$HT_\tau(\rho) = \{\lambda_{\iota\tau,1} + n - 1, \lambda_{\iota\tau,2} + n - 2, \dots, \lambda_{\iota\tau,n}\}.$$

- (c) If $v \mid p$ is a place of F , then π_v is unramified.

Then ρ is automorphic: there exists a cuspidal automorphic representation Π of $\text{GL}_n(\mathbf{A}_F)$ of weight λ , such that $\rho \cong r_\iota(\Pi)$. Moreover, if v is a finite place of F and either $v \mid p$ or both ρ and π are unramified at v , then Π_v is unramified.

Theorem 1.3. *Let F be an imaginary CM or totally real field, let $c \in \text{Aut}(F)$ be complex conjugation and let p be a prime. Suppose given a continuous representation $\rho : G_F \rightarrow \text{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

1. ρ is unramified almost everywhere.
2. Let $\mathbf{Z}_+^n = \{(\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\}$. For each place $v \mid p$ of F , the representation $\rho|_{G_{F_v}}$ is potentially semistable, ordinary with regular Hodge-Tate weights. In other words, there exists a weight $\lambda \in (\mathbf{Z}_+^n)^{\text{Hom}(F, \overline{\mathbf{Q}}_p)}$, such that for each place $v \mid p$, there is an isomorphism

$$\rho|_{G_{F_v}} \sim \begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \psi_{v,n} \end{pmatrix},$$

where for each $i = 1, \dots, n$ the character $\psi_{v,i} : G_{F_v} \rightarrow \overline{\mathbf{Q}}_p^\times$ agrees with the character

$$\sigma \in I_{F_v} \mapsto \prod_{\tau \in \text{Hom}(F_v, \overline{\mathbf{Q}}_p)} \tau(\text{Art}_{F_v}^{-1}(\sigma))^{-(\lambda_{\tau,n-i+1} + i - 1)}$$

on an open subgroup of the inertia group I_{F_v} .

3. $\bar{\rho}$ is absolutely irreducible and decomposed generic. The image of $\bar{\rho}|_{G_{F(\zeta_p)}}$ is adequate.
4. There exists $\sigma \in G_F - G_F(\zeta_p)$, such that $\bar{\rho}(\sigma)$ is a scalar. We have $p > n$.
5. There exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\iota : \bar{\mathbf{Q}}_p \rightarrow \mathbf{C}$, such that π is ι -ordinary and $\bar{\rho} \cong \overline{r_\iota(\pi)}$.

Then ρ is ordinarily automorphic of weight $\iota\lambda$: there exists a ι -ordinary cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$, such that $\rho \cong r_\iota(\Pi)$. Moreover, if $v \nmid p$ is a finite place of F and both ρ and π are unramified at v , then Π_v is unramified.

The theorems above are very similar to [ACC⁺18, Theorem 6.1.1] and [ACC⁺18, Theorem 6.1.2], respectively. The only difference is replacing the *enormous* condition on image of $\bar{\rho}|_{G_{F(\zeta_p)}}$ with *adequate*. This is a useful improvement, particularly in light of [GHTT12], which proves that when $p > 2(n+1)$, adequacy is equivalent to absolute irreducibility. This makes it a condition easy to work with in the context of automorphy of compatible systems, which we hope would help generalise [BLGGT14] to the context of [ACC⁺18] and this paper. We now give a brief overview of the argument. The main change in comparison to [ACC⁺18] is the usage of parahoric-level subgroups at Taylor-Wiles primes instead of Iwahori-level, the idea first introduced to relax the big image assumption in the setting of automorphy lifting theorems to ‘adequate’ in [Tho12]. To make the argument work in the parahoric setting, we need to analyse the representations of $\mathrm{GL}_n(F_v)$ with fixed vectors under various parahoric subgroups and their interactions with the local Langlands correspondence. A notable difficulty in comparison to [Tho12] is that we can no longer restrict to working with generic local representations, since they arise as components of cuspidal automorphic representations of unitary groups instead of GL_n . The local computations allow us to prove the necessary local-global compatibility results for Galois representations landing in Hecke algebras acting on cohomology of locally symmetric spaces with parahoric level. Another novel component is a proof of a ‘growth of the space of cusp forms’-type result when adding Taylor-Wiles primes with parahoric level, which requires an investigation of representations of $\mathrm{GL}_n(F_v)$ over fields of finite characteristic.

1.1. Notation

We write GL_n for the usual general linear group (viewed as a reductive group scheme over $\mathbf{Z}$) and $T_n \subset B_n \subset GL_n$ for its subgroups of diagonal and of upper triangular matrices, respectively. We identify $X^*(T)$ with $\mathbf{Z}^n$ in the usual way and write $\mathbf{Z}_+^n \subset \mathbf{Z}^n$ for the subset of B_n -dominant weights. If R is a local ring, we write $\mathfrak{m}_R$ for the maximal ideal of R . If Γ is a profinite group and $\rho : \Gamma \rightarrow GL_n(\bar{\mathbf{Q}}_p)$ is a continuous homomorphism, then we will let $\bar{\rho} : \Gamma \rightarrow GL_n(\bar{\mathbf{F}}_p)$ denote the *semisimplification* of its reduction, which is well defined up to conjugacy (by the Brauer-Nesbitt theorem). If M is a topological abelian group with a continuous action of Γ , then by $H^i(\Gamma, M)$, we shall mean the continuous cohomology. If G is a locally profinite group, $U \subset G$ is an open compact subgroup and R is a commutative ring, then we write $\mathcal{H}_R(G, U)$ for the algebra of compactly supported, U -biinvariant functions $f : G \rightarrow R$, with multiplication given by convolution with respect to the Haar measure on G which gives U volume 1. If $X \subset G$ is a compact U -biinvariant subset, then we write $[X]$ for the characteristic function of X , an element of $\mathcal{H}_R(G, U)$. When R is omitted from the notation, we take $R = \mathbf{Z}$. We write $\iota_{\mathcal{H}}$ for the anti-involution given by $\iota_{\mathcal{H}}(f)(g) = f(g^{-1})$.

If F is a perfect field, we let $\bar{F}$ denote an algebraic closure of F and G_F the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. We will use ζ_n to denote a primitive n -th root of unity when it exists. Let ϵ_l denote the l -adic cyclotomic character. We will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $GL_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Frobenius semisimple Weil-Deligne representation of the Weil group W_K . If K is a finite extension of $\mathbf{Q}_p$ for some p , we write K^{ur} for its maximal unramified extension, I_K for the inertia subgroup of G_K , $\mathrm{Frob}_K \in G_K/I_K$ for the geometric Frobenius and W_K for the Weil group. We will write $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ for the Artin map normalised to send uniformisers to geometric Frobenius elements.

We will write rec for rec_K when the choice of K is clear. We write rec_K^T for the normalisation of the local Langlands correspondence as defined in, for example [CT14, Section 2.1]; it is defined on irreducible admissible representations of $GL_n(K)$ defined over any field which is abstractly isomorphic to $\mathbf{C}$ (e.g. $\overline{\mathbf{Q}}_l$). If (r, N) is a Weil-Deligne representation of W_K , we will write $(r, N)^{F-ss}$ for its Frobenius semisimplification. If ρ is a continuous representation of G_K over $\overline{\mathbf{Q}}_l$ with $l \neq p$, then we will write $WD(\rho)$ for the corresponding Weil-Deligne representation of W_K . By a Steinberg representation of $GL_n(K)$, we will mean a representation $Sp_n(\psi)$ (in the notation of Section 1.3 of [HT01]), where ψ is an unramified character of $K^\times$.

If G is a reductive group over K and P is a parabolic subgroup with unipotent radical N and Levi component L , and if π is a smooth representation of $L(K)$, then we define $\text{Ind}_{P(K)}^{G(K)} \pi$ to be the set of locally constant functions $f : G(K) \rightarrow \pi$, such that $f(hg) = \pi(hN(K))f(g)$ for all $h \in P(K)$ and $g \in G(K)$. It is a smooth representation of $G(K)$, where $(g_1 f)(g_2) = f(g_2 g_1)$. This is sometimes referred to as ‘un-normalised’ induction. We let δ_P denote the determinant of the action of L on $\text{Lie } N$. Then we define the ‘normalised’ induction $\text{ind}_{P(K)}^{G(K)} \pi$ to be $\text{Ind}_{P(K)}^{G(K)} (\pi \otimes |\delta_P|_K^{1/2})$. We also define a parabolic restriction functor $r_{G(K)}^{P(K)}$ from $G(K)$ -representations to $L(K)$ -representations to be the composition of restriction to $P(K)$ and taking $N(K)$ -coinvariants. If F is a CM number field and π is an automorphic representation of $\text{GL}_n(\mathbf{A}_F)$, we say that π is regular algebraic if π_∞ has the same infinitesimal character as an irreducible algebraic representation W of $(\text{Res}_{F/\mathbf{Q}} \text{GL}_n)_{\mathbf{C}}$. If $W^\vee$ has highest weight $\lambda \in (\mathbf{Z}_+^n)^{\text{Hom}(F, \mathbf{C})}$, then we say π has weight λ .

If $P(X) \in A[X]$ is a polynomial of degree n over any ring A , such that $P(0) \in A^\times$, we write $P^\vee(X)$ for $P(0)^{-1} X^n P(X^{-1})$. For two polynomials $P, Q \in A[X]$, we write $\text{Res}(P, Q)$ to denote their resultant.

Given a Galois representation $\rho : G_{F,S} \rightarrow \text{GL}_n(A)$, we will write $\rho^\perp := \rho^{c,\vee} \otimes \epsilon^{1-2n}$, and given a $G_{F,S}$ -group determinant D , we will denote by $D^\perp$ the corresponding dual.

2. Representation theory of $\text{GL}_n(F_v)$ in characteristic p

Let p be a rational prime and $k = \overline{\mathbf{F}}_p$. Let $F/\mathbf{Q}$ be a finite extension, and let x be a prime in F with residue field k_x of order q satisfying $q \equiv 1 \pmod{p}$ and the corresponding ring of integers $\mathcal{O}_x = \mathcal{O}_{F_x}$. Set $G_x = \text{Gal}(\overline{F}_x/F_x)$. Also set $G = \text{GL}_n$ with $p > n$, and let $T \subset B \subset G$ be the maximal torus and the corresponding Borel and $U \subset G$ be the unipotent subgroup. Let $K^1(x) \subset G(\mathcal{O}_x)$ be the full congruence subgroup. We also let $\text{Iw}, \text{Iw}_1 \subset G(\mathcal{O}_x)$ be the Iwahori and the Iwahori-1, respectively, and let $\text{Iw}_1 \subset \text{Iw}^p \subset \text{Iw}$ be the subgroup, such that $[\text{Iw}^p : \text{Iw}_1]$ has order prime to p and $[\text{Iw} : \text{Iw}^p]$ has p -power order. Let $\mathfrak{p}(x)$ be a two-block parahoric subgroup of $G(\mathcal{O}_x)$ with blocks of sizes $n_1 + n_2 = n$ and P the corresponding parabolic. Let $W \cong S_n$ be the Weyl group for GL_n , and for a given parabolic subgroup $Q \subset G$, let $W_Q \subset W$ be the Weyl group of its Levi factor. Set $T_0 := T(\mathcal{O}_x)$ and $T_1 := \ker(T_0 \rightarrow T(\mathcal{O}_x/\varpi))$. Fix $\overline{\rho} : G_x \rightarrow \text{GL}_n(k)$ —a continuous unramified semisimple representation. We say that an irreducible admissible representation π of G over k is associated to $\overline{\rho}$ if π is a subquotient of $\text{Ind}_B^G \chi_1 \otimes \dots \otimes \chi_n$, where χ_i are unramified characters, such that $\{\chi_1(\varpi), \dots, \chi_n(\varpi)\}$ is the set of eigenvalues of $\overline{\rho}(\text{Frob}_x)$. We write $I(\chi)$ for $\text{Ind}_B^G \chi_1 \otimes \dots \otimes \chi_n$. The following lemma shows that if we do not fix the ordering of χ_i , then we can always consider π to be a subrepresentation of $I(\chi)$.

Proposition 2.1. *Let π be an irreducible admissible $k[G]$ -module associated to $\overline{\rho}$. Then there exists an ordering of $\chi_1, \dots, \chi_n$, such that π is a subrepresentation of $I(\chi)$.*

Proof. We use the adjunction between Ind_B^G and the parabolic restriction r_B^G to get an isomorphism

$$\text{Hom}(\pi, I(\chi)) \cong \text{Hom}(r_G^B(\pi), \chi).$$

Since π is associated to $\overline{\rho}$, we know that $r_G^B(\pi) \neq 0$. Since $r_G^B(\pi)$ is a representation of the torus, there exists a 1-dimensional quotient given by some character $\chi : T \rightarrow k^\times$. Then we get that $\text{Hom}(\pi, I(\chi)) \neq 0$, and since π is irreducible, this implies that π is a subrepresentation of $I(\chi)$. Then χ forms the

supercuspidal support of π and in fact has to be a permutation of the original $\chi_1, \dots, \chi_n$. For the notion of supercuspidal support in positive characteristic, see [Vig96, II.2.6]. We would also like to remark, here, that in the case $q \equiv 1 \pmod{p}$, $p > n$, the notions of cuspidal and supercuspidal representations coincide (see [Vig96, II.3.9]). $\square$

We now describe the Bernstein presentation of Iwahori-Hecke algebra $\mathcal{H}_k(G, \text{Iw})$, following [Vig96, I.3.14]. Let

$$t_j = \text{diag}(\underbrace{\varpi, \dots, \varpi}_j, 1, \dots, 1),$$

and set $T_j = [\text{Iw } t_j \text{Iw}]$ and $X^j = T_j(T_{j-1})^{-1}$. We also let s_j be the permutation matrix corresponding to the transposition $(j, j+1)$ and set $S^j = [\text{Iw } s_j \text{Iw}]$. The elements X^j for $1 \leq j \leq n$ generate the group algebra $k[\mathbf{Z}^n]$ on which S_j acts by permuting the indices. The Bernstein presentation states that

$$\mathcal{H}_k(G, \text{Iw}) \cong k[S_n \ltimes \mathbf{Z}^n]$$

under the action described above.

Now we introduce some useful Hecke operators. For any ring R , $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$ let $V^{j,2} \in \mathcal{H}_R(G, \mathfrak{p}(x))$ be the Hecke operator associated to the double coset

$$[\mathfrak{p}(x) \text{diag}(\underbrace{1, \dots, 1}_{n_1}, \underbrace{\varpi, \dots, \varpi}_j, \underbrace{1, \dots, 1}_{n_2-j}) \mathfrak{p}(x)]$$

and let $V^{i,1}$ be associated to

$$[\mathfrak{p}(x) \text{diag}(\underbrace{\varpi, \dots, \varpi}_i, 1, \dots, 1) \mathfrak{p}(x)].$$

The following is part of [CHT08, Theorem B.1]:

Proposition 2.2. *Let V be an irreducible admissible $k[G]$ -module, which is generated by its Iwahori-invariant vectors. Then $V^{\text{Iw}} = V^{\text{Iw}_1}$.*

Under the conditions of 2.2, we thus get an isomorphism

$$\begin{aligned} H^1(\text{Iw}, V) &\cong H^1(B(k), V^{K^1(x)}) \cong H^1(T(k), V^{\text{Iw}_1}) \\ &\cong H^1(T(k), V^{\text{Iw}}) \cong \text{Hom}(T(k), V^{\text{Iw}}). \end{aligned} \quad (2.3)$$

Both sides of 2.3 can be endowed with the action of $\mathcal{H}_k(G, \text{Iw})$. On $H^1(\text{Iw}, V)$, we take the derived $\mathcal{H}_k(G, \text{Iw})$ -action, and on $\text{Hom}(T(k), V^{\text{Iw}})$, we consider the natural action on the target.

Proposition 2.4. *The isomorphism 2.3 is equivariant with respect to X^i for all $1 \leq i \leq n$.*

Proof. The action of X^i on $[f] \in H^1(\text{Iw}, V)$ can be described as follows. Write

$$\text{Iw } t_i \text{Iw} = \bigsqcup_j g_{i,j} \text{Iw}.$$

We now give an explicit description for $g_{i,j}$. Fix a set of representatives $S \subset \mathcal{O}_F$ for k . For each $m \in M_{i \times (n-i)}(S)$, let $g_{i,m}$ be the matrix, such that $g_{i,m}(k, k) = \varpi$ for $k \leq i$, $g_{i,m}(k, k) = 1$ for $k > i$ and $g_{i,m}(k, \ell) = m(k, \ell - i)$ for $k \leq i, \ell > i$. The rest of the entries are set to 0. Let us show that this is

a full set of representatives. First we show that $g_{i,m}$ represent distinct cosets, that is that $g_{i,m}^{-1}g_{i,m'} \notin \text{Iw}$ for $m \neq m'$. Suppose $m(k, \ell) \neq m'(k, \ell)$. Then

$$(g_{i,m}^{-1}g_{i,m'})(k, \ell + i) = \varpi^{-1}(m'(k, \ell) - m(k, \ell))$$

which is not in $\mathcal{O}_F$. Now we just need to verify that the number of cosets is $q^{i(n-i)}$. Indeed,

$$[\text{Iw } t_i \text{ Iw} : \text{Iw}] = [\text{Iw} : \text{Iw} \cap t_i \text{ Iw } t_i^{-1}] = q^{i(n-i)}$$

since $\text{Iw} \cap t_i \text{ Iw } t_i^{-1}$ are just the elements of the Iwahori whose (k, ℓ) -coordinates for $k \leq i, \ell > i$ vanish mod ϖ .

Then

$$(X^i[f])(x) = \sum_j g_{i,\sigma(j)} f(g_{i,\sigma(j)}^{-1} x g_{i,j}),$$

where σ is the unique permutation, such that

$$g_{i,\sigma(j)}^{-1} x g_{i,j} \in \text{Iw}$$

for all j . Denote by $\bar{\cdot} : \text{Iw} \rightarrow T(k)$ the reduction map. Let s be the inverse of 2.3. For $[\tau] \in \text{Hom}(T(k), V^{\text{Iw}})$, we get

$$\begin{aligned} (X^i[s(\tau)])(x) &= \sum_j g_{i,\sigma(j)} s(\tau)(\overline{g_{i,\sigma(j)}^{-1} x g_{i,j}}) \\ &= \sum_j g_{i,\sigma(j)} s(\tau)(\bar{x}) = s(X^i[\tau])(x). \end{aligned}$$

The second equality is due to all the $g_{i,j}$ being in the Borel and having the same diagonal. $\square$

Definition 2.5. A G -modules V over k is *locally admissible* if it is smooth, and for every $v \in V$ the subrepresentation generated by v is admissible. Let $\mathcal{C}$ denote the abelian category of locally admissible G -modules V over k , such that every irreducible subquotient of V is associated to $\bar{\rho}$.

The following is analogous to [CG18, Lemma 9.14]:

Proposition 2.6. *The category $\mathcal{C}$ has enough injectives, and the inclusion functor from $\mathcal{C}$ to locally admissible G -modules is exact.*

Proof. Inside the category of G -modules, the category $\mathcal{C}$ is fully contained inside the unipotent block (the block containing the trivial representation). By part 4) of [CHT08, Theorem B.1], the unipotent block is equivalent to the category of $\mathcal{H}_k(G, \text{Iw}^p)$ -modules. Via the Bernstein embedding¹, such modules can naturally be viewed as $\mathcal{H}_k(G, G(\mathcal{O}_x))$ -modules, where $\mathcal{H}_k(G, G(\mathcal{O}_x))$ can be explicitly described via the Satake isomorphism as $k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^W$. Here, we use the Satake isomorphism twisted by $|\det|^{(1-n)/2}$, which is defined over $\mathbf{Z}[q^{-1}]$. If V is any locally admissible element of the unipotent block, the associated Hecke module V^{Iw^p} is locally finite-dimensional over k , and thus we can write

$$V^{\text{Iw}^p} = \bigoplus_{\mathfrak{m}} V_{\mathfrak{m}}^{\text{Iw}^p},$$

where the sum is taken over all maximal ideals of $\mathcal{H}_k(G, G(\mathcal{O}_x))$. Let $\mathcal{D}$ denote the category of locally admissible representations in the unipotent block. Then we can write $\mathcal{D} = \bigoplus_{\mathfrak{m}} \mathcal{D}_{\mathfrak{m}}$, where $\mathcal{D}_{\mathfrak{m}}$ consists

¹For the details on the Bernstein embedding $k[\mathbf{Z}^n] \rightarrow \mathcal{H}_k(G, I)$ in the case of an arbitrary open compact subgroup $I \subset \text{Iw}$, such that $\text{Iw}_1 \subset I$, see [ACC⁺18, Section 2.2.4]. We note that there the authors are working over some p -adic ring $\mathcal{O}$, but the results are valid over k as well since $q \equiv 1 \pmod{p}$.

of representations whose associated $\mathcal{H}_k(G, G(\mathcal{O}_x))$ -module is supported only at $\mathfrak{m}$. The maximal ideals of $\mathcal{H}_k(G, G(\mathcal{O}_x))$ have the form $(t_1 - \alpha_1, \dots, t_n - \alpha_n)$, where $\alpha_i \in k$ and $t_i = e_i(X_1, \dots, X_n)$ is the i -th elementary symmetric polynomial of $X_1, \dots, X_n$. If we now let $\mathfrak{n}$ be the ideal defined by $\alpha_i = e_i(\chi_1(\varpi), \dots, \chi_n(\varpi))$, then it is clear that $\mathcal{C} = \mathcal{D}_{\mathfrak{n}}$. The exactness is now clear, and to show that $\mathcal{C}$ has enough injectives, it is enough to check that the category $\text{Mod}_G^{\text{l.adm.}}(k)$ of locally admissible G -modules has enough injectives. The full category $\text{Mod}_G(k)$ certainly has enough injectives, and the functor $\mathcal{L} : \text{Mod}_G(k) \rightarrow \text{Mod}_G^{\text{l.adm.}}(k)$ taking a module to its smooth locally admissible vectors is right adjoint to the natural embedding $\text{Mod}_G^{\text{l.adm.}}(k) \rightarrow \text{Mod}_G(k)$. This proves the claim. $\square$

From now on, fix $\alpha = \chi_i(\varpi)$ for some $1 \leq i \leq n$. Let

$$P(X) = \prod_{i=1}^n (X - \chi_i(\varpi)).$$

For $1 \leq j \leq n_2$, let P_j be a polynomial whose roots with multiplicities are precisely

$$\sum_{\substack{J \subset S \\ \#J=j}} \prod_{a \in J} \chi_a(\varpi).$$

Factor $P_j = Q_j R_j$, where

$$R_j(X) = \left(X - \binom{n_2}{j} \alpha^j \right)^{k_j}$$

and Q_j, R_j are coprime. Set

$$e_\alpha := \lim_{m \rightarrow \infty} \left(\prod_{i=1}^{n_2} Q_j(V^{j,2}) \right)^{m!}.$$

Here, we consider e_α as an operator acting on $V^{\mathfrak{p}(x)}$ for $V \in \mathcal{C}$. Since objects in $\mathcal{C}$ are locally admissible, the limit makes sense.

We now define two functors $F, G : \mathcal{C} \rightarrow k\text{-Vect}$. On objects, we set

$$F(V) := V^{G(\mathcal{O}_x)}, \quad G(V) := e_\alpha V^{\mathfrak{p}(x)}.$$

Note that F, G are both left-exact and e_α is exact. Then we can form derived functors $R^k F, R^k G$ and identify

$$R^k F(V) = H^k(G(\mathcal{O}_x), V), \quad R^k G(V) = e_\alpha H^k(\mathfrak{p}(x), V).$$

We have a natural transformation $\iota : F \rightarrow G$ given by composing the inclusion $V^{G(\mathcal{O}_x)} \hookrightarrow V^{\mathfrak{p}(x)}$ with e_α . We will make use of the following simple algebraic fact.

Lemma 2.7. *Let G be a profinite group and $H \triangleleft G$ be a normal subgroup. Let A be a p -torsion G -module for some positive integer p , and let H have pro- q order for a prime q satisfying $q \equiv 1 \pmod{p}$. Then the inflation map*

$$\text{inf} : H^1(G/H, A^H) \rightarrow H^1(G, A)$$

is an isomorphism whose inverse sends a cocycle $[f] \in H^1(G, A)$ to

$$g \mapsto f(g) + (1 - g)a_f$$

for some $a_f \in A$.

Proof. The condition $q \equiv 1 \pmod{p}$ ensures that $H^1(H, A)$ vanishes. Then it is enough to take $(g-1)a_f$ to be the coboundary trivialising the restriction of $[f]$ to H . $\square$

Proposition 2.8. *Let π be an irreducible admissible $k[G]$ -module associated to $\bar{\rho}$. Then the map*

$$f : H^1(G(k), \pi^{K^1(x)}) \rightarrow e_\alpha H^1(P(k), \pi^{K^1(x)})$$

is injective.

Proof. Both cohomology groups in question inject into $H^1(B(k), \pi^{K^1(x)})$ since

$$[G(k) : B(k)] \equiv n! \not\equiv 0 \pmod{p}$$

when $p > n$, so let us analyse that group. Since $q \equiv 1 \pmod{p}$, by inflation-restriction, we get

$$H^1(B(k), \pi^{K^1(x)}) \cong H^1(T(k), \pi^{\text{Iw}_1}).$$

As a special case of 2.3, we have

$$H^1(\text{Iw}, \pi) \cong H^1(B(k), \pi^{K^1(x)}) \cong \text{Hom}(T(k), \pi^{\text{Iw}}) \cong (\pi^{\text{Iw}})^{\oplus n}. \quad (2.9)$$

The isomorphism above is equivariant with respect to the natural actions of $\{X^i\}$ on both sides arising from the actions of $\mathcal{H}_k(G, \text{Iw})$ by Proposition 2.4. The space π^{Iw} injects into $I(\chi)^{\text{Iw}}$, which has a basis $\{\varphi_w\}$ for $w \in W$, where φ_w is supported on $Bw \text{Iw}$ and satisfies $\varphi_w(w) = 1$. It follows from the proof of [Tho12, Lemma 5.10], that on each component of $(I(\chi)^{\text{Iw}})^{\oplus n}$, the operator e_α acts as a projection onto the space spanned by $\{\varphi_{w'} \mid w' \in W'\}$, where W' is the subset of W consisting of permutations which send $\{n_1 + 1, \dots, n\}$ to the positions of α -s in the sequence $\chi_1(\omega), \dots, \chi_n(\omega)$. On the level of cocycles, the isomorphism 2.9 sends $[s] \in H^1(B(k), \pi^{K^1(x)})$ to the map

$$g \mapsto s(g) + (1 - g)\psi$$

for some $\psi \in I(\chi)$ (Lemma 2.7). Thus, a cocycle $[s] \in H^1(G(k), I(\chi)^{K^1(x)})$ being in the kernel of f means that for all $t \in T(k)$ and $w_0 \in W'$, we have

$$(s(t) + (1 - t)\psi)(w_0) = 0. \quad (2.10)$$

For any $w \in W$, we have

$$(t\psi)(w) = \psi(w\tilde{t}) = \psi(w(\tilde{t})w) = \psi(w).$$

Here, $\tilde{t}$ is a lift of t to T_0 and w acts on the torus in a natural way. Note that here, we used that χ is unramified. Thus

$$((1 - t)\psi)(w) = 0. \quad (2.11)$$

Combining 2.10 and 2.11 applied to w_0 , we get

$$s(t)(w_0) = 0.$$

Now let us conjugate t by an arbitrary $w \in W$. Since the result is again in T , we use the cocycle condition and the transformation law of $I(\chi)$ with respect to the Borel to write

$$0 = s(wtw^{-1})(w_0) = (s(w) + w(s(t) + ts(w^{-1}))) (w_0) \quad (2.12)$$

$$(wts(w^{-1}))(w_0) = ws(w^{-1})(w_0) = -s(w)(w_0). \quad (2.13)$$

Combining 2.12 and 2.13, we get

$$0 = (ws(t))(w_0) = s(t)(w_0w).$$

In other words, we now have $s(t)(w) = 0$ for all $t \in T(k)$ and for all $w \in W$. By 2.11, this implies that $[s] = 0$ since $\{\varphi_w\}$ make a basis for $I(\chi)^{Iw}$. $\square$

Theorem 2.14. *The natural transformation $\iota : F \rightarrow G$ given by $V^{G(\mathcal{O}_x)} \mapsto e_\alpha V^{\mathfrak{p}(x)}$ on objects is an isomorphism of functors. In particular, we get functorial isomorphisms*

$$\iota_* : H^k(G(\mathcal{O}_x), V) \xrightarrow{\sim} e_\alpha H^k(\mathfrak{p}(x), V)$$

for all $k \geq 0$.

Proof. In the proof of Proposition 2.6, we have identified $\mathcal{C}$ with a subcategory of $\mathcal{H}_k(G, Iw^P)\text{-Mod}$. Thus, every element of $\mathcal{C}$ is a direct limit of finite length elements of $\mathcal{C}$, and it is, therefore, enough to establish the isomorphism for finite length V . The first step will be to show that $\iota(V)$ is an isomorphism for all $V \in \mathcal{C}$. For an irreducible subrepresentation $\pi \subset V$, consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(\pi) & \longrightarrow & F(V) & \longrightarrow & F(V/\pi) \longrightarrow R^1F(\pi) \\ & & \downarrow \iota(\pi) & & \downarrow \iota(V) & & \downarrow \iota(V/\pi) & \downarrow f \\ 0 & \longrightarrow & G(\pi) & \longrightarrow & G(V) & \longrightarrow & G(V/\pi) \longrightarrow R^1G(\pi). \end{array} \quad (2.15)$$

To show that $\iota(V)$ is injective, we can use the four lemmas and induct on the length of V . Thus, we only need to show that $\iota(\pi)$ is injective for irreducible π . This is done in [Tho12, Lemma 5.10].

Now we would like to show that $\iota(\pi)$ is an isomorphism. Consider the injection $\pi \subset I(\chi)$ and the associated diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(\pi) & \longrightarrow & F(I(\chi)) & \longrightarrow & F(I(\chi)/\pi) \\ & & \downarrow \iota(\pi) & & \downarrow \iota(I(\chi)) & & \downarrow \iota(I(\chi)/\pi) \\ 0 & \longrightarrow & G(\pi) & \longrightarrow & G(I(\chi)) & \longrightarrow & G(I(\chi)/\pi). \end{array} \quad (2.16)$$

We already know that $\iota(I(\chi)/\pi)$ is injective. Then to show that $\iota(\pi)$ is surjective by the four lemmas, we need to know that $\iota(I(\chi))$ is surjective. This follows once again from the proof of [Tho12, Lemma 5.10].

Finally, we are ready to see that $\iota(V)$ is an isomorphism for all $V \in \mathcal{C}$. We induct on the length of V using Eq. 2.15. Since f is injective by Proposition 2.8, the result follows. $\square$

3. Representation theory of $\mathrm{GL}_n(F_v)$ in characteristic 0

Fix a finite extension $E/\mathbf{Q}_p$ in $\overline{\mathbf{Q}_p}$ which contains the images of all embeddings $F \rightarrow \overline{\mathbf{Q}_p}$. We write $\mathcal{O}$ for the ring of integers of E and $\varpi \in \mathcal{O}$ for a choice of uniformiser. If v is a finite place of F prime to p , we write $\Xi_v := \mathbf{Z}^n$ and $\Xi_{v,1} := \langle \tau_v \rangle \times \mathbf{Z}^n$, where τ_v is the generator of $k_v^\times(p)$ —the maximal p -power order quotient of $k_v^\times$. We have a natural homomorphism $\mathcal{O}_{F_v}^\times \rightarrow \mathbf{Z}[\Xi_{v,1}]$ induced by the homomorphism $\mathcal{O}_{F_v}^\times \rightarrow k_v^\times \rightarrow k_v^\times(p)$, which we denote by $\langle \cdot \rangle$. Consider a standard parabolic subgroup $P \subset \mathrm{GL}_n(F_v)$ corresponding to a partition $n = n_1 + \dots + n_m$ which we will denote as μ . Given a partition of n , we will always let $s_{\mu,i} = n_1 + \dots + n_i$, with $s_{\mu,0} = 0$. Let $P = MN$ and $\bar{P} = M\bar{N}$ be the Levi decompositions of P and its opposite parabolic. Let $\mathfrak{m}$ be the hyperspecial maximal compact subgroup of M . Define the subgroup of the symmetric group $S_\mu = S_{n_1} \times \dots \times S_{n_m}$. For any positive integer k , let

$$S_k : \mathcal{H}_{\mathbf{Z}[q_v^{1/2}]}(\mathrm{GL}_k(F_v), \mathrm{GL}_k(\mathcal{O}_{F_v})) \rightarrow \mathbf{Z}[q_v^{\pm 1/2}][X_1^{\pm 1}, \dots, X_k^{\pm 1}]^{S_k}$$

denote the (normalised) Satake isomorphism. We use those isomorphisms to identify

$$\mathcal{S}_\mu = \mathcal{S}_{n_1} \otimes \dots \otimes \mathcal{S}_{n_k} : \mathcal{H}_{\mathbf{Z}[q_v^{1/2}]}(M, \mathfrak{m}) \xrightarrow{\sim} \mathbf{Z}[q_v^{1/2}][\Xi_v]^{S_\mu}.$$

Consider any open compact subgroup $\mathfrak{q}$ of $\mathrm{GL}_n(F_v)$, and set

$$\mathfrak{q}_M = \mathfrak{q} \cap M, \quad \mathfrak{q}^+ = \mathfrak{q} \cap N, \quad \mathfrak{q}^- = \mathfrak{q} \cap \overline{N}.$$

From now on, assume that $\mathfrak{q}$ has an Iwahori decomposition with respect to P , which means that $\mathfrak{q} = \mathfrak{q}^- \mathfrak{q}_M \mathfrak{q}^+$. We define a submonoid $M^+ \subset M$ of *positive* elements to consist of elements $m \in M$, such that

$$m \mathfrak{q}^+ m^{-1} \subset \mathfrak{q}^+, \quad m^{-1} \mathfrak{q}^- m \subset \mathfrak{q}^-.$$

Inside M^+ , we have a further submonoid M^{++} of *strictly positive* elements consisting of $m \in M^+$ satisfying the following conditions:

- For any compact open subgroups $\mathfrak{n}_1, \mathfrak{n}_2$ of N , there exists a positive integer $x \geq 0$, such that

$$m^x \mathfrak{n}_1 m^{-x} \subset \mathfrak{n}_2.$$

- For any compact open subgroups $\overline{\mathfrak{n}}_1, \overline{\mathfrak{n}}_2$ of $\overline{N}$, there exists a positive integer $x \geq 0$, such that

$$m^{-x} \overline{\mathfrak{n}}_1 m^x \subset \overline{\mathfrak{n}}_2.$$

We denote by $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)^+$ the elements of $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)$ whose support is contained in M^+ . From now on, we also assume that q_v has a square root in $\mathcal{O}$ and fix such square root.

Proposition 3.1.

1. The map $t_\mu^+ : \mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)^+ \rightarrow \mathcal{H}_\mathcal{O}(G, \mathfrak{q})$ given by

$$[\mathfrak{q}_M m \mathfrak{q}_M] \mapsto \delta_P^{1/2}(m)[q m q]$$

is an algebra homomorphism.

2. The map t_μ^+ extends to a homomorphism $t_\mu : \mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M) \rightarrow \mathcal{H}_\mathcal{O}(G, \mathfrak{q})$ if and only if there exists a strictly positive element $a \in Z(M)$, such that $[q a q]$ is invertible in $\mathcal{H}_\mathcal{O}(G, \mathfrak{q})$.
3. Assuming the existence of the extension in (2), for any smooth $\mathbf{C}[\mathrm{GL}_n(F_v)]$ -module π , the canonical map $\pi^\mathfrak{q} \rightarrow \pi_N^{\mathfrak{q}_M}$ is a homomorphism of $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)$ -modules, where $\mathcal{H}_\mathcal{O}(M, \mathfrak{q}_M)$ acts on $\pi^\mathfrak{q}$ via the map t_μ .

Proof. For the first two claims, see [Vig98, II.6]. For the third, see [Vig98, II.10.1]. $\square$

Now we record some results about smooth admissible representations of $\mathrm{GL}_n(F_v)$ in characteristic 0. Let $\mathfrak{p}$ be a parahoric corresponding to the partition $n = n_1 + \dots + n_k$ which we call μ , and let P be the underlying parabolic with the Levi decomposition $P = MN$. Let $\mathfrak{m} = M(\mathcal{O}_{F_v})$. We also let $\mathfrak{p}_1, \mathfrak{m}_1$ denote the kernels of the homomorphisms

$$\mathfrak{p} \rightarrow P(k_v) \rightarrow \mathrm{GL}_{n_k}(F_v) \xrightarrow{\det} k_v^\times \rightarrow k_v^\times(\mathfrak{p})$$

$$\mathfrak{m} \rightarrow M(k_v) \rightarrow \mathrm{GL}_{n_k}(F_v) \xrightarrow{\det} k_v^\times \rightarrow k_v^\times(\mathfrak{p}).$$

Finally, let $\mathrm{Iw}' = \mathfrak{p}_1 \cap \mathrm{Iw}$.

Lemma 3.2. *The condition in part (2) of Proposition 3.1 is satisfied for $\mathfrak{q} = \mathfrak{p}, \mathfrak{p}_1$.*

Proof. This is a special case of [Whi22, Proposition 5.7]. $\square$

Fix a uniformiser ϖ_c of F_v . For any $1 \leq j \leq k$ and $1 \leq i \leq n_j$, consider the operators in $\mathcal{H}_O(G, \mathfrak{p})$ given by

$$V^{i,j} = t_\mu(S_\mu^{-1}(e_i(X_{s_{\mu,j-1}+1}, \dots, X_{s_{\mu,j}}))).$$

We will also consider operators in $\mathcal{H}_O(G, \mathfrak{p}_1)$, such that their actions on $\pi^{\mathfrak{p}} \subset \pi^{\mathfrak{p}_1}$ agree with the action of $V^{i,j}$ for any smooth representation π . They can be constructed in the same way as $V^{i,j}$ above by replacing S_μ with the Satake isomorphism for $\mathfrak{m}_1$ from [Whi22, Theorem 5.1]. These operators will also be denoted $V^{i,j}$. We also define operators $T^{i,j}$ representing the images of the same elements under S_μ^{-1} in $\mathcal{H}_O(M, \mathfrak{m})$ and the corresponding operators on $\mathcal{H}_O(M, \mathfrak{m}_1)$.

The following lemmas are straightforward generalisations of the lemmas in [Tho12, Section 5]. Given a parabolic subgroup Q of $\mathrm{GL}_n(F_v)$, we write $W_Q \subset W$ for the Weyl group of its Levi factor. Recall from [Cas] that the space $W_Q \backslash W / W_P$ has a canonical set of representatives $[W_Q \backslash W / W_P]$, consisting of minimal length elements from each double coset.

Lemma 3.3. *Let Q be a parabolic corresponding to the partition $n = m_1 + \dots + m_r$. Then $[W_Q \backslash W / W_P]$ is isomorphic to the set of partitions*

$$m_i = n_1^i + \dots + n_k^i, 1 \leq i \leq r,$$

such that

$$\sum_i n_j^i = n_j \text{ for all } 1 \leq j \leq k.$$

With Q as in the last lemma, let L_i denote the i -th component of the corresponding Levi subgroup. For $w \in [W_Q \backslash W / W_P]$ corresponding to the partition $n_1^i + \dots + n_k^i$, let $\mathfrak{p}_i^w$ denote the parahoric subgroup of L_i corresponding to this partition, and let $\mathfrak{p}_{i,1}^w$ be the kernel of

$$\mathfrak{p}_i^w \rightarrow \mathrm{GL}_{n_k^i}(F_v) \xrightarrow{\det} k_v^\times \rightarrow k_v^\times(p).$$

Let $\mathfrak{q}$ be the parahoric corresponding to the partition $\{n_1^1, \dots, n_k^1, n_1^2, \dots, n_k^r\}$, and let $\mathfrak{n}$ be the hyperspecial maximal compact of the corresponding Levi subgroup. We define $\mathfrak{p}_{1,w}$ as a subgroup of $\mathfrak{q}$ defined by the conditions $\mathrm{im}(\det N_k^j \rightarrow k_v^\times(p)) = 1$ for all j , where N_k^j is the block corresponding to n_k^j .

Lemma 3.4. *For each $1 \leq i \leq r$, let π_i be a smooth representation of L_i . Then*

1. *For any $w \in [W_Q \backslash W / W_P]$, we have $L_i \cap w \mathfrak{p} w^{-1} = \mathfrak{p}_i^w$.*
2. *For any $w \in [W_Q \backslash W / W_P]$, we have $Q \cap w \mathfrak{p}_1 w^{-1} \supset \mathfrak{p}_{1,w}$.*
- 3.

$$(\mathrm{ind}_Q^G \pi_1 \otimes \dots \otimes \pi_r)^{\mathfrak{p}} \cong \bigoplus_{w \in [W_Q \backslash W / W_P]} \pi_1^{\mathfrak{p}_1^w} \otimes \dots \otimes \pi_r^{\mathfrak{p}_r^w}.$$

- 4.

$$(\mathrm{ind}_Q^G \pi_1 \otimes \dots \otimes \pi_r)^{\mathfrak{p}_1} \subset \bigoplus_{w \in [W_Q \backslash W / W_P]} \pi_1^{\mathfrak{p}_{1,1}^w} \otimes \dots \otimes \pi_r^{\mathfrak{p}_{r,1}^w}.$$

Let π be an irreducible admissible representation of G , such that $\pi^{\mathfrak{p}_1} \neq 0$. Since $\mathrm{Iw}' \subset \mathfrak{p}_1$, supercuspidal support of π consists of tamely ramified characters. We will now use the Bernstein-Zelevinsky classification [BZ77], following the conventions of [Rod82], as they are best suited for applications to local Langlands correspondence. We can write π as a quotient of

$$\mathrm{ind}_G^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \dots \otimes \mathrm{Sp}_{k_r}(\chi_r),$$

where $\mathrm{Sp}_n(\chi)$ for a tamely ramified character $\chi : F_v^\times \rightarrow \mathbf{C}^\times$ is the unique irreducible quotient of $\mathrm{ind}_B^{\mathrm{GL}_n} \chi \otimes \chi|\cdot| \otimes \dots \otimes \chi|\cdot|^{n-1}$. The twisted Steinberg factors $\mathrm{Sp}_{k_i}(\chi_i)$ correspond to Zelevinsky segments $\Delta_i = (\chi, \chi(1), \dots, \chi(k_i - 1))$.

Let $\mathcal{A}$ index the partitions of $sc(\pi)$ into k labeled subsets $S_1, \dots, S_k$ satisfying the following conditions:

- $|S_i| = n_i$ for all i .
- characters from the same Zelevinsky segment always belong to different subsets.
- if $\chi \in S_i, \chi' \in S_j$ share a segment and $\chi' = \chi(a)$ for $a > 0$, then $i < j$.

For each partition $\alpha \in \mathcal{A}$, let $r(\alpha)$ be the representation of $T(F)$ given by tensoring the characters of $sc(\pi)$ in the following order: characters in S_i precede characters in S_j when $i < j$, and the ordering of characters within each S_i is induced by the ordering of Zelevinsky segments.

Lemma 3.5. *For each $1 \leq i \leq r$, let π_i be a smooth representation of L_i . Then*

$$(\mathrm{ind}_Q^G \pi_1 \otimes \dots \otimes \pi_r)_N^{ss} = \bigoplus_{w \in [W_Q \backslash W/W_P]} \mathrm{ind}_{w^{-1}Qw \cap M}^M w^{-1}(\pi_1 \otimes \dots \otimes \pi_r)_{L \cap wNw^{-1}}.$$

Lemma 3.6. *Let π be an irreducible admissible $\mathrm{GL}_n(F_v)$ -module, such that $\pi^{\mathfrak{p}_1} \neq 0$. Consider $\pi^{\mathfrak{p}_1}$ as a $\mathbf{Z}[\Xi_v]^{S_\mu}$ -module via the map $t_\mu \circ S_\mu^{-1}$. Then $(\pi^{\mathfrak{p}_1})^{ss}$ is a direct sum of 1-dimensional submodules indexed by a subset of $\mathcal{A}$. For a finite set S of characters and positive integer $k \leq |S|$, let $e_k(S(\varpi))$ denote the k -th symmetric polynomial of elements of S evaluated at ϖ . Then on the component associated to $(S_1, \dots, S_k) \in \mathcal{A}$, the action of $V^{i,j}$ is given by $e_i(S_j)$ for all $1 \leq i \leq n_j$.*

Proof. We have a surjection

$$\mathrm{ind}_Q^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \dots \otimes \mathrm{Sp}_{k_r}(\chi_r) \twoheadrightarrow \pi,$$

and the induced map

$$(\mathrm{ind}_Q^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \dots \otimes \mathrm{Sp}_{k_r}(\chi_r))^{\mathfrak{p}_1} \rightarrow \pi^{\mathfrak{p}_1}$$

is also surjective. By Lemma 3.5, we can write

$$(\mathrm{ind}_Q^G \mathrm{Sp}_{k_1}(\chi_1) \otimes \dots \otimes \mathrm{Sp}_{k_r}(\chi_r))_N^{ss} = \sigma \oplus \bigoplus_{(S_1, \dots, S_k) \in \mathcal{A}} \mathrm{ind}_{B \cap M}^M \left(\bigotimes_{\psi_1 \in S_1} \psi_1 \otimes \dots \otimes \bigotimes_{\psi_k \in S_k} \psi_k \right).$$

Here, the summands indexed by $\mathcal{A}$ correspond to $w \in [W_Q \backslash W/W_P]$ represented by partitions $\{n_j^i\}$ satisfying $n_j^i \leq 1$ for all i, j (cf. Lemma 3.3) and σ represents all other summands. We will now show that σ does not have $\mathfrak{m}_1$ -invariants. Let $\mathfrak{m}_{i,1}^w \subset \mathfrak{p}_{i,1}^w$ be the subgroups of the Levi subgroup of L_i defined analogously to $\mathfrak{p}_{i,1}^w$.

Suppose $\sigma^{\mathfrak{m}_1}$ is nonzero. Let θ be a representation of $\mathrm{GL}_{n_j^i}(F_v)$ which is a tensor factor of $(\mathrm{Sp}_{k_1}(\chi_1) \otimes \dots \otimes \mathrm{Sp}_{k_r}(\chi_r))_{L \cap wNw^{-1}}$ for some $w \in [W_Q \backslash W/W_P]$ contributing to σ . Then θ has to be spherical if $j < k$ and has to have a fixed vector by $\ker(\mathrm{GL}_{n_j^i}(\mathcal{O}_{F_v}) \rightarrow \mathrm{GL}_{n_j^i}(k_v) \xrightarrow{\det} k_v^\times \rightarrow k_v^\times(p))$ if $j = k$. This would imply that $\mathrm{Sp}_{k_i}(\chi_i)^{\mathfrak{p}_{i,1}^w} \neq 0$ for all $1 \leq i \leq r$ and all w representing partitions $m_i = n_1^i + \dots + n_k^i$, such that there exists at least one $1 \leq i \leq r$ for which $k_i > 1$ and $n_j^i > 1$ for some $1 \leq j \leq k$. To get a contradiction, it is therefore enough to show that $\mathrm{Sp}_{k_i}(\chi_i)^{\mathfrak{p}_{i,1}^w} = 0$.

Define the subgroup $\mathrm{Iw}'_i \subset \mathfrak{p}_{i,1}^w$ to be a subgroup of the L_i -Iwahori with 1's mod ϖ on the diagonal at indices $n_{k-1}^i + 1$ through n_k^i . There are two possibilities: either $\mathfrak{p}_{i,1}^w = \mathrm{GL}_{m_i}(\mathcal{O}_{F_v})$, or Iw'_i has at least

one $\ast \bmod \varpi$ on the diagonal. In the former case, we are done since $\mathrm{Sp}_{k_i}(\chi_i)$ is never spherical. In the latter case, let $\mathfrak{t}'$ be the diagonal component of Iw'_i . Then

$$\mathrm{Sp}_{k_i}(\chi_i)^{\mathrm{Iw}'_i} = \mathrm{Sp}_{k_i}(\chi_i)^{\mathfrak{t}'_U} = (\chi_i \otimes \dots \otimes \chi_i | \cdot |^{k_i-1})^{\mathfrak{t}'},$$

where U is the unipotent radical of the Borel. Since $\mathfrak{t}'$ has at least one $\mathcal{O}_{F_v}^\times$ factor, if this is nonzero, χ_i must be unramified. But in this case, any $\mathfrak{p}_{i,1}^w$ -fixed vector would be automatically fixed by the parahoric $\mathfrak{p}_i^w$, which properly contains the Iwahori, and hence, does not fix any vector in $\mathrm{Sp}_{k_i}(\chi_i)$. $\square$

For a partition $n = n_1 + \dots + n_k$ which we call μ , define elements

$$\begin{aligned} P_{\mu,i} &= \prod_{j=s_{\mu,i-1}+1}^{s_{\mu,i}} (T - X_j) \\ \mathrm{Res}_\mu &= \prod_{i < j} \mathrm{Res}(P_{\mu,i}, P_{\mu,j}) \in \mathbf{Z}[\Xi_v]^{S_\mu} \\ \mathrm{Res}_{q_v, \mu} &= \prod_{i < j} \mathrm{Res}(P_{\mu,i}(q_v T), P_{\mu,j}) \in \mathbf{Z}[\Xi_v]^{S_\mu}. \end{aligned}$$

Then there exist unique polynomials $Q_{\mu,i} \in \mathbf{Z}[\Xi_v]^{S_\mu}[T]$, such that $\deg Q_{\mu,i} < n_i$ and

$$\sum_{i=1}^n Q_{\mu,i} \prod_{j \neq i} P_{\mu,j} = \mathrm{Res}_\mu.$$

Define

$$E_{\mu,i} = Q_{\mu,i} \prod_{j \neq i} P_{\mu,j}.$$

The following statement is elementary.

Lemma 3.7. *Take any $A \in M_n(\mathbf{C})$ with a factorisation*

$$\det(T - A) = \prod_{i=1}^k p_{\mu,i}(T),$$

where $p_{\mu,i} \in \mathbf{C}[T]$ are pairwise coprime and $\deg p_{\mu,i} = n_i$. Consider the homomorphism $\varphi : \mathbf{Z}[\Xi_v]^{S_\mu} \rightarrow \mathbf{C}$ defined by the polynomials $p_{\mu,i}$. By this, we mean the homomorphism sending $e_j(X_{s_{\mu,i-1}+1}, \dots, X_{s_{\mu,i}})$ to $(-1)^j$ times the coefficient of T^j in $p_{\mu,i}$. This homomorphism can be extended to $\varphi : \mathbf{Z}[\Xi_v]^{S_\mu}[T, \mathrm{Res}_\mu^{-1}] \rightarrow \mathbf{C}[T]$. Then $\varphi(E_{\mu,i}/\mathrm{Res}_\mu)(A)$ projects $\mathbf{C}^n$ onto the sum of generalised eigenspaces of A corresponding to the roots of $p_{\mu,i}$.

Proposition 3.8. *Let π be an irreducible admissible $GL_n(F_v)$ -module. Then either $\mathrm{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1} = 0$, or*

$$\mathrm{rec}_{F_v}(\pi) = (\chi_1 \oplus \dots \oplus \chi_n, 0),$$

where $\chi_1, \dots, \chi_{n_1+\dots+n_{k-1}}$ are unramified and the rest are tamely ramified with equal restriction to inertia.

Proof. Using the notation from the discussion preceding Lemma 3.5, if there exists some $k_i > 1$, then $\mathrm{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1} = 0$ follows from Lemma 3.6. Otherwise, we can apply the proof of [CHT08, Lemma 3.1.6] for the second conclusion. $\square$

Proposition 3.9. *Let π be an irreducible admissible $GL_n(F_v)$ -module. Let $(r, N) = \text{rec}_{F_v}(\pi)$. Then either $(S_\mu \circ t_\mu^{-1} \circ \iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1})(\text{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1}) = 0$ or $N = 0$ and*

$$r^\vee = \chi_1 \oplus \dots \oplus \chi_n,$$

where $\chi_1, \dots, \chi_{n_1+\dots+n_{k-1}}$ are unramified and the rest are tamely ramified with equal restriction to inertia.

Proof. Let $\pi^\vee$ be the contragradient of π . Then $\text{rec}_{F_v}(\pi^\vee) = (r^\vee, -^t N)$. We have a perfect pairing $(\pi^\vee)^{\mathfrak{p}_1} \times \pi^{\mathfrak{p}_1} \rightarrow \mathbf{C}$ which is antisymmetric with respect to action of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ and $S_\mu \circ t_\mu^{-1} \circ \iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1}$. Therefore, $(S_\mu \circ t_\mu^{-1} \circ \iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1})(\text{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1}) = 0$ if and only if $\text{Res}_{q_v, \mu}^{n!} (\pi^\vee)^{\mathfrak{p}_1} = 0$. Thus, we can assume both of these are nonzero, in which case, by Proposition 3.8, we get the desired result. $\square$

Let $\varphi_v \in G_{F_v}$ be any lift of Frobenius.

Proposition 3.10. *Let π be an irreducible admissible $GL_n(F_v)$ -module. Let $(r, N) = \text{rec}_{F_v}(\pi)$. Let R be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\text{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ under the map $t_\mu \circ S_\mu^{-1}$. Then either $\text{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1} = 0$ or the following relation holds over R : for all $\tau \in I_{F_v}$*

$$\text{Res}_\mu^{n!} \left(\sum_{i=1}^{k-1} E_{\mu,i}(r(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu,k}(r(\varphi_v)) - \text{Res}_\mu r(\tau) \right) = 0.$$

Proof. Assume $\text{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1} \neq 0$. It is enough to check our relation for each localisation of R at a maximal ideal $\mathfrak{m}$. If $\text{Res}_\mu \in \mathfrak{m}$, then $\text{Res}_\mu^{n!} = 0$ in $R_{\mathfrak{m}}$. Otherwise, $R_{\mathfrak{m}} = \mathbf{C}$ by [Sta18, Tag 00UA] and the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $R/\mathfrak{m}$ corresponds to the polynomials $\prod_{j=s_{\mu,i-1}+1}^{s_{\mu,i}} (T - \chi_j(\varphi_v))$ for $i = 1, \dots, k$. Then the image of

$$\text{Res}_\mu^{-1} \left(\sum_{i=1}^{k-1} E_{\mu,i}(r(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu,k}(r(\varphi_v)) \right)$$

in $M_n(R_{\mathfrak{m}})$ is a diagonal matrix with $n - n_k$ first entries equal to 1 and the rest equal to $\chi_n(\tau)$. This concludes the proof. $\square$

Proposition 3.11. *Let π be an irreducible admissible $GL_n(F_v)$ -module. Let $(r, N) = \text{rec}_{F_v}(\pi)$. Let R' be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\text{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ via the map $\iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1}$. Then either $(\iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1})(\text{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1}) = 0$ or the following relation holds over R' : for all $\tau \in I_{F_v}$*

$$(\iota_{\mathcal{H}} \circ t_\mu \circ S_\mu^{-1}) \left(\text{Res}_\mu^{n!} \left(\sum_{i=1}^{k-1} E_{\mu,i}(r^\vee(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu,k}(r^\vee(\varphi_v)) - \text{Res}_\mu r^\vee(\tau) \right) \right) = 0.$$

Proof. This follows from Proposition 3.9 in the same way as Proposition 3.10 follows from Proposition 3.8. $\square$

In what follows, we will use a twisted version of the propositions above. Define a map $\Sigma^T : \mathcal{O}[\Xi_{v,1}]^{S_\mu} \rightarrow \mathcal{H}_{\mathcal{O}}(GL_n(F_v), \mathfrak{p}_{v,1})$ given by

$$\Sigma^T(f)(g) = t_\mu(S_\mu^{-1}(f))(g) |\det(g)|^{(1-n)/2}.$$

Let us show that this map is in fact defined over $\mathbf{Z}[q_v^{-1}]$ and thus does not depend on the choice of square root of q_v^{-1} . Note that t_μ is defined over $\mathbf{Z}[q_v^{-1}]$ up to $\delta_{p_\mu}^{1/2}$ and S_μ is defined over $\mathbf{Z}[q_v^{-1}]$ up to

$\prod_{i=1}^k \det(m_i)^{(1-n_i)/2}$, where $(m_i) \in M_\mu(F_v)$ with $m_i \in \mathrm{GL}_{n_i}(F_v)$. Thus, the desired rationality over $\mathbf{Z}[q_v^{-1}]$ follows from the fact that

$$\prod_{i=1}^k |\det(m_i)|^{(1-n)/2} \prod_{i=1}^k |\det(m_i)|^{(1-n_i)/2} \prod_{1 \leq i < j \leq k} |\det(m_i)|^{n_j/2} |\det(m_j)|^{-n_i/2}$$

lies in $\mathbf{Z}[q_v^{-1}]$. Now let us restate Proposition 3.10 and Proposition 3.11.

Proposition 3.12. *Let π be an irreducible admissible $\mathrm{GL}_n(F_v)$ -module. Let $(r, N) = \mathrm{rec}_{F_v}^T(\pi)$. Let R be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\mathrm{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ under the map Σ^T . Then either $\mathrm{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1} = 0$ or the following relation holds over R : for all $\tau \in I_{F_v}$*

$$\mathrm{Res}_{\mu}^{n!} \left(\sum_{i=1}^{k-1} E_{\mu,i}(r(\varphi_v)) + \langle \mathrm{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu,k}(r(\varphi_v)) - \mathrm{Res}_{\mu} r(\tau) \right) = 0.$$

Proposition 3.13. *Let π be an irreducible admissible $\mathrm{GL}_n(F_v)$ -module. Let $(r, N) = \mathrm{rec}_{F_v}^T(\pi)$. Let R' be the image of $\mathcal{O}[\Xi_{v,1}]^{S_\mu}$ in $\mathrm{End}_{\mathcal{O}}(\pi^{\mathfrak{p}_1})$ via the map $\iota_{\mathcal{H}} \circ \Sigma^T$. Then either $(\iota_{\mathcal{H}} \circ \Sigma^T)(\mathrm{Res}_{q_v, \mu}^{n!} \pi^{\mathfrak{p}_1}) = 0$ or the following relation holds over R' : for all $\tau \in I_{F_v}$*

$$(\iota_{\mathcal{H}} \circ \Sigma^T) \left(\mathrm{Res}_{\mu}^{n!} \left(\sum_{i=1}^{k-1} E_{\mu,i}(r^{\vee}(\varphi_v)) + \langle \mathrm{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu,k}(r^{\vee}(\varphi_v)) - \mathrm{Res}_{\mu} r^{\vee}(\tau) \right) \right) = 0.$$

4. Setup

Let F/F^+ be an imaginary CM-field with ring of integers $\mathcal{O}$. Let Ψ_n be the matrix with 1-s on the antidiagonal and 0-s elsewhere, and let

$$J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}.$$

Define $\widetilde{G}$ to be the group scheme over $\mathcal{O}_{F^+}$ defined by the functor of points

$$\widetilde{G}(R) = \{g \in \mathrm{GL}_{2n}(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F) \mid {}^t g J_n g^c = J_n\}.$$

Then $\widetilde{G}$ is a quasisplit reductive group over F^+ . It is a form of GL_{2n} which becomes split after the quadratic base change F/F^+ . If v is a place of F lying above a place $\bar{v}$ of F^+ which splits in F , then we have a canonical isomorphism $\iota_v : \widetilde{G}(F_v^{\pm}) \cong \mathrm{GL}_{2n}(F_v)$. There is an isomorphism $F_v^{\pm} \otimes_{F^+} F \cong F_v \times F_{v^c}$ and ι_v is given by composition

$$\widetilde{G}(F_v^+) \hookrightarrow \mathrm{GL}_{2n}(F_v) \times \mathrm{GL}_{2n}(F_{v^c}) \rightarrow \mathrm{GL}_{2n}(F_v),$$

where the second map is the projection on the first factor. We write $T \subset B \subset G$ for the subgroups consisting, respectively, of the diagonal and upper-triangular matrices in $\widetilde{G}$. Similarly, we write $G \subset P \subset \widetilde{G}$ for the Levi and parabolic subgroups consisting, respectively, of the block upper diagonal and block upper-triangular matrices with blocks of size $n \times n$. Then $P = U \rtimes G$, where U is the unipotent radical of P , and we can identify G with $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_n$ via the map

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mapsto D \in \mathrm{GL}_n(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F).$$

An element $(g_v)_v \in G(\mathbf{A}_{F^+}^\infty) = GL_n(\mathbf{A}_F^\infty)$ is called *neat* if the intersection $\cap_v \Gamma_v$ is trivial, where $\Gamma_v \subset \overline{\mathbf{Q}}^\times$ is the torsion subgroup of the subgroup of $\overline{F}_v^\times$ generated by the eigenvalues of g_v acting via some faithful representation of G . We call a neat open compact subgroup $K \subset G(\mathbf{A}_{F^+}^\infty)$ *good* if it has the form $K = \prod_v K_v$, where the product is running over the finite places of F . We make similar definitions with $\tilde{G}$ in place of G .

After extending scalars to F^+ , T and B form a maximal torus and a Borel subgroup, respectively, of $\tilde{G}$ and G is the unique Levi subgroup of the parabolic subgroup P of $\tilde{G}$ which contains T . We call an open compact subgroup $\tilde{K}$ of $\tilde{G}(\mathbf{A}_{F^+}^\infty)$ *decomposed* with respect to the Levi decomposition $P = GU$ if $\tilde{K} = \tilde{K}_G \ltimes \tilde{K}_U$, where $\tilde{K}_G$ is the image of $\tilde{K}$ in G and $\tilde{K}_U = \tilde{K} \cap U(\mathbf{A}_{F^+}^\infty)$.

If K is a good subgroup of G , we let X_K be the corresponding locally symmetric space. Similarly, if $\tilde{K}$ is a good open compact subgroup of $\tilde{G}$, then $\tilde{X}_{\tilde{K}}$ denotes the locally symmetric space. More generally, if H is a connected algebraic group over a number field L and $K_H \subset H(\mathbf{A}_M^\infty)$ is a good subgroup, then we write $X_{K_H}^H$ for the locally symmetric space of H of level K_H .

Fix a rational prime p and a finite extension $E/\mathbf{Q}_p$ which contains the images of all embeddings $F \hookrightarrow \overline{\mathbf{Q}}_p$. We write $\mathcal{O}$ for the ring of integers of E and $\varpi \in \mathcal{O}$ for a choice of uniformiser. For $\lambda \in (\mathbf{Z}_+^n)^{\text{Hom}}(F, E)$, we define an $\mathcal{O}[\prod_{v|p} GL_n(\mathcal{O}_{F_v})]$ -module $\mathcal{V}_\lambda$ as in [ACC⁺18, Section 2.2.1]. Similarly for $\tilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\text{Hom}}(F^+, E)$, we have an $\mathcal{O}[\prod_{\bar{v}|p} \tilde{G}(\mathcal{O}_{F_v^+})]$ -module $\mathcal{V}_{\tilde{\lambda}}$. Both $\mathcal{V}_\lambda$ and $\mathcal{V}_{\tilde{\lambda}}$ are finite free $\mathcal{O}$ -modules.

Let S be a set of places of F , such that $S = S^c$ and, such that S contains all places above p and all places of F which are ramified over F^+ . Let $\bar{S}$ be the set of places of F^+ lying below a place in S . Let $K \subset G(\mathbf{A}_{F^+}^\infty)$ be a good subgroup, such that $K_{\bar{v}} = G(\mathcal{O}_{F_v^+})$ for $\bar{v} \notin \bar{S}$, and similarly, let $\tilde{K} \subset \tilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup, such that $\tilde{K}_{\bar{v}} = \tilde{G}(\mathcal{O}_{F_v^+})$ for $\bar{v} \notin \bar{S}$. Additionally, we define $\tilde{\Xi}_{\bar{v}} := \Xi_v \times \Xi_{v^c}$ and $\tilde{\Xi}_{\bar{v},1} := \Xi_{v,1} \times \Xi_{v^c}$.

Define the Hecke algebras

$$\mathcal{H}^S = \mathcal{H}_{\mathcal{O}}(G(\mathbf{A}_{F^+}^{\infty, \bar{S}}), K^{\bar{S}})$$

$$\tilde{\mathcal{H}}^S = \mathcal{H}_{\mathcal{O}}(\tilde{G}(\mathbf{A}_{F^+}^{\infty, \bar{S}}), \tilde{K}^{\bar{S}})$$

$$\mathbf{T}^S \cong \bigotimes_{v \notin S}^{\prime} \mathcal{O}[\Xi_v]^{S_n}$$

$$\tilde{\mathbf{T}}^S \cong \bigotimes_{\bar{v} \notin \bar{S}}^{\prime} \mathcal{O}[\tilde{\Xi}_{\bar{v}}]^{S_{2n}}.$$

Using the isomorphism

$$G(\mathcal{O}_{F_v^+}) \cong GL_n(\mathcal{O}_{F_v})$$

together with the Satake isomorphisms, as well as the homomorphism

$$\mathcal{O}[\tilde{\Xi}_{\bar{v}}]^{S_{2n}} \rightarrow \mathcal{H}_{\mathcal{O}}(\tilde{G}(F_v^+), \tilde{G}(\mathcal{O}_{F_v^+}))$$

given by the polynomial $\tilde{P}_v(X)$ defined in [ACC⁺18, Equation 2.2.6], we get homomorphisms $\mathbf{T}^S \rightarrow \mathcal{H}^S$ and $\tilde{\mathbf{T}}^S \rightarrow \tilde{\mathcal{H}}^S$. We also have homomorphisms

$$\mathbf{T}^S \rightarrow \text{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}_\lambda))$$

$$\tilde{\mathbf{T}}^S \rightarrow \text{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(\tilde{X}_{\tilde{K}}, \mathcal{V}_{\tilde{\lambda}}))$$

defined in [ACC⁺18, Section 2.1.2], and we can denote by $\mathbf{T}^S(K, \lambda)$, $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})$, respectively, the images of those homomorphisms. The functor H^* induces $\mathcal{O}$ -algebra homomorphisms

$$\mathbf{T}^S(K, \lambda) \rightarrow \mathrm{End}_{\mathcal{O}}(H^*(X_K, \mathcal{V}_\lambda))$$

$$\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda}) \rightarrow \mathrm{End}_{\mathcal{O}}(H^*(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})).$$

5. Boundary cohomology

Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a neat compact open subgroup decomposed with respect to the Levi decomposition $P = GU$. We also assume that $\widetilde{K}_v = \widetilde{G}(\mathcal{O}_{F_v^+})$ for $v \notin \overline{S}$. Define K as the image of $\widetilde{K}$ in $G(\mathbf{A}_{F^+}^\infty)$, $\widetilde{K}_P = \widetilde{K} \cap P(\mathbf{A}_{F^+}^\infty)$ and $K_U = \widetilde{K} \cap U(\mathbf{A}_{F^+}^\infty)$. Both K and $\widetilde{K}_P$ are neat. We recall from [NT16, Section 3.1.2] that the boundary $\partial \widetilde{X}_{\widetilde{K}} = \widetilde{X}_{\widetilde{K}}$ of the Borel-Serre compactification has a $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ -equivariant stratification indexed by the standard parabolic subgroups of $\widetilde{G}$. For each standard parabolic subgroup Q , label the corresponding stratum $\widetilde{X}_{\widetilde{K}}^Q$. We can write

$$\widetilde{X}_{\widetilde{K}}^Q = Q(F^+) \backslash (X^Q \times \widetilde{G}(\mathbf{A}_{F^+}^\infty) / \widetilde{K}).$$

From now on, we will focus on the stratum $\widetilde{X}_{\widetilde{K}}^P$ corresponding to the Siegel parabolic. Let us establish some useful maps between the manifolds introduced above. The stratum $\widetilde{X}_{\widetilde{K}}^P$ can be described as a union of connected components indexed by the set $P(F^+) \backslash \widetilde{G}(\mathbf{A}_{F^+}^\infty) / \widetilde{K}$. The locally symmetric space $X_{\widetilde{K}}^P$ is a union of the same components indexed by the set $P(F^+) \backslash P(\mathbf{A}_{F^+}^\infty) / \widetilde{K}_P$. Thus, we have a natural open immersion $i : X_{\widetilde{K}}^P \rightarrow \widetilde{X}_{\widetilde{K}}^P$, such that $i^* : H^*(\widetilde{X}_{\widetilde{K}}^P, \mathcal{O}) \rightarrow H^*(X_{\widetilde{K}}^P, \mathcal{O})$ is a split epimorphism. We also have a proper map $j : X_{\widetilde{K}_P}^P \rightarrow X_K$ which has a section by [NT16, Section 3.1.1]. Thus, we get a split monomorphism $j^* : H^*(X_K, \mathcal{O}) \rightarrow H^*(X_{\widetilde{K}}^P, \mathcal{O})$. We also recall the ‘restriction to P’ and ‘integration along N’ homomorphisms:

$$r_P : \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}_{F^+}^\infty, \overline{S}), \widetilde{K} \overline{S}) \rightarrow \mathcal{H}_{\mathcal{O}}(\widetilde{P}(\mathbf{A}_{F^+}^\infty, \overline{S}), \widetilde{K}_P \overline{S})$$

$$r_G : \mathcal{H}_{\mathcal{O}}(\widetilde{P}(\mathbf{A}_{F^+}^\infty, \overline{S}), \widetilde{K}_P \overline{S}) \rightarrow \mathcal{H}_{\mathcal{O}}(G(\mathbf{A}_{F^+}^\infty, \overline{S}), K \overline{S})$$

defined in [NT16, Section 2.2]. We record the following proposition, which follows from the discussion above:

Proposition 5.1.

1. For all $t \in \widetilde{\mathbf{T}}^S$ and $h \in H^*(\widetilde{X}_{\widetilde{K}}^P, \mathcal{O})$, we have $i^*(th) = r_P(t)i^*(h)$.
2. For all $t \in \mathcal{H}_{\mathcal{O}}(\widetilde{P}(\mathbf{A}_{F^+}^\infty, \overline{S}), \widetilde{K}_P \overline{S})$ and $h \in H^*(X_K, \mathcal{O})$, we have $j^*(r_G(t)h) = tj^*(h)$.

Consider the composite

$$S = r_G \circ r_P : \mathcal{H}_{\mathcal{O}}(\widetilde{G}(\mathbf{A}_{F^+}^\infty, \overline{S}), \widetilde{K} \overline{S}) \rightarrow \mathcal{H}_{\mathcal{O}}(G(\mathbf{A}_{F^+}^\infty, \overline{S}), K \overline{S}).$$

By [NT16, Proposition-Definition 5.3], this map coincides with the tensor product of maps $\mathcal{O}[\widetilde{\Xi}_v]^{S_{2n}} \rightarrow \mathcal{O}[\Xi_v]^{S_n}$ determined by the polynomial $S_n(P_v(X)q_v^{n(2n-1)}P_{v^c}^\vee(q_v^{1-2n}X))$.

Let $\mathfrak{m} \subset \mathbf{T}^S$ be a non-Eisenstein maximal ideal of Galois type with residue field k . We have an associated continuous semisimple representation $\bar{\rho}_{\mathfrak{m}} : G_{F,S} \rightarrow \mathrm{GL}_n(k)$, such that $\det(X - \bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)) \equiv P_v(X) \pmod{\mathfrak{m}}$. Fix a tuple $(Q, (\alpha_v)_{v \in Q})$, where

- $Q \subset S$ and $Q \cap Q^c = \emptyset$.
- Each place $v \in Q$ is split over F^+ . Moreover, for each place $v \in Q$, there exists an imaginary quadratic subfield $F_0 \subset F$, such that q_v splits in F_0 .
- For each place $v \in Q$, $\bar{\rho}_{\mathfrak{m}}$ is unramified at v and v^c and α_v is a root of $\det(X - \bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))$.

For each $v \in Q$, let d_v be multiplicity of α_v as a root of $\det(X - \bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v))$. Fix the partitions

$$\mu_v : 2n = d_v + (n - d_v) + n$$

$$\nu_v : n = d_v + (n - d_v).$$

Let

$$\Delta_v = \bigsqcup_{m \in M_{\mu_v}^+} [\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}] \subset \mathrm{GL}_n(F_v).$$

nonumber

Now we recall the theory of Hecke algebras of a monoid from [ACC⁺18, Section 2.1.9]. Specifically, we consider the restriction from $\widetilde{G}$ to P

$$r_P : \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) \rightarrow \mathcal{H}(P(F_v^+), P(F_v^+) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))$$

and integration along fibres

$$r_G : \mathcal{H}(P(F_v^+), P(F_v^+) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) \rightarrow \mathcal{H}(G(F_v^+), G(F_v^+) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))$$

combined with the isomorphism

$$\mathcal{H}(G(F_v^+), G(F_v^+) \cap \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) \cong \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})),$$

we get a map

$$S_v^+ : \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) \rightarrow \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})).$$

Write $P_{n,n} = M_{n,n}L_{n,n}$ for the parabolic subgroup of $GL_{2n}(F_v)$ corresponding to the partition $2n = n+n$, together with its Levi decomposition. For a given $m \in M^{++}$, from [ACC⁺18, Section 2.1.9], we know that

$$S_v^+(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}])) = |\delta_P(m)|^{-1} \iota_v^{-1}([\mathfrak{p}_{\mu_v,1} \cap M_{n,n}) m (\mathfrak{p}_{\mu_v,1} \cap M_{n,n})]).$$

By the same argument as in the proof of Lemma 3.2, we see that there exists $m \in M^{++}$, such that the right-hand side is invertible in $\mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}))$. Thus, we can extend the homomorphism to

$$\begin{aligned} \mathcal{S}_v : \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})) & [(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1} m \mathfrak{p}_{\mu_v,1}]))^{-1}] \\ & \rightarrow \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})). \end{aligned}$$

This homomorphism fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{O}[\tilde{\Xi}_{\bar{v},1}]^{S_{\mu_v}} & \longrightarrow & \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))[(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1}m\mathfrak{p}_{\mu_v,1}]))^{-1}] \\ \downarrow \mathcal{S}_v^f & & \downarrow \mathcal{S}_v \\ \mathcal{O}[\Xi_{v,1}]^{S_{\nu_v}} \otimes_{\mathcal{O}} \mathcal{O}[\Xi_{v^c}]^{S_n} & \longrightarrow & \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})), \end{array}$$

where $\mathcal{S}_v^f$ is the unique homomorphism which corresponds the polynomial $\prod_{i=1}^{2n}(T - X_i)$ to the tuple of polynomials $\prod_{i=1}^{d_v}(T - X_i)$, $\prod_{i=d_v+1}^n(T - X_i)$, $\mathcal{S}_n(q_v^{n(2n-1)}P_{v^c}^\vee(q_v^{1-2n}X))$ and maps $\tau_{\bar{v}}$ to τ_v .

We can define global Hecke algebras associated to our Taylor-Wiles data:

$$\begin{aligned} \tilde{\mathcal{H}}_Q^S &= \tilde{\mathcal{H}}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{H}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v,1}))[(\iota_v^{-1}([\mathfrak{p}_{\mu_v,1}m\mathfrak{p}_{\mu_v,1}]))^{-1}] \\ \tilde{\mathbf{T}}_Q^S &= \tilde{\mathbf{T}}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{O}[\tilde{\Xi}_{\bar{v},1}]^{S_{\mu_v}} \\ \mathcal{H}_Q^S &= \mathcal{H}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{H}(\mathrm{GL}_n(F_v) \times \mathrm{GL}_n(F_{v^c}), \mathfrak{p}_{\nu_v,1} \times \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})) \\ \mathbf{T}_Q^S &= \mathbf{T}^S \otimes_{\mathbf{Z}} \bigotimes_{v \in Q} \mathcal{O}[\Xi_{v,1}]^{S_{\nu_v}} \otimes_{\mathcal{O}} \mathcal{O}[\Xi_{v^c}]^{S_n}. \end{aligned}$$

The following proposition follows from the discussion above:

Proposition 5.2. *There exist homomorphisms $\mathcal{S}_Q^f : \tilde{\mathbf{T}}_Q^S \rightarrow \mathbf{T}_Q^S$ and $\mathcal{S}_Q : \tilde{\mathcal{H}}_Q^S \rightarrow \mathcal{H}_Q^S$ fitting into a commutative diagram*

$$\begin{array}{ccc} \tilde{\mathbf{T}}_Q^S & \longrightarrow & \tilde{\mathcal{H}}_Q^S \\ \downarrow \mathcal{S}_Q^f & & \downarrow \mathcal{S}_Q \\ \mathbf{T}_Q^S & \longrightarrow & \mathcal{H}_Q^S, \end{array}$$

where $\mathcal{S}_Q^f$ coincides with $\mathcal{S}_v^f$ at places $v \in Q$ and with the Satake isomorphism from [NT16, Proposition-Definition 5.3] at places $v \notin S$.

Let $\tilde{K}$ be a good subgroup of $\tilde{G}(\mathbf{A}_{F^+}^\infty)$, such that $\tilde{K}^S = \tilde{G}(\hat{\mathcal{O}}_{F^+}^\vee)$ and $\tilde{K}$ is decomposed with respect to P . We can define subgroups $\tilde{K}_1(Q) \subset \tilde{K}_0(Q) \subset \tilde{K}$ as follows:

- If $\bar{v} \notin \underline{Q}$, then $\tilde{K}_1(Q)_{\bar{v}} = \tilde{K}_0(Q)_{\bar{v}} = \tilde{K}_{\bar{v}}$.
- If $\bar{v} \in \underline{Q}$, then $\tilde{K}_1(Q)_{\bar{v}} = \iota_v^{-1}(\mathfrak{p}_{\mu_v,1})$ and $\tilde{K}_0(Q)_{\bar{v}} = \iota_v^{-1}(\mathfrak{p}_{\mu_v})$.

Let $K_1(Q)$, $K_0(Q)$, K be the images in $G(\mathbf{A}_{F^+}^\infty)$ of the intersections of $\tilde{K}_1(Q)$, $\tilde{K}_0(Q)$, $\tilde{K}$ with $P(\mathbf{A}_{F^+}^\infty)$. From the definition, we can see that all the subgroups from the previous sentence are decomposed with respect to P .

Proposition 5.3. *For $i = 0, 1$, we have*

1. *The open immersion $i : X_{\tilde{K}_i(Q)}^P \rightarrow \tilde{X}_{\tilde{K}_i(Q)}^P$ yields a split epimorphism*

$$i^* : H^*(\tilde{X}_{\tilde{K}_i(Q)}^P, \mathcal{O}) \rightarrow H^*(X_{\tilde{K}_i(Q)}^P, \mathcal{O}).$$
2. *The proper map $j : X_{\tilde{K}_i(Q)_P}^P \rightarrow X_{K_i(Q)}$ yields a split monomorphism*

$$j^* : H^*(X_{K_i(Q)}, \mathcal{O}) \rightarrow H^*(X_{\tilde{K}_i(Q)}^P, \mathcal{O}).$$

3. For all $t \in \mathcal{H}_{\mathcal{O}}(\iota_v^{-1}(\Delta_v), \iota_v^{-1}(\mathfrak{p}_{\mu_v, 1}))$ and $h \in H^*(\tilde{X}_{\tilde{K}_i(Q)}^P, \mathcal{O})$, we have

$$i^*(th) = r_P(t)i^*(h).$$

4. For all $t \in \mathcal{H}_{\mathcal{O}}(\tilde{P}(\mathbf{A}_{F^+}^{\infty, \bar{S}}), \tilde{K}_i(Q)_{\bar{P}}^{\bar{S}})$ and $h \in H^*(X_{K_i(Q)}, \mathcal{O})$, we have

$$j^*(r_G(t)h) = tj^*(h).$$

Proof. This follows from the discussion above Proposition 5.1 and [ACC⁺18, Lemma 2.1.14]. $\square$

Now let $\mathfrak{m}_Q \subset \mathbf{T}_Q^S$ be the maximal ideal generated by $\mathfrak{m}$ and the kernels of the maps $\mathcal{O}[\tilde{\Xi}_{\bar{v}, 1}]^{S_{\mu_v}} \rightarrow k$ associated to the polynomials $(x - \alpha_v)^{d_v}, \det(X - \bar{\rho}_{\mathfrak{m}}(\text{Frob}_v))/(x - \alpha_v)^{d_v}, \det(X - \bar{\rho}_{\mathfrak{m}}(\text{Frob}_{v^c}))$ for $v \in Q$. Also, let $\tilde{\mathfrak{m}}_Q = S_Q^{f^{-1}}(\mathfrak{m}_Q)$.

Proposition 5.4. For $i = 0, 1$, the map $S_Q^f : \tilde{\mathbf{T}}_Q^S \rightarrow \mathbf{T}_Q^S$ descends to homomorphisms

$$\tilde{\mathbf{T}}_Q^S(H^*(\tilde{X}_{\tilde{K}_i(Q)}^P, \mathcal{O})) \rightarrow \mathbf{T}_Q^S(H^*(X_{K_i(Q)}, \mathcal{O}))$$

$$\tilde{\mathbf{T}}_Q^S(H^*(\partial \tilde{X}_{\tilde{K}_i(Q)}, \mathcal{O})_{\tilde{\mathfrak{m}}}) \rightarrow \mathbf{T}_Q^S(H^*(X_{K_i(Q)}, \mathcal{O})_{\mathfrak{m}}).$$

Proof. To prove the first statement, we need to show that for $t \in \text{Ann}_{\tilde{\mathbf{T}}_Q^S}(H^*(\tilde{X}_{\tilde{K}_i(Q)}^P, \mathcal{O}))$, we have $S_Q(t) \in \text{Ann}_{\mathbf{T}_Q^S}(H^*(X_{K_i(Q)}, \mathcal{O}))$. Let α be the right inverse of i^* and β be the left inverse of j^* . Take any $h \in H^*(X_{K_i(Q)}, \mathcal{O})$. Then we can write

$$\begin{aligned} S_Q(t)h &= r_G(r_P(t))h = \beta(j^*(r_G(r_P(t))h)) = \beta(r_P(t)j^*(h)) \\ &= \beta(r_P(t)i^*(\alpha(j^*(h)))) = \beta(i^*(t\alpha(j^*(h)))) = \beta(i^*(0)) = 0. \end{aligned}$$

To prove the second statement, it is enough to note that $H^*(\tilde{X}_{\tilde{K}_i(Q)}^P, \mathcal{O})_{\tilde{\mathfrak{m}}} \cong H^*(\partial \tilde{X}_{\tilde{K}_i(Q)}, \mathcal{O})_{\tilde{\mathfrak{m}}}$ by [ACC⁺18, Theorem 2.4.2]. $\square$

6. Galois deformation theory

Let $E \subset \bar{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$, with valuation ring $\mathcal{O}$, uniformiser ϖ and residue field k . Given a complete Noetherian local $\mathcal{O}$ -algebra Λ with residue field k , we let CNL_{Λ} denote the category of complete Noetherian local Λ -algebras with residue field k . We refer to an object in CNL_{Λ} as a CNL_{Λ} -algebra. We fix a number field F and let S_p be the set of places of F above p . We assume that E contains the images of all embeddings of F in $\mathbf{Q}_p$. We also fix a continuous absolutely irreducible homomorphism $\bar{\rho} : G_F \rightarrow \text{GL}_n(k)$. We assume throughout that $p \nmid 2n$.

Following [ACC⁺18, Definition 6.2.2], we call a global deformation problem a tuple

$$\mathcal{S} = (\bar{\rho}, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S}),$$

where

- S is a finite set of finite places of F containing S_p and all the places at which $\bar{\rho}$ is ramified.
- Λ_v is an object of $\text{CNL}_{\mathcal{O}}$ for each $v \in S$.
- $\mathcal{D}_v$ is a local deformation problem ([ACC⁺18, Section 6.2.1]) for each $v \in S$.

Associated to this global deformation problem, we have a completed tensor product $\Lambda = \widehat{\otimes}_{v \in S} \Lambda_v$. A global deformation problem determines a representable functor $\mathcal{D}_{\mathcal{S}} : \text{CNL}_{\Lambda} \rightarrow \mathbf{Set}$ which takes an object $A \in \text{CNL}_{\Lambda}$ to the set of deformations $\rho : G_F \rightarrow \text{GL}_n(A)$ of type $\mathcal{S}$.

Let v be a finite place of F , such that $v \notin S$ and $q_v \equiv 1 \pmod{p}$. We let $\mathcal{D}_v^1$ denote the local deformation problem consisting of all lifts which associate $A \in \text{CNL}_{\Lambda_v}$ to the set of lifts which are $1 + M_n(\mathfrak{m}_A)$ -conjugate to a lift of the form $s_v \oplus \psi_v$, where s_v is unramified and the image of ψ_v under

inertia is contained in the set of scalar matrices. This is indeed a local deformation problem, as is shown in [Tho12, Lemma 4.2].

Lemma 6.1. *Let $\bar{r} : G_{F_v} \rightarrow \mathrm{GL}_n(k)$ be an unramified continuous representation and A is a complete Noetherian local $\mathcal{O}$ -algebra with residue field k and a principal maximal ideal $\mathfrak{m}_A$. Suppose further that $\bar{r}$ may be written in the form $\bar{r} = \bar{r}_1 \oplus \bar{r}_2$, where $\det(X - \bar{r}_1(\mathrm{Frob}_v))$ and $\det(X - \bar{r}_2(\mathrm{Frob}_v))$ are relatively prime. Also suppose that $q_v = 1$ in k . Then any lift $r : G_{F_v} \rightarrow \mathrm{GL}_n(A)$ of $\bar{r}$ is $1 + M_n(\mathfrak{m}_A)$ -conjugate to one of the form $r = r_1 \oplus r_2$, where r_1 and r_2 are lifts of $\bar{r}_1$ and $\bar{r}_2$, respectively.*

Proof. Let $n_i = \dim \bar{r}_i$. Suppose we have a lift $r_m : G_{F_v} \rightarrow \mathrm{GL}_n(A)$ of $\bar{r}$, such that $r_m \bmod \mathfrak{m}_A^m$ can be written in the form $r_1 \oplus r_2$. We will show that there exists a matrix $X_m \in 1 + M_n(\mathfrak{m}_A^m)$, such that $r_{m+1} := X_m r_m X_m^{-1}$ satisfies the same condition $\bmod \mathfrak{m}_A^{m+1}$. Write

$$X_m = \begin{pmatrix} 1 & Y \\ Z & 1 \end{pmatrix} \quad r_m = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $Y \in M_{n_1 \times n_2}(\mathfrak{m}_A^m)$ and $Z \in M_{n_2 \times n_1}(\mathfrak{m}_A^m)$. Then the condition on r_{m+1} transforms into

$$YD - AY + B = 0 \bmod \mathfrak{m}_A^{m+1} \quad (6.2)$$

$$ZA - DZ + C = 0 \bmod \mathfrak{m}_A^{m+1}. \quad (6.3)$$

We will focus on the first condition, the second is similar. We know that $r_m \bmod \mathfrak{m}_A^m$ is block-diagonal, so we can consider $\bar{b}, \bar{y}$ to be the images of B and Y , respectively, in $\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}$,

$$\bar{b} \bar{r}_2^{-1} = \bar{r}_1 \bar{y} \bar{r}_2^{-1} - \bar{y} \quad (6.4)$$

in $M_n(\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}) = M_n(k) \otimes_k \mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}$. Using the fact that r is a homomorphism, for $\sigma, \tau \in G_{F_v}$, we can write

$$A(\sigma)B(\tau) + B(\sigma)D(\tau) = B(\sigma\tau).$$

Rewriting and reducing $\bmod \mathfrak{m}_A^{n+1}$, we get

$$\bar{r}_1(\sigma) \bar{b}(\tau) + \bar{b}(\sigma) \bar{r}_2(\tau) = \bar{b}(\sigma\tau)$$

$$\bar{b}(\sigma\tau) \bar{r}_2^{-1}(\sigma\tau) = \bar{r}_1(\sigma) \bar{b}(\tau) \bar{r}_2^{-1}(\tau) \bar{r}_2^{-1}(\sigma) + \bar{b}(\sigma) \bar{r}_2^{-1}(\sigma). \quad (6.5)$$

Give $M_{n_1 \times n_2}(\mathfrak{m}_A^m/\mathfrak{m}_A^{m+1})$ the structure of a G_{F_v} -module via $\bar{r}_1(-)\bar{r}_2^{-1}$, and denote this module $\mathrm{ad}(\bar{r}_1, \bar{r}_2)$. Then the last equation implies that $\bar{b} \bar{r}_2^{-1}$ is in $Z^1(G_{F_v}, \mathrm{ad}(\bar{r}_1, \bar{r}_2))$. Since $\bar{r}_1, \bar{r}_2$ have coprime characteristic polynomials, we know that $H^1(G_{F_v}, \mathrm{ad}(\bar{r}_1, \bar{r}_2)) = 0$ by local Tate duality (here, we are using that $q_v = 1$ in k), which means $\bar{b} \bar{r}_2^{-1} \in B^1(G_{F_v}, \mathrm{ad}(\bar{r}_1, \bar{r}_2))$, and thus we can find y satisfying Eq. 6.4. $\square$

Now we define our version of the Taylor-Wiles datum, analogous to the one appearing in [ACC⁺18, Section 6.2.27].

Definition 6.6. Let

$$S = (\bar{\rho}, S, \{\Lambda_v\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S})$$

be a global deformation problem. A Taylor-Wiles datum of level $N \geq 1$ for $\mathcal{S}$ consists of a tuple $(Q, \alpha_{v \in Q})$, where

- A finite set Q of places of F , disjoint from S , such that $q_v \equiv 1 \pmod{p^N}$ for each $v \in Q$.
- For each $v \in Q$, α_v is an eigenvalue of $\bar{\rho}(\text{Frob}_v)$.

Given a Taylor-Wiles datum $(Q, (\alpha_v))$, we define a global deformation problem

$$\mathcal{S}_Q = (\bar{\rho}, S \cup Q, \{\Lambda_v\}_{v \in S} \cup \{\mathcal{O}_{F_v}\}_{v \in Q}, \{\mathcal{D}_v\}_{v \in S} \cup \{\mathcal{D}_v^1\}_{v \in Q}).$$

Define $\Delta_Q = \prod_{v \in Q} \Delta_v$. The representing object $R_{\mathcal{S}_Q}$ has a structure of a $\mathcal{O}[\Delta_Q]$ -algebra satisfying $R_{\mathcal{S}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R_S$.

Proposition 6.7. *Take $T = S$, and let $q > h_{S^\perp, T}^1(\text{ad } \bar{\rho}(1))$. Assume that $F = F^+ F_0$, where F_0 is an imaginary quadratic field, that $\zeta_p \notin F$ and that $\bar{\rho}(G_{F(\zeta_p)})$ is adequate. Then for every $N \geq 1$, there exists a choice of Taylor-Wiles datum $(Q_N, (\alpha_v)_{v \in Q})$ of level N satisfying the following:*

1. $|Q_N| = q$.
2. For each $v \in Q_N$, the rational prime below v splits in F_0 and $v^c \notin Q_N$.
3. Let $g = q - n^2[F^+ : \mathbf{Q}]$. Then there is a surjective morphism

$$R_S^{T, \text{loc}}[[X_1, \dots, X_g]] \rightarrow R_{\mathcal{S}_Q}^T,$$

in CNL_Λ .

Proof. The proof is very similar to the proof of [ACC⁺18, Proposition 6.2.32] (cf. [Tho12, Proposition 4.4]), we omit the details. $\square$

7. Representations into Hecke algebras

In this section, we construct the necessary Galois representations into the Hecke algebras associated to G . From Proposition 5.4, we know that we can create representations valued in the Hecke algebra acting on $H^*(X_{K_i(Q)}, \mathcal{O})_{\mathfrak{m}_Q}$ from representations valued in the Hecke algebra acting on $H^*(\partial \tilde{X}_{\tilde{K}_i(Q)}, \mathcal{O})_{\mathfrak{m}_Q}$. The latter representations will be constructed by glueing together Galois representations associated to cuspidal cohomological automorphic representations of $\tilde{G}(\mathbf{A}_{F^+}^\infty)$ as in [Sch15] and using the local computations of Section 3.

7.1. Hecke algebras for $\tilde{G}$

Theorem 7.1. *Suppose that $\tilde{K} \subset \tilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup which is decomposed with respect to P . Then there exists a $2n$ -dimensional $\tilde{\mathbf{T}}_Q^S(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I$ -valued group determinant $D_{c, Q}$ of $G_{F, S}$ for some ideal I of nilpotence degree depending only on n and $[F : \mathbf{Q}]$, such that the following properties hold:*

1. If $v \notin S$ is a place of F , then $D_{c, Q}(X - \text{Frob}_v)$ is equal to the image of $\tilde{P}_v(X)$ in $\tilde{\mathbf{T}}_Q^S(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I[X]$.
2. If $v \in Q$, then for any $\sigma \in G_{F, S}$ and $\tau \in I_{F_v}$, we have the relation

$$\text{Tr}_{D_{c, Q}} \left(\sigma \text{Res}_{q_v, \mu_v}^{(2n)!} \text{Res}_{\mu_v}^{(2n)!} \left(\sum_{i=1}^{k-1} E_{\mu_v, i}(\varphi_v) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\varphi_v) - \text{Res}_{\mu_v} \tau \right) \right) = 0.$$

Proof. This follows from Proposition 3.12 by using [ACC⁺18, Theorem 2.3.3] and [Sch15, Corollary 5.1.11] (see proof of [ACC⁺18, Proposition 3.2.2]). $\square$

Now we prove the version of the previous proposition for noncompactly supported cohomology:

Theorem 7.2. Suppose that $\tilde{K} \subset \tilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup which is decomposed with respect to P . Then there exists a $2n$ -dimensional $\tilde{\mathbf{T}}_Q^S(H^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I$ -valued group determinant D_Q of $G_{F,S}$ for some ideal I of nilpotence degree depending only on n and $[F : \mathbf{Q}]$, such that the following properties hold:

1. If $v \notin S$ is a place of F , then $D_Q(X - \text{Frob}_v)$ is equal to the image of $\tilde{P}_v(X)$ in $\tilde{\mathbf{T}}_Q^S(H^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I[X]$.
2. If $v \in Q$, then for any $\sigma \in G_{F,S}$ and $\tau \in I_{F_v}$, we have the relation

$$\text{Tr}_{D_Q} \left(\sigma \text{Res}_{q_v, \mu_v}^{(2n)!} \text{Res}_{\mu_v}^{(2n)!} \left(\sum_{i=1}^{k-1} E_{\mu_v, i}(\varphi_v) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\varphi_v) - \text{Res}_{\mu_v} \tau \right) \right) = 0.$$

Proof. Denote by $\tilde{\mathbf{T}}_{Q, \iota}^S(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))$ the image of $\tilde{\mathbf{T}}_Q^S$ under the homomorphism

$$\tilde{\mathbf{T}}_Q^S \rightarrow \mathcal{H}_O(\tilde{G}(\mathbf{A}_{F^+}^\infty), \tilde{K}_1(Q)) \xrightarrow{\iota_{\mathcal{H}}} \mathcal{H}_O(\tilde{G}(\mathbf{A}_{F^+}^\infty), \tilde{K}_1(Q)) \rightarrow \text{End}_{\mathbf{D}(O)}(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O})).$$

The same argument as in the proof of Theorem 7.1 shows that there exists a group determinant D_ι valued in $\tilde{\mathbf{T}}_{Q, \iota}^S(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I$ satisfying the following properties:

1. If $v \notin S$ is a place of F , then $D_Q(X - \text{Frob}_v)$ is equal to the image of $\tilde{P}_v(X)$ in $\tilde{\mathbf{T}}_{Q, \iota}^S(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I[X]$.
2. If $v \notin Q$, then for any $\sigma \in G_{F,S}$ and $\tau \in I_{F_v}$, we have the relation

$$\text{Tr}_{D_\iota} \left(\sigma \text{Res}_{q_v, \mu_v}^{(2n)!} \text{Res}_{\mu_v}^{(2n)!} \left(\sum_{i=1}^{k-1} E_{\mu_v, i}(\varphi_v) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\varphi_v) - \text{Res}_{\mu_v} \tau \right) \right) = 0.$$

By [NT16, Proposition 3.7], we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_O(\tilde{G}(\mathbf{A}_{F^+}^\infty), \tilde{K}_1(Q)) & \longrightarrow & \text{End}_{\mathbf{D}(O)}(R\Gamma(X_{\tilde{K}_1(Q)}, \mathcal{O})) \\ \downarrow \iota_{\mathcal{H}} & & \downarrow \wr \\ \mathcal{H}_O(\tilde{G}(\mathbf{A}_{F^+}^\infty), \tilde{K}_1(Q)) & \longrightarrow & \text{End}_{\mathbf{D}(O)}(R\Gamma_c(X_{\tilde{K}_1(Q)}, \mathcal{O})), \end{array} \quad (7.3)$$

where the right vertical arrow is induced by Poincaré duality. Then we get an isomorphism

$$\tilde{\mathbf{T}}_{Q, \iota}^S(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I_1 \xrightarrow{\sim} \tilde{\mathbf{T}}_Q^S(H^*(X_{\tilde{K}_1(Q)}, \mathcal{O}))/I_2$$

over $\tilde{\mathbf{T}}_Q^S$ for some ideals $I_{1,2}$ of nilpotence degrees depending only on n and $[F : \mathbf{Q}]$. Moreover, we can choose I_1 , such that it contains I . We can conclude by making D_Q the image of D_ι under this homomorphism. $\square$

Lemma 7.4. Let k be a field, and let $\bar{\rho}_1, \bar{\rho}_2 : G \rightarrow GL_n(k)$ be two nonisomorphic absolutely irreducible representations. Then the extended map $k[G] \rightarrow M_n(k) \oplus M_n(k)$ defined by $\bar{\rho}_1 \oplus \bar{\rho}_2$ is surjective.

Proof. We may pass to the algebraic closure of k (which we still denote k). Let $\ell_i : k[G] \rightarrow M_n(k)$ be the linear extension of $\bar{\rho}_i$ for $i = 1, 2$. The two maps ℓ_i are surjective by Burnside's theorem. Let A be the image of $\ell_1 \oplus \ell_2$, and let $I_i = \ker(A \rightarrow M_n(k))$, where $i = 1, 2$ corresponds to projecting on the first and second factor. Since ℓ_i are surjective, I_i are in fact two-sided ideals of $M_n(k)$. Then $I_i = M_n(k)$ or $I_i = 0$. If $I_i = M_n(k)$ for some i , then $\ell_1 \oplus \ell_2$ is surjective. Suppose then that $I_1 = I_2 = 0$. Then we have an automorphism f of $M_n(k)$ defined by $(v, f(v)) \in A$ for all $v \in M_n(k)$. Since all the automorphisms

of $M_n(k)$ are inner, we conclude that there exists $u \in \mathrm{GL}_n(k)$, such that $A = \{(v, uvu^{-1}) \mid v \in M_n(k)\}$. But this is impossible since $\bar{\rho}_1$ and $\bar{\rho}_2$ are nonisomorphic. $\square$

Theorem 7.5. *Suppose that $\tilde{K} \subset \tilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup which is decomposed with respect to P and that for each $v \in Q$, we have $\mathrm{Res}_{\mu_v} \notin \tilde{\mathfrak{m}}_Q$. Then there exists a continuous representation*

$$\rho_{\mathfrak{m}_Q} : G_{F, S \cup Q} \rightarrow \mathrm{GL}_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I)$$

satisfying the conditions below for some ideal $I \subset \mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})$ of nilpotence degree depending only on n and $[F : \mathbf{Q}]$.

1. If $v \notin S$ is a place of F , the characteristic polynomial of $\rho_{\mathfrak{m}_Q}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I[X]$.
2. If $v \in Q$, then $\rho_{\mathfrak{m}_Q}|_{G_{F_v}}$ is unramified.
3. If $v \in Q$, then $\rho_{\mathfrak{m}_Q}|_{G_{F_v}} = s \oplus \psi$, where s is unramified and $\tau \in I_{F_v}$ acts on ψ as a scalar $\langle \mathrm{Art}_{F_v}^{-1}(\tau) \rangle$.

Proof. Using Theorem 7.1 and Theorem 7.2, we can construct a $\tilde{\mathbf{T}}_Q^S(H_c^*(X_{\tilde{K}_1(Q)}, \mathcal{O})_{\tilde{\mathfrak{m}}_Q} \oplus H^*(X_{\tilde{K}_1(Q)}, \mathcal{O})_{\tilde{\mathfrak{m}}_Q})/I$ -valued group determinant D_Q of $G_{F, S \cup Q}$. Consider the long exact sequence

$$\dots \rightarrow H_c^i(\tilde{X}_{\tilde{K}_1(Q)}, \mathcal{O}) \rightarrow H^i(\tilde{X}_{\tilde{K}_1(Q)}, \mathcal{O}) \rightarrow H^i(\partial \tilde{X}_{\tilde{K}_1(Q)}, \mathcal{O}) \rightarrow H_c^{i+1}(\tilde{X}_{\tilde{K}_1(Q)}, \mathcal{O}) \rightarrow \dots$$

Using this sequence and Proposition 5.4, we know that S_Q^f descends to a homomorphism

$$\tilde{\mathbf{T}}_Q^S(H_c^*(\tilde{X}_{\tilde{K}_1(Q)}, \mathcal{O})_{\tilde{\mathfrak{m}}_Q} \oplus H^*(\tilde{X}_{\tilde{K}_1(Q)}, \mathcal{O})_{\tilde{\mathfrak{m}}_Q}) \rightarrow \mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I_0$$

for some ideal I_0 with square 0. We can use this to construct a $2n$ -dimensional group determinant D_Q^0 valued in $\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I$, such that:

1. For $v \notin S$, we have $D_Q^0(X - \mathrm{Frob}_v) = P_v(X)q_v^{n(2n-1)}P_{v^c}^\vee(q_v^{1-2n}X)$.
2. For $v \in Q$, we have

$$\mathrm{Tr}_{D_Q^0} \left(S_Q^f \left(\sigma \mathrm{Res}_{q_v, \mu_v}^{(2n)!} \mathrm{Res}_{\mu_v}^{(2n)!} \left(\sum_{i=1}^{k-1} E_{\mu_v, i}(\varphi_v) + \langle \mathrm{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\varphi_v) - \mathrm{Res}_{\mu_v} \tau \right) \right) \right) = 0,$$

and I has nilpotence degree depending only on n and $[F : \mathbf{Q}]$. By [ACC⁺18, Theorem 2.3.7], there also exists an n -dimensional group determinant D_Q^1 of $G_{F, S \cup Q}$ valued in $\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I$, such that $D_Q^1(X - \mathrm{Frob}_v) = P_v(X)$ for $v \notin S$. Then the group determinants $D_Q^1 \oplus D_Q^{1, \perp}$ and D_Q^0 are equal. Moreover, since $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible, there exists a continuous representation

$$\rho_{\mathfrak{m}_Q} : G_{F, S \cup Q} \rightarrow \mathrm{GL}_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q)}, \mathcal{O})_{\mathfrak{m}_Q})/I),$$

such that the characteristic polynomial of $\rho_{\mathfrak{m}_Q}$ is associated to D_Q^1 . Let $\rho'_{\mathfrak{m}_Q} := \rho_{\mathfrak{m}_Q} \oplus \rho_{\mathfrak{m}_Q}^\perp$. Writing out the relation at places $v \in Q$, we get

$$\begin{aligned} & \mathrm{Tr}(\rho'_{\mathfrak{m}_Q}(\sigma) S_Q^f(\mathrm{Res}_{q_v, \mu_v}^{(2n)!} \mathrm{Res}_{\mu_v}^{(2n)!} (\sum_{i=1}^{k-1} E_{\mu_v, i}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) \\ & + \langle \mathrm{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) - \mathrm{Res}_{\mu_v} \rho'_{\mathfrak{m}_Q}(\tau))) = 0. \end{aligned}$$

Since $\text{Res}_{\mu_v} \notin \widetilde{\mathfrak{m}}_Q$, we know that $\bar{\rho}_{\mathfrak{m}}$ and $\bar{\rho}_{\mathfrak{m}}^{-1}$ are not isomorphic. Applying Nakayama's lemma and Lemma 7.4, we see that the extended map

$$\mathbf{T}_Q^S[G_{F,S \cup Q}] \rightarrow M_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q), \mathcal{O}})_{\mathfrak{m}_Q})/I) \oplus M_n(\mathbf{T}_Q^S(H^*(X_{K_1(Q), \mathcal{O}})_{\mathfrak{m}_Q})/I)$$

given by $\rho_{\mathfrak{m}_Q} \oplus \rho_{\mathfrak{m}_Q}^{-1}$ is surjective. Considering the trace relation above with σ replaced by an arbitrary element of $\mathbf{T}_Q^S[G_{F,S \cup Q}]$, we conclude that

$$\begin{aligned} S_Q^f(\text{Res}_{q_v, \mu_v}^{(2n)!} \text{Res}_{\mu_v}^{(2n)!} (\sum_{i=1}^{k-1} E_{\mu_v, i}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) \\ + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) - \text{Res}_{\mu_v} \rho'_{\mathfrak{m}_Q}(\tau))) = 0. \end{aligned}$$

Since $q_v \equiv 1 \pmod{p}$, we know that $\text{Res}_{q_v, \mu_v} \notin \widetilde{\mathfrak{m}}_Q$. Thus

$$S_Q^f\left(\sum_{i=1}^{k-1} E_{\mu_v, i}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle E_{\mu_v, k}(\rho'_{\mathfrak{m}_Q}(\varphi_v)) - \text{Res}_{\mu_v} \rho'_{\mathfrak{m}_Q}(\tau)\right) = 0.$$

This implies that

$$\rho_{\mathfrak{m}_Q}(\tau) = S_Q^f\left(\sum_{i=1}^{k-1} \text{Res}_{\mu_v}^{-1} E_{\mu_v, i}(\rho_{\mathfrak{m}_Q}(\varphi_v))\right) + S_Q^f(\langle \text{Art}_{F_v}^{-1}(\tau) \rangle \text{Res}_{\mu_v}^{-1} E_{\mu_v, k}(\rho_{\mathfrak{m}_Q}(\varphi_v))).$$

Using Proposition 5.2, we can transform the equation above into

$$\rho_{\mathfrak{m}_Q}(\tau) = \text{Res}_{\nu_v}^{-1} E_{\nu_v, 1}(\rho_{\mathfrak{m}_Q}(\varphi_v)) + \langle \text{Art}_{F_v}^{-1}(\tau) \rangle \text{Res}_{\nu_v}^{-1} E_{\nu_v, 2}(\rho_{\mathfrak{m}_Q}(\varphi_v)).$$

Let $\mathbf{T} := \mathbf{T}_Q^S(H^*(X_{K_1(Q), \mathcal{O}})_{\mathfrak{m}_Q})/I$. Consider the decomposition $\bar{\rho}_{\mathfrak{m}} = \bar{r}_1 \oplus \bar{r}_2$, corresponding to the Frobenius generalised eigenspaces of all eigenvalues not equal to α_v and α_v , respectively. Then

$$\mathbf{T}^n = \text{Res}_{\nu_v}^{-1} E_{\nu_v, 1}(\rho_{\mathfrak{m}_Q}(\varphi_v)) \mathbf{T}^n \oplus \text{Res}_{\nu_v}^{-1} E_{\nu_v, 2}(\rho_{\mathfrak{m}_Q}(\varphi_v)) \mathbf{T}^n$$

is the unique $\rho_{\mathfrak{m}_Q}(\varphi_v)$ -invariant lift of $\bar{r}_1 \oplus \bar{r}_2$, and we are done by Lemma 6.1. $\square$

7.2. Hecke algebras for G

Let $\lambda \in (\mathbf{Z}_+^n)^{\text{Hom}(F, E)}$. Further let S be a finite set of finite places of F containing the p -adic places and stable under complex conjugation satisfying the following condition:

1. Let l be a rational prime, such that there exists a place above l in S or l is ramified in F . Then there exists an imaginary quadratic subfield $F_0 \subset F$, such that l splits in F_0 .

Let $K \subset \text{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, such that for all $v \notin S$, we have $K_v = \text{GL}_n(\mathcal{O}_{F_v})$. Let $\mathfrak{m} \subset \mathbf{T}^S(K, \lambda)$ be a non-Eisenstein maximal ideal with residue field k . By [ACC⁺18, Theorem 2.3.5], there exists an associated residual representation $\bar{\rho}_{\mathfrak{m}} : G_{F, S} \rightarrow \text{GL}_n(\mathbf{T}^S(K, \lambda)/\mathfrak{m})$. By [ACC⁺18, Theorem 2.3.7], there exists an ideal $I \subset \mathbf{T}^S(K, \lambda)$ of nilpotence degree depending only on n and $[F : \mathbf{Q}]$ and a continuous lift $\rho_{\mathfrak{m}} : G_{F, S} \rightarrow \text{GL}_n(\mathbf{T}^S(K, \lambda)_{\mathfrak{m}}/I)$, such that for each $v \in S$, $\det(X - \rho_{\mathfrak{m}}(\text{Frob}_v))$ is the image of $P_v(X)$ in $\mathbf{T}^S(K, \lambda)_{\mathfrak{m}}/I[X]$. We consider the following Taylor-Wiles datum: a tuple $(Q, (\alpha_v)_{v \in Q})$ consisting of

- A finite set Q of places of F , disjoint from Q^c , such that $q_v \equiv 1 \pmod{p}$ for each $v \in Q$.
- Each $v \in Q$ is split in F^+ , and there exists an imaginary quadratic subfield $F_0 \subset F$, such that v is split in F_0 . Moreover, $\bar{\rho}_{\mathfrak{m}}$ is unramified at v and v^c .
- α_v is a root of $\det(X - \bar{\rho}_{\mathfrak{m}}(\text{Frob}_v))$.

Consider the partition $\nu_v : n = d_v + (n - d_v)$, where d_v is the multiplicity of α_v as a root of $\det(X - \bar{\rho}_m(\text{Frob}_v))$.

We define auxiliary level subgroups $K_1(Q) \subset K_0(Q) \subset K$. They are good subgroups of $\text{GL}_n(\mathbf{A}_F^\infty)$ defined by the following conditions:

- if $v \notin Q$, then $K_1(Q)_v = K_0(Q)_v = K_v$.
- if $v \in Q$, then $K_0(Q)_v = \mathfrak{p}_{\nu_v}$ and $K_1(Q)_v = \mathfrak{p}_{\nu_v, 1}$.

We have a natural isomorphism $K_0(Q)/K_1(Q) \cong \Delta_Q = \prod_{v \in Q} \Delta_v$. Let $S' = S \cup Q \cup Q^c$. We define $\mathbf{T}_Q^{S'} = \mathbf{T}^{S \cup Q} \otimes_{\mathbf{Z}} \mathbf{Z}[\Xi_{v,1}]^{S_{\nu_v}}$. Let $\mathbf{T}_Q^{S'}(K_0(Q), \lambda)$ and $\mathbf{T}_Q^{S'}(K_0(Q)/K_1(Q), \lambda)$ be the images of $\mathbf{T}_Q^{S'}$ in $\text{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_{K_0(Q)}, V_\lambda))$ and $\text{End}_{\mathbf{D}(\mathcal{O}[\Delta_Q])}(R\Gamma(X_{K_1(Q)}, V_\lambda))$, respectively. Let $\mathfrak{m}_Q$ be the maximal ideal of $\mathbf{T}_Q^{S'}$ generated by $\mathfrak{m}$ and the kernels of the homomorphisms $\mathbf{Z}[\Xi_{v,1}]^{S_{\nu_v}} \rightarrow k$ given by the coefficients of polynomials $(X - \alpha_v)^{d_v}$, $\det(X - \bar{\rho}_m(\text{Frob}_v))/(X - \alpha_v)^{d_v}$.

Theorem 7.6. *We have natural isomorphisms*

$$R\Gamma(X_K, V_\lambda)_{\mathfrak{m}} \simeq R\Gamma(X_{K_0(Q)}, V_\lambda)_{\mathfrak{m}_Q}$$

$$R\Gamma(X_{K_0(Q)}, V_\lambda)_{\mathfrak{m}_Q} \simeq R\Gamma(\Delta_Q, R\Gamma(X_{K_1(Q)}, V_\lambda))_{\mathfrak{m}_Q}$$

in $\mathbf{D}(\mathcal{O})$.

Proof. The second isomorphism is straightforward. For the first, we can check on the level of cohomology. It is enough to check that it is an isomorphism in $\mathbf{D}(k)$ after applying the functor $- \otimes^{\mathbf{L}} k$. Thus, we need to show that the map

$$H^*(X_K, V_\lambda/\varpi)_{\mathfrak{m}} \rightarrow H^*(X_{K_0(Q)}, V_\lambda/\varpi)_{\mathfrak{m}_Q}$$

is an isomorphism. We can do this one prime at a time, so we can assume $Q = \{v\}$. For each j , let

$$M_j := \lim_{m \rightarrow \infty} H^j(X_{K(v^m)}, V_\lambda/\varpi)_{\mathfrak{m}},$$

where $K(v^m)_w = K_w$ for places $w \neq v$ and $K(v^m)_v$ is the principal congruence subgroup of level v^m . We have two Hochschild-Serre spectral sequences:

$$H^i(\text{GL}_n(\mathcal{O}_{F_v}), M_j) \Rightarrow H^{i+j}(X_K, V_\lambda/\varpi)_{\mathfrak{m}}$$

$$e_{\alpha_v} H^i(\mathfrak{p}_{\nu_v}, M_j) \Rightarrow e_{\alpha_v} H^{i+j}(X_{K_0(Q)}, V_\lambda/\varpi) = H^{i+j}(X_{K_0(Q)}, V_\lambda/\varpi)_{\mathfrak{m}_Q}.$$

There is a natural map ι^* between these spectral sequences, which arises from deriving the map

$$M_j^{\text{GL}_n(\mathcal{O}_{F_v})} \rightarrow M_j^{\mathfrak{p}_{\nu_v}} \rightarrow e_{\alpha_v} M_j^{\mathfrak{p}_{\nu_v}}.$$

Thus, it is enough to show that ι^* is an isomorphism. M_j is admissible, and we can use [Vig98, Theorem III.6] to write M_j as a direct sum of $\text{GL}_n(F_v)$ -modules, each belonging to a single block. Let $N \subset M_j$ be a summand from a nonunipotent block. Let $T_p(k)$ be the p -power part of $T(k)$. We note that both $H^i(\text{GL}_n(\mathcal{O}_{F_v}), N)$ and $H^i(\mathfrak{p}_{\nu_v}, N)$ inject into $H^i(\text{Iw}, N)$, which in turn is equal to $H^i(T_p(k), N^{\text{Iw}^p})$. Since N is from a nonunipotent block, we know that $N^{\text{Iw}^p} = 0$, and so

$$H^i(\text{GL}_n(\mathcal{O}_{F_v}), N) = H^i(\mathfrak{p}_{\nu_v}, N) = 0.$$

Thus, we can restrict to the summand $M_j^1 \subset M_j$ from the unipotent block, and it is enough to prove that

$$\iota^* : H^i(\text{GL}_n(\mathcal{O}_{F_v}), M_j^1) \rightarrow e_{\alpha_v} H^i(\mathfrak{p}_{\nu_v}, M_j^1)$$

is an isomorphism. By [CHT08, Theorem B.1], the unipotent block in our case consists of representations generated by their Iw^P -invariant vectors. Therefore, every irreducible subrepresentation $\pi \subset M_j^1$ has a Iw^P -invariant vector. It follows from the argument similar to the proof of Proposition 2.1 that

$$\pi \subset \text{Ind}_B^{GL_n} \chi_1 \otimes \dots \otimes \chi_n,$$

where χ_i are tamely ramified characters whose restriction to $\mathcal{O}_{F_v}/(1 + \varpi \mathcal{O}_{F_v})$ has p -power order. But these characters are valued in $k^\times$ which has order coprime to p , which means χ_i are in fact unramified.

We can now select the smallest number $d > 0$, such that π embeds into $M_j[\mathfrak{m}^d]$. Since π is irreducible, it must then embed into $M_j[\mathfrak{m}^d]/M_j[\mathfrak{m}^{d-1}]$ and local-global compatibility for Iwahori level ([ACC⁺18, Theorem 3.1.1]) then implies that $\{\chi_i(\varpi)\}_{i=1, \dots, n}$ is the set of eigenvalues of $\bar{\rho}_{\mathfrak{m}}(\text{Frob}_v)$. Thus, we have shown that $M_j \in \mathcal{C}$, and we are done by Theorem 2.14. $\square$

Theorem 7.7. *There exists an ideal $I \subset \mathbf{T}'_Q(K_0(Q)/K_1(Q), \lambda)_{\mathfrak{m}_Q}$ of nilpotence degree depending only on n and $[F : \mathbf{Q}]$, together with a continuous homomorphism*

$$\rho_{\mathfrak{m}, Q} : G_{F, S \cup Q} \rightarrow \text{GL}_n(\mathbf{T}'_Q(K_0(Q)/K_1(Q), \lambda)_{\mathfrak{m}_Q}/I)$$

lifting $\bar{\rho}_{\mathfrak{m}}$ and satisfying the following conditions:

1. For a finite place $v \notin S \cup Q$ of F , $\det(X - \rho_{\mathfrak{m}, Q}(\text{Frob}_v))$ equals to the image of $P_v(X)$ in $\mathbf{T}'_Q(K_0(Q)/K_1(Q), \lambda)_{\mathfrak{m}_Q}/I[X]$.
2. For $v \in Q$, $\rho_{\mathfrak{m}, Q}|_{G_{F_v^c}}$ is unramified and $\rho_{\mathfrak{m}, Q}|_{G_{F_v}}$ is a lifting of type $\mathcal{D}_v$, and the induced map $\mathcal{O}[\Delta_Q] \rightarrow \mathbf{T}'_Q(K_0(Q)/K_1(Q), \lambda)_{\mathfrak{m}_Q}/I$ is a homomorphism of $\mathcal{O}[\Delta_Q]$ -algebras.

Proof. We first make a few reductions. Let us show that we can reduce to the situation where $\det(X - \bar{\rho}_{\mathfrak{m}}(\text{Frob}_v))$ and $\det(X - \bar{\rho}_{\mathfrak{m}}(\text{Frob}_{v^c}))$ are coprime for each $v \in Q$. To achieve this, we will use twisting. Pick an odd prime $l \neq p$ and consider a character $\psi : G_F \rightarrow \mathcal{O}^\times$ of order ℓ , such that $\det(X - (\bar{\rho}_{\mathfrak{m}} \otimes \bar{\psi})(\text{Frob}_v))$ and $\det(X - (\bar{\rho}_{\mathfrak{m}} \otimes \bar{\psi})(\text{Frob}_{v^c}))$ are coprime. Let S_ψ denote the places of F at which ψ is ramified. We will further require that S_ψ is disjoint from S' . Define a good subgroup $K^\psi \subset K$ given by $K_v^\psi = K_v$ at places v at which ψ is unramified and $K_v^\psi = \ker(\text{GL}_n(\mathcal{O}_{F_v}) \rightarrow k(v)^\times/(k(v)^\times)^l)$ at places v , where ψ is ramified. Following the discussion above [ACC⁺18, Proposition 2.2.22], we have a homomorphism $f_\psi : \mathbf{T}^{S' \cup S_\psi}(K^\psi, \lambda) \rightarrow \mathbf{T}^{S' \cup S_\psi}(K^\psi, \lambda)$ given by

$$f_\psi([K^\psi{}^{S' \cup S_\psi} g K^\psi{}^{S' \cup S_\psi}]) = \psi^{-1}(\text{Art}(\det(g)))[K^\psi{}^{S' \cup S_\psi} g K^\psi{}^{S' \cup S_\psi}]. \quad (7.8)$$

We have a maximal ideal $\mathfrak{m}_\psi = f_\psi(\mathfrak{m})$ of $\mathbf{T}^{S' \cup S_\psi}(K^\psi, \lambda)$. [ACC⁺18, Proposition 2.2.22] implies an isomorphism $\bar{\rho}_{\mathfrak{m}} \otimes \bar{\psi} \cong \bar{\rho}_{\mathfrak{m}_\psi}$. Similarly to Eq. 7.8, we have an isomorphism

$$\mathbf{T}_Q^{S' \cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q), \lambda)_{\mathfrak{m}_\psi Q} \cong \mathbf{T}_Q^{S' \cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q), \lambda)_{\mathfrak{m}_Q},$$

where $\mathfrak{m}_\psi Q$ is the maximal ideal of $\mathbf{T}_Q^{S' \cup S_\psi}$ generated by $\mathfrak{m}_\psi$ and the kernels of the homomorphisms $\mathbf{Z}[\Xi_{v,1}]^{S_{v^c}} \rightarrow k$ given by the coefficients of polynomials $(X - \psi(\text{Frob}_v)\alpha_v)^{d_v}$, $\det(X - \bar{\rho}_{\mathfrak{m}_\psi}(\text{Frob}_v))/(X - \psi(\text{Frob}_v)\alpha_v)^{d_v}$. We have a surjective map of $\mathbf{T}^{S' \cup S_\psi}$ -algebras

$$\mathbf{T}_Q^{S' \cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q), \lambda)_{\mathfrak{m}_Q} \rightarrow \mathbf{T}_Q^{S' \cup S_\psi}(K_0(Q)/K_1(Q), \lambda)_{\mathfrak{m}_Q}.$$

Thus, if the theorem holds for representations into $\mathbf{T}_Q^{S' \cup S_\psi}(K_0^\psi(Q)/K_1^\psi(Q), \lambda)_{\mathfrak{m}_Q}$, it will hold for representations into $\mathbf{T}_Q^{S' \cup S_\psi}(K_0(Q)/K_1(Q), \lambda)_{\mathfrak{m}_Q}$. Since there are infinitely many ψ satisfying the conditions we require, we can vary them to conclude that the theorem holds for $\mathbf{T}'_Q(K_0(Q)/K_1(Q), \lambda)_{\mathfrak{m}_Q}$, which is our target Hecke algebra.

Let $\tilde{K} \subset \tilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup satisfying the following conditions:

1. $\tilde{K}$ is decomposed with respect to P .
2. $\tilde{K} \cap G(\mathbf{A}_{F^+}^\infty) \subset K$.
3. if $\bar{v}$ is a finite place of F^+ , such that $\bar{v} \notin \bar{S}$, then $\tilde{K}_{\bar{v}} = \tilde{G}(\mathcal{O}_{F^+}^{\bar{v}})$.

We can use the Hochschild-Serre spectral sequence to reduce to the case where $K = \tilde{K} \cap G(\mathbf{A}_{F^+}^\infty)$. We can further reduce our theorem to the case $\lambda = 0$, by a standard use of the Hochschild-Serre spectral sequence to trivialise the weight modulo some power m at the expense of shrinking the level at p . Now the theorem follows from Theorem 7.5. $\square$

8. Proof of Theorem 1.2 and Theorem 1.3

Let us recall the proof structure of [ACC⁺18, Theorem 6.1.1]. The theorem is reduced in [ACC⁺18] to [ACC⁺18, Corollary 6.5.5], which is proved using [ACC⁺18, Theorem 6.5.4]. The reduction does not use the ‘enormous’ assumption on the image of $\bar{\rho}$. Thus, it will be sufficient for us to prove an analog of [ACC⁺18, Theorem 6.5.4], replacing ‘enormous’ by ‘adequate’ in the hypotheses.

Let F be an imaginary CM number field, and fix the following data:

1. An integer $n \geq 2$ and a prime $p > n^2$.
2. A finite set S of finite places of F , including the places above p .
3. A (possibly empty) subset $R \subset S$ of places which are prime to p .
4. A cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, which is regular algebraic of some weight λ .
5. A choice of isomorphism $\iota : \bar{\mathbf{Q}}_p \cong \mathbf{C}$.

We assume that the following conditions are satisfied:

6. If l is a prime lying below an element of S , or which is ramified in F , then F contains an imaginary quadratic field in which l splits. In particular, each place of S is split over F^+ and the extension F/F^+ is everywhere unramified.
7. The prime p is unramified in F .
8. For each embedding $\tau : F \hookrightarrow \mathbf{C}$, we have

$$\lambda_{\tau,1} + \lambda_{\tau\mathbf{C},1} - \lambda_{\tau,n} - \lambda_{\tau\mathbf{C},n} < p - 2n.$$

9. For each $v \in S_p$, let $\bar{v}$ denote the place of F^+ lying below v . Then there exists a place $\bar{v}' \neq \bar{v}$ of F^+ , such that $\bar{v}' \mid p$ and

$$\sum_{\bar{v}'' \neq \bar{v}, \bar{v}'} [F_{\bar{v}''}^+ : \mathbf{Q}_p] > \frac{1}{2} [F^+ : \mathbf{Q}].$$

10. The residual representation $\overline{r_\iota(\pi)}$ is absolutely irreducible.
11. If v is a place of F lying above p , then π_v is unramified.
12. If $v \in R$, then $\pi_v^{\mathrm{Iw}_v} \neq 0$.
13. If $v \in S - (R \cup S_p)$, then π_v is unramified and $H^2(F_v, \mathrm{ad} \overline{r_\iota(\pi)}) = 0$.
Moreover, v is absolutely unramified and of residue characteristic $q > 2$.
14. $S - (R \cup S_p)$ contains at least two places with distinct residue characteristics.
15. If $v \notin S$ is a finite place of F , then π_v is unramified.
16. If $v \in R$, then $q_v \equiv 1 \pmod{p}$ and $\overline{r_\iota(\pi)}|_{G_{F_v}}$ is trivial.
17. The representation $\overline{r_\iota(\pi)}$ is decomposed generic in the sense of [ACC⁺18, Definition 4.3.1] and the image of $\overline{r_\iota(\pi)}|_{G_{F(\zeta_p)}}$ is adequate.

We define an open compact subgroup $K = \prod_v K_v$ of $\mathrm{GL}_n(\widehat{\mathcal{O}}_F)$ as follows:

- If $v \notin S$, or $v \in S_p$, then $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.
- If $v \in R$, then $K_v = \mathrm{Iw}_v$.
- If $v \in S - (R \cup S_p)$, then $K_v = \mathrm{Iw}_{v,1}$.

By [ACC⁺18, Theorem 2.4.10], we can find a coefficient field $E \subset \overline{\mathbf{Q}}_p$ and a maximal ideal $\mathfrak{m} \subset \mathbf{T}^S(K, \mathcal{V}_\lambda)$, such that $\overline{\rho}_\mathfrak{m} \cong \overline{r}_l(\pi)$. After possibly enlarging E , we can and do assume that the residue field of $\mathfrak{m}$ is equal to k . For each tuple $(\chi_{v,i})_{v \in R, i=1, \dots, n}$ of characters $\chi_{v,i} : k(v)^\times \rightarrow \mathcal{O}^\times$ which are trivial modulo ϖ , we define a global deformation problem by the formula

$$S_\chi = (\overline{\rho}_\mathfrak{m}, S, \{\mathcal{O}\}_{v \in S}, \{\mathcal{D}_v^{\mathrm{FL}}\}_{v \in S_p} \cup \{\mathcal{D}_v^\chi\}_{v \in R}) \cup \{\mathcal{D}_v^\square\}_{v \in S - (R \cup S_p)}.$$

We fix representatives ρ_{S_χ} of the universal deformations which are identified modulo ϖ via the identifications $R_{S_\chi}/\varpi \cong R_{S_1}/\varpi$. We define an $\mathcal{O}[K_S]$ -module $\mathcal{V}_\lambda(\chi^{-1}) = \mathcal{V}_\lambda \otimes_{\mathcal{O}} \mathcal{O}(\chi^{-1})$, where K_S acts on $\mathcal{V}_\lambda$ by projection to K_p and on $\mathcal{O}(\chi^{-1})$ by the projection $K_S \rightarrow K_R = \prod_{v \in R} \mathrm{Iw}_v \rightarrow \prod_{v \in R} (k(v)^\times)^n$.

Theorem 8.1. *Under assumptions (1)–(17) above, $H^*(X_K, \mathcal{V}_\lambda(1))_\mathfrak{m}$ is a nearly faithful R_{S_1} -module. In other words, $\mathrm{Ann}_{R_{S_1}}(H^*(X_K, \mathcal{V}_\lambda(1))_\mathfrak{m})$ is nilpotent.*

The rest of the paper is devoted to the proof of Theorem 8.1.

Consider the Taylor-Wiles datum $(Q, \{\alpha_v\}_{v \in Q})$ satisfying the following conditions:

- For each place $v \in Q$ of residue characteristic l , there exists an imaginary quadratic subfield $F_0 \subset F$, such that l splits in F_0 .
- Q and Q^c are disjoint.

We have the following result, combining [ACC⁺18, Proposition 6.5.3] and Theorem 7.7:

Proposition 8.2. *There exists an integer $\delta \geq 1$ depending only on n and $[F : \mathbf{Q}]$, an ideal $J \subset \mathbf{T}'_Q(R\Gamma(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}_Q})$, such that $J^\delta = 0$ and a continuous surjection of $\mathcal{O}[\Delta_Q]$ -algebras $f_{S_\chi, Q} : R_{\chi, Q} \rightarrow \mathbf{T}'_Q(R\Gamma(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}_Q})/J$, such that for each finite place $v \notin S \cup Q$, the characteristic polynomial of $f_{S_\chi, Q} \circ \rho_{S_\chi, Q}$ equals the image of $P_v(X)$.*

Let

$$q = h^1(F_S/F, \mathrm{ad} \overline{\rho}_\mathfrak{m}(1)) \quad \text{and} \quad g = q - n^2[F^+ : \mathbf{Q}],$$

and set $\Delta_\infty = \mathbf{Z}_p^g$. Let $\mathcal{T}$ be a power series ring over $\mathcal{O}$ in $n^2|S| - 1$ variables, and let $S_\infty = \mathcal{T}[[\Delta_\infty]]$. Let $\mathfrak{a}_\infty$ be the augmentation ideal of S_∞ viewed as an augmented $\mathcal{O}$ -algebra. Since $p > n$, for each $v \in R$, we can choose a tuple of pairwise distinct characters $\chi_v = (\chi_{v,1}, \dots, \chi_{v,n})$, with $\chi_{v,i} : \mathcal{O}_{F_v}^\times \rightarrow \mathcal{O}^\times$ trivial modulo ϖ . We write χ for the tuple $(\chi_v)_{v \in R}$ as well as for the induced character $\prod_{v \in R} I_v \rightarrow \mathcal{O}^\times$. Fix an imaginary quadratic subfield $F_0 \subset F$. Then for each $N \geq 1$, we fix a choice of Taylor-Wiles datum $(Q, \{\alpha_v\}_{v \in Q})$ for S_1 of level N using Proposition 6.7. For $N = 0$, we set $Q_0 = \emptyset$. For each $N \geq 1$, we set $\Delta_N = \Delta_{Q_N}$ and fix a surjection $\Delta_\infty \rightarrow \Delta_N$. We let Δ_0 be the trivial group, viewed as a quotient of Δ_∞ . For each $N \geq 0$, we set $R_N = R_{S_1, Q_N}$ and $R'_N = R_{S_\chi, Q_N}$. Let $R^{\mathrm{loc}} = R_{S_1}^{S, \mathrm{loc}}$ and $R'^{\mathrm{loc}} = R_{S_\chi}^{S, \mathrm{loc}}$ denote the local deformation rings. We let R_∞ and R'_∞ be formal power series rings in g variables over R^{loc} and R'^{loc} , respectively. We also have canonical isomorphisms $R_N/\varpi \cong R'_N/\varpi$ and $R^{\mathrm{loc}}/\varpi \cong R'^{\mathrm{loc}}/\varpi$. Using [ACC⁺18, Proposition 6.2.24] and [ACC⁺18, Proposition 6.2.31], we have local $\mathcal{O}$ -algebra surjections $R_\infty \rightarrow R_N$ and $R'_\infty \rightarrow R'_N$ for $N \geq 0$. We can and do assume that these are compatible with the fixed identifications modulo ϖ and with the isomorphisms $R_N \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R_0$ and $R'_N \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R'_0$.

Define $\mathcal{C}_0 = R \mathrm{Hom}_{\mathcal{O}}(R\Gamma(X_K, \mathcal{V}_\lambda(1))_\mathfrak{m}, \mathcal{O})[-d] \in \mathbf{D}(\mathcal{O})$ and $T_0 = \mathbf{T}^S(\mathcal{C}_0)$. Similarly, we define $\mathcal{C}'_0 = R \mathrm{Hom}_{\mathcal{O}}(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_\mathfrak{m}, \mathcal{O})[-d]$ and $T'_0 = \mathbf{T}^S(\mathcal{C}'_0)$. For any $N \geq 1$, we let

$$\mathcal{C}_N = R \mathrm{Hom}_{\mathcal{O}}(R\Gamma(X_{K_1(Q)}, \mathcal{V}_\lambda(1))_{\mathfrak{m}_{Q_N}}, \mathcal{O})[-d],$$

and

$$T_N = \mathbf{T}_Q^{S'}(C_N).$$

Similarly, we let

$$C'_N = R \operatorname{Hom}_{\mathcal{O}}(R\Gamma(X_{K_1}(\mathcal{Q}), V_\lambda(\chi^{-1}))_{\mathfrak{m}_{\mathcal{Q}_N}}, \mathcal{O})[-d]$$

and

$$T'_N = \mathbf{T}_Q^{S'}(C'_N).$$

For any $N \geq 0$, there are canonical isomorphisms

$$C_N \otimes_{\mathcal{O}[\Delta_N]}^{\mathbf{L}} k[\Delta_N] \cong C'_N \otimes_{\mathcal{O}[\Delta_N]}^{\mathbf{L}} k[\Delta_N]$$

in $\mathbf{D}(k[\Delta_N])$. These yield the identification

$$\operatorname{End}_{\mathbf{D}(\mathcal{O})}(C_N \otimes_{\mathcal{O}}^{\mathbf{L}} k) \cong \operatorname{End}_{\mathbf{D}(\mathcal{O})}(C'_N \otimes_{\mathcal{O}}^{\mathbf{L}} k).$$

Thus, we can write $\bar{T}_N$ for the image of both T_N and T'_N in the identified endomorphism algebras. By Theorem 7.6, there are canonical isomorphisms $C_N \otimes_{\mathcal{O}[\Delta_N]}^{\mathbf{L}} \mathcal{O} \cong C_0$ and $C'_N \otimes_{\mathcal{O}[\Delta_N]}^{\mathbf{L}} \mathcal{O} \cong C'_0$ in $\mathbf{D}(\mathcal{O})$, which are compatible with the reductions modulo ϖ . By Proposition 8.2, we can find an integer $\delta \geq 1$ and for each $N \geq 0$ ideals I_N of T_N and I'_N of T'_N of nilpotence degree $\leq \delta$, such that there exist local $\mathcal{O}[\Delta_N]$ -algebra surjections $R_N \rightarrow T_N/I_N$ and $R'_N \rightarrow T'_N/I'_N$. Denoting by $\bar{I}_N$ and $\bar{I}'_N$ the images of I_N and I'_N , respectively, in $\bar{T}_N$, we get maps $R_N/\varpi \rightarrow \bar{T}_N/(\bar{I}_N + \bar{I}'_N)$ and $R'_N/\varpi \rightarrow \bar{T}_N/(\bar{I}_N + \bar{I}'_N)$ which are compatible with the identification $R_N/\varpi \cong R'_N/\varpi$. The objects constructed above satisfy the setup described in [ACC⁺18, Section 6.4.1]. Thus, we can apply the results of [ACC⁺18, Section 6.4.2] as in the second part of the proof of [ACC⁺18, Theorem 6.4.4] to conclude that $H^*(C_0)$ is a nearly faithful R_{S_1} -module, which implies Theorem 8.1.

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