Every Steiner Triple System Contains an Almost Spanning d-Ary Hypertree

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Abstract

In this paper we make a partial progress on the following conjecture: for every $\mu > 0$ and a large enough n, every Steiner triple system S on at least $(1+\mu)n$ vertices contains every hypertree T on n vertices. We prove that the conjecture holds if T is a perfect d-ary hypertree.

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1 Introduction

In this paper we study the following conjecture, raised by the second author and Bradley Elliot [4].

Conjecture 1. Given $\mu > 0$ there is n_0, such that for any $n \geqslant n_0$, any hypertree T on n vertices and any Steiner triple system S on at least $(1+\mu)n$ vertices, S contains T as a subhypergraph.

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Note that any hypertree T can be embedded into any Steiner triple system S, provided $|V(T)| \leqslant \frac{1}{2}(|V(S)|+3)$. The problem becomes more interesting if the size of the tree is larger. In [3] (see also [4] Section 5) Conjecture 1 was verified for some special classes of hypertrees. In this paper we verify the conjecture for another class of hypertrees – perfect d-ary hypertrees.

Definition 2. A perfect d-ary hypertree T of height h is a hypertree T with $V(T) = \bigcup_{i=0}^{h} V_i$, where $|V_i| = (2d)^i$ for all $i \in [0,h]$, such that for every $i \in [0, h-1]$ and $v \in V_i$ there are $2d$ vertices $\{u_1, \ldots, u_{2d}\} \subseteq V_{i+1}$ such that $\{v, u_{2j-1}, u_{2j}\}$ is a hyperedge of T for all $j \in [d]$.

In other words, T is a perfect d-ary hypertree if every non-leaf vertex has $2d$ children (or a forward degree d). The main result of this paper is the following theorem.

Theorem 3. *For any real $\mu > 0$ there is n_0 such that the following holds for all $n \geqslant n_0$ and any positive integer d. If S is a Steiner triple system with at least $(1+\mu)n$ vertices and T is a perfect d-ary hypertree on at most n vertices, then $T \subseteq S$.*

2 Preliminaries

For a positive integer k let $[k] = \{1, \ldots, k\}$ and for positive integers $k < \ell$ let $[k, \ell] = \{k, k+1, \ldots, \ell\}$. We write $x = y \pm z$ if $x \in [y-z, y+z]$. We write $A = B \sqcup C$ if A is a union of disjoint sets B and C.

A hypertree is a connected, simple (linear) 3uniform hypergraph in which every two vertices are joined by a unique path. A hyperstar S of size a centered at v is a hypertree on the vertex set $v, v_1, v_2, \ldots, v_{2a}$ with the edge set $E(S) = \{\{v, v_{2i-1}, v_{2i}\} : i \in [a]\}$. A Steiner triple system (STS) is a 3uniform hypergraph in which every pair of vertices is contained in exactly one edge.

If H is a hypergraph and $v \in V(H)$, then $d_H(v)$ (or $d(v)$ when the context is clear) is the degree of a vertex v in H.

For $V(H) = X \sqcup Y$ we denote by $H[X,Y]$ the spanning subhypergraph of H with

$$E(H[X,Y]) = \{e \in E(H) : |e \cap X| = 1, |e \cap Y| = 2\}.$$

The proof of Theorem 3 relies on the application of an existence of an almost perfect matching in an almost regular 3-uniform simple hypergraph. We will use two versions of such results. In the first version the degrees of a small proportion of vertices are allowed to deviate from the average degree. We will use Theorem 4.7.1 from [2] (see [5] and [8] for earlier versions).

Theorem 4. *For any $\delta > 0$ and $k > 0$ there exists ε and D_0 such that the following holds. Let H be a 3-uniform simple hypergraph on N vertices and $D \geqslant D_0$ be such that*

(i) for all but at most εN vertices x of H the degree of x

$$d(x) = (1 \pm \varepsilon)D.$$

(ii) for all $x \in V(H)$ we have

$$d(x) \leqslant kD.$$

Then H contains a matching on at least $N(1-\delta)$ vertices.

A second version is a result by Alon, Kim and Spencer [1], where under the assumption that all degrees are concentrated near the average, a stronger conclusion may be drawn. We use a version of this result as stated in [7]*.

Theorem 5. *For any $K > 0$ there exists D_0 such that the following holds. Let H be a 3-uniform simple hypergraph on N vertices and $D \geqslant D_0$ be such that $deg(x) = D \pm K\sqrt{D \ln D}$ for all $x \in V(H)$. Then H contains a matching on $N - O(ND^{-1/2} \ln^{3/2} D)$ vertices.*

Here the constant in $O()$-notation is depending on K only and is independent of N and D.

In Lemma 9 we consider a random partition of the vertex set of Steiner triples system S and heavily use the following version of Chernoff's bound (this is Corollary 2.3 of Janson, Łuczak, Rucinski [6]).

Theorem 6. *Let $X \sim Bi(n,p)$ be a binomial random variable with the expectation μ, then for $t \leqslant \frac{3}{2}\mu$*

$$\mathbb{P}(|X-\mu| > t) \leqslant 2e^{-t^2/(3\mu)}.$$

In particular for $K \leqslant \frac{3}{2}\sqrt{\frac{\mu}{\ln \mu}}$ and $t = K\sqrt{\mu \ln \mu}$

$$\mathbb{P}(|X-\mu| > K\sqrt{\mu \ln \mu}) \leqslant 2(\mu)^{-K^2/3}. \tag{1}$$

If $\varepsilon > 0$ is fixed and $\mu > \mu(\varepsilon)$, then

$$\mathbb{P}(X = (1 \pm \varepsilon)\mu) = 1 - e^{-\Omega(\mu)}. \tag{2}$$

3 Proof of Theorem 3

3.1 Proof Idea

Assume that S is an STS on at least $(1+\mu)n$ vertices. We will choose a small constant $\varepsilon \ll \mu$.

Let T be a perfect d-ary tree on at most n vertices with levels V_i, $i \in [0,h]$ and let $i_0 = \max\{i, |V_i| \leqslant \varepsilon n\}$. Let T_0 be a subhypertree of T induced on $\bigcup_{i=0}^{i_0} V_i$. To simplify our notation we set $t = h - i_0$ and for all $i \in [0,t]$ we set $L_i = V_{i_0+i}$. Our goal is to find $L \subset V(S)$ with $|L| = |V(T)|$ such that $S[L]$, the subhypergraph induced by L, contains a spanning copy of T. In particular we would find such an L, level by level, first embedding T_0, and then $L_1, \ldots, L_t$.

To start we consider a partition $\mathcal{P} = \{C_0, \ldots, C_t, R\}$ of $V(S)$ with "random-like" properties (see Lemma 9 for the description of $\mathcal{P}$) in the following way:

*We refer to Theorem 3 from that paper. There is a typo in the conclusion part of that theorem, where instead of $O(ND^{1/2} \ln^{3/2} D)$ there should be $O(ND^{-1/2} \ln^{3/2} D)$

- The first few levels of T, constituting T_0 with the subset of leafs L_0, will be embedded greedily into $S[C_0]$.
- Then Lemma 7 and Lemma 8 will be used to find star forests in $S[L_0, C_1]$ and $S[C_{i-1}, C_i]$ for $i \in [2, t]$. The union of these star forests will establish embedding of almost all vertices of T (see Claim 12).
- Finally, reservoir vertices R will be used to complete the embedding (see Claim 13).

3.2 Auxiliary Lemmas

We start our proof with the following Lemma that will later allow us to verify that $S[L_0, C_1]$ contains an almost perfect packing of hyperstars, each of size at most d, centered at the vertices of L_0

Lemma 7. *For any positive real δ, $k > 1$ there are $\varepsilon > 0$ and D_0 such that the following holds for all $D \geqslant D_0$ and all positive integers d. Let $G = (V, E)$ be a 3-uniform simple hypergraph on N vertices such that $V = X \sqcup Y$ and*

(i) for all $e \in E$, $|e \cap X| = 1$ and $|e \cap Y| = 2$.

(ii) for all vertices $v \in X$ we have

$$d(v) = dD(1 \pm \varepsilon),$$

and for all but at most εN vertices $v \in Y$ we have

$$d(v) = D(1 \pm \varepsilon).$$

(iii) $d(v) \leqslant kD$ for all $v \in Y$.

Then G contains a packing of hyperstars of size at most d centered at vertices of X that covers all but at most δN vertices.

Proof. For given δ and k set $\delta_{2.1} = 2\delta/3$ and $k_{2.1} = k$. With these parameters as an input, Theorem 4 yields $\varepsilon_{2.1}$ and D_0. Set $\varepsilon = \varepsilon_{2.1}/2$ and note that if Theorem 4 holds for some $\varepsilon_{2.1}$ and D_0 then it also holds for smaller values of $\varepsilon_{2.1}$ and larger values of D_0. Therefore we may assume that ε is sufficiently small with respect to δ and k.

Let G that satisfies conditions (i)–(iii) be given. We start with constructing an auxiliary hypergraph H that is obtained from G by repeating the following splitting procedure for each vertex $v \in X$: split the hyperedges incident to v into d disjoint groups, each of size $D(1 \pm 2\varepsilon) = D(1 \pm \varepsilon_{2.1})$, and then replace v with new vertices $v_1, \ldots, v_d$ and each hyperedge $\{v, u, w\}$ that belongs to the group j with a hyperedge $\{v_j, u, w\}$.

First, we show that $|V(H)| \leqslant \frac{3}{2}N$. Note that due to conditions (i)-(iii)

$$|E(G)| \sim |X|dD \sim \frac{1}{2}|Y|D.$$

Provided ε is small enough and N is large enough compared to δ we can guarantee $|X| \leqslant \frac{N}{2d}$. Finally, $|V(H)| \leqslant d|X| + |Y|$ by the construction of H, so

$$N < |V(H)| \leqslant (d-1)|X| + |X| + |Y| \leqslant (d-1)|X| + N \leqslant \frac{3}{2}N.$$

Hypergraph H satisfies the assumptions of Theorem 4 with parameters $\delta_4 = \frac{2}{3}\delta$, $k_{2.1} = k$, $\varepsilon_{2.1} = 2\varepsilon$, $D_{2.1} = D$ and $N_{2.1} = |V(H)|$. Indeed, all of the vertices in H still have degrees at most $k_{2.1}D_{2.1}$, and for all but at most $\varepsilon_{2.1}N_{2.1}$ vertices we have $d_H(v) = D_{2.1}(1 \pm \varepsilon_{2.1})$. Therefore, there is a matching M in H that omits at most $\delta_4 N_{2.1} \leqslant \delta N$ vertices.

Now, the matching M in H corresponds to a collection of hyperstars $S_1, \dots, S_k$ in G with centers at vertices of X, and such that the size of each S_i is at most d. Indeed, recall that during the construction of H some vertices $v \in X$ were replaced by d vertices $v_1, \dots, v_d$, hyperedges incident to v were split into d almost equal in size disjoint groups, and then each hyperedge $\{v, u, w\}$ in j-th group was replaced with $\{v_j, u, w\}$. Consequently, a matching in H that covers some vertices v_i gives a rise to a hyperstar centered at v of size at most d in G.

Moreover since M in H omits at most δN vertices, the union of hyperstars $S_1, \dots, S_k$ also omits at most δN vertices. □

The following Lemma will later allow us to verify that for all $i \in [t-1]$ the subhypergraph $S[C_i, C_{i+1}]$ contains an almost perfect packing of hyperstars, each of size at most d, centered at the vertices of C_i.

Lemma 8. *For any positive real K there is D_0 such that the following holds for all $D \geqslant D_0$, $\Delta = K\sqrt{D \ln D}$ and any positive integer d. Let $G = (V, E)$ be a 3-uniform simple hypergraph on N vertices such that $V = X \sqcup Y$ and*

(i) for all $e \in E$, $|e \cap X| = 1$ and $|e \cap Y| = 2$.

(ii) $d(v) = d(D \pm \Delta)$ for all $v \in X$ and $d(v) = D \pm \Delta$ for all $v \in Y$.

Then G contains a packing of hyperstars of size at most d centered at the vertices of X that covers all but at most $O(ND^{-1/2} \ln^{3/2} D)$ vertices.

Here the constant in $O()$-notation depends on K only.

Proof. The proof is almost identical to the proof of Lemma 7. For a given K let D_0 be the number guaranteed by Theorem 5 with $2K$ as input.

Let G that satisfies conditions (i),(ii) be given. We start with constructing an auxiliary hypergraph H that is obtained from G by splitting every vertex $v \in X$ into d new vertices $v_1, \dots, v_d$ that have degrees $D \pm 2\Delta$.

First, we will show that $|V(H)| = \Theta(N)$. Note that due to conditions (i) and (ii), we have

$$\frac{|Y|(D \pm \Delta)}{2} = |E(G)| = |X|d\left(D \pm \frac{\Delta}{2}\right).$$

In particular,

$$|Y| = |X|d\left(\frac{2D \pm \Delta}{D \pm \Delta}\right).$$

As $|X| + |Y| = N$ we have that $|Y| = \Theta(N)$ and hence $d|X| = \Theta(N)$. Then $|V(H)| = d|X| + |Y|$ by construction of H, so $|V(H)| = \Theta(N)$ as well.

Hypergraph H satisfies the assumptions of the Theorem 5 with parameters $2K$ and $D \geqslant D_0$. Therefore, there is a matching M in H that omits at most $O(ND^{-1/2}\ln^{3/2} D)$ vertices.

Now, matching M in H corresponds to a collection of hyperstars $S_1, \ldots, S_k$ in G with centers at the vertices of X. Each hyperstar S_i contains at most d hyperedges and hyperstars $S_1, \ldots, S_k$ cover all but at most $O(ND^{-1/2}\ln^{3/2} D)$ vertices of G, which finishes the proof. □

3.3 Formal Proof

We start by defining constants, proving some useful inequalities and proving Lemma 9.

Let S be a Steiner triple system on $m \geqslant (1+\mu)n$ vertices and let T be the largest perfect d-ary hypertree with at most n vertices. Our goal is to show that $T \subset S$.

We make few trivial observations. First, if $d > \sqrt{n}$ and T is perfect d-ary hypertree with $|V(T)| \leqslant n$, then T is just a hyperstar which S clearly contains. Second, if $m > 2n$, then T can be found in S greedily. Finally, if Theorem 3 holds for some value of μ, then Theorem 3 holds for larger values of μ. Hence we may assume without loss of generality that $d \leqslant \sqrt{n}$, $m \leqslant 2n$ and $\mu \leqslant \frac{1}{4}$.

Constants. We will choose new constant $\varepsilon < \delta < \rho < \mu$ independent of m, n:

$$\rho = \left(\frac{3\mu - \mu^2}{8(1+\mu)}\right)^2, \qquad \delta = \frac{(1+\mu)\rho}{20}. \tag{3}$$

Let $\varepsilon_{3.1}$ be a constant guaranteed by Lemma 7 with δ and $k = 2$ as an input. We choose ε to be small enough, in particular we want

$$\varepsilon < \min\{\delta^2, (\mu/16)^2, (\varepsilon_{3.1})^{10}, 1/10^{100}\}. \tag{4}$$

Properties of T. Here we define the levels of T and prove some useful inequalities. Recall that V_i, $i \in [0, h]$ denoted the levels of T. For $i_0 = \max\{i, |V_i| \leqslant \varepsilon n\}$ let T_0 be a subhypertree of T induced on $\bigcup_{i=0}^{i_0} V_i$. To simplify our notation we also set $t = h - i_0$ and for all $i \in [0, t]$, $L_i = V_{i_0+i}$ and $\ell_i = |L_i|$. Then we have for $i \in [t]$

$$\ell_i = (2d)^i \ell_0, \qquad \varepsilon n \geqslant \ell_0 > \frac{\varepsilon}{2d} n. \tag{5}$$

Since $n \geqslant \ell_t$, (5) implies $n \geqslant (2d)^{t-1}\varepsilon n$, and consequently

$$t \leqslant 1 + \frac{\log \frac{1}{\varepsilon}}{\log(2d)} \leqslant 1 + \log \frac{1}{\varepsilon}. \tag{6}$$

Finally, $T_0 = \bigcup_{i=0}^{i_0} V_i$, where $|V_{i_0}| = (2d)^{i_0} = \ell_0$, so

$$|V(T_0)| = \frac{(2d)^{i_0+1} - 1}{2d-1} \leqslant \frac{(2d)\ell_0}{2d-1} \overset{(5)}{\leqslant} 2\varepsilon n. \tag{7}$$

Partition Lemma.

For a given Steiner triple system S with m vertices our goal will be to find a partition $\mathcal{P} = \{C_1, \dots, C_t, R\}$ of $V(S)$ so that $S[C_0]$ contains a copy of T_0 (and L_0), sets $C_1, \dots, C_t$ are the "candidates" for levels $L_1, \dots, L_t$ of T, and R is a reservoir. Such a partition will be guaranteed by Lemma 9.

In the proof we will consider a random partition $\mathcal{P}$, where each vertex $v \in V(S)$ ends up in C_i with probability p_i and in R with probability γ independently of other vertices.

To that end set

$$p_0 = 4\sqrt{\varepsilon}, \tag{8}$$

then by (7) and (4)

$$\frac{p_0^2}{4}(m-1) \geqslant |V(T_0)|, \qquad p_0 \leqslant \frac{\mu}{4}. \tag{9}$$

Now, for all $i \in [t]$ define

$$p_i = \frac{\ell_i}{m} \overset{(5)}{\geqslant} \frac{\varepsilon n}{m} \geqslant \frac{\varepsilon}{2}. \tag{10}$$

Finally let $\gamma = 1 - \sum_{i=0}^{t} p_i$. Then

$$\gamma \geqslant 1 - \frac{\mu}{4} - \sum_{i=1}^{t} p_i = 1 - \frac{\mu}{4} - \frac{\sum_{i=1}^{t} \ell_i}{m} \geqslant 1 - \frac{\mu}{4} - \frac{|V(T)|}{m},$$

and so

$$\gamma \geqslant 1 - \frac{\mu}{4} - \frac{n}{m} \geqslant 1 - \frac{\mu}{4} - \frac{1}{1+\mu} = \frac{3\mu - \mu^2}{4(1+\mu)} \overset{(3)}{=} 2\sqrt{\rho}. \tag{11}$$

Hence $\gamma \in (0,1)$.

Lemma 9. *Let ε, $\ell_0, \dots, \ell_t$, $p_0, \dots, p_t$, γ and ρ be defined as above. Then for some $m_0 = m_0(\varepsilon)$ and $K = 8$ the following is true for any $m \geqslant m_0$. If S is a STS on m vertices, then there is a partition $\mathcal{P} = C_0 \sqcup C_1 \cdots \sqcup C_t \sqcup R$ of $V(S)$ with the following properties:*

(a) $|C_i| = \ell_i \pm K\sqrt{\ell_i \ln \ell_i}$ *for all* $i \in [t]$.

(b) for all $i \in [t]$ *and all* $v \in C_{i-1}$

$$d_{S[C_{i-1}, C_i]}(v) = d\left(p_i \ell_{i-1} \pm K\sqrt{p_i \ell_{i-1} \ln p_i \ell_{i-1}}\right).$$

(c) for all $i \in [2,t]$ *and all* $v \in C_i$

$$d_{S[C_{i-1}, C_i]}(v) = p_i \ell_{i-1} \pm K\sqrt{p_i \ell_{i-1} \ln p_i \ell_{i-1}}.$$

(d) for all $v \in V(S)$, $d_{S[v\cup R]}(v) \geqslant \rho m$.

(e) $|C_0| = p_0 m \pm K\sqrt{p_0 m \ln p_0 m}$ *and* $S[C_0]$ *contains a copy of a hypertree* T_0 *with* L_0 *as its last level. Moreover for all but at most* $\varepsilon^{0.1}|C_1|$ *vertices* $v \in C_1$

$$d_{S[L_0,C_1]}(v) = (1 \pm \varepsilon^{0.1})p_1\ell_0,$$

and for all vertices $v \in C_1$

$$d_{S[L_0,C_1]}(v) \leqslant 2p_1\ell_0.$$

Proof. Recall that $\sum_{i=0}^{t} p_i + \gamma = 1$. Consider a random partition $\mathcal{P} = \{C_0, \dots, C_t, R\}$, where vertices $v \in V(S)$ are chosen into partition classes independently so that $\mathbb{P}[v \in C_i] = p_i$ for $i \in [0,t]$ while $\mathbb{P}[v \in R] = \gamma$. For $j \in \{a,b,c,d,e\}$ let $X^{(j)}$ be the event that the corresponding part of Lemma 9 fails. We will prove that $\mathbb{P}[X^{(j)}] = o(1)$ for each $j \in \{a,b,c,d,e\}$.

Proof of Property (a). For all $i \in [t]$ let $X_i^{(a)}$ be the event that

$$||C_i| - p_i m| > K\sqrt{p_i m \ln p_i m}.$$

Then since $|C_i| \sim \mathrm{Bi}(m,p_i)$ and $\mathbb{E}(|C_i|) = p_i m \overset{(10)}{=} \ell_i \overset{(5)}{=} \Omega(m)$, Theorem 6 implies that

$$\mathbb{P}[X_i^{(a)}] \leqslant 2(\ell_i)^{-K^2/3} = o(m^{-20}).$$

Since by (6), $t \leqslant 1 + \log\frac{1}{\varepsilon} \ll m$ we infer that

$$\mathbb{P}[X^{(a)}] = \mathbb{P}[\bigcup_{i=1}^{t} X_i^{(a)}] \leqslant \sum_{i=1}^{t} \mathbb{P}[X_i^{(a)}] = o(1).$$

Proof of Property (b). For all $i \in [t]$ and $v \in V(G)$ let $X_{i,v}^{(b)}$ be the event

$$\left|d_{S[C_{i-1},C_i]}(v) - dp_i\ell_{i-1}\right| > Kd\sqrt{p_i\ell_{i-1}\ln p_i\ell_{i-1}},$$

and $Y_{i,v}^{(b)}$ be the event

$$\left|d_{S[C_{i-1},C_i]}(v) - (m-1)p_i^2/2\right| > \frac{K}{2}\sqrt{(m-1)p_i^2/2\ln(m-1)p_i^2/2}.$$

Then for $i \in [t]$ and $v \in C_{i-1}$ we have $d_{S[C_{i-1},C_i]}(v) \sim \mathrm{Bi}(\frac{m-1}{2}, p_i^2)$ and

$$\mathbb{E}(d_{S[C_{i-1},C_i]}(v)) = (m-1)p_i^2/2 \overset{(10),(5)}{=} dp_i\ell_{i-1} \pm 1.$$

Therefore $X_{i,v}^b \subseteq Y_{i,v}^b$ for a large enough m. Moreover, Theorem 6 implies that

$$\mathbb{P}[X_{i,v}^{(b)}] \leqslant \mathbb{P}[Y_{i,v}^{(b)}] \leqslant 2((m-1)p_i^2/2)^{-K^2/12} \overset{(10)}{=} O(m^{-2}).$$

Finally, the union bound yields

$$\mathbb{P}[X^{(b)}] \leqslant \sum_{i\in[t],v\in C_i} \mathbb{P}[X^{(b)}_{i,v}] = o(1).$$

Proof of Property (c). Proof follows the lines of the proof of part (b), since for $i \in [2,t]$ and $v \in C_i$ we have $d_{S[C_{i-1},C_i]}(v) \sim \mathrm{Bi}(\frac{m-1}{2}, 2p_ip_{i-1})$ and $\mathbb{E}(d_{S[C_{i-1},C_i]}(v)) = (m-1)p_ip_{i-1} = p_i\ell_{i-1} \pm 1$. Hence we have $\mathbb{P}[X^{(c)}] = o(1)$

Proof of Property (d). Proof follows the lines of the proof of part (b), since for all $v \in V(S)$ we have $d_{S[v\cup R]}(v) \sim \mathrm{Bi}(\frac{m-1}{2}, \gamma^2)$ and $\mathbb{E}(d_{S[v\cup R]}(v)) = \frac{m-1}{2}\gamma^2 \overset{(11)}{\geqslant} 2\rho(m-1)$. Hence we have $\mathbb{P}[X^{(d)}] = o(1)$

Proof of Property (e)

We say that a set $C \subseteq V(S)$ is *typical* if $|C| = p_0m \pm K\sqrt{p_0m\ln p_om}$ and $S[C]$ contains a copy of T_0. For a partition $\mathcal{P} = \{C_0,\dots,C_t,R\}$ set $C_i(\mathcal{P}) = C_i$ for all $i \in [0,t]$.

Next we will show that the first statement of (e), namely that $C_0(\mathcal{P})$ is typical, holds asymptotically almost surely.

Claim 10.

$$\mathbb{P}[C_0(\mathcal{P}) \textit{ is typical }] = 1 - o(1).$$

Proof. Let X be the event that $||C_0| - p_0m| \leqslant K\sqrt{p_0m\ln p_0m}$, and Y be the event that $S[C_0(\mathcal{P})]$ contains a copy of T_0. Since $|C_0| \sim \mathrm{Bi}(m,p_0)$ and $\mathbb{E}(|C_0|) = p_0m \overset{(5)}{=} \Omega(m)$, Theorem 6 implies $\mathbb{P}[X] = 1 - o(1)$.

For $v \in V(S)$ let Z_v denote the event that $d_{S[C_0]}(v) \geqslant |V(T_0)|$, then $\bigcap_{v\in V(S)} Z_v \subseteq Y$. Indeed, if every vertex has degree at least $|V(T_0)|$ is $S[C_0]$, then T_0 can be found in $S[C_0]$ greedily, adding one hyperedge at a time.

Following the lines of proof of (d), we have $d_{S[C_0]}(v) \sim \mathrm{Bi}(\frac{m-1}{2}, p_0^2)$, and

$$\mathbb{E}(d_{S[C_0]}(v)) = \frac{m-1}{2}p_0^2 \overset{(9)}{\geqslant} 2|V(T_0)|,$$

so by Theorem 6 for all $v \in V(S)$ we have $\mathbb{P}[Z_v] \geqslant 1 - o(m^{-20})$. Finally,

$$\mathbb{P}[Y] \geqslant \mathbb{P}[\bigcap_{v\in V(S)} Z_v] \geqslant 1 - m\cdot o(m^{-20}) \geqslant 1 - o(1).$$

Therefore $\mathbb{P}[X] = \mathbb{P}[Y] = 1 - o(1)$ and hence $\mathbb{P}[X \cap Y] = \mathbb{P}[C_0(\mathcal{P})$ is typical $] = 1 - o(1)$. □

Now, for every typical set C, we fix one copy of T_0 in $S[C]$.

We first show that there are not many vertices in $\overline{C} = V(S) \setminus C$ that have low degree in $S[L_0, \overline{C}]$.

Claim 11. *For any $\alpha > 0$ and any typical set C all but at most $|C|/\alpha$ vertices in $v \in \overline{C}$ satisfy*

$$d_{S[L_0,\overline{C}]}(v) = (1 \pm \alpha)\ell_0.$$

Proof of Claim. For $v \in \overline{C}$ and $x \in L_0$ there is a unique $w \in V(S)$ such that $\{v, x, w\} \in E(S)$. Consequently, $d_{S[L_0,\overline{C}]}(v) \leqslant \ell_0$ holds for any $v \in \overline{C}$.

Let $A = \{\{x, v, w\} : x \in L_0, v \in \overline{C}, w \in \overline{C}\}$. Since for every $x \in L_0$ there are at most $|C|$ edges $\{v, x, w\}$ with $v \in \overline{C}$ and $w \in C$

$$|A| \geqslant \ell_0(|\overline{C}| - |C|). \tag{12}$$

On the other hand, let b be the number of "bad" vertices $v \in \overline{C}$, i.e., vertices v with $d_{S[L_0,\overline{C}]}(v) < (1-\alpha)\ell_0$. Then we have

$$|A| \leqslant b(1-\alpha)\ell_0 + (|\overline{C}| - b)\ell_0. \tag{13}$$

Comparing (12) and (13) yields that $b \leqslant |C|/\alpha$. $\square$

Let E be the event that property (e) holds. Next we will show that

$$\mathbb{P}[E|C_0(\mathcal{P}) = C] = 1 - o(1) \text{ for every typical } C. \tag{14}$$

This implies that E holds with probability $1 - o(1)$.

Indeed, by Claim 10, $\mathbb{P}[C_0(\mathcal{P}) \text{ is typical}] = \sum_{C \text{ is typical}} \mathbb{P}[C_0(\mathcal{P}) = C] = (1 - o(1))$ and so

$$\begin{aligned}\mathbb{P}[E] &\geqslant \sum_{C \text{ is typical}} \mathbb{P}(C_0(\mathcal{P}) = C)\mathbb{P}[E|C_0(\mathcal{P}) = C] \\ &\overset{(14)}{\geqslant} (1 - o(1)) \sum_{C \text{ is typical}} \mathbb{P}(C_0(\mathcal{P}) = C) \geqslant 1 - o(1).\end{aligned}$$

It remains to prove (14).

Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the space of all partitions of $V(S)$ with $\mathbb{P}[v \in C_i] = p_i$ for $i \in [0, t]$ and $\mathbb{P}[v \in R] = \gamma$, and for fixed C let $(\Omega, \mathcal{F}, \mathbb{P}_C)$ to be the space of all partitions of $V(S)$ with the probability function $\mathbb{P}_C(A) = \mathbb{P}(A|C_0(\mathcal{P}) = C)$.

With this notation we need to show that $\mathbb{P}_C(E) = 1 - o(1)$ for every typical C.

Recall that $\overline{C} = V(S)/C$ and for all $v \in \overline{C}$ let

$$\chi(v) = \begin{cases} 1, & \text{if } v \in C_1 \\ 0, & \text{otherwise.} \end{cases}$$

Note that for all $v \in \overline{C}$

$$\begin{aligned}\mathbb{P}_C(\chi(v) = 1) = \mathbb{P}_C(v \in C_1) = \mathbb{P}(v \in C_1|C_0(\mathcal{P}) = C) &= \frac{\mathbb{P}(v \in C_1 \wedge C_0(\mathcal{P}) = C)}{\mathbb{P}(C_0(\mathcal{P}) = C)} \\ = \frac{p_1 \cdot p_0^{|C|}(1-p_0)^{m-|C|-1}}{p_0^{|C|}(1-p_0)^{m-|C|}} = \frac{p_1}{1-p_0} = q.\end{aligned}$$

Then by (8)

$$q = (1 \pm \varepsilon^{0.3})p_1. \tag{15}$$

Moreover, since for fixed $v \in V(S)$ the event $\{v \in C_1\}$ was, in the "initial" space $(\Omega, \mathcal{F}, \mathbb{P})$, independent of the outcome of a random experiment for the remaining vertices $w \in V(S) \setminus \{v\}$, we infer that the random variables $\{\chi(v) : v \in \overline{C}\}$ are mutually independent.

Therefore for the rest of the proof we assume that typical C with $L_0 \subset C$ is fixed and all events and random variables are considered in the space $(\Omega, \mathcal{F}, \mathbb{P}_C)$.

For a typical C define

$$M = M(C) = \{v \in \overline{C} : d_{S[L_0,\overline{C}]}(v) = (1 \pm \varepsilon^{0.2})\ell_0\}. \tag{16}$$

Recall that since C is typical we have

$$|C| = (1 + o(1))p_0 m, \text{ and } |\overline{C}| = (1 - o(1))(1 - p_0)m. \tag{17}$$

Then by Claim 11 with $\alpha = \varepsilon^{0.2}$

$$|M| \geqslant \overline{C} - \frac{|C|}{\varepsilon^{0.2}} \overset{(17)}{=} |\overline{C}| - \frac{|\overline{C}|p_0}{\varepsilon^{0.2}(1-p_0)}(1 - o(1)) \overset{(8)}{\geqslant} (1 - \varepsilon^{0.2})|\overline{C}|. \tag{18}$$

Note that M is independent of the choice of C_1 and is fully determined by C and S.

Next we verify that certain events $E^{(1)}$, $E^{(2)}$, $E^{(3)}$ hold asymptotically almost surely and that $E^{(1)} \wedge E^{(2)} \wedge E^{(3)} \subseteq E$. Let event $E^{(1)}$ be defined as

$$E^{(1)} : |M \cap C_1| \geqslant (1 - \varepsilon^{0.1})|C_1|.$$

Since $|C_1| \sim \mathrm{Bi}(|\overline{C}|, q)$ and $|M \cap C_1| \sim \mathrm{Bi}(|M|, q)$, we have

$$\mathbb{E}(|C_1|) = |\overline{C}|q \text{ and } \mathbb{E}(|M \cap C_1|) \overset{(18)}{\geqslant} (1 - \varepsilon^{0.2})|\overline{C}|q.$$

Hence Theorem 6 implies that with probability $1 - o(1)$ we have $|M \cap C_1|/|C_1| \geqslant 1 - \varepsilon^{0.1}$ and so $\mathbb{P}_C[E^{(1)}] = 1 - o(1)$.

Now for every $v \in \overline{C}$ let $N(v)$ be the random variable that equals to the number of hyperedges $\{v, x, w\}$, where $x \in L_0$ and $w \in C_1$. Then $N(v) \sim \mathrm{Bi}(d_{S[L_0,\overline{C}]}(v), q)$ for all $v \in \overline{C}$.

Let $E^{(2)}$ be the event

$$E^{(2)} : N(v) = (1 \pm \varepsilon^{0.1})\ell_0 p_1 \text{ for all } v \in M.$$

For every $v \in M$, we have $N(v) \sim \mathrm{Bi}(d_{S[L_0,\overline{C}]}(v), q)$ and so $\mathbb{E}(N(v)) = (1 \pm 2\varepsilon^{0.2})\ell_0 p_1$ by (16) and (15). Then Theorem 6 combined with the union bound implies $\mathbb{P}_C[E^{(2)}] = 1 - o(1)$.

Let $E^{(3)}$ be the event

$$E^{(3)} : N(v) \leqslant 2\ell_0 p_1 \text{ for all } v \in \overline{C}.$$

For every $v \in \overline{C}$ we have $N(v) \sim \mathrm{Bi}(d_{S[L_0,\overline{C}]}(v), q)$ and $d_{S[L_0,\overline{C}]} \leqslant \ell_0$, hence we always have $\mathbb{E}(N(v)) \overset{(15)}{\leqslant} (1+\varepsilon^{0.3})\ell_0 p_1$. Therefore by Theorem 6 and the union bound we have $\mathbb{P}_C[E^{(3)}] = 1 - o(1)$.

It remains to notice that for $v \in C_1$ we have $d_{S[L_0,C_1]}(v) = N(v)$ and so $E^{(1)} \wedge E^{(2)} \wedge E^{(3)} \subseteq E$. Therefore, $\mathbb{P}_C[E] \geqslant 1 - o(1)$, finishing the proof of (14). □

Embedding of T. We start with applying Lemma 9 to S obtaining a partition $\mathcal{P} = \{C_0, \ldots, C_t, R\}$ of $V(S)$ that satisfies properties (a)-(e) of Lemma 9. To simplify our notation we set $G_1 = S[L_0, C_1]$ and for $i \in [2,t]$ $G_i = S[C_{i-1}, C_i]$.

1) We first verify that Lemma 9 guarantees that the assumptions of Lemma 7 and Lemma 8 are satisfied. These Lemmas then yield systems of stars $\mathcal{S}_i = \{S_i^1, \ldots, S_i^{p_i}\}$ for $i \in [2,t]$, such that each $\mathcal{S}_i$ covers almost all vertices of the respective G_i. (See Figure 1, where each star S_i^j is represented by a single grey edge.)

2) Let F be the union of T_0 with $\mathcal{S}_i$'s. The "almost cover" property of $\mathcal{S}_i$'s then allows us to show that hyperforest F contains a large connected component T_1 which contains almost all vertices of T. (See Figure 1, green and grey edges form T_1.)

3) Finally, we extend T_1 into a full copy of T in a greedy procedure using the vertices of R. (See Figure 1, vertices of R are blue.)

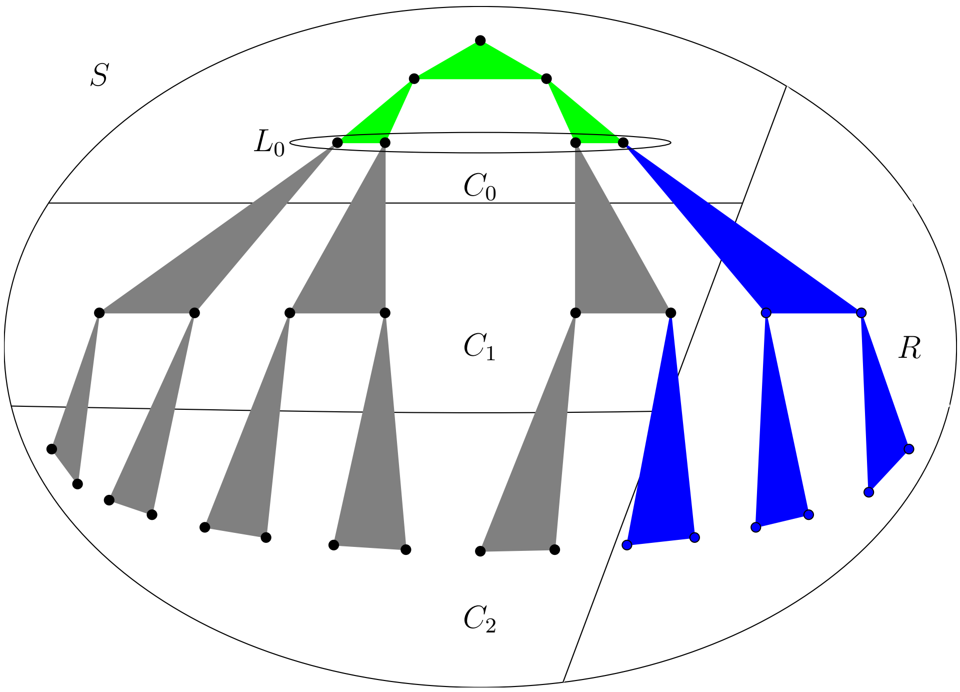

Figure 1: Case $d = 1$ and $t = 2$. Green edges form T_0, grey edges are hyperstars S_i^j, blue edges are constructed by using vertices in reservoir R.

Step 1. Construction of the hyperforest F.

We start with applying Lemma 9 to S and obtaining a partition $\mathcal{P} = \{C_0, \ldots, C_t, R\}$ of $V(S)$ that satisfies properties (a)-(e) of Lemma 9. Recall that $G_1 = S[L_0, C_1]$ and for $i \in [2, t]$ let $G_i = S[C_{i-1}, C_i]$.

Let $N_i = |V(G_i)|$, then $N_1 = \ell_0 + |C_1|$ and $N_i = |C_{i-1}| + |C_i|$ for $i \in [2, t]$. Due to property (a) of Lemma 9 we have that for a sufficiently large m and for all $i \in [t]$

$$N_i = (1 \pm \varepsilon)(\ell_{i-1} + \ell_i) \leqslant (1 \pm \varepsilon)\left(\frac{\ell_i}{2d} + \ell_i\right) \leqslant 2\ell_i. \tag{19}$$

In what follows we will show that G_1 satisfies the assumptions of Lemma 7 and for $i \in [2, t]$, G_i satisfies the assumptions of Lemma 8.

We start with G_1. Recall that for a given μ we have defined δ (see (3)), and $\varepsilon_{3.1}$ was a constant guaranteed by Lemma 7 with δ and $k = 2$ as input. We then defined ε (see (4)) to be small enough, in particular such that $\varepsilon_{3.1} \geqslant \varepsilon^{0.1}$. We will now show that G_1 satisfies conditions of Lemma 7 with $D = D_1 = p_1\ell_0$ and $N = N_1$.

To verify condition (i) of Lemma 7 it is enough to recall that $G_1 = S[L_0, C_1]$ and so (i) holds with $X = L_0$ and $Y = C_1$.

Note that condition (iii) is guaranteed by property (e), since for all $v \in C_1 = Y$ we have $d_{G_1}(v) \leqslant 2D$.

We now verify condition (ii) of Lemma 7. Property (b) with $i = 1$ guarantees that for all $v \in X = L_0$

$$d_{G_1}(v) = d\left(D \pm K\sqrt{D \ln D}\right).$$

Since $D = p_1\ell_0 \overset{(5),(10)}{=} \Omega(\sqrt{n})$, for a large enough n we have for all $v \in X = L_0$

$$d_{G_1}(v) = dD(1 \pm \varepsilon_{3.1}). \tag{20}$$

Property (e) in turn guarantees that for all but at most $\varepsilon^{0.1}|C_1| \leqslant \varepsilon_{3.1}N_1$ vertices $v \in Y = C_1$ we have

$$d_{G_1}(v) = (1 \pm \varepsilon^{0.1})p_1\ell_0 = D(1 \pm \varepsilon_{3.1}). \tag{21}$$

Now, (20) and (21) imply that G_1 satisfies condition (ii) of Lemma 7.

Lemma 7 produces a collection $\{S_1^1, \ldots, S_{p_1}^1\}$ of disjoint hyperstars of G_1 centered at vertices of L_0, each S_j^1 has at most d hyperedges, and

$$\text{the star forest } \mathcal{S}_1 = \bigcup_{j=1}^{p_1} S_j^1 \text{ covers all but at most } \delta N_1 \text{ vertices of } G_1. \tag{22}$$

Similarly for all $i \in [2, t]$, the hypergraph G_i satisfies the assumptions of Lemma 8 with the parameters $K_{3.2} = 8, d_{3.2} = d$ and $D = D_i = p_i\ell_{i-1}$. Indeed, condition (i) of Lemma 8 is guaranteed by taking $X = C_{i-1}$ and $Y = C_i$, and condition (ii) is guaranteed by properties (b) and (c) of Lemma 9.

Then for $i \in [2, t]$, Lemma 8 when applied to G_i yields a collection $\{S_1^i, \ldots, S_{p_i}^i\}$ of disjoint hyperstars centered at vertices of C_{i-1}, each S_j^i has at most d hyperedges, and the star forest $\mathcal{S}_i = \bigcup_{j=1}^{p_i} S_j$ covers all but at most $O(N_i D_i^{-1/2} \ln^{3/2} D_i)$ vertices of G_i. Once again recall that for $i \in [2,t]$ we have $D_i = p_i \ell_{i-1} \overset{(5),(10)}{=} \Omega(n)$ and so for a large enough n,

$$\mathcal{S}_i \text{ covers all but at most } \varepsilon N_i \text{ vertices of } G_i. \tag{23}$$

Now, let $F = T_0 \cup \bigcup_{i=1}^{t} \mathcal{S}_i$, then F is a hyperforest and we will now find a hypertree $T_1 \subseteq F$ such that T_1 is an almost spanning subhypertree of T.

We first will estimate $|V(F)|$. Since for $i \in [2,t]$ a star forest $\mathcal{S}_i$ misses at most εN_i vertices of C_i and $\mathcal{S}_1$ misses at most δN_1 vertices of C_1 we have:

$$|V(F)| \geqslant |V(T_0)| + |C_1| - \delta N_1 + \sum_{i=2}^{t} (|C_i| - \varepsilon N_i).$$

Now, by property (a) of Lemma 9, for large enough m and for any $i \in [t]$, we have $|C_i| = (1 \pm \varepsilon)\ell_i$. Therefore

$$|V(F)| \geqslant |V(T_0)| + \sum_{i=1}^{t}(1-\varepsilon)\ell_i - \delta N_1 - \varepsilon \sum_{i=2}^{t} N_i \overset{(19)}{\geqslant} |V(T_0)| + \sum_{i=1}^{t} \ell_i - (\varepsilon + 2\delta)\ell_1 - 3\varepsilon \sum_{i=2}^{t} \ell_i.$$

Since $|V(T_0)| + \sum_{i=1}^{t} \ell_i = |V(T)|$ and $\varepsilon < \delta$ we have

$$|V(F)| \geqslant |V(T)| - 3\delta \sum_{i=1}^{t} \ell_i \geqslant (1 - 3\delta)|V(T)|. \tag{24}$$

Step 2. Embedding the most of T.

Claim 12. *S contains a hypertree $T_1 \subseteq F$ such that T_1 is a subhypertree of T and $|E(T_1)| \geqslant (1 - 20\delta)|E(T)|$.*

Proof. Recall that while $V(T) = \bigcup_{i=0}^{h} V_i$, the forest F has the vertex set $V(F) = \bigcup_{i=0}^{i_0} V_i \cup \bigcup_{i=1}^{t} C_i$, where we have set $t = h - i_0$ and $V_{i_0} = L_0$. Also recall that $V_0 = \{v_0\}$ was a root of T (and F).

For a non-root vertex $v \in V_i$ (or $v \in C_i$) a parent of v is a unique vertex $u \in V_{i-1}$ (or $u \in C_{i-1}$) such that $\{u, v, w\} \in F$. If a vertex v has a parent u we will write $p(v) = u$, if a vertex v has no parent we will say that v is an orphan.

For each $v \in V(F) \setminus \{v_0\}$ consider "a path of ancestors" $v = a_i, a_{i-1}, \ldots, a_1 = a^*$, i.e., a path satisfying $p(a_j) = a_{j-1}$, $j \in [2,i]$ and such that $a^* = a^*(v)$ is an orphan in F.

Let $T_1 \subseteq F$ be a subtree of F induced on a set $\{v \in V(F) : a^*(v) = v_0\}$. Note that if for some $v \in V(F)$ we have $a^*(v) \neq v_0$, then $a^*(v) \notin V(T_0)$.

For $i \in [t]$ let $U_i \subseteq C_i$ be the set of vertices not covered by $\mathcal{S}_i$. Then

$$|U_1| \overset{(22)}{\leqslant} \delta N_1 \text{ and } |U_i| \overset{(23)}{\leqslant} \varepsilon N_i \text{ for all } i \in [2,t]. \tag{25}$$

Note that all orphan vertices, except of v_0, belong to $\bigcup_{i=1}^{t} U_i$ as they were not covered by some $\mathcal{S}_i$. In particular, for every $v \notin V(T_1)$ its ancestor $a^*(v)$ is an orphan and hence belongs to $\bigcup_{i=1}^{t} U_i$

For an orphan vertex a^* let $T(a^*)$ be a subtree of F rooted at a^*. Then for every orphan vertex $a^* \in U_i$ we have

$$|V(T(a^*))| \leqslant 1 + 2d + \cdots + (2d)^{t-i} \leqslant 3(2d)^{t-i}.$$

Finally, every $v \notin V(T_1)$ is in $T(a^*(v))$, where $a^*(v) \in U_i$ for some $i \in [t]$, and therefore we have

$$|V(T_1)| \geqslant |V(F)| - \sum_{i=1}^{t} |U_i| \cdot 3(2d)^{t-i} \overset{(24),(25)}{\geqslant} (1-3\delta)|V(T)| - 3\delta N_1 (2d)^{t-1} - 3\varepsilon \sum_{i=2}^{t} N_i (2d)^{t-i}.$$

Now, by (19), we have $N_i \leqslant 2\ell_i$ and, by (5), $\ell_i(2d)^{t-i} = \ell_t$ for all $i \in [t]$, and so

$$|V(T_1)| \geqslant (1-3\delta)|V(T)| - 6\delta\ell_t - 6\varepsilon(t-1)\ell_t.$$

Recall that ℓ_t is the size of the last level of T, so $\ell_t \leqslant |V(T)|$. Also $t \leqslant 1 + \log(\frac{1}{\varepsilon})$, and since ε is sufficiently small $6\varepsilon(t-1) \leqslant \sqrt{\varepsilon} \leqslant \delta$ and so

$$|V(T_1)| \geqslant (1-10\delta)|V(T)|.$$

For every hypertree T' we have $|V(T')| = 2|E(T')| + 1$, and so

$$|E(T_1)| \geqslant (1-20\delta)|E(T)|,$$

finishing the proof of the Claim. □

Step 3. Finally, we complete T_1 to a full copy of T by using the reservoir R.

Claim 13. *Assume that $\mathcal{P}$ is a partition guaranteed by Lemma 9 and T_1 be a hypertree guaranteed by Claim 12. Then T_1 can be extended to a copy of T in S.*

Proof. Since T_1 is a subhypertree of T, there is a sequence of hyperedges $\{e_1, \ldots, e_{p-1}\}$ such that each $T_i = T_1 \bigcup_{j=1}^{i-1} e_j$ for $i \in [p]$ is a hypertree an $T_p \cong T$. Every vertex $v \in V(S)$ has degree at least ρm in R (by Lemma 9) and

$$p - 1 = |E(T)| - |E(T_1)| \overset{\text{Claim 12}}{\leqslant} 20\delta|E(T)| \leqslant 10\delta n \overset{(3)}{\leqslant} \frac{\rho m}{2}.$$

Hence we can greedily embed edges $e_1, \ldots, e_{p-1}$. Indeed, having embedded the edges $e_1, \ldots, e_{i-1}$ for some $i \in [p-1]$, for set $R_i = R \setminus \bigcup_{j=1}^{i-1} e_j$ and all $v \in V(S)$ we have

$$d_{S[v \cup R_i]} \geqslant \rho m - 2(i-1) > 0$$

allowing the greedy embeding to continue. Then the last hypertree T_p is, by the construction, isomorphic to T. □

4 Concluding remarks

We notice that with a similar proof one can verify Conjecture 1 for some other types of hypertrees.

Let $D = \{d_1, \dots, d_k\}$ be a sequence of integers. Let T be a tree rooted at v_0 and let $V(T) = V_0 \sqcup V_1 \sqcup \dots \sqcup V_h$ be a partition of $V(T)$ into levels, so that V_i consist of vertices distance i from v_0. We say that T is *D-ary hypertree* if for every $i \in [0, h-1]$ there is $j \in [k]$ such that for every $v \in V_i$ the forward degree of v is d_j. In other words, forward degree of every non-leaf vertex of T is in D, and depends only on the height of a vertex in T.

Following the lines of the proof of Theorem 3 one can conclude that for any finite set $D \subset \mathbb{N}$, and any μ, any large enough STS S contains any D-ary hypertree T, provided $|V(T)| \leqslant |V(S)|/(1+\mu)$.

Another type of hypertrees for which Conjecture 1 holds are truncated d-ary hypertrees. In a perfect d-ary hypertree label children of every vertex with numbers $\{1, \dots, 2d\}$. Then every leaf can be identified with a sequence $\{a_1, \dots, a_h\} \in [2d]^h$ based on the way that leaf is reached from the root, and all leafs can be ordered by a lexicographic order. We say that T is a *truncated perfect d-ary hypertree* if, for some integer t, the hypertree T is obtained from a perfect d-ary hypertree by removing the smallest $2t$ leafs (according to the lexicographic order).

With an essentially same proof as of Theorem 3, for every $\mu > 0$ and d, any sufficiently large Steiner triple system S contains any truncated d-ary hypertree T with $|V(T)| \leqslant |V(S)|/(1+\mu)$.

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