

# Passage from the Boltzmann equation with diffuse boundary to the incompressible Euler equation with heat convection

Yunbai Cao<sup>a,\*</sup>, Juhi Jang<sup>b</sup>, Chanwoo Kim<sup>c</sup>

<sup>a</sup> Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA

<sup>b</sup> Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA

<sup>c</sup> Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

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## Abstract

We derive the incompressible Euler equations with heat convection with the no-penetration boundary condition from the Boltzmann equation with the diffuse boundary in the hydrodynamic limit for the scale of large Reynold number. Inspired by the recent framework in [30], we consider the Navier-Stokes-Fourier system with no-slip boundary conditions as an intermediary approximation and develop a Hilbert-type expansion of the Boltzmann equation around the global Maxwellian that allows the nontrivial heat transfer by convection in the limit. To justify our expansion and the limit, a new direct estimate of the heat flux and its derivatives in the Navier-Stokes-Fourier system is established adopting a recent Green's function approach in the study of the inviscid limit.

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\* Corresponding author.

E-mail addresses: [yc1157@math.rutgers.edu](mailto:yc1157@math.rutgers.edu) (Y. Cao), [juhijang@usc.edu](mailto:juhijang@usc.edu) (J. Jang), [chanwoo.kim@wisc.edu](mailto:chanwoo.kim@wisc.edu) (C. Kim).

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## 1. Introduction

Building the connection between kinetic theory and macroscopic fluid dynamics has been an important subject over the past decades. Kinetic equations study the time evolution of the distribution function  $F(t, x, v) \geq 0$  representing the density of particles of some rarified gas with position  $x$  and velocity  $v$  in the phase space  $\Omega \times \mathbb{R}^3$  at time  $t$ . The interaction of particles through collisions is often modeled by some binary collision operator  $\mathcal{Q}(F, F)$ .

When the gas is dense enough such that particles go through many collisions, the hydrodynamic limits are obtained. A small parameter  $\mathcal{Kn}$  called the *Knudsen number*, which represents the ratio of the mean free path of particles between collisions to the characteristic length, is a key dimensionless number in describing such phenomena. On the other hand, the velocity scale that some macroscopic portion of the gas is transported, described by the kinetic *Strouhal number*  $St$ , also affect the limits. The dimensionless Boltzmann equation takes the form of

$$St \partial_t F + v \cdot \nabla_x F = \frac{1}{\mathcal{Kn}} \mathcal{Q}(F, F), \text{ on } [0, \infty) \times \Omega \times \mathbb{R}^3, \quad (1.1)$$

where  $\Omega$  is an open subset of  $\mathbb{R}^3$ . Throughout this paper, we assume the hard sphere Boltzmann collision operator:

$$\begin{aligned} \mathcal{Q}(F, G) = & \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot u| \{ F(v') G(v'_*) + G(v') F(v'_*) \\ & - F(v) G(v_*) - G(v) F(v_*) \} du dv_*, \end{aligned} \quad (1.2)$$

where  $v' := v - ((v - v_*) \cdot u)u$  and  $v'_* := v_* + ((v - v_*) \cdot u)u$ . This collision operator enjoys the collision invariance property: for any  $F(v)$  and  $G(v)$ ,

$$\int_{\mathbb{R}^3} \mathcal{Q}(F, G)(v) \left( 1, v, \frac{|v|^2 - 3}{\sqrt{6}} \right) dv = (0, 0, 0). \quad (1.3)$$

We denote the global Maxwellian, which satisfies  $Q(\cdot, \cdot) = 0$ , by  $\mu$

$$\mu(v) := \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}}. \quad (1.4)$$

Another parameter considered in hydrodynamic limits is the *Mach number*  $\mathcal{M}a$ , which can be viewed as the scale of fluctuations around some reference flow. In this paper, we take the scale that the *Mach number* is equal to the *Strouhal number*:

$$St = \mathcal{M}a = \varepsilon. \quad (1.5)$$

The reciprocal viscosity of a fluid can be measured in terms of the Reynold number  $\mathcal{R}e = \mathcal{M}a/\mathcal{K}n$ . In this paper, we consider the scale of large Reynold number with

$$\mathcal{K}n = \kappa \varepsilon, \quad (1.6)$$

where  $\kappa = \kappa(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Under such scaling, we will derive the incompressible Euler equations with heat convection with the no-penetration boundary condition in the limit

$$\partial_t u_E + u_E \cdot \nabla_x u_E + \nabla_x p_E = 0 \quad \text{in } \Omega, \quad (1.7)$$

$$\nabla_x \cdot u_E = 0 \quad \text{in } \Omega, \quad (1.8)$$

$$u_E \cdot n = 0 \quad \text{on } \partial\Omega, \quad (1.9)$$

$$\partial_t \theta_E + u_E \cdot \nabla_x \theta_E = 0 \quad \text{in } \Omega. \quad (1.10)$$

Here  $n = n(x)$  is the unit outward normal vector at  $x \in \partial\Omega$ . Note that the boundary value of the heat convection  $\theta_E$  is completely determined by the transport equation (1.10) and the initial data, therefore no boundary condition on  $\theta_E$  is imposed.

In many important physical applications, e.g. turbulence theory, boundary effect plays an important role in global dynamics, and it is of both physical and mathematical interests to take the boundary into consideration in the hydrodynamic limit. In this paper we consider one of the physical boundary conditions, the so-called *diffuse boundary condition*, which models the ideal situation that gas particles reflected from the boundary reach an instantaneous thermal equilibration (see [13]): for  $(x, v) \in \{\partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}$ ,

$$F(t, x, v) = c_\mu \mu(v) \int_{n(x) \cdot v > 0} F(t, x, v)(n(x) \cdot v) dv. \quad (1.11)$$

Here, the constant  $c_\mu$  is chosen to be  $\sqrt{2\pi}$  so that  $\int_{n(x) \cdot v > 0} \mu(v)(n(x) \cdot v) dv = 1$ , which ensures the null flux condition  $\int_{\mathbb{R}^3} F(t, x, v)(n(x) \cdot v) dv = 0$  for  $x \in \partial\Omega$ .

The first mathematical studies of hydrodynamic limits of the Boltzmann equation may date back to Hilbert in [28], where he introduced the method of Hilbert expansion to obtain the derivations at the formal level. Since then there have been many results on the rigorous justifications of hydrodynamic limits based on the truncated asymptotic expansions method. For instance, the compressible Euler limits with heat transfer are derived in [6,44,57]; incompressible

Navier-Stokes limits are achieved in [5,11,19]; and diffusive limits from the Vlasov-Maxwell-Boltzmann system have been obtained in [29]; and hydrodynamic limit toward singular fluid solutions [40,41]. On the other hand, since Bardos, Golse, and Levermore [1,2] have proposed the derivation of weak solutions of fluid equations from the renormalized solutions of the Boltzmann equation [12] in the early nineties, extensive studies have been made in this direction and a lot of important results have been obtained (see [17,33,52] for the references in this direction). Notably, the Leray solutions of the incompressible Navier-Stokes equations have been successfully derived in [18,45,46]; and dissipative solutions of the incompressible Euler have been obtained in [4,45,46,50].

It should be noted that all the results mentioned above deal with domains that do not contain boundaries. In general, however, solutions of the Boltzmann equation with physical boundary conditions behave differently; in particular, high regularity may not be expected (see [24,25,36]). To overcome this an  $L^2 - L^\infty$  framework was developed in [20] to study global solutions of the Boltzmann equation with various boundary conditions, which prompted substantial development in various directions including [7–10,13–15,21–23,26,31,32,37–39,42,59]. Among them in [13,14], the hydrodynamic limit of the incompressible Navier-Stokes-Fourier system was derived from the Boltzmann equation under the diffuse boundary condition.

In terms of the incompressible Euler limit, based on the relative entropy method, the convergence of renormalized solutions of the Boltzmann equation to dissipative solutions of the incompressible Euler equations was first obtained in [4] assuming the local conservation of momentum which is not guaranteed for renormalized solutions of the Boltzmann equation, some nonlinear estimate, and the initial data to be well-prepared, in particular, the initial temperature fluctuation  $\theta_{in} = 0$ , so there is no heat transfer. Later, the local conservation of momentum assumption was removed in [45] by using the local momentum conservation with matrix-valued defect measure satisfied by renormalized solutions of the Boltzmann equation. The nonlinear estimate assumption in [4] was further removed in [50] using refined dissipation estimates which were developed first in the framework of the BGK equation in [49]. When considering physical boundaries, the incompressible Euler limit with no heat transfer has been derived in [3] from the Boltzmann equation under the Maxwell boundary condition. Finally, in [51] the well-prepared initial data assumption in [4] was removed by the construction of refined approximate solutions with converging modulated entropy, and the incompressible Euler limit with heat transfer has been justified under specular reflection boundary condition. We refer to [52] for detailed results and discussions in this direction. As far as we know, all the existing results on the incompressible Euler limit using the relative entropy methods are limited to the Maxwell-boundary conditions whose accommodation coefficients essentially vanish when  $\varepsilon \rightarrow 0$ . In particular, the sole diffuse boundary condition is excluded.

To the best of our knowledge, the incompressible Euler limit with heat transfer from the Boltzmann equation with diffuse boundary has not been established in any framework yet. The main goal of the paper is to rigorously justify the incompressible Euler limit with heat convection under the no-penetration boundary condition (1.7)–(1.10) from the Boltzmann equation (1.1) in the scale of (1.5), (1.6) with the diffuse boundary condition (1.11).

Under the scaling (1.5), (1.6), it is well-known that a mismatch exists between the diffuse boundary condition (1.11) of the Boltzmann equation and the no-penetration boundary condition (1.9) of the Euler equation. To overcome such difficulty, adopting the recent framework in [30], we study the Euler limit with heat transfer from the Boltzmann equation through the Navier-Stokes-Fourier system with the no-slip boundary condition

$$\partial_t u + u \cdot \nabla_x u - \kappa \eta_v \Delta u + \nabla_x p = 0 \quad \text{in } \Omega, \quad (1.12)$$

$$\nabla_x \cdot u = 0 \quad \text{in } \Omega, \quad (1.13)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (1.14)$$

$$\partial_t \theta + u \cdot \nabla_x \theta - \kappa \eta_c \Delta \theta = 0 \quad \text{in } \Omega, \quad (1.15)$$

$$\theta = 0 \quad \text{on } \partial\Omega, \quad (1.16)$$

$$\text{Boussinesq relation} \quad \rho + \theta = 0 \quad \text{in } \Omega, \quad (1.17)$$

where  $\eta_v$  and  $\eta_c$  are physical constants that can be computed explicitly by the Boltzmann theory as in (2.17), (2.17).

We expand the Boltzmann solution  $F$  around a global Maxwellian  $\mu$  plus the first and second correction associated with a Navier-Stokes-Fourier flow (1.12)-(1.17):

$$F = \mu + \varepsilon f_1 \sqrt{\mu} + \varepsilon^2 f_2 \sqrt{\mu} + \varepsilon^{3/2} f_R \sqrt{\mu}, \quad (1.18)$$

where  $f_1 = (\rho + u \cdot v + \theta \frac{|v|^2 - 3}{2}) \sqrt{\mu}$ , and  $f_2$ , which also determined by  $u, \rho, \theta$ , will be specified in (2.2)-(2.4). Here  $F = F^\varepsilon$ ,  $f_R = f_R^\varepsilon$  depend on  $\varepsilon$ , but we drop the superscript  $\varepsilon$  for the sake of simplicity. Then the equation for the Boltzmann remainder  $f_R$  is

$$\begin{aligned} & \partial_t f_R + \frac{1}{\varepsilon} v \cdot \nabla_x f_R + \frac{1}{\varepsilon^2 \kappa} L f_R \\ &= -\frac{1}{\varepsilon^{5/2} \kappa} L f_1 \end{aligned} \quad (1.19)$$

$$- \frac{1}{\varepsilon^{3/2}} \left\{ v \cdot \nabla_x f_1 - \frac{1}{\kappa} \Gamma(f_1, f_1) + \frac{1}{\kappa} L f_2 \right\} \quad (1.20)$$

$$- \frac{1}{\varepsilon^{1/2}} \left\{ \partial_t f_1 + v \cdot \nabla_x f_2 - \frac{2}{\kappa} \Gamma(f_1, f_2) \right\} \quad (1.21)$$

$$- \varepsilon^{1/2} \partial_t f_2 + \frac{2}{\varepsilon \kappa} \Gamma(f_1, f_R) + \frac{2}{\kappa} \Gamma(f_2, f_R) + \frac{\varepsilon^{1/2}}{\kappa} \Gamma(f_2, f_2) + \frac{1}{\varepsilon^{1/2} \kappa} \Gamma(f_R, f_R) \quad (1.22)$$

where

$$L f = \frac{-2}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} f), \quad \Gamma(f, g) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} g). \quad (1.23)$$

The operators  $L$  and  $\Gamma$  can be written as

$$L f(v) = v f(v) - K f(v) = v(v) f(v) - \int_{\mathbb{R}^3} \mathbf{k}(v, v_*) f(v_*) dv_*, \quad (1.24)$$

$$\begin{aligned} \Gamma(f, g)(t, v) &= \Gamma_+(f, g)(t, v) - \Gamma_-(f, g)(t, v) \\ &= \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot u| \sqrt{\mu(v_*)} (f(t, v') g(t, v'_*) + g(t, v') f(t, v'_*)) du dv_* \end{aligned} \quad (1.25)$$

$$- \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot u| \sqrt{\mu(v_*)} (f(t, v)g(t, v_*) + g(t, v)f(t, v_*)) \, du dv_*$$

where the collision frequency  $\nu$  is defined as

$$\nu(v) := \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot u| \mu_0(v_*) \, du dv_* \sim \langle v \rangle := \sqrt{1 + |v|^2}, \quad (1.26)$$

and

$$\mathbf{k}(v, v_*) = C_1 |v - v_*| e^{-\frac{|v|^2 + |v_*|^2}{4}} - \frac{C_2}{|v - v_*|} e^{-\frac{|v - v_*|^2}{8} - \frac{1}{8} \frac{(|v|^2 - |v_*|^2)^2}{|v - v_*|^2}} \quad (1.27)$$

for some constant  $C_1, C_2 > 0$ . The null space of  $L$ , denoted by  $\mathcal{N}$ , is a subspace of  $L^2(\mathbb{R}^3)$  spanned by orthogonal bases  $\{\varphi_i \sqrt{\mu}\}_{i=0}^4$  with

$$\varphi_0 := 1, \quad \varphi_i := v_i \quad \text{for } i = 1, 2, 3, \quad \varphi_4 := \frac{|v|^2 - 3}{2}. \quad (1.28)$$

We define a hydrodynamic projection  $\mathbf{P}$  as an  $L_v^2$ -projection on  $\mathcal{N}$  such as

$$\mathbf{P}g := \sum (P_j g) \varphi_j \sqrt{\mu}, \quad P_j g := \langle g, \varphi_j \sqrt{\mu} \rangle, \quad \text{and} \quad Pg := (P_0 g, P_1 g, P_2 g, P_3 g, P_4 g), \quad (1.29)$$

where  $\langle \cdot, \cdot \rangle$  stands for an  $L_v^2$ -inner product. It is well-known that the operators enjoy  $\mathbf{P}L = L\mathbf{P} = \mathbf{P}\Gamma = 0$ . Importantly the linear operator  $L$  enjoys a coercivity away from the kernel  $\mathcal{N}$ : for  $\nu(v) \geq 0$  defined in (1.24),

$$\langle Lf, f \rangle \geq \sigma_0 \|\sqrt{\nu}(\mathbf{I} - \mathbf{P})f\|_{L^2(\mathbb{R}^3)}^2 \quad \text{for some } \sigma_0 > 0. \quad (1.30)$$

From the no-slip boundary condition (1.14), (1.16),  $f_1 = 0$  on  $\partial\Omega$  and hence  $\mu + \varepsilon f_1 \sqrt{\mu}$  satisfies the diffuse reflection boundary condition (1.11). By plugging (1.18) into the boundary condition, we arrive at

$$(\varepsilon^2 f_2 + \varepsilon^{3/2} f_R)|_{\gamma_-} = c_\mu \sqrt{\mu(v)} \int_{n(x) \cdot v > 0} (\varepsilon^2 f_2 + \varepsilon^{3/2} f_R) \sqrt{\mu(v)} (n(x) \cdot v) \, dv.$$

Letting  $P_{\gamma_+}$  be an  $L^2(\{v : n(x) \cdot v > 0\})$ -projection of  $\sqrt{c_\mu \mu}$ , we derive that

$$\begin{aligned} f_R(t, x, v)|_{\gamma_-} &= P_{\gamma_+} f_R(t, x, v) - \varepsilon^{1/2} (1 - P_{\gamma_+}) f_2(t, x, v) \\ &:= \sqrt{c_\mu \mu(v)} \int_{n(x) \cdot v > 0} f_R(t, x, v) \sqrt{c_\mu \mu(v)} (n(x) \cdot v) \, dv - \varepsilon^{1/2} (1 - P_{\gamma_+}) f_2(t, x, v). \end{aligned} \quad (1.31)$$

Note that  $\int_{n(x) \cdot v > 0} c_\mu \mu(v) (n(x) \cdot v) \, dv = 1$ . And  $\partial_t f_R$  satisfies

$$\partial_t f_R|_{\gamma_-} = P_{\gamma_+} \partial_t f_R - \varepsilon^{1/2} (1 - P_{\gamma_+}) \partial_t (\mathbf{I} - \mathbf{P}) f_2. \quad (1.32)$$

We are now ready to state the main result of this paper (informal version); the precise statement can be found in Theorem 10 in Section 4.

**Theorem 1** (Informal statement). *Let  $\Omega$  be an upper half space in  $3D$ :*

$$\Omega := \mathbb{T}^2 \times \mathbb{R}_+ \ni (x_1, x_2, x_3), \text{ where } \mathbb{R}_+ := \{x_3 \in \mathbb{R} : x_3 > 0\}, \quad (1.33)$$

*and  $\mathbb{T}$  is the periodic interval  $(-\pi, \pi)$ .*

*Suppose an initial velocity field  $u_{in}$  is divergence-free, the corresponding initial vorticity  $\omega_{in} = \nabla_x \times u_{in}$  and the initial heat  $\theta_{in}$  live in some real analytic space and satisfy certain compatibility conditions. Choosing the proper correction terms  $f_1$  and  $f_2$ , then there exists a large set of initial data  $f_{R,in}$  such that for some  $T > 0$ , there exists a unique solution  $F(t, x, v)$  of the form (1.18) to the Boltzmann equation (1.1) with the diffuse reflection boundary condition (1.11) under the scale of (1.5), (1.6) on  $[0, T]$  such that for some choice of  $\varepsilon$  and  $\kappa(\varepsilon)$ ,*

$$\sup_{0 \leq t \leq T} \left\| \frac{F(t, x, v) - \mu(v)}{\varepsilon \sqrt{\mu(v)}} - \left( -\theta_E(t, x) + u_E(t, x) \cdot v + \theta_E(t, x) \frac{|v|^2 - 3}{2} \right) \sqrt{\mu(v)} \right\|_{L^2(\Omega \times \mathbb{R}^3)} \rightarrow 0$$

*as  $\varepsilon \rightarrow 0$ , where  $u_E$  and  $\theta_E$  are the solutions of the incompressible Euler equations with no-penetration boundary condition under heat convection (1.7)-(1.10).*

For the rest of this section, we present the strategy and key ideas to the theorems for rigorously justifying the limit. The core of Section 2 is to establish an  $L^2$  estimate for the Boltzmann remainder  $f_R$  uniformly in  $\varepsilon$ . From our Hilbert expansion around the global Maxwellian  $\mu$  in (1.18), the equation for  $f_R$  is (1.19)-(1.22). In order to have a desired bound on  $f_R$  it is necessary to dispose of the highly singular terms in (1.19)-(1.21). We show in Section 2.1 that by the choice of the first and second order correction  $f_1, f_2$  associated with a Navier-Stokes-Fourier flow as in (2.1)-(2.4), we obtain (1.19) = (1.20) = 0, and  $\mathbf{P}(1.21) = 0$ . The remaining terms in (1.19)-(1.22) are of low enough singularity that can be controlled eventually.

The  $L^2$  estimate for  $f_R$  shares the same framework as in [30]. A trilinear forcing term comes up as  $\frac{1}{\kappa \varepsilon^{1/2}} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \Gamma(f_R, f_R)(\mathbf{I} - \mathbf{P}) f_R$  in the energy estimate of  $f_R$ . Utilizing the dissipation from the linearized Boltzmann operator (1.30) we can bound this term as

$$\frac{\varepsilon^{1/2}}{\kappa^{1/2}} \|P f_R\|_{L_t^\infty L_x^6} \|P f_R\|_{L_t^2 L_x^3} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2}.$$

The  $L_t^2 L_x^3$ -estimate for  $P f_R$  is achieved by extending  $f_R$  into a particularly designed domain first and then into the whole space using special cutoff functions, followed by Duhamel's formula and employing the  $TT^*$ -method developed in [15, 27, 34]. Detailed estimates can be found in section 2.3. An  $L^6$  integrability gain for  $\mathbf{P} f_R$  is obtained by employing the micro-macro decomposition method as in [14]. We use the test function method as in [13] to invert the operator  $v \cdot \nabla_x \mathbf{P}$  and bound the  $L^6$  norm of  $P f_R$  by the dissipation and the  $L^2$  bound of  $\partial_t f_R$ . Now the temporal derivative of  $f_R$  comes into play and we need to include the estimates of  $\partial_t f_R$  in a

parallel way to those of  $f_R$ . The  $L_{t,x}^\infty$  of  $f_R$  and the  $L_t^2 L_x^\infty$  of  $\partial_t f_R$  will be controlled using Duhamel's formula and analyzing the particle's interaction with the diffuse boundary condition. Note that the space  $L_t^2 L_x^\infty$  for  $\partial_t f_R$  is needed as the forcing term  $\nabla_x \partial_t^2 u$  processes an initial-boundary layer. Now, the derivatives of the fluid part will need to be considered in order to close the estimate. From Theorem 8, the control of the kinetic remainder requires the control of fluid parts as in (2.193), (2.194). This part is explored in Section 3.

We point out one major difference between our estimate for  $f_R$  and the corresponding one in [30]. Here in (1.18) we expand around the *global Maxwellian*  $\mu$  and have two correction terms  $f_1, f_2$ ; while in [30] the expansion was performed around the local Maxwellian

$$\tilde{\mu} = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v - \varepsilon u(t,x)|^2}{2}} \text{ as}$$

$$F = \tilde{\mu} + \varepsilon^2 f_2 \sqrt{\tilde{\mu}} + \varepsilon^{3/2} f_R \sqrt{\tilde{\mu}},$$

where  $u(t, x)$  is the solution of the Navier-Stokes equation, and the expansion only have one correction term. Now, in the presence of heat transfer  $\theta$ , such an expansion around local Maxwellian would lead to an undesirable propagation of  $|v|^3$  in the estimates. Which makes the remainder analysis more complicated and harder.

In both cases, effective cancellations on the singular terms are obtained by making the correct choice of the correction terms. One benefit in our global Maxwellian setting is that whenever the derivatives  $\partial_t, \nabla_x$  hit  $\mu$ , it vanishes. This reduces the amount of computation and makes the estimates more straightforward compared to the local Maxwellian setting. However, such a setting also brings an extra singular term  $\frac{2}{\varepsilon \kappa} \Gamma(f_1, f_R)$  in the equation for  $f_R$  in (1.22), which can be compared to the term  $\frac{(\partial_t + \varepsilon^{-1} v \cdot \nabla_x) \sqrt{\tilde{\mu}}}{\sqrt{\tilde{\mu}}} f_R$  in [30]. This singular term will result in a bound

$$\frac{1}{\kappa} \int_0^t \|P f_R(s)\|_{L_x^2}^2 ds$$

in the  $L^2$  estimate of  $f_R$ , which would essentially give rise to the growth of  $e^{\frac{1}{\kappa}}$  by the Gronwall's inequality. Fortunately, the expansion leaves enough room so that we can find a range of  $\varepsilon$  in terms of  $\kappa$  in a scale of large Reynolds number to absorb the growth.

In Section 3 we present the inviscid limit of the Navier-Stokes-Fourier system (1.12)-(1.17). It is well known that due to the mismatch of the boundary conditions (1.9) and (1.14), boundary layers will form. In [53,54] the famous Prandtl expansion was introduced to justify the limit with analytic initial data. In a recent work [56], the authors proved the inviscid limit using a new Green's function approach based on the boundary vorticity formulation established in [48] (See also [43]). In this paper, we employ the same Green's function approach using the vorticity formulation (3.2)-(3.6) for the Navier-Stokes-Fourier system. As mentioned above, temporal derivatives of the solution to the fluid equations also need to be controlled. To this end, we follow the same strategy in [30] by setting the compatibility as in (3.25) and deriving a similar integral representation formula for  $\partial_t \omega$  without any initial layer.

It turns out the equation for the heat flow  $\theta$  resembles the one for the vertical component of the vorticity  $\omega_3$ . Using Green's formula (3.60) we can establish the analytic bounds (3.27) for  $\theta$  in the same way for  $\omega_3$ . However, this does not fulfill our need since the conormal derivative in the analytic norm (3.16) allows a  $\frac{1}{x_3}$  singularity of the normal derivative of  $\theta$  in the boundary layer,

and thus prevents us from getting the  $L^\infty$  bound. In order to tackle the issue we design a similar, yet different  $\|\cdot\|_z$  norm in (3.17) to directly target the normal derivative of  $\theta$  and its temporal derivatives. Taking  $\partial_{x_3}$  derivative directly to (3.60) and using integration by parts we derive the integral formula for  $\partial_{x_3}\theta$ . From there we carefully bound the bilinear integrand and applying the analytic recovery lemma to obtain the desired bound in (3.28). Note that this estimate leads to the highest singularity as

$$\|\nabla_x^2 \theta\|_{L_{t,x}^\infty} \sim \frac{1}{\kappa},$$

which is of the same order as that of  $\nabla_x^2 u$ . The results of these estimates are summarized in Theorem 9. Because of our choice of  $\varepsilon$  with respect to  $\kappa$ , such growth is affordable in the Hilbert expansion. Therefore, combining Theorem 8 and Theorem 9, we prove the main result Theorem 10 in section 4. This concludes the rigorous justification of the incompressible Euler limit with heat flow.

## 2. Boltzmann estimate

### 2.1. Hilbert expansion

In this section we prove the following proposition:

**Proposition 2.** Suppose that  $F$  of (1.18), solves (1.1) and (1.11) with (1.5), (1.6), and that  $(\rho, u, \theta)$  solves (1.12)–(1.17). We choose  $f_1$  and  $f_2$  as

$$f_1 = (\rho + u \cdot v + \theta \frac{|v|^2 - 3}{2})\sqrt{\mu}, \quad (2.1)$$

$$(\mathbf{I} - \mathbf{P})f_2 = L^{-1}(-\kappa v \cdot \nabla_x f_1 + \Gamma(f_1, f_1)), \quad (2.2)$$

$$\mathbf{P}f_2 = (\rho_2 + u_2 \cdot v + \theta_2 \frac{|v|^2 - 3}{2})\sqrt{\mu}, \quad (2.3)$$

where  $\rho_2, u_2$ , and  $\theta_2$  satisfy

$$\begin{aligned} \nabla \cdot u_2 &= -\partial_t \rho, \\ \nabla_x \left( \rho_2 + \theta_2 - \frac{1}{3}|u|^2 \right) &= \nabla_x p. \end{aligned} \quad (2.4)$$

Then  $f_R$  in (1.18) satisfies

$$\begin{aligned} \left[ \partial_t + \frac{1}{\varepsilon} v \cdot \nabla_x + \frac{1}{\varepsilon^2 \kappa} L \right] f_R &= -\frac{1}{\varepsilon^{1/2}} \left\{ (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) - \frac{2}{\kappa} \Gamma(f_1, f_2) \right\} \\ &\quad + \varepsilon^{1/2} \left( -\partial_t f_2 + \frac{1}{\kappa} \Gamma(f_2, f_2) \right) + \frac{2}{\varepsilon \kappa} \Gamma(f_1 + \varepsilon f_2, f_R) + \frac{1}{\varepsilon^{1/2} \kappa} \Gamma(f_R, f_R), \end{aligned} \quad (2.5)$$

and  $\partial_t f_R$  satisfies

$$\begin{aligned}
& \left[ \partial_t + \frac{1}{\varepsilon} v \cdot \nabla_x + \frac{1}{\varepsilon^2 \kappa} L \right] \partial_t f_R \\
&= -\frac{1}{\varepsilon^{1/2}} \left\{ (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x \partial_t f_2) - \frac{2}{\kappa} \Gamma(\partial_t f_1, f_2) - \frac{2}{\kappa} \Gamma(f_1, \partial_t f_2) \right\} \\
&\quad + \varepsilon^{1/2} \left( -\partial_t^2 f_2 + \frac{2}{\kappa} \Gamma(\partial_t f_2, f_2) \right) \\
&\quad + \frac{2}{\varepsilon \kappa} (\Gamma(\partial_t f_1 + \varepsilon \partial_t f_2, f_R) + \Gamma(f_1 + \varepsilon f_2, \partial_t f_R)) \\
&\quad + \frac{2}{\varepsilon^{1/2} \kappa} \Gamma(\partial_t f_R, f_R).
\end{aligned} \tag{2.6}$$

**Remark 1.** There is some freedom to choose  $u_2, \rho_2, \theta_2$  satisfying (2.4). One possible choice could be

$$u_2(x_h, x_3) = \begin{bmatrix} 0 \\ 0 \\ \int_0^{x_3} \partial_t \theta(x_h, y) dy \end{bmatrix}, \quad \rho_2 = \frac{1}{3} |u|^2 + p, \quad \theta_2 = 0. \tag{2.7}$$

**Proof.** To show (2.5), it suffices to prove (1.19) = 0, (1.20) = 0 and  $\mathbf{P}(1.21) = 0$  from the remainder equation (1.19)-(1.22), and (2.6) follows from a direct differentiation of (2.5).

From our choice (2.1), we have  $(\mathbf{I} - \mathbf{P})f_1 = 0$ , therefore (1.19) = 0. Next, let's show (2.2) is well-defined, which implies (1.20) = 0. By the Fredholm alternative, the inverse operator of  $L$  maps

$$L^{-1} : \mathcal{N}^\perp \rightarrow \mathcal{N}^\perp, \text{ where } \mathcal{N}^\perp \text{ stands an } L_v^2 - \text{orthogonal complement of } \mathcal{N}.$$

Hence in order to show (2.2) is well-defined, all we need is to show

$$\mathbf{P}(-\kappa v \cdot \nabla_x f_1 + \Gamma(f_1, f_1)) = -\kappa \mathbf{P}(v \cdot \nabla_x f_1) = 0,$$

or equivalently

$$\left\langle v \cdot \nabla_x f_1, \left[ 1, v, \frac{|v|^2}{2} \right] \sqrt{\mu} \right\rangle = 0. \tag{2.8}$$

By direction computation, from the oddness in the  $v$ -integral and the incompressible condition (1.13) we have

$$\begin{aligned}
\langle v \cdot \nabla_x f_1, \sqrt{\mu} \rangle &= \langle v \cdot \nabla_x \rho, \mu \rangle + \langle v \cdot \nabla_x u \cdot v, \mu \rangle + \langle v \cdot \nabla_x \theta, \frac{|v|^2 - 3}{2} \mu \rangle \\
&= \sum_{i=1}^3 \partial_i u^i \left( \int_{\mathbb{R}^3} v_i^2 \mu dv \right) = \nabla \cdot u = 0.
\end{aligned} \tag{2.9}$$

Next, for fixed  $i = 1, 2, 3$ , again from the oddness in the  $v$ -integral and (1.17) we have

$$\begin{aligned}
\langle v \cdot \nabla_x f_1, v_i \sqrt{\mu} \rangle &= \langle v \cdot \nabla_x \rho, v_i \mu \rangle + \langle v \cdot \nabla_x u \cdot v, v_i \mu \rangle + \langle v \cdot \nabla_x \theta, \frac{|v|^2 - 3}{2} v_i \mu \rangle \\
&= \partial_i \rho \left( \int_{\mathbb{R}^3} v_i^2 \mu \, dv \right) + \partial_i \theta \left( \int_{\mathbb{R}^3} \frac{|v|^2 - 3}{2} v_i^2 \mu \, dv \right) = \partial_i (\rho + \theta) = 0.
\end{aligned} \tag{2.10}$$

Finally similar to (2.9), we have

$$\begin{aligned}
\left\langle v \cdot \nabla_x f_1, \frac{|v|^2}{2} \sqrt{\mu} \right\rangle &= \left\langle v \cdot \nabla_x \rho, \frac{|v|^2}{2} \mu \right\rangle + \left\langle v \cdot \nabla_x u \cdot v, \frac{|v|^2}{2} \mu \right\rangle + \left\langle v \cdot \nabla_x \theta, \frac{|v|^2 - 3}{2} \frac{|v|^2}{2} \mu \right\rangle \\
&= \sum_{i=1}^3 \partial_i u^i \left( \int_{\mathbb{R}^3} v_i^2 \frac{|v|^2}{2} \mu \, dv \right) = \frac{5}{2} \nabla \cdot u = 0.
\end{aligned} \tag{2.11}$$

Therefore from (2.9), (2.10), and (2.11) we prove (2.8).

The only thing left to get (2.5) is then to prove  $\mathbf{P}(1.21) = 0$ . It suffices to prove

$$\left\langle \partial_t f_1 + v \cdot \nabla_x f_2, \left[ 1, v, \frac{|v|^2}{2} \right] \sqrt{\mu} \right\rangle = 0. \tag{2.12}$$

This is equivalent to

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot u_2 = 0, \\
&\partial_t u + \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}) f_2, v \sqrt{\mu} \rangle + \nabla_x (\rho_2 + \theta_2) = 0, \\
&\frac{3}{2} \partial_t (\rho + \theta) + \frac{1}{2} \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}) f_2, |v|^2 \sqrt{\mu} \rangle + \frac{5}{2} \nabla \cdot u_2 = 0.
\end{aligned} \tag{2.13}$$

Let's consider the second equation of (2.13). Using (2.2) we compute

$$\begin{aligned}
\langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}) f_2, v \sqrt{\mu} \rangle &= \langle v \cdot \nabla_x L^{-1} (-\kappa v \cdot \nabla_x f_1 + \Gamma(f_1, f_1)), v \sqrt{\mu} \rangle \\
&= \underbrace{\langle v \cdot \nabla_x L^{-1} (-\kappa v \cdot \nabla_x f_1), v \sqrt{\mu} \rangle}_{(2.14)_1} + \underbrace{\langle v \cdot \nabla_x L^{-1} (\Gamma(f_1, f_1)), v \sqrt{\mu} \rangle}_{(2.14)_2}.
\end{aligned} \tag{2.14}$$

We first consider (2.14)<sub>1</sub>. It suffices to compute for fixed  $i$ ,

$$\sum_j \left\langle v_i v_j \partial_j L^{-1} (-\kappa v \cdot \nabla_x f_1), \sqrt{\mu} \right\rangle. \tag{2.15}$$

Since  $(v_i v_j - \frac{|v|^2}{3} \delta_{ij}) \sqrt{\mu} \in \mathcal{N}^\perp$ . Define

$$A_{ij}(v) := L^{-1} \left( (v_i v_j - \frac{|v|^2}{3} \delta_{ij}) \sqrt{\mu} \right) (v). \tag{2.16}$$

By splitting  $v_i v_j = (v_i v_j - \frac{1}{3}|v|^2 \delta_{ij}) + \frac{1}{3}|v|^2 \delta_{ij}$  in (2.15), and using the self-adjointness of  $L^{-1}$  we have

$$\begin{aligned}
 (2.15) &= -\kappa \sum_j \left\langle L^{-1} \left( \sum_k v_k \partial_{jk} f_1 \right), \left( v_i v_j - \frac{1}{3}|v|^2 \delta_{ij} \right) \sqrt{\mu} \right\rangle \\
 &\quad + \partial_i \left( \frac{\kappa}{3} \left\langle L^{-1} (-v \cdot \nabla_x f_1), |v|^2 \sqrt{\mu} \right\rangle \right) \\
 &= -\kappa \sum_j \left\langle L^{-1} \left( \sum_k v_k \partial_{jk} f_1 \right), \left( v_i v_j - \frac{1}{3}|v|^2 \delta_{ij} \right) \sqrt{\mu} \right\rangle \\
 &= -\kappa \sum_{j,k,l} \partial_{jk} u^\ell \langle v_k v_l \sqrt{\mu}, A_{ij} \rangle = -\kappa \sum_{j,k,l} \partial_{jk} u^\ell \langle L A_{kl}, A_{ij} \rangle,
 \end{aligned}$$

where the  $\theta$  and  $\rho$  terms vanish because of the oddness in the  $v$ -integration. Next by the computation (Lemma 4.4 in [2]), the above term is equal to

$$\begin{aligned}
 -\kappa \sum_{j,k,l} \langle A_{ij}, L A_{kl} \rangle \partial_{jk} u^l &= -\kappa \eta_v \sum_{j,k,l} \left( (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \partial_{jk} u^l \\
 &= -\kappa \eta_v \{ \Delta u^i - \partial_i \nabla \cdot u - \frac{2}{3} \partial_i \nabla \cdot u \} = -\kappa \eta_v \Delta u^i \quad \text{for } i = 1, 2, 3.
 \end{aligned} \tag{2.17}$$

Here we have used the incompressible condition (1.12). Therefore we get

$$(2.14)_1 = -\kappa \eta_v \Delta u. \tag{2.18}$$

Next we consider (2.14)<sub>2</sub>. First by direct computation we have

$$\begin{aligned}
 &\Gamma(f_1, f_1) - L \left( \frac{f_1^2}{2\sqrt{\mu}} \right) \\
 &= \frac{1}{\sqrt{\mu}} \left( Q(\sqrt{\mu} f_1, \sqrt{\mu} f_1) + Q(\mu, f_1^2) \right) \\
 &= \frac{1}{2\sqrt{\mu}} \iint |(v - v_*) \cdot u| \left( 2\sqrt{\mu}(v') f_1(v') \sqrt{\mu}(v'_*) f_1(v'_*) - 2\sqrt{\mu}(v) f_1(v) \sqrt{\mu}(v_*) f_1(v_*) \right) dudv_* \\
 &\quad + \frac{1}{2\sqrt{\mu}} \iint |(v - v_*) \cdot u| \left( \mu(v') f_1^2(v'_*) + \mu(v'_*) f_1^2(v') - \mu(v) f_1^2(v_*) - \mu(v_*) f_1^2(v) \right) dudv_* \\
 &= \frac{1}{2\sqrt{\mu}} \iint |(v - v_*) \cdot u| \left( \sqrt{\mu}(v') f_1(v'_*) + \sqrt{\mu}(v'_*) f_1(v') - \sqrt{\mu}(v) f_1(v_*) - \sqrt{\mu}(v_*) f_1(v) \right) \\
 &\quad \times \left( \sqrt{\mu}(v') f_1(v'_*) + \sqrt{\mu}(v'_*) f_1(v') + \sqrt{\mu}(v) f_1(v_*) + \sqrt{\mu}(v_*) f_1(v) \right) dudv_* \\
 &= \frac{1}{2} \iint \sqrt{\mu}(v_*) |(v - v_*) \cdot u| \left( \frac{f_1(v'_*)}{\sqrt{\mu}(v'_*)} + \frac{f_1(v')}{\sqrt{\mu}(v')} - \frac{f_1(v_*)}{\sqrt{\mu}(v_*)} - \frac{f_1(v)}{\sqrt{\mu}(v)} \right)
 \end{aligned}$$

$$\times (\sqrt{\mu}(v')f_1(v'_*) + \sqrt{\mu}(v'_*)f_1(v') + \sqrt{\mu}(v)f_1(v_*) + \sqrt{\mu}(v_*)f_1(v)) dudv_*.$$

And since  $f_1 \in \mathcal{N}$ ,  $\frac{f_1(v'_*)}{\sqrt{\mu}(v'_*)} + \frac{f_1(v')}{\sqrt{\mu}(v')} - \frac{f_1(v_*)}{\sqrt{\mu}(v_*)} - \frac{f_1(v)}{\sqrt{\mu}(v)} = 0$ . Thus

$$\Gamma(f_1, f_1) = L \left( \frac{f_1^2}{2\sqrt{\mu}} \right), \text{ therefore } L^{-1}(\Gamma(f_1, f_1)) = (\mathbf{I} - \mathbf{P}) \left( \frac{f_1^2}{2\sqrt{\mu}} \right). \quad (2.19)$$

From (2.1),

$$(\mathbf{I} - \mathbf{P}) \left( \frac{f_1^2}{2\sqrt{\mu}} \right) = \frac{1}{2}(\mathbf{I} - \mathbf{P}) \left( \rho + u \cdot v + \theta \frac{|v|^2 - 3}{2} \right)^2 \sqrt{\mu} = \frac{1}{2}(\mathbf{I} - \mathbf{P}) \left( u \cdot v + \theta \frac{|v|^2 - 3}{2} \right)^2 \sqrt{\mu}. \quad (2.20)$$

Now for fixed  $i$ , from (2.19), (2.20), and using  $|v|^2 \sqrt{\mu} \in \mathcal{N}$ ,

$$\begin{aligned} & \langle v \cdot \nabla_x L^{-1}(\Gamma(f_1, f_1)), v_i \sqrt{\mu} \rangle \\ &= \frac{1}{2} \sum_j \left\langle \partial_j \left( (\mathbf{I} - \mathbf{P}) \left( u \cdot v + \theta \frac{|v|^2 - 3}{2} \right)^2 \sqrt{\mu} \right), v_j v_i \sqrt{\mu} \right\rangle \\ &= \frac{1}{2} \sum_j \left\langle \partial_j \left( (\mathbf{I} - \mathbf{P}) \left( u \cdot v + \theta \frac{|v|^2 - 3}{2} \right)^2 \sqrt{\mu} \right), (v_j v_i - \frac{1}{3} \delta_{ij} |v|^2) \sqrt{\mu} \right\rangle. \end{aligned} \quad (2.21)$$

Now using  $(v_j v_i - \frac{1}{3} \delta_{ij} |v|^2) \sqrt{\mu} \in \mathcal{N}^\perp$ , we have

$$\begin{aligned} & \langle v \cdot \nabla_x L^{-1}(\Gamma(f_1, f_1)), v_i \sqrt{\mu} \rangle \\ &= \frac{1}{2} \sum_j \left\langle \partial_j \left( (u \cdot v + \theta \frac{|v|^2 - 3}{2})^2 \sqrt{\mu} \right), (v_j v_i - \frac{1}{3} \delta_{ij} |v|^2) \sqrt{\mu} \right\rangle \\ &= \frac{1}{2} \sum_j \underbrace{\left\langle \partial_j \left( (u \cdot v)^2 \sqrt{\mu} \right), (v_j v_i - \frac{1}{3} \delta_{ij} |v|^2) \sqrt{\mu} \right\rangle}_{(2.22)_1} \\ & \quad + \frac{1}{2} \sum_j \underbrace{\left\langle \partial_j \left( \theta^2 \left( \frac{|v|^2 - 3}{2} \right)^2 \sqrt{\mu} \right), (v_j v_i - \frac{1}{3} \delta_{ij} |v|^2) \sqrt{\mu} \right\rangle}_{(2.22)_2} \\ & \quad + \underbrace{\sum_j \left\langle (u \cdot v) \theta (|v|^2 - 3) \sqrt{\mu}, (v_j v_i - \frac{1}{3} \delta_{ij} |v|^2) \sqrt{\mu} \right\rangle}_{(2.22)_3}. \end{aligned} \quad (2.22)$$

First we have  $(2.22)_3 = 0$  from oddness of the integration. By direct computation

$$\begin{aligned}
 (2.22)_2 &= \frac{1}{2} \sum_j \left\langle \left( \frac{|v|^2 - 3}{2} \right)^2 \partial_j(\theta^2) \sqrt{\mu}, \left( v_j v_i - \frac{1}{3} \delta_{ij} |v|^2 \right) \sqrt{\mu} \right\rangle \\
 &= \frac{1}{2} \partial_i(\theta^2) \int \left( \frac{|v|^2 - 3}{2} \right)^2 \left( v_i^2 - \frac{1}{3} |v|^2 \right) \mu \, dv = 0.
 \end{aligned}$$

Lastly for  $(2.22)_1$ , we have from computation

$$\begin{aligned}
 (2.22)_1 &= \frac{1}{2} \int \left( \sum_{j,k,\ell} v_i v_j v_k v_\ell \partial_j(u^k u^\ell) \right) \mu \, dv - \frac{1}{6} \int \left( \sum_{k,\ell} \partial_i(u^k u^\ell) v_k v_\ell |v|^2 \right) \mu \, dv \\
 &= \frac{1}{2} \int \left( \sum_{j \neq i} 2v_i^2 v_j^2 \partial_j(u^i u^j) \right) \mu \, dv + \frac{1}{2} \int \left( \sum_k v_i^2 v_k^2 \partial_i(u^k)^2 \right) \mu \, dv \\
 &\quad - \frac{1}{6} \int \left( \sum_k v_k^2 |v|^2 \partial_i(u^k)^2 \right) \mu \, dv \\
 &= \sum_{j \neq i} \partial_j(u^i u^j) + \frac{3}{2} \partial_i(u^i)^2 + \frac{1}{2} \sum_{k \neq i} \partial_i(u^k)^2 - \frac{5}{6} \partial_i(|u|^2) \\
 &= \sum_j \partial_j(u^i u^j) + \frac{1}{2} \partial_i(|u|^2) - \frac{5}{6} \partial_i(|u|^2) = \sum_j [(\partial_j u^i) u^j] - \frac{1}{3} \partial_i(|u|^2),
 \end{aligned}$$

where we've used  $\nabla \cdot u = 0$ . Therefore we get

$$(2.14)_2 = u \cdot \nabla_x u - \frac{1}{3} \nabla_x(|u|^2).$$

Combining with (2.18) we get the second equation of (2.13) is equivalent to

$$\partial_t u + u \cdot \nabla_x u - \kappa v_v \Delta u + \nabla_x \left( \rho_2 + \theta_2 - \frac{1}{3} |u|^2 \right) = 0,$$

which is guaranteed by (2.4).

We now consider the third equation of (2.13) subtracting  $\frac{5}{2}$  of the first equation of (2.13):

$$\frac{5}{2} \partial_t \theta + \frac{1}{2} \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}) f_2, |v|^2 \sqrt{\mu} \rangle = 0. \quad (2.23)$$

Using (2.2) we compute

$$\begin{aligned}
 \frac{1}{2} \langle v \cdot \nabla_x (\mathbf{I} - \mathbf{P}) f_2, |v|^2 \sqrt{\mu} \rangle &= \frac{1}{2} \langle v \cdot \nabla_x L^{-1} (-\kappa v \cdot \nabla_x f_1 + \Gamma(f_1, f_1)), |v|^2 \sqrt{\mu} \rangle \\
 &= \underbrace{\frac{1}{2} \langle v \cdot \nabla_x L^{-1} (-\kappa v \cdot \nabla_x f_1), |v|^2 \sqrt{\mu} \rangle}_{(2.24)_1} + \underbrace{\frac{1}{2} \langle v \cdot \nabla_x L^{-1} (\Gamma(f_1, f_1)), |v|^2 \sqrt{\mu} \rangle}_{(2.24)_2}.
 \end{aligned} \quad (2.24)$$

Since  $(\frac{|v|^2-5}{2})v_i\sqrt{\mu} \in \mathcal{N}^\perp$ . Define

$$B_i = L^{-1} \left( \left( \frac{|v|^2-5}{2} \right) v_i \sqrt{\mu} \right).$$

By the self-adjointness of  $L^{-1}$ , we compute

$$\begin{aligned} (2.24)_1 &= -\frac{\kappa}{2} \sum_{j,k} \left\langle \partial_j L^{-1}(v_k \partial_k f_1), \frac{v_j |v|^2}{2} \sqrt{\mu} \right\rangle \\ &= -\frac{\kappa}{2} \sum_{j,k} \left\langle v_k \partial_{jk} \left( \rho + v \cdot u + \frac{|v|^2-3}{2} \theta \right), B_j \right\rangle \\ &= \underbrace{-\frac{\kappa}{2} \sum_{j,k} \langle v_k \partial_{jk} (v \cdot u) \sqrt{\mu}, B_j \rangle}_{(2.25)_1} - \underbrace{\frac{\kappa}{2} \sum_{j,k} \left\langle v_k \left( \frac{|v|^2-5}{2} \partial_{jk} \theta \right) \sqrt{\mu}, B_j \right\rangle}_{(2.25)_2} \\ &\quad - \underbrace{\frac{\kappa}{2} \sum_{j,k} \langle v_k \partial_{jk} (\rho + \theta) \sqrt{\mu}, B_j \rangle}_{(2.25)_3}. \end{aligned} \quad (2.25)$$

From [19],  $B_j$  is odd in  $v_j$ , thus  $(2.25)_1 = 0$ . Also  $(2.25)_3 = 0$  as  $\rho + \theta = \text{constant}$ . Now by lemma 4.4 in [2], we have for some constant  $\eta_c$ ,

$$\langle L B_k, B_j \rangle = 5\eta_c \delta_{kj}. \quad (2.26)$$

Thus

$$(2.25)_2 = \frac{\kappa}{2} \sum_{j,k} \left\langle v_k \left( \frac{|v|^2-5}{2} \partial_{jk} \theta \right) \sqrt{\mu}, B_j \right\rangle = \frac{\kappa}{2} \sum_{j,k} \langle L B_k, B_j \rangle \partial_{jk} \theta = \frac{5}{2} \kappa \eta_c \Delta \theta.$$

Finally we compute  $\frac{1}{2} \langle v \cdot \nabla_x L^{-1}(\Gamma(f_1, f_1)), |v|^2 \sqrt{\mu} \rangle$ . From (2.19), (2.20) and using  $v_j(|v|^2 - 5)\sqrt{\mu} \in \mathcal{N}^\perp$ ,

$$\begin{aligned} &\frac{1}{2} \langle v \cdot \nabla_x L^{-1}(\Gamma(f_1, f_1)), |v|^2 \sqrt{\mu} \rangle \\ &= \frac{1}{4} \sum_j \left\langle \partial_j \left( (u \cdot v + \theta \frac{|v|^2-3}{2})^2 \sqrt{\mu} \right), v_j (|v|^2 - 5) \sqrt{\mu} \right\rangle \\ &= \frac{1}{4} \sum_j \left\langle \partial_j \left( (u \cdot v) \theta (|v|^2 - 3) \sqrt{\mu} \right), v_j (|v|^2 - 5) \sqrt{\mu} \right\rangle \\ &= \frac{1}{4} \sum_j \partial_j (u^j \theta) \int \left( (|v|^2 - 3) v_j^2 (|v|^2 - 5) \mu \right) dv \end{aligned}$$

$$= \frac{5}{2} u \cdot \nabla \theta.$$

Therefore (2.23) gives

$$\frac{5}{2} (\partial_t \theta + u \cdot \nabla \theta - \kappa \eta_c \Delta \theta) = 0,$$

which is guaranteed by (1.15).  $\square$

## 2.2. $L^2$ -energy estimate

We start with an  $L^2$ -energy estimate for  $f_R$  and  $\partial_t f_R$ . From (2.5), we define

$$\begin{aligned} \partial_t f_{R,0} = & -\frac{1}{\varepsilon} v \cdot \nabla_x f_{R,in} - \frac{1}{\varepsilon^2 \kappa} L f_{R,in} - \frac{1}{\varepsilon^{1/2}} \left\{ (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_{2,in}) - \frac{2}{\kappa} \Gamma(f_{1,in}, f_{2,in}) \right\} \\ & - \varepsilon^{1/2} \partial_t f_{2,in} + \frac{2}{\varepsilon \kappa} \Gamma(f_{1,in} + \varepsilon f_{2,in}, f_{R,in}) + \frac{\varepsilon^{1/2}}{\kappa} \Gamma(f_{2,in}, f_{2,in}) \\ & + \frac{1}{\varepsilon^{1/2} \kappa} \Gamma(f_{R,in}, f_{R,in}). \end{aligned} \quad (2.27)$$

Here  $\partial_t f_{2,in}$  is defined through solving the fluid equations for  $\partial_t$  part and evaluate at  $t = 0$ .

For the sake of notational simplicity, let's denote  $\mathcal{I}, \mathcal{I}_2 : [0, T] \times \Omega \rightarrow \mathbb{R}^5$ ,

$$\mathcal{I} = [\rho, u^1, u^2, u^3, \theta], \text{ and } \mathcal{I}_2 = [\rho_2, u_2^1, u_2^2, u_2^3, \theta_2]. \quad (2.28)$$

The following trace theorem is useful to control the boundary terms.

**Lemma 1** (Trace theorem).

$$\frac{1}{\varepsilon} \int_0^t \int_{\gamma_+^N} |h| d\gamma ds \lesssim_N \iint_{\Omega \times \mathbb{R}^3} |h(0)| + \int_0^t \iint_{\Omega \times \mathbb{R}^3} |h| + \int_0^t \iint_{\Omega \times \mathbb{R}^3} |\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h|, \quad (2.29)$$

where  $\gamma_+^N := \{(x, v) \in \gamma_+ : |n(x) \cdot v| > 1/N \text{ and } 1/N < |v| < N\}$ .

**Proof.** The proof is standard (for example see Lemma 3.2 in [14] or Lemma 7 in [8]).  $\square$

**Proposition 3.** Under the same assumptions in Proposition 2, we have

$$\begin{aligned}
& \|f_R(t)\|_2^2 + d_2 \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}^2 + \int_0^t |\varepsilon^{-\frac{1}{2}} f_R|_{L_\gamma^2}^2 \\
& \lesssim \|f_R(0)\|_{L_{x,v}^2}^2 + \frac{1}{\kappa} (1 + \|\mathcal{I}\|_{L_{t,x}^\infty}^2) \int_0^t \|Pf_R(s)\|_{L_x^2}^2 ds \\
& \quad + \frac{\varepsilon}{\kappa^3} \|\kappa^{1/2} Pf_R(s)\|_{L_t^\infty L_x^6}^2 \|\kappa^{1/2} Pf_R\|_{L_t^2 L_x^3}^2 \\
& \quad + \varepsilon \kappa \|(2.50)\|_{L_{t,x,v}^2}^2 + \frac{\varepsilon}{\kappa} \|\kappa(2.51)\|_{L_{t,x,v}^2}^2 + \|(2.35)\|_{L_t^2 L^2(\partial\Omega)},
\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
d_2 &:= \frac{\sigma_0}{2} - \varepsilon^{3/2} \|\mathfrak{w} f_R\|_{L_{t,x,v}^\infty} - \left( \varepsilon^2 \|(2.36)\|_{L_{t,x}^\infty} + \varepsilon \|\nabla_x \mathcal{I}\|_{L_{t,x}^\infty} + \frac{\varepsilon^2}{\kappa} \|(2.36)\|_{L_{t,x}^\infty}^2 \right) \\
& \quad - \kappa^{1/2} \varepsilon^{1/2} \|(2.46)\|_{L_{t,x}^2} - \frac{\varepsilon^{1/2}}{\kappa^{1/2}} \|\kappa(2.47)\|_{L_{t,x}^2} - \kappa^{1/2} \varepsilon^{3/2} \|(2.50)\|_{L_{t,x,v}^2} \\
& \quad - \frac{\varepsilon^{3/2}}{\kappa^{1/2}} \|\kappa(2.51)\|_{L_{t,x,v}^2}.
\end{aligned} \tag{2.31}$$

We also have

$$\begin{aligned}
& \|\partial_t f_R(t)\|_{L_{x,v}^2}^2 + d_{2,t} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})\partial_t f_R\|_{L_{t,x,v}^2}^2 + |\varepsilon^{-\frac{1}{2}} \partial_t f_R|_{L_t^2 L_\gamma^2}^2 \\
& \lesssim \|\partial_t f_R(0)\|_{L_{x,v}^2}^2 + \varepsilon \kappa \|(2.68)\|_{L_{t,x,v}^2}^2 + \frac{\varepsilon}{\kappa} \|\kappa(2.69)\|_{L_{t,x,v}^2}^2 \\
& \quad + \frac{\varepsilon}{\kappa^3} \|\kappa^{1/2} Pf_R(s)\|_{L_t^\infty L_x^6}^2 \|\kappa^{1/2} P\partial_t f_R\|_{L_t^2 L_x^3}^2 + \frac{1}{\kappa} \int_0^t \|P\partial_t f_R(s)\|_{L_x^2}^2 ds \\
& \quad + \left\{ 1 + \frac{1}{\kappa} \left( \|\mathcal{I}\|_{L_{t,x}^\infty}^2 + \|\partial_t \mathcal{I}\|_{L_{t,x}^\infty}^2 \right) \right\} \times \int_0^t \|Pf_R(s)\|_{L_x^2}^2 ds \\
& \quad + \left\{ 1 + \varepsilon^2 \|\sqrt{v} \partial_t f_1\|_{L_{t,x,v}^\infty}^2 \right\} \times \|\varepsilon^{-1} \kappa^{-1/2} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}^2,
\end{aligned} \tag{2.32}$$

where  $\partial_t f_R(0, x, v) := f_{R,t}(0, x, v)$  is defined in (2.27). Here

$$\begin{aligned}
d_{2,t} &:= \frac{\sigma_0}{2} - \frac{\varepsilon^2}{\kappa} \|(2.49)\|_{L_{t,x,v}^\infty}^2 - \frac{\varepsilon^2}{\kappa} \|\mathcal{I}_2\|_{L_{t,x}^\infty} - \frac{\varepsilon^2}{\kappa} \|(\mathbf{I} - \mathbf{P})f_2\|_{L_{t,x}^\infty} \\
& \quad - \varepsilon \|\mathcal{I}\|_{L_{t,x}^\infty} - \varepsilon^{3/2} \|\mathfrak{w} f_R\|_{L_{t,x,v}^\infty} \\
& \quad - \kappa^{1/2} \varepsilon^{1/2} \|(2.65)\|_{L_{t,x}^2} - \frac{\varepsilon^{1/2}}{\kappa^{1/2}} \|\kappa(2.66)\|_{L_{t,x}^2} - \kappa^{1/2} \varepsilon^{3/2} \|(2.68)\|_{L_{t,x,v}^2} \\
& \quad - \frac{\varepsilon^{3/2}}{\kappa^{1/2}} \|\kappa(2.69)\|_{L_{t,x,v}^2}
\end{aligned} \tag{2.33}$$

**Proof.** We fix a  $0 < \varrho < \frac{1}{4}$ . Define

$$\mathfrak{w}_\varrho(x, v) = \mathfrak{w} := e^{\varrho|v|^2}. \quad (2.34)$$

We will use both  $\mathfrak{w}_\varrho$  and  $\mathfrak{w}$  equivalently.

From (2.3) we have  $|\mathbf{P}f_2| \lesssim e^{-\varrho|v|^2}|\mathcal{I}_2|$ . And from (2.2), by [1] we have

$$\begin{aligned} |(\mathbf{I} - \mathbf{P})f_2| &= |L^{-1}(-\kappa v \cdot \nabla_x f_1 + \Gamma(f_1, f_1))| \\ &\lesssim \kappa e^{-\varrho|v|^2}|\nabla_x \mathcal{I}| + e^{-\varrho|v|^2}|\mathcal{I}|^2. \end{aligned} \quad (2.35)$$

Thus

$$|f_2| \lesssim e^{-\varrho|v|^2} \left( |\mathcal{I}|^2 + |\mathcal{I}_2| + \kappa |\nabla_x \mathcal{I}| \right). \quad (2.36)$$

An energy estimate to (2.5) and (1.31) reads as

$$\frac{1}{2} \|f_R(t)\|_{L_{x,v}^2}^2 - \frac{1}{2} \|f_R(0)\|_{L_{x,v}^2}^2 + \frac{1}{\kappa \varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^3} f_R L f_R \quad (2.37)$$

$$+ \frac{1}{2\varepsilon} \int_0^t \int_{\gamma_+} |f_R|^2 - \frac{1}{2\varepsilon} \int_0^t \int_{\gamma_-} |P_{\gamma_+} f_R - \varepsilon^{1/2}(1 - P_{\gamma_+})(\mathbf{I} - \mathbf{P})f_2|^2 \quad (2.38)$$

$$= \frac{1}{\kappa \varepsilon^{1/2}} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \Gamma(f_R, f_R)(\mathbf{I} - \mathbf{P})f_R \quad (2.39)$$

$$+ \frac{2}{\kappa \varepsilon} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \Gamma(f_1 + \varepsilon f_2, f_R)(\mathbf{I} - \mathbf{P})f_R \quad (2.40)$$

$$+ \frac{1}{\varepsilon^{1/2}} \int_0^t \iint_{\Omega \times \mathbb{R}^3} - \left\{ (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) - \frac{2}{\kappa} \Gamma(f_1, f_2) \right\} (\mathbf{I} - \mathbf{P})f_R \quad (2.41)$$

$$+ \varepsilon^{1/2} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \left( -\partial_t f_2 + \frac{1}{\kappa} \Gamma(f_2, f_2) \right) f_R \quad (2.42)$$

We start with (2.37). From the spectral gap estimate in (1.30), we have

$$(2.37) \geq \frac{1}{2} \|f_R(t)\|_{L_{x,v}^2}^2 - \frac{1}{2} \|f_R(0)\|_{L_{x,v}^2}^2 + \sigma_0 \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}^2. \quad (2.43)$$

Now we consider (2.39), in which we need integrability gain of  $\mathbf{P}f_R$  in  $L_x^6$  of the next sections. From decomposition  $f_R = \mathbf{P}f_R + (\mathbf{I} - \mathbf{P})f_R$  and  $\Gamma = \Gamma_+ - \Gamma_-$  in (1.25), we derive

$$\begin{aligned}
|(2.39)| &\lesssim \frac{1}{\kappa \varepsilon^{1/2}} \sum_{i=\pm} \int_0^t \iint_{\Omega \times \mathbb{R}^3} |v^{-\frac{1}{2}} \Gamma_i(|f_R|, (\mathbf{I} - \mathbf{P})f_R)| |\sqrt{v}(\mathbf{I} - \mathbf{P})f_R| \\
&\quad + \frac{\delta}{\kappa \varepsilon^{1/2}} \sum_{i=\pm} \int_0^t \iint_{\Omega \times \mathbb{R}^3} |v^{-\frac{1}{2}} \Gamma_i(|\mathbf{P}f_R|, |\mathbf{P}f_R|)| |\sqrt{v}(\mathbf{I} - \mathbf{P})f_R| \\
&\lesssim \varepsilon^{3/2} \|\mathfrak{w} f_R\|_{L_{x,v}^\infty} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}^2 \\
&\quad + \frac{\varepsilon^{1/2}}{\kappa^{3/2}} \|\kappa^{1/2} P f_R\|_{L_t^\infty L_x^6} \|\kappa^{1/2} P f_R\|_{L_t^2 L_x^3} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}.
\end{aligned} \tag{2.44}$$

Next we consider (2.40), we have

$$\begin{aligned}
|(2.40)| &\leq \frac{1}{\kappa^{1/2}} \left( \varepsilon \| (2.36) \|_{L_{t,x}^\infty} + \|\mathcal{I}\|_{L_{t,x}^\infty} \right) \{ \|P f_R\|_{L_{t,x}^2} + \kappa^{\frac{1}{2}} \varepsilon \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2} \} \\
&\quad \times \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2} \\
&\lesssim \{ \varepsilon^2 \| (2.36) \|_{L_{t,x}^\infty} + \varepsilon \|\mathcal{I}\|_{L_{t,x}^\infty} + \frac{\varepsilon^2}{\kappa} \| (2.36) \|_{L_{t,x}^\infty}^2 \} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}^2 \\
&\quad + \frac{\sigma_0}{10} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}^2 + \frac{10}{\sigma_0} \frac{1}{\kappa} \|\mathcal{I}\|_{L_{t,x}^\infty}^2 \|P f_R\|_{L_t^2 L_x^2}^2.
\end{aligned} \tag{2.45}$$

From (2.35) and (2.36)

$$\begin{aligned}
&\left| (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) - \frac{2}{\kappa} \Gamma(f_1, f_2) \right| \\
&\lesssim |(\mathbf{I} - \mathbf{P})(v \cdot \nabla_x \mathbf{P} f_2)| + \left| (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x L^{-1}(-\kappa v \cdot \nabla_x f_1 + \Gamma(f_1, f_1))) \right| + \frac{1}{\kappa} |\Gamma(f_1, f_2)| \\
&\lesssim e^{-\varrho|v|^2} |\nabla_x \mathcal{I}_2| + \kappa e^{-\varrho|v|^2} |\nabla_x^2 \mathcal{I}| + e^{-\varrho|v|^2} |\mathcal{I}|^2
\end{aligned} \tag{2.46}$$

$$+ \frac{1}{\kappa} e^{-\varrho|v|^2} |\mathcal{I}| \left( |\mathcal{I}|^2 + |\mathcal{I}_2| + \kappa |\nabla_x \mathcal{I}| \right), \tag{2.47}$$

we derive that

$$|(2.41)| \lesssim \left( \kappa^{1/2} \varepsilon^{1/2} \| (2.46) \|_{L_{t,x}^2} + \frac{\varepsilon^{1/2}}{\kappa^{1/2}} \|\kappa (2.47)\|_{L_{t,x}^2} \right) \|\kappa^{-1/2} \varepsilon^{-1} (\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2}. \tag{2.48}$$

Next we have from (2.19)

$$\begin{aligned}
\partial_t f_2 &= \partial_t \mathbf{P} f_2 + \partial_t (\mathbf{I} - \mathbf{P}) f_2 \\
&= (\partial_t \rho_2 + \partial_t u_2 \cdot v + \partial_t \theta_2 \frac{|v|^2 - 3}{2}) \sqrt{\mu} + L^{-1} (-\kappa v \cdot \nabla_x \partial_t f_1 + \Gamma(\partial_t f_1, f_1) + \Gamma(f_1, \partial_t f_1)) \\
&= (\partial_t \rho_2 + \partial_t u_2 \cdot v + \partial_t \theta_2 \frac{|v|^2 - 3}{2}) \sqrt{\mu} + L^{-1} (-\kappa v \cdot \nabla_x \partial_t f_1) + (\mathbf{I} - \mathbf{P}) \left( \frac{f_1 \cdot \partial_t f_1}{\sqrt{\mu}} \right).
\end{aligned}$$

So we can bound

$$|\partial_t f_2| \lesssim e^{-\varrho|v|^2} (|\partial_t \mathcal{I}_2| + \kappa |\nabla_x \partial_t \mathcal{I}|) + e^{-\varrho|v|^2} |\mathcal{I}| |\partial_t \mathcal{I}|. \quad (2.49)$$

Together with (2.36) we have

$$\left| -\partial_t f_2 + \frac{1}{\kappa} \Gamma(f_2, f_2) \right| \lesssim e^{-\varrho|v|^2} (|\partial_t \mathcal{I}_2| + \kappa |\nabla_x \partial_t \mathcal{I}| + |\mathcal{I}| |\partial_t \mathcal{I}|) \quad (2.50)$$

$$+ \frac{1}{\kappa} \left( e^{-\varrho|v|^2} (|\mathcal{I}|^2 + |\mathcal{I}_2| + \kappa |\nabla_x \mathcal{I}|) \right)^2 \quad (2.51)$$

Thus

$$\begin{aligned}
|(2.42)| &\lesssim \varepsilon \kappa \|(2.50)\|_{L_{t,x,v}^2}^2 + \frac{\varepsilon}{\kappa} \|\kappa(2.51)\|_{L_{t,x,v}^2}^2 + \frac{1}{\kappa} \|\mathbf{P} f_R\|_{L_{t,x,v}^2}^2 \\
&\quad + \left( \kappa^{1/2} \varepsilon^{3/2} \|(2.50)\|_{L_{t,x,v}^2} + \frac{\varepsilon^{3/2}}{\kappa^{1/2}} \|\kappa(2.51)\|_{L_{t,x,v}^2} \right) \|\kappa^{-1/2} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2} \\
&\quad (2.52)
\end{aligned}$$

Finally we control the boundary term (2.38) using a trace theorem (2.29). First we have, from (1.31),

$$\begin{aligned}
(2.38) &= \frac{1}{2\varepsilon} \int_0^t \int_{\gamma_+} \{|f_R|^2 - |P_{\gamma_+} f_R|^2\} - \frac{1}{2} \int_0^t \int_{\gamma_-} |(1 - P_{\gamma_+})(\mathbf{I} - \mathbf{P}) f_2|^2 \\
&\quad - \int_0^t \int_{\gamma_-} \frac{1}{\varepsilon^{1/2}} P_{\gamma_+} f_R (1 - P_{\gamma_+})(\mathbf{I} - \mathbf{P}) f_2 \\
&\quad (2.53) \\
&\geq \frac{1}{2} |\varepsilon^{-1/2} (1 - P_{\gamma_+}) f_R|_{L_t^2 L_{\gamma_+}^2}^2 - \frac{1}{8C} |\varepsilon^{-1/2} P_{\gamma_+} f_R|_{L_t^2 L_{\gamma_+}^2}^2 \\
&\quad - \left( \frac{1}{2} + 2C \right) \int_0^t \int_{\gamma_-} |(1 - P_{\gamma_+})(\mathbf{I} - \mathbf{P}) f_2|^2 \text{ for } C \gg 1,
\end{aligned}$$

where we have used the fact  $|P_{\gamma_+} f_R|_{L_{\gamma_+}^2} = |P_{\gamma_+} f_R|_{L_{\gamma_-}^2}$  from  $P_{\gamma_+} f_R(t, x, v)$  being a function of  $(t, x, |v|)$  due to  $u|_{\partial\Omega} = 0$ .

Now we estimate  $P_{\gamma_+} f_R$ . Since  $P_{\gamma_+}$  in (1.31) is a projection of  $c_\mu \sqrt{\mu}$  on  $\gamma_+$ , it follows  $\int_{\gamma_+} |P_{\gamma_+} f|^2 \leq 2 \int_{\gamma_+^N} |P_{\gamma_+} f|^2$  for large enough  $N > 0$ , where  $\gamma_+^N := \{(x, v) \in \gamma_+ : |n(x) \cdot v| > 1/N \text{ and } 1/N < |v| < N\}$ . Setting  $h = |f_R|^2$  in (2.29) and using (2.5) we derive

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^t \int_{\gamma_+^N} |f_R|^2 d\gamma ds &\leq C_N \iint_{\Omega \times \mathbb{R}^3} |f_R(0)|^2 + \int_0^t \iint_{\Omega \times \mathbb{R}^3} |f_R|^2 \\ &+ \int_0^t \iint_{\Omega \times \mathbb{R}^3} \left| \left[ -\frac{1}{\varepsilon^2 \kappa} L f_R + \frac{2}{\kappa \varepsilon} \Gamma(f_1 + \varepsilon f_2, f_R) + \frac{1}{\varepsilon^{1/2} \kappa} \Gamma(f_R, f_R) \right. \right. \\ &\quad \left. \left. - \frac{1}{\varepsilon^{1/2}} \left\{ (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) - \frac{2}{\kappa} \Gamma(f_1, f_2) \right\} + \varepsilon^{1/2} \left( -\partial_t f_2 + \frac{1}{\kappa} \Gamma(f_2, f_2) \right) \right] f_R \right| \\ &\leq C_N \left\{ \|f_R(0)\|_{L_{x,v}^2}^2 + \|\mathbf{P} f_R\|_{L_{t,x,v}^2}^2 + \|\varepsilon^{-1} \kappa^{-1/2} \sqrt{v} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2}^2 \right. \\ &\quad \left. + (2.44) + (2.45) + (2.48) + (2.52) \right\}. \end{aligned} \quad (2.54)$$

Furthermore from (1.31) and (2.54)

$$|f_R|_{L_t^2 L_{\gamma_-}^2}^2 \lesssim |f_R|_{L_t^2 L_{\gamma_+}^2}^2 + \varepsilon \|(1 - P_{\gamma_+})(\mathbf{I} - \mathbf{P}) f_2\|_{L_t^2 L_{\gamma_-}^2}^2 = |f_R|_{L_t^2 L_{\gamma_+}^2}^2 + \varepsilon \|(2.35)\|_{L_t^2 L^2(\partial\Omega)}^2. \quad (2.55)$$

Finally we collect the terms as

$$\begin{aligned} \text{r.h.s of (2.43) + (2.53)} &+ \frac{1}{4C} |\varepsilon^{-1/2} P_{\gamma_+} f_R|_{L_t^2 L_{\gamma_+}^2}^2 + \frac{\varepsilon^{-1}}{16C} |f_R|_{L_t^2 L_{\gamma_-}^2}^2 \\ &\leq \text{r.h.s of (2.44) + (2.45) + (2.48) + (2.52)} + \frac{1}{4C} \times \text{r.h.s of (2.54)} + \frac{\varepsilon^{-1}}{16C} \times \text{r.h.s of (2.55)}. \end{aligned}$$

We choose large  $N$  and then large  $C$  so that  $\frac{C_N}{4C} \ll \sigma_0$ . Using Young's inequality for products, and then moving contributions of  $\|\kappa^{-1/2} \varepsilon^{-1} \sqrt{v} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2}^2$  to l.h.s., we derive (2.30).

Next we prove (2.32). An energy estimate to (2.6) and (1.32) lead to (2.32)

$$\frac{1}{2} \|\partial_t f_R(t)\|_2^2 - \frac{1}{2} \|\partial_t f_R(0)\|_2^2 + \frac{1}{\kappa \varepsilon^2} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \partial_t f_R L \partial_t f_R \quad (2.56)$$

$$\begin{aligned} &+ \frac{1}{2\varepsilon} \int_0^t \int_{\gamma_+} |\partial_t f_R|^2 \\ &- \frac{1}{2\varepsilon} \int_0^t \int_{\gamma_-} |P_{\gamma_+} \partial_t f_R - \varepsilon^{1/2} (1 - P_{\gamma_+}) \partial_t (\mathbf{I} - \mathbf{P}) f_2|^2 \end{aligned} \quad (2.57)$$

$$= \frac{2}{\kappa \varepsilon^{1/2}} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \Gamma(\partial_t f_R, f_R)(\mathbf{I} - \mathbf{P}) \partial_t f_R \quad (2.58)$$

$$+ \frac{2}{\varepsilon \kappa} \int_0^t \iint_{\Omega \times \mathbb{R}^3} (\Gamma(\partial_t f_1 + \varepsilon \partial_t f_2, f_R) + \Gamma(f_1 + \varepsilon f_2, \partial_t f_R)) \partial_t f_R \quad (2.59)$$

$$+ \frac{1}{\varepsilon^{1/2}} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \left\{ -(\mathbf{I} - \mathbf{P})(v \cdot \nabla_x \partial_t f_2) + \frac{2}{\kappa} \Gamma(\partial_t f_1, f_2) + \frac{2}{\kappa} \Gamma(f_1, \partial_t f_2) \right\} (\mathbf{I} - \mathbf{P}) \partial_t f_R \quad (2.60)$$

$$+ \varepsilon^{1/2} \int_0^t \iint_{\Omega \times \mathbb{R}^3} \left( -\partial_t^2 f_2 + \frac{2}{\kappa} \Gamma(\partial_t f_2, f_2) \right) \partial_t f_R \quad (2.61)$$

We control the terms similarly as in the proof of (2.30):

$$(2.56) \geq \frac{1}{2} \|\partial_t f_R(t)\|_{L_{x,v}^2}^2 - \frac{1}{2} \|\partial_t f_R(0)\|_{L_{x,v}^2}^2 + \sigma_0 \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2}^2, \quad (2.62)$$

$$|(2.58)| \lesssim (\varepsilon^{3/2} \|\mathfrak{W} f_R\|_{L_{t,x,v}^\infty}) \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2}^2 \\ + \frac{\varepsilon^{1/2}}{\kappa^{3/2}} \|\kappa^{1/2} P f_R\|_{L_t^\infty L_x^6} \|\kappa^{1/2} P \partial_t f_R\|_{L_t^2 L_x^3} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2} \quad (2.63)$$

$$|(2.59)| \\ \lesssim \left( \kappa^{-\frac{1}{2}} \varepsilon \|\sqrt{v} \partial_t f_2\|_{L_{t,x,v}^\infty} + \kappa^{-\frac{1}{2}} \|\sqrt{v} \partial_t f_1\|_{L_{t,x,v}^\infty} \right) \\ \times \{ \|P f_R\|_{L_{t,x}^2} + \kappa^{\frac{1}{2}} \varepsilon \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2} \} \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2} \\ + \left( \frac{\varepsilon}{\kappa^{1/2}} \|P f_2\|_{L_{t,x}^\infty} + \frac{\varepsilon}{\kappa^{1/2}} \|(\mathbf{I} - \mathbf{P}) f_2\|_{L_{t,x}^\infty} + \frac{1}{\kappa^{1/2}} \|v f_1\|_{L_{t,x}^\infty} \right) \\ \times \left( \|P \partial_t f_R\|_{L_{t,x}^2} + \kappa^{\frac{1}{2}} \varepsilon \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2} \right) \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2} \quad (2.64) \\ \lesssim \left( \frac{\sigma_0}{10} + \frac{\varepsilon^2}{\kappa} \|\sqrt{v} \partial_t f_2\|_{L_{t,x,v}^2}^2 + \frac{\varepsilon^2}{\kappa} \|P f_2\|_{L_{t,x}^\infty}^2 + \frac{\varepsilon^2}{\kappa} \|(\mathbf{I} - \mathbf{P}) f_2\|_{L_{t,x}^\infty}^2 + \varepsilon \|v f_1\|_{L_{t,x}^\infty}^2 \right) \\ \times \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2}^2 \\ + \left( 1 + \frac{1}{\kappa} \|\sqrt{v} \partial_t f_1\|_{L_{t,x,v}^\infty}^2 + \frac{1}{\kappa} \|\sqrt{v} f_1\|_{L_{t,x,v}^\infty}^2 \right) \|P f_R\|_{L_{t,x}^2}^2 \\ + \left( 1 + \varepsilon^2 \|\sqrt{v} \partial_t f_1\|_{L_{t,x,v}^\infty}^2 \right) \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2}^2$$

We have

$$\begin{aligned} & \left| -(\mathbf{I} - \mathbf{P})(v \cdot \nabla_x \partial_t f_2) + \frac{2}{\kappa} \Gamma(\partial_t f_1, f_2) + \frac{2}{\kappa} \Gamma(f_1, \partial_t f_2) \right| \\ & \lesssim |(\mathbf{I} - \mathbf{P})(v \cdot \nabla_x \mathbf{P} \partial_t f_2)| + \left| (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x L^{-1}(-\kappa v \cdot \nabla_x \partial_t f_1 + 2\Gamma(\partial_t f_1, f_1))) \right| \\ & \quad + \frac{1}{\kappa} |\Gamma(\partial_t f_1, f_2)| + \frac{1}{\kappa} |\Gamma(f_1, \partial_t f_2)| \\ & \lesssim e^{-\varrho|v|^2} |\nabla_x \partial_t \mathcal{I}_2| + \kappa e^{-\varrho|v|^2} |\nabla_x^2 \partial_t \mathcal{I}| + e^{-\varrho|v|^2} |\mathcal{I}| |\partial_t \mathcal{I}| \end{aligned} \quad (2.65)$$

$$+ \frac{1}{\kappa} e^{-\varrho|v|^2} |\partial_t \mathcal{I}| \left( |\mathcal{I}|^2 + |\mathcal{I}_2| + \kappa |\nabla_x \mathcal{I}| \right) + \frac{1}{\kappa} e^{-\varrho|v|^2} |\mathcal{I}| (|\partial_t \mathcal{I}_2| + \kappa |\nabla_x \partial_t \mathcal{I}| + |\mathcal{I}| |\partial_t \mathcal{I}|). \quad (2.66)$$

So we derive that

$$|(2.60)| \lesssim \left( \kappa^{1/2} \varepsilon^{1/2} \|(2.65)\|_{L^2_{t,x,v}} + \frac{\varepsilon^{1/2}}{\kappa^{1/2}} \|\kappa(2.66)\|_{L^2_{t,x,v}} \right) \|\kappa^{-1/2} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L^2_{t,x,v}}. \quad (2.67)$$

And we have

$$\begin{aligned} & \left| -\partial_t^2 f_2 + \frac{2}{\kappa} \Gamma(\partial_t f_2, f_2) \right| \lesssim \\ & e^{-\varrho|v|^2} \left( |\partial_t^2 \mathcal{I}_2| + \kappa |\nabla_x \partial_t^2 \mathcal{I}| \right) + e^{-\varrho|v|^2} \left( |\mathcal{I}| |\partial_t^2 \mathcal{I}| + |\partial_t \mathcal{I}|^2 \right) \end{aligned} \quad (2.68)$$

$$+ \frac{1}{\kappa} e^{-\varrho|v|^2} \left( |\mathcal{I}|^2 + |\mathcal{I}_2| + \kappa |\nabla_x \mathcal{I}| \right) (|\partial_t \mathcal{I}_2| + \kappa |\nabla_x \partial_t \mathcal{I}| + |\mathcal{I}| |\partial_t \mathcal{I}|) \quad (2.69)$$

Thus

$$\begin{aligned} |(2.61)| & \lesssim \varepsilon \kappa \|(2.68)\|_{L^2_{t,x,v}}^2 + \frac{\varepsilon}{\kappa} \|\kappa(2.69)\|_{L^2_{t,x,v}}^2 + \frac{1}{\kappa} \|\mathbf{P} \partial_t f_R\|_{L^2_{t,x,v}}^2 \\ & \quad + \left( \kappa^{1/2} \varepsilon^{3/2} \|(2.68)\|_{L^2_{t,x,v}} + \frac{\varepsilon^{3/2}}{\kappa^{1/2}} \|\kappa(2.69)\|_{L^2_{t,x,v}} \right) \|\kappa^{-1/2} \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L^2_{t,x,v}} \end{aligned} \quad (2.70)$$

Lastly we estimate (2.57). As in (2.53) we derive that (2.57) is bounded from below by

$$\begin{aligned} & \frac{1}{2} |\varepsilon^{-\frac{1}{2}} (1 - P_{\gamma_+}) \partial_t f_R|_{L^2((0,T); L^2_{\gamma_+})}^2 - \frac{1}{8C} |\varepsilon^{-\frac{1}{2}} P_{\gamma_+} \partial_t f_R|_{L^2((0,T); L^2_{\gamma_+})}^2 \\ & \quad - C |(1 - P_{\gamma_+}) (\mathbf{I} - \mathbf{P}) \partial_t f_2|_{L^2((0,T); L^2_{\gamma_-})}^2 \\ & \geq \frac{1}{2} |\varepsilon^{-\frac{1}{2}} (1 - P_{\gamma_+}) \partial_t f_R|_{L^2((0,T); L^2_{\gamma_+})}^2 - \frac{1}{8C} |\varepsilon^{-\frac{1}{2}} P_{\gamma_+} \partial_t f_R|_{L^2((0,T); L^2_{\gamma_+})}^2 \\ & \quad - C \left\{ \|e^{-\varrho|v|^2} (\kappa |\nabla_x \partial_t \mathcal{I}| + |\mathcal{I}| |\partial_t \mathcal{I}|) \|_{L^2_t L^2(\partial\Omega)} \right\} \end{aligned} \quad (2.71)$$

for some large  $C$ .

Now we bound  $P_{\gamma_+} \partial_t f_R$  using (2.29). Following the argument arriving at (2.54) and setting  $h = |\partial_t f_R|^2$  we derive

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^t \int_{\gamma_+} |\partial_t f_R|^2 d\gamma ds \\ & \lesssim_N \|\partial_t f_R(0)\|_{L_{x,v}^2} + \|\partial_t f_R\|_{L_{t,x,v}^2} + \int_0^t \iint_{\Omega \times \mathbb{R}^3} \left| \left( -\frac{1}{\varepsilon^2 \kappa} L \partial_t f_R + \text{r.h.s of (2.6)} \right) \partial_t f_R \right| \quad (2.72) \\ & \lesssim_N \|\partial_t f_R(0)\|_{L_{x,v}^2}^2 + \|P \partial_t f_R\|_{L_{t,x}^2}^2 + \|\varepsilon^{-1} \kappa^{-1/2} \sqrt{v} (\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2}^2 \\ & \quad + (2.63) + (2.64) + (2.67) + (2.70). \end{aligned}$$

We conclude (2.32) by collecting the terms.  $\square$

### 2.3. $L_t^2 L_x^3$ -integrability gain for $\mathbf{P} f_R$

The goal for this section is the following proposition:

**Proposition 4.** Assume the same assumptions in Proposition 2. Then we have, for  $2 < p < 3$ ,

$$\begin{aligned} & d_3 \|P f_R\|_{L_t^2 L_x^p} \\ & \lesssim \|f_R\|_{L_t^\infty L_{x,v}^2} + \|f_R(0)\|_{L_\gamma^2} + \frac{1}{\kappa} \|\mathcal{I}\|_{L_{t,x}^\infty} \|P f_R\|_{L_{t,x}^2} \\ & \quad + \left\{ \frac{1}{\varepsilon \kappa} + \frac{1}{\kappa} \left( \|\mathcal{I}\|_{L_{t,x}^\infty} + \varepsilon \|(2.36)\|_{L_{t,x}^\infty} \right) + \frac{\varepsilon^{1/2}}{\kappa} \|\mathfrak{w}_Q f_R\|_{L_{t,x,v}^\infty} \right\} \|\sqrt{v} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2} \quad (2.73) \\ & \quad + \varepsilon^{1/2} \|(2.46)\|_{L_{t,x}^2} + \frac{\varepsilon^{1/2}}{\kappa} \|\kappa(2.47)\|_{L_{t,x}^2} + \varepsilon^{3/2} \|(2.50)\|_{L_{t,x,v}^2} \\ & \quad + \frac{\varepsilon^{3/2}}{\kappa} \|\kappa(2.51)\|_{L_{t,x,v}^2}, \end{aligned}$$

with

$$d_3 := 1 - \frac{\varepsilon}{\kappa} \|(2.36)\|_{L_t^\infty L_x^{\frac{2p}{p-2}}} - \frac{\varepsilon^{1/2}}{\kappa} \|P f_R\|_{L_t^\infty L_x^{\frac{3(p-2)}{p}}} \|\mathfrak{w}_Q f_R\|_{L_{t,x,v}^{\frac{6-2p}{p}}}, \quad (2.74)$$

and for  $Q' < Q$

$$\begin{aligned}
& d_{3,t} \|P \partial_t f_R\|_{L_t^2 L_x^p} \\
& \lesssim \left\{ \frac{1}{\kappa} \left( \|\partial_t \mathcal{I}\|_{L_{t,x,v}^\infty} + \varepsilon \|(2.49)\|_{L_{t,x,v}^\infty} \right) \right\} \left\{ \|Pf_R\|_{L_{t,x}^2} + \|\sqrt{v}(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2} \right\} \\
& + \left\{ \frac{1}{\kappa \varepsilon} + \frac{\varepsilon^{1/2}}{\kappa} \|\mathbf{w} f_R\|_{L_{t,x,v}^\infty} + \frac{1}{\kappa} \left( \|\mathcal{I}\|_{L_{t,x}^\infty} + \varepsilon \|(2.36)\|_{L_{t,x}^\infty} \right) \right\} \|\sqrt{v}(\mathbf{I} - \mathbf{P})\partial_t f_R\|_{L_{t,x,v}^2} \\
& + (\kappa \varepsilon)^{\frac{2}{p-2}} \|\mathbf{w}_{\mathcal{Q}'} \partial_t f_R\|_{L_t^2 L_{x,v}^\infty} + \|\partial_t f_R\|_{L_t^\infty L_{x,v}^2} + \|\partial_t f_R(0)\|_{L_y^2} + \frac{1}{\kappa} \|\mathcal{I}\|_{L_{t,x}^\infty} \|P \partial_t f_R\|_{L_t^2 L_x^2} \\
& + \varepsilon^{1/2} \|(2.65)\|_{L_{t,x,v}^2} + \frac{\varepsilon^{1/2}}{\kappa} \|\kappa(2.66)\|_{L_{t,x,v}^2} + \varepsilon^{3/2} \|(2.68)\|_{L_{t,x,v}^2}^2 + \frac{\varepsilon^{3/2}}{\kappa} \|\kappa(2.69)\|_{L_{t,x,v}^2}^2, \tag{2.75}
\end{aligned}$$

with

$$d_{3,t} := 1 - \frac{\varepsilon}{\kappa} \|(2.36)\|_{L_t^\infty L_x^{\frac{2p}{p-2}}} - \frac{\varepsilon^{1/2}}{\kappa} \|Pf_R\|_{L_t^\infty L_x^{\frac{3(p-2)}{p}}} \|\mathbf{w}_{\mathcal{Q}'} f_R\|_{L_{t,x,v}^{\frac{6-2p}{p}}}, \tag{2.76}$$

where both bounds are uniform-in- $p$  for  $2 < p < 3$ .

We prove the proposition by several steps.

**Step 1: Extension.** We define a subset

$$\tilde{\Omega} := (0, 2\pi) \times (0, 2\pi) \times (0, \infty) \subset \mathbb{R}^3. \tag{2.77}$$

We regard  $\tilde{\Omega}$  as an open subset but not a periodic domain as  $\Omega$ . Without loss of generality we may assume that  $f_R(0, x, v)$  is defined in  $\mathbb{R}^3 \times \mathbb{R}^3$  and  $\|f_R(0)\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} \lesssim \|f_R(0)\|_{L^p(\tilde{\Omega} \times \mathbb{R}^3)}$  for all  $1 \leq p \leq \infty$ . Then we extend a solution for whole time  $t \in \mathbb{R}$  as

$$f_I(t, x, v) := \mathbf{1}_{t \geq 0} f_R(t, x, v) + \mathbf{1}_{t \leq 0} \chi_1(t) f_R(0, x, v), \tag{2.78}$$

where a smooth non-negative function  $\chi_1$  satisfies  $\chi_1(t) \equiv 1$  for  $t \in [-1, 0]$ ,  $\chi_1(t) \equiv 0$  for  $t < -2$ , and  $0 \leq \frac{d}{dt} \chi_1 \leq 4$ .

A closure of  $\tilde{\Omega}$  is given as  $cl(\tilde{\Omega}) = [0, 2\pi] \times [0, 2\pi] \times [0, \infty)$ . Let us define  $\tilde{t}_B(x, v) \in \mathbb{R}$  for  $(x, v) \in (\mathbb{R}^3 \setminus \tilde{\Omega}) \times \mathbb{R}^3$ . We consider  $\tilde{B}(x, v) := \{s \in \mathbb{R} : x + sv \in \mathbb{R}^3 \setminus cl(\tilde{\Omega})\}$ . Clearly if  $\tilde{B}(x, v) \neq \emptyset$  then  $\{s > 0\} \subset \tilde{B}(x, v)$  or  $\{s < 0\} \subset \tilde{B}(x, v)$  exclusively.

If  $\{s > 0\} \subset \tilde{B}(x, v)$ , let  $I_+$  be the largest interval such that  $\{s > 0\} \subset I_+ \subset \tilde{B}(x, v)$ . And if  $\{s < 0\} \subset \tilde{B}(x, v)$ , let  $I_-$  be the largest interval such that  $\{s > 0\} \subset I_- \subset \tilde{B}(x, v)$ . We define

$$\tilde{t}_B(x, v) = \begin{cases} 0 & \text{if } x \in \partial \tilde{\Omega}, \\ \inf I_+ & \text{if } x \in \mathbb{R}^3 \setminus cl(\tilde{\Omega}) \text{ and } \tilde{B}(x, v) \neq \emptyset \text{ and } \{s > 0\} \subset I_+ \subset \tilde{B}(x, v), \\ \sup I_- & \text{if } x \in \mathbb{R}^3 \setminus cl(\tilde{\Omega}) \text{ and } \tilde{B}(x, v) \neq \emptyset \text{ and } \{s < 0\} \subset I_- \subset \tilde{B}(x, v), \\ -\infty & \text{if } \tilde{B}(x, v) = \emptyset \text{ and } x \notin \partial \tilde{\Omega}. \end{cases} \tag{2.79}$$

Using (2.79) we define

$$f_E(t, x, v) := \mathbf{1}_{(x,v) \in (\mathbb{R}^3 \setminus \tilde{\Omega}) \times \mathbb{R}^3} f_I(t + \varepsilon \tilde{t}_B(x, v), \tilde{x}_B(x, v), v) \quad \text{with} \quad \tilde{x}_B(x, v) := x + \tilde{t}_B(x, v)v. \quad (2.80)$$

It is easy to see that  $\varepsilon \partial_t f_E + v \cdot \nabla_x f_E = 0$  in the sense of distributions.

Next we define two cutoff functions. For any  $N > 0$  we define smooth non-negative functions as

$$\chi_2(x) \equiv 1 \text{ for } x \in [-\pi, 3\pi] \times [-\pi, 3\pi] \times [-\pi, \infty), \quad (2.81)$$

$$\chi_2(x) \equiv 0 \text{ for } x \notin [-2\pi, 4\pi] \times [-2\pi, 4\pi] \times [-2\pi, \infty), \quad |\nabla_x \chi_2| \leq 10,$$

$$\chi_3(v) \equiv 1 \text{ for } |v| \leq N-1, \text{ and } |v_i| \geq 2/N \text{ for all } i = 1, 2, 3, \quad (2.82)$$

$$\chi_3(v) \equiv 0 \text{ for } |v| \geq N \text{ or } |v_i| \leq 1/N \text{ for any } i = 1, 2, 3, \quad |\nabla_v \chi_3| \leq 10.$$

We denote

$$U := [-2\pi, 4\pi] \times [-2\pi, 4\pi] \times [-2\pi, \infty), \quad (2.83)$$

$$V := \{v \in \mathbb{R}^3 : |v| \leq N\} \cap \bigcap_{i=1,2,3} \{v \in \mathbb{R}^3 : |v_i| \geq 1/N\}$$

We define an extension of cut-offed solutions

$$\bar{f}_R(t, x, v) := \chi_2(x) \chi_3(v) \{ \mathbf{1}_{\tilde{\Omega}}(x) f_I(t, x, v) + f_E(t, x, v) \} \quad \text{for } (t, x, v) \in (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (2.84)$$

We note that in the sense of distributions  $\bar{f}_R$  solves

$$\varepsilon \partial_t \bar{f}_R + v \cdot \nabla_x \bar{f}_R = \bar{g} \quad \text{in } (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3, \quad (2.85)$$

$$\bar{g} := \frac{v \cdot \nabla_x \chi_2}{\chi_2} \bar{f}_R + \mathbf{1}_{t \geq 0} \mathbf{1}_{\tilde{\Omega}}(x) \chi_2(x) \chi_3(v) [\varepsilon \partial_t + v \cdot \nabla_x] f_R$$

$$+ \mathbf{1}_{t \leq 0} \{ \varepsilon \partial_t \chi_1(t) f_R(0, x, v) + \chi_1(t) v \cdot \nabla_x f_R(0, x, v) \}$$

Here we have used the fact that  $\bar{f}_R$  in (2.85) is continuous along the characteristics across  $\partial \tilde{\Omega}$  and  $\{t = 0\}$ . We derive that, using (2.85),

$$\bar{f}_R(t, x, v) = \frac{1}{\varepsilon} \int_{-\infty}^t \bar{g}(s, x - \frac{t-s}{\varepsilon} v, v) ds \quad \text{for } (t, x, v) \in (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (2.86)$$

Recall  $\varphi_i \in \{\varphi_0, \dots, \varphi_4\}$  in (1.28). From (2.84) we note that

$$\left\| \int_{\mathbb{R}^3} \bar{f}_R(t, x, v) \varphi_i(v) \sqrt{\mu(v)} dv \right\|_{L_t^2((0,T); L_x^p(\tilde{\Omega}))} \quad (2.87)$$

$$= \left\| \int_{\mathbb{R}^3} \chi_2(x) \chi_3(v) f_R(t, x, v) \tilde{\varphi}_i(v) \sqrt{\mu(v)} dv \right\|_{L_t^2((0,T); L_x^p(\tilde{\Omega}))}$$

From (1.29), we decompose

$$\begin{aligned}
 (2.87) &\geq \left\| \sum_j \chi_2(x) P_j f_R(t, x) \int_{\mathbb{R}^3} \chi_3(v) \varphi_j(v) \varphi_i(v) \mu(v) dv \right\|_{L_t^2((0, T); L_x^p(\tilde{\Omega}))} \\
 &\quad - \left\| \int_{\mathbb{R}^3} \chi_3(v) (\mathbf{I} - \mathbf{P}) f_R(t, x, v) \varphi_i(v) \sqrt{\mu(v)} dv \right\|_{L_t^2((0, T); L_x^p(\tilde{\Omega}))} \\
 &\geq \|Pf_R\|_{L_t^2((0, T); L_x^p(\tilde{\Omega}))} - C_{T, N} \|(\mathbf{I} - \mathbf{P}) f_R\|_{L^p((0, T) \times \tilde{\Omega} \times \mathbb{R}^3)} \\
 &\geq \|Pf_R\|_{L_t^2((0, T); L_x^p(\tilde{\Omega}))} \\
 &\quad - C_{T, N} \|\mathfrak{w}_{\mathcal{Q}, \mathbb{B}} f_R(t)\|_{L^\infty((0, T) \times \tilde{\Omega} \times \mathbb{R}^3)}^{\frac{p-2}{p}} \|(\mathbf{I} - \mathbf{P}) f_R\|_{L^2((0, T) \times \tilde{\Omega} \times \mathbb{R}^3)}^{\frac{2}{p}},
 \end{aligned} \tag{2.88}$$

where for the second inequality we use  $L_t^2(0, T) \subset L_t^p(0, T)$ , and  $L^1(\{|v| \leq N\}) \subset L^p(\{|v| \leq N\})$ .

**Step 2: Average lemma.** Recall  $\varphi_i \in \{\varphi_0, \dots, \varphi_4\}$  in (1.28). We choose  $\varphi(v)$  such that

$$\begin{aligned}
 &\chi_3(v) |\varphi_i(v)| \sqrt{\mu_0(v)} \leq \varphi(v), \quad \varphi(v) \in C_c^\infty(\mathbb{R}^3) \\
 &\text{and } \varphi(v) \equiv 0 \text{ for } |v| \geq N \text{ or } |v_i| \leq 1/N \text{ for any } i = 1, 2, 3.
 \end{aligned} \tag{2.89}$$

**Lemma 2.** We define

$$S(\bar{g})(t, x) := \frac{1}{\varepsilon} \int_{-\infty}^t \int_{\mathbb{R}^3} |\bar{g}(s, x - \frac{t-s}{\varepsilon} v, v)| \varphi(v) dv ds \text{ for } (t, x) \in (-\infty, T] \times \mathbb{R}^3. \tag{2.90}$$

Then, for  $p < 3$  and  $1 \ll N$ ,

$$\|S(\bar{g})\|_{L_t^2((0, T); L_x^p(\mathbb{T}^2 \times \mathbb{R}))} \lesssim_N \|\mathbf{1}_{(t, x, v) \in \mathcal{D}_T \bar{g}}\|_{L^2((0, T) \times (\mathbb{T}^2 \times \mathbb{R}) \times \{|v| \leq N\})}, \tag{2.91}$$

where the bound (2.91) only depends on  $N$  but can be independent on  $p < 3$ .

We remark that from (2.86) and (2.90)  $\int_{\mathbb{R}^3} \bar{f}_R(t, x, v) \varphi_i(v) dv \leq S(\bar{g})(t, x)$ . For the proof we refer to Lemma 6 in [30].

Now we apply Lemma 2 to (2.86) and derive that

$$\begin{aligned}
 &\left\| \int_{\mathbb{R}^3} \bar{f}_R(t, x, v) \varphi(v) dv \right\|_{L_t^2((-1, T]; L_x^p(\tilde{\Omega}))} \\
 &\lesssim \|\mathbf{1}_{(t, x, v) \in \mathcal{D}_T \bar{g}}\|_{L^2((-1, T] \times U \times V)} \\
 &\lesssim \|f_R(t, x, v)\|_{L^2((0, T) \times \tilde{\Omega} \times V)} + \|f_R(0, x, v)\|_{L^2(\tilde{\Omega} \times V)} + \|\nabla_x f_R(0, x, v)\|_{L^2(\tilde{\Omega} \times V)}
 \end{aligned}$$

$$+ \|\mathbf{1}_{(t,x,v) \in \mathfrak{D}_T} f_I(t + \varepsilon \tilde{t}_B(x, v), \tilde{x}_B(x, v), v)\|_{L^2((-1, T] \times (U \setminus \tilde{\Omega}) \times V)} \quad (2.92)$$

$$+ \|\varepsilon \partial_t + v \cdot \nabla_x f_R\|_{L^2((0, T] \times \tilde{\Omega} \times V)}, \quad (2.93)$$

where we have used (2.84), (2.78), (2.80), and the fact that  $|v \cdot \nabla_x \chi_2(x)| \lesssim_N 1$  on  $v \in V$ .

First we consider (2.92). We split the cases of (2.92) according to (2.79). For  $x \in \partial \tilde{\Omega}$ , which has a zero measure in  $L^2((-1, T] \times (U \setminus \tilde{\Omega}) \times V)$ , we have  $\tilde{t}_B(x, v) = 0$  from the first line of (2.79). If  $\tilde{B}(x, v) = \emptyset$  and  $x \notin \partial \tilde{\Omega}$  then  $\tilde{t}_B(x, v) = -\infty$  from the last line of (2.79) and hence  $\tilde{f}_R(-\infty) = 0$  since  $\chi_1(-\infty) = 0$  in (2.78). Therefore we derive that

$$(2.92) \leq \|\mathbf{1}_{\{s < 0\} \subset \tilde{B}(x, v)} \mathbf{1}_{(t,x,v) \in \mathfrak{D}_T} f_I(t + \varepsilon \tilde{t}_B(x, v), \tilde{x}_B(x, v), v)\|_{L^2((-1, T] \times (U \setminus \tilde{\Omega}) \times V)} \quad (2.94)$$

$$+ \|\mathbf{1}_{\{s > 0\} \subset \tilde{B}(x, v)} \mathbf{1}_{(t,x,v) \in \mathfrak{D}_T} f_I(t + \varepsilon \tilde{t}_B(x, v), \tilde{x}_B(x, v), v)\|_{L^2((-1, T] \times (U \setminus \tilde{\Omega}) \times V)}. \quad (2.95)$$

We need a special attention to (2.94). Since  $(t, x, v) \in \mathfrak{D}_T$  we know that  $\inf\{\tau \geq t : x + \frac{\tau-t}{\varepsilon}v \in cl(\tilde{\Omega})\} \leq T$ . If  $\{s < 0\} \subset \tilde{B}(x, v)$  then, from the third line of (2.79),  $\tilde{t}_B(x, v) = \sup \tilde{B}(x, v) = \sup\{s \in \mathbb{R} : x + sv \in \mathbb{R}^3 \setminus cl(\tilde{\Omega})\} \leq (T - t)/\varepsilon$ . Therefore the argument of  $f_I$  in (2.94) is confined as

$$(t + \varepsilon \tilde{t}_B(x, v), \tilde{x}_B(x, v), v) \in (-\infty, T] \times \partial \tilde{\Omega} \times V. \quad (2.96)$$

For (2.95), from the second line of (2.79),  $\tilde{t}_B(x, v) = \inf \tilde{B}(x, v) = \inf\{s \in \mathbb{R} : x + sv \in \mathbb{R}^3 \setminus cl(\tilde{\Omega})\} \leq 0$ . Therefore  $t + \varepsilon \tilde{t}_B(x, v) \leq t \leq T$  and hence the argument of  $f_I$  in (2.95) is confined as in (2.96). Now we apply the Minkowski's inequality in time, change of variables  $t + \varepsilon \tilde{t}_B(x, v) \mapsto t$ , and use (2.96) to derive that

$$(2.94) + (2.95) \lesssim \left\| \|f_I(t, \tilde{x}_B(x, v), v)\|_{L^2_t((-1, T])} \right\|_{L^2_{x,v}((U \setminus \tilde{\Omega}) \times V)}. \quad (2.97)$$

Let us define an outward normal  $\tilde{n}(x)$  on  $\partial \tilde{\Omega}$ . More precisely

$$\tilde{n}(x) = \begin{cases} (0, 0, -1) & \text{if } x_3 = 0 \text{ and } x \in \partial \tilde{\Omega}, \\ ((-1)^{\frac{x_1}{2\pi}+1}, 0, 0) & \text{if } x_1 \in \{0, 2\pi\} \text{ and } x \in \partial \tilde{\Omega}, \\ (0, (-1)^{\frac{x_2}{2\pi}+1}, 0) & \text{if } x_2 \in \{0, 2\pi\} \text{ and } x \in \partial \tilde{\Omega}. \end{cases} \quad (2.98)$$

From (2.83) we have therefore  $(x, v) \in (U \setminus \tilde{\Omega}) \times V$  then  $|\tilde{n}(\tilde{x}_B(x, v)) \cdot v| \geq 1/N$ . We consider maps

$$\begin{aligned} (x_1, x_3) &\mapsto \tilde{x}_B(x, v) \in (0, 2\pi) \times (0, 2\pi) \times \{x_3 = 0\}, \\ &\text{with } \left| \det \left( \frac{\partial(\tilde{x}_{B,1}(x, v), \tilde{x}_{B,2}(x, v))}{\partial(x_1, x_3)} \right) \right| = \left| \frac{v_2}{v \cdot \tilde{n}} \right|, \\ (x_i, x_3) &\mapsto (\tilde{x}_{B,i}(x, v), \tilde{x}_{B,3}(x, v)) \in (0, 2\pi) \times (0, \infty), \\ &\text{with } \left| \det \left( \frac{\partial(\tilde{x}_{B,i}(x, v), \tilde{x}_{B,3}(x, v))}{\partial(x_1, x_3)} \right) \right| = \left| \frac{v_i}{v \cdot \tilde{n}} \right|, \text{ for } i = 1, 2. \end{aligned} \quad (2.99)$$

Note that if  $v \in V$  of (2.83) then  $|v_i| \geq 1/N$  for all  $i = 1, 2, 3$ . We define

$$\tilde{\gamma} := \partial\tilde{\Omega} \times \mathbb{R}^3, \quad \tilde{\gamma}^N := \partial\tilde{\Omega} \times (\mathbb{R}^3 \setminus V). \quad (2.100)$$

We apply the change of variables (2.99) to (2.97):

$$\begin{aligned} (2.97) &= \left\| \left[ \int_{-2\pi}^{4\pi} \int_{-2\pi}^{\infty} \int_{-2\pi}^{4\pi} \|f_I(t, \tilde{x}_B(x, v), v)\|_{L^2_t((-1, T])}^2 dx_1 dx_3 dx_2 \right]^{1/2} \right\|_{L^2_v(V)} \\ &\leq \left\| \left[ 5 \times 6\pi N \int_{\partial\tilde{\Omega}-1}^T \int |f_I(t, y, v)|^2 |v \cdot \tilde{n}(y)| dy dt \right]^{1/2} \right\|_{L^2_v(V)} \\ &\lesssim \|f_R\|_{L^2((0, T) \times \tilde{\gamma} \setminus \tilde{\gamma}^N)} + \|f_R(0)\|_{L^2(\tilde{\gamma} \setminus \tilde{\gamma}^N)}. \end{aligned} \quad (2.101)$$

We recall the trace theorem:

$$\begin{aligned} \int_0^T \int_{\tilde{\gamma} \setminus \tilde{\gamma}^N} |h| dy ds &\lesssim \sup_{t \in [0, T]} \|h(t)\|_{L^1(\tilde{\Omega} \times V)} + \int_0^T \|h(s)\|_{L^1(\tilde{\Omega} \times V)} ds \\ &\quad + \int_0^T \|[\varepsilon \partial_t + v \cdot \nabla_x]h\|_{L^1(\tilde{\Omega} \times V)} ds. \end{aligned} \quad (2.102)$$

We apply (2.102) with  $h = f^2$  and derive an estimate

$$\begin{aligned} &\|f_R\|_{L^2((0, T) \times \tilde{\gamma} \setminus \tilde{\gamma}^N)}^2 \\ &\lesssim \sup_{t \in [0, T]} \|f_R(t)\|_{L^2(\tilde{\Omega} \times V)}^2 + \int_0^T \|f_R(s)\|_{L^2(\tilde{\Omega} \times V)}^2 ds + \int_0^T \iint_{\tilde{\Omega} \times V} |f_R[\varepsilon \partial_t + v \cdot \nabla_x]f_R| dx dv ds \\ &\lesssim_T \|f_R\|_{L^\infty([0, T]; L^2(\Omega \times \mathbb{R}^3))}^2 + \|[\varepsilon \partial_t + v \cdot \nabla_x]f_R\|_{L^2([0, T] \times \Omega \times \mathbb{R}^3)}. \end{aligned} \quad (2.103)$$

Finally we conclude a bound of (2.92) as below via (2.94), (2.95), (2.97), (2.101), and (2.103)

$$(2.92) \lesssim \|f_R(0)\|_{L^2_\gamma} + \|f_R\|_{L^\infty([0, T]; L^2(\Omega \times \mathbb{R}^3))} + \underbrace{\|[\varepsilon \partial_t + v \cdot \nabla_x]f_R\|_{L^2([0, T] \times \Omega \times \mathbb{R}^3)}}_{(2.104)_*}. \quad (2.104)$$

Next we estimate (2.93) (and (2.104)<sub>\*</sub>). Using (2.85) and (2.5) we conclude that

$$\begin{aligned}
 & (2.93) + (2.104)_* \\
 & \lesssim \left\| -\frac{1}{\varepsilon\kappa} Lf_R + \frac{2}{\kappa} \Gamma(f_1 + \varepsilon f_2, f_R) + \frac{\varepsilon^{1/2}}{\kappa} \Gamma(f_R, f_R) \right. \\
 & \quad \left. - \varepsilon^{1/2} \left\{ (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) - \frac{2}{\kappa} \Gamma(f_1, f_2) \right\} + \varepsilon^{3/2} \left( -\partial_t f_2 + \frac{1}{\kappa} \Gamma(f_2, f_2) \right) \right\|_{L^2((0,T] \times \Omega \times V)}.
 \end{aligned}$$

Following the arguments of (2.44)–(2.52), we derive that

$$\begin{aligned}
 & (2.93) + (2.104)_* \\
 & \lesssim \left\{ \frac{\varepsilon}{\kappa} \|(2.36)\|_{L_t^\infty((0,T); L_x^{\frac{2p}{p-2}}(\Omega))} + \frac{\varepsilon^{1/2}}{\kappa} \|Pf_R\|_{L_t^\infty((0,T); L_x^{\frac{2p}{p-2}}(\Omega))} \right\} \|Pf_R\|_{L_t^2((0,T); L_x^p(\Omega))} \\
 & \quad + \frac{1}{\kappa} \|Pf_1\|_{L_{t,x}^\infty} \|Pf_R\|_{L_{t,x}^2} \\
 & \quad + \left\{ \frac{1}{\varepsilon\kappa} + \frac{1}{\kappa} \left( \|Pf_1\|_{L_{t,x}^\infty} + \varepsilon \|(2.36)\|_{L_{t,x}^\infty} \right) + \frac{\varepsilon^{1/2}}{\kappa} \|\mathfrak{w}_\varrho f_R\|_{L_t^\infty((0,T) \times \Omega \times \mathbb{R}^3)} \right\} \\
 & \quad \times \|(\mathbf{I} - \mathbf{P})f_R\|_{L_t^2((0,T) \times \Omega \times \mathbb{R}^3)} \\
 & \quad + \varepsilon^{1/2} \|(2.46)\|_{L_{t,x}^2} + \frac{\varepsilon^{1/2}}{\kappa} \|\kappa(2.47)\|_{L_{t,x}^2} + \varepsilon^{3/2} \|(2.50)\|_{L_{t,x,v}^2} + \frac{\varepsilon^{3/2}}{\kappa} \|\kappa(2.51)\|_{L_{t,x,v}^2},
 \end{aligned} \tag{2.105}$$

where we further bound

$$\|Pf_R\|_{L_x^{\frac{2p}{p-2}}(\Omega)} \leq \|Pf_R\|_{L_x^{\frac{p}{p-2}}(\Omega)}^{\frac{3(p-2)}{p}} \|\mathfrak{w}_\varrho f_R\|_{L_x^\infty(\Omega)}^{\frac{6-2p}{p}}. \tag{2.106}$$

**Step 3. Proof of (2.73).** First we use (2.88) and then (2.92) and (2.93). We bound (2.92) via (2.97) and (2.101), which are bounded by (2.103) and (2.105) respectively. These conclude that, for  $p < 3$ ,

$$\begin{aligned}
 & \|Pf_R\|_{L_t^2((0,T); L_x^p(\tilde{\Omega}))} - C_{T,N} \|\mathfrak{w}_{\varrho, \mathfrak{B}} f_R(t)\|_{L^\infty((0,T) \times \tilde{\Omega} \times \mathbb{R}^3)}^{\frac{p-2}{p}} \|(\mathbf{I} - \mathbf{P})f_R\|_{L^2((0,T) \times \tilde{\Omega} \times \mathbb{R}^3)}^{\frac{2}{p}} \\
 & \leq \left\| \int_{\mathbb{R}^3} \bar{f}_R(t, x, v) \varphi_i(v) \sqrt{\mu_0(v)} dv \right\|_{L_t^2((0,T); L_x^p(\tilde{\Omega}))} \\
 & \leq \left\| \int_{\mathbb{R}^3} \bar{f}_R(t, x, v) \varphi(v) dv \right\|_{L_t^2((0,T); L_x^p(\tilde{\Omega}))} \\
 & \lesssim \|f_R\|_{L^\infty([0,T]; L^2(\Omega \times \mathbb{R}^3))} + \|f_R(0)\|_{L_\gamma^2} + \text{r.h.s. of (2.105) with (2.106)}.
 \end{aligned} \tag{2.107}$$

Then we move a contribution of  $\|Pf_R\|_{L^2_t((0,T);L^p_x(\Omega))}$  to the l.h.s. and use (2.106). This concludes (2.73).

**Step 4: Sketch of proof for (2.75).** We follow the same argument for (2.73). Thereby we only pin point the difference of the proof of (2.75). Recall  $\partial_t f_R(0, x, v) = f_{R,t}(0, x, v)$  from (2.27). We regard  $\tilde{\Omega}$  as an open subset but not a periodic domain as  $\Omega$ . Without loss of generality we may assume that  $f_{R,t}(0, x, v)$  is defined in  $\mathbb{R}^3 \times \mathbb{R}^3$  and  $\|f_{R,t}(0)\|_{L^p(\mathbb{R}^3) \times \mathbb{R}^3} \lesssim \|f_{R,t}(0)\|_{L^p(\tilde{\Omega}) \times \mathbb{R}^3}$  for all  $1 \leq p \leq \infty$ . Then we extend a solution for whole time  $t \in \mathbb{R}$  as

$$f_{I,t}(t, x, v) := \mathbf{1}_{t \geq 0} \partial_t f_R(t, x, v) + \mathbf{1}_{t \leq 0} \chi_1(t) f_{R,t}(0, x, v). \quad (2.108)$$

Using  $\tilde{t}_B(x, v)$  in (2.79) we define

$$f_{E,t}(t, x, v) := \mathbf{1}_{(x,v) \in (\mathbb{R}^3 \setminus \tilde{\Omega}) \times \mathbb{R}^3} f_{I,t}(t + \varepsilon \tilde{t}_B(x, v), \tilde{x}_B(x, v), v). \quad (2.109)$$

We define an extension of cut-offed solutions

$$\begin{aligned} \bar{f}_{R,t}(t, x, v) &:= \chi_2(x) \chi_3(v) \{ \mathbf{1}_{\tilde{\Omega}}(x) f_{I,t}(t, x, v) + f_{E,t}(t, x, v) \} \\ &\quad \text{for } (t, x, v) \in (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned} \quad (2.110)$$

We note that in the sense of distributions  $\bar{f}_{R,t}$  solves

$$\begin{aligned} \varepsilon \partial_t \bar{f}_{R,t} + v \cdot \nabla_x \bar{f}_{R,t} &= \bar{g}_t \text{ in } (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3, \text{ where} \\ \bar{g}_t &:= \frac{v \cdot \nabla_x \chi_2}{\chi_2} \bar{f}_{R,t} + \mathbf{1}_{t \geq 0} \mathbf{1}_{\tilde{\Omega}}(x) \chi_2(x) \chi_3(v) [\varepsilon \partial_t + v \cdot \nabla_x] \partial_t f_R \\ &\quad + \mathbf{1}_{t \leq 0} \chi_2(x) \chi_3(v) \{ \varepsilon \partial_t \chi_1(t) f_{R,t}(0, x, v) + \chi_1(t) v \cdot \nabla_x f_{R,t}(0, x, v) \}. \end{aligned} \quad (2.111)$$

Here we have used the fact that  $\bar{f}_{R,t}$  in (2.111) is continuous along the characteristics across  $\partial \tilde{\Omega}$  and  $\{t = 0\}$ . We derive that, using (2.111),

$$\bar{f}_{R,t}(t, x, v) = \frac{1}{\varepsilon} \int_{-\infty}^t \bar{g}_t(s, x - \frac{t-s}{\varepsilon} v, v) ds \text{ for } (t, x, v) \in (-\infty, T] \times \mathbb{R}^3 \times \mathbb{R}^3. \quad (2.112)$$

Now we apply Lemma 2 to (2.112) and derive that, for  $p < 3$ ,

$$\begin{aligned} &\|S(\bar{g}_t)\|_{L^2_t((0,T);L^p_x(\mathbb{T}^2 \times \mathbb{R}))} \\ &\lesssim \|\mathbf{1}_{(t,x,v) \in \mathcal{D}_T} \bar{g}_t\|_{L^2((0,T) \times (\mathbb{T}^2 \times \mathbb{R}) \times \{|v| \leq N\})} \\ &\lesssim \|f_{R,t}(0)\|_{L^2(\Omega \times \mathbb{R}^3)} + \|\varepsilon \partial_t f_R + v \cdot \nabla_x f_R\|_{L^2((0,T) \times \tilde{\Omega} \times V)} \\ &\quad + \|\mathbf{1}_{(t,x,v) \in \mathcal{D}_T} f_{I,t}(t + \varepsilon \tilde{t}_B(x, v), \tilde{x}_B(x, v), v)\|_{L^2((-1,T] \times (U \setminus \tilde{\Omega}) \times V)}. \end{aligned} \quad (2.113)$$

Following the same argument of (2.104)-(2.105) we deduce that

$$\begin{aligned}
 (2.113) &\lesssim \|\partial_t f_R\|_{L^\infty([0,T];L^2(\Omega\times\mathbb{R}^3))} + \|\partial_t f_R(0)\|_{L^2_V} \\
 &\quad + \left\| -\frac{1}{\varepsilon\kappa}L(\mathbf{I}-\mathbf{P})\partial_t f_R + \varepsilon \times \text{r.h.s. of (2.6)} \right\|_{L^2((0,T]\times\Omega\times V)}.
 \end{aligned}
 \tag{2.114}$$

From (2.63)–(2.70), the last term of (2.114) is bounded above by

$$\begin{aligned}
 &\left\{ \frac{1}{\kappa} \left( \|\partial_t \mathcal{I}\|_{L^\infty_{t,x,v}} + \varepsilon \|(2.49)\|_{L^\infty_{t,x,v}} \right) \right\} \left\{ \|Pf_R\|_{L^2_{t,x}} + \|\sqrt{v}(\mathbf{I}-\mathbf{P})f_R\|_{L^2_{t,x,v}} \right\} \\
 &+ \left\{ \frac{1}{\kappa\varepsilon} + \frac{\varepsilon^{1/2}}{\kappa} \|\mathfrak{w}f_R\|_{L^\infty_{t,x,v}} + \frac{1}{\kappa} \left( \|Pf_1\|_{L^\infty_{t,x}} + \varepsilon \|(2.36)\|_{L^\infty_{t,x}} \right) \right\} \|\sqrt{v}(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{t,x,v}} \\
 &+ \left\{ \frac{\varepsilon^{1/2}}{\kappa} \|Pf_R\|_{L^\infty_{t,x}}^{\frac{3(p-2)}{p}} \|\mathfrak{w}f_R\|_{L^\infty_{t,x,v}}^{\frac{6-2p}{p}} + \frac{\varepsilon}{\kappa} \|(2.36)\|_{L^\infty_{t,x}}^{\frac{2p}{p-2}} \right\} \|P\partial_t f_R\|_{L^2_{t,x}L^p_x} \\
 &+ \frac{1}{\kappa} \|Pf_1\|_{L^\infty_{t,x}} \|P\partial_t f_R\|_{L^2_{t,x}L^2_x} \\
 &+ \varepsilon^{1/2} \|(2.65)\|_{L^2_{t,x,v}} + \frac{\varepsilon^{1/2}}{\kappa} \|\kappa(2.66)\|_{L^2_{t,x,v}} + \varepsilon^{3/2} \|(2.68)\|_{L^2_{t,x,v}}^2 + \frac{\varepsilon^{3/2}}{\kappa} \|\kappa(2.69)\|_{L^2_{t,x,v}}^2.
 \end{aligned}
 \tag{2.115}$$

On the other hand from (2.110) and the argument of (2.87) we derive

$$\begin{aligned}
 &\|S(\bar{g}_t)\|_{L^2_t((0,T);L^p_x(\mathbb{T}^2\times\mathbb{R}))} \gtrsim \left\| \int_{\mathbb{R}^3} \bar{f}_{R,t}(t,x,v)\varphi_i(v)\sqrt{\mu_0(v)}dv \right\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))} \\
 &\gtrsim \|P\partial_t f_R\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))} - (\kappa\varepsilon)^{\frac{2}{p-2}} \|\mathfrak{w}'\partial_t f_R\|_{L^2_t((0,T);L^\infty_{x,v}(\Omega\times\mathbb{R}^3))} \\
 &\quad - \frac{1}{\kappa\varepsilon} \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}.
 \end{aligned}
 \tag{2.116}$$

Here we have used

$$\begin{aligned}
 &\left\| \int_{\mathbb{R}^3} \chi_2(x)\chi_3(v)(\mathbf{I}-\mathbf{P})\partial_t f_R(t,x,v)\varphi_i(v)\sqrt{\mu_0(v)}dv \right\|_{L^2_t((0,T);L^p_x(\tilde{\Omega}))} \\
 &\leq \|(\mathbf{I}-\mathbf{P})\partial_t f_R(t,x,v)\|_{L^2_t((0,T);L^p_{x,v}(\tilde{\Omega}\times\mathbb{R}^3))} \\
 &\lesssim \left\| \|\mathfrak{w}'\partial_t f_R\|_{L^\infty_{x,v}(\Omega\times\mathbb{R}^3)}^{\frac{p-2}{p}} \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{x,v}(\Omega\times\mathbb{R}^3)}^{\frac{2}{p}} \right\|_{L^2_t((0,T))} \\
 &\lesssim \left\| \|\mathfrak{w}'\partial_t f_R\|_{L^\infty_{x,v}(\Omega\times\mathbb{R}^3)}^{\frac{p-2}{p}} \right\|_{L^{\frac{2p}{p-2}}_t((0,T))} \left\| \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2_{x,v}(\Omega\times\mathbb{R}^3)}^{\frac{2}{p}} \right\|_{L^p_t((0,T))} \\
 &\lesssim (\kappa\varepsilon)^{\frac{2}{p}} \|\mathfrak{w}'\partial_t f_R\|_{L^2_t((0,T);L^\infty_{x,v}(\Omega\times\mathbb{R}^3))}^{\frac{p-2}{p}} (\kappa\varepsilon)^{-\frac{2}{p}} \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}^{\frac{2}{p}} \\
 &\lesssim (\kappa\varepsilon)^{\frac{2}{p-2}} \|\mathfrak{w}'\partial_t f_R\|_{L^2_t((0,T);L^\infty_{x,v}(\Omega\times\mathbb{R}^3))} + (\kappa\varepsilon)^{-1} \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}.
 \end{aligned}
 \tag{2.117}$$

Combining (2.116), (2.113), (2.114), and (2.115) and choosing  $N \gg 1$  we conclude (2.75).

#### 2.4. $L_x^6$ -integrability gain for $\mathbf{P}f_R$

We prove an important  $L_x^6$ -integrability gain for  $\mathbf{P}f_R$  in the next proposition.

**Proposition 5.** *Under the same assumptions in Proposition 2, we have for all  $t \in [0, T]$*

$$\begin{aligned}
 & d_6 \|Pf_R(t)\|_{L_x^6} \\
 & \lesssim (\varepsilon \kappa^{-1} \|(2.36)\|_{L_{t,x}^\infty}) \|f_R(t)\|_{L_{x,v}^2} + \varepsilon \|\partial_t f_R(t)\|_{L_{x,v}^2} + o(1)(\kappa \varepsilon)^{1/2} \|\mathfrak{w} f_R(t)\|_{L_{x,v}^\infty} \\
 & \quad + \varepsilon^{1/2} |(2.36)|_{L^4(\partial\Omega)} \\
 & \quad + \varepsilon^{1/2} \|(2.46)\|_{L_{t,x}^2} + \frac{\varepsilon^{1/2}}{\kappa} \|\kappa(2.47)\|_{L_{t,x}^2} + \varepsilon^{3/2} \|(2.50)\|_{L_{t,x,v}^2} + \frac{\varepsilon^{3/2}}{\kappa} \|\kappa(2.51)\|_{L_{t,x,v}^2} \\
 & \quad + \left( \frac{1}{\varepsilon \kappa} + \frac{1}{\kappa} \|\mathcal{I}\|_\infty + \frac{\varepsilon}{\kappa} \|(2.36)\|_\infty + \frac{\varepsilon^{1/2}}{\kappa} \|\mathfrak{w}_\varrho f_R(t)\|_{L_{x,v}^\infty} \right) \\
 & \quad \times \left\{ \|(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2} + \|(\mathbf{I} - \mathbf{P})\partial_t f_R\|_{L_{t,x,v}^2} \right\} \\
 & \quad + \left( \frac{1}{\kappa} \|\mathcal{I}\|_\infty + \frac{\varepsilon}{\kappa} \|(2.36)\|_\infty \right) \|\mathbf{P}f_R\|_{L_{x,v}^2} \\
 & \quad + \|\mathfrak{w}_\varrho f_R(t)\|_{L_{x,v}^\infty}^{1/2} \left\{ |f_R|_{L_t^2 L^2(\gamma_+)}^{1/2} + |\partial_t f_R|_{L_t^2 L^2(\gamma_+)}^{1/2} \right\},
 \end{aligned} \tag{2.118}$$

where

$$d_6 := 1 - \left[ \frac{\varepsilon^{1/2}}{\kappa} \|Pf_R(t)\|_{L_x^6}^{1/2} \|Pf_R(t)\|_{L_x^2}^{1/2} \right]^{1/6}. \tag{2.119}$$

**Proof.** We view (2.5) as a weak formulation for a test function  $\psi$

$$\begin{aligned}
 & \underbrace{\iint_{\Omega \times \mathbb{R}^3} f_R v \cdot \nabla_x \psi}_{(2.120)_1} - \underbrace{\int_\gamma f_R \psi}_{(2.120)_2} - \underbrace{\iint_{\Omega \times \mathbb{R}^3} \varepsilon \partial_t f_R \psi}_{(2.120)_3} \\
 & = \iint_{\Omega \times \mathbb{R}^3} \psi \left\{ \frac{1}{\varepsilon \kappa} L f_R + \varepsilon^{1/2} \left( (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) - \frac{2}{\kappa} \Gamma(f_1, f_2) \right) - \varepsilon^{3/2} \left( -\partial_t f_2 + \frac{1}{\kappa} \Gamma(f_2, f_2) \right) \right. \\
 & \quad \left. - \frac{2}{\kappa} \Gamma(f_1 + \varepsilon f_2, f_R) + \frac{\varepsilon^{1/2}}{\kappa} \Gamma(f_R, f_R) \right\}.
 \end{aligned} \tag{2.120}$$

The proof of the lemma is based on a recent test function method in the weak formulation ([13, 14]). We define

$$\tilde{\mathbf{P}}f_R := \left\{ a + b \cdot v + c \frac{|v|^2 - 3}{\sqrt{6}} \right\} \sqrt{\mu} \text{ and } \tilde{P}f_R := (a, b, c), \tag{2.121}$$

where  $a := \langle f_R, \sqrt{\mu} \rangle$ ,  $b := \langle f_R, v\sqrt{\mu} \rangle$ , and  $c := \langle f_R, \frac{|v|^2-3}{\sqrt{6}}\sqrt{\mu} \rangle$ . We choose a family of test functions as

$$\psi_a := (|v|^2 - \beta_a)v\sqrt{\mu_0} \cdot \nabla_x \varphi_a, \quad (2.122)$$

$$\psi_{b,1}^{i,j} := (v_i^2 - \beta_b)\sqrt{\mu_0}\partial_j\varphi_b^i, \quad i, j = 1, 2, 3, \quad (2.123)$$

$$\psi_{b,2}^{i,j} := |v|^2 v_i v_j \sqrt{\mu_0} \partial_j \varphi_b^i, \quad i \neq j, \quad (2.124)$$

$$\psi_c := (|v|^2 - \beta_c)v\sqrt{\mu_0} \cdot \nabla_x \varphi_c, \quad (2.125)$$

where we choose  $\beta_a = 10$ ,  $\beta_b = 1$ ,  $\beta_c = 5$  such that

$$0 = \int_{\mathbb{R}^3} (|v|^2 - \beta_a) \frac{|v|^2 - 3}{\sqrt{6}} (v_1)^2 \mu_0(v) dv = \int_{\mathbb{R}} (v_1^2 - \beta_b) \mu_0(v_1) dv_1 = \int_{\mathbb{R}^3} (|v|^2 - \beta_c) v_i^2 \mu_0(v) dv. \quad (2.126)$$

Here,

$$-\Delta_x \varphi_a = a^5 \quad \text{with} \quad \frac{\partial \varphi_a}{\partial n} \Big|_{\partial\Omega} = 0, \quad (2.127)$$

$$-\Delta_x \varphi_b^j = b_j^5 \quad \text{with} \quad \varphi_b^j|_{\partial\Omega} = 0, \quad (2.128)$$

$$-\Delta_x \varphi_c = c^5 \quad \text{with} \quad \varphi_c|_{\partial\Omega} = 0. \quad (2.129)$$

A unique solvability to the above Poisson equations when  $(a, b, c) \in L^6(\Omega)$  and an estimate

$$\|\nabla_x^2 \varphi_{(a,b,c)}\|_{L^{6/5}(\Omega)} + \|\nabla_x \varphi_{(a,b,c)}\|_{L^2(\Omega)} + \|\varphi_{(a,b,c)}\|_{L^6(\Omega)} \lesssim \|\tilde{\mathbf{P}} f_R\|_{L^{6/5}(\Omega)}^5 \lesssim \|\tilde{\mathbf{P}} f_R\|_{L^6(\Omega)}^5. \quad (2.130)$$

is a direct consequence of Lax-Milgram and suitable extension (extend  $a^5$  of (2.127) evenly in  $x_3 \in \mathbb{R}$ , and  $b^5$  and  $c^5$  of (2.128) and (2.129) oddly in  $x_3 \in \mathbb{R}$ , then solve the Poisson equation, and then restrict the whole space solutions to the half space  $x_3 > 0$ ) and a standard elliptic estimate ( $L^{\frac{6}{5}}(\Omega) \rightarrow \dot{W}^{2,\frac{6}{5}}(\Omega) \cap \dot{W}^{1,2}(\Omega) \cap L^6(\Omega)$ ).

Clearly to prove the lemma and (2.118) it suffices to prove the same bound for  $\|\tilde{\mathbf{P}} f_R\|_{L_{x,v}^6} := \|(a, b, c)\|_{L_x^6}$ .

Following the direct computations in the proof of Lemma 2.12 in [14] we derive that

$$(2.120)_1 = \begin{cases} 5\|a(t)\|_6^6 + o(1)\|\tilde{\mathbf{P}} f_R(t)\|_6^6 + O(1)\|(\mathbf{I} - \mathbf{P})f_R(t)\|_6^6 & \text{if } \psi = \psi_a, \\ -2 \int_{\Omega} b_i \partial_i \partial_j \varphi_b^j + o(1)\|\tilde{\mathbf{P}} f_R(t)\|_6^6 + O(1)\|(\mathbf{I} - \mathbf{P})f_R(t)\|_6^6 & \text{if } \psi = \psi_{b,1}^{i,j}, \\ \int_{\Omega} b_j \partial_i \partial_j \varphi_b^i + \int_{\Omega} b_i \partial_j \partial_j \varphi_b^i + O(1)\|(\mathbf{I} - \mathbf{P})f_R(t)\|_6^6 & \text{if } \psi = \psi_{b,2}^{i,j} \text{ and } i \neq j, \\ 5\|c(t)\|_6^6 + o(1)\|\tilde{\mathbf{P}} f_R(t)\|_6^6 + O(1)\|(\mathbf{I} - \mathbf{P})f_R(t)\|_6^6 & \text{if } \psi = \psi_c. \end{cases} \quad (2.131)$$

For  $\|b_i\|_6^6$ , using the second and third estimate of (2.131) we deduce that

$$\begin{aligned}
\|b_i\|_{L^6(\Omega)}^6 &= - \int_{\Omega} b_i \Delta_x \varphi_b^i dx = - \int_{\Omega} b_i \partial_i^2 \varphi_b^i dx - \sum_{j(\neq i)} \int_{\Omega} b_i \partial_j^2 \varphi_b^i dx \\
&= \frac{1}{2} \sum_j (2.120)_1|_{\psi_{b,1}^{i,j}} - \sum_{j(\neq i)} (2.120)_1|_{\psi_{b,2}^{i,j}} + o(1) \|\tilde{\mathbf{P}} f_R(t)\|_6^6 + O(1) \|(\mathbf{I} - \mathbf{P}) f_R(t)\|_6^6.
\end{aligned} \tag{2.132}$$

Now we consider the boundary term  $(2.120)_2$ . From (2.122)-(2.125) and (2.126)

$$\int_{\gamma} \psi P_{\gamma+} f_R = \begin{cases} \int_{\partial\Omega} \partial_n \varphi_a \int_{\mathbb{R}^3} (|v|^2 - \beta_a)(v \cdot n)^2 \mu_0 dv dS_x = 0 & \text{if } \psi = \psi_a, \\ 0 & \text{if } \psi = \psi_{b,1}^{i,j} \text{ or } \psi_{b,2}^{i,j}, \\ \int_{\partial\Omega} \partial_n \varphi_c \int_{\mathbb{R}^3} (|v|^2 - \beta_c)(v \cdot n)^2 \mu_0 dv dS_x = 0 & \text{if } \psi = \psi_c. \end{cases} \tag{2.133}$$

Here we have used the Neumann boundary condition of (2.127) for  $\psi_a$ , and the last identity in (2.126) for  $\psi_c$ . For  $\psi_{b,1}^{i,j}$  or  $\psi_{b,2}^{i,j}$  we used the fact that the integrands are odd in  $v$ . From (1.31), we decompose  $f_R|_{\gamma} = P_{\gamma+} f_R + \mathbf{1}_{\gamma+}(1 - P_{\gamma+})f_R - \mathbf{1}_{\gamma-}\varepsilon^{1/2}(1 - P_{\gamma+})f_2$ . From (2.133) we have

$$\begin{aligned}
|(2.120)_2| &= \left| \cancel{\int_{\gamma} \psi P_{\gamma+} f_R} + \int_{\gamma} \psi \{\mathbf{1}_{\gamma+}(1 - P_{\gamma+})f_R - \mathbf{1}_{\gamma-}\varepsilon^{1/2}(1 - P_{\gamma+})f_2\} \right| \\
&\lesssim |\nabla_x \varphi|_{L^{4/3}(\partial\Omega)} \{ |(1 - P_{\gamma+})f_R|_{4,\gamma+} + \varepsilon^{1/2} |(2.36)|_{L^4(\partial\Omega)} \}
\end{aligned} \tag{2.134}$$

where we have used  $|\int_{\gamma+} \psi (1 - P_{\gamma+})f_R| \lesssim |\nabla_x \varphi|_{L^{4/3}(\partial\Omega)} |(1 - P_{\gamma+})f_R|_{4,\gamma+}$  at the last line. Here  $\varphi \in \{\varphi_a, \varphi_b, \varphi_c\}$ . For the first term of (2.134) we interpolate

$$|(1 - P_{\gamma+})f_R|_{4,\gamma+} \lesssim |\varepsilon^{-\frac{1}{2}}(1 - P_{\gamma+})f_R|_{2,\gamma+}^{\frac{1}{2}} \varepsilon^{\frac{1}{4}} \|\mathfrak{w}_{\varrho} f_R\|_{\infty}^{\frac{1}{2}}. \tag{2.135}$$

For the second term of (2.134), we use (2.130) and a trace theorem  $(\dot{W}^{1,\frac{6}{5}}(\mathbb{T}^2 \times \mathbb{R}_+) \cap L^2(\mathbb{T}^2 \times \mathbb{R}_+)) \rightarrow W^{1-\frac{1}{6},\frac{6}{5}}(\mathbb{T}^2)$ , and the Sobolev embedding  $(W^{\frac{1}{6},\frac{6}{5}}(\mathbb{T}^2) \rightarrow L^{4/3}(\mathbb{T}^2))$  to conclude that

$$|\nabla_x \varphi|_{L^{\frac{4}{3}}(\mathbb{T}^2)} \lesssim |\nabla_x \varphi|_{W^{\frac{1}{6},\frac{6}{5}}(\mathbb{T}^2)} \lesssim \|\nabla_x \varphi\|_{\dot{W}^{1,\frac{6}{5}}(\mathbb{T}^2 \times \mathbb{R}_+) \cap L^2(\Omega)} \lesssim \|\tilde{P} f_R\|_{L^6(\mathbb{T}^2 \times \mathbb{R}_+)}^5. \tag{2.136}$$

Next we consider  $(2.120)_3$ . For  $\psi$  of (2.122)-(2.125) and  $\varphi$  of (2.127)-(2.129), using (2.130), it follows that

$$\begin{aligned}
|(2.120)_3| &\lesssim \varepsilon \|\partial_t f_R\|_{L_{x,v}^2} \|\psi\|_{L_{x,v}^2} \lesssim \varepsilon \|\partial_t f_R\|_{L_{x,v}^2} \|\nabla_x \varphi\|_{L_x^2} \lesssim \varepsilon \|\partial_t f_R\|_{L_{x,v}^2} \|\tilde{P} f_R\|_{L_x^6}^5 \\
&\leq O(1) [\varepsilon \|\partial_t f_R\|_{L_{x,v}^2}]^6 + o(1) \|\tilde{P} f_R\|_{L_x^6}^6.
\end{aligned} \tag{2.137}$$

Lastly we consider the right hand side of (2.120). From (2.130), it follows

$$\begin{aligned}
 & \left| \iint_{\Omega \times \mathbb{R}^3} \psi \frac{1}{\varepsilon \kappa} L f_R \right| = \left| \iint_{\Omega \times \mathbb{R}^3} \psi \frac{1}{\varepsilon \kappa} L(\mathbf{I} - \mathbf{P}) f_R \right| \\
 & \lesssim \frac{1}{\varepsilon \kappa} \int_{\Omega} \int_{\mathbb{R}^3} |\nabla_x \varphi(a, b, c)(x)| \mu(v)^{1/4} \left[ v(v) |(\mathbf{I} - \mathbf{P}) f_R(x, v)| \right. \\
 & \quad \left. + \int_{\mathbb{R}^3} k_{\vartheta}(v, v_*) |(\mathbf{I} - \mathbf{P}) f_R(x, v_*)| dv_* \right] dv dx \\
 & \lesssim \frac{1}{\varepsilon \kappa} \|\nabla_x \varphi(a, b, c)\|_{L_x^2} \|(\mathbf{I} - \mathbf{P}) f_R\|_{L_{x,v}^2} \lesssim \frac{1}{\varepsilon \kappa} \|\tilde{P} f\|_{L_x^6}^5 \|(\mathbf{I} - \mathbf{P}) f_R\|_{L_{x,v}^2} \\
 & \leq o(1) \|\tilde{P} f\|_{L_x^6}^6 + [\varepsilon^{-1} \kappa^{-1} \|(\mathbf{I} - \mathbf{P}) f_R\|_{L_{x,v}^2}]^6.
 \end{aligned} \tag{2.138}$$

We have

$$\begin{aligned}
 & \left| \iint_{\Omega \times \mathbb{R}^3} \psi \frac{1}{\kappa} \Gamma(f_1 + \varepsilon f_2, f_R) \right| \\
 & \lesssim \|\nabla_x \varphi(a, b, c)\|_{L_x^2} \left( \frac{1}{\kappa} \|\mathcal{I}\|_{\infty} + \frac{\varepsilon}{\kappa} \|(2.36)\|_{\infty} \right) \|f_R\|_{L_{x,v}^2} \\
 & \leq o(1) \|\tilde{P} f\|_{L_x^6}^6 + \left[ \left( \frac{1}{\kappa} \|\mathcal{I}\|_{\infty} + \frac{\varepsilon}{\kappa} \|(2.36)\|_{\infty} \right) (\|P f_R\|_{L_{x,v}^2} + \|(\mathbf{I} - \mathbf{P}) f_R\|_{L_{x,v}^2}) \right]^6.
 \end{aligned} \tag{2.139}$$

For the contribution of  $\Gamma(f_R, f_R)$  we decompose  $f_R = \mathbf{P} f_R + (\mathbf{I} - \mathbf{P}) f_R$ , and we have

$$\begin{aligned}
 & |\Gamma(f_R, f_R)(v)| \\
 & \lesssim |\Gamma(\mathbf{P} f_R, \mathbf{P} f_R)(v)| + |\Gamma((\mathbf{I} - \mathbf{P}) f_R, (\mathbf{I} - \mathbf{P}) f_R)(v)| \\
 & \lesssim v(v) |P f_R|^2 \\
 & \quad + \|\mathfrak{w}_{\varrho} f_R\|_{\infty} \left\{ v(v) |(\mathbf{I} - \mathbf{P}) f_R(v)| + \int_{\mathbb{R}^3} k_{\vartheta}(v, v_*) |(\mathbf{I} - \mathbf{P}) f_R(v_*)| dv_* \right\}.
 \end{aligned} \tag{2.140}$$

Then from (2.122)–(2.125), and the Hölder's inequality ( $1 = 1/2 + 1/3 + 1/6$ )

$$\begin{aligned}
 & \left| \iint_{\Omega \times \mathbb{R}^3} \psi \frac{\varepsilon^{1/2}}{\kappa} \Gamma(f_R, f_R) \right| \\
 & \lesssim \frac{\varepsilon^{1/2}}{\kappa} \|\nabla_x \varphi(a, b, c)\|_{L_x^2} \left\{ \|P f_R\|_{L_x^3} \|P f_R\|_{L_x^6} + \|\mathfrak{w}_{\varrho} f_R\|_{L_{x,v}^{\infty}} \|(\mathbf{I} - \mathbf{P}) f_R\|_{L_{x,v}^2} \right\} \\
 & \lesssim \frac{\varepsilon^{1/2}}{\kappa} \|\tilde{P} f\|_{L_x^6}^5 \|P f_R\|_{L_x^6}^{3/2} \|P f_R\|_{L_x^2}^{1/2} + \frac{\varepsilon^{3/2}}{\kappa^{1/2}} \|\tilde{P} f\|_{L_x^6}^5 \|\mathfrak{w}_{\varrho} f_R\|_{L_{x,v}^{\infty}} \|\varepsilon^{-1} \kappa^{-1/2} (\mathbf{I} - \mathbf{P}) f_R\|_{L_{x,v}^2},
 \end{aligned} \tag{2.141}$$

where we have used an interpolation  $\|Pf_R\|_{L^3} \leq \|Pf_R\|_{L^6}^{1/2} \|Pf_R\|_{L^2}^{1/2}$  and (2.130) at the last step. A contribution of the rest of terms in the r.h.s. of (2.120) can be easily bounded as

$$\begin{aligned} & \iint_{\Omega \times \mathbb{R}^3} |\psi| \left| \varepsilon^{1/2} \left( (\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) - \frac{2}{\kappa} \Gamma(f_1, f_2) \right) - \varepsilon^{3/2} \left( -\partial_t f_2 + \frac{1}{\kappa} \Gamma(f_2, f_2) \right) \right| \\ & \lesssim \|Pf_R\|_{L_x^6}^5 \left\{ \varepsilon^{1/2} \|(2.46)\|_{L_{t,x}^2} + \frac{\varepsilon^{1/2}}{\kappa} \|\kappa(2.47)\|_{L_{t,x}^2} + \varepsilon^{3/2} \|(2.50)\|_{L_{t,x,v}^2} + \frac{\varepsilon^{3/2}}{\kappa} \|\kappa(2.51)\|_{L_{t,x,v}^2} \right\}. \end{aligned} \quad (2.142)$$

Finally, from a standard 1D embedding (see appendix (A.1) in [30] for the proof): for  $T > 0$ ,

$$|g(t)|^2 \lesssim_T \int_0^T |g(s)|^2 ds + \int_0^T |g'(s)|^2 ds \quad \text{for } t \in [0, T], \quad (2.143)$$

we collect the terms from (2.131) with (2.132), (2.134) with (2.135) and (2.136), (2.137), (2.138), (2.139), (2.141), (2.142) to conclude

$$\begin{aligned} \sup_{0 \leq s \leq t} \|(\mathbf{I} - \mathbf{P})f_R(s)\|_{L_{x,v}^2} & \lesssim \|(\mathbf{I} - \mathbf{P})f_R\|_{L_{t,x,v}^2} + \|(\mathbf{I} - \mathbf{P})\partial_t f_R\|_{L_{t,x,v}^2}, \\ \sup_{0 \leq s \leq t} \|\mathbf{P}f_R(s)\|_{L_{x,v}^2} & \lesssim \|\mathbf{P}f_R\|_{L_{t,x,v}^2} + \|\mathbf{P}\partial_t f_R\|_{L_{t,x,v}^2}, \\ \sup_{0 \leq s \leq t} |(1 - P_{\gamma_+})f_R(s)|_{L^2(\gamma_+)} & \lesssim \sup_{0 \leq s \leq t} |f_R(s)|_{L^2(\gamma_+)} \lesssim |f_R|_{L_t^2 L^2(\gamma_+)} + |\partial_t f_R|_{L_t^2 L^2(\gamma_+)}. \end{aligned} \quad (2.144)$$

This proves (2.118).  $\square$

## 2.5. $L^\infty$ -estimate

In this section we develop a unified  $L^\infty$ -estimate.

Recall the weight  $\mathfrak{w}$  in (2.34). We consider

$$h(t, x, v) = \mathfrak{w}(v) f_R(t, x, v). \quad (2.145)$$

An equation for  $h$  can be written from (2.5) and (1.31) as

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h + \frac{v}{\varepsilon^2 \kappa} h = \frac{1}{\varepsilon^2 \kappa} K_{\mathfrak{w}} h + \mathcal{S}_h, \quad (2.146)$$

$$h|_{\gamma_-} = \mathfrak{w} P_{\gamma_+} \left( \frac{h}{\mathfrak{w}} \right) + r. \quad (2.147)$$

For (2.145) and (2.5), we have  $r = -\varepsilon^{1/2} \mathfrak{w}(1 - P_{\gamma_+}) f_2$  and

$$\begin{aligned} \mathcal{S}_h &:= \frac{1}{\kappa \varepsilon^{1/2}} \Gamma_{\mathfrak{w}}(h, h) + \frac{2}{\kappa \varepsilon} \Gamma_{\mathfrak{w}}(\mathfrak{w} f_1 + \varepsilon \mathfrak{w} f_2, h) \\ &\quad + \frac{1}{\varepsilon^{1/2}} \left\{ -\mathfrak{w}(\mathbf{I} - \mathbf{P})(v \cdot \nabla_x f_2) + \frac{2}{\kappa} \Gamma_{\mathfrak{w}}(\mathfrak{w} f_1, \mathfrak{w} f_2) \right\} \\ &\quad + \varepsilon^{1/2} \left( -\mathfrak{w} \partial_t f_2 + \frac{1}{\kappa} \Gamma_{\mathfrak{w}}(\mathfrak{w} f_2, \mathfrak{w} f_2) \right), \end{aligned}$$

where we denote  $\Gamma_{\mathfrak{w}}(\cdot, \cdot)(v) := \mathfrak{w}(v) \Gamma(\frac{\cdot}{\mathfrak{w}}, \frac{\cdot}{\mathfrak{w}})(v)$  and  $K_{\mathfrak{w}}(\cdot) := \mathfrak{w} K(\frac{\cdot}{\mathfrak{w}})$ .

From (1.25)

$$\begin{aligned} &|\mathfrak{w}(v) \Gamma(\frac{h}{\mathfrak{w}}, \frac{h}{\mathfrak{w}})(v)| \\ &\leq \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot u| \sqrt{\mu(v_*)} e^{-\varrho |v_*|^2} \{ |h(v')| |h(v'_*)| + |h(v)| |h(v_*)| \} du dv_* \\ &\lesssim_{\varrho} v(v) \|h\|_{L_v^\infty}^2. \end{aligned} \quad (2.148)$$

From (expression of  $\mathbf{k}$ ) clearly we have

$$\mathbf{k}(v, v_*) \frac{\mathfrak{w}_{\varrho}(v)}{\mathfrak{w}_{\varrho}(v_*)} \leq \mathbf{k}_{\mathfrak{w}}(v, v_*) := \frac{2C_2}{|v - v_*|} e^{-\frac{|v - v_*|^2}{8} - \frac{1}{8} \frac{(|v - \varepsilon u|^2 - |v_* - \varepsilon u|^2)^2}{|v - v_*|^2}} \frac{\mathfrak{w}_{\varrho}(v)}{\mathfrak{w}_{\varrho}(v_*)}. \quad (2.149)$$

As in (estimate for  $\mathbf{k}$ ) we derive

$$\int_{\mathbb{R}^3} \mathbf{k}_{\mathfrak{w}}(v, v_*) dv_* \lesssim \frac{1}{1 + |v|}. \quad (2.150)$$

**Proposition 6.** Recall  $\mathfrak{w}_{\varrho}$  in (2.34). Assume the same assumptions in Proposition 2. Then

$$\begin{aligned} &d_{\infty} \|\mathfrak{w}_{\varrho} f_R\|_{L_{t,x,v}^\infty} \\ &\lesssim \|\mathfrak{w}_{\varrho} f(0)\|_{L_{x,v}^\infty} + \varepsilon^{1/2} \|(2.36)\|_{L_{t,x}^\infty} \\ &\quad + \varepsilon^{3/2} \kappa \left( \|(2.46)\|_{\infty} + \frac{1}{\kappa} \|\kappa(2.47)\|_{\infty} \right) + \varepsilon^{5/2} \kappa \left( \|(2.50)\|_{\infty} + \frac{1}{\kappa} \|\kappa(2.51)\|_{\infty} \right) \\ &\quad + \frac{1}{\varepsilon^{1/2} \kappa^{1/2}} \|P f_R\|_{L_t^\infty L_x^6} + \frac{1}{\varepsilon^{3/2} \kappa^{3/2}} \left\{ \|\sqrt{v}(\mathbf{I} - \mathbf{P}) f_R\|_{L_{t,x,v}^2} + \|\sqrt{v}(\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L_{t,x,v}^2} \right\}, \end{aligned} \quad (2.151)$$

where

$$d_{\infty} := 1 - \varepsilon^2 \|(2.36)\|_{L_{t,x}^\infty} - \varepsilon \|\mathcal{I}\|_{L_{t,x}^\infty} - \varepsilon^{3/2} \|\mathfrak{w}_{\varrho} f_R\|_{L_{t,x,v}^\infty}. \quad (2.152)$$

**Proposition 7.** Assume the same assumptions of Proposition 6. We denote

$$\mathfrak{w}'(v) := \mathfrak{w}_{\varrho'}(v) \text{ for } \varrho' < \varrho. \quad (2.153)$$

Let  $p < 3$ . Then

$$\begin{aligned}
 & d_{\infty,t} \|\mathfrak{w}' \partial_t f_R\|_{L_t^2((0,T); L_{x,v}^\infty(\Omega \times \mathbb{R}^3))} \\
 & \lesssim \varepsilon \kappa^{1/2} \|\mathfrak{w}' \partial_t f_R(0)\|_{L_{x,v}^\infty} + \frac{1}{\varepsilon^{3/p} \kappa^{3/p}} \|P \partial_t f\|_{L_t^2 L_x^p} + \frac{1}{\varepsilon^{3/2} \kappa^{3/2}} \|\sqrt{v}(\mathbf{I} - \mathbf{P}) \partial_t f\|_{L_{t,x,v}^2} \\
 & \quad + \varepsilon^{3/2} \left( \kappa \|(2.65)\|_{L_t^2 L_x^\infty} + \|\kappa(2.66)\|_{L_t^2 L_x^\infty} \right) + \varepsilon^{5/2} \left( \kappa \|(2.68)\|_{L_t^2 L_x^\infty} + \|\kappa(2.69)\|_{L_t^2 L_x^\infty} \right) \\
 & \quad + \left( \varepsilon \|\partial_t \mathcal{I}\|_{L_{t,x}^\infty} + \varepsilon^2 \|(2.49)\|_{L_{t,x}^\infty} \right) \|\mathfrak{w} f_R\|_{L_{t,x,v}^\infty},
 \end{aligned} \tag{2.154}$$

with

$$d_{\infty,t} := 1 - \varepsilon \|\mathcal{I}\|_{L_{t,x}^\infty} - \varepsilon^2 \|(2.36)\|_{L_{t,x}^\infty} - \varepsilon^{3/2} \|\mathfrak{w} f_R\|_{L_{t,x,v}^\infty}. \tag{2.155}$$

In the proof of propositions, for simplicity, we often use  $\|\cdot\|_\infty$  for  $\|\cdot\|_{L_{t,x,v}^\infty}$ ,  $\|\cdot\|_{L_{x,v}^\infty}$  or  $\|\cdot\|_{L_x^\infty}$  if there would be no confusion.

**Proof of Proposition 6.** We define backward exit time and position as

$$t_{\mathbf{b}}(x, v) := \varepsilon \frac{x_3}{v_3}, \quad x_{\mathbf{b}}(x, v) := x - \frac{x_3}{v_3} v \quad \text{for } (x, v) \in \Omega \times \mathbb{R}^3. \tag{2.156}$$

Since the characteristics for (2.146) are given by  $(x - \frac{t-s}{\varepsilon} v, v)$ , we have, for  $0 \leq t-s < t_{\mathbf{b}}(x, v)$ ,

$$\frac{d}{ds} \left\{ e^{-\int_s^t \frac{v}{\varepsilon^2 \kappa}} h(s, x - \frac{t-s}{\varepsilon} v, v) \right\} = e^{-\int_s^t \frac{v}{\varepsilon^2 \kappa}} \left\{ \frac{1}{\varepsilon^2 \kappa} K_{\mathfrak{w}} h + \mathcal{S}_h \right\} (s, x - \frac{t-s}{\varepsilon} v, v). \tag{2.157}$$

Here  $e^{-\int_s^t \frac{v}{\varepsilon^2 \kappa}} = e^{-\int_s^t \frac{v(\tau, x - \frac{t-\tau}{\varepsilon} v, v)}{\varepsilon^2 \kappa}} d\tau$ . We regard  $(x_1 - \frac{t-s}{\varepsilon} v_1, x_2 - \frac{t-s}{\varepsilon} v_2) \in \mathbb{R}^2$  belongs to  $\mathbb{T}^2$  without redefining them in  $[-\pi, \pi]^2$ .

Now we represent  $h$  using (2.157) and (2.147) as

$$\begin{aligned}
 h(t, x, v) &= \mathbf{1}_{t-t_{\mathbf{b}}(x,v) < 0} e^{-\int_0^t \frac{v}{\varepsilon^2 \kappa}} h(0, x - \frac{t}{\varepsilon} v, v) \\
 &+ \int_{\max\{0, t-t_{\mathbf{b}}(x,v)\}}^t e^{-\int_s^t \frac{v}{\varepsilon^2 \kappa}} \frac{1}{\varepsilon^2 \kappa} K_{\mathfrak{w}} h(s, x - \frac{t-s}{\varepsilon} v, v) ds
 \end{aligned} \tag{2.158}$$

$$\begin{aligned}
 &+ \int_{\max\{0, t-t_{\mathbf{b}}(x,v)\}}^t e^{-\int_s^t \frac{v}{\varepsilon^2 \kappa}} \mathcal{S}_h(s, x - \frac{t-s}{\varepsilon} v, v) ds \\
 &+ \mathbf{1}_{t-t_{\mathbf{b}}(x,v) \geq 0} e^{-\int_{t-t_{\mathbf{b}}(x,v)}^t \frac{v}{\varepsilon^2 \kappa}} h(t - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v).
 \end{aligned} \tag{2.159}$$

Since the integrand of (2.159) reads on the boundary, using the boundary condition (2.147) and (2.157) again, we represent it as

$$\begin{aligned}
& h(t - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v) \\
&= \mathfrak{w}(v) c_{\mu} \sqrt{\mu(v)} \int_{v_3 < 0} h(t - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), \mathfrak{v}) \frac{\sqrt{\mu(\mathfrak{v})} |\mathfrak{v}_3|}{\mathfrak{w}(\mathfrak{v})} d\mathfrak{v} + r(t - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v) \\
&= \mathfrak{w}(v) c_{\mu} \sqrt{\mu(v)} \int_{v_3 < 0} e^{-\int_0^{t-t_{\mathbf{b}}(x,v)} \frac{v}{\varepsilon^2 \kappa}} h(0, x_{\mathbf{b}}(x, v) - \frac{t - t_{\mathbf{b}}(x, v)}{\varepsilon} \mathfrak{v}, \mathfrak{v}) \frac{\sqrt{\mu(\mathfrak{v})} |\mathfrak{v}_3|}{\mathfrak{w}(\mathfrak{v})} d\mathfrak{v} \\
&\quad + \mathfrak{w}(v) c_{\mu} \sqrt{\mu(v)} \int_{v_3 < 0} \\
&\quad \times \int_0^{t-t_{\mathbf{b}}(x,v)} e^{-\int_s^{t-t_{\mathbf{b}}(x,v)} \frac{v}{\varepsilon^2 \kappa}} \frac{1}{\varepsilon^2 \kappa} K_{\mathfrak{w}} h(s, x_{\mathbf{b}}(x, v) - \frac{t - t_{\mathbf{b}}(x, v) - s}{\varepsilon} \mathfrak{v}, \mathfrak{v}) \frac{\sqrt{\mu(\mathfrak{v})} |\mathfrak{v}_3|}{\mathfrak{w}(\mathfrak{v})} ds d\mathfrak{v} \\
&\quad + \mathfrak{w}(v) c_{\mu} \sqrt{\mu(v)} \int_{v_3 < 0} \\
&\quad \times \int_0^{t-t_{\mathbf{b}}(x,v)} e^{-\int_s^{t-t_{\mathbf{b}}(x,v)} \frac{v}{\varepsilon^2 \kappa}} \mathcal{S}_h(s, x_{\mathbf{b}}(x, v) - \frac{t - t_{\mathbf{b}}(x, v) - s}{\varepsilon} \mathfrak{v}, \mathfrak{v}) \frac{\sqrt{\mu(\mathfrak{v})} |\mathfrak{v}_3|}{\mathfrak{w}(\mathfrak{v})} ds d\mathfrak{v} \\
&\quad + r(t - t_{\mathbf{b}}(x, v), x_{\mathbf{b}}(x, v), v),
\end{aligned} \tag{2.160}$$

where  $r = -\varepsilon^{1/2} \mathfrak{w}(1 - P_{\gamma_+}) f_2$  and  $e^{-\int_0^{t-t_{\mathbf{b}}(x,v)} \frac{v}{\varepsilon^2 \kappa}} := e^{-\int_0^{t-t_{\mathbf{b}}(x,v)} \frac{1}{\varepsilon^2 \kappa} v(\tau, x - \frac{t_{\mathbf{b}}(x,v)}{\varepsilon} v - \frac{t-t_{\mathbf{b}}(x,v)-s}{\varepsilon} \mathfrak{v}, \mathfrak{v}) d\tau}$ .

Note that, from (2.46), (2.47), (2.50), (2.51), and (2.36),

$$\begin{aligned}
& |\mathcal{S}_h(s, x - \frac{t-s}{\varepsilon} v, v)| \\
&\lesssim v(v) \frac{1}{\kappa \varepsilon^{1/2}} \|h\|_{\infty}^2 + \frac{v(v)}{\kappa \varepsilon} (\|\mathcal{I}\|_{\infty} + \varepsilon \|(2.36)\|_{\infty}) \|h\|_{\infty} \\
&\quad + \frac{1}{\varepsilon^{1/2}} \left( \|(2.46)\|_{\infty} + \frac{1}{\kappa} \|\kappa(2.47)\|_{\infty} \right) + \varepsilon^{1/2} \left( \|(2.50)\|_{\infty} + \frac{1}{\kappa} \|\kappa(2.51)\|_{\infty} \right), \\
&|\mathfrak{w}(1 - P_{\gamma_+}) f_2| \lesssim \|(2.36)\|_{\infty}.
\end{aligned} \tag{2.161}$$

We derive a preliminary estimate as

$$\begin{aligned}
& |h(t, x, v)| \lesssim e^{-\frac{v}{2\varepsilon^2 \kappa} t} \|h(0)\|_{\infty} \\
&\quad + \varepsilon^{3/2} \sup_{0 \leq s \leq t} \|h(s)\|_{\infty}^2 + \varepsilon^2 \sup_{0 \leq s \leq t} \|(2.36)\|_{\infty} \|h(s)\|_{\infty} + \varepsilon \sup_{0 \leq s \leq t} \|\mathcal{I}\|_{\infty} \|h(s)\|_{\infty} \\
&\quad + \varepsilon^{1/2} \sup_{0 \leq s \leq t} \|(2.36)\|_{\infty} + \varepsilon^{3/2} \kappa \left( \|(2.46)\|_{\infty} + \frac{1}{\kappa} \|\kappa(2.47)\|_{\infty} \right)
\end{aligned}$$

$$+ \varepsilon^{5/2} \kappa \left( \| (2.50) \|_{\infty} + \frac{1}{\kappa} \| \kappa (2.51) \|_{\infty} \right) \quad (2.162)$$

$$+ \int_0^t \frac{e^{-\frac{v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}}(v, v_*) |h(s, x - \frac{t-s}{\varepsilon}, v_*)| dv_* ds \quad (2.163)$$

$$+ \mathbf{w}(v) c_{\mu} \sqrt{\mu(v)} \int_{v_3 < 0} \int_0^{t-t_{\mathbf{b}}(x,v)} \frac{e^{-\frac{v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \\ \times \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}}(v, v_*) |h(s, x_{\mathbf{b}}(x, v) - \frac{t-t_{\mathbf{b}}(x, v)-s}{\varepsilon} v, v_*)| dv_* ds \frac{\sqrt{\mu(v)} |v_3|}{\mathbf{w}(v)} dv. \quad (2.164)$$

We note that  $|h(s, x - \frac{t-s}{\varepsilon}, v_*)|$  has the same upper bound. Then we bound (2.163) by a summation of (2.162) and

$$\sup_{\substack{(x_{\mathbf{b}}, v) \in \partial\Omega \times \mathbb{R}^3 \\ t-t_{\mathbf{b}} \geq 0}} \mathbf{w}(x_{\mathbf{b}}, v) c_{\mu} \sqrt{\mu(v)} \int_{v_3 < 0} \int_0^{t-t_{\mathbf{b}}} \frac{e^{-\frac{v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \\ \times \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}}(v, v_*) |h(s, x_{\mathbf{b}} - \frac{t-t_{\mathbf{b}}-s}{\varepsilon} v, v_*)| dv_* ds \frac{\sqrt{\mu(v)} |v_3|}{\mathbf{w}(x_{\mathbf{b}}, v)} dv, \quad (2.165)$$

and importantly

$$\int_0^t \frac{e^{-\frac{v(v)}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}}(v, v_*) \int_0^s \frac{e^{-\frac{v(v_*)}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \\ \times \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}}(v_*, v_{**}) |h(s, x - \frac{t-s}{\varepsilon} v - \frac{s-\tau}{\varepsilon} v_*, v_{**})| dv_{**} d\tau dv_* ds. \quad (2.166)$$

We consider (2.166). We decompose the integration of  $\tau \in [0, s] = [0, s - o(1)\varepsilon^2\kappa] \cup [s - o(1)\varepsilon^2\kappa, s]$ . The contribution of  $\int_{s-o(1)\varepsilon^2\kappa}^s \cdots d\tau$  is bounded as

$$\frac{2}{v(v)} (1 - e^{-\frac{v(v)}{2\varepsilon^2\kappa}}) \| \mathbf{k}_{\mathbf{w}}(v, \cdot) \|_{L^1} \frac{o(1)\varepsilon^2\kappa}{\varepsilon^2\kappa} \| \mathbf{k}_{\mathbf{w}}(v_*, \cdot) \|_{L^1} \sup_{0 \leq s \leq t} \| h(s) \|_{\infty} \leq o(1) \sup_{0 \leq s \leq t} \| h(s) \|_{\infty}. \quad (2.167)$$

For the rest of term we decompose  $\mathbf{k}_{\mathbf{w}}(v_*, v_{**}) = \mathbf{k}_{\mathbf{w}, N}(v_*, v_{**}) + \{\mathbf{k}_{\mathbf{w}}(v_*, v_{**}) - \mathbf{k}_{\mathbf{w}, N}(v_*, v_{**})\}$  where  $\mathbf{k}_{\mathbf{w}, N}(v_*, v_{**}) := \mathbf{k}_{\mathbf{w}}(v_*, v_{**}) \times \mathbf{1}_{\frac{1}{N} < |v_* - v_{**}| < N \text{ \& \& } |v_*| < N}$ . From (2.150),

$$\int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}}(v_*, v_{**}) \mathbf{1}_{|v_*| \geq N} dv_{**} \lesssim 1/N.$$

Also from the fact  $\mathbf{k}_{\mathbf{w}}(v_*, v_{**}) \leq \frac{e^{-C|v_*-v_{**}|^2}}{|v_*-v_{**}|} \in L^1(\{v_* - v_{**} \in \mathbb{R}^3\})$ ,  $\sup_{v_*} \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}}(v_*, v_{**}) \{\mathbf{1}_{\frac{1}{N} \geq |v_*-v_{**}|} + \mathbf{1}_{|v_*-v_{**}| \geq N}\} dv_{**} \downarrow 0$  as  $N \rightarrow \infty$ . Hence for  $N \gg 1$

$$\begin{aligned}
 (2.166) &\leq \int_0^t \frac{e^{-\frac{v(v)}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w},N}(v, v_*) \int_0^{s-o(1)\varepsilon^2\kappa} \frac{e^{-\frac{v(v_*)}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \\
 &\quad \times \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w},N}(v_*, v_{**}) |h(s, x - \frac{t-s}{\varepsilon}v - \frac{s-\tau}{\varepsilon}v_*, v_{**})| dv_{**} d\tau dv_* ds \\
 &\leq C_N \int_0^t \frac{e^{-\frac{v(v)}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_{|v_*| \leq 2N} \int_0^{s-o(1)\varepsilon^2\kappa} \frac{e^{-\frac{v(v_*)}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \\
 &\quad \times \int_{|v_{**}| < 2N} |f_R(s, x - \frac{t-s}{\varepsilon}v - \frac{s-\tau}{\varepsilon}v_*, v_{**})| dv_{**} d\tau dv_* ds \\
 &\quad + o(1) \sup_{0 \leq s \leq t} \|h(s)\|_{L_{x,v}^\infty},
 \end{aligned} \tag{2.168}$$

where we have used the fact  $\sup_x \mathbf{k}_{\mathbf{w}}(v_*, v_{**}) \mathbf{w}_Q(v_{**}) \leq C_N < \infty$  when  $\frac{1}{N} < |v_* - v_{**}| < N$  and  $|v_*| < N$  (then  $|v_{**}| < 2N$ ).

Now we decompose  $f_R = \mathbf{P}f_R + (\mathbf{I} - \mathbf{P})f_R$ . We first take integrations (2.168) over  $v_*$  and  $v_{**}$  and use Holder's inequality with  $p = 6$ ,  $p = 2$  in  $1/p + 1/p' = 1$  for  $\mathbf{P}f_R$ ,  $(\mathbf{I} - \mathbf{P})f_R$  respectively to derive

(2.168)

$$\begin{aligned}
 &\leq (4N)^3 C_N \frac{1}{v(v)} \\
 &\quad \times \sup_{\substack{0 \leq s \leq t \\ 0 \leq \tau \leq s-o(1)\varepsilon^2\kappa}} \left( \iint_{|v_*| \leq N, |v_{**}| \leq 2N} |\mathbf{P}f_R(s, x - \frac{t-s}{\varepsilon}v - \frac{s-\tau}{\varepsilon}v_*, v_{**})|^6 dv_{**} dv_* \right)^{1/6} \\
 &\quad + (4N)^3 C_N \frac{1}{v(v)} \\
 &\quad \times \sup_{\substack{0 \leq s \leq t \\ 0 \leq \tau \leq s-o(1)\varepsilon^2\kappa}} \left( \iint_{|v_*| \leq N, |v_{**}| \leq 2N} |(\mathbf{I} - \mathbf{P})f_R(s, x - \frac{t-s}{\varepsilon}v - \frac{s-\tau}{\varepsilon}v_*, v_{**})|^2 dv_{**} dv_* \right)^{1/2}.
 \end{aligned} \tag{2.169}$$

Now we consider a map

$$v_* \in \{\mathbb{R}^3 : |v_*| \leq N\} \mapsto y := x - \frac{t-s}{\varepsilon}v - \frac{s-\tau}{\varepsilon}v_* \in \Omega, \quad \text{where } \left| \frac{\partial y}{\partial v_*} \right| = \left| \frac{s-\tau}{\varepsilon} \right|^3 \gtrsim \varepsilon^3 \kappa^3. \tag{2.170}$$

We note that this mapping is not one-to-one and the image can cover  $\Omega$  at most  $N$  times. Therefore we have

$$\begin{aligned}
 & \left( \iint_{|v_*| \leq N, |v_{**}| \leq N} |\mathbf{P} f_R(s, x - \frac{t-s}{\varepsilon} v - \frac{s-\tau}{\varepsilon} v_*, v_{**})|^6 dv_{**} dv_* \right)^{1/6} \\
 & \leq N^{1/6} \left( \iint_{|v_*| \leq N, |v_{**}| \leq N} |\mathbf{P} f_R(s, y, v_{**})|^6 dv_{**} \frac{dy}{\varepsilon^3 \kappa^3} \right)^{1/6} \leq \frac{N^{1/6}}{\varepsilon^{1/2} \kappa^{1/2}} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}}, \\
 & \left( \iint_{|v_*| \leq N, |v_{**}| \leq N} |(\mathbf{I} - \mathbf{P}) f_R(s, x - \frac{t-s}{\varepsilon} v - \frac{s-\tau}{\varepsilon} v_*, v_{**})|^6 dv_{**} dv_* \right)^{1/6} \\
 & \leq \frac{N^{1/2}}{\varepsilon^{3/2} \kappa^{3/2}} \|(\mathbf{I} - \mathbf{P}) f_R(s)\|_{L^2_{x,v}}.
 \end{aligned}$$

Therefore we conclude that

$$\begin{aligned}
 (2.166) \quad & \leq (4N)^3 C_N(2.169) + o(1) \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty_{x,v}} \\
 & \leq (4N)^4 C_N \left\{ \frac{1}{\varepsilon^{1/2} \kappa^{1/2}} \sup_{0 \leq s \leq t} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}} + \frac{1}{\varepsilon^{3/2} \kappa^{3/2}} \sup_{0 \leq s \leq t} \|(\mathbf{I} - \mathbf{P}) f_R(s)\|_{L^2_{x,v}} \right\} \\
 & \quad + o(1) \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty_{x,v}} \\
 & \lesssim_N \frac{1}{\varepsilon^{1/2} \kappa^{1/2}} \sup_{0 \leq s \leq t} \|\mathbf{P} f_R(s)\|_{L^6_{x,v}} + \frac{1}{\varepsilon^{3/2} \kappa^{3/2}} \left\{ \|(\mathbf{I} - \mathbf{P}) f_R\|_{L^2_{t,x,v}} + \|(\mathbf{I} - \mathbf{P}) \partial_t f_R\|_{L^2_{t,x,v}} \right\} \\
 & \quad + o(1) \sup_{0 \leq s \leq t} \|h(s)\|_{L^\infty_{x,v}}, \tag{2.171}
 \end{aligned}$$

where we have used (2.143) the Sobolev embedding in 1D at the last line.

Now we consider (2.164) and (2.165). We decompose  $s \in [0, t - t_b] = [0, t - t_b - o(1)\varepsilon^2\kappa] \cup [t - t_b - o(1)\varepsilon^2\kappa, t - t_b]$ . The contribution of  $\int_{t-t_b-o(1)\varepsilon^2\kappa}^{t-t_b} \cdots$  is bounded as

$$\frac{o(1)\varepsilon^2\kappa}{\varepsilon^2\kappa} \|\mathbf{k}_{\mathbf{w}}(\mathbf{v}, \cdot)\|_{L^1} \sup_{0 \leq s \leq t} \|h(s)\|_\infty \leq o(1) \sup_{0 \leq s \leq t} \|h(s)\|_\infty. \tag{2.172}$$

For  $s \in [0, t - t_b - o(1)\varepsilon^2\kappa]$  we consider a map as (2.170)

$$\mathbf{v} \in \{\mathbf{v} \in \mathbb{R}^3 : v_3 < 0\} \mapsto y := x_b - \frac{t - t_b - s}{\varepsilon} \mathbf{v} \in \Omega, \quad \text{where} \quad \left| \frac{\partial y}{\partial \mathbf{v}} \right| = \left| \frac{t - t_b - s}{\varepsilon} \right|^3 \gtrsim \varepsilon^3 \kappa^3. \tag{2.173}$$

Following the argument to have (2.169) we bound

$$\begin{aligned} \text{the contribution of } & \int_0^{t-t_b-o(1)\varepsilon^2\kappa} \cdots \text{ of (2.165)} \\ & \lesssim_N \frac{1}{\varepsilon^{1/2}\kappa^{1/2}} \|\mathbf{P}f_R(s)\|_{L_{x,v}^6} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \left\{ \|(\mathbf{I}-\mathbf{P})f_R\|_{L_{t,x,v}^2} + \|(\mathbf{I}-\mathbf{P})\partial_t f_R\|_{L_{t,x,v}^2} \right\}. \end{aligned} \quad (2.174)$$

In conclusion, we bound  $|h(t, x, v)|$  by (2.162), (2.171), (2.172), (2.174) and conclude (2.151) by choosing small enough  $o(1)$  in (2.171) and (2.172).  $\square$

**Proof of Proposition 7.** Since many parts of the proof are overlapped with the proof of Proposition 6 we only pin point the differences. An equation for  $\mathfrak{w}'\partial_t f_R$  takes the similar form of (2.146) and (2.147). We can read (2.6) for

$$h(t, x, v) = \mathfrak{w}'(x, v)\partial_t f_R(t, x, v), \quad \text{for } \varrho' < \varrho, \quad (2.175)$$

as (2.146) and (2.147) replacing

$$\begin{aligned} S_h &= \frac{2}{\kappa\varepsilon} \Gamma_{\mathfrak{w}'}(\mathfrak{w}'f_1 + \varepsilon\mathfrak{w}'f_2, h) + \frac{2}{\varepsilon^{1/2}\kappa} \Gamma_{\mathfrak{w}'}(\mathfrak{w}'f_R, h) + \frac{2}{\kappa\varepsilon} \Gamma_{\mathfrak{w}'}(\mathfrak{w}'\partial_t f_1 + \varepsilon\mathfrak{w}'\partial_t f_2, \mathfrak{w}'f_R) \\ &\quad - \frac{1}{\varepsilon^{1/2}} \left\{ \mathfrak{w}'(\mathbf{I}-\mathbf{P})(v \cdot \nabla_x \partial_t f_2) - \frac{2}{\kappa} \Gamma_{\mathfrak{w}'}(\mathfrak{w}'\partial_t f_1, \mathfrak{w}'f_2) - \frac{2}{\kappa} \Gamma_{\mathfrak{w}'}(\mathfrak{w}'f_1, \mathfrak{w}'\partial_t f_2) \right\} \\ &\quad + \varepsilon^{1/2} \left( -\mathfrak{w}'\partial_t^2 f_2 + \frac{2}{\kappa} \Gamma_{\mathfrak{w}'}(\mathfrak{w}'\partial_t f_2, \mathfrak{w}'f_2) \right), \\ r &= -\varepsilon^{1/2} \mathfrak{w}'(1 - P_{\gamma_+})\partial_t f_2. \end{aligned} \quad (2.176)$$

We have the same equality of (2.158), (2.159) with (2.160) for  $h$  of (2.175) but replacing  $S_h$  and  $r$  of (2.176). From (2.49), (2.65), (2.66), (2.68), (2.69), and (2.148), we bound terms of (2.176)

$$\begin{aligned} |S_h| &\lesssim v(v) \left\{ \frac{1}{\kappa\varepsilon} (\|\mathcal{I}\|_\infty + \varepsilon\|(2.36)\|_\infty) + \frac{1}{\kappa^{1/2}\varepsilon} \|\mathfrak{w}'f_R\|_\infty \right\} \|h\|_\infty \\ &\quad + \frac{v(v)}{\kappa\varepsilon} \left\{ \|\partial_t \mathcal{I}\|_\infty + \varepsilon\|(2.49)\|_\infty \right\} \|\mathfrak{w}'f_R\|_\infty \end{aligned} \quad (2.177)$$

$$+ \frac{1}{\varepsilon^{1/2}} \left( \|(2.65)\|_\infty + \frac{1}{\kappa} \|\kappa(2.66)\|_\infty \right) + \varepsilon^{1/2} \left( \|(2.68)\|_\infty + \frac{1}{\kappa} \|\kappa(2.69)\|_\infty \right),$$

$$|r| \lesssim \varepsilon^{1/2} \|(2.49)\|_\infty. \quad (2.178)$$

Then as in (2.162)–(2.166) we derive a preliminary estimate as

$$|h(t, x, v)| \lesssim e^{-\frac{v}{2\varepsilon^2\kappa}t} \|h(0)\|_\infty + \frac{\varepsilon^2\kappa}{v(v)} (2.177) + (2.178) \quad (2.179)$$

$$+ \int_0^t \frac{e^{-\frac{v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}'}(v, v_*) |h(s, x - \frac{t-s}{\varepsilon}, v_*)| dv_* ds \quad (2.180)$$

$$+ \mathbf{w}'(x_{\mathbf{b}}(x, v), v) c_\mu \sqrt{\mu(v)} \int_{v_3 < 0} \int_0^{t-t_{\mathbf{b}}(x, v)} \frac{e^{-\frac{v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \times \int_{\mathbb{R}^3} \mathbf{k}_{\mathbf{w}'}(v, v_*) |h(s, x_{\mathbf{b}}(x, v) - \frac{t-t_{\mathbf{b}}(x, v)-s}{\varepsilon} \mathbf{v}, v_*)| dv_* ds \frac{\sqrt{\mu(\mathbf{v})} |\mathbf{v}_3|}{\mathbf{w}'(x_{\mathbf{b}}(x, v), \mathbf{v})} d\mathbf{v}. \quad (2.181)$$

As (2.165) and (2.166), we bound (2.180) by a summation of (2.179) and

$$\int_0^t \frac{e^{-\frac{C_{\mathbf{v}}}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_0^{s-o(1)\varepsilon^2\kappa} \frac{e^{-\frac{C_{\mathbf{v}}}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \int_{|v_*| \leq 2N} |h(s, x - \frac{t-s}{\varepsilon} v - \frac{s-\tau}{\varepsilon} v_*, v_{**})| dv_{**} dv_* d\tau ds, \quad (2.182)$$

$$+ \sup_{\substack{(x_{\mathbf{b}}, v) \in \partial\Omega \times \mathbb{R}^3 \\ t-t_{\mathbf{b}} \geq 0}} \mathbf{w}'(x_{\mathbf{b}}, v) c_\mu \sqrt{\mu(v)} \int_{v_3 < 0} \int_0^{t-t_{\mathbf{b}}-o(1)\varepsilon^2\kappa} \frac{e^{-\frac{v}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \times \int_{|v_*| \leq 2N} |h(s, x_{\mathbf{b}} - \frac{t-t_{\mathbf{b}}-s}{\varepsilon} \mathbf{v}, v_*)| dv_* ds \frac{\sqrt{\mu(\mathbf{v})} |\mathbf{v}_3|}{\mathbf{w}'(x_{\mathbf{b}}, \mathbf{v})} d\mathbf{v} \quad (2.183)$$

$$+ o(1) \sup_{0 \leq s \leq t} \|h(s)\|_{L_{x,v}^\infty}. \quad (2.184)$$

Then we follow the argument of (2.169)-(2.171) to derive that, for  $p < 3$ ,

$$|(2.182)| \lesssim \int_0^t \frac{e^{-\frac{C_{\mathbf{v}}}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_0^{s-o(1)\varepsilon^2\kappa} \frac{e^{-\frac{C_{\mathbf{v}}}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \frac{N^{1/3}}{\varepsilon^{3/p}\kappa^{3/p}} \|\mathbf{P}\partial_t f(\tau)\|_{L_{x,v}^p} d\tau ds \quad (2.185)$$

$$+ \int_0^t \frac{e^{-\frac{C_{\mathbf{v}}}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \int_0^{s-o(1)\varepsilon^2\kappa} \frac{e^{-\frac{C_{\mathbf{v}}}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \frac{N^{1/2}}{\varepsilon^{3/2}\kappa^{3/2}} \|\mathbf{P}\partial_t f(\tau)\|_{L_{x,v}^2} d\tau ds. \quad (2.186)$$

Now we use the Young's inequality for temporal convolution twice to derive that, for  $p < 3$ ,

$$\begin{aligned}
& \| (2.182) \|_{L_t^2(0,T)} \\
& \lesssim \left\| \frac{e^{-\frac{C_V}{2\varepsilon^2\kappa}|s|}}{\varepsilon^2\kappa} \right\|_{L_s^1(\mathbb{R})} \left\| \int_0^s \frac{e^{-\frac{C_V}{2\varepsilon^2\kappa}(s-\tau)}}{\varepsilon^2\kappa} \left( \frac{N^{1/3}}{\varepsilon^{3/p}\kappa^{3/p}} \|\mathbf{P}\partial_t f(\tau)\|_{L_{x,v}^p} + \frac{N^{1/2}}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f(\tau)\|_{L_{x,v}^2} \right) d\tau \right\|_{L_s^2(\mathbb{R})} \\
& \lesssim \left\| \frac{e^{-\frac{C_V}{2\varepsilon^2\kappa}|s|}}{\varepsilon^2\kappa} \right\|_{L_s^1(\mathbb{R})} \left\| \frac{e^{-\frac{C_V}{2\varepsilon^2\kappa}|\tau|}}{\varepsilon^2\kappa} \right\|_{L_t^1(\mathbb{R})} \\
& \quad \times \left( \frac{N^{1/3}}{\varepsilon^{3/p}\kappa^{3/p}} \|P\partial_t f\|_{L_t^2((0,T);L_x^p(\Omega))} + \frac{N^{1/2}}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)} \right) \\
& \lesssim_N \frac{1}{\varepsilon^{3/p}\kappa^{3/p}} \|P\partial_t f\|_{L_t^2((0,T);L_x^p(\Omega))} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}. \tag{2.187}
\end{aligned}$$

As in (2.174), for (2.183) we use (2.173) to derive that, for  $p < 3$ ,

$$\begin{aligned}
& \| (2.183) \|_{L_t^2(0,T)} \\
& \lesssim \left\| \int_0^t \frac{e^{-\frac{C_V}{2\varepsilon^2\kappa}(t-s)}}{\varepsilon^2\kappa} \left( \frac{1}{\varepsilon^{3/p}\kappa^{3/p}} \|\mathbf{P}\partial_t f(s)\|_{L_{x,v}^p} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f(s)\|_{L_{x,v}^2} \right) ds \right\|_{L_t^2(0,T)} \\
& \lesssim \left\| \frac{e^{-\frac{C_V}{2\varepsilon^2\kappa}|s|}}{\varepsilon^2\kappa} \right\|_{L_s^1(\mathbb{R})} \left\{ \frac{1}{\varepsilon^{3/p}\kappa^{3/p}} \|P\partial_t f\|_{L_t^2((0,T);L_x^p(\Omega))} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)} \right\} \\
& \lesssim \frac{1}{\varepsilon^{3/p}\kappa^{3/p}} \|P\partial_t f\|_{L_t^2((0,T);L_x^p(\Omega))} + \frac{1}{\varepsilon^{3/2}\kappa^{3/2}} \|(\mathbf{I}-\mathbf{P})\partial_t f\|_{L^2((0,T)\times\Omega\times\mathbb{R}^3)}, \tag{2.188}
\end{aligned}$$

where we have used the Young's inequality for temporal convolution.

In conclusion, we bound  $\|h\|_{L_t^2 L_{x,v}^\infty}$  by  $\|(2.179)\|_{L_t^2 L_{x,v}^\infty}$ , (2.187), (2.184), (2.188) and conclude (2.154) by choosing small enough  $o(1)$  in (2.184).  $\square$

## 2.6. Uniform controls of the Boltzmann remainder $f_R$ (Theorem 8)

Inspired by the energy structure of the PDE and the coercivity of the linear operator  $L$  in (1.30), we define an energy and a dissipation as

$$\begin{aligned}
\mathcal{E}(t) &:= \|f_R(t)\|_{L^2(\Omega \times \mathbb{R}^3)}^2 + \|\partial_t f_R(t)\|_{L^2(\Omega \times \mathbb{R}^3)}^2, \\
\mathcal{D}(t) &:= \int_0^t \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P}) f_R(s)\|_{L^2(\Omega \times \mathbb{R}^3)}^2 ds \\
&\quad + \int_0^t \|\kappa^{-\frac{1}{2}} \varepsilon^{-1} \sqrt{v}(\mathbf{I} - \mathbf{P}) \partial_t f_R(s)\|_{L^2(\Omega \times \mathbb{R}^3)}^2 ds \\
&\quad + \int_0^t \left( |\varepsilon^{-\frac{1}{2}} f_R(s)|_{L_Y^2}^2 + |\varepsilon^{-\frac{1}{2}} \partial_t f_R(s)|_{L_Y^2}^2 \right) ds.
\end{aligned} \tag{2.189}$$

As explained in Section 1.2, the temporal derivative gets involved mainly in order to access the  $L^6$ -bound of the hydrodynamic part  $\mathbf{P}f_R$ , while we will control the following auxiliary norm to be used in order to handle the nonlinearity: for  $p < 3$  and  $t > 0$

$$\begin{aligned}
\mathcal{F}_p(t) &:= \sup_{0 \leq s \leq t} \left\{ \|\kappa^{1/2} P f_R(s)\|_{L^6(\Omega)}^2 + \|\kappa^{1/2} P f_R\|_{L^2((0,s); L^p(\Omega))}^2 \right. \\
&\quad + \|\kappa^{\mathfrak{P}+1/2} P \partial_t f_R\|_{L^2((0,s); L^p(\Omega))}^2 + \|\varepsilon^{1/2} \kappa \mathfrak{w}_\varrho f_R(s)\|_{L^\infty(\Omega \times \mathbb{R}^3)}^2 \\
&\quad \left. + \|(\varepsilon \kappa)^{3/p} \kappa^{\frac{1}{2} + \mathfrak{P}} \mathfrak{w}_{\varrho'} f_R(s)\|_{L^2((0,s); L^\infty(\Omega \times \mathbb{R}^3))}^2 \right\}.
\end{aligned} \tag{2.190}$$

We will use the norms of the initial data:

$$\mathcal{E}(0) := \mathcal{E}(f_{R,0}) := \|f_{R,0}\|_{L^2(\Omega \times \mathbb{R}^3)}^2 + \|\partial_t f_{R,0}\|_{L^2(\Omega \times \mathbb{R}^3)}^2, \tag{2.191}$$

$$\begin{aligned}
\mathcal{F}_p(0) &:= \left\{ \kappa^{\frac{1}{2}} \|f_{R,0}\|_{L_Y^2}^2 + \kappa^{\mathfrak{P}+\frac{1}{2}} \|\partial_t f_{R,0}\|_{L_Y^2}^2 \right. \\
&\quad \left. + \varepsilon^{\frac{1}{2}} \kappa \|\mathfrak{w} f_{R,0}\|_{L^\infty(\bar{\Omega} \times \mathbb{R}^3)}^2 + (\varepsilon \kappa)^{1+\frac{3}{p}} \kappa^{\mathfrak{P}} \|\mathfrak{w}' \partial_t f_{R,0}\|_{L^\infty(\bar{\Omega} \times \mathbb{R}^3)}^2 \right\}^2.
\end{aligned} \tag{2.192}$$

**Theorem 8** (Uniform controls of the Boltzmann remainder  $f_R$ ). Recall  $\mathcal{I}$  in (2.28). Suppose for  $T > 0$  and  $\mathfrak{P} \geq 1/2$

$$\sum_{\ell=0,1} \|\nabla_x \partial_t^\ell \mathcal{I}\|_{L^\infty([0,T] \times \bar{\Omega})} + \frac{1}{\kappa^{1/2}} \sum_{\ell=0,1,2} \|\partial_t^\ell \mathcal{I}\|_{L^\infty([0,T] \times \bar{\Omega})} + \frac{1}{\kappa^{1/2}} \|P\|_{L^\infty([0,T] \times \bar{\Omega})} \lesssim \frac{1}{\kappa^{\mathfrak{P}}}. \tag{2.193}$$

We further assume that, for  $0 \leq \mathfrak{P}' < \mathfrak{P}$ ,

$$\begin{aligned}
&\sum_{\ell=1,2} \|\partial_t^\ell \mathcal{I}\|_{L^\infty([0,T]; L^\infty(\bar{\Omega}) \cap L^2(\Omega))} + \sum_{\substack{0 \leq \ell \leq 1 \\ 1 \leq |\beta| \leq 2}} \|\nabla_x^\beta \partial_t^\ell \mathcal{I}\|_{L^\infty([0,T]; L^\infty(\bar{\Omega}) \cap L^2(\Omega))} \\
&+ \sum_{|\beta|=1} \|\nabla_x^\beta \partial_t^2 \mathcal{I}\|_{L^2([0,T]; L^\infty(\bar{\Omega}) \cap L^2(\Omega))} \\
&+ \|\partial_t^2 P\|_{L^2([0,T]; L^\infty(\bar{\Omega}) \cap L^2(\Omega))} + \sum_{|\beta|=0,1} \|\nabla_x^\beta \partial_t P\|_{L^\infty([0,T]; L^\infty(\bar{\Omega}) \cap L^2(\Omega))} \lesssim \exp\left(\frac{1}{\kappa^{\mathfrak{P}'}}\right).
\end{aligned} \tag{2.194}$$

For given such  $T > 0$ , let us choose  $\varepsilon$  and  $\kappa$  as, for some  $\mathfrak{C} \gg 1$ ,

$$\varepsilon = \exp\left(\frac{-\mathfrak{C}(T+1)}{\kappa^{2\mathfrak{P}}}\right). \quad (2.195)$$

Assume that an initial datum for the remainder  $f_{R,in}$  satisfies, for some  $p < 3$  and  $|p-3| \ll 1$ ,

$$\sqrt{\mathcal{E}(0)} + \sqrt{\mathcal{F}_p(0)} \lesssim 1. \quad (2.196)$$

Then we construct a unique solution  $f_R(t, x, v)$  of the form of

$$F = \mu + \varepsilon f_1 \sqrt{\mu} + \varepsilon^2 f_2 \sqrt{\mu} + \varepsilon^{3/2} f_R \sqrt{\mu} \text{ in } [0, T] \times \Omega \times \mathbb{R}^3,$$

which solves the Boltzmann equation (1.1) and the diffuse reflection boundary condition (1.11) with the scale of (1.5), (1.6) and (2.195), and satisfies the initial condition  $F|_{t=0} = \mu + \varepsilon f_1|_{t=0} \sqrt{\mu} + \varepsilon^2 f_2|_{t=0} \sqrt{\mu} + \varepsilon^{3/2} f_R|_{t=0} \sqrt{\mu} + \delta \varepsilon f_{R,in}$ , in a time interval  $t \in [0, T]$ . Moreover, we have

$$\varepsilon^{\frac{1}{4} - \frac{3}{2p}(1-\frac{p}{3})} \sup_{0 \leq t \leq T} \{\sqrt{\mathcal{E}(t)} + \sqrt{\mathcal{D}(t)} + \sqrt{\mathcal{F}_p(t)}\} \lesssim 1. \quad (2.197)$$

**Proof of Theorem 8.** An existence of a unique global solution  $F$  for each  $\varepsilon > 0$  can be found in [13–16]. Thereby we only focus on the (a priori) estimates (2.197).

**Step 1.** Fix  $\delta_0 > 0$ ,  $\mathfrak{C}_1 > 0$ ,  $\mathfrak{C}_2 > 0$  such that  $0 < \frac{1}{\sqrt{\mathfrak{C}_1}} \ll \delta_0 \ll 1$ . And we choose  $\varepsilon$  with

$$\varepsilon \leq \left[ \frac{\kappa^{\frac{5}{2}+3\mathfrak{P}+\frac{3}{p}(1-\frac{p}{3})}}{\mathfrak{C}_1(\mathcal{E}(0) + \mathcal{F}_p(0) + 1)} \exp\left(\frac{-2\mathfrak{C}_2(T+1)}{\kappa^{2\mathfrak{P}}}\right) \right]^{\frac{1}{\frac{1}{2}-\frac{3}{p}(1-\frac{p}{3})}}. \quad (2.198)$$

Then we define  $T_* > 0$  as

$$\begin{aligned} T_* = \sup \left\{ t \geq 0 : \min\{d_2, d_{2,t}, d_6, d_3, d_{3,t}, d_\infty, d_{\infty,t}\} \geq \frac{\sigma_0}{4}, \right. \\ \frac{\varepsilon}{\kappa} \sqrt{\mathcal{D}(s)} + \varepsilon^{3/2} \|\mathfrak{w}_Q f(s)\|_{L_{x,v}^\infty} + \frac{\varepsilon}{\kappa^{1+\mathfrak{P}}} \|Pf_R(s)\|_{L_x^2} < \delta_0, \text{ and} \\ \varepsilon^{\frac{1}{4}-\frac{3}{2p}(1-\frac{p}{3})} \kappa^{-\frac{3}{4}-\frac{\mathfrak{P}}{2}-\frac{3}{2p}(1-\frac{p}{3})} \left( \sqrt{\mathcal{E}(s)} + \sqrt{\mathcal{D}(s)} + \frac{1}{\kappa^{1/2+\mathfrak{P}}} \|f_R(s)\|_{L_{x,v}^2} \right) < \delta_0, \\ \left. \text{for all } 0 \leq s \leq t \right\}, \end{aligned} \quad (2.199)$$

where  $d_2, d_{2,t}, d_6, d_3, d_{3,t}, d_\infty, d_{\infty,t}$  are defined in (2.31), (2.33), (2.119), (2.74), (2.76), (2.152) and (2.155). Note that from (2.198) we have  $\varepsilon^{\frac{1}{4}-\frac{3}{2p}(1-\frac{p}{3})} \kappa^{-\frac{3}{4}-\frac{\mathfrak{P}}{2}-\frac{3}{2p}(1-\frac{p}{3})} < \delta_0$ , so  $T_* > 0$  is well defined.

First from (2.193) and (2.194),

$$\begin{aligned} & \| (2.36) \|_{L_{t,x}^\infty} + \| (2.46) \|_{L_{t,x}^\infty} + \| \kappa (2.47) \|_{L_{t,x}^\infty} + \| (2.50) \|_{L_{t,x}^\infty} + \| \kappa (2.51) \|_{L_{t,x}^\infty} \\ & + \| (2.65) \|_{L_{t,x}^\infty} + \| \kappa (2.66) \|_{L_{t,x}^\infty} + \| (2.68) \|_{L_{t,x}^\infty} + \| \kappa (2.69) \|_{L_{t,x}^\infty} + \| (2.49) \|_{L_{t,x}^\infty} \lesssim \exp \left( \frac{2}{\kappa^{\mathfrak{P}'}} \right). \end{aligned}$$

Now from (2.194) and (2.199) we read all the estimates of Proposition 3, Proposition 5, Proposition 6, Proposition 4, and Proposition 7 in terms of  $\mathcal{E}(t)$  and  $\mathcal{D}(t)$  as follows.

From (2.151), (2.199), and (2.194)

$$\begin{aligned} \sup_{0 \leq s \leq t} \| \mathfrak{w}_\varrho f_R(s) \|_{L_{x,v}^\infty} & \lesssim \frac{1}{\varepsilon^{1/2} \kappa^{1/2}} \sup_{0 \leq s \leq t} \| \mathbf{P} f_R(s) \|_{L_{x,v}^6} + \frac{1}{\varepsilon^{1/2} \kappa} \sqrt{\mathcal{D}(t)} + \frac{1}{\varepsilon^{1/2} \kappa^{1+\mathfrak{P}}} \| P f_R \|_{L_{t,x}^2} \\ & + \| \mathfrak{w}_\varrho f(0) \|_\infty + \varepsilon^{1/2} \exp \left( \frac{3}{\kappa^{\mathfrak{P}'}} \right). \end{aligned} \quad (2.200)$$

Now applying (2.200) to (2.118) we derive that

$$\begin{aligned} \sup_{0 \leq s \leq t} \| P f_R(s) \|_{L_x^6} & \lesssim \frac{\varepsilon}{\kappa} \exp \left( \frac{1}{\kappa^{\mathfrak{P}'}} \right) \sup_{0 \leq s \leq t} \sqrt{\mathcal{E}(s)} + \frac{1}{\kappa^{1/2}} \sqrt{\mathcal{D}(t)} + \frac{1}{\kappa^{1/2+\mathfrak{P}}} \| P f_R \|_{L_{t,x}^2} \\ & + (\varepsilon \kappa)^{\frac{1}{2}} \| \mathfrak{w}_\varrho f(0) \|_{L_{x,v}^\infty} + \varepsilon^{1/2} \exp \left( \frac{3}{\kappa^{\mathfrak{P}'}} \right) + \frac{1}{\kappa^{1+\mathfrak{P}}} \sup_{0 \leq s \leq t} \| f_R \|_{L_{x,v}^2}. \end{aligned} \quad (2.201)$$

From (2.200), (2.201), and (2.199) and (2.194) we conclude that

$$\begin{aligned} & \sup_{0 \leq s \leq t} \left\{ \kappa^{\frac{1}{2}} \| P f_R(s) \|_{L_x^6} + \varepsilon^{\frac{1}{2}} \kappa \| \mathfrak{w}_\varrho f_R(s) \|_{L_{x,v}^\infty} \right\} \\ & \lesssim \underbrace{\varepsilon^{1/2} \kappa^{1/2} \exp \left( \frac{3}{\kappa^{\mathfrak{P}'}} \right) + \sqrt{\mathcal{F}_p(0)} + \sup_{0 \leq s \leq t} \left\{ \sqrt{\mathcal{E}(s)} + \sqrt{\mathcal{D}(s)} \right\} + \frac{1}{\kappa^{\mathfrak{P}}} \| P f_R \|_{L_{t,x}^2} + \frac{1}{\kappa^{1/2+\mathfrak{P}}} \sup_{0 \leq s \leq t} \| f_R \|_{L_{x,v}^2}}_{(2.202)_*}. \end{aligned} \quad (2.202)$$

From (2.73), (2.202), (2.199) and (2.194)

$$\kappa^{\frac{1}{2}} \| P f_R \|_{L_t^2((0,t); L_x^p)} \lesssim \underbrace{(2.202)_* \left( 1 + \frac{\varepsilon}{\kappa} (2.202)_* \right)}_{(2.203)_*}. \quad (2.203)$$

Using (2.202) and (2.193), from (2.75) and (2.154), we deduce that, for  $p < 3$  and  $\varrho' < \varrho$ ,

$$\kappa^{\frac{1}{2}+\mathfrak{B}} \| P \partial_t f_R \|_{L_t^2((0,t); L_x^p)} + (\varepsilon \kappa)^{3/p} \kappa^{\frac{1}{2}+\mathfrak{B}} \| \mathfrak{w}_{\varrho'} \partial_t f_R \|_{L_t^2((0,t); L_{x,v}^\infty)} \lesssim (2.203)_* \quad (2.204)$$

**Step 2.** Using the estimates of the previous step we will close the estimate ultimately in the basic energy estimates (2.30) and (2.32) via the Gronwall's inequality. We note that from (2.193) the

multipliers of  $\int_0^t \|Pf_R(s)\|_{L_x^2}^2 ds$  in (2.30) and  $\int_0^t \|P\partial_t f_R(s)\|_{L_x^2}^2 ds$  in (2.32) are bounded above by

$$O(1)\kappa^{-2\mathfrak{P}}\left(1 + \varepsilon\kappa^{\frac{1}{2}-\mathfrak{P}} + (\varepsilon\kappa^{\frac{1}{2}-\mathfrak{P}})^2\right) \lesssim \kappa^{-2\mathfrak{P}}, \quad (2.205)$$

where we have used (2.195).

In (2.30) and (2.32) we bound

$$\begin{aligned} \|\kappa^{1/2}Pf_R\|_{L_t^2 L_x^3} &\lesssim \kappa^{\frac{1}{2}(1-\frac{p}{3})} \|Pf_R\|_{L_t^2 L_x^\infty}^{1-\frac{p}{3}} \|\kappa^{1/2}Pf_R\|_{L_t^2 L_x^p}^{\frac{p}{3}} \\ &\lesssim_T (\varepsilon\kappa)^{-\frac{1}{2}(1-\frac{p}{3})} |(2.202)_*|^{1-\frac{p}{3}} |(2.203)_*|^{\frac{p}{3}}, \\ \|P\partial_t f_R\|_{L_t^2 L_x^3} &\lesssim \|P\partial_t f_R\|_{L_t^2 L_x^\infty}^{1-\frac{p}{3}} \|P\partial_t f_R\|_{L_t^2 L_x^p}^{\frac{p}{3}} \lesssim \varepsilon^{-\frac{3}{p}(1-\frac{p}{3})} \kappa^{-\frac{1}{2}-\mathfrak{P}-\frac{3}{p}(1-\frac{p}{3})} |(2.203)_*|. \end{aligned} \quad (2.206)$$

Applying (2.202), (2.203), (2.204), (2.206) to (2.30) +  $o(1)$ (2.32), using the above bound and (2.195), and collecting the terms, we derive that

$$\begin{aligned} &\sup_{0 \leq s \leq t} \mathcal{E}(s) + (d_2 - o(1))\mathcal{D}(t) \\ &\lesssim \mathcal{E}(0) + \mathcal{F}(0) + \exp\left(\frac{6}{\kappa^{\mathfrak{P}'}}\right) + (2.205) \int_0^T \mathcal{E}(s) ds \\ &\quad + \varepsilon^{1-(1-\frac{p}{3})} \kappa^{-4+\frac{p}{3}} |(2.202)_*|^{4-\frac{2p}{3}} |(2.203)_*|^{\frac{2p}{3}} \\ &\quad + \varepsilon^{1-\frac{6}{p}(1-\frac{p}{3})} \kappa^{-3-2\mathfrak{P}-\frac{6}{p}(1-\frac{p}{3})} |(2.202)_*|^2 |(2.203)_*|^2. \end{aligned} \quad (2.207)$$

Now from the last inequality in (2.199),

$$[\varepsilon^{1-(1-\frac{p}{3})} \kappa^{-4+\frac{p}{3}}]^{1/4} (2.202)_* \ll 1, \quad [\varepsilon^{1-\frac{6}{p}(1-\frac{p}{3})} \kappa^{-3-2\mathfrak{P}-\frac{6}{p}(1-\frac{p}{3})}]^{1/4} (2.202)_* \ll 1, \quad (2.208)$$

thus we derive that, for  $\mathfrak{C}_1 > 0$  and  $\mathfrak{C}_2 > 0$  large enough,

$$\sup_{0 \leq s \leq t} \mathcal{E}(s) + \mathcal{D}(t) \leq \mathfrak{C}_1 \left( \mathcal{E}(0) + \mathcal{F}_p(0) + \exp\left(\frac{6}{\kappa^{\mathfrak{P}'}}\right) \right) + \mathfrak{C}_2 \kappa^{-2\mathfrak{P}} \int_0^t \mathcal{E}(s) ds. \quad (2.209)$$

Applying the Gronwall's inequality to (2.209) (we may redefine  $\mathcal{E}(t)$  as  $\sup_{0 \leq s \leq t} \mathcal{E}(s)$  if necessary), and using the fact  $\mathfrak{P}' < \mathfrak{P}$  we derive that, after redefining  $\mathfrak{C}_1$  if necessary,

$$\sup_{0 \leq s \leq t} \mathcal{E}(s) + \mathcal{D}(t) + \mathcal{F}_p(t) \leq \mathfrak{C}_1 (\mathcal{E}(0) + \mathcal{F}_p(0) + 1) \exp\left(\frac{2\mathfrak{C}_2(t+1)}{\kappa^{2\mathfrak{P}}}\right) \quad \text{for all } t \leq T_*, \quad (2.210)$$

under the assumptions of (2.194), (2.199), and (2.208).

Now from (2.210) and (2.195) we derive (2.197), which implies

$$\sup_{0 \leq s \leq t} \left\{ \|\kappa^{1/2} P f_R(s)\|_{L_x^6} + \|\varepsilon^{1/2} \kappa \mathfrak{w}_{\varrho, \mathbb{B}} f_R(s)\|_{L_{x,v}^\infty} + \|(\varepsilon \kappa)^{3/p} \kappa^{\frac{1}{2} + \mathfrak{P}} \mathfrak{w}_{\varrho', \mathbb{B}} f_R(s)\|_{L^2((0,s); L_{x,v}^\infty)} \right\} \lesssim \varepsilon^{-\frac{1}{4} + \frac{3}{2p}(1-\frac{p}{3})}.$$

These imply  $\min\{d_2, d_{2,t}, d_6, d_3, d_{3,t}, d_\infty, d_{\infty,t}\} \geq \frac{1}{4}$  and  $\frac{\varepsilon}{\kappa} \sqrt{\mathcal{D}(t)} \ll \delta_0$  from (2.31), (2.33), (2.119), (2.74), (2.76), (2.152) and (2.155). Moreover, we have

$$\varepsilon^{\frac{1}{4} - \frac{3}{2p}(1-\frac{p}{3})} \kappa^{-\frac{3}{4} - \frac{\mathfrak{P}}{2} - \frac{3}{2p}(1-\frac{p}{3})} \left( \sqrt{\mathcal{E}(s)} + \sqrt{\mathcal{D}(s)} + \frac{1}{\kappa^{1/2+\mathfrak{P}}} \|f_R(s)\|_{L_{x,v}^2} \right) < \frac{1}{\sqrt{\mathfrak{C}}} \ll \delta_0.$$

Then by the standard continuation argument we can verify all assumptions (2.199) up to  $t \leq T$  and  $T = T_*$ . The estimate (2.197) follows easily.  $\square$

### 3. Fluid estimate

We denote the vorticity by

$$\omega = \nabla \times u, \quad u = \nabla \times (-\Delta)^{-1} \omega, \quad (3.1)$$

while the second identity is the famous Biot-Savart law. Here  $(-\Delta)^{-1}$  denotes the inverse of  $-\Delta$  with the zero Dirichlet boundary condition on  $\partial\Omega$ .

Our analysis of the Navier-Stokes-Fourier system is based on the vorticity formulation of the velocity field in 3D ([47,48]):

$$\partial_t \omega - \kappa \eta_0 \Delta \omega = -u \cdot \nabla \omega + \omega \cdot \nabla u \quad \text{in } \Omega, \quad (3.2)$$

$$\omega|_{t=0} = \omega_{in} \quad \text{in } \Omega, \quad (3.3)$$

$$\kappa \eta_0 (\partial_{x_3} + \sqrt{-\Delta_h}) \omega_h = [\partial_{x_3} (-\Delta)^{-1} (-u \cdot \nabla \omega_h + \omega \cdot \nabla u_h)], \quad \omega_3 = 0 \quad \text{on } \partial\Omega, \quad (3.4)$$

$$\partial_t \theta + u \cdot \nabla_x \theta - \kappa \eta_c \Delta \theta = 0 \quad \text{in } \Omega, \quad (3.5)$$

$$\theta = 0 \quad \text{on } \partial\Omega \quad (3.6)$$

where  $\sqrt{-\Delta_h} = |\nabla_h|$  is defined as

$$\sqrt{-\Delta_h} g(x_h, x_3) = \sum_{\xi \in \mathbb{Z}^2} |\xi| g_\xi(x_3) e^{i x_h \cdot \xi}. \quad (3.7)$$

Here,  $g_\xi(x_3) = \frac{1}{(2\pi)^2} \iint_{\mathbb{T}^2} e^{-i x_h \cdot \xi} g(x_h, x_3) dx_h \in \mathbb{C}$  with  $\xi = (\xi_1, \xi_2) \in \mathbb{Z}^2$  denotes the Fourier transform in the horizontal variables, which satisfies  $g(x_1, x_2, x_3) = \sum_{\xi \in \mathbb{Z}^2} g_\xi(x_3) e^{i x_h \cdot \xi}$ . The Fourier transform can be regarded as a function  $g_\xi(z)$  where  $z$  is sitting in a pencil-like complex domain: for any  $\lambda > 0$ ,

$$\mathcal{H}_\lambda := \left\{ z \in \mathbb{C} : \operatorname{Re} z \geq 0, |\operatorname{Im} z| < \lambda \min\{\operatorname{Re} z, 1\} \right\}. \quad (3.8)$$

### 3.1. Higher regularity of Navier-Stokes-Fourier system in the inviscid limit (Theorem 9)

In this section, we will use the following notations:  $x = (x_h, x_3) = (x_1, x_2, x_3) \in \mathbb{T}^2 \times \mathbb{R}_+ = \Omega$ ,  $\nabla_x = \nabla = (\nabla_h, \partial_3) = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ ; for a vector valued function  $g \in \mathbb{R}^3$ ,  $g = (g_h, g_3) = (g_1, g_2, g_3)$ .

We define analytic function spaces without the boundary layer,  $\mathfrak{L}^{p,\lambda}$ , for holomorphic functions with a finite norm, for  $p \geq 1$ ,

$$\|g\|_{p,\lambda} := \sum_{\xi \in \mathbb{Z}^2} e^{\lambda|\xi|} \|g_\xi\|_{\mathcal{L}_\lambda^p} \quad \text{where} \quad \|g_\xi\|_{\mathcal{L}_\lambda^p} := \sup_{0 \leq \sigma \leq \lambda} \left( \int_{\partial \mathcal{H}_\sigma} |g_\xi(z)|^p |dz| \right)^{1/p}. \quad (3.9)$$

Next we introduce an  $L^\infty$ -based analytic boundary layer function space, for  $\lambda > 0$  and  $\kappa \geq 0$ , that consists of holomorphic functions in  $\mathcal{H}_\lambda$  with a finite norm

$$\|g\|_{\infty,\lambda,\kappa} = \sum_{\xi \in \mathbb{Z}^2} e^{\lambda|\xi|} \|g_\xi\|_{\mathcal{L}_{\lambda,\kappa}^\infty}, \quad (3.10)$$

where  $\|g_\xi\|_{\mathcal{L}_{\lambda,0}^\infty} := \|e^{\bar{\alpha}\text{Re } z} g_\xi(z)\|_{\mathcal{L}_\lambda^\infty} := \sup_{z \in \mathcal{H}_\lambda} e^{\bar{\alpha}\text{Re } z} g_\xi(z)$  and

$$\|g_\xi\|_{\mathcal{L}_{\lambda,\kappa}^\infty} := \left\| \frac{e^{\bar{\alpha}\text{Re } z}}{1 + \phi_\kappa(z)} g_\xi(z) \right\|_{\mathcal{L}_\lambda^\infty} := \sup_{z \in \mathcal{H}_\lambda} \frac{e^{\bar{\alpha}\text{Re } z}}{1 + \phi_\kappa(z)} |g_\xi(z)|.$$

Here, a boundary layer weight function is defined as

$$\phi_\kappa(z) := \frac{1}{\sqrt{\kappa}} \phi\left(\frac{z}{\sqrt{\kappa}}\right) \quad \text{with} \quad \phi(z) = \frac{1}{1 + |\text{Re } z|^\tau} \quad \text{for some } \tau > 1. \quad (3.11)$$

We define  $\mathfrak{B}^{\lambda,\kappa}$  for holomorphic functions  $g = (g_1, g_2, g_3)$  with a finite norm

$$[[g]]_{\infty,\lambda,\kappa} = \sum_{i=1,2} \|g_i\|_{\infty,\lambda,\kappa} + \|g_3\|_{\infty,\lambda,0}. \quad (3.12)$$

And we say a scalar valued function  $\tilde{g} : \Omega \rightarrow \mathbb{R}$  is in  $\mathfrak{B}^{\lambda,\kappa}$  if  $\|\tilde{g}\|_{\infty,\lambda,\kappa} < \infty$ .

We note that  $\mathfrak{B}^{\lambda,\kappa} \subset \mathfrak{L}^{1,\lambda}$ , but  $\mathfrak{B}^{\lambda,0} \not\subset \mathfrak{L}^{\infty,\lambda}$  if  $\bar{\alpha} > 0$ .

Due to its singular nature of the Navier-Stokes flow in the inviscid limit, we introduce the conormal derivatives

$$D = (D_h, D_3) = (\nabla_h, \zeta(x_3)\partial_3) \quad \text{where} \quad \zeta(z) = \frac{z}{1+z}. \quad (3.13)$$

With the multi-indices  $\beta = (\beta_h, \beta_3) := (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$ , the higher derivatives are denoted by  $D^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3}$  and  $D_\xi^\beta = (i\xi_1)^{\beta_1} (i\xi_2)^{\beta_2} D_3^{\beta_3}$ .

Now we define, for  $\lambda_0 > 0$ ,  $\gamma_0 > 0$ ,  $\alpha > 0$ ,  $\kappa \geq 0$ , and  $t \in (0, \frac{\lambda_0}{2\gamma_0})$

$$\|g\|_{\infty, \kappa} = \sup_{\lambda < \lambda_0 - \gamma_0 t} \left\{ \sum_{0 \leq |\beta| \leq 1} [[D^\beta g]]_{\infty, \lambda, \kappa} + \sum_{|\beta|=2} (\lambda_0 - \lambda - \gamma_0 t)^\alpha [[D^\beta g]]_{\infty, \lambda, \kappa} \right\}, \quad (3.14)$$

$$\begin{aligned} \|g\|_1 = \sup_{\lambda < \lambda_0 - \gamma_0 t} \left\{ \sum_{0 \leq |\beta| \leq 1} \|D^\beta (1 + |\nabla_h|)g\|_{1, \lambda} \right. \\ \left. + (\lambda_0 - \lambda - \gamma_0 t)^\alpha \sum_{|\beta|=2} \|D^\beta (1 + |\nabla_h|)g\|_{1, \lambda} \right\}. \end{aligned} \quad (3.15)$$

And for a scalar valued function  $\tilde{g}$ , define

$$\|\tilde{g}\|_{\infty, 0} = \sup_{\lambda < \lambda_0 - \gamma_0 t} \left\{ \sum_{0 \leq |\beta| \leq 1} \|D^\beta \tilde{g}\|_{\infty, \lambda, 0} + \sum_{|\beta|=2} (\lambda_0 - \lambda - \gamma_0 t)^\alpha \|D^\beta \tilde{g}\|_{\infty, \lambda, 0} \right\}, \quad (3.16)$$

$$\|\tilde{g}\|_z := \sup_{\lambda < \lambda_0 - \gamma_0 t} \left\{ (\lambda_0 - \lambda - \gamma_0 t)^\alpha \sum_{0 \leq |\beta| \leq 1} \|D_h^{\beta_h} \partial_{x_3} g\|_{\infty, \lambda, 0} \right\}. \quad (3.17)$$

With an initial-boundary layer weight function as in [56]

$$\phi_{\kappa t}(z) = \frac{1}{\sqrt{\kappa t}} \phi\left(\frac{z}{\sqrt{\kappa t}}\right), \quad (3.18)$$

we define an initial-boundary layer function space  $\mathfrak{B}^{\lambda, \kappa t}$  for holomorphic functions  $g = (g_1, g_2, g_3)$  with a finite norm

$$[[g]]_{\infty, \lambda, \kappa t} = \sum_{i=1,2} \|g_i\|_{\infty, \lambda, \kappa t} + \|g_3\|_{\infty, \lambda, 0}, \quad (3.19)$$

where an  $L^\infty$ -based analytic norm with the initial-boundary layer is defined as

$$\|g\|_{\infty, \lambda, \kappa t} = \sum_{\xi \in \mathbb{Z}^2} e^{\lambda|\xi|} \|g_\xi\|_{\mathcal{L}_{\lambda, \kappa t}^\infty}, \quad \|g_\xi\|_{\mathcal{L}_{\lambda, \kappa t}^\infty} = \left\| \frac{e^{\bar{\alpha} \operatorname{Re} z}}{1 + \phi_\kappa(z) + \phi_{\kappa t}(z)} g_\xi(z) \right\|_{\mathcal{L}_\lambda^\infty}. \quad (3.20)$$

We finally define, for  $t \in (0, \frac{\lambda_0}{2\gamma_0})$ ,

$$\|g\|_{\infty, \kappa t} = \sup_{\lambda < \lambda_0 - \gamma_0 t} \left\{ \sum_{0 \leq |\beta| \leq 1} [[D^\beta g]]_{\infty, \lambda, \kappa t} + \sum_{|\beta|=2} (\lambda_0 - \lambda - \gamma_0 t)^\alpha [[D^\beta g]]_{\infty, \lambda, \kappa t} \right\}. \quad (3.21)$$

In this section,  $\alpha, \bar{\alpha}$  are given positive small constants,  $\lambda_0$  is a given positive constant, and  $\gamma_0$  is a sufficiently large constant to be determined in Theorem 9.

Next we discuss the initial data  $u_{in}$ ,  $\theta_{in}$ , and the corresponding vorticity  $\omega_{in} = \nabla_x \times u_{in}$ . Inspired by the PDEs, let

$$\begin{aligned}
\omega_0 &:= \omega_{in}, \quad \partial_t \omega_0 := \kappa \eta_0 \Delta \omega_0 - u_0 \cdot \nabla \omega_0 + \omega_0 \cdot \nabla u_0, \\
u_0 &:= \nabla \times (-\Delta)^{-1} \omega_0, \quad \partial_t u_0 := \nabla \times (-\Delta)^{-1} \partial_t \omega_0, \\
\partial_t^2 \omega_0 &:= \kappa \eta_0 \Delta \partial_t \omega_0 - u_0 \cdot \nabla \partial_t \omega_0 - \partial_t u_0 \cdot \nabla \omega_0 + \omega_0 \cdot \nabla \partial_t u_0 + \partial_t \omega_0 \cdot \nabla u_0, \\
\partial_t \theta_0 &:= \kappa \eta_c \Delta \theta_0 - u_0 \cdot \nabla \theta_0, \\
\partial_t^2 \theta_0 &:= \kappa \eta_c \Delta \partial_t \theta_0 - u_0 \cdot \nabla \partial_t \theta_0 - \partial_t u_0 \cdot \nabla \theta_0.
\end{aligned} \tag{3.22}$$

**Theorem 9.** Let  $\lambda_0 > 0$  and  $\omega_{in}, \theta_{in} \in \mathfrak{B}^{\lambda_0, \kappa}$  with (3.22) satisfy for  $\ell = 0, 1, 2$ ,

$$\sum_{0 \leq |\beta| \leq 2} \|D^\beta \partial_t^\ell \omega_0\|_{1, \lambda_0} + \sum_{0 \leq |\beta| \leq 2} \|D^\beta \partial_t^\ell \omega_0\|_{\infty, \lambda_0, \kappa} < \infty, \tag{3.23}$$

$$\sum_{0 \leq |\beta| \leq 2} \|D^\beta \partial_t^\ell \theta_0\|_{1, \lambda_0} + \sum_{0 \leq |\beta| \leq 2} \|D^\beta \partial_t^\ell \theta_0\|_{\infty, \lambda_0, 0} + \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t^\ell \theta_0(t, x_3)\|_{\infty, \lambda_0, 0} < \infty. \tag{3.24}$$

Further assume that  $\omega_{in} = \omega_0, \theta_{in} = \theta_0$ , and (3.22) satisfies the compatibility conditions on  $\partial\Omega$

$$\begin{aligned}
\kappa \eta_0 (\partial_{x_3} + \sqrt{-\Delta_h}) \omega_{0,h} &= [\partial_{x_3} (-\Delta)^{-1} (-u_0 \cdot \nabla \omega_{0,h} + \omega_0 \cdot \nabla u_{0,h})], \\
\omega_{0,3} &= 0, \quad \partial_t \omega_{0,3} = 0, \quad \theta_0 = 0, \quad \partial_t \theta_0 = 0, \quad \partial_t^2 \theta_0 = 0.
\end{aligned} \tag{3.25}$$

Then there exist a constant  $\gamma_0 > 0$  and a time  $T > 0$  depending only on  $\lambda_0$  and the size of the initial data such that the solution  $\omega(t)$  to the vorticity formulation of the Navier-Stokes equations (3.2)-(3.4) exists in  $C^1([0, T]; \mathfrak{B}^{\lambda, \kappa})$  with  $\partial_t^\ell \omega$  in  $C(0, T; \mathfrak{B}^{\lambda, \kappa t})$  for  $0 < \lambda < \lambda_0$  satisfying

$$\sup_{t \in [0, T]} \left[ \sum_{\ell=0}^2 \left\| \partial_t^\ell \omega(t) \right\|_1 + \sum_{\ell=0}^1 \left\| \partial_t^\ell \omega(t) \right\|_{\infty, \kappa} + \left\| \partial_t^2 \omega(t) \right\|_{\infty, \kappa t} \right] < \infty. \tag{3.26}$$

And the solution  $\theta(t)$  to (3.5)-(3.6) exists in  $C^2([0, T]; \mathfrak{B}^{\lambda, 0})$  satisfying

$$\sup_{t \in [0, T]} \left[ \sum_{\ell=0}^2 \left\| \partial_t^\ell \theta(t) \right\|_1 + \sum_{\ell=0}^2 \left\| \partial_t^\ell \theta(t) \right\|_{\infty, 0} \right] < \infty, \tag{3.27}$$

and

$$\sqrt{\kappa} \left( \sup_{t \in [0, T]} \sum_{\ell=0}^2 \left\| \partial_t^\ell \theta(t) \right\|_z \right) < \infty. \tag{3.28}$$

Furthermore, for each  $(t, x) \in [0, T] \times \Omega$ ,

(1) (Bounds on the vorticity and its derivatives)  $\omega(t, x)$  enjoys the following bounds:

$$|\nabla_h^i \partial_t^\ell \omega_h(t, x)| \lesssim e^{-\bar{\alpha} x_3} (1 + \phi_\kappa(x_3)), \quad |\nabla_h^i \partial_t^\ell \omega_3(t, x)| \lesssim e^{-\bar{\alpha} x_3} \text{ for } i, \ell = 0, 1, \tag{3.29}$$

$$|\partial_t^2 \omega_h(t, x)| \lesssim e^{-\bar{\alpha} x_3} (1 + \phi_\kappa(x_3) + \phi_{\kappa t}(x_3)), \quad |\partial_t^2 \omega_3(t, x)| \lesssim e^{-\bar{\alpha} x_3}, \tag{3.30}$$

$$|\partial_{x_3} \partial_t^\ell \omega_h(t, x)| \lesssim \kappa^{-1} e^{-\bar{\alpha} x_3}, \quad |\partial_{x_3} \partial_t^\ell \omega_3(t, x)| \lesssim e^{-\bar{\alpha} x_3} (1 + \phi_\kappa(x_3)) \text{ for } \ell = 0, 1. \tag{3.31}$$

(2) (Bounds on the velocity and its derivatives) The corresponding velocity field  $u(t, x)$  satisfies the following:

$$|\partial_t^\ell u(t, x)| \lesssim 1 \text{ for } \ell = 0, 1, 2, \quad (3.32)$$

$$\sum_{1 \leq |\beta| \leq 2} |\nabla^\beta \partial_t^\ell u(t, x)| \lesssim (1 + \phi_\kappa(x_3) + (|\beta| - 1)\kappa^{-1})e^{-\min(1, \frac{\tilde{\alpha}}{2})x_3} \text{ for } \ell = 0, 1, \quad (3.33)$$

$$\sum_{|\beta|=1} |\nabla^\beta \partial_t^2 u(t, x)| \lesssim (1 + \phi_\kappa(x_3) + \phi_{\kappa t}(x_3))e^{-\min(1, \frac{\tilde{\alpha}}{2})x_3}. \quad (3.34)$$

Moreover, we have the decay estimate for  $\partial_t^\ell u$ :

$$|\partial_t^\ell u| \lesssim \kappa^{-\frac{1}{2}}e^{-\min(1, \frac{\tilde{\alpha}}{2})x_3} \text{ for } \ell = 1, 2. \quad (3.35)$$

(3) (Bounds on the temperature and its derivatives) The temperature  $\theta(t, x)$  satisfies the following:

$$\sum_{0 \leq \beta_h \leq 2} |\nabla_h^{\beta_h} \partial_t^\ell \theta(t, x)| \lesssim e^{-\tilde{\alpha}x_3} \text{ for } \ell = 0, 1, 2, \quad (3.36)$$

$$\sum_{0 \leq |\beta_h| \leq 1} |\nabla_h^{\beta_h} \partial_{x_3} \partial_t^\ell \theta(t, x)| \lesssim \kappa^{-\frac{1}{2}}e^{-\tilde{\alpha}x_3} \text{ for } \ell = 0, 1, 2, \quad (3.37)$$

$$|\partial_{x_3}^2 \partial_t^\ell \theta(t, x)| \lesssim \kappa^{-1}e^{-\tilde{\alpha}x_3} \text{ for } \ell = 0, 1. \quad (3.38)$$

(4) (Bounds on the pressure and its derivatives) Choosing the pressure such that  $p(t, x) \rightarrow 0$  as  $x_3 \rightarrow \infty$ , then  $p$  satisfies the following:

$$|\partial_t^\ell p(t, x)| \lesssim 1 \text{ for } \ell = 0, 1, 2, \quad (3.39)$$

$$\sum_{0 \leq |\beta| \leq 1} |\nabla^\beta \partial_t^\ell p(t, x)| \lesssim \kappa^{-\frac{1}{2}}e^{-\min(1, \frac{\tilde{\alpha}}{2})x_3} \text{ for } \ell = 0, 1, \quad (3.40)$$

$$|\partial_t^2 p| \lesssim (\kappa^{-\frac{1}{2}} + \phi_{\kappa t}(x_3))e^{-\min(1, \frac{\tilde{\alpha}}{2})x_3}. \quad (3.41)$$

The proof of the theorem relies on the integral representation of the solution to the Navier-Stokes-Fourier system using the Green's function for the Stokes problem in the same spirit of [56].

### 3.2. Elliptic estimates and nonlinear estimates

**Lemma 3** ([56, 58], Embeddings and Cauchy estimates). The following holds

- (1)  $\mathfrak{B}^{\lambda, \kappa t} \subset \mathfrak{L}^{1, \lambda}$  and  $\mathfrak{B}^{\lambda, \kappa} \subset \mathfrak{L}^{1, \lambda}$ .
- (2)  $\|g_1 g_2\|_{*, \lambda} \lesssim \|g_1\|_{\infty, \lambda} \|g_2\|_{*, \lambda}$ .
- (3)  $\sum_{|\beta|=1} \|D^\beta g\|_{*, \lambda} \lesssim \frac{\|g\|_{*, \tilde{\lambda}}}{\lambda - \tilde{\lambda}}$ , for any  $0 < \lambda < \tilde{\lambda}$ .

For (2) and (3),  $\|\cdot\|_{*,\lambda}$  can be either  $\|\cdot\|_{\infty,\lambda,\kappa}$  or  $\|\cdot\|_{\infty,\lambda,\kappa t}$  or  $\|\cdot\|_{\infty,\lambda,0}$  or  $\|\cdot\|_{1,\lambda}$ .

**Lemma 4** ([56,58], Elliptic estimates). Let  $\phi$  be the solution of  $-\Delta\phi = \omega$  with the zero Dirichlet boundary condition, and let  $u = \nabla \times \phi$ . Then

$$\begin{aligned} \|u\|_{\infty,\lambda} + \|\nabla u\|_{1,\lambda} &\lesssim \|\omega\|_{1,\lambda}, \\ \|\nabla_h u\|_{\infty,\lambda} + \|\nabla u_3\|_{\infty,\lambda} &\lesssim \sum_{0 \leq |\beta| \leq 1} \|\nabla_h^\beta \omega\|_{1,\lambda}, \\ \|\partial_3 u_h\|_{\infty,\lambda} &\lesssim \sum_{0 \leq |\beta| \leq 1} \|\nabla_h^\beta \omega\|_{1,\lambda} + \|\omega_h\|_{\infty,\lambda}, \\ \|\zeta^{-1} \nabla_h^{\beta'} u_3\|_{\infty,\lambda} &\lesssim \sum_{0 \leq |\beta| \leq 1} \|\nabla_h^{\beta+\beta'} \omega_h\|_{1,\lambda}. \end{aligned} \quad (3.42)$$

As a consequence of Lemma 4, we have the following nonlinear estimates.

**Lemma 5** ([56,58]). Let  $u$  and  $\tilde{u}$  be the velocity field associated with  $\omega = \nabla_x \times u$  and  $\tilde{\omega} = \nabla_x \times \tilde{u}$  respectively. Then

$$\begin{aligned} \|u \cdot \nabla \tilde{\omega}\|_{1,\lambda} &\lesssim \|\omega\|_{1,\lambda} \|\nabla_h \tilde{\omega}\|_{1,\lambda} + \|(1 + |\nabla_h|)\omega\|_{1,\lambda} \|\zeta \partial_z \tilde{\omega}\|_{1,\lambda}, \\ \|\omega \cdot \nabla \tilde{u}_3\|_{1,\lambda} &\lesssim \|\omega_h\|_{1,\lambda} \|\nabla_h \tilde{u}_3\|_{\infty,\lambda} + \|\omega_3\|_{1,\lambda} \|\partial_3 \tilde{u}_3\|_{\infty,\lambda} \lesssim \|\omega\|_{1,\lambda} \|(1 + |\nabla_h|)\tilde{\omega}\|_{1,\lambda}, \\ \|\omega \cdot \nabla \tilde{u}_h\|_{1,\lambda} &\lesssim \|\omega_h\|_{1,\lambda} \|\nabla_h \tilde{u}_h\|_{\infty,\lambda} + \|\omega_3\|_{\infty,\lambda} \|\partial_3 \tilde{u}_h\|_{1,\lambda} \\ &\lesssim \|\omega\|_{1,\lambda} (\|\tilde{\omega}_3\|_{\infty,\lambda} + \|(1 + |\nabla_h|)\omega\|_{1,\lambda}). \end{aligned} \quad (3.43)$$

Moreover

$$\begin{aligned} \|u \cdot \nabla \tilde{\omega}_h\|_{*,\lambda} &\lesssim \|\omega\|_{1,\lambda} \|\nabla_h \tilde{\omega}_h\|_{*,\lambda} + (\|(1 + |\nabla_h|)\omega\|_{1,\lambda} + \|\zeta \partial_z \omega_3\|_{\infty,\lambda}) \|\zeta \partial_z \tilde{\omega}_h\|_{*,\lambda}, \\ \|\omega \cdot \nabla \tilde{u}_h\|_{*,\lambda} &\lesssim \|\omega_3\|_{\infty,\lambda,0} (\|(1 + |\nabla_h|)\tilde{\omega}\|_{1,\lambda} + \|\tilde{\omega}_h\|_{*,\lambda}) + \|\omega_h\|_{*,\lambda} \sum_{0 \leq |\beta| \leq 1} \|\nabla_h^\beta \tilde{\omega}\|_{1,\lambda}, \end{aligned} \quad (3.44)$$

where  $\|\cdot\|_{*,\lambda}$  can be either  $\|\cdot\|_{\infty,\lambda,\kappa}$  or  $\|\cdot\|_{\infty,\lambda,\kappa t}$ .

Furthermore

$$\begin{aligned} \|u \cdot \nabla \tilde{\omega}_3\|_{\infty,\lambda,0} &\lesssim \|\omega\|_{1,\lambda} \|\nabla_h \tilde{\omega}_3\|_{\infty,\lambda,0} + \|(1 + |\nabla_h|)\omega\|_{1,\lambda} \|\zeta \partial_3 \tilde{\omega}_3\|_{\infty,\lambda,0}, \\ \|\omega \cdot \nabla \tilde{u}_3\|_{\infty,\lambda,0} &\lesssim \|\omega_h\|_{*,\lambda} \|(1 + |\nabla_h|^2)\tilde{\omega}_h\|_{1,\lambda} + \|\omega_3\|_{\infty,\lambda,0} \|(1 + |\nabla_h|)\tilde{\omega}_h\|_{1,\lambda}, \end{aligned} \quad (3.45)$$

where  $\|(1 + |\nabla_h|^k)g\|_* = \sum_{\ell=0}^k \|\nabla_h^\ell g\|_*$ .

We finally record the crucial estimate of nonlinear forcing terms  $N = -u \cdot \nabla \omega + \omega \cdot \nabla u$ , as an outcome of Lemma 5, that will be also crucially used to control  $B = [\partial_{x_3}(-\Delta)^{-1}(-u \cdot \nabla \omega + \omega \cdot \nabla u)]|_{x_3=0}$  in the vorticity formulation (1.12) and (1.14).

**Lemma 6** ([56,58], Nonlinear estimate). Let  $\lambda \in (0, \lambda_0 - \gamma s)$  be given. We have the following:

$$\begin{aligned} \|(1 + |\nabla_h|)N\|_{1,\lambda} &\lesssim (\|(1 + |\nabla_h|)\omega\|_{1,\lambda} + \|(1 + |\nabla_h|)\omega_3\|_{\infty,\lambda,0})\|(1 + |\nabla_h|^2)\omega\|_{1,\lambda} \\ &\quad + \sum_{|\beta|=1} \|(1 + |\nabla_h|)D^\beta \omega\|_{1,\lambda} \|(1 + |\nabla_h|^2)\omega\|_{1,\lambda}, \end{aligned} \quad (3.46)$$

$$\begin{aligned} &\sum_{|\beta|=1} \|D^\beta (1 + |\nabla_h|)N\|_{1,\lambda} \\ &\lesssim \sum_{|\beta|\leq 1} \|D^\beta (1 + |\nabla_h|)\omega\|_{1,\lambda} \left( \sum_{|\beta|\leq 2} \|D^\beta (1 + |\nabla_h|)\omega\|_{1,\lambda} + \|(1 + |\nabla_h|)\omega\|_{\infty,\lambda,0} \right) \\ &\quad + \sum_{|\beta|\leq 1} \|D^\beta (1 + |\nabla_h|)\omega_3\|_{\infty,\lambda,0} \|(1 + |\nabla_h|)^2\omega\|_{1,\lambda}. \end{aligned} \quad (3.47)$$

For  $[\cdot]_{*,\lambda}$  to be either  $[\cdot]_{\infty,\lambda,\kappa}$  or  $[\cdot]_{\infty,\lambda,\kappa t}$ ,

$$[[N]]_{*,\lambda} \lesssim \|(1 + |\nabla_h|^2)\omega\|_{1,\lambda} [[\omega]]_{*,\lambda} + \|(1 + |\nabla_h|)\omega\|_{1,\lambda} [[D\omega]]_{*,\lambda}, \quad (3.48)$$

$$\begin{aligned} \sum_{|\beta|=1} [[D^\beta N]]_{*,\lambda} &\lesssim \sum_{|\beta|=1} \|(1 + |\nabla_h|^{|\beta|+2})\omega\|_{1,\lambda} [[\omega]]_{*,\lambda} \\ &\quad + \sum_{|\beta|=1} [[D^\beta \omega]]_{*,\lambda} (\|(1 + |\nabla_h|^2)\omega\|_{1,\lambda} + \beta_3 [[D_3^{\beta_3} \omega]]_{*,\lambda}) \\ &\quad + \sum_{|\beta|=2} [[D^\beta \omega]]_{*,\lambda} \|(1 + |\nabla_h|)\omega\|_{1,\lambda}. \end{aligned} \quad (3.49)$$

The proof relies on Lemma 5. We refer to Lemma 4.2 and Lemma 4.5 in [58] for the detailed proof.

### 3.3. Green's function and integral representation

By taking the Fourier transform of (1.12)-(1.13) in  $x_h \in \mathbb{T}^2$ , we obtain

$$\partial_t \omega_\xi - \kappa \eta_0 \Delta_\xi \omega_\xi = N_\xi \quad \text{in } \mathbb{R}_+, \quad (3.50)$$

$$\kappa \eta_0 (\partial_{x_3} + |\xi|) \omega_{\xi,h} = B_\xi, \quad \omega_{\xi,3} = 0 \quad \text{on } x_3 = 0, \quad (3.51)$$

with  $\omega_\xi|_{t=0} = \omega_{0\xi}$  for  $\xi \in \mathbb{Z}^2$ . Here

$$\Delta_\xi = -|\xi|^2 + \partial_{x_3}^2, \quad (3.52)$$

and

$$N_\xi = N_\xi(t, x_3) := (-u \cdot \nabla \omega + \omega \cdot \nabla u)_\xi(t, x_3), \quad B_\xi = B_\xi(t) := (\partial_{x_3} (-\Delta_\xi)^{-1} N_{\xi,h}(t))|_{x_3=0}. \quad (3.53)$$

Here  $(-\Delta_\xi)^{-1}$  denotes the inverse of  $-\Delta_\xi$  with the zero Dirichlet boundary condition at  $x_3 = 0$ .

We give the integral representation and present key estimates on Green's function for the Stokes problem. As shown in [56,58], letting  $G_\xi(t, x_3, y)$  be the Green's function for (3.50)-(3.51), the solution can be represented by the integral formula via Duhamel's principle:

$$\begin{aligned} \omega_\xi(t, x_3) = & \int_0^\infty G_\xi(t, x_3, y) \omega_{0\xi}(y) dy + \int_0^t \int_0^\infty G_\xi(t-s, x_3, y) N_\xi(s, y) dy ds \\ & - \int_0^t G_\xi(t-s, x_3, 0) (B_\xi(s), 0) ds, \end{aligned} \quad (3.54)$$

where

$$G_\xi = \begin{bmatrix} G_{\xi h} & 0 & 0 \\ 0 & G_{\xi h} & 0 \\ 0 & 0 & G_{\xi 3} \end{bmatrix}, \quad (3.55)$$

with  $G_{\xi h}$  of (3.62) and  $G_{\xi 3}$  of (3.65); for  $G_{\xi*}$  can be either  $G_{\xi h}$  or  $G_{\xi 3}$

$$\partial_t G_{\xi*}(t, x_3, y) - \kappa \eta_0 \Delta_\xi G_{\xi*}(t, x_3, y) = 0, \quad x_3 > 0, \quad (3.56)$$

$$\kappa \eta_0 (\partial_{x_3} + |\xi|) G_{\xi h}(t, x_3, y) = 0, \quad x_3 = 0, \quad (3.57)$$

$$G_{\xi 3}(t, x_3, y) = 0, \quad x_3 = 0. \quad (3.58)$$

Similarly for  $\theta$ , by taking the Fourier transform of (3.5), (3.6) in  $x_h$ , we have  $\theta_\xi(t, x_3)$  solves

$$\begin{aligned} \partial_t \theta_\xi - \kappa \eta_c \Delta_\xi \theta_\xi &= M_\xi \text{ in } \mathbb{R}_+, \\ \theta_\xi &= 0 \text{ on } x_3 = 0, \end{aligned} \quad (3.59)$$

with  $\theta_\xi|_{t=0} = \theta_{0\xi}$  for  $\xi \in \mathbb{Z}^2$ . Here  $M_\xi = M_\xi(t, x_3) := (-u \cdot \nabla \theta)_\xi(t, x_3)$ . Thus the integral representation is

$$\theta_\xi(t, x_3) = \int_0^\infty G_{\xi 3}(t, x_3, y) \theta_{0\xi}(y) dy + \int_0^t \int_0^\infty G_{\xi 3}(t-s, x_3, y) M_\xi(s, y) dy ds, \quad (3.60)$$

here by abuse of notation  $G_{\xi 3}$  is the same Green's function solving (3.56) and (3.58), with  $\eta_0$  replaced by  $\eta_c$ . Note that since  $u = 0$  on  $\partial\Omega$ , we have

$$\partial_t^\ell M_\xi(t, 0) = 0, \quad \ell = 0, 1, 2. \quad (3.61)$$

The following estimates and properties for  $G_\xi$  will be useful to show the propagation of analytic norms of  $\partial_t^\ell \omega$ ,  $\partial_t^\ell \theta$ , for  $\ell = 0, 1, 2$ .

**Lemma 7** ([56,58]).

- (1) (Bounds on  $G_{\xi h}$ ) The Green's function  $G_{\xi h}$  for the Stokes problem (3.56) and (3.57) is given by

$$G_{\xi h} = \tilde{H}_\xi + R_\xi, \quad (3.62)$$

where  $\tilde{H}_\xi$  is the one dimensional Heat kernel in the half-space with the homogeneous Neumann boundary condition which takes the form of

$$\tilde{H}_\xi(t, x_3, y) = H_\xi(t, x_3 - y) + H_\xi(t, x_3 + y) = \frac{1}{\sqrt{\kappa\eta_0 t}} \left( e^{-\frac{|x_3-y|^2}{4\kappa\eta_0 t}} + e^{-\frac{|x_3+y|^2}{4\kappa\eta_0 t}} \right) e^{-\kappa\eta_0 |\xi|^2 t}, \quad (3.63)$$

and the residual kernel  $R_\xi$  due to the boundary condition satisfies

$$|\partial_{x_3}^k R_\xi(t, x_3, y)| \lesssim b^{k+1} e^{-\theta_0 b(x_3+y)} + \frac{1}{(\kappa\eta_0 t)^{(k+1)/2}} e^{-\theta_0 \frac{|x_3+y|^2}{\kappa\eta_0 t}} e^{-\frac{\kappa\eta_0 |\xi|^2 t}{8}}, \quad (3.64)$$

with  $b = |\xi| + \frac{1}{\sqrt{\kappa\eta_0}}$  and  $R_\xi(t, x_3, y) = R_\xi(t, x_3 + y)$ .

- (2) (Formula of  $G_{\xi 3}$ ) The Green's function  $G_{\xi 3}$  for the Stokes problem (3.56) and (3.58) is given by one dimensional Heat kernel in the half-space with the homogeneous Dirichlet boundary condition as

$$G_{\xi 3}(t, x_3, y) = H_\xi(t, x_3 - y) - H_\xi(t, x_3 + y) = \frac{1}{\sqrt{\kappa\eta_0 t}} \left( e^{-\frac{|x_3-y|^2}{4\kappa\eta_0 t}} - e^{-\frac{|x_3+y|^2}{4\kappa\eta_0 t}} \right) e^{-\kappa\eta_0 |\xi|^2 t}. \quad (3.65)$$

- (3) (Complex extension) The Green's function  $G_\xi$  has a natural extension to the complex domain  $\mathcal{H}_\lambda$  for small  $\lambda > 0$  with similar bounds in terms of  $\operatorname{Re} y$  and  $\operatorname{Re} z$  (cf. (3.16) in [56]). The solution  $\omega_\xi$  to (3.50)–(3.51) in  $\mathcal{H}_\lambda$  has a similar representation: for any  $z \in \mathcal{H}_\lambda$ , let  $\sigma$  be the positive constant so that  $z \in \partial\mathcal{H}_\lambda$ , then  $\omega_\xi$  satisfies

$$\begin{aligned} \omega_\xi(t, z) &= \int_{\partial\mathcal{H}_\sigma} G_\xi(t, z, y) \omega_{0\xi}(y) dy + \int_0^t \int_{\partial\mathcal{H}_\sigma} G_\xi(t-s, z, y) N_\xi(s, y) dy ds \\ &\quad - \int_0^t G_\xi(t-s, z, 0) (B_\xi(s), 0) ds. \end{aligned}$$

The proof of Lemma 7 can be found in Proposition 3.3 and Section 3.3 of [56]. The next lemma concerns the convolution estimates.

**Lemma 8.** Let  $T > 0$  be given. For any  $0 \leq s < t \leq T$  and  $k \geq 0$ , there exists a constant  $C_T > 0$  so that the following estimates hold: for  $G_{\xi*}$  can be either  $G_{\xi h}$  or  $G_{\xi 3}$

(1) ( $\mathcal{L}_\lambda^1$  estimates)

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi*}(t, z, y) g_\xi(y) dy \right\|_{\mathcal{L}_\lambda^1} \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}_\lambda^1}, \quad (3.66)$$

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi*}(t-s, z, y) g_\xi(y) dy \right\|_{\mathcal{L}_\lambda^1} \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}_\lambda^1}. \quad (3.67)$$

(2) ( $\mathcal{L}_{\lambda, \kappa t}^\infty$  estimates)

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi*}(t, z, y) g_\xi(y) dy \right\|_{\mathcal{L}_{\lambda, \kappa t}^\infty} \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}_{\lambda, \kappa}^\infty}, \quad (3.68)$$

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi*}(t-s, z, y) g_\xi(y) dy \right\|_{\mathcal{L}_{\lambda, \kappa t}^\infty} \leq C_T \sum_{j=0}^k \sqrt{\frac{t}{s}} \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}_{\lambda, \kappa s}^\infty}. \quad (3.69)$$

(3) ( $\mathcal{L}_{\lambda, \kappa}^\infty$  estimates) For either  $\kappa = 0$  or  $\kappa > 0$ 

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi*}(t, z, y) g_\xi(y) dy \right\|_{\mathcal{L}_{\lambda, \kappa}^\infty} \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}_{\lambda, \kappa}^\infty}, \quad (3.70)$$

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j \int_0^\infty G_{\xi*}(t-s, z, y) g_\xi(y) dy \right\|_{\mathcal{L}_{\lambda, \kappa}^\infty} \leq C_T \sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j g_\xi \right\|_{\mathcal{L}_{\lambda, \kappa}^\infty}. \quad (3.71)$$

The proof of (1) and (2) can be found in Propositions 3.7 and 3.8 of [56]. For the proof of (3) we refer to Lemma 12 of [30].

The next result concerns the estimates for the trace kernel.

**Lemma 9.** Let  $a_\xi(s) = [\partial_z(-\Delta_\xi)^{-1} g_\xi] |_{z=0}$ . Then for any  $0 \leq s < t \leq T$  and  $k \geq 0$ , we have the following

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j G_{\xi h}(t-s, z, 0) a_\xi(s) \right\|_{\mathcal{L}_\lambda^1} \lesssim \|g_\xi\|_{\mathcal{L}_\lambda^1}, \quad (3.72)$$

$$\sum_{j=0}^k \left\| (\zeta(z)\partial_z)^j G_{\xi h}(t-s, z, 0) a_\xi(s) \right\|_{\mathcal{L}_{\lambda, \kappa}^\infty} \lesssim \frac{1}{\sqrt{t-s}} \|g_\xi\|_{\mathcal{L}_\lambda^1}. \quad (3.73)$$

We refer the proof to Lemma 13 of [30].

### 3.4. Proof of Theorem 9

Our goal is to show that  $\omega(t)$  indeed belongs to  $C^1([0, T]; \mathfrak{B}^{\lambda, \kappa})$  without the initial layer under the compatibility condition (3.25), and that  $\partial_t^2 \omega$  in  $\mathfrak{B}^{\lambda, \kappa t}$  with the initial layer. And  $\theta(t)$  belongs to  $C^2([0, T]; \mathfrak{B}^{\lambda, 0})$  under the compatibility condition (3.25). The existence of  $\omega(t)$  and  $\theta(t)$  in their corresponding spaces under the assumption of Theorem 9 can be proved by following the argument of [56] and [58]. For the 2D case, Theorem 1.1 of [56] indeed ensures the existence of  $\omega(t)$  in  $C^1([0, T]; \mathfrak{B}^{\lambda, \kappa t})$  under the assumption of Theorem 9. Such a result follows from Lemma 3, Lemma 7, Lemma 8, Lemma 4, Lemma 5. A 3D result can be obtained analogously. Hence, it suffices to show the propagation of the analytic norms in (3.26).

The propagation estimate for the vorticity  $\omega$  is analogous to [30]; we provide steps for reader's convenience and refer to [30] for the detailed proof of some estimates when the same proof holds. The detailed proof will be given for the new estimate for the temperature fluctuation  $\theta(t)$ .

**Step 1: Propagation of analytic norms for  $\omega$ .** It is convenient to define

$$\|\omega(t)\|_t := \|\omega(t)\|_{\infty, \kappa} + \|\omega(t)\|_1. \quad (3.74)$$

From the nonlinear iteration using the representation formula (3.54), by using Lemma 3 and Lemma 6, we obtain for sufficiently large  $\gamma_0$

$$\sup_{0 \leq t < \frac{\lambda_0}{2\gamma_0}} \|\omega(t)\|_t \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta \omega_0\|_{\infty, \lambda_0, \kappa} + \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) \omega_0\|_{1, \lambda_0}. \quad (3.75)$$

We refer to [30] for the detailed proof.

**Step 2: Propagation of analytic norms for  $\partial_t \omega$ .** The continuity of  $\omega(t)$  in  $t$  follows from the mild solution form (3.54) of  $\omega_\xi(t)$ . We claim that  $\omega(t) \in C^1([0, T]; \mathfrak{B}^{\lambda, \kappa})$  and moreover  $\|\partial_t \omega(t)\|_t$  is bounded. To this end, we first derive the mild form of  $\partial_t \omega_\xi$  from (3.54):

$$\begin{aligned} \partial_t \omega_\xi(t, x_3) &= \int_0^\infty G_\xi(t, x_3, y) \partial_t \omega_{0\xi}(y) dy + \int_0^t \int_0^\infty G_\xi(t-s, x_3, y) \partial_s N_\xi(s, y) dy ds \\ &\quad - \int_0^t G_\xi(t-s, x_3, 0) (\partial_s B_\xi(s), 0) ds, \end{aligned} \quad (3.76)$$

where we recall  $\partial_t \omega_0$  in (3.22). To justify this formula, we first recall (3.56)–(3.58). We start with the horizontal part of the formula (3.76) for  $\partial_t \omega_{\xi, h}$ . From Lemma 7,  $G_{\xi h}(t, x_3, y) = H_\xi(t, x_3 - y) + H_\xi(t, x_3 + y) + R_\xi(t, x_3 + y)$ . Then by using the fact that  $H'_\xi(t, \cdot)$  is an odd function, we see  $\partial_{x_3} G_{\xi h}(t, x_3, y)|_{x_3=0} = R'_\xi(t, y)$ . Now we read (3.57) as

$$\kappa \eta_0 R'_\xi(t, y) + \kappa \eta_0 |\xi| G_{\xi h}(t, 0, y) = 0, \quad \kappa \eta_0 R'_\xi(t, x_3) + \kappa \eta_0 |\xi| G_{\xi h}(t, x_3, 0) = 0, \quad (3.77)$$

where we have used that  $H_\xi(t, \cdot)$  is an even function for the second relation. On the other hand, since we also have  $\partial_{y_3} G_{\xi h}(t, x_3, y)|_{y=0} = R'_\xi(t, x_3)$ , we deduce that

$$\kappa \eta_0 (\partial_{y_3} + |\xi|) G_{\xi h}(t, x_3, y_3) = 0, \quad y_3 = 0. \quad (3.78)$$

It is straightforward to see  $\Delta_\xi G_{\xi h} = \partial_{x_3}^2 G_{\xi h} - |\xi|^2 G_{\xi h} = \partial_y^2 G_{\xi h} - |\xi|^2 G_{\xi h}$ .

We now take  $\partial_t$  of (3.54) to obtain

$$\begin{aligned} \partial_t \int_0^\infty G_{\xi h}(t, x_3, y) \omega_{0\xi, h}(y) dy &= \int_0^\infty \partial_t G_{\xi, h}(t, x_3, y) \omega_{0\xi, h}(y) dy \\ &= -\kappa \eta_0 \partial_y G_{\xi h}(t, x_3, 0) \omega_{0\xi, h}(0) + \kappa \eta_0 G_{\xi h}(t, x_3, 0) \partial_y \omega_{0\xi, h}(0) \\ &\quad + \int_0^\infty G_{\xi h}(t, x_3, y) \kappa \eta_0 \Delta_\xi \omega_{0\xi, h}(y) dy, \end{aligned}$$

and

$$\begin{aligned} \partial_t \int_0^t \int_0^\infty G_{\xi h}(t-s, x_3, y) N_{\xi, h}(s, y) dy ds &= \int_0^\infty G_{\xi h}(t, x_3, y) N_{\xi, h}(0, y) dy \\ &\quad + \int_0^t \int_0^\infty G_{\xi h}(s, x_3, y) \partial_t N_{\xi, h}(t-s, y) dy ds, \\ \partial_t \int_0^t G_{\xi h}(t-s, x_3, 0) B_\xi(s) ds &= G_{\xi h}(t, x_3, 0) B_\xi(0) + \int_0^t G_{\xi h}(t-s, x_3, 0) \partial_s B_\xi(s) ds. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \partial_t \omega_{\xi, h}(t, x_3) &= -\kappa \eta_0 \partial_y G_{\xi h}(t, x_3, 0) \omega_{0\xi, h}(0) + \kappa \eta_0 G_{\xi h}(t, x_3, 0) \partial_y \omega_{0\xi, h}(0) \\ &\quad - G_{\xi h}(t, x_3, 0) B_\xi(0) + \int_0^\infty G_{\xi h}(t, x_3, y) \{ \kappa \eta_0 \Delta_\xi \omega_{0\xi, h}(y) + N_{\xi, h}(0, y) \} dy \\ &\quad + \int_0^t \int_0^\infty G_{\xi h}(t-s, x_3, y) \partial_s N_{\xi, h}(s, y) dy ds - \int_0^t G_{\xi h}(t-s, x_3, 0) \partial_s B_\xi(s) ds. \end{aligned} \quad (3.79)$$

Next we show that the first line in the right-hand side is 0. From (3.78)

$$\begin{aligned} &-\kappa \eta_0 \partial_y G_{\xi h}(t, x_3, 0) \omega_{0\xi, h}(0) + \kappa \eta_0 G_{\xi h}(t, x_3, 0) \partial_y \omega_{0\xi, h}(0) \\ &= G_{\xi h}(t, x_3, 0) \kappa \eta_0 (|\xi| + \partial_y) \omega_{0\xi, h}(0), \end{aligned}$$

and hence the first line of (3.79) reads

$$G_{\xi h}(t, x_3, 0) [\kappa \eta_0(|\xi| + \partial_{x_3}) \omega_{0\xi, h}(0) - B_{\xi}(0)], \quad (3.80)$$

which is zero due to the first compatibility condition of (3.25). Recalling  $\partial_t \omega_0$  in (3.22), the formula (3.76) for  $\partial_t \omega_{\xi, h}$  has been established. We may follow the same procedure to verify the vertical part of the formula (3.76) for  $\partial_t \omega_{\xi, 3}$  by noting that the second compatibility condition of (3.25) removes the term  $-\kappa \eta_0 \partial_y G_{\xi 3}(t, x_3, 0) \omega_{0\xi, 3}(0)$  which would create the initial layer otherwise because  $\partial_y G_{\xi 3}(t, x_3, 0)$  does not vanish.

We may now repeat Step 1 for  $\partial_t \omega$  using the representation formula (3.76). The estimates are obtained in the same fashion. For the nonlinear terms, since  $\partial_t N = -u \cdot \nabla \partial_t \omega - \partial_t u \cdot \nabla \omega + \omega \cdot \nabla \partial_t u + \partial_t \omega \cdot \nabla u$ , the structure of  $\partial_t N$  with respect to  $\partial_t \omega$  is consistent with the one of  $N$  with respect to  $\omega$  and we can use the bilinear estimates (3.43) and (3.44). In summary, one can derive that for  $t < \frac{\lambda_0}{2\gamma_0}$

$$\|\partial_t \omega(t)\|_1 \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^{\beta} (1 + |\nabla_h|) \partial_t \omega_0\|_{1, \lambda_0} + (t + \frac{1}{\gamma_0}) \sup_{0 \leq s \leq t} \|\omega(s)\|_s \sup_{0 \leq s \leq t} \|\partial_t \omega(s)\|_s, \quad (3.81)$$

$$\|\partial_t \omega(t)\|_{\infty, \kappa} \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^{\beta} \partial_t \omega_0\|_{\infty, \lambda_0, \kappa} + (\sqrt{t} + \frac{1}{\gamma_0}) \sup_{0 \leq s \leq t} \|\omega(s)\|_s \sup_{0 \leq s \leq t} \|\partial_t \omega(s)\|_s, \quad (3.82)$$

which lead to the desired bounds for  $\partial_t \omega(t)$  by choosing sufficiently large  $\gamma_0$ .

**Step 3: Propagation of analytic norms for  $\partial_t^2 \omega_{\xi}$ .** As a consequence of Step 2,  $\partial_t \omega_{\xi}(t, x_3)$  solves the following system

$$\partial_t^2 \omega_{\xi} - \kappa \eta_0 \Delta_{\xi} \partial_t \omega_{\xi} = \partial_t N_{\xi} \quad \text{in } \mathbb{R}_+, \quad (3.83)$$

$$\kappa \eta_0 (\partial_{x_3} + |\xi|) \partial_t \omega_{\xi, h} = \partial_t B_{\xi} \quad \text{on } x_3 = 0, \quad (3.84)$$

$$\partial_t \omega_{\xi, 3} = 0 \quad \text{on } x_3 = 0, \quad (3.85)$$

with  $\partial_t \omega_{\xi}|_{t=0} = \partial_t \omega_{0\xi}$  for  $\xi \in \mathbb{Z}^2$  where  $\partial_t \omega_0$  is defined in (3.22). Then as done in Step 2, by using the properties of  $G_{\xi}$  and integration by parts and by the last compatibility condition of (3.25), we can derive the representation formula for  $\partial_t^2 \omega$ :

$$\begin{aligned} \partial_t^2 \omega_{\xi}(t, x_3) &= (G_{\xi h}(t, x_3, 0) [\kappa \eta_0(|\xi| + \partial_{x_3}) \partial_t \omega_{0\xi, h}(0) - \partial_t B_{\xi}(0)], 0) \\ &+ \int_0^{\infty} G_{\xi}(t, x_3, y) \partial_t^2 \omega_{0\xi}(y) dy + \int_0^t \int_0^{\infty} G_{\xi}(t-s, x_3, y) \partial_s^2 N_{\xi}(s, y) dy ds \\ &- \int_0^t G_{\xi}(t-s, x_3, 0) (\partial_s^2 B_{\xi}(s), 0) ds, \end{aligned} \quad (3.86)$$

where we recall  $\partial_t^2 \omega_0$  in (3.22).

Similar to the estimate in Step 2, the  $L^1$ -based analytic norm is easily obtained as

$$\begin{aligned}
 & \left\| \partial_t^2 \omega(t) \right\|_1 \\
 & \lesssim \kappa \eta_0 \left\| (1 + |\nabla_h|^3) \nabla \partial_t \omega_0 \right\|_{1,\lambda} + \left\| (1 + |\nabla_h|^4) \partial_t \omega_0 \right\|_{1,\lambda} \sum_{0 \leq |\beta| \leq 1} \|D^\beta (1 + |\nabla_h|^3) \partial_t \omega_0\|_{1,\lambda} \\
 & + \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) \partial_t^2 \omega_0\|_{1,\lambda_0} + \left(t + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \left\| \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t^2 \omega(s) \right\|_s \\
 & + \left(t + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \left\| \partial_t \omega(s) \right\|_s^2.
 \end{aligned} \tag{3.87}$$

For the  $L^\infty$ -based analytic norm bound, as we do not require higher order compatibility condition for the horizontal vorticity, a new term representing the initial-boundary layer emerges and we obtain

$$\begin{aligned}
 & \left\| \partial_t^2 \omega(t) \right\|_{\infty, \kappa t} \\
 & \lesssim \kappa \eta_0 \sum_{0 \leq |\beta| \leq 2} \left\| \nabla_h^\beta \nabla \partial_t \omega_0 \right\|_{\infty, \lambda} + \left\| (1 + |\nabla_h|^3) \partial_t \omega_0 \right\|_{1,\lambda} \sum_{0 \leq |\beta| \leq 1} \|D^\beta (1 + |\nabla_h|^2) \partial_t \omega_0\|_{1,\lambda} \\
 & + \sum_{0 \leq |\beta| \leq 2} \|D^\beta \partial_t^2 \omega_0\|_{\infty, \lambda_0, \kappa t} + \left(\sqrt{t} + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \left\| \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t^2 \omega(s) \right\|_s \\
 & + \left(\sqrt{t} + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \left\| \partial_t \omega(s) \right\|_s^2.
 \end{aligned} \tag{3.88}$$

We refer the detailed proof to [30].

Finally combining (3.87) and (3.88) and then choosing sufficiently large  $\gamma_0$  we derive a desired estimate for  $\left\| \partial_t^2 \omega(t) \right\|_t$  for  $t \in (0, \frac{\lambda_0}{2\gamma_0})$ .

Altogether from (3.75), (3.81), (3.82), (3.87), and (3.88), we finish the proof of the estimate (3.26).

#### Step 4: Propagation of analytic $\left\| \cdot \right\|_t$ norms for $\partial_t^\ell \theta$ , $\ell = 0, 1, 2$ .

Recall the norm in (3.16). We define

$$\left\| \theta(t) \right\|_t := \left\| \theta(t) \right\|_{\infty, 0} + \left\| \theta(t) \right\|_1. \tag{3.89}$$

Recall (3.60). We now follow the argument in Step 1, from (3.66), (3.67), and Lemma 6 we have

$$\begin{aligned}
 & \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) \theta(s)\|_{1,\lambda} \\
 & \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) \theta_0\|_{1,\lambda} + \int_0^t \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) M(s)\|_{1,\lambda} ds.
 \end{aligned} \tag{3.90}$$

Similar to the computation in Lemma 6, and using Lemma 5 we bound the nonlinear  $M$  term as

$$\begin{aligned}
 & \sum_{0 \leq |\beta| \leq 1} \|D^\beta (1 + |\nabla_h|) M(s)\|_{1,\lambda} \\
 & \lesssim \sum_{0 \leq |\beta| \leq 1} (\|\nabla_h (1 + |\nabla_h|) \theta\|_{1,\lambda} \|D^\beta (1 + |\nabla_h|) u_h\|_{\infty,\lambda} \\
 & \quad + \|\nabla_h D^\beta (1 + |\nabla_h|) \theta\|_{1,\lambda} \|(1 + |\nabla_h|) u_h\|_{\infty,\lambda} \\
 & \quad + \|\zeta \partial_z (1 + |\nabla_h|) \theta\|_{1,\lambda} \|D^\beta (1 + |\nabla_h|) \frac{u^3}{\zeta}\|_{\infty,\lambda} \\
 & \quad + \|D^\beta \zeta \partial_z (1 + |\nabla_h|) \theta\|_{1,\lambda} \|(1 + |\nabla_h|) \frac{u^3}{\zeta}\|_{\infty,\lambda}).
 \end{aligned} \tag{3.91}$$

Using Lemma 4 and the definition of  $\|\cdot\|$  we have

$$\begin{aligned}
 \int_0^t \sum_{0 \leq |\beta| \leq 1} \|D^\beta (1 + |\nabla_h|) M(s)\|_{1,\lambda} ds & \lesssim \int_0^t \|\theta(s)\|_s \|\omega(s)\|_s (1 + (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha}) ds \\
 & \lesssim \left(t + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \|\theta(s)\|_s \sup_{0 \leq s \leq t} \|\omega(s)\|_s.
 \end{aligned} \tag{3.92}$$

Next, by applying the analyticity recovery estimate (3) of Lemma 3, and choosing  $\tilde{\lambda} = \frac{\lambda + \lambda_0 - \gamma_0 s}{2}$  we get for  $|\beta| = 2$ ,

$$\begin{aligned}
 \sum_{|\beta|=2} \|D^\beta (1 + |\nabla_h|) M(s)\|_{1,\lambda} & \lesssim \frac{1}{\tilde{\lambda} - \lambda} \sum_{0 \leq |\beta| \leq 1} \|D^\beta (1 + |\nabla_h|) M(s)\|_{1,\tilde{\lambda}} \\
 & \lesssim \left(1 + (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)}\right) \|\theta(s)\|_s \|\omega(s)\|_s,
 \end{aligned} \tag{3.93}$$

Therefore we derive that for  $t < \frac{\lambda_0}{2\gamma_0}$  and  $\lambda < \lambda_0 - \gamma_0 t$

$$\begin{aligned}
 & \int_0^t \sum_{|\beta|=2} \|D^\beta (1 + |\nabla_h|) M(s)\|_{1,\lambda} ds \\
 & \lesssim \int_0^t \left(1 + (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)}\right) \|\theta(s)\|_s \|\omega(s)\|_s ds \\
 & \lesssim \left((\lambda_0 - \lambda - \gamma_0 t)^{-\alpha} \frac{1}{\gamma_0} + t\right) \sup_{0 \leq s \leq t} \|\theta(s)\|_s \sup_{0 \leq s \leq t} \|\omega(s)\|_s ds.
 \end{aligned} \tag{3.94}$$

Therefore, we conclude that, from (3.90) with (3.92), and (3.94), for  $t < \frac{\lambda_0}{2\gamma_0}$ ,

$$\|\theta(t)\|_1 \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|)\theta_0\|_{1,\lambda_0} + \left(t + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \|\theta(s)\|_s \sup_{0 \leq s \leq t} \|\omega(s)\|_s. \quad (3.95)$$

The propagation of the  $L^\infty$ -based norm  $\|\theta(t)\|_\infty$  can be shown analogously. Again follow the argument in Step 1, from (3.70), (3.71), and Lemma 6 we have

$$\sum_{0 \leq |\beta| \leq 2} \|D^\beta \theta(s)\|_{\infty,\lambda,0} \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta \theta_0\|_{\infty,\lambda,0} + \int_0^t \sum_{0 \leq |\beta| \leq 2} \|D^\beta M(s)\|_{\infty,\lambda,0} ds. \quad (3.96)$$

Similar to the computation in Lemma 6, and using Lemma 5 we again bound

$$\begin{aligned} \sum_{0 \leq |\beta| \leq 1} \|D^\beta M(s)\|_{\infty,\lambda,0} &\lesssim \sum_{0 \leq |\beta| \leq 1} \left( \|\nabla_h \theta\|_{\infty,\lambda,0} \|D^\beta u_h\|_{\infty,\lambda} + \|\nabla_h D^\beta \theta\|_{\infty,\lambda,0} \|u_h\|_{\infty,\lambda} \right. \\ &\quad \left. + \|\zeta \partial_z \theta\|_{\infty,\lambda,0} \|D^\beta \frac{u_3}{\zeta}\|_{\infty,\lambda} + \|D^\beta \zeta \partial_z \theta\|_{\infty,\lambda,0} \|\frac{u_3}{\zeta}\|_{\infty,\lambda} \right). \end{aligned} \quad (3.97)$$

Using Lemma 4 and the definition of  $\|\cdot\|$  we have

$$\begin{aligned} \int_0^t \sum_{0 \leq |\beta| \leq 1} \|D^\beta M(s)\|_{\infty,\lambda,0} ds &\lesssim \int_0^t \|\theta(s)\|_s \|\omega(s)\|_s (1 + (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha}) ds \\ &\lesssim \left(t + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \|\theta(s)\|_s \sup_{0 \leq s \leq t} \|\omega(s)\|_s. \end{aligned} \quad (3.98)$$

Next, again by applying the analyticity recovery estimate (3) of Lemma 3, and choosing  $\tilde{\lambda} = \frac{\lambda + \lambda_0 - \gamma_0 s}{2}$  we get for  $|\beta| = 2$ ,

$$\begin{aligned} \sum_{|\beta|=2} \|D^\beta M(s)\|_{\infty,\lambda,0} &\lesssim \frac{1}{\tilde{\lambda} - \lambda} \sum_{0 \leq |\beta| \leq 1} \|D^\beta M(s)\|_{\infty,\tilde{\lambda},0} \\ &\lesssim \left(1 + (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)}\right) \|\theta(s)\|_s \|\omega(s)\|_s, \end{aligned} \quad (3.99)$$

Therefore we derive that for  $t < \frac{\lambda_0}{2\gamma_0}$  and  $\lambda < \lambda_0 - \gamma_0 t$

$$\begin{aligned} \int_0^t \sum_{|\beta|=2} \|D^\beta M(s)\|_{\infty,\lambda,0} ds &\lesssim \int_0^t \left(1 + (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)}\right) \|\theta(s)\|_s \|\omega(s)\|_s ds \\ &\lesssim \left((\lambda_0 - \lambda - \gamma_0 t)^{-\alpha} \frac{1}{\gamma_0} + t\right) \sup_{0 \leq s \leq t} \|\theta(s)\|_s \sup_{0 \leq s \leq t} \|\omega(s)\|_s ds. \end{aligned} \quad (3.100)$$

Therefore, we conclude that, from (3.96) with (3.98), and (3.100), for  $t < \frac{\lambda_0}{2\gamma_0}$ ,

$$\|\theta(t)\|_{\infty,0} \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta \theta_0\|_{\infty, \lambda_0, 0} + (t + \frac{1}{\gamma_0}) \sup_{0 \leq s \leq t} \|\theta(s)\|_s \sup_{0 \leq s \leq t} \|\omega(s)\|_s. \quad (3.101)$$

From (3.95) and (3.101) and a standard continuity argument we conclude for sufficiently large  $\gamma_0$ ,

$$\sup_{0 \leq t < \frac{\lambda_0}{2\gamma_0}} \|\theta(t)\|_t \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta \theta_0\|_{\infty, \lambda_0, 0} + \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) \theta_0\|_{1, \lambda_0}. \quad (3.102)$$

**Step 5: Propagation of analytic  $\|\cdot\|_z$  norms for  $\partial_t^\ell \theta$ ,  $\ell = 0, 1, 2$ .**

Next let's investigate  $\partial_{x_3} \theta$ . Taking  $\partial_{x_3}$  to (3.60) gives

$$\partial_{x_3} \theta_\xi(t, x_3) = \int_0^\infty \partial_{x_3} G_{\xi 3}(t, x_3, y) \theta_{0\xi}(y) dy + \int_0^t \int_0^\infty \partial_{x_3} G_{\xi 3}(t-s, x_3, y) M_\xi(s, y) dy ds. \quad (3.103)$$

Now using  $G_{\xi 3}(t, x_3, y) = H_\xi(t, x_3 - y) - H_\xi(t, x_3 + y)$  as in (3.65) (with  $\eta_0$  replaced by  $\eta_c$ ), we have  $\partial_{x_3} G_{\xi 3}(t, x_3, y) = -\partial_y H_\xi(t, x_3 - y) - \partial_y H_\xi(t, x_3 + y)$ . From the compatibility condition (3.25),  $\theta_{0\xi}(0) = 0$ . And from (3.61),  $M_\xi(s, 0) = 0$ . Thus from integration by parts we get

$$\begin{aligned} \partial_{x_3} \theta_\xi(t, x_3) &= \int_0^\infty (H_\xi(t, x_3 - y) + H_\xi(t, x_3 + y)) \partial_y \theta_{0\xi}(y) dy \\ &\quad + \int_0^t \int_0^\infty (H_\xi(t-s, x_3 - y) + H_\xi(t-s, x_3 + y)) \partial_y M_\xi(s, y) dy ds. \end{aligned} \quad (3.104)$$

From the inequality

$$e^{-\frac{|y-z|^2}{2M\kappa(t-s)}} e^{-\tilde{\alpha}y} = e^{-\frac{1}{2}|\frac{y-z}{\sqrt{M\kappa(t-s)}} + \tilde{\alpha}\sqrt{M\kappa(t-s)}|^2} e^{\frac{M}{2}\tilde{\alpha}^2\kappa(t-s)} e^{-\tilde{\alpha}z} \leq e^{\frac{M}{2}\tilde{\alpha}^2\kappa(t-s)} e^{-\tilde{\alpha}z} \lesssim e^{-\tilde{\alpha}z}, \quad (3.105)$$

we bound

$$\begin{aligned} &\left| \int_0^\infty (H_\xi(t, x_3 - y) + H_\xi(t, x_3 + y)) \partial_y \theta_{0\xi}(y) dy \right| \\ &\lesssim \int_0^\infty \frac{1}{\sqrt{\kappa t}} e^{-\frac{|y-z|^2}{M\kappa t}} e^{-\tilde{\alpha}y} e^{\tilde{\alpha}y} \partial_y \theta_{0\xi}(y) dy \\ &\lesssim e^{-\tilde{\alpha}x_3} \|\partial_{x_3} \theta_{0\xi}\|_{L_{\lambda,0}^\infty} \int_0^\infty \frac{1}{\sqrt{\kappa t}} e^{-\frac{|y-z|^2}{2M\kappa t}} dy \lesssim e^{-\tilde{\alpha}x_3} \|\partial_{x_3} \theta_{0\xi}\|_{L_{\lambda,0}^\infty}. \end{aligned} \quad (3.106)$$

And similarly

$$\left| \int_0^\infty (H_\xi(t-s, x_3-y) + H_\xi(t-s, x_3+y)) \partial_y M_\xi(s, y) dy \right| \lesssim e^{-\tilde{\alpha}x_3} \|\partial_{x_3} M_\xi(s)\|_{L_{\lambda,0}^\infty} \quad (3.107)$$

Therefore from (3.104), (3.106), (3.107), and taking summation in  $\xi \in \mathbb{Z}^2$ , we get  $\|\partial_{x_3} \theta(t, x_3)\|_{\infty, \lambda, 0} \lesssim \|\partial_{x_3} \theta_0\|_{\infty, \lambda, 0} + \int_0^t \|\partial_{x_3} M(s)\|_{\infty, \lambda, 0} ds$ . And similarly

$$\begin{aligned} \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \theta(t, x_3)\|_{\infty, \lambda, 0} &\lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \theta_0\|_{\infty, \lambda, 0} \\ &+ \int_0^t \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} M(s)\|_{\infty, \lambda, 0} ds. \end{aligned} \quad (3.108)$$

We have

$$\begin{aligned} \|D_h \partial_{x_3} M(s)\|_{\infty, \lambda, 0} &\leq \underbrace{\|D_h \partial_{x_3} u \cdot \nabla \theta + \partial_{x_3} u \cdot D_h \nabla \theta\|_{\infty, \lambda, 0}}_{(3.109)_1} \\ &+ \underbrace{\|D_h u \cdot \nabla \partial_{x_3} \theta + u \cdot D_h \nabla \partial_{x_3} \theta\|_{\infty, \lambda, 0}}_{(3.109)_2}, \end{aligned} \quad (3.109)$$

and we bound the two parts separately.

From (3.42), and that  $\|\nabla_h \omega_h\|_{\infty, \lambda} \lesssim (1 + \frac{1}{\sqrt{\kappa}}) \|\nabla_h \omega_h\|_{\infty, \lambda, \kappa} \lesssim \frac{1}{\sqrt{\kappa}}$ , we have

$$\begin{aligned} (3.109)_1 &\lesssim \left( \sum_{0 \leq |\beta| \leq 2} \|\nabla_h^\beta \omega(s)\|_{1, \lambda} + \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \omega_h(s)\|_{\infty, \lambda} \right) \sup_{0 \leq s \leq t} \|\theta(s)\|_s \\ &+ \left( \sum_{0 \leq |\beta| \leq 2} \|\nabla_h^\beta \omega(s)\|_{1, \lambda} \right) (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha} \sup_{0 \leq s \leq t} \|\theta(s)\|_z \\ &\lesssim \frac{1}{\sqrt{\kappa}} \sup_{0 \leq s \leq t} \|\omega(s)\|_s \sup_{0 \leq s \leq t} \|\theta(s)\|_s + (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha} \sup_{0 \leq s \leq t} \|\omega(s)\|_s \sup_{0 \leq s \leq t} \|\theta(s)\|_z \\ &\lesssim \frac{1}{\sqrt{\kappa}} + (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha} \sup_{0 \leq s \leq t} \|\theta(s)\|_z, \end{aligned} \quad (3.110)$$

where we have used (3.26), (3.27), and from previous steps  $\sup_{0 \leq s \leq t} \|\omega(s)\|_s < \infty$ ,  $\sup_{0 \leq s \leq t} \|\theta(s)\|_z < \infty$ . Next for (3.109)<sub>2</sub>, we use (3.42), and the analytic recovery lemma (Lemma 3 (3)) to get

$$\begin{aligned}
(3.109)_2 &\lesssim \sum_{i=1,2} (\|(D_h u_i \partial_{x_i} \partial_{x_3} \theta)(s)\|_{\infty, \lambda, 0} + \|(u_i D_h \partial_{x_i} \partial_{x_3} \theta)(s)\|_{\infty, \lambda, 0}) \\
&\quad + \|(\zeta^{-1} D_h u_3 \zeta \partial_{x_3} \partial_{x_3} \theta)(s)\|_{\infty, \lambda, 0} + \|(\zeta^{-1} u_3 \zeta \partial_{x_3} D_h \partial_{x_3} \theta)(s)\|_{\infty, \lambda, 0} \\
&\lesssim \left( \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} u_h(s)\|_{\infty, \lambda} \right) \left( \sum_{0 \leq |\beta_h| \leq 1} \|\nabla_h D_h^{\beta_h} \partial_{x_3} \theta(s)\|_{\infty, \lambda, 0} \right) \\
&\quad + \left( \sum_{0 \leq |\beta_h| \leq 1} \|\zeta^{-1} D_h^{\beta_h} u_3(s)\|_{\infty, \lambda} \right) \left( \sum_{0 \leq |\beta_h| \leq 1} \|\zeta \partial_{x_3} D_h^{\beta_h} \partial_{x_3} \theta(s)\|_{\infty, \lambda, 0} \right) \\
&\lesssim \left( \sum_{0 \leq |\beta| \leq 2} \|\nabla_h^\beta \omega_h(s)\|_{1, \lambda} + \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \omega(s)\|_{1, \lambda} \right) \frac{1}{\tilde{\lambda} - \lambda} \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \theta(s)\|_{\infty, \tilde{\lambda}, 0} \\
&\lesssim (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)} \sup_{0 \leq s \leq t} \|\omega(s)\|_s \sup_{0 \leq s \leq t} \|\theta(s)\|_z \\
&\lesssim (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)} \sup_{0 \leq s \leq t} \|\theta(s)\|_z,
\end{aligned} \tag{3.111}$$

where as before, we have chosen  $\tilde{\lambda} = \frac{\lambda + \lambda_0 - \gamma_0 s}{2}$ , so  $\tilde{\lambda} - \lambda = \frac{\lambda_0 - \gamma_0 s - \lambda}{2} = \lambda_0 - \gamma_0 s - \tilde{\lambda}$ . Thus combining (3.108), (3.110), and (3.111) we get

$$\begin{aligned}
&\sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \theta(t, x_3)\|_{\infty, \lambda, 0} \\
&\lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \theta_0\|_{\infty, \lambda, 0} + \int_0^t \frac{1}{\sqrt{\kappa}} ds + \left( \int_0^t (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha} ds \right) \sup_{0 \leq s \leq t} \|\theta(s)\|_z \\
&\quad + \left( \int_0^t (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)} ds \right) \sup_{0 \leq s \leq t} \|\theta(s)\|_z \\
&\lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \theta_0\|_{\infty, \lambda, 0} + \frac{t}{\sqrt{\kappa}} + \left( \frac{1}{\gamma_0} + (\lambda_0 - \lambda - \gamma_0 t)^{-\alpha} \frac{1}{\gamma_0} \right) \sup_{0 \leq s \leq t} \|\theta(s)\|_z.
\end{aligned} \tag{3.112}$$

Therefore we conclude that from (3.17), (3.112), for sufficiently large  $\gamma_0$ ,

$$\sup_{0 \leq t \leq \frac{\lambda_0}{2\gamma_0}} \|\theta(t)\|_z \lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \theta_0\|_{\infty, \lambda_0, 0} + \frac{1}{\sqrt{\kappa}}. \tag{3.113}$$

This proves the bound for  $\partial_{x_3} \theta$  in (3.28).

Next, we look at  $\partial_t \theta$ . For the propagation of analytic norms  $\|\partial_t \theta\|_t$ , we follow the argument in Step 2. Taking  $\partial_t$  of (3.60) and using integration by parts, from (3.56), (3.58), we achieve

$$\begin{aligned} \partial_t \theta_\xi(t, x_3) = & \kappa \eta_c \partial_y G_{\xi 3}(t, x_3, 0) \theta_{0\xi}(0) + \int_0^\infty G_{\xi 3}(t, x_3, y) \left( \kappa \eta_c (-|\xi|^2 + \partial_y^2) \theta_{0\xi} + M_\xi(0, y) \right) dy \\ & + \int_0^t \int_0^\infty G_{\xi 3}(t-s, x_3, y) \partial_s M_\xi(s, y) dy ds. \end{aligned} \quad (3.114)$$

Recalling  $\partial_t \theta_0$  in (3.22), and the compatibility condition (3.25), we get

$$\partial_t \theta_\xi(t, x_3) = \int_0^\infty G_{\xi 3}(t, x_3, y) \partial_t \theta_{0\xi} dy + \int_0^t \int_0^\infty G_{\xi 3}(t-s, x_3, y) \partial_s M_\xi(s, y) dy ds. \quad (3.115)$$

With this representation formula we can repeat the estimate earlier in this step in the same fashion. For the nonlinear terms, since  $\partial_t M = -u \cdot \nabla \partial_t \theta - \partial_t u \cdot \nabla \theta$ , we can use the bilinear estimates (3.43), (3.110), and (3.111) in the same way to derive that for  $t < \frac{\lambda_0}{2\gamma_0}$

$$\begin{aligned} \|\partial_t \theta(t)\|_1 & \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) \partial_t \theta_0\|_{1, \lambda_0} \\ & + \left( t + \frac{1}{\gamma_0} \right) \left( \sup_{0 \leq s \leq t} \|\omega(s)\|_s \sup_{0 \leq s \leq t} \|\partial_t \theta(s)\|_s + \sup_{0 \leq s \leq t} \|\partial_t \omega(s)\|_s \sup_{0 \leq s \leq t} \|\theta(s)\|_s \right), \end{aligned} \quad (3.116)$$

$$\begin{aligned} \|\partial_t \theta(t)\|_{\infty, 0} & \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta \partial_t \theta_0\|_{\infty, \lambda_0, 0} \\ & + \left( t + \frac{1}{\gamma_0} \right) \left( \sum_{\ell=0,1} \sup_{0 \leq s \leq t} \left\| \partial_t^\ell \omega(s) \right\|_s \right) \left( \sum_{\ell=0,1} \sup_{0 \leq s \leq t} \left\| \partial_t^\ell \theta(s) \right\|_s \right), \end{aligned} \quad (3.117)$$

which lead to the desired bounds for  $\partial_t \theta(t)$  in (3.27) by choosing sufficiently large  $\gamma_0$ .

For the propagation of the norm  $\|\partial_t \theta\|_z$ , we follow the argument in the estimate of  $\|\theta\|_z$ . Taking  $\partial_{x_3}$  to (3.115) and from (3.25), (3.61), we have  $\partial_t \theta_{0\xi}(0) = 0$ ,  $\partial_t M_\xi(0) = 0$ . Thus from integration by parts we can follow the argument from (3.103)–(3.108) to get

$$\begin{aligned} \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t \theta(t, x_3)\|_{\infty, \lambda, 0} & \lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t \theta_0\|_{\infty, \lambda, 0} \\ & + \int_0^t \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t M(s)\|_{\infty, \lambda, 0} ds. \end{aligned}$$

Then following the argument from (3.110)–(3.112) we get

$$\begin{aligned}
& \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t \theta(t, x_3)\|_{\infty, \lambda, 0} \\
& \lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t \theta_0\|_{\infty, \lambda, 0} + \int_0^t \frac{1}{\sqrt{\kappa}} ds \left( \sup_{0 \leq s \leq t} \sum_{\ell=0,1} \left\| \partial_t^\ell \omega(s) \right\|_s \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0,1} \left\| \partial_t^\ell \theta(s) \right\|_s \right) \\
& \quad + \left( \int_0^t (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha} ds \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0,1} \left\| \partial_t^\ell \omega(s) \right\|_s \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0,1} \left\| \partial_t^\ell \theta(s) \right\|_z \right) \\
& \quad + \left( \int_0^t (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)} ds \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0,1} \left\| \partial_t^\ell \omega(s) \right\|_s \right) \left( \sum_{\ell=0,1} \sup_{0 \leq s \leq t} \left\| \partial_t^\ell \theta(s) \right\|_z \right) \\
& \lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t \theta_0\|_{\infty, \lambda, 0} + \frac{t}{\sqrt{\kappa}} + \left( \frac{1}{\gamma_0} + (\lambda_0 - \lambda - \gamma_0 t)^{-\alpha} \frac{1}{\gamma_0} \right) \sup_{0 \leq s \leq t} \left\| \partial_t \theta(s) \right\|_z,
\end{aligned}$$

which lead to the desired bounds for  $\partial_{x_3} \partial_t \theta(t)$  in (3.28) by choosing sufficiently large  $\gamma_0$ .

The propagation of analytic norms for  $\partial_t^2 \theta$  follows in the same way. From the compatibility condition (3.25) we get

$$\partial_t^2 \theta(t, x_3) = \int_0^\infty G_{\xi 3}(t, x_3, y) \partial_t^2 \theta_{0\xi} dy + \int_0^t \int_0^\infty G_{\xi 3}(t-s, x_3, y) \partial_s^2 M_\xi(s, y) dy ds. \quad (3.118)$$

Using the bilinear estimate on  $\partial_t^2 M$  we derive that for  $t < \frac{\lambda_0}{2\gamma_0}$

$$\begin{aligned}
\left\| \partial_t^2 \theta(t) \right\|_1 & \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta (1 + |\nabla_h|) \partial_t^2 \theta_0\|_{1, \lambda_0} + \left(t + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \left\| \partial_t \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t \theta(s) \right\|_s \\
& \quad + \left(t + \frac{1}{\gamma_0}\right) \left( \sup_{0 \leq s \leq t} \left\| \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t^2 \theta(s) \right\|_s + \sup_{0 \leq s \leq t} \left\| \partial_t \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t \theta(s) \right\|_s \right), \quad (3.119)
\end{aligned}$$

$$\begin{aligned}
\left\| \partial_t^2 \theta(t) \right\|_{\infty, 0} & \lesssim \sum_{0 \leq |\beta| \leq 2} \|D^\beta \partial_t^2 \theta_0\|_{\infty, \lambda_0, 0} + \left(t + \frac{1}{\gamma_0}\right) \sup_{0 \leq s \leq t} \left\| \partial_t \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t \theta(s) \right\|_s \\
& \quad + \left(t + \frac{1}{\gamma_0}\right) \left( \sup_{0 \leq s \leq t} \left\| \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t^2 \theta(s) \right\|_s + \sup_{0 \leq s \leq t} \left\| \partial_t \omega(s) \right\|_s \sup_{0 \leq s \leq t} \left\| \partial_t \theta(s) \right\|_s \right), \quad (3.120)
\end{aligned}$$

which lead to the desired bounds for  $\partial_t^2 \theta(t)$  in (3.27) by choosing sufficiently large  $\gamma_0$ .

Taking  $\partial_{x_3}$  to (3.118) and using (3.25), (3.61), we have  $\partial_t^2 \theta_{0\xi}(0) = 0$ ,  $\partial_t^2 M_\xi(0) = 0$ . Thus from integration by parts we again follow the argument from (3.103)-(3.108) to get

$$\begin{aligned} \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t^2 \theta(t, x_3)\|_{\infty, \lambda, 0} &\lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t^2 \theta_0\|_{\infty, \lambda, 0} \\ &+ \int_0^t \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t^2 M(s)\|_{\infty, \lambda, 0} ds. \end{aligned}$$

Then following the argument from (3.110)–(3.112) we get

$$\begin{aligned} &\sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t^2 \theta(t, x_3)\|_{\infty, \lambda, 0} \\ &\lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t^2 \theta_0\|_{\infty, \lambda, 0} \\ &+ \int_0^t \left( \frac{1}{\sqrt{\kappa}} + \frac{1}{\sqrt{\kappa s}} \right) ds \left( \sup_{0 \leq s \leq t} \sum_{\ell=0}^2 \left\| \partial_t^\ell \omega(s) \right\|_s \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0}^2 \left\| \partial_t^\ell \theta(s) \right\|_s \right) \\ &+ \left( \int_0^t (\lambda_0 - \lambda - \gamma_0 s)^{-\alpha} ds \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0}^2 \left\| \partial_t^\ell \omega(s) \right\|_s \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0}^2 \left\| \partial_t^\ell \theta(s) \right\|_z \right) \\ &+ \left( \int_0^t (\lambda_0 - \lambda - \gamma_0 s)^{-(\alpha+1)} ds \right) \left( \sup_{0 \leq s \leq t} \sum_{\ell=0}^2 \left\| \partial_t^\ell \omega(s) \right\|_s \right) \left( \sum_{\ell=0}^2 \sup_{0 \leq s \leq t} \left\| \partial_t^\ell \theta(s) \right\|_z \right) \\ &\lesssim \sum_{0 \leq |\beta_h| \leq 1} \|D_h^{\beta_h} \partial_{x_3} \partial_t^2 \theta_0\|_{\infty, \lambda, 0} + \frac{t + \sqrt{t}}{\sqrt{\kappa}} + \left( \frac{1}{\gamma_0} + (\lambda_0 - \lambda - \gamma_0 t)^{-\alpha} \frac{1}{\gamma_0} \right) \sup_{0 \leq s \leq t} \left\| \partial_t^2 \theta(s) \right\|_z. \end{aligned}$$

which lead to the desired bounds for  $\partial_{x_3} \partial_t^2 \theta(t)$  in (3.28) by choosing sufficiently large  $\gamma_0$ .

**Step 6: Estimate (1), vorticity estimates.** Both (3.29) and (3.30) are direct consequences of (3.26). To show (3.31), we first note that the boundedness of  $\omega(t)$  norms implies  $|\partial_{x_3} \omega_\xi(t, x_3)| \lesssim e^{-\tilde{\alpha} x_3} e^{-\lambda |\xi|}$  for all  $|\xi|$  and  $x_3 \geq 1$  (away from the boundary). When  $x_3 \leq 1$ , we draw on the equation (3.50) to rewrite  $\partial_{x_3}^2 \omega_{\xi, h} = \frac{1}{\kappa \eta_0} \{ \partial_t \omega_{\xi, h} + \kappa \eta_0 |\xi|^2 \omega_{\xi, h} - N_{\xi, h} \}$  and the boundary condition (3.51):

$$\begin{aligned} \partial_{x_3} \omega_{\xi, h}(t, x_3) &= \partial_{x_3} \omega_{\xi, h}(t, 0) + \int_0^{x_3} \partial_{x_3}^2 \omega_{\xi, h}(t, y) dy \\ &= -|\xi| \omega_{\xi, h}(t, 0) + \frac{1}{\kappa \eta_0} B_\xi(t) + \int_0^{x_3} \frac{1}{\kappa \eta_0} [\partial_t \omega_{\xi, h} + \kappa \eta_0 |\xi|^2 \omega_{\xi, h} - N_{\xi, h}](t, y) dy. \end{aligned} \tag{3.121}$$

We now appeal to  $|B_\xi(t)| \leq \|N_\xi(t)\|_{\mathcal{L}_\lambda^1}$  and  $\sum_{0 \leq \ell \leq 1} (\left\| \partial_t^\ell \omega(t) \right\|_{\infty, \kappa} + \left\| \partial_t^\ell \omega(t) \right\|_1) < \infty$  to obtain that for all  $x_3 \in \mathbb{R}_+$

$$|\partial_{x_3} \omega_{\xi,h}(t, x_3)| \lesssim \frac{1}{\kappa} e^{-\bar{\alpha} x_3} e^{-\lambda|\xi|} \text{ for } 0 < \lambda < \lambda_0, \quad (3.122)$$

which proves (3.31) for  $\omega_h$  and  $\ell = 0$ . The remaining case can be estimated similarly. Near  $O(1)$  boundary, from (3.83) and (3.84), we derive

$$\begin{aligned} & \partial_{x_3} \partial_t \omega_{\xi,h}(t, x_3) \\ &= -|\xi| \partial_t \omega_{\xi,h}(t, 0) + \frac{1}{\kappa \eta_0} \partial_t B_{\xi}(t) + \int_0^{x_3} \frac{1}{\kappa \eta_0} [\partial_t^2 \omega_{\xi,h} + \kappa \eta_0 |\xi|^2 \partial_t \omega_{\xi,h} - \partial_t N_{\xi,h}](t, y) dy. \end{aligned} \quad (3.123)$$

Together with  $\sum_{0 \leq \ell \leq 1} \|\partial_t^\ell \omega(t)\|_{\infty, \kappa} + \sum_{0 \leq \ell \leq 2} \|\partial_t^\ell \omega(t)\|_1 < \infty$  we deduce (3.31) for  $\omega_h$  and  $\ell = 1$ . For  $\omega_3$  we use  $\nabla \cdot \omega = 0$  to write  $\partial_3 \omega_3 = -\partial_1 \omega_1 - \partial_2 \omega_2$ . Now (3.31) for  $\omega_3$  follows from (3.29).

**Step 7: Estimate (2), velocity estimates, except (3.35).** From Lemma 4, the estimation of the velocity follows from  $\phi$  estimate. From  $(|\xi|^2 - \partial_z^2) \phi_{\xi} = \omega_{\xi}$  and  $\phi_{\xi}(0) = 0$

$$\begin{aligned} \phi_{\xi}(z) &= \int_0^z G_{-}(y, z) \omega_{\xi}(y) dy + \int_z^{\infty} G_{+}(y, z) \omega_{\xi}(y) dy, \\ \text{with } G_{\pm}(y, z) &:= \frac{-1}{2|\xi|} \left( e^{\pm|\xi|(z-y)} - e^{-|\xi|(y+z)} \right), \end{aligned} \quad (3.124)$$

and we have

$$|\xi|^{\beta_h} |\partial_z^{\beta_3} \partial_t^{\ell} \phi_{\xi}(t, z)| \lesssim \int_{\partial \mathcal{H}_{\lambda}} |\xi|^{|\beta|-1} e^{-|\xi||y-z|} |\partial_t^{\ell} \omega_{\xi}(t, y)| |dy| \text{ for } \beta_3 \leq 1. \quad (3.125)$$

For  $|\beta| = |\beta_h| + \beta_3 = 1$  we bound (3.125) by  $e^{-\lambda|\xi|} \|\partial_t^{\ell} \omega(t)\|_{1, \lambda}$ . Then from (3.26) we conclude (3.32).

For  $|\beta| \geq 2$  and  $\beta_3 \leq 1$ , we bound (3.125) by

$$\begin{aligned} (3.125) &\lesssim \int_{\partial \mathcal{H}_{\lambda}} |\xi|^{|\beta|-2} |\xi| e^{-|\xi||y-z|} e^{-\bar{\alpha} \operatorname{Re} y} e^{-\lambda|\xi|} (1 + \phi_{\kappa}(y) + \phi_{\kappa t}(y)) |dy| \\ &\lesssim |\xi|^{|\beta|-2} e^{-\lambda|\xi|} e^{-\min(1, \frac{\bar{\alpha}}{2})x_3} \int_{\partial \mathcal{H}_{\lambda}} e^{-\frac{\bar{\alpha}}{2} \operatorname{Re} y} (1 + \phi_{\kappa}(y) + \phi_{\kappa t}(y)) |dy| \\ &\lesssim |\xi|^{|\beta|-2} e^{-\lambda|\xi|} e^{-\min(1, \frac{\bar{\alpha}}{2})x_3} \text{ for } |\beta| \geq 2, \text{ and } \beta_3 \leq 1, \text{ and } \ell = 0, 1, 2, \text{ and } t \in [0, T], \end{aligned} \quad (3.126)$$

where we have used  $|\xi||y-z| + \frac{\bar{\alpha}}{2} \operatorname{Re} y \geq \min(1, \frac{\bar{\alpha}}{2})x_3$  for  $|\xi| \geq 1$  and (3.26).

For  $\beta_3 = 2, 3$  we use  $\partial_z^2 \partial_t^{\ell} \phi_{\xi} = |\xi|^2 \partial_t^{\ell} \phi_{\xi} + \partial_t^{\ell} \omega_{\xi}$ . Then following the same argument of (3.126), we derive

$$\begin{aligned}
& |\xi|^{\beta_h} |\partial_z^{\beta_3} \partial_t^\ell \phi_\xi(t, z)| \\
& \lesssim |\xi|^{|\beta_h|+2} |\partial_z^{\beta_3-2} \partial_t^\ell \phi_\xi(t, z)| + |\xi|^{\beta_h} |\partial_z^{\beta_3-2} \partial_t^\ell \omega_\xi(t, z)| \\
& \lesssim \begin{cases} (|\xi|^{|\beta|-2} + |\xi|^{\beta_h}) e^{-\lambda|\xi|} e^{-\min(1, \frac{\tilde{\alpha}}{2}) \operatorname{Re} z} (1 + \phi_\kappa(z)) & \text{for } \ell = 0, 1, \text{ and } \beta_3 = 2, \\ (|\xi|^{|\beta|-2} + |\xi|^{\beta_h}) e^{-\lambda|\xi|} e^{-\min(1, \frac{\tilde{\alpha}}{2}) \operatorname{Re} z} \kappa^{-1} & \text{for } \ell = 0, 1, \text{ and } \beta_3 = 3, \\ (|\xi|^{|\beta|-2} + |\xi|^{\beta_h}) e^{-\lambda|\xi|} e^{-\min(1, \frac{\tilde{\alpha}}{2}) \operatorname{Re} z} (1 + \phi_\kappa(z) + \phi_{\kappa t}(z)) & \text{for } \ell = 2, \text{ and } \beta_3 = 2. \end{cases}
\end{aligned} \tag{3.127}$$

Finally from (3.126) and (3.127) we conclude (3.33) and (3.34).

**Step 8: Estimate (3), temperature estimates.** Both (3.36) and (3.37) are direct consequences of (3.27). To show (3.38), we use the equation (3.5) to get

$$\begin{aligned}
\partial_{x_3}^2 \theta &= \frac{1}{\kappa \eta_c} (\partial_t \theta + u \cdot \nabla_x \theta) - \partial_{x_1}^2 \theta - \partial_{x_2}^2 \theta. \\
&= \frac{1}{\kappa \eta_c} \left( \partial_t \theta + u_1 \partial_{x_1} \theta + u_2 \partial_{x_2} \theta + \left( \frac{u_3}{\zeta} \right) (\zeta \partial_{x_3} \theta) \right) - \partial_{x_1}^2 \theta - \partial_{x_2}^2 \theta
\end{aligned} \tag{3.128}$$

Then from (3.32), (3.36), and (3.37) we get

$$\begin{aligned}
\sum_{\ell=0}^1 |\partial_t^\ell \partial_{x_3}^2 \theta| &\lesssim \frac{1}{\kappa \eta_c} \left[ \sum_{\ell=1}^2 |\partial_t^\ell \theta| + \left( \sum_{\ell=0}^1 (|\partial_t^\ell u_1| + |\partial_t^\ell u_2| + |\partial_t^\ell (\frac{u_3}{\zeta})|) \right) \right. \\
&\quad \times \left. \left( \sum_{\ell=0}^1 (|\partial_t^\ell \partial_{x_1} \theta| + |\partial_t^\ell \partial_{x_2} \theta| + |\partial_t^\ell (\zeta \partial_{x_3} \theta)|) \right) \right] \\
&\quad + \sum_{\ell=0}^1 (|\partial_t^\ell \partial_{x_1}^2 \theta| + |\partial_t^\ell \partial_{x_2}^2 \theta|) \\
&\lesssim \frac{1}{\kappa} \left( e^{-\tilde{\alpha} x_3} \right) + e^{-\tilde{\alpha} x_3} \lesssim \kappa^{-\frac{3}{2}} e^{-\tilde{\alpha} x_3}.
\end{aligned}$$

This proves (3.38).

The pressure estimate (3.39)-(3.41) can be found in Theorem 3 of [30]. The last estimate for  $\partial_t^\ell u$  for  $\ell = 1, 2$  follows from the equation:  $\partial_t u = \kappa \eta_0 \Delta u - u \cdot \nabla u - \nabla p$  and  $\partial_t^2 u = \kappa \eta_0 \Delta \partial_t u - u \cdot \nabla \partial_t u - \partial_t u \cdot \nabla u - \nabla \partial_t p$ . This finishes the proof.

#### 4. Main theorem

In the last section we state and prove the precise statement of the main theorem which was informally stated as Theorem 1. Before that we first show the convergence of heat flow  $\theta$  to  $\theta_E$  in  $L^\infty([0, T]; L^2(\Omega))$  in the following lemma.

**Lemma 10** (Convergence of  $\theta$  to  $\theta_E$ ). *Let  $u_E$  be the solution of the Euler equation (1.7), (1.8), (1.9), and  $u$  be the solution of the Navier-Stokes equation (1.12), (1.13), (1.14). For the following equations*

$$\begin{aligned}\partial_t \theta_E + u_E \cdot \nabla_x \theta_E &= 0 \text{ in } \Omega, \\ \partial_t \theta + u \cdot \nabla_x \theta - \kappa \eta_c \Delta \theta &= 0 \text{ in } \Omega, \quad \theta = 0 \text{ on } \partial\Omega.\end{aligned}\tag{4.1}$$

Assume  $\theta_E(0, x) = \theta(0, x) = \theta_0(x)$ , and

$$\theta_0(x) = 0 \text{ on } \partial\Omega.\tag{4.2}$$

Then

$$\sup_{0 \leq t \leq T} \|\theta - \theta_E\|_{L^2(\Omega)} \rightarrow 0, \text{ as } \kappa \rightarrow 0.\tag{4.3}$$

**Proof.** From (4.1) we have

$$\theta_E(t, x) = \theta_E(0, X(0; t, x)),\tag{4.4}$$

where  $X(s; t, x)$  satisfies

$$\frac{d}{ds} X(s; t, x) = u_E(s, X(s; t, x)), \quad X(t; t, x) = x.$$

From assumption (4.2) and  $u_E \cdot n = 0$  on  $\partial\Omega$ , we have  $\{X(0; t, x) : x \in \partial\Omega\} \subset \partial\Omega$ . Therefore

$$\theta_E(t, x) = 0 \text{ on } \partial\Omega.\tag{4.5}$$

For the solution  $u_E(t, x)$  to the Euler equation (1.7), (1.8), (1.9), it is shown in [55] that for smooth initial data  $u_0$  satisfying  $\nabla \cdot u_0 = 0$  and  $u_0 \cdot n = 0$  on  $\partial\Omega$ , there exists a time  $T > 0$  such that  $u_E$  is smooth on  $(0, T)$ . In particular,

$$\sup_{0 < s < T} \|u_E(s)\|_{C^2(\Omega)} < \infty.\tag{4.6}$$

Thus taking  $\nabla_x$  derivative to

$$X(s; t, x) = x - \int_s^t u_E(\tau, X(\tau; t, x)) d\tau,\tag{4.7}$$

we have

$$|\nabla_x X(s; t, x)| \lesssim 1 + \int_s^t \|\nabla_x u_E(\tau)\|_{L^\infty} |\nabla_x X(\tau; t, x)| d\tau.$$

From Gronwall's inequality

$$\sup_{0 \leq s \leq t \leq T^*} \sup_{x \in \Omega} |\nabla_x X(s; t, x)| \lesssim e^{T \sup_{0 < s < T^*} \|\nabla_x u_E(s)\|_{L^\infty}} < C.\tag{4.8}$$

Similarly, taking  $\nabla_x^2$  to (4.7) and using (4.6), (4.8) we get

$$\begin{aligned} |\nabla_x^2 X(s; t, x)| &\lesssim \int_s^t \left( \|\nabla_x^2 u_E(\tau)\|_{L^\infty} |\nabla_x X(\tau; t, x)| + \|\nabla_x u_E(\tau)\|_{L^\infty} |\nabla_x^2 X(\tau; t, x)| \right) d\tau \\ &\lesssim TC + \int_s^t |\nabla_x^2 X(\tau; t, x)| d\tau \end{aligned}$$

Using Gronwall we get

$$\sup_{0 \leq s \leq t \leq T^*} \sup_{x \in \Omega} |\nabla_x^2 X(s; t, x)| \lesssim TCe^T < C_1. \quad (4.9)$$

Combining (4.4), (4.8), and (4.9) we get

$$\begin{aligned} &\sup_{0 \leq t \leq T^*} \sup_{x \in \Omega} \left( |\nabla_x \theta_E(t, x)| + |\nabla_x^2 \theta_E(t, x)| \right) \\ &\lesssim \left( \|\nabla_x \theta_0\|_{L^\infty} + \|\nabla_x^2 \theta_0\|_{L^\infty} \right) \sup_{0 \leq t \leq T^*} \sup_{x \in \Omega} \left( |\nabla_x X(s; t, x)| + |\nabla_x^2 X(s; t, x)| \right) < C_2. \end{aligned} \quad (4.10)$$

Now let  $w = \theta - \theta_E$ , then from (4.1), the equation for  $w$  is

$$\partial_t w + u \cdot \nabla w + (u - u_E) \cdot \nabla \theta_E - \kappa \eta_c \Delta w = \kappa \eta_c \Delta \theta_E. \quad (4.11)$$

From (4.1) and (4.5),  $w = 0$  on  $\partial\Omega$ . Thus from integration by parts one gets  $\int_\Omega (\Delta w) w \, dx = -\int_\Omega |\nabla w|^2 \, dx$  without the boundary term. Therefore the standard energy estimate  $\int_\Omega (4.11) \cdot w \, dx$  gives

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \kappa \eta_c \|\nabla w\|_{L^2}^2 + \int_\Omega (u \cdot \nabla w) w \, dx \\ &= - \int_\Omega (u - u_E) \cdot \nabla \theta_E w \, dx + \kappa \eta_c \int_\Omega \Delta \theta_E w \, dx. \end{aligned} \quad (4.12)$$

From  $\nabla \cdot u = 0$ , and  $u = 0$  on  $\partial\Omega$ , we have

$$\int_\Omega (u \cdot \nabla w) w \, dx = \frac{1}{2} \int_\Omega u \cdot \nabla (w^2) \, dx = -\frac{1}{2} \int_\Omega (\nabla \cdot u) w^2 \, dx = 0.$$

Now we bound the terms on the RHS of (4.12). Using (4.10) we have

$$\int_\Omega |(u - u_E) \cdot \nabla \theta_E w| \, dx \lesssim \|\nabla \theta_E\|_{L^\infty} \left( \|u - u_E\|_{L^2}^2 + \|w\|_{L^2}^2 \right) \lesssim \|u - u_E\|_{L^2}^2 + \|w\|_{L^2}^2, \quad (4.13)$$

and

$$\kappa \eta_c \int_{\Omega} |\Delta \theta_E w| dx \lesssim (\kappa \eta_c)^2 \|\Delta \theta_E\|_{L^2}^2 + \|w\|_{L^2}^2 \lesssim \kappa^2 + \|w\|_{L^2}^2. \quad (4.14)$$

Now collect (4.13), (4.14), (4.12) gives

$$\frac{d}{dt} \|w\|_{L^2}^2 \lesssim \|w\|_{L^2}^2 + \|u - u_E\|_{L^2}^2 + \kappa^2,$$

Therefore by Gronwall's inequality

$$\sup_{0 \leq t \leq T} \|w\|_{L^2}^2 \lesssim e^T \left( \|u - u_E\|_{L^2}^2 + \kappa^2 \right) \rightarrow 0, \text{ as } \kappa \rightarrow 0. \quad \square$$

Finally we prove the main theorem of the paper through combining separate estimates we established in different sections of the paper.

**Theorem 10 (Main Theorem).** *Let  $\Omega$  be a upper half space in 3D:*

$$\Omega := \mathbb{T}^2 \times \mathbb{R}_+ \ni (x_1, x_2, x_3), \text{ where } \mathbb{R}_+ := \{x_3 \in \mathbb{R} : x_3 > 0\}, \quad (4.15)$$

*and  $\mathbb{T}$  is the periodic interval  $(-\pi, \pi)$ .*

*Suppose an initial velocity field  $u_{in}$  is divergence-free*

$$\nabla_x \cdot u_{in} = 0 \quad \text{in } \Omega, \quad (4.16)$$

*the corresponding initial vorticity  $\omega_{in} = \nabla_x \times u_{in}$  and the initial heat  $\theta_{in}$  satisfy*

$$\omega_{in}, \theta_{in} \in \mathfrak{B}^{\lambda_0, \kappa}, \quad (4.17)$$

*with the real analytic space  $\mathfrak{B}^{\lambda_0, \kappa}$  defined in (3.12) for some  $\lambda_0 > 0$ , such that (3.23), (3.24) hold. Further we assume that  $\omega_{in}, \theta_{in}$  satisfies the compatibility conditions (3.25) on  $\partial\Omega$ . Then there exists a unique real analytic solution  $(u(t, x), \theta(t, x))$  to the Navier-Stokes-Fourier flow (1.12)-(1.17) in  $[0, T] \times \Omega$ , while  $T > 0$  only depends on  $\lambda_0$  and the size of the initial data as in (3.23), (3.24).*

*Choosing a pressure  $p(t, x)$  such that  $p(t, x) \rightarrow 0$  as  $x_3 \rightarrow \infty$ , and setting the first order and the second order correction terms  $f_1, f_2$  as (2.1)-(2.4), we also choose  $\varepsilon$  and  $\kappa$  in the relation of (2.195). Assume that an initial datum for the remainder  $f_{R,in}$  satisfies (2.196), then for the given  $T > 0$ , we construct a unique solution  $f_R(t, x, v)$  of the form of*

$$F = \mu + \varepsilon f_1 \sqrt{\mu} + \varepsilon^2 f_2 \sqrt{\mu} + \varepsilon^{3/2} f_R \sqrt{\mu} \quad \text{in } [0, T] \times \Omega \times \mathbb{R}^3,$$

*which solves the Boltzmann equation (1.1) and the diffuse reflection boundary condition (1.11) with the scale of (1.5), (1.6), (2.195), and satisfies the initial condition*

$$F|_{t=0} = \mu + \varepsilon \sqrt{\mu} f_1|_{t=0} + \varepsilon^2 \sqrt{\mu} f_2|_{t=0} + \varepsilon^{3/2} \sqrt{\mu} f_{R,in},$$

*such that, for each  $\varepsilon$  and  $\kappa$  of (2.195),*

$$\sup_{0 \leq t \leq T} \left\| \frac{F(t, x, v) - \mu(v)}{\varepsilon \sqrt{\mu(v)}} - f_1 \right\|_{L^2(\Omega \times \mathbb{R}^3)} \lesssim \exp\left(\frac{-\mathfrak{C}(T+1)}{4\kappa}\right) \text{ for } \kappa \ll 1. \quad (4.18)$$

Moreover, let  $u_E(t, x), \theta_E(t, x)$  be the unique solution of the incompressible Euler equations under heat transfer (1.7)–(1.10) with the initial condition  $(u_E, \theta_E)|_{t=0} = (u_{in}, \theta_{in})$  satisfying (4.16), (4.17). Then

$$\sup_{0 \leq t \leq T} \left\| \frac{F(t, x, v) - \mu(v)}{\varepsilon \sqrt{\mu(v)}} - \left( -\theta_E(t, x) + u_E(t, x) \cdot v + \theta_E(t, x) \frac{|v|^2 - 3}{2} \right) \sqrt{\mu(v)} \right\|_{L^2(\Omega \times \mathbb{R}^3)} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ .

**Proof.** The existence of the Navier-Stokes-Fourier system follows from Theorem 9. For the remaining assertions, we note that all the estimates (3.32)–(3.35) of Theorem 9 ensure the conditions of Theorem 8 with  $\mathfrak{P} = \frac{1}{2}$ . Therefore (4.18) follows directly as a consequence of Theorem 8 and Theorem 9. As for the incompressible Euler limit, note that

$$\begin{aligned} & \frac{F(t, x, v) - \mu(v)}{\varepsilon \sqrt{\mu(v)}} - \left( -\theta_E + u_E \cdot v + \theta_E \frac{|v|^2 - 3}{2} \right) \sqrt{\mu} \\ &= \left[ \frac{F(t, x, v) - \mu(v)}{\varepsilon \sqrt{\mu(v)}} - f_1 \right] + \left[ f_1 - \left( -\theta_E + u_E \cdot v + \theta_E \frac{|v|^2 - 3}{2} \right) \sqrt{\mu} \right]. \end{aligned}$$

The first term can be bounded as in (4.18). The second term converges to 0 as  $\kappa \downarrow 0$  from Theorem 9, Lemma 10, and the famous Kato's condition for vanishing viscosity limit in [35].  $\square$

## Data availability

No data was used for the research described in the article.

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## References

- [1] C. Bardos, F. Golse, D. Levermore, Fluid dynamical limits of kinetic equations I. Formal derivations, *J. Stat. Phys.* 63 (1–2) (1991) 323–344.
- [2] C. Bardos, F. Golse, D. Levermore, Fluid dynamical limits of kinetic equations, II: Convergence proofs for the Boltzmann equation, *Commun. Pure Appl. Math.* 46 (5) (1993) 667–753.
- [3] C. Bardos, F. Golse, L. Paillard, The incompressible Euler limit of the Boltzmann equation with accommodation boundary condition, *Commun. Math. Sci.* 10 (1) (2012) 159–190.
- [4] F. Bouchut, F. Golse, M. Pulvirenti, in: L. Desvillettes, B. Perthame (Eds.), *Kinetic Equations and Asymptotic Theory*, Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, 2000.
- [5] M. Briant, S. Merino-Aceituno, C. Mouhot, From Boltzmann to incompressible Navier-Stokes in Sobolev spaces with polynomial weight, *Anal. Appl.* 17 (1) (2019) 85–116.

- [6] R. Caflisch, The fluid dynamic limit of the nonlinear Boltzmann equation, *Commun. Pure Appl. Math.* 33 (5) (1980) 651–666.
- [7] Y. Cao, C. Kim, On some recent progress in the Vlasov-Poisson-Boltzmann system with diffuse reflection boundary, in: F. Salvarani (Ed.), *Recent Advances in Kinetic Equations and Applications*, in: Springer INdAM Series, vol 48, Springer, Cham, [https://doi.org/10.1007/978-3-030-82946-9\\_4](https://doi.org/10.1007/978-3-030-82946-9_4).
- [8] Y. Cao, C. Kim, D. Lee, Global strong solutions of the Vlasov-Poisson-Boltzmann system in bounded domains, *Arch. Ration. Mech. Anal.* 233 (3) (2019) 1027–1130.
- [9] H. Chen, C. Kim, Regularity of stationary Boltzmann equation in convex domains, *Arch. Ration. Mech. Anal.* 244 (2022) 1099–1222, <https://doi.org/10.1007/s00205-022-01781-5>.
- [10] H. Chen, C. Kim, Q. Li, Local well-posedness of Vlasov-Poisson-Boltzmann equation with generalized diffuse boundary condition, *J. Stat. Phys.* 179 (2) (2020) 535–631.
- [11] A. de Masi, R. Esposito, J.L. Lebowitz, Incompressible Navier-Stokes and Euler limits of the Boltzmann equation, *Commun. Pure Appl. Math.* (1989) 1189–1214.
- [12] R.J. DiPerna, P.L. Lions, On the Cauchy problem for the Boltzmann equation: global existence and weak stability results, *Ann. Math.* 130 (1990) 321–366.
- [13] R. Esposito, Y. Guo, C. Kim, R. Marra, Non-isothermal boundary in the Boltzmann theory and Fourier law, *Commun. Math. Phys.* 323 (1) (2003) 177–239.
- [14] R. Esposito, Y. Guo, C. Kim, R. Marra, Stationary solutions to the Boltzmann equation in the hydrodynamic limit, *Ann. PDE* 4 (1) (2018) 1–119.
- [15] R. Esposito, Y. Guo, C. Kim, R. Marra, Diffusive limits of the Boltzmann equation in bounded domain, *Ann. Appl. Math.* 36 (2020) 111–185.
- [16] R. Esposito, Y. Guo, R. Marra, Hydrodynamic limit of a kinetic gas flow past an obstacle, *Commun. Math. Phys.* 364 (2018) 765–823.
- [17] F. Golse, *Hydrodynamic Limits*, Eur. Math. Soc., Zurich, 2005, pp. 699–717.
- [18] F. Golse, L. Saint-Raymond, The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels, *Invent. Math.* 155 (2004) 81–161.
- [19] Y. Guo, Boltzmann diffusive limit beyond the Navier-Stokes approximation, *Commun. Pure Appl. Math.* 59 (2006) 626–687.
- [20] Y. Guo, Decay and continuity of the Boltzmann equation in bounded domains, *Arch. Ration. Mech. Anal.* 197 (3) (2010) 713–809.
- [21] Y. Guo, J. Jang, N. Jiang, Acoustic limit for the Boltzmann equation in optimal scaling, *Commun. Pure Appl. Math.* 63 (3) (2010) 337–361.
- [22] Y. Guo, J. Jang, Global Hilbert expansion for the Vlasov-Poisson-Boltzmann system, *Commun. Math. Phys.* 299 (2) (2010) 469–501.
- [23] Y. Guo, J. Jang, N. Jiang, Local Hilbert expansion for the Boltzmann equation, *Kinet. Relat. Models* (1) (2009) 205–214.
- [24] Y. Guo, C. Kim, D. Tonon, A. Trescases, Regularity of the Boltzmann equation in convex domains, *Invent. Math.* 207 (2017) 115–290.
- [25] Y. Guo, C. Kim, D. Tonon, A. Trescases, BV-regularity of the Boltzmann equation in non-convex domains, *Arch. Ration. Mech. Anal.* 220 (2016) 1045–1093.
- [26] Y. Guo, L. Wu, Geometric correction in diffusive limit of neutron transport equation in 2D convex domains, *Arch. Ration. Mech. Anal.* 226 (1) (2017) 321–403.
- [27] J. Ginibre, G. Velo, Smoothing properties and retarded estimates for some dispersive evolution equations, *Commun. Math. Phys.* 144 (1) (1992) 163–188.
- [28] D. Hilbert, Begründung der kinetischen Gastheorie, *Math. Ann.* 72 (4) (1912) 562–577.
- [29] J. Jang, Vlasov-Maxwell-Boltzmann diffusive limit, *Arch. Ration. Mech. Anal.* 194 (2) (2009) 531–584.
- [30] J. Jang, C. Kim, Incompressible Euler limit from Boltzmann equation with diffuse boundary condition for analytic data, <https://arxiv.org/abs/2005.12192>, *Ann. PDE* 7 (2021) 22, <https://doi.org/10.1007/s40818-021-00108-z>.
- [31] J. Jang, C. Kim, A note on acoustic limit for the Boltzmann equation, in: F. Salvarani (Ed.), *Recent Advances in Kinetic Equations and Applications*, in: Springer INdAM Series, vol 48, Springer, Cham, [https://doi.org/10.1007/978-3-030-82946-9\\_10](https://doi.org/10.1007/978-3-030-82946-9_10).
- [32] J. Jin, C. Kim, Damping of kinetic transport equation with diffuse boundary condition, *SIAM J. Math. Anal.* 54 (5) (2022), <https://doi.org/10.1137/21M1455358>.
- [33] N. Jiang, N. Masoudi, Boundary layers and incompressible Navier-Stokes-Fourier limit of the Boltzmann equation in bounded domain I, *Commun. Pure Appl. Math.* 70 (1) (2017) 90–171.
- [34] P.-E. Jabin, L. Vega, A real space method for averaging lemmas, *J. Math. Pures Appl.* 83 (2004) 1309–1351.

- [35] T. Kato, Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary, in: *Seminar on Nonlinear Partial Differential Equations*, Berkeley, Calif., 1983, in: *Math. Sci. Res. Inst. Publ.*, vol. 2, Springer, New York, 1984, pp. 85–98.
- [36] C. Kim, Formation and propagation of discontinuity for Boltzmann equation in non-convex domains, *Commun. Math. Phys.* 308 (2011) 641–701.
- [37] C. Kim, Boltzmann equation with a large potential in a periodic box, *Commun. Partial Differ. Equ.* 39 (8) (2014) 1393–1423.
- [38] C. Kim, D. Lee, The Boltzmann equation with specular boundary condition in convex domains, *Commun. Pure Appl. Math.* 71 (2018) 411–504.
- [39] C. Kim, D. Lee, Decay of the Boltzmann equation with the specular boundary condition in non-convex cylindrical domains, *Arch. Ration. Mech. Anal.* 230 (1) (2018) 49–123.
- [40] C. Kim, J. La, Vorticity convergence from Boltzmann to 2D incompressible Euler equations below Yudovich class, *arXiv:2206.00543*.
- [41] C. Kim, T. Nguyen, Asymptotics of Helmholtz-Kirchhoff point-vortices in the phase space, *arXiv:2303.1225*.
- [42] C. Kim, S. Yun, The Boltzmann equation near a rotational local Maxwellian, *SIAM J. Math. Anal.* 44 (4) (2012) 2560–2598.
- [43] I. Kukavica, V. Vicol, F. Wang, The inviscid limit for the Navier-Stokes equations with data analytic only near the boundary, *Arch. Ration. Mech. Anal.* 237 (2020) 779–827.
- [44] M. Lachowicz, On the initial layer and the existence theorem for the nonlinear Boltzmann equation, *Math. Methods Appl. Sci.* 9 (3) (1987) 342–366.
- [45] P.L. Lions, N. Masmoudi, From the Boltzmann equations to the equations of incompressible fluid mechanics, I, *Arch. Ration. Mech. Anal.* 158 (2001) 173–193.
- [46] P.L. Lions, N. Masmoudi, From the Boltzmann equations to the equations of incompressible fluid mechanics, II, *Arch. Ration. Mech. Anal.* 158 (2001) 195–211.
- [47] Y. Maekawa, Solution formula for the vorticity equations in the half plane with application to high vorticity creation at zero viscosity limit, *Adv. Differ. Equ.* 18 (1/2) (2013) 101–146.
- [48] Y. Maekawa, On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane, *Commun. Pure Appl. Math.* 67 (7) (2014) 1045–1128.
- [49] L. Saint-Raymond, From the BGK Boltzmann model to the Euler equations of incompressible fluids, *Bull. Sci. Math.* 126 (6) (2002) 493–506.
- [50] L. Saint-Raymond, Convergence of solutions to the Boltzmann equation in the incompressible Euler limit, *Arch. Ration. Mech. Anal.* 166 (2003) 47–80.
- [51] L. Saint-Raymond, Hydrodynamic limits: some improvements of the relative entropy method, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 26 (3) (2009) 705–744.
- [52] L. Saint-Raymond, *Hydrodynamic Limits of the Boltzmann Equation*, Springer, Berlin, Heidelberg, 2009.
- [53] M. Sammartino, R.E. Caflisch, Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Commun. Math. Phys.* 192 (2) (1998) 433–461.
- [54] M. Sammartino, R.E. Caflisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution, *Commun. Math. Phys.* 192 (2) (1998) 463–491.
- [55] R. Temam, On the Euler equations of incompressible perfect fluids, *J. Funct. Anal.* 20 (1) (1975) 32–43.
- [56] T.T. Nguyen, T.T. Nguyen, The inviscid limit of Navier-Stokes equations for analytic data on the half-space, *Arch. Ration. Mech. Anal.* 230 (3) (2018) 1103–1129.
- [57] S. Ukai, K. Asano, The Euler limit and initial layer of the nonlinear Boltzmann equation, *Hokkaido Math. J.* 12 (3, part 1) (1983) 311–332.
- [58] F. Wang, The 3D inviscid limit problem with data analytic near the boundary, *SIAM J. Math. Anal.* 52 (4) (2020) 3520–3545.
- [59] L. Wu, Hydrodynamic limit with geometric correction of stationary Boltzmann equation, *J. Differ. Equ.* 260 (10) (2016) 7152–7249.