STABLE HOMOTOPY GROUPS OF SPHERES: FROM DIMENSION 0 TO 90*

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ABSTRACT

Using techniques in motivic homotopy theory, especially the theorem of Gheorghe, the second and the third author on the isomorphism between motivic Adams spectral sequence for $C\tau$ and the algebraic Novikov spectral sequence for BP_*, we compute the classical and motivic stable homotopy groups of spheres from dimension 0 to 90, except for some carefully enumerated uncertainties.

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1. Introduction

The computation of stable homotopy groups of spheres is one of the most fundamental and important problems in homotopy theory. It informs on many topics in topology, such as the cobordism theory of framed manifolds, the classification of smooth structures on spheres, obstruction theory, the theory of topological modular forms, algebraic K-theory, motivic homotopy theory, and equivariant homotopy theory.

Despite their simple definition, which was available eighty years ago, these groups are notoriously hard to compute. All known methods only give a complete answer through a range, but they eventually stall. Further progress requires the introduction of a new method. The standard approach to computing stable stems is to use an Adams spectral sequence (based on a generalized cohomology theory E) that converges from algebra to homotopy. In turn, to identify the algebraic E_2-pages, one needs algebraic spectral sequences that converge from simpler algebra to more complicated algebra. For any spectral sequence, difficulties arise in computing differentials and in solving extension problems. Different methods lead to trade-offs. One method may compute some types of differentials and extension problems efficiently, but leave other types unanswered, perhaps even unsolvable by that technique. To obtain complete computations, one must be eclectic, applying and combining different methodologies. Even so, combining all known methods, there are eventually some problems that have not been solved. Mahowald's uncertainty principle states that no finite collection of methods can completely compute the stable homotopy groups of spheres.

Because stable stems are finite abelian groups (except for the 0-stem), the computation is most easily accomplished by working one prime at a time. At odd primes, the Adams-Novikov spectral sequence and the chromatic spectral sequence, which are based on complex cobordism and formal groups, have yielded a wealth of data [53]. As the prime grows, so does the range of computation. For example, at the primes 3 and 5, we have complete knowledge up to around 100 and 1000 stems respectively [53].

The prime 2, being the smallest prime, remains the most difficult part of the computation. This entire manuscript considers exclusively the 2-completed stable homotopy groups. In this case, the Adams spectral sequence is the most effective tool. The manuscript [30] presents a careful analysis of the Adams spectral sequence, in both the classical and $\mathbb{C}$-motivic contexts, that is essentially complete through the 59-stem. This includes a verification of the details in the classical literature [2, 3, 10, 45]. Subsequently, the second and third authors computed the 60-stem and 61-stem [60].

We also mention [38, 40], which take an entirely different approach to computing stable homotopy groups. However, the computations in [38, 40] are now known to contain several errors. See [60, Section 2] for a more detailed discussion.

The goal of this manuscript is to continue the analysis of the Adams spectral sequence into higher stems at the prime 2. We will present information up to the 90-stem. While we have not been able to resolve all of the possible differentials in this range, we enumerate the handful of uncertainties explicitly within Table 9.

The charts in [33] and [34] are an essential companion to this manuscript. They present the same information in an easily interpretable graphical format.

Our analysis uses various methods and techniques, including machine-generated homological algebra computations, a deformation of homotopy theories that connects **C**-motivic and classical stable homotopy theory, and the theory of motivic modular forms. Here is a quick summary of our approach:

(1) Compute the cohomology of the **C**-motivic Steenrod algebra by machine. These groups serve as the input to the **C**-motivic Adams spectral sequence.
(2) Compute by machine the algebraic Novikov spectral sequence that converges to the cohomology of the Hopf algebroid (BP_*, BP_*BP). This includes all differentials, and the multiplicative structure of the cohomology of (BP_*, BP_*BP).
(3) Identify the **C**-motivic Adams spectral sequence for the cofiber of τ with the algebraic Novikov spectral sequence [21]. This includes an identification of the cohomology of (BP_*, BP_*BP) with the homotopy groups of the cofiber of τ.
(4) Pull back and push forward Adams differentials for the cofiber of τ to Adams differentials for the **C**-motivic sphere, along the inclusion of the bottom cell and the projection to the top cell.
(5) Deduce additional Adams differentials for the **C**-motivic sphere with a variety of ad hoc arguments. The most important methods are Toda bracket shuffles and comparison to the motivic modular forms spectrum *mmf* [20].
(6) Deduce hidden τ extensions in the **C**-motivic Adams spectral sequence for the sphere, using a long exact sequence in homotopy groups.
(7) Obtain the classical Adams spectral sequence and the classical stable homotopy groups by inverting τ.

The machine-generated data that we obtain in steps (1) and (2) are available at [33] and [34]. See also [59] for a discussion of the implementation of the machine computation.

Much of this process is essentially automatic. The exception occurs in step (5) where ad hoc arguments come into play.

This document describes the results of this systematic program through the 90-stem. We anticipate that our approach will allow us to compute into even higher stems, especially towards the last unsolved Kervaire invariant problem in dimension 126. However, we have not yet carried out a careful analysis.

1.1. *New ingredients*

We discuss in more detail several new ingredients that allow us to carry out this program.

1.1.1. *Machine-generated algebraic data.* — The Adams-Novikov spectral sequence has been used very successfully to carry out computations at odd primes. However, at the prime 2, its usage has not been fully exploited in stemwise computations. This is due to the difficulty of computing its E_2-page. The first author predicted in [30] that "the next major breakthrough in computing stable stems will involve machine computation of the Adams-Novikov E_2-page."

The second author achieved this machine computation; the resulting data is available at [34]. The process goes roughly like this. Start with a minimal resolution that computes the cohomology of the Steenrod algebra. Lift this resolution to a resolution of BP_*BP. Finally, use the Curtis algorithm to compute the homology of the resulting complex, and to compute differentials in the associated algebraic spectral sequences, such as the algebraic Novikov spectral sequence and the Bockstein spectral sequence. See [59] for further details.

1.1.2. *Motivic homotopy theory.* — The **C**-motivic stable homotopy category gives rise to new methods to compute stable stems. These ideas are used in a critical way in [30] to compute stable stems up to the 59-stem.

The key insight of this article that distinguishes it significantly from [30] is that **C**-motivic cellular stable homotopy theory is a deformation of classical stable homotopy theory [21]. From this perspective, the "generic fiber" of **C**-motivic stable homotopy theory is classical stable homotopy theory, and the "special fiber" has an entirely algebraic description. The special fiber is Hovey's stable derived category of BP_*BP-comodules [26], or equivalently, the stable derived category of quasicoherent sheaves on the moduli stack of 1-dimensional formal groups.

In more concrete terms, let $C\tau$ be the cofiber of the **C**-motivic stable map τ. The cofiber sequence $S^{0,0} \to C\tau \to S^{1,-1}$ induces maps

$$\begin{array}{ccccc} E_2(S^{0,0}) & \longrightarrow & E_2(C\tau) & \longrightarrow & E_2(S^{1,-1}) \\ \Downarrow & & \Downarrow & & \Downarrow \\ \pi_{*,*}(S^{0,0}) & \longrightarrow & \pi_{*,*}(C\tau) & \longrightarrow & \pi_{*,*}(S^{1,-1}) \end{array}$$

of spectral sequences, in which each vertical column represents a **C**-motivic Adams spectral sequence.

The homotopy category of $C\tau$-modules has an algebraic structure [21]. In particular, the **C**-motivic Adams spectral sequence for $C\tau$ is isomorphic to the algebraic Novikov spectral sequence that computes the E_2-page of the Adams-Novikov spectral sequence for BP_*. This means that the middle spectral sequence in the above diagram can be computed by machine. Naturality then yields information about the **C**-motivic Adams spectral sequence for the **C**-motivic sphere spectrum in two different ways, since the latter spectral sequence appears on both the left and right side of the diagram. Finally, the Betti realization functor produces differentials in the classical Adams spectral sequence.

Our use of **C**-motivic stable homotopy theory appears to rely on the fundamental computations, due to Voevodsky [56] [57], of the motivic cohomology of a point and of the motivic Steenrod algebra. In fact, recent progress has determined that our results do not depend on this deep and difficult work. There are now purely topological constructions of homotopy categories that have identical computational properties to the cellular stable **C**-motivic homotopy category [20, 52]. In these homotopy categories, one can obtain from first principles the fundamental computations of the cohomology of a point and of the Steenrod algebra, using only well-known classical computations. Therefore, the material in this manuscript does not logically depend on Voevodsky's work, even though the methods were very much inspired by his groundbreaking computations.

1.1.3. *Motivic modular forms.* — In classical chromatic homotopy theory, the theory of topological modular forms, introduced by Hopkins and Mahowald [18], plays a central role in the computations of the K(2)-local sphere.

Using a topological model of the cellular stable **C**-motivic homotopy category, one can construct a "motivic modular forms" spectrum *mmf* [20], whose motivic $\mathbf{F}_2$-cohomology is the quotient of the **C**-motivic Steenrod algebra by its subalgebra generated by Sq^1, Sq^2, and Sq^4. Just as *tmf* plays an essential role in studies of the classical Adams spectral sequence [5, 8], *mmf* is an essential tool for motivic computations. The **C**-motivic Adams spectral sequence for *mmf* can be analyzed completely [31], and naturality of Adams spectral sequences along the unit map of *mmf* provides much information about the behavior of the **C**-motivic Adams spectral sequence for the **C**-motivic sphere spectrum.

1.2. *Main results*

We summarize our main results in the following theorem and corollaries.

Theorem **1.1.** — *The* **C**-*motivic Adams spectral sequence for the* **C**-*motivic sphere spectrum is displayed in the charts in [33], up to the* 90-*stem.*

The proof of Theorem 1.1 consists of a series of specific computational facts, which are verified throughout this manuscript.

Corollary **1.2.** — *The classical Adams spectral sequence for the sphere spectrum is displayed in the charts in [33], up to the* 90-*stem.*

Corollary 1.2 follows immediately from Theorem 1.1. One simply inverts τ, or equivalently ignores τ-torsion.

Theorem 1.1 could also be used to completely determine the E_2-page and all differentials of the Adams-Novikov spectral sequence for the sphere spectrum. As described in [30, Chapter 6], the Adams-Novikov spectral sequence can be reverse-engineered from information about **C**-motivic stable homotopy groups.

Corollary **1.3.** — *Table 1 describes the stable homotopy groups π_k for all values of k up to* 90.

We adopt the following notation in Table 1. An integer n stands for the cyclic abelian group $\mathbf{Z}/n$; the symbol $\cdot$ by itself stands for the trivial group; the expression $n{\cdot}m$ stands for the direct sum $\mathbf{Z}/n \oplus \mathbf{Z}/m$; and n^j stands for the direct sum of j copies of $\mathbf{Z}/n$. The horizontal line after dimension 61 indicates the range in which our computations are new information.

Table 1 describes each group π_k as the direct sum of three subgroups: the 2-primary v_1-torsion, the odd primary v_1-torsion, and the v_1-periodic subgroups.

The last column of Table 1 describes the groups of homotopy spheres that classify smooth structures on spheres in dimensions at least 5. See Section 1.4 and Theorem 1.6 for more details.

Starting in dimension 84, there remain some uncertainties in the 2-primary v_1-torsion. In most cases, these uncertainties mean that the order of some stable homotopy groups are known only up to factors of 2. In a few cases, the additive group structures are also undetermined.

These uncertainties have two causes. First, there are a handful of differentials that remain unresolved; they are listed in Table 9. Second, there are some possible hidden 2 extensions that remain unresolved.

Figure 1 displays the 2-primary stable homotopy groups in a graphical format that is a modification by Allen Hatcher of Adams spectral sequence charts [23] (color figure online). Vertical chains of n dots indicate $\mathbf{Z}/2^n$. The non-vertical lines indicate multiplications by η and ν. The blue dots represent the v_1-periodic subgroups. The green dots are associated to the topological modular forms spectrum *tmf*. These elements are detected by the unit map from the sphere spectrum to *tmf*, either in homotopy or in the algebraic Ext groups that serve as Adams E_2-pages.

Finally, the red dots indicate uncertainties. In addition, in higher stems, there are possible extensions by 2, η, and ν that are not indicated in Figure 1. See Tables 17, 20, and 23 for more details about these possible extensions.

The orders of individual 2-primary stable homotopy groups do not follow a clear pattern, with large increases and decreases seemingly at random. However, an empirically observed pattern emerges if we consider the cumulative size of the groups, i.e., the product of the orders of all 2-primary stable homotopy groups from dimension 1 to dimension k.

Our data strongly suggest that asymptotically, there is a linear relationship between k^2 and the logarithm of this product of orders. In other words, the number of dots in Figure 1 in stems 1 through k is linearly proportional to k^2. Correspondingly, the number of dots in the classical Adams E_∞-page in stems 1 through k is linearly proportional to k^2. Thus, in extending from dimension 60 to dimension 90, the overall size of the computation more than doubles. Specifically, through dimension 60, the cumulative rank of the Adams E_∞-page is 199, and is 435 through dimension 90. Similarly, through dimension 60, the cumulative rank of the Adams E_2-page is 488, and is 1,461 through dimension 90.

TABLE 1. — Stable homotopy groups up to dimension 90. n stands for $\mathbf{Z}/n$; $n{\cdot}m$ stands for $\mathbf{Z}/n \oplus \mathbf{Z}/m$; and n^j stands for $(\mathbf{Z}/n)^j$

k	v_1-torsion at the prime 2	v_1-torsion at odd primes	v_1-periodic	Group of smooth structures
1	·	·	2	·
2	·	·	2	·
3	·	·	$8\cdot 3$	·
4	·	·	·	?
5	·	·	·	·
6	2	·	·	·
7	·	·	$16\cdot 3\cdot 5$	$\underline{b_2}$
8	2	·	2	2
9	2	·	2^2	$\underline{2}\cdot 2^2$
10	·	3	2	$2\cdot 3$
11	·	·	$8\cdot 9\cdot 7$	$\underline{b_3}$
12	·	·	·	·
13	·	3	·	3
14	$2\cdot 2$	·	·	2
15	2	·	$32\cdot 3\cdot 5$	$\underline{b_4}\cdot 2$
16	2	·	2	2
17	2^2	·	2^2	$\underline{2}\cdot 2^3$
18	8	·	2	$2\cdot 8$
19	2	·	$8\cdot 3\cdot 11$	$\underline{b_5}\cdot 2$
20	8	3	·	$8\cdot 3$
21	2^2	·	·	$\underline{2}\cdot 2^2$
22	2^2	·	·	2^2
23	$2\cdot 8$	3	$16\cdot 9\cdot 5\cdot 7\cdot 13$	$\underline{b_6}\cdot 2\cdot 8\cdot 3$
24	2	·	2	2
25	·	·	2^2	$\underline{2}\cdot 2$
26	2	3	2	$2^2\cdot 3$
27	·	·	$8\cdot 3$	$\underline{b_7}$
28	2	·	·	2
29	·	3	·	3
30	2	3	·	3
31	2^2	·	$64\cdot 3\cdot 5\cdot 17$	$\underline{b_8}\cdot 2^2$
32	2^3	·	2	2^3
33	2^3	·	2^2	$\underline{2}\cdot 2^4$
34	$2^2\cdot 4$	·	2	$2^3\cdot 4$
35	2^2	·	$8\cdot 27\cdot 7\cdot 19$	$\underline{b_9}\cdot 2^2$
36	2	3	·	$2\cdot 3$
37	2^2	3	·	$\underline{2}\cdot 2^2\cdot 3$
38	$2\cdot 4$	$3\cdot 5$	·	$2\cdot 4\cdot 3\cdot 5$
39	2^5	3	$16\cdot 3\cdot 25\cdot 11$	$\underline{b_{10}}\cdot 2^5\cdot 3$
40	$2^4\cdot 4$	3	2	$2^4\cdot 4\cdot 3$
41	2^3	·	2^2	$\underline{2}\cdot 2^4$
42	$2\cdot 8$	3	2	$2^2\cdot 8\cdot 3$
43	·	·	$8\cdot 3\cdot 23$	$\underline{b_{11}}$
44	8	·	·	8
45	$2^3\cdot 16$	$9\cdot 5$	·	$\underline{2}\cdot 2^3\cdot 16\cdot 9\cdot 5$
46	2^4	3	·	$2^4\cdot 3$
47	$2^3\cdot 4$	3	$32\cdot 9\cdot 5\cdot 7\cdot 13$	$\underline{b_{12}}\cdot 2^3\cdot 4\cdot 3$
48	$2^3\cdot 4$	·	2	$2^3\cdot 4$
49	·	3	2^2	$\underline{2}\cdot 2\cdot 3$
50	2^2	3	2	$2^3\cdot 3$
51	$2\cdot 8$	·	$8\cdot 3$	$\underline{b_{13}}\cdot 2\cdot 8$
52	2^3	3	·	$2^3\cdot 3$
53	2^4	·	·	$\underline{2}\cdot 2^4$
54	$2\cdot 4$	·	·	$2\cdot 4$

k	v_1-torsion at the prime 2	v_1-torsion at odd primes	v_1-periodic	Group of smooth structures
55	·	3	$16\cdot3\cdot5\cdot29$	$\underline{b_{14}}\cdot3$
56	·	·	2	·
57	2	·	2^2	$\underline{2}\cdot2^2$
58	2	·	2	2^2
59	2^2	·	$8\cdot9\cdot7\cdot11\cdot31$	$\underline{b_{15}}\cdot2^2$
60	4	·	·	4
61	·	·	·	·
62	2^4	3	·	$2^3\cdot3$
63	$2^2\cdot4$	·	$128\cdot3\cdot5\cdot17$	$\underline{b_{16}}\cdot2^2\cdot4$
64	$2^5\cdot4$	·	2	$2^5\cdot4$
65	$2^7\cdot4$	3	2^2	$\underline{2}\cdot2^8\cdot4\cdot3$
66	$2^5\cdot8$	·	2	$2^6\cdot8$
67	$2^3\cdot4$	·	$8\cdot3$	$\underline{b_{17}}\cdot2^3\cdot4$
68	2^3	3	·	$2^3\cdot3$
69	2^4	·	·	$\underline{2}\cdot2^4$
70	$2^5\cdot4^2$	·	·	$2^5\cdot4^2$
71	$2^6\cdot4\cdot8$	·	$16\cdot27\cdot5\cdot7\cdot13\cdot19\cdot37$	$\underline{b_{18}}\cdot2^6\cdot4\cdot8$
72	2^7	3	2	$2^7\cdot3$
73	2^5	·	2^2	$\underline{2}\cdot2^6$
74	4^3	3	2	$2\cdot4^3\cdot3$
75	2	9	$8\cdot3$	$\underline{b_{19}}\cdot2\cdot9$
76	$2^2\cdot4$	5	·	$2^2\cdot4\cdot5$
77	$2^5\cdot4$	·	·	$\underline{2}\cdot2^5\cdot4$
78	$2^3\cdot4^2$	3	·	$2^3\cdot4^2\cdot3$
79	$2^6\cdot4$	·	$32\cdot3\cdot25\cdot11\cdot41$	$\underline{b_{20}}\cdot2^6\cdot4$
80	2^8	·	2	2^8
81	$2^3\cdot4\cdot8$	3^2	2^2	$\underline{2}\cdot2^4\cdot4\cdot8\cdot3^2$
82	$2^5\cdot8$	$3\cdot7$	2	$2^6\cdot8\cdot3\cdot7$ or $2^4\cdot4\cdot8\cdot3\cdot7$
83	$2^3\cdot8$	5	$8\cdot9\cdot49\cdot43$	$\underline{b_{21}}\cdot2^3\cdot8\cdot5$
84	2^6 or 2^5	3^2	·	$2^6\cdot3^2$ or $2^5\cdot3^2$
85	$2^6\cdot4^2$ or $2^5\cdot4^2$ or $2^4\cdot4^3$ or $2^7\cdot4$	3^2	·	$2^6\cdot4^2\cdot3^2$ or $2^5\cdot4^2\cdot3^2$ or $2^4\cdot4^3\cdot3^2$ or $2^7\cdot4\cdot3^2$
86	$2^4\cdot8^2$ or $2^2\cdot4\cdot8^2$	$3\cdot5$	·	$2^4\cdot8^2\cdot3\cdot5$ or $2^2\cdot4\cdot8^2\cdot3\cdot5$
87	$2^5\cdot4$	·	$16\cdot3\cdot5\cdot23$	$\underline{b_{22}}\cdot2^5\cdot4$
88	$2^4\cdot4$	·	2	$2^4\cdot4$
89	2^3	·	2^2	$\underline{2}\cdot2^4$
90	$2^3\cdot8$ or $2^2\cdot8$	3	2	$2^4\cdot8\cdot3$ or $2^3\cdot8\cdot3$

Conjecture **1.4.** — *Let $f(k)$ be the product of the orders of the 2-primary stable homotopy groups in dimensions 1 through k. There exists a non-zero constant C such that*

$$\lim_{k\to\infty}\frac{\log_2 f(k)}{k^2}=C.$$

One interpretation of this conjecture is that the expected value of the logarithm of the order of the 2-primary component of π_k grows linearly in k. We have only data to support the conjecture, and we have not formulated a mathematical rationale. It is possible that in higher stems, new phenomena occur that alter the growth rate of the stable homotopy groups.

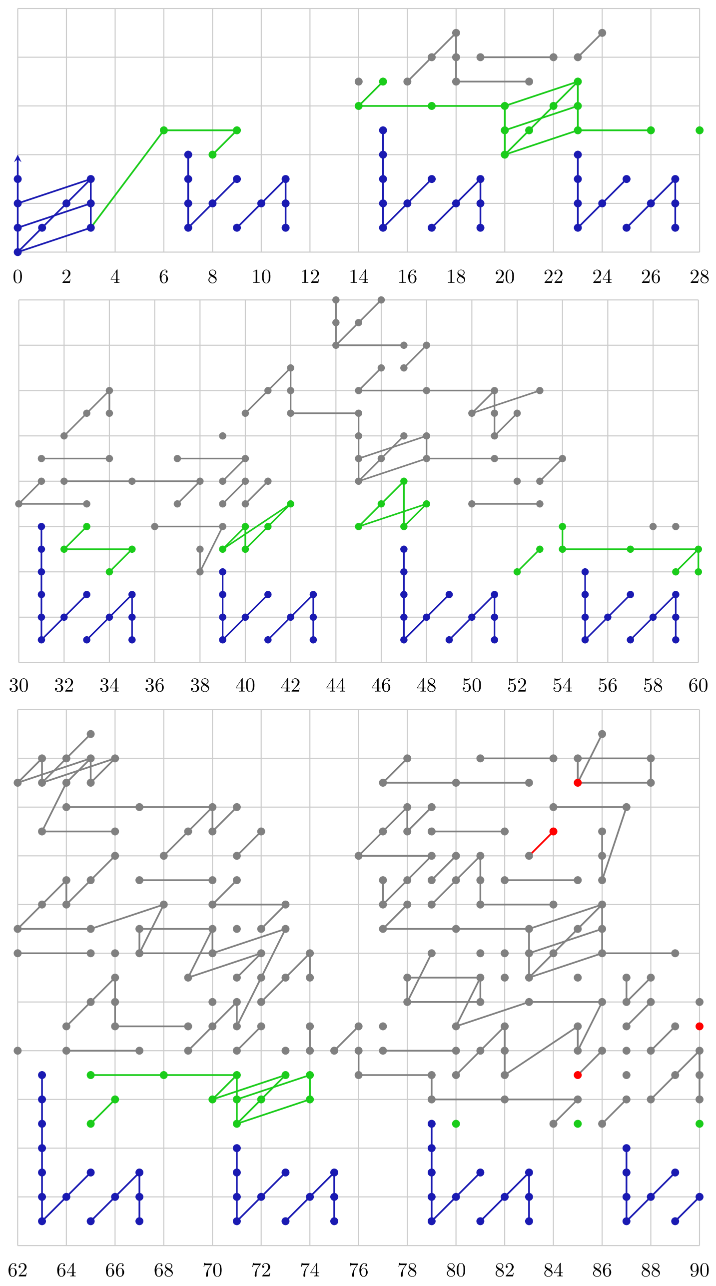

FIG. 1. — 2-primary stable homotopy groups (Color figure online)

By comparison, data indicates that the growth rate of the Adams E_2-page is qualitatively greater than the growth rate of the Adams E_∞-page. This apparent mismatch has implications for the frequency of Adams differentials.

1.3. *Remaining uncertainties*

Some uncertainties remain in the analysis of the first 90 stable stems. All undetermined possible differentials in this range are mentioned within Table 9. All of these uncertainties concern the Adams differentials d_r for $r \geq 9$. This means that the orders of some of the stable homotopy groups are known only up to factors of 2.

In addition, there are some possible hidden extensions by 2, η, and ν that remain unresolved. Tables 17, 20, and 23 summarize these possibilities. The presence of unknown hidden extensions by 2 means that the group structures of some stable homotopy groups are not known, even though their orders are known.

1.4. *Groups of homotopy spheres*

An important application of stable homotopy group computations is to the work of Kervaire and Milnor [36] on the classification of smooth structures on spheres in dimensions at least 5. Let Θ_n be the group of h-cobordism classes of homotopy n-spheres. This group classifies the differential structures on S^n for $n \geq 5$. It has a subgroup Θ_n^{bp}, which consists of homotopy spheres that bound parallelizable manifolds. The relation between Θ_n and the stable homotopy group π_n is summarized in Theorem 1.5. See also [49] for a survey on this subject.

Theorem **1.5** *(Kervaire-Milnor [36]). — Suppose that $n \geq 5$.*

(1) *The subgroup Θ_n^{bp} is cyclic, and has the following order:*

$$|\Theta_n^{bp}| = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 1 \text{ or } 2, & \text{if } n = 4k+1, \\ b_k, & \text{if } n = 4k-1. \end{cases}$$

Here b_k is $2^{2k-2}(2^{2k-1}-1)$ times the numerator of $8\zeta(1-2k)$, where ζ is the Riemann zeta function.

(2) *For $n \not\equiv 2 \pmod 4$, there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \longrightarrow 0.$$

Here π_n/J is the cokernel of the J-homomorphism.

(3) *For $n \equiv 2 \pmod 4$, there is an exact sequence*

$$0 \longrightarrow \Theta_n^{bp} \longrightarrow \Theta_n \longrightarrow \pi_n/J \xrightarrow{\Phi} \mathbf{Z}/2 \longrightarrow \Theta_{n-1}^{bp} \longrightarrow 0.$$

Here the map Φ is the Kervaire invariant.

The first few values, and then estimates, of the numbers b_k (for $k \geq 2$) are given by the sequence

$$28,\ 992,\ 8128,\ 261632,\ 1.45\times 10^9,\ 6.71\times 10^7,$$
$$1.94\times 10^{12},\ 7.54\times 10^{14},\ldots.$$

Theorem **1.6.** — *The last column of Table 1 describes the groups Θ_n for $n \leq 90$, with the exception of $n = 4$. The underlined symbols denote the contributions from Θ_n^{bp}.*

The cokernel of the J-homomorphism is slightly different than the v_1-torsion part of π_n at the prime 2. In dimensions $8m+1$ and $8m+2$, there are classes detected by $P^m h_1$ and $P^m h_1^2$ in the Adams spectral sequence. These classes are v_1-periodic, in the sense that they are detected by the K(1)-local sphere. However, they are also in the cokernel of the J-homomorphism.

We restate the following conjecture from [60], which is based on the current knowledge of stable stems and a problem proposed by Milnor [49].

Conjecture **1.7.** — *In dimensions greater than 4, the only spheres with unique smooth structures are* S^5, S^6, S^{12}, S^{56}, *and* S^{61}.

Uniqueness in dimensions 5, 6 and 12 was known to Kervaire and Milnor [36]. Uniqueness in dimension 56 is due to the first author [30], and uniqueness in dimension 61 is due to the second and the third authors [60].

Conjecture 1.7 is equivalent to the claim that the group Θ_n is not of order 1 for dimensions greater than 61. This conjecture has been confirmed in all odd dimensions by the second and the third authors [60] based on the work of Hill, Hopkins, and Ravenel [24], and in more than half of the even dimensions by Behrens, Hill, Hopkins, Mahowald and Quigley [7, 8].

1.5. *Notation*

The cohomology of the Steenrod algebra is highly irregular, so consistent naming systems for elements presents a challenge. A list of multiplicative generators appears in Table 4. To a large extent, we rely on the traditional names for elements, as used in [11, 30, 54], and elsewhere. However, we have adopted some new conventions in order to partially systematize the names of elements.

First, we use the symbol Δx to indicate an element that is represented by $v_2^4 x$ in the May spectral sequence. This use of Δ is consistent with the role that v_2^4 plays in the homotopy of *tmf*, where it detects the discriminant element Δ. For example, instead of the traditional symbol r, we use the name Δh_2^2.

Second, the symbol M indicates the Massey product operator $\langle -, h_0^3, g_2\rangle$. For example, instead of the traditional symbol B_1, we use the name Mh_1.

Similarly, the symbol g indicates the Massey product operator $\langle -, h_1^4, h_4\rangle$. For example, we write $h_2 g$ for the indecomposable element $\langle h_2, h_1^4, h_4\rangle$.

Eventually, we encounter elements that neither have traditional names, nor can be named using symbols such as P, Δ, M, and g. In these cases, we use arbitrary names of the form $x_{s,f}$, where s and f are the stem and Adams filtration of the element.

The last column of Table 4 gives alternative names, if any, for each multiplicative generator. These alternative names appear in at least one of [11, 30, 54].

Remark **1.8.** — One specific element deserves further discussion. In the cohomology of the motivic Steenrod algebra, we define τQ_3 to be the unique non-zero element in degree $(67, 5, 35)$ such that $h_3 \cdot \tau Q_3 = 0$. This choice is not compatible with the notation of [30]. The element τQ_3 from [30] equals the element $\tau Q_3 + \tau n_1$ in this manuscript.

We shall also extensively study the Adams spectral sequence for the cofiber of τ. See Section 3.1 for more discussion of the names of elements in this spectral sequence, and how they relate to the Adams spectral sequence for the sphere.

Table 1 gives some notation for elements in $\pi_{*,*}$. Many of these names follow standard usage, but we have introduced additional non-standard elements such as κ_1 and $\overline{\kappa}_2$. These elements are defined by the classes in the Adams E_∞-page that detect them. In some cases, this style of definition leaves indeterminacy because of the presence of elements in the E_∞-page in higher filtration. In some of these cases, Table 1 provides additional defining information. Beware that this additional defining information does not completely specify a unique element in $\pi_{*,*}$ in all cases. For the purposes of our computations, these remaining indeterminacies are not consequential.

Here is a list of the key notation that we use extensively:

- Because we have completed at 2, we have a map $\tau : S^{0,-1} \to S^{0,0}$ [27, Lemma 25]. We write $C\tau$ for its cofiber. We can also write S/τ for this **C**-motivic spectrum, but the latter notation is more cumbersome.
- $\mathrm{Ext} = \mathrm{Ext}_{\mathbf{C}}$ is the cohomology of the **C**-motivic Steenrod algebra. It is graded in the form (s, f, w), where s is the stem (i.e., the total degree minus the Adams filtration), f is the Adams filtration (i.e., the homological degree), and w is the motivic weight.
- $\mathrm{Ext}_{\mathrm{cl}}$ is the cohomology of the classical Steenrod algebra. It is graded in the form (s, f), where s is the stem (i.e., the total degree minus the Adams filtration), and f is the Adams filtration (i.e., the homological degree).
- $\pi_{*,*}$ are the 2-completed **C**-motivic stable homotopy groups.
- $H^*(S; BP)$ is the Adams-Novikov E_2-page for the classical sphere spectrum, i.e., $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$.
- $H^*(S/2; BP)$ is the Adams-Novikov E_2-page for the classical mod 2 Moore spectrum, i.e., $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/2)$.

1.6. *How to use this manuscript*

The manuscript is oriented around a series of tables to be found in Section 8. In a sense, the rest of the manuscript consists of detailed arguments for establishing each of the computations listed in the tables. We have attempted to give references and cross-references within these tables, so that the reader can more easily find the specific arguments pertaining to each computation.

We have attempted to make the arguments accessible to users who do not intend to read the manuscript in its entirety. To some extent, with an understanding of how the manuscript is structured, it is possible to extract information about a particular homotopy class in isolation. A secondary goal is to offer a guide to the computational techniques in use in stable homotopy theory today.

We assume that the reader is also referring to the Adams charts in [33] and [31]. These charts describe the same information as the tables, except in graphical form. Especially when there are multiple elements in a single degree, the charts can be somewhat ambiguous. In such cases, we encourage readers to use the associated spreadsheets [33]. These spreadsheets are more cumbersome than charts, but they are entirely explicit.

The style of this manuscript is very much similar to [30]. We will frequently refer to discussions in [30], rather than repeat that same material here in an essentially redundant way. This is especially true for the first parts of Chapters 2, 3, and 4 of [30], which discuss respectively the general properties of Ext, the May spectral sequence, and Massey products; the Adams spectral sequence and Toda brackets; and hidden extensions.

Section 2 provides some additional miscellaneous background material not already covered in [30]. Section 3 discusses the nature of the machine-generated data that we rely on. In particular, it describes our data on the algebraic Novikov spectral sequence, which is equal to the Adams spectral sequence for the cofiber of τ. Section 4 provides some tools for computing Massey products in Ext, and gives some specific computations. Section 5 carries out a detailed analysis of Adams differentials. Section 6 computes some miscellaneous Toda brackets that are needed for various specific arguments elsewhere. Section 7 methodically studies hidden extensions by τ, 2, η, and ν in the E_∞-page of the $\mathbf{C}$-motivic Adams spectral sequence. This section also gives some information about other miscellaneous hidden extensions. Finally, Section 8 includes the tables that summarize the multitude of specific computations that contribute to our study of stable homotopy groups.

2. Background

2.1. *Associated graded objects*

Definition **2.1.** — *A filtered abelian group* A *consists of a finite chain*

$$A = F_0A \supseteq F_1A \supseteq F_2A \supseteq \cdots \supseteq F_{p-1}A \supseteq F_pA = 0$$

of inclusions descending from A *to* 0.

We will only consider finite chains because these are the examples that arise in our Adams spectral sequences. Thus we do not need to refer to "exhaustive" and "Hausdorff" conditions on filtrations, and we avoid subtle convergence issues associated with infinite filtrations.

Example **2.2.** — The $\mathbf{C}$-motivic stable homotopy group $\pi_{14,8} = \mathbf{Z}/2 \oplus \mathbf{Z}/2$ is a filtered abelian group under the Adams filtration. The generators of this group are σ^2 and κ. The subgroup F_5 is zero, the subgroup $F_3 = F_4$ is generated by κ, and the subgroup $F_0 = F_1 = F_2$ is generated by σ^2 and κ.

Definition **2.3.** — *Let* A *be a filtered abelian group. The associated graded object* Gr A *is the sequence*

$$\left\{ \mathrm{Gr}_i\, \mathrm{A} = \frac{\mathrm{F}_i \mathrm{A}}{\mathrm{F}_{i+1} \mathrm{A}} \right\}_{i=0}^{p-1}$$

of successive quotients.

If a is an element of one of the quotients $\mathrm{Gr}_i\,\mathrm{A}$, then we say that i is the filtration of a. We will frequently refer to elements in "higher filtration" and "lower filtration". These comparisons refer to the numerical values of filtrations in the sense described here.

Similarly, if α is an element of $\mathrm{F}_i\mathrm{A} - \mathrm{F}_{i+1}\mathrm{A}$, then we say that α has filtration i or that α is detected in filtration i.

If a is an element of $\mathrm{Gr}_i\,\mathrm{A}$, then we write $\{a\}$ for the set of elements of A that are detected by a. In general, $\{a\}$ consists of more than one element of A, unless a happens to have highest filtration. In other words, the element a is a coset $\alpha + \mathrm{F}_{i+1}\mathrm{A}$ for some α in A, and $\{a\}$ is another name for this coset. In this situation, we say that a detects α.

In this manuscript, the main example of a filtered abelian group is a $\mathbf{C}$-motivic homotopy group $\pi_{p,q}$, equipped with its Adams filtration.

Example **2.4.** — Consider the $\mathbf{C}$-motivic stable homotopy group $\pi_{14,8}$ with its Adams filtration, as described in Example 2.2. The associated graded object is non-trivial only in degrees 2 and 4, and it is generated by h_3^2 and d_0 respectively.

Definition **2.5.** — *Let* A *and* B *be filtered abelian groups, perhaps with filtrations of different lengths. A map* $f : \mathrm{A} \to \mathrm{B}$ *is filtration preserving if* $f(\mathrm{F}_i\mathrm{A})$ *is contained in* $\mathrm{F}_i\mathrm{B}$ *for all* i.

Let $f : \mathrm{A} \to \mathrm{B}$ be a filtration preserving map of filtered abelian groups. We write $\mathrm{Gr} f : \mathrm{Gr}\,\mathrm{A} \to \mathrm{Gr}\,\mathrm{B}$ for the induced map on associated graded objects.

Definition **2.6.** — *Let* a *and* b *be elements of* $\mathrm{Gr}_i\,\mathrm{A}$ *and* $\mathrm{Gr}_j\,\mathrm{B}$ *respectively. The element* b *is the (not hidden) value of* a *under* f *if* $\mathrm{Gr}_i f(a) = b$.

The element b is a hidden value of a under f if:

(1) $\mathrm{Gr}_i f(a) = 0$.
(2) *there exists an element α of $\{a\}$ in* A *such that:*
  (a) *$f(\alpha)$ is contained in $\{b\}$ in* B*, and*
  (b) *there is no element γ of* A *in filtration strictly higher than i such that $f(\gamma)$ is contained in $\{b\}$.*

Alternatively, condition (2b) can be restated to say that $f(\mathrm{F}_{i+1})$ does not intersect $\{b\}$.

The motivation for condition (2b) may not be obvious. The point is to avoid situations in which condition (2a) is satisfied trivially. Suppose that there is an element γ such that $f(\gamma)$ is contained in $\{b\}$. Let a be any element of Gr A whose filtration is strictly lower than the filtration of γ. Now let α be any element of $\{a\}$ such that $f(\alpha) = 0$. (It may not be possible to choose such an α in general, but sometimes it is possible.) Then $\alpha + \gamma$ is another element of $\{a\}$ such that $f(\alpha + \gamma)$ is contained in $\{b\}$. Thus f takes some element of $\{a\}$ into $\{b\}$, but only because of the presence of γ. Condition (2b) is designed to exclude this situation.

Example **2.7.** — We illustrate the role of condition (2b) in Definition 2.6 with a specific example. Consider the map $\eta : \pi_{14,8} \to \pi_{15,9}$. The associated graded map $\mathrm{Gr}(\eta)$ takes h_3^2 to 0 and takes d_0 to $h_1 d_0$.

The coset $\{h_3^2\}$ in $\pi_{14,8}$ consists of two elements σ^2 and $\sigma^2 + \kappa$. One of these elements is non-zero after multiplying by η. (In fact, $\eta\sigma^2$ equals zero, and $\eta(\sigma^2 + \kappa) = \eta\kappa$ is non-zero, but that is not relevant here.) Conditions (1) and (2a) of Definition 2.6 are satisfied, but condition (2b) fails because of the presence of κ in higher filtration.

Suppose that b is a hidden value of a under f. It is typically the case that $f(\alpha)$ is contained in $\{b\}$ for every α in A. However, an even more complicated situation can occur in which this is not true.

Suppose that b_0 is a hidden value of a_0 under f, and suppose that b_1 is a (hidden or not hidden) value of a_1 under f. Moreover, suppose that the filtration of a_0 is strictly lower than the filtration of a_1, and the filtration of b_0 is strictly higher than the filtration of b_1. In this situation, we say that the value of a_0 under f crosses the value of a_1 under f.

The terminology arises from the usual graphical calculus, in which elements of higher filtration are drawn above elements of lower filtration, and values of maps are indicated by line segments, as in Figure 2.

Example **2.8.** — For any map X → Y of **C**-motivic spectra, naturality of the Adams spectral sequence induces a filtration preserving map $\pi_{p,q}\mathrm{X} \to \pi_{p,q}\mathrm{Y}$. We are often interested in inclusion $\mathrm{S}^{0,0} \to \mathrm{C}\tau$ of the bottom cell into $\mathrm{C}\tau$, and in projection $\mathrm{C}\tau \to \mathrm{S}^{1,-1}$ from $\mathrm{C}\tau$ to the top cell. We also consider the unit map $\mathrm{S}^{0,0} \to$ *mmf*.

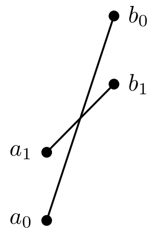

FIG. 2. — Crossing values

2.1.1. *Indeterminacy in hidden values.* — Definition 2.6 allows for the possibility that a fixed element a could have more than one hidden value under f. In order to manage this complication, we introduce indeterminacy into our definition.

Suppose, as in Definition 2.6, that b is a hidden value of a under f, so there exists some α in $\{a\}$ such that $f(\alpha)$ is contained in $\{b\}$. Suppose also that there is another element a' in $\mathrm{Gr}\,\mathrm{A}$ in filtration strictly higher than the filtration of a, such that $f(\alpha')$ is contained in $\{b'\}$, where α' is in $\{a'\}$ and b' has the same filtration as b. Then $b+b'$ is also a hidden value of a under f, since $\alpha+\alpha'$ is contained in $\{a\}$ and $f(\alpha+\alpha')$ is contained in $\{b+b'\}$. In this case, we say that b' belongs to the target indeterminacy of the hidden value.

Example **2.9.** — Consider the map $\eta : \pi_{63,33} \to \pi_{64,34}$. The element $h_3 Q_2$ is a hidden value of $\tau h_1 H_1$ under this map. This hidden value has target indeterminacy generated by $\tau h_1 X_2 = h_1 \cdot (\tau X_2 + \tau C')$.

2.1.2. *Hidden extensions.* — Let α be an element of $\pi_{a,b}$ with Adams filtration i. Then multiplication by α induces a map $\pi_{p,q} \to \pi_{p+a,q+b}$ that takes elements of Adams filtration f to elements of Adams filtration $f+i$ or higher. In other words, the map $\pi_{p,q} \to \pi_{p+a,q+b}$ is filtration-preserving if we add i to all of the filtration values in $\pi_{p,q}$. A hidden value of this map (with shifted filtration values on the source) is precisely the same as a hidden extension by α in the sense of [30, Definition 4.2]. For clarity, we repeat the definition here.

Definition **2.10.** — *Let α be an element of $\pi_{*,*}$ that is detected by an element a of the E_∞-page of the $\mathbf{C}$-motivic Adams spectral sequence. A hidden extension by α is a pair of elements b and c of E_∞ such that:*

(1) *$ab = 0$ in the E_∞-page.*
(2) *There exists an element β of $\{b\}$ such that $\alpha\beta$ is contained in $\{c\}$.*
(3) *If there exists an element β' of $\{b'\}$ such that $\alpha\beta'$ is contained in $\{c\}$, then the Adams filtration of b' is lower than or equal to the Adams filtration of b.*

A crossing value for the map $\alpha : \pi_{p,q} \to \pi_{p+a,q+b}$ is precisely the same as a crossing extension in the sense of [30, Examples 4.6 and 4.7].

The discussion of target indeterminacy applies to the case of hidden extensions. For example, the hidden η extension from $h_3 Q_2$ to $\tau h_1 H_1$ has target indeterminacy generated by $\tau h_1 X_2$.

In later sections, we will thoroughly explore hidden extensions by 2, η, and ν. We warn the reader that a complete understanding of such hidden extensions does not necessarily lead to a complete understanding of multiplication by 2, η, and ν in the **C**-motivic stable homotopy groups.

For example, in the 45-stem, there exists an element $\theta_{4.5}$ that is detected by $h_3^2 h_5$ such that $4\theta_{4.5}$ is detected by $h_0 h_5 d_0$. This is an example of a hidden 4 extension. However, there is no hidden 2 extension from $h_0 h_3^2 h_5$ to $h_0 h_5 d_0$; condition (2b) of Definition 2.6 is not satisfied.

In fact, a complete understanding of *all* hidden extensions leads to a complete understanding of the multiplicative structure of the **C**-motivic stable homotopy groups, but the process is perhaps more complicated than expected.

For example, we mentioned in Example 2.7 that either $\eta(\sigma^2 + \kappa)$ or $\eta\sigma^2$ is non-zero, but these cases cannot be distinguished by a study of hidden η extensions. However, we can express that $\eta\sigma^2$ is zero by observing that there is no hidden σ extension from $h_1 h_3$ to $h_1 d_0$.

There are even further complications. For example, the equation $h_2^3 + h_1^2 h_3 = 0$ does not prove that $\nu^3 + \eta^2\sigma$ equals zero because it could be detected in higher filtration. In fact, this does occur. Toda's relation [55] says that

$$\eta^2\sigma + \nu^3 = \eta\epsilon,$$

where $\eta\epsilon$ is detected by $h_1 c_0$.

We can express Toda's relation in terms of a "matric hidden extension". We have a map $[\nu \quad \eta] : \pi_{6,4} \oplus \pi_{8,5} \to \pi_{9,6}$. The associated graded map takes $(h_2^2, h_1 h_3)$ to zero, but $h_1 c_0$ is a hidden value of $(h_2^2, h_1 h_3)$ under this map, in the sense of Definition 2.6.

2.2. *Motivic modular forms*

Over **C**, a "motivic modular forms" spectrum *mmf* has recently been constructed [20]. From our computational perspective, *mmf* is a ring spectrum whose cohomology is $A//A(2)$, i.e., the quotient of the **C**-motivic Steenrod algebra by the subalgebra generated by Sq^1, Sq^2, and Sq^4. By the usual change-of-rings isomorphism, this implies that the homotopy groups of *mmf* are computed by an Adams spectral sequence whose E_2-page is the cohomology of **C**-motivic A(2) [28]. The Adams spectral sequence for *mmf* has been completely computed [31].

By naturality, the unit map $S^{0,0} \to mmf$ yields a map of Adams spectral sequences. This map allows us to transport information from the thoroughly understood spectral sequence for *mmf* to the less well understood spectral sequence for $S^{0,0}$. This comparison technique is essential at many points throughout our computations.

We rely on notation from [28] and [31] for the Adams spectral sequence for *mmf*, except that we use a and n instead of α and ν respectively.

For the most part, the map $\pi_{*,*} \to \pi_{*,*}mmf$ is detected on Adams E_∞-pages. However, this map does have some hidden values.

Theorem **2.11.** — *Through dimension 90, Table 2 lists all hidden values of the map $\pi_{*,*} \to \pi_{*,*}mmf$.*

Proof. — Most of these hidden values follow from hidden τ extensions in the Adams spectral sequences for $S^{0,0}$ and for *mmf*. For example, for $S^{0,0}$, there is a hidden τ extension from h_1h_3g to d_0^2. For *mmf*, there is a hidden τ extension from cg to d^2. This implies that cg is a hidden value of h_1h_3g.

A few cases are slightly more difficult. The hidden values of Δh_1h_3 and h_0h_5i follow from the Adams-Novikov spectral sequences for $S^{0,0}$ and for *mmf*. These two values are detected on Adams-Novikov E_∞-pages in filtration 2.

Next, the hidden value on Ph_2h_5j follows from multiplying the hidden value on h_0h_5i by d_0. Finally, the hidden values on $\Delta h_1^2h_3$, $h_0h_2h_5i$, and Ph_5j follow from already established hidden values, relying on h_1 extensions and h_2 extensions. □

Remark **2.12.** — Through the 90-stem, there are no crossing values for the map $\pi_{*,*} \to \pi_{*,*}mmf$. Moreover, in this range, there is only one hidden value that has target indeterminacy. Namely, Δ^2h_2d is the hidden value of Ph_5j, with target indeterminacy generated by $\tau^3\Delta h_1g^2$.

2.3. *The cohomology of the* **C**-*motivic Steenrod algebra*

We have implemented machine computations of Ext, i.e., the cohomology of the **C**-motivic Steenrod algebra, in detail through the 110-stem. We take this computational data for granted. It is depicted graphically in the chart of the E_2-page shown in [33]; the data is also available there. See [59] for a discussion of the implementation.

In addition to the additive structure of Ext, we also have complete information about multiplications by h_0, h_1, h_2, and h_3. We do not have complete multiplicative information. Occasionally we must deduce some multiplicative information on an ad hoc basis.

Similarly, we do not have systematic machine-generated Massey product information about Ext. We deduce some of the necessary information about Massey products in Section 4.

In the classical situation, Bruner has carried out extensive machine computations of the cohomology of the classical Steenrod algebra [11]. More recently, Bruner and Rognes have extended these computations to total degree 184 [13]. This data includes complete primary multiplicative information, but no higher Massey product structure.

We rely heavily on this information. Our reliance on this data is so ubiquitous that we will not give repeated citations. Very recent work of Joey Beauvais-Feisthauer, Hood Chatham, and Dexter Chua [6] and of Weinan Lin [42] [43] extends these machine computations of classical Ext to significantly higher stems.

The May spectral sequence is the key tool for a conceptual computation of Ext. See [30] for full details. In this manuscript, we use the May spectral sequence to compute some Massey products that we need for various specific purposes; see Remark 2.26 for more details.

For convenience, we restate the following structural theorem about a portion of $\mathrm{Ext}_{\mathbf{C}}$ [30, Theorem 2.19].

Theorem **2.13.** — *There is a highly structured isomorphism from* $\mathrm{Ext}_{\mathrm{cl}}$ *to the subalgebra of* Ext *consisting of elements in degrees* (s, f, w) *with* $s+f-2w=0$. *This isomorphism takes classical elements of degree* (s, f) *to motivic elements of degree* $(2s+f, f, s+f)$.

2.4. *Toda brackets*

Toda brackets are an essential computational tool for understanding stable homotopy groups [38, Chapter 2] [55].

Brackets appear throughout the various stages of the computations, including in the analysis of Adams differentials and in the resolution of hidden extensions.

It is well-known that the stable homotopy groups form a ring under the composition product. The higher Toda bracket structure is an extension of this ring structure that is much deeper and more intricate. Our philosophy is that the stable homotopy groups are not really understood until the Toda bracket structure is revealed.

A complete analysis of all Toda brackets (even in a range) is not a practical goal. There are simply too many possibilities to take into account methodically, especially when including matric Toda brackets (and possibly other more exotic non-linear types of brackets). In practice, we compute only the Toda brackets that we need for our specific computational purposes.

2.4.1. *The Moss Convergence Theorem.* — We next discuss the Moss Convergence Theorem [50], which is the essential tool for computing Toda brackets in stable homotopy groups. See also [9] for a modern proof of the Moss Convergence Theorem that applies not only in classical stable homotopy theory but also to a wide variety of stable homotopy theories including **C**-motivic stable homotopy theory.

In order to state the Moss Convergence Theorem precisely, we must clarify the various types of bracket operations that arise. First, the Adams E_2-page has Massey products arising from the fact that it is the homology of the cobar complex, which is a differential graded algebra. We typically refer to these simply as "Massey products", although we write "Massey products in the E_2-page" for clarification when necessary.

Next, each higher E_r-page also has Massey products, since it is the homology of the E_{r-1}-page, which is a differential graded algebra. We always refer to these as "Massey products in the E_r-page" to avoid confusion with the more familiar Massey products in the E_2-page. This type of bracket appears only occasionally throughout the manuscript.

Beware that the higher E_r-pages do not inherit Massey products from the preceding pages. For example, τh_1^2 equals the Massey product $\langle h_0, h_1, h_0 \rangle$ in the E_2-page. However, in the E_3-page, the bracket $\langle h_0, h_1, h_0 \rangle$ equals zero, since the product $h_0 h_1$ is already equal to zero in the E_2-page before taking homology to obtain the E_3-page.

On the other hand, the Massey product $\langle h_1, h_0, h_3^2 \rangle$ is not a well-defined Massey product in the E_2-page since $h_0 h_3^2$ is non-zero, while $\langle h_1, h_0, h_3^2 \rangle$ in the E_3-page equals $h_1 h_4$ because of the differential $d_2(h_4) = h_0 h_3^2$.

Finally, we have Toda brackets in the stable homotopy groups $\pi_{*,*}$. The point of the Moss Convergence Theorem is to relate these various kinds of brackets.

Definition **2.14.** — *Given r and a degree (s, f, w), a crossing differential is a nonzero differential $d_{r+n}(x) = y$ in the $\mathbf{C}$-motivic Adams spectral sequence such that y has degree (s, f', w) with $f' > f$ and x has degree $(s+1, f'', w)$ with $f'' < f - r$.*

Here is the idea behind Definition 2.14. Consider the d_r differential restricted to source degree $(s+1, f-r, w)$ and target degree (s, f, w). A crossing differential is a longer differential whose source degree lies strictly "below" $(s+1, f-r, w)$ and whose target degree lies strictly "above" (s, f, w).

Remark **2.15.** — In practice, we will never use Definition 2.14. Rather, we will consider elements a and b in the E_r-page of the $\mathbf{C}$-motivic Adams spectral sequence such that $ab = 0$. We informally call a nonzero differential

$$d_{r+n} x = y$$

a crossing differential for the product ab if it satisfies Definition 2.14 for the degree of ab.

Figure 3 depicts the situation of a crossing differential in a chart for the E_r-page. Typically, the product ab is zero in the E_r-page because it was hit by a d_{r-1} differential, as shown by the dashed arrow in the figure. However, it may very well be the case that the product ab is already zero in the E_{r-1}-page (or even in an earlier page), in which case the dashed d_{r-1} differential is actually $d_{r-1}(0) = 0$.

Theorem **2.16** *(Moss Convergence Theorem).* — *Suppose that a, b, and c are permanent cycles in the E_r-page of the $\mathbf{C}$-motivic Adams spectral sequence that detect homotopy classes α, β, and γ in $\pi_{*,*}$ respectively. Suppose further that*

(1) *the Massey product $\langle a, b, c \rangle$ is defined in the E_r-page, i.e., $ab = 0$ and $bc = 0$ in the E_r-page.*

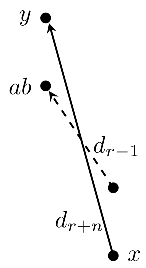

FIG. 3. — Crossing differentials

(2) *the Toda bracket* $\langle \alpha, \beta, \gamma \rangle$ *is defined in* $\pi_{*,*}$, *i.e.,* $\alpha\beta = 0$ *and* $\beta\gamma = 0$.
(3) *there are no crossing differentials for the products* ab *and* bc *in the* E_r*-page.*

Then there exists an element e *contained in the Massey product* $\langle a, b, c \rangle$ *in the* E_r*-page, such that*

(1) *the element* e *is a permanent cycle.*
(2) *the element* e *detects a homotopy class in the Toda bracket* $\langle \alpha, \beta, \gamma \rangle$.

Remark **2.17.** — The homotopy classes α, β, and γ are usually not unique. The presence of elements in higher Adams filtration implies that a, b, and c detect more than one homotopy class. Moreover, it may be the case that $\langle \alpha, \beta, \gamma \rangle$ is defined for only some choices of α, β, and γ, while the Toda bracket is not defined for other choices.

Remark **2.18.** — The Moss Convergence Theorem 2.16 says that a certain Massey product $\langle a, b, c \rangle$ in the E_r-page contains an element with certain properties. The theorem does not claim that every element of $\langle a, b, c \rangle$ has these properties. In the presence of indeterminacies, there can be elements in $\langle a, b, c \rangle$ that do not satisfy the given properties.

Remark **2.19.** — Beware that the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ may have non-zero indeterminacy. In this case, we only know that e detects one element of the Toda bracket. Other elements of the Toda bracket could possibly be detected by other elements of the Adams E_∞-page; these occurrences must be determined by inspection.

Remark **2.20.** — In practice, one computes a Toda bracket $\langle \alpha, \beta, \gamma \rangle$ by first studying its corresponding Massey product $\langle a, b, c \rangle$ in a certain page of the Adams spectral sequence. In the case that the Massey product $\langle a, b, c \rangle$ equals zero in the E_r-page in Adams filtration f, the Moss Convergence Theorem 2.16 does not imply that the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ contains zero. Rather, the Toda bracket contains an element (possibly zero) that is detected in Adams filtration at least $f + 1$.

Example **2.21.** — Consider the Toda bracket $\langle \nu, \eta, \nu \rangle$. The elements h_1 and h_2 are permanent cycles that detect η and ν, and the product $\eta\nu$ is zero. We have that $\langle h_2, h_1, h_2 \rangle$

equals h_1h_3, with no indeterminacy, in the E_2-page. There are no crossing differentials for the product $h_1h_2 = 0$ in the E_2-page, so the Moss Convergence Theorem 2.16 implies that h_1h_3 detects a homotopy class in $\langle \nu, \eta, \nu \rangle$.

Note that h_1h_3 detects the homotopy class $\eta\sigma$ because h_3 is a permanent cycle that detects σ. However, we cannot conclude that $\eta\sigma$ is contained in $\langle \nu, \eta, \nu \rangle$. The presence of the permanent cycle c_0 in higher filtration means that h_1h_3 detects both $\eta\sigma$ and $\eta\sigma + \epsilon$, where ϵ is the unique homotopy class that is detected by c_0. The Moss Convergence Theorem 2.16 implies that either $\eta\sigma$ or $\eta\sigma + \epsilon$ is contained in the Toda bracket $\langle \nu, \eta, \nu \rangle$. In fact, $\eta\sigma + \epsilon$ is contained in the Toda bracket, but determining this requires further analysis.

Example **2.22.** — Consider the Toda bracket $\langle \sigma^2, 2, \eta \rangle$. The elements h_3^2, h_0, and h_1 are permanent cycles that detect σ^2, 2, and η respectively, and the products $2\sigma^2$ and 2η are both zero. Due to the Adams differential $d_2(h_4) = h_0h_3^2$, the Massey product $\langle h_3^2, h_0, h_1 \rangle$ equals h_1h_4 in the E_3-page, with zero indeterminacy. There are no crossing differentials for the products $h_0h_3^2 = 0$ and $h_0h_1 = 0$ in the E_3-page. The Moss Convergence Theorem 2.16 implies that h_1h_4 detects a homotopy class in the Toda bracket $\langle \sigma^2, 2, \eta \rangle$.

The element h_3^2 also detects $\sigma^2 + \kappa$, where κ is the unique homotopy class that is detected by d_0, and the product $2(\sigma^2 + \kappa)$ is zero. The Moss Convergence Theorem 2.16 also implies that h_1h_4 detects a homotopy class in the Toda bracket $\langle \sigma^2 + \kappa, 2, \eta \rangle$.

Example **2.23.** — Consider the Toda bracket $\langle \kappa, 2, \eta \rangle$. The elements d_0, h_0, and h_1 are permanent cycles that detect κ, 2, and η respectively, and the products 2κ and 2η are both zero. Due to the Adams differential $d_3(h_0h_4) = h_0d_0$, the Massey product $\langle d_0, h_0, h_1 \rangle$ equals $h_0h_4 \cdot h_1 = 0$ in Adams filtration 3 in the E_4-page, with zero indeterminacy. There are no crossing differentials for the products $h_0d_0 = 0$ and $h_0h_1 = 0$ in the E_4-page. The Moss Convergence Theorem 2.16 implies that the Toda bracket $\langle \kappa, 2, \eta \rangle$ either contains zero, or it contains a non-zero element detected in Adams filtration higher than 3.

The only possible detecting element is Pc_0. There is a hidden η extension from $h_0^3h_4$ to Pc_0, so Pc_0 detects an element in the indeterminacy of $\langle \kappa, 2, \eta \rangle$. Consequently, the Toda bracket is $\{0, \eta\rho_{15}\}$, where ρ_{15} is detected by $h_0^3h_4$.

Example **2.24.** — The Massey product $\langle h_2, h_3^2, h_0^2 \rangle$ equals $\{f_0, f_0 + h_0^2h_2h_4\}$ in the E_2-page. The elements h_2, h_3^2, and h_0^2 are permanent cycles that detect ν, σ^2, and 4 respectively, and the products $\nu\sigma^2$ and $4\sigma^2$ are both zero. However, the product $h_0^2h_3^2$ has a crossing differential $d_3(h_0h_4) = h_0d_0$. The Moss Convergence Theorem 2.16 does not apply, and we cannot conclude anything about the Toda bracket $\langle \nu, \sigma^2, 4 \rangle$. In particular, we cannot conclude that $\{f_0, f_0 + h_0^2h_2h_4\}$ contains a permanent cycle. In fact, both elements support Adams d_2 differentials.

Remark **2.25.** — There is a version of the Moss Convergence Theorem 2.16 for computing fourfold Toda brackets $\langle \alpha, \beta, \gamma, \delta \rangle$ in terms of fourfold Massey products $\langle a, b, c, d \rangle$ in the E_r-page. In this case, the crossing differential condition applies not only to the products ab, bc, and cd, but also to the subbrackets $\langle a, b, c \rangle$ and $\langle b, c, d \rangle$.

Remark **2.26.** — Just as the Moss Convergence Theorem 2.16 is the key tool for computing Toda brackets with the Adams spectral sequence, the May Convergence Theorem is the key tool for computing Massey products with the May spectral sequence. The statement of the May Convergence Theorem is entirely analogous to the statement of the Moss Convergence Theorem, with Adams differentials replaced by May differentials; Adams E_r-pages replaced by May E_r-pages; $\pi_{*,*}$ replaced by Ext; and Toda brackets replaced by Massey products. An analogous crossing differential condition applies. See [30, Section 2.2] [46] for more details. We will use the May Convergence Theorem to compute various Massey products that we need for specific purposes.

2.4.2. *Moss's higher Leibniz rule.* — Occasionally, we will use Moss's higher Leibniz rule [50], which describes how Massey products in the E_r-page interact with the Adams d_r differential. This theorem is a direct generalization of the usual Leibniz rule $d_r(ab) = d_r(a)b + ad_r(b)$ for twofold products.

Theorem **2.27** *([50]). — Suppose that a, b, and c are elements in the E_r-page of the **C**-motivic Adams spectral sequence such that $ab = 0$, $bc = 0$, $d_r(b)a = 0$, and $d_r(b)c = 0$. Then*

$$d_r\langle a, b, c\rangle \subseteq \langle d_r(a), b, c\rangle + \langle a, d_r(b), c\rangle + \langle a, b, d_r(c)\rangle,$$

where all brackets are computed in the E_r-page.

Remark **2.28.** — By the Leibniz rule, the conditions $d_r(b)a = 0$ and $d_r(b)c = 0$ imply that $d_r(a)b = 0$ and $d_r(c)b = 0$. Therefore, all of the Massey products in Theorem 2.27 are well-defined.

Remark **2.29.** — The Massey products in Moss's higher Leibniz rule 2.27 may have indeterminacies, so the statement involves an inclusion of sets, rather than an equality.

Remark **2.30.** — Beware that Moss's higher Leibniz rule 2.27 cannot be applied to Massey products in the E_r-page to study differentials in higher pages. For example, we cannot use it to compute the d_3 differential on a Massey product in the E_2-page. In fact, there are versions of the higher Leibniz rule that apply to higher differentials [37, Theorem 8.2] [46, Theorem 4.3], but these results have strong vanishing conditions that often do not hold in practice.

Example **2.31.** — Consider the element $\tau\Delta_1 h_1^2$, which was called G in [54]. Table 4 shows that there is an Adams differential $d_2(\tau\Delta_1 h_1^2) = \mathrm{M}h_1h_3$, which follows by comparison to $\mathrm{C}\tau$. To illustrate Moss's higher Leibniz rule 2.27, we shall give an independent derivation of this differential.

Table 3 shows that $\tau\Delta_1 h_1^2$ equals the Massey product $\langle h_1, h_0, \mathrm{D}_1\rangle$, with no indeterminacy. By Moss's higher Leibiz rule 2.27, the element $d_2(\tau\Delta_1 h_1^2)$ is contained in

$$\langle 0, h_0, \mathrm{D}_1\rangle + \langle h_1, 0, \mathrm{D}_1\rangle + \langle h_1, h_0, d_2(\mathrm{D}_1)\rangle.$$

By inspection, the first two terms vanish. Also, Table 4 shows that $d_2(\mathrm{D}_1)$ equals $h_0^2h_3g_2$.

Therefore, $d_2(\tau\Delta_1 h_1^2)$ is contained in the bracket $\langle h_1, h_0, h_0^2h_3g_2\rangle$, which equals $\langle h_1, h_0, h_0^2g_2\rangle h_3$. Finally, Table 3 shows that $\langle h_1, h_0, h_0^2g_2\rangle$ equals $\mathrm{M}h_1$. This shows that $d_2(\tau\Delta_1 h_1^2)$ equals $\mathrm{M}h_1h_3$.

Example **2.32.** — Consider the element $\tau e_0 g$ in the Adams E_3-page. Because of the Adams differential $d_2(e_0) = h_1^2 d_0$, we have that $\tau e_0 g$ equals $\langle d_0, h_1^2, \tau g\rangle$ in the Adams E_3-page. The higher Leibniz rule 2.27 implies that $d_3(\tau e_0 g)$ is contained in

$$\langle 0, h_1^2, \tau g\rangle + \langle d_0, 0, \tau g\rangle + \langle d_0, h_2^2, 0\rangle,$$

which equals $\{0, c_0 d_0^2\}$. In this case, the higher Leibniz rule 2.27 does not help to determine the value of $d_3(\tau e_0 g)$ because the indeterminacy is too large. (In fact, $d_3(\tau e_0 g)$ does equal $c_0 d_0^2$, but we need a different argument.)

Example **2.33.** — Lemma 5.32 shows that $d_3(\Delta h_2^2 h_6)$ equals $h_1 h_6 d_0^2$. This argument uses that $\Delta h_2^2 h_6$ equals $\langle \Delta h_2^2, h_5^2, h_0\rangle$ in the E_3-page, because of the Adams differential $d_2(h_6) = h_0 h_5^2$.

2.4.3. *Shuffling formulas for Toda brackets.* — Toda brackets satisfy various types of formal relations that we will use extensively. The most important example of such a relation is the shuffle formula

$$\alpha\langle\beta, \gamma, \delta\rangle = \langle\alpha, \beta, \gamma\rangle\delta,$$

which holds whenever both Toda brackets are defined. Note the equality of sets here; the indeterminacies of both expressions are the same.

The following theorem states some formal properties of threefold Toda brackets that we will use later. We apply these results so frequently that we typically use them without further mention.

Theorem **2.34** *([55, p. 33]). — Let α, α', β, γ, and δ be homotopy classes in $\pi_{*,*}$. Each of the following relations involving threefold Toda brackets holds up to signs, whenever the Toda brackets are defined:*

(1) $\langle \alpha + \alpha', \beta, \gamma \rangle \subseteq \langle \alpha, \beta, \gamma \rangle + \langle \alpha', \beta, \gamma \rangle$.
(2) $\langle \alpha, \beta, \gamma \rangle = \langle \gamma, \beta, \alpha \rangle$.
(3) $\alpha \langle \beta, \gamma, \delta \rangle \subseteq \langle \alpha\beta, \gamma, \delta \rangle$.
(4) $\langle \alpha\beta, \gamma, \delta \rangle \subseteq \langle \alpha, \beta\gamma, \delta \rangle$.
(5) $\alpha \langle \beta, \gamma, \delta \rangle = \langle \alpha, \beta, \gamma \rangle \delta$.
(6) $0 \in \langle \alpha, \beta, \gamma \rangle + \langle \beta, \gamma, \alpha \rangle + \langle \gamma, \alpha, \beta \rangle$.
(7) $0 \in \langle \langle \alpha, \beta, \gamma \rangle, \delta, \epsilon \rangle + \langle \alpha, \langle \beta, \gamma, \delta \rangle, \epsilon \rangle + \langle \alpha, \beta, \langle \gamma, \delta, \epsilon \rangle \rangle$.

Part (7) of Theorem 2.34 requires some further explanation. In the expression $\langle \langle \alpha, \beta, \gamma \rangle, \delta, \epsilon \rangle$, we have a set $\langle \alpha, \beta, \gamma \rangle$ as the first input to a threefold Toda bracket. The expression $\langle \langle \alpha, \beta, \gamma \rangle, \delta, \epsilon \rangle$ is defined to be the union of all sets of the form $\langle \zeta, \delta, \epsilon \rangle$, where ζ ranges over all elements of $\langle \alpha, \beta, \gamma \rangle$ such that $\zeta\delta = 0$. Similar remarks apply to the other terms in Part (7).

We next turn our attention to fourfold Toda brackets. New complications arise in this context. If $\alpha\beta = 0$, $\beta\gamma = 0$, $\gamma\delta = 0$, $\langle \alpha, \beta, \gamma \rangle$ contains zero, and $\langle \beta, \gamma, \delta \rangle$ contains zero, then the fourfold bracket $\langle \alpha, \beta, \gamma, \delta \rangle$ is not necessarily defined. Problems can arise when both threefold subbrackets have indeterminacy. See [29] for a careful analysis of this problem in the analogous context of Massey products.

However, when at least one of the threefold subbrackets is strictly zero, then these difficulties vanish. Every fourfold bracket that we use has at least one threefold subbracket that is strictly zero.

Another complication with fourfold Toda brackets lies in the description of the indeterminacy. If at least one threefold subbracket is strictly zero, then the indeterminacy of $\langle \alpha, \beta, \gamma, \delta \rangle$ is the linear span of the sets $\langle \alpha, \beta, \epsilon \rangle$, $\langle \alpha, \epsilon, \delta \rangle$, and $\langle \epsilon, \gamma, \delta \rangle$, where ϵ ranges over all possible values in the appropriate degree for which the Toda bracket is defined.

The following theorem states some formal properties of fourfold Toda brackets that we will use later. We apply these results so frequently that we typically use them without further mention.

Theorem **2.35** *([38, Chapter 2]). — Let α, α', β, γ, δ, and ϵ be homotopy classes in $\pi_{*,*}$. Each of the following relations involving fourfold Toda brackets holds up to sign, whenever the Toda brackets are defined:*

(1) $\langle \alpha + \alpha', \beta, \gamma, \delta \rangle \subseteq \langle \alpha, \beta, \gamma, \delta \rangle + \langle \alpha', \beta, \gamma, \delta \rangle$.
(2) $\langle \alpha, \beta, \gamma, \delta \rangle = \langle \delta, \gamma, \beta, \alpha \rangle$.
(3) $\alpha \langle \beta, \gamma, \delta, \epsilon \rangle \subseteq \langle \alpha\beta, \gamma, \delta, \epsilon \rangle$.
(4) $\langle \alpha\beta, \gamma, \delta, \epsilon \rangle \subseteq \langle \alpha, \beta\gamma, \delta, \epsilon \rangle$.
(5) $\alpha \langle \beta, \gamma, \delta, \epsilon \rangle = \langle \alpha, \beta, \gamma, \delta \rangle \epsilon$.
(6) $\alpha \langle \beta, \gamma, \delta, \epsilon \rangle \subseteq \langle \langle \alpha, \beta, \gamma \rangle, \delta, \epsilon \rangle$.

As in Part (7) of Theorem 2.34, Part (6) of Theorem 2.35 requires some further explanation. The expression $\langle \langle \alpha, \beta, \gamma \rangle, \delta, \epsilon \rangle$ is defined to be the union of all sets of the form $\langle \zeta, \delta, \epsilon \rangle$, where ζ ranges over all elements of $\langle \alpha, \beta, \gamma \rangle$ such that $\zeta\delta = 0$.

We will make occasional use of matric Toda brackets. We will not describe their shuffling properties in detail, except to observe that they obey analogous matric versions of the properties in Theorems 2.34 and 2.35. These properties can be proved with the same techniques that apply to matric Massey product [46]; see [37, 39] for examples of this style of argument.

3. The algebraic Novikov spectral sequence

Consider the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1},$$

where $C\tau$ is the cofiber of τ. The inclusion i of the bottom cell and projection p to the top cell are tools for comparing the **C**-motivic Adams spectral sequence for $S^{0,0}$ to the **C**-motivic Adams spectral sequence for $C\tau$. In [30], the first author analyzed both spectral sequences simultaneously, playing the structure of each against the other in order to obtain more detailed information about both. Then the structure of the homotopy of $C\tau$ was used to reverse-engineer the structure of the classical Adams-Novikov spectral sequence.

In this manuscript, we use $C\tau$ in a different, much more powerful way, because we have a deeper understanding of the connection between the homotopy of $C\tau$ and the structure of the classical Adams-Novikov spectral sequence. Namely, the **C**-motivic spectrum $C\tau$ is an E_∞-ring spectrum [19]. Here we are referring to the classical E_∞-operad that parametrizes homotopy-coherent commutative multiplications.

Moreover, the homotopy category of $C\tau$-modules is equivalent to Hovey's stable derived category of BP_*BP-comodules [21, 41]. By considering endomorphisms of unit objects, this comparison of homotopy categories gives a structured explanation for the identification of the homotopy of $C\tau$ and the classical Adams-Novikov E_2-page.

From a computational perspective, there is an even better connection. Namely, the algebraic Novikov spectral sequence for computing the Adams-Novikov E_2-page [48, 51] is identical to the **C**-motivic Adams spectral sequence for computing the homotopy of $C\tau$ [21]. This rather shocking, and incredibly powerful, identification of spectral sequences allows us to transform purely algebraic computations directly into information about Adams differentials for $C\tau$. Finally, naturality along the inclusion i of the bottom cell and along the projection p to the top cell allows us to deduce information about Adams differentials for $S^{0,0}$.

Due to the large quantity of data, we do not explicitly describe the structure of the Adams spectral sequence for $C\tau$ in this manuscript. We refer the interested reader to the charts in [34], which provide details in a graphical form.

3.1. *Naming conventions*

Our naming convention for elements of the algebraic Novikov spectral sequence (and for elements of the Adams-Novikov spectral sequence) differs from previous approaches. Our names are chosen to respect the inclusion i of the bottom cell and the projection p to the top cell. Specifically, if x is an element of the $\mathbf{C}$-motivic Adams E_2-page for $S^{0,0}$, then we use the same letter x to indicate its image $i_*(x)$ in the Adams E_2-page for $C\tau$. It is certainly possible that $i_*(x)$ is zero, but we will only use this convention in cases where $i_*(x)$ is non-zero, i.e., when x is not a multiple of τ.

On the other hand, if x is an element of the $\mathbf{C}$-motivic Adams E_2-page for $S^{0,0}$ such that τx is zero, then we use the symbol $\overline{x}$ to indicate an element of $p_*^{-1}(x)$ in the Adams E_2-page for $C\tau$. There is often more than one possible choice for $\overline{x}$, and the indeterminacy in this choice equals the image of i_* in the appropriate degree. We will not usually be explicit about these choices. However, potential confusion can arise in this context. For example, it may be the case that one choice of $\overline{x}$ supports an h_1 extension, while another choice of $\overline{x}$ supports an h_2 extension, but there is no possible choice of $\overline{x}$ that simultaneously supports both extensions. (The authors dwell on this point because this precise issue has generated confusion about specific computations.)

3.2. *Machine computations*

We have analyzed the algebraic Novikov spectral sequence by computer in a large range. Roughly speaking, our algorithm computes a Curtis table for a minimal resolution. Significant effort went into optimizing the linear algebra algorithms to complete the computation in a reasonable amount of time. The data is available at [34]. See [59] for a discussion of the implementation.

Our machine computations give us a full description of the additive structure of the algebraic Novikov E_2-page, together with all d_r differentials for $r \geq 2$. It thus yields a full description of the additive structure of the algebraic Novikov E_∞-page.

Moreover, the data also gives full information about multiplication by 2, h_1, and h_2 in the Adams-Novikov E_2-page for the classical sphere spectrum, which we denote by $H^*(S; BP)$.

We have also conducted machine computations of the Adams-Novikov E_2-page for the classical cofiber of 2, which we denote by $H^*(S/2; BP)$. Note that $H^*(S; BP)$ is the homology of a differential graded algebra (i.e., the cobar complex) that is free as a $\mathbf{Z}_2$-module. Therefore, $H^*(S/2; BP)$ is the homology of this differential graded algebra modulo 2. We have computed this homology by machine, including full information about multiplication by h_1, h_2, and h_3. These computations are related by a long exact sequence

$$\cdots \longrightarrow H^*(S; BP) \xrightarrow{j} H^*(S/2; BP) \xrightarrow{q} H^*(S; BP) \longrightarrow \cdots$$

Because h_2^2, h_3^2, h_4^2, and h_5^2 are annihilated by 2 in $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$, there are classes $\widetilde{h_2^2}$, $\widetilde{h_3^2}$, $\widetilde{h_4^2}$, and $\widetilde{h_5^2}$ in $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$ such that $q(\widetilde{h_i^2})$ equals h_i^2 for $2 \leq i \leq 5$. We also have full information about multiplication by $\widetilde{h_2^2}$, $\widetilde{h_3^2}$, $\widetilde{h_4^2}$, and $\widetilde{h_5^2}$ in $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$.

This multiplicative information allows use to determine some of the Massey product structure in the Adams-Novikov E_2-page for the sphere spectrum. There are several cases to consider.

First, let x and y be elements of $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$. If the product $j(x)j(y)$ is non-zero in $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$, then xy must also be non-zero in $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$.

In the second case, let x be an element of $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$, and let $\widetilde{y}$ be an element of $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$ such that $q(\widetilde{y}) = y$. If the product $x \cdot \widetilde{y}$ is non-zero in $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$ and equals $j(z)$ for some z in $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$, then z belongs to the Massey product $\langle 2, y, x\rangle$. This follows immediately from the relationship between Massey products and the multiplicative structure of a cofiber, as discussed in [30, Section 3.1.1].

Third, let $\widetilde{x}$ and $\widetilde{y}$ be elements of $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$ such that $q(\widetilde{x}) = x$, $q(\widetilde{y}) = y$, and $q(\widetilde{x} \cdot \widetilde{y}) = z$. Then z belongs to the Massey product $\langle x, 2, y\rangle$ in $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$. This follows immediately from the multiplicative snake lemma 3.3.

Example **3.1.** — Computer data shows that the product $\widetilde{h_4^2} \cdot \widetilde{h_5^2}$ does not equal zero in $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$. This implies that the Massey product $\langle h_4^2, 2, h_5^2\rangle$ does not contain zero in $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$, which in turn implies that the Toda bracket $\langle \theta_4, 2, \theta_5\rangle$ does not contain zero in $\pi_{93,48}$.

Remark **3.2.** — let $\widetilde{x}$ and $\widetilde{y}$ be elements of $\mathrm{H}^*(\mathrm{S}/2;\mathrm{BP})$ such that $q(\widetilde{x})$ and $q(\widetilde{y})$ equal x and y, and such that $\widetilde{x} \cdot \widetilde{y}$ equals $j(z)$ for some z in $\mathrm{H}^*(\mathrm{S};\mathrm{BP})$. It appears that z has some relationship to the fourfold Massey product $\langle 2, x, 2, y\rangle$, but we have not made this precise.

Lemma **3.3** *(Multiplicative snake lemma).* — *Let* A *be a differential graded algebra that has no* 2*-torsion, and let* $\mathrm{H}(\mathrm{A})$ *be its homology. Also let* $\mathrm{H}(\mathrm{A}/2)$ *be the homology of* $\mathrm{A}/2$*, and let* $\delta : \mathrm{H}(\mathrm{A}/2) \to \mathrm{H}(\mathrm{A})$ *be the boundary map associated to the short exact sequence*

$$0 \longrightarrow \mathrm{A} \xrightarrow{2} \mathrm{A} \longrightarrow \mathrm{A}/2 \longrightarrow 0.$$

Suppose that a and b are elements of $\mathrm{H}(\mathrm{A}/2)$ *such that* $2\delta(a) = 0$ *and* $2\delta(b) = 0$ *in* $\mathrm{H}(\mathrm{A})$. *Then the Massey product* $\langle \delta(a), 2, \delta(b)\rangle$ *in* $\mathrm{H}(\mathrm{A})$ *contains* $\delta(ab)$.

Proof. — We carry out a diagram chase in the spirit of the snake lemma. Write ∂ for the boundary operators in A and A/2.

Let x and y be cycles in A/2 that represent a and b respectively. Let x' and y' be elements in A that reduce to x and y. Then $\partial x'$ and $\partial y'$ reduce to zero in A/2 because x and y are cycles. Therefore, $\partial x' = 2\widetilde{x}$ and $\partial y' = 2\widetilde{y}$ for some $\widetilde{x}$ and $\widetilde{y}$ in A.

By definition of the boundary map, $\delta(a)$ and $\delta(b)$ are represented by $\widetilde{x}$ and $\widetilde{y}$. By the definition of Massey products, the cycle $\widetilde{x}y' + x'\widetilde{y}$ is contained in $\langle \delta(a), 2, \delta(b) \rangle$.

Now we compute $\delta(ab)$. Note that $x'y'$ is an element of A that reduces to ab. Then

$$\partial(x'y') = \partial(x')y' + x'\partial(y') = 2(\widetilde{x}y' + x'\widetilde{y}).$$

This shows that $\delta(ab)$ is represented by $\widetilde{x}y' + x'\widetilde{y}$. □

3.3. h_1-*Bockstein spectral sequence*

The charts in [34] show graphically the algebraic Novikov spectral sequence, i.e., the Adams spectral sequence for $C\tau$. Essentially all of the information in the charts can be read off from machine-generated data. This includes hidden extensions in the E_∞-page.

One aspect of these charts requires further explanation. The **C**-motivic Adams E_2-page for $C\tau$ contains a large number of h_1-periodic elements, i.e., elements that support infinitely many h_1 multiplications. The behavior of these elements is entirely understood [22], at least up to many multiplications by h_1, i.e., in an h_1-periodic sense.

On the other hand, it takes some work to "delocalize" this information. For example, we can immediately deduce from [22] that $d_2(h_1^k e_0) = h_1^{k+2} d_0$ for large values of k, but that does not necessarily determine the behavior of Adams differentials for small values of k.

The behavior of these elements is a bit subtle in another sense, as illustrated by Example 3.4.

Example **3.4.** — Consider the h_1-periodic element $\overline{c_0 e_0}$ in the algebraic Novikov spectral sequence. Machine computations tell us that this element supports a d_2 differential, but there is more than one possible value for $d_2(\overline{c_0 e_0})$ because of the presence of both $h_1^2 \overline{c_0 d_0}$ and Pe_0.

In fact, $d_2(\overline{c_0 d_0})$ equals Pd_0, and $d_2(Pe_0)$ equals $Ph_1^2 d_0$. Therefore, $Pe_0 + h_1^2 \overline{c_0 d_0}$ is the only non-zero d_2 cycle, and it follows that $d_2(\overline{c_0 e_0})$ must equal $Pe_0 + h_1^2 \overline{c_0 d_0}$.

The careful reader will note that $d_2(\overline{c_0 e_0})$ is not shown on the algebraic Novikov chart in [34]. As discussed in [34, Section 4], the h_1-periodic differentials are not shown for legibility. Instead, the differential is shown in the h_1-Bockstein spectral sequence chart of [34], up to higher powers of h_1.

In higher stems, it becomes more and more difficult to determine the exact values of the Adams d_2 differentials on h_1-periodic classes. Eventually, these complications become unmanageable because they involve sums of many monomials.

Fortunately, we only need concern ourselves with the Adams d_2 differential in this context. The h_1-periodic E_3-page equals the h_1-periodic E_∞-page, and the only non-zero classes are well-understood v_1-periodic families running along the top of the Adams chart.

Our solution to this problem, as usual, is to introduce a filtration that hides the higher order terms. In this case, we filter by powers of h_1. The effect is that terms involving higher powers of h_1 are ignored, and the formulas become much more manageable.

This h_1-Bockstein spectral sequence starts with an E_0-page, because there are some differentials that do not increase h_1 divisibility. For example, we have Bockstein differentials $d_0(\overline{h_1^2 e_0}) = \overline{h_1^4 d_0}$ and $d_0(\overline{c_0 d_0}) = Pd_0$, reflecting the Adams differentials $d_2(\overline{h_1^2 e_0}) = \overline{h_1^4 d_0}$ and $d_2(\overline{c_0 d_0}) = Pd_0$.

There are also plenty of higher h_1-Bockstein differentials, such as $d_2(e_0) = h_1^2 d_0$, and $d_7(e_0^2 g) = Mh_1^8$.

Remark **3.5.** — Beware that filtering by powers of h_1 changes the multiplicative structure in perhaps unexpected ways. For example, Ph_1 and d_0 are not h_1-multiples, so their h_1-Bockstein filtration is zero. One might expect their product to be $Ph_1 d_0$, but the h_1-Bockstein filtration of this element is 1. Therefore, $Ph_1 \cdot d_0$ equals 0 in the h_1-Bockstein spectral sequence.

But not all Ph_1 multiplications are trivial in the h_1-Bockstein spectral sequence. For example, we have $Ph_1 \cdot \overline{c_0 d_0} = \overline{Ph_1 c_0 d_0}$ because the h_1-Bockstein filtrations of all three elements are zero.

In Example 3.4, we explained that there is an Adams differential $d_2(\overline{c_0 e_0}) = Pe_0 + h_1^2 \overline{c_0 d_0}$. When we throw out higher powers of h_1, we obtain the h_1-Bockstein differential $d_0(\overline{c_0 e_0}) = Pe_0$. We also have an h_1-Bockstein differential $d_0(\overline{c_0 d_0}) = Pd_0$.

The first four charts in [34] show graphically how this h_1-Bockstein spectral sequence plays out in practice. The main point is that the h_1-Bockstein E_∞-page reveals which (formerly) h_1-periodic classes contribute to the Adams E_3-page for $C\tau$.

4. Massey products

The purpose of this section is to provide some general tools, and to give some specific computations, of Massey products in Ext. This material contributes to Table 3, which lists a number of Massey products in Ext that we need for various specific purposes. Most commonly, these Massey products yield information about Toda brackets via the Moss Convergence Theorem 2.16.

We begin with a $\mathbb{C}$-motivic version of a classical theorem of Adams about symmetric Massey products.

Theorem **4.1.**

(1) *If $h_0 x$ is zero, then $\langle h_0, x, h_0 \rangle$ contains $\tau h_1 x$.*
(2) *If $n \geq 1$ and $h_n x$ is zero, then $\langle h_n, x, h_n \rangle$ contains $h_{n+1} x$.*

Proof. — The element $\mathrm{Sq}^0(h_n)x$ is contained in $\langle h_n, x, h_n\rangle$ [25], where Sq^0 is an algebraic Steenrod operation [47]. We compute that $\mathrm{Sq}^0(h_0)$ equals τh_1 and $\mathrm{Sq}^0(h_n)$ equals h_{n+1} for $n \geq 1$. These motivic computations follow from the analogous classical computations [1, Lemma 2.5.4]. □

Remark **4.2.** — The two parts of Theorem 4.1 are less different than they appear. Because of the specific values of the motivic weights, we have a factor of τ in $\mathrm{Sq}^0(h_0)$, while no τ appears in $\mathrm{Sq}^0(h_n)$ for $n \geq 1$.

4.1. *The operator g*

The projection map $p : \mathrm{A}_* \to \mathrm{A}(2)_*$ induces a map $p_* : \mathrm{Ext}_{\mathbf{C}} \to \mathrm{Ext}_{\mathrm{A}(2)}$. Because $\mathrm{Ext}_{\mathrm{A}(2)}$ is completely known [28], this map is useful for detecting structure in $\mathrm{Ext}_{\mathbf{C}}$. Proposition 4.3 provides a tool for using p_* to compute certain types of Massey products.

Proposition **4.3.** — *Let x be an element of $\mathrm{Ext}_{\mathbf{C}}$ such that $h_1^4 x = 0$. Then $p_*\left(\langle h_4, h_1^4, x\rangle\right)$ equals the element $gp_*(x)$ in $\mathrm{Ext}_{\mathrm{A}(2)}$.*

Proof. — The idea of the proof is essentially the same as in [32, Proposition 3.1]. The $\mathrm{Ext}_{\mathbf{C}}$-module $\mathrm{Ext}_{\mathrm{A}(2)}$ is a "Toda module", in the sense that Massey products $\langle x, a, b\rangle$ are defined for all x in $\mathrm{Ext}_{\mathrm{A}(2)}$ and all a and b in $\mathrm{Ext}_{\mathbf{C}}$ such that $x \cdot a = 0$ and $ab = 0$. In particular, the bracket $\langle 1, h_4, h_1^4\rangle$ is defined in $\mathrm{Ext}_{\mathrm{A}(2)}$. We wish to compute this bracket.

We use the May Convergence Theorem in order to compute the bracket. The crossing differentials condition on the theorem is satisfied because there are no possible differentials that could interfere.

The key point is the May differential $d_4(b_{21}^2) = h_1^4 h_4$. This shows that g is contained in $\langle 1, h_4, h_1^4\rangle$. Also, the bracket has no indeterminacy by inspection.

Now suppose that x is an element of $\mathrm{Ext}_{\mathbf{C}}$ such that $h_1^4 x = 0$. Then

$$p_*\left(\langle h_4, h_1^4, x\rangle\right) = 1 \cdot \langle h_4, h_1^4, x\rangle = \langle 1, h_4, h_1^4\rangle \cdot x = gp_*(x).$$ □

Example **4.4.** — We illustrate the practical usefulness of Proposition 4.3 with a specific example. Consider the Massey product $\langle h_1^3 h_4, h_1, h_2\rangle$. The proposition says that

$$p_*\left(\langle h_1^3 h_4, h_1, h_2\rangle\right) = h_2 g$$

in $\mathrm{Ext}_{\mathrm{A}(2)}$. This implies that $\langle h_1^3 h_4, h_1, h_2\rangle$ equals $h_2 g$ in $\mathrm{Ext}_{\mathbf{C}}$.

Remark **4.5.** — The Massey product computation in Example 4.4 is in relatively low dimension, and it can be computed using other more direct methods. Table 3 lists additional examples, including some that cannot be determined by more elementary methods.

4.2. *The Mahowald operator*

We recall some results from [32] about the Mahowald operator. The Mahowald operator is defined to be $\mathrm{M}x = \langle x, h_0^3, g_2 \rangle$ for all x such that $h_0^3 x$ equals zero. As always, one must be cautious about indeterminacy in $\mathrm{M}x$.

There exists a subalgebra B of the **C**-motivic Steenrod algebra whose cohomology $\mathrm{Ext}_{\mathrm{B}}(\mathbf{M}_2, \mathbf{M}_2)$ equals $\mathbf{M}_2[v_3] \otimes_{\mathbf{M}_2} \mathrm{Ext}_{\mathrm{A}(2)}$. The inclusion of B into the **C**-motivic Steenrod algebra induces a map $p_* : \mathrm{Ext}_{\mathbf{C}} \to \mathrm{Ext}_{\mathrm{B}}$.

Proposition **4.6** *([32, Theorem 1.1]). — The map* $p_* : \mathrm{Ext}_{\mathbf{C}} \to \mathrm{Ext}_{\mathrm{B}}$ *takes* $\mathrm{M}x$ *to the product* $(e_0 v_3^2 + h_1^3 v_3^3) p_*(x)$*, whenever* $\mathrm{M}x$ *is defined.*

Proposition 4.6 is useful in practice for detecting certain Massey products of the form $\langle x, h_0^3, g_2 \rangle$. For example, if x is an element of $\mathrm{Ext}_{\mathbf{C}}$ such that $h_0^3 x$ equals zero and $e_0 p_*(x)$ is non-zero in $\mathrm{Ext}_{\mathrm{A}(2)}$, then $\langle x, h_0^3, g_2 \rangle$ is non-zero.

Example **4.7.** — Proposition 4.6 shows that $\langle h_1, h_0, h_0^2 g_2 \rangle$ is non-zero. There is only one non-zero element in the appropriate degree, so we have identified the Massey product. We give this element the name $\mathrm{M}h_1$.

Example **4.8.** — Expanding on Example 4.7, Proposition 4.6 also shows that $\langle \mathrm{M}h_1, h_0, h_0^2 g_2 \rangle$ is non-zero. Again, there is only one non-zero element in the appropriate degree, so we have identified the Massey product. We give this element the name $\mathrm{M}^2 h_1$.

4.3. *Additional computations*

Lemma **4.9.** — **(66, 6, 36)** *The Massey product* $\langle h_1^2, h_4^2, h_1^2, h_4^2 \rangle$ *equals* $\Delta_1 h_3^2$.

Proof. — Table 3 shows that Δh_2^2 equals the Massey product $\langle h_0^2, h_3^2, h_0^2, h_3^2 \rangle$. Recall the isomorphism between classical Ext groups and **C**-motivic Ext groups in degrees satisfying $s + f - 2w = 0$, as described in Theorem 2.13. This shows that $\Delta_1 h_3^2$ equals $\langle h_1^2, h_4^2, h_1^2, h_4^2 \rangle$. □

Lemma **4.10.** — **(66, 7, 35)** *The Massey product* $\langle \mathrm{A}', h_1, h_2 \rangle$ *equals* $\tau \mathrm{G}_0$.

Proof. — Consider the shuffle

$$\mathrm{A}' \langle h_1, h_2, h_1 \rangle = \langle \mathrm{A}', h_1, h_2 \rangle h_1.$$

Table 3 shows that the left side equals $h_2^2 \mathrm{A}'$, which equals $h_1 \cdot \tau \mathrm{G}_0$. This implies that $\langle \mathrm{A}', h_1, h_2 \rangle$ contains $\tau \mathrm{G}_0$.

The indeterminacy is zero by inspection. □

Lemma **4.11.** — **(71, 13, 40)** *The Massey product* $\langle h_1^3 h_4, h_1, \tau gn\rangle$ *equals* $\tau g^2 n$*, with indeterminacy generated by* $\mathrm{M} h_0 h_2^2 g$.

Proof. — We start by analyzing the indeterminacy. The product $\mathrm{M}c_0 \cdot h_1^3 h_4$ equals

$$\langle g_2, h_0^3, c_0\rangle h_1^3 h_4 = \langle g_2, h_0^3, h_1^3 h_4 c_0\rangle = \langle g_2, h_0^3, h_0 h_2 \cdot h_2 g\rangle = \langle g_2, h_0^3, h_2 g\rangle h_0 h_2,$$

which equals $\mathrm{M} h_0 h_2^2 g$. The equalities hold because the indeterminacies are zero, and the first and last brackets in this computation are given by Table 3. This shows that $\mathrm{M} h_0 h_2^2 g$ belongs to the indeterminacy.

Table 3 shows that

$$\langle h_2, h_1^3 h_4, h_1\rangle = \langle h_2, h_1, h_1^3 h_4\rangle$$

equals $h_2 g$. Then

$$h_2\langle h_1^3 h_4, h_1, \tau gn\rangle = \langle h_2, h_1^3 h_4, h_1\rangle \tau gn = \tau h_2 g^2 n.$$

This implies that $\langle h_1^3 h_4, h_1, \tau gn\rangle$ contains either $\tau g^2 n$ or $\Delta h_3 g^2$. However, the shuffle

$$h_1\langle h_1^3 h_4, h_1, \tau gn\rangle = \langle h_1, h_1^3 h_4, h_1\rangle \tau gn = 0$$

eliminates $\Delta h_3 g^2$. □

Lemma **4.12.** — **(80, 5, 42)** *The Massey product* $\langle h_3, p', h_2\rangle$ *equals* $h_0 e_2$*, with no indeterminacy.*

Proof. — We have

$$\langle h_3, p', h_2\rangle h_4^2 = h_3\langle p', h_2, h_4^2\rangle = p'\langle h_2, h_4^2, h_3\rangle.$$

Table 3 shows that the last Massey product equals c_2. Observe that $p' c_2$ equals $h_0 h_4^2 e_2$.

Since $h_4^2 \cdot h_6 e_0$ is zero (as usual, we rely on complete information about classical products in a large range [11, 13]), this shows that $\langle h_3, p', h_2\rangle$ equals either $h_0 e_2$ or $h_0 e_2 + h_6 e_0$. However, shuffle to obtain

$$\langle h_3, p', h_2\rangle h_1 = h_3\langle p', h_2, h_1\rangle,$$

which must equal zero because multiplication by h_3 is zero in the appropriate degree. Since $h_1(h_0 e_2 + h_6 e_0)$ is non-zero, it cannot equal $\langle h_3, p', h_2\rangle$.

The indeterminacy is zero by inspection. □

Remark **4.13.** — The Massey product of Lemma 4.11 cannot be established with Proposition 4.3 because $p_*(\tau gn) = 0$ in $\mathrm{Ext}_{\mathrm{A}(2)}$.

Lemma **4.14.** — **(82, 12, 45)** *The Massey product* $\langle \Delta e_1 + C_0, h_1^3, h_1 h_4 \rangle$ *equals* $(\Delta e_1 + C_0)g$*, with no indeterminacy.*

Proof. — Consider the Massey product $\langle \tau(\Delta e_1 + C_0), h_1^4, h_4 \rangle$. By inspection, this Massey product has no indeterminacy. Therefore,

$$\langle \tau(\Delta e_1 + C_0), h_1^4, h_4 \rangle = (\Delta e_1 + C_0)\langle \tau, h_1^4, h_4 \rangle.$$

Table 3 shows that the latter bracket equals τg, so the expression equals $\tau(\Delta e_1 + C_0)g$.

On the other hand, it also equals $\tau\langle \Delta e_1 + C_0, h_1^3, h_1 h_4 \rangle$. Therefore, the bracket $\langle \Delta e_1 + C_0, h_1^3, h_1 h_4 \rangle$ must contain $(\Delta e_1 + C_0)g$. Finally, the indeterminacy can be computed by inspection. □

Lemma **4.15.** — **(93, 13, 49)** *The Massey product* $\langle \tau^3 g G_0, h_0 h_2, h_2 \rangle$ *has indeterminacy generated by* $\tau M^2 h_2$*, and it either contains zero or* $\tau e_0 x_{76,9}$*. In particular, it does not contain any linear combination of* $\Delta^2 h_1 g_2$ *with other elements.*

Proof. — The indeterminacy can be computed by inspection.

The only possible elements in the Massey product $\langle \tau^2 g G_0, h_0 h_2, h_2 \rangle$ are linear combinations of $e_0 x_{76,9}$ and $M^2 h_2$. The inclusion

$$\tau\langle \tau^2 g G_0, h_0 h_2, h_2 \rangle \subseteq \langle \tau^3 g G_0, h_0 h_2, h_2 \rangle$$

gives the desired result. □

5. Adams differentials

The goal of this section is to describe the values of the Adams differentials in the motivic Adams spectral sequence. These values are given in Tables 4, 6, 7, 8, and 9. See also the Adams charts in [33] for a graphical representation of the computations. For easy reference, the many lemmas in this section are labelled with degrees that match the degrees given in the tables.

5.1. *The Adams d_2 differential*

Table 4 lists all of the multiplicative generators of the Adams E_2-page through the 95-stem. The third column indicates the value of the d_2 differential, if it is non-zero. A blank entry in the third column indicates that the d_2 differential is zero. The fourth column indicates the proof. A blank entry in the fourth column indicates that there are no possible values for the differential. The fifth column gives alternative names for the element, as used in [11, 54], and [30].

Theorem **5.1.** — *Table 4 lists the values of the Adams d_2 differential on all multiplicative generators through the 95-stem.*

Remark **5.2.** — A previous version of this manuscript left uncertain the value of d_2 on three multiplicative generators. These three values have since been determined by Dexter Chua [17]. We have included those values here, but we defer to [17] and [4] for their proofs. Also, we are grateful to Joey Beauvais-Feisthauer [4] for discovering an error in our previous calculation of $d_2(x_{85,6})$.

Proof. — The fourth column of Table 4 gives information on the proof of each differential. Most follow immediately by comparison to the Adams spectral sequence for $C\tau$ [34]. A few additional differentials follow by comparison to the classical Adams spectral sequence for *tmf* [14].

If an element is listed in the fourth column of Table 4, then the corresponding differential can be deduced from a straightforward argument using a multiplicative relation. For example, it is possible that $d_2(\Delta h_1 h_3)$ equals $\tau d_0 e_0$. However, $h_0 \cdot \Delta h_1 h_3$ is zero, while $h_0 \cdot \tau d_0 e_0$ is non-zero. Therefore, $d_2(\Delta h_1 h_3)$ must equal zero.

In some cases, it is necessary to combine these different techniques to establish the differential.

The remaining more difficult computations are carried out in the following lemmas. We refer to [17] and [4] in a few cases. □

Lemma **5.3.** — **(61, 9, 32)** $d_2(\Delta x) = h_0^2 B_4 + \tau M h_1 d_0$.

Proof. — We have a differential $d_2(\Delta x) = h_0^2 B_4$ in the Adams spectral sequence for $C\tau$. Therefore, $d_2(\Delta x)$ equals either $h_0^2 B_4$ or $h_0^2 B_4 + \tau M h_1 d_0$.

We have the relation $h_1^2 \cdot \Delta x = P h_1 \cdot \tau \Delta_1 h_1^2$ (as usual, we rely on complete information about classical products in a large range [11, 13]), so $h_1^2 d_2(\Delta x) = P h_1 d_2(\tau \Delta_1 h_1^2) = P h_1 h_3 \cdot M h_1 = \tau M h_1^3 d_0$. Therefore, $d_2(\Delta x)$ must equal $h_0^2 B_4 + \tau M h_1 d_0$. □

Remark **5.4.** — The proof of [30, Lemma 3.50] is incorrect. We claimed that $h_1^2 \cdot \Delta x$ equals $h_3 \cdot \Delta^2 h_1 h_3$, when in fact $h_1^2 \cdot \Delta x$ equals $\tau h_3 \cdot \Delta^2 h_1 h_3$.

Lemma **5.5.** — **(77, 7, 40)** $d_2(x_{77,7}) = \tau M h_1 h_4^2$.

The following proof was suggested to us by Dexter Chua.

Proof. — This follows from the interaction between algebraic squaring operations and classical Adams differentials [10, Theorem 2.2], applied to the element x in the 37-stem. The theorem says that

$$d_* \, \mathrm{Sq}^2 x = \mathrm{Sq}^3 d_2 x \dotplus h_0 \mathrm{Sq}^3 x.$$

The notation means that there is an Adams differential on $\mathrm{Sq}^2 x$ hitting either $\mathrm{Sq}^3 d_2 x = 0$ or $h_0 \mathrm{Sq}^3 x$, depending on which element has lower Adams filtration. Therefore $d_2 \mathrm{Sq}^2 x = h_0 \mathrm{Sq}^3 x$.

Next, observe from [12] that $\mathrm{Sq}^3 x = h_0^2 x_{76,6} + \tau^2 d_1 g_2$, so

$$h_0 \mathrm{Sq}^3 x = h_0^3 x_{76,6} = \tau \mathrm{M} h_1 h_4^2.$$

Therefore, there is a d_2 differential whose value is $\tau \mathrm{M} h_1 h_4^2$, and the possibility is that $d_2(x_{77,7})$ equals $\tau \mathrm{M} h_1 h_4^2$. □

Lemma **5.6.** — **(86, 14, 47)** $d_2(\tau \mathrm{B}_5 g) = \tau \mathrm{M} h_0^2 g^2$.

Proof. — We use the Mahowald operator methods of Section 4.2. According to [32, Table 1], the map $p_* : \mathrm{Ext}_{\mathbf{C}} \to \mathrm{Ext}_{\mathrm{B}}$ takes $d_0 \cdot \tau \mathrm{B}_5 g$ to $\tau h_0 a g^3 v_3^2$, which is non-zero. We deduce that the product $d_0 \cdot \tau \mathrm{B}_5 g$ is non-zero in Ext. By inspection of motivic weights, the only possibility is that it equals $\tau \mathrm{M} g \cdot h_0 m$.

Now $d_2(\tau \mathrm{M} g \cdot h_0 m)$ equals $\tau \mathrm{M} g \cdot h_0^2 e_0^2$, which we also know is non-zero since it maps to the non-zero element $\tau h_0^2 d e g^2 v_3^2$ of $\mathrm{Ext}_{\mathrm{B}}$ by [32, Theorem 1]. It follows that $d_2(\tau \mathrm{B}_5 g)$ is non-zero. By inspection of motivic weights, the only possibility is $\tau \mathrm{M} h_0^2 g^2$. □

5.2. *The Adams d_3 differential*

Table 6 lists the multiplicative generators of the Adams E_3-page through the 95-stem whose d_3 differentials are non-zero, or whose d_3 differentials are zero for non-obvious reasons.

Theorem **5.7.** — *Table 6 lists some values of the Adams d_3 differential on multiplicative generators. Through the 95-stem, the Adams d_3 differential is zero on all multiplicative generators not listed in the table.*

Remark **5.8.** — A previous version of this manuscript left uncertain the value of d_3 on several multiplicative generators. These values have since been determined by Dexter Chua [17]. We have included those values here, but we defer to [17] for their proofs. We are also grateful to Dexter Chua for correcting a few mistakes in the values of the d_3 differential.

Proof. — The d_3 differential on many multiplicative generators is zero. A few of these multiplicative generators appear in Table 6 because their proofs require further explanation. For the remaining majority of such multiplicative generators, the d_3 differential is zero because there are no possible non-zero values, because of comparison to the Adams spectral sequence for $\mathrm{C}\tau$, or because the element is already known to be a permanent cycle as shown in Table 5. These cases do not appear in Table 6.

The last column of Table 6 gives information on the proof of each differential. Most follow immediately by comparison to the Adams spectral sequence for $C\tau$. A few additional differentials follow by comparison to the classical Adams spectral sequence for *tmf*, or by comparison to the **C**-motivic Adams spectral sequence for *mmf*.

If an element is listed in the last column of Table 6, then the corresponding differential can be deduced from a straightforward argument using a multiplicative relation. For example,

$$d_3(h_1 \cdot \tau \mathrm{P} d_0 e_0) = \mathrm{P}h_1 \cdot d_3(\tau d_0 e_0) = \mathrm{P}^2 h_1 c_0 d_0,$$

so $d_3(\tau \mathrm{P} d_0 e_0)$ must equal $\mathrm{P}^2 c_0 d_0$.

If a d_4 differential is listed in the last column of Table 6, then the corresponding differential is forced by consistency with that later differential. In each case, a d_3 differential on an element x is forced by the existence of a later d_4 differential on τx. For example, Table 7 shows that there is a differential $d_4(\tau^2 e_0 g) = \mathrm{P} d_0^2$. Therefore, $\tau e_0 g$ cannot survive to the E_4-page. It follows that $d_3(\tau e_0 g) = c_0 d_0^2$.

In some cases, it is necessary to combine these different techniques to establish the differential.

The remaining more difficult computations are carried out in the following lemmas. We refer to [17] in a few cases. □

Proposition **5.9.** — *Some permanent cycles in the* **C***-motivic Adams spectral sequence are shown in Table 5.*

Proof. — The third column of the table gives information on the proof for each element. If a Toda bracket is given in the third column, then the Moss Convergence Theorem 2.16 implies that the element must survive to detect that Toda bracket (see Table 10 for more information on how each Toda bracket is computed). If a product is given in the third column, then the element must survive to detect that product (see Table 24 for more information on how each product is computed). In a few cases, the third column refers to a specific lemma that gives a more detailed argument. □

Lemma **5.10.**

(1) **(34, 2, 18)** $d_3(h_2 h_5) = \tau h_1 d_1$.
(2) **(74, 6, 38)** $d_3(\mathrm{P} h_2 h_6) = \tau h_1 h_4 \mathrm{Q}_2$.

Proof. — In the Adams spectral sequence for $C\tau$, there is an η extension from $\overline{h_2 h_5}$ to $\overline{h_1^2 d_1}$. The element $\overline{h_1^2 d_1}$ maps to $h_1^2 d_1$ under projection from $C\tau$ to the top cell, so $\overline{h_2 h_5}$ must also map non-trivially under projection from $C\tau$ to the top cell. The only possibility is that $\overline{h_2 h_5}$ maps to $h_1 d_1$. Therefore, $\tau h_1 d_1$ must be hit by a differential. This establishes the first differential.

The proof for the second differential is identical, using that there is an η extension from Ph_2h_6 to $\overline{h_1^2 h_4 Q_2}$ in the Adams spectral sequence for $C\tau$. □

Lemma **5.11.** — **(54, 6, 28)** $d_3(\tau^2 \Delta_1 h_1^2) = \tau M c_0$.

Proof. — The element MP maps to zero under inclusion of the bottom cell into $C\tau$. Therefore, MP is either hit by a differential, or it is the target of a hidden τ extension. If it is the target of a hidden τ extension, then the only possibility is that $\tau M c_0$ is zero in the E_∞-page, and that there is a hidden τ extension from Mc_0 to MP.

It remains to show that MP cannot be hit by a differential. The only possibility is that $d_4(\tau^2 \Delta_1 h_1^2)$ might equal MP. Note that $Ph_1 \cdot \tau^2 \Delta_1 h_1^2$ equals $h_1(\tau h_1 \cdot \Delta x)$ in the E_4-page. This means that $d_4(Ph_1 \cdot \tau^2 \Delta_1 h_1^2)$ cannot equal $MP^2 h_1$ since $MP^2 h_1$ is not divisible by h_1. In turn, $d_4(\tau^2 \Delta_1 h_1^2)$ cannot equal MP. □

Lemma **5.12.** — **(68, 11, 35)** $d_3(\tau h_0^3 \cdot \Delta g_2) = \tau^3 \Delta h_2^2 e_0 g$.

Proof. — Table 2 shows that the element $h_0 h_5 i$ maps to $\Delta^2 h_2^2$ in the Adams spectral sequence for *tmf*.

Now $\Delta^2 h_2^2 d_0$ is not zero and not divisible by 2 in *tmf*. Therefore, $\kappa\{h_0 h_5 i\}$ must be non-zero and not divisible by 2 in $\pi_{68,36}$. The only possibility is that $\kappa\{h_0 h_5 i\}$ is detected by $Ph_2h_5j = d_0 \cdot h_0 h_5 i$, and that Ph_2h_5j is not an h_0 multiple in the E_∞-page. Therefore, $\tau \Delta g_2 \cdot h_0^3$ cannot survive to the E_∞-page. □

Lemma **5.13.** — **(69, 8, 36)** $d_3(\tau D_3') = \tau^2 M h_2 g$.

Proof. — Table 10 shows that the Toda bracket $\langle 2, 8\sigma, 2, \sigma^2 \rangle$ contains $\tau \nu \overline{\kappa}$, which is detected by $\tau^2 h_2 g$. Table 21 shows that Mh_2 detects $\nu\alpha$ for some α in $\pi_{45,24}$ detected by $h_3^2 h_5$. (Beware that there is a crossing extension, Mh_2 does not detect $\nu\alpha$ for every α that is detected by $h_3^2 h_5$.) It follows that $\tau^2 M h_2 g$ detects $\langle 2, 8\sigma, 2, \sigma^2 \rangle \alpha$.

This expression is contained in $\langle 2, 8\sigma, \langle 2, \sigma^2, \alpha \rangle \rangle$. Lemma 6.15 shows that the inner bracket equals $\{0, 2\tau\overline{\kappa}^3\}$.

The Toda bracket $\langle 2, 8\sigma, 0 \rangle$ in $\pi_{68,36}$ consists entirely of multiples of 2. The Toda bracket $\langle 2, 8\sigma, 2\tau\overline{\kappa}^3 \rangle$ contains $\langle 2, 8\sigma, 2 \rangle \tau \overline{\kappa}^3$. This last expression equals zero because

$$\langle 2, 8\sigma, 2 \rangle = \tau\eta \cdot 8\sigma = 0$$

by Corollary 6.2. Therefore, $\langle 2, 8\sigma, 2\tau\overline{\kappa}^3 \rangle$ equals its indeterminacy, which consists entirely of multiples of 2 in $\pi_{68,36}$.

We conclude that $\tau^2 M h_2 g$ is either hit by a differential, or is the target of a hidden 2 extension. Lemma 7.23 shows that there is no hidden 2 extension from $h_3 A'$ to $\tau^2 M h_2 g$, and there are no other possible extensions to $\tau^2 M h_2 g$.

Therefore, $\tau^2 M h_2 g$ must be hit by a differential, and the only possible source of this differential is $\tau D_3'$. □

Lemma **5.14.** — $(\mathbf{77},\mathbf{14},\mathbf{40})$ $d_3(\tau^2 \mathrm{M} h_0 l) = \Delta^2 h_0 d_0^2$.

Proof. — Table 7 shows that $d_4(\tau^2 d_0 e_0 + h_0^7 h_5)$ equals $\mathrm{P}^2 d_0$, so $d_4(\tau^3 \mathrm{M} h_1 d_0 e_0)$ equals $\tau \mathrm{MP}^2 h_1 d_0$. We have the relation $h_0 \cdot \tau^2 \mathrm{M} h_0 l = \tau^3 \mathrm{M} h_1 d_0 e_0$, but the element $\tau \mathrm{MP}^2 h_1 d_0$ is not divisible by h_0. Therefore, $\tau^2 \mathrm{M} h_0 l$ cannot survive to the E_4-page.

By comparison to the Adams spectral sequence for *tmf*, the value of $d_3(\tau^2 \mathrm{M} h_0 l)$ cannot be $\tau^3 \Delta h_1 e_0^3 + \Delta^2 h_0 d_0^2$ or $\tau^3 \Delta h_1 e_0^3$. The only remaining possibility is that $d_3(\tau^2 \mathrm{M} h_0 l)$ equals $\Delta^2 h_0 d_0^2$. □

Lemma **5.15.** — $(\mathbf{78},\mathbf{13},\mathbf{40})$ $d_3(h_0^3 x_{78,10}) = \tau^6 e_0 g^3$.

Proof. — Suppose that $h_0^3 x_{78,10}$ were a permanent cycle. Then it would map under inclusion of the bottom cell to the element $h_0^3 x_{78,10}$ in the Adams E_∞-page for $\mathrm{C}\tau$.

There is a hidden ν extension from $h_0^3 x_{78,10}$ to $\Delta^3 h_1^2 h_3$ in the Adams E_∞-page for $\mathrm{C}\tau$. Then $\Delta^3 h_1^2 h_3$ would also have to be in the image of inclusion of the bottom cell. The only possible pre-image is the element $\Delta^3 h_1^2 h_3$ in the Adams spectral sequence for the sphere, but this element does not survive by Lemma 5.47.

By contradiction, we have shown that $h_0^3 x_{78,10}$ must support a differential. The only possibility is that $d_3(h_0^3 x_{78,10})$ equals $\tau^6 e_0 g^3$. □

Lemma **5.16.** — $(\mathbf{79},\mathbf{5},\mathbf{42})$ $d_3(x_1) = \tau h_1 m_1$.

Proof. — This follows from the interaction between algebraic squaring operations and classical Adams differentials [10, Theorem 2.2]. The theorem says that

$$d_* \operatorname{Sq}^1 e_1 = \operatorname{Sq}^3 d_3 e_1 \dotplus h_1 \operatorname{Sq}^3 e_1.$$

The notation means that there is an Adams differential on $\operatorname{Sq}^1 e_1$ hitting either $\operatorname{Sq}^3 d_3 e_1$ or $h_1 \operatorname{Sq}^3 e_1$, depending on which element has lower Adams filtration. Therefore $d_3 \operatorname{Sq}^1 e_1 = h_1 \operatorname{Sq}^3 e_1$.

Finally, we observe from [12] that $\operatorname{Sq}^1 e_1 = x_1$ and $\operatorname{Sq}^3 e_1 = m_1$. □

Lemma **5.17.** — $(\mathbf{80},\mathbf{12},\mathbf{42})$ *The element* $\Delta^2 d_1$ *is a permanent cycle.*

Proof. — The element $\Delta^2 d_1$ in the Adams E_∞-page for $\mathrm{C}\tau$ must map to zero under the projection from $\mathrm{C}\tau$ to the top cell. The only possible value in sufficiently high filtration is $\tau^2 \Delta h_1 e_0^2 g$. However, comparison to *mmf* shows that this element is not annihilated by τ, and therefore cannot be in the image of projection to the top cell.

Therefore, $\Delta^2 d_1$ must be in the image of the inclusion of the bottom cell into $\mathrm{C}\tau$. The element $\Delta^2 d_1$ is the only possible pre-image in the Adams E_∞-page for the sphere in sufficiently low filtration. □

Lemma **5.18.** — **(80, 14, 41)** *$d_3(\Delta^3 h_1 h_3)$ equals either $\tau^4 \Delta h_1 e_0^2 g$, $\tau \Delta^2 h_0 d_0 e_0$, or $\tau^4 \Delta h_1 e_0^2 g + \tau \Delta^2 h_0 d_0 e_0$; and it is not equal to $d_3(\tau^2 d_0 B_5)$.*

Proof. — There is a relation $Ph_1 \cdot \Delta^3 h_1 h_3 = \tau \Delta^3 h_1^3 d_0$ in the Adams E_2-page. Because of the differential $d_2(\Delta^3 h_1 e_0) = \Delta^3 h_1^3 d_0 + \tau^5 e_0^2 gm$, we have the relation $Ph_1 \cdot \Delta^3 h_1 h_3 = \tau^6 e_0^2 gm$ in the E_3-page.

There is a differential $d_4(\tau^6 e_0^2 gm) = \tau^4 d_0^4 l$. But $\tau^4 d_0^4 l$ is not divisible by Ph_1, so $\tau^6 e_0^2 gm$ cannot be divisible by Ph_1 in the E_4-page. Therefore, $d_3(\Delta^3 h_1 h_3)$ must be non-zero.

The same argument shows that $d_3(\Delta^3 h_1 h_3 + \tau^3 d_0 B_5)$ must also be non-zero. □

Remark **5.19.** — In fact, Chua has determined that $d_3(\Delta^3 h_1 h_3)$ equals $\tau^4 \Delta h_1 e_0^2 g$ [17].

Lemma **5.20.** — **(80, 14, 42)** *$d_3(\tau^2 d_0 B_5)$ equals either $\Delta^2 h_0 d_0 e_0$ or $\Delta^2 h_0 d_0 e_0 + \tau^3 \Delta h_1 e_0^2 g$.*

Proof. — The element $\Delta^2 h_0 d_0 e_0$ is a permanent cycle because there are no possible differentials that it could support. Moreover, it must map to zero under the inclusion of the bottom cell into $C\tau$ because there are no elements in the Adams E_∞-page for $C\tau$ of sufficiently high filtration. Therefore, $\Delta^2 h_0 d_0 e_0$ is either hit by a differential, or it is the target of a hidden τ extension, or it is the target of a non-hidden τ extension.

The only possible hidden τ extension has source $h_1^3 x_{76,6}$. However, Table 13 shows that $h_1^3 x_{76,6}$ is in the image of projection from $C\tau$ to the top cell. Therefore, it cannot support a hidden extension.

We now know that $\Delta^2 h_0 d_0 e_0$ must be hit by a differential, or it is τ-divisible in the E_∞-page. Lemma 5.17 rules out one possible source for the differential. The only remaining possibilities are that $d_3(\tau^2 d_0 B_5)$ equals $\Delta^2 h_0 d_0 e_0$ or $\Delta^2 h_0 d_0 e_0 + \tau^3 \Delta h_1 e_0^2 g$. □

Remark **5.21.** — We are grateful to Dexter Chua for pointing out an error in a previous version of Lemma 5.20. In fact, Chua has determined that the value of $d_3(\tau^2 d_0 B_5)$ is $\Delta^2 h_0 d_0 e_0 + \tau^3 \Delta h_1 e_0^2 g$ [17].

Lemma **5.22.**

(1) **(81, 3, 42)** $d_3(h_2 h_4 h_6) = 0$.
(2) **(82, 10, 42)** $d_3(P^2 h_2 h_6) = 0$.

Proof. — The value of $d_3(h_2 h_4 h_6)$ is not $h_2 h_6 d_0$ nor $h_2 h_6 d_0 + \tau h_1 x_1$ by comparison to the Adams spectral sequence for $C\tau$.

It remains to show that $d_3(h_2 h_4 h_6)$ cannot equal $\tau h_1 x_1$. Suppose that the differential did occur. Then there would be no possible targets for a hidden τ extension on $h_1 x_1$, so the η extension from h_1x_1 to $h_1^2x_1$ would be detected by projection from $C\tau$ to the top cell. But there is no such η extension in the homotopy groups of $C\tau$. This establishes the first formula.

The proof of the second formula is essentially the same, using that the η extension from $\Delta^2h_1d_1$ to $\Delta^2h_1^2d_1$ cannot be detected by projection from $C\tau$ to the top cell. □

Lemma **5.23.** — **(81, 12, 42)** $d_3(\Delta^2 p) = 0$.

Proof. — Suppose that $d_3(\Delta^2 p)$ were equal to $\tau^3h_1Me_0^2$. In the Adams E_4-page, the Massey product $\langle \tau^2Mg, \tau h_1d_0, d_0\rangle$ would equal $\tau^2M\Delta h_2^2g$, with no indeterminacy, because of the Adams differential $d_3(\Delta h_2^2) = \tau h_1d_0^2$ and because $d_0 \cdot \Delta^2 p = 0$. By Moss's higher Leibniz rule 2.27, $d_4(\tau^2M\Delta h_2^2g)$ would be a linear combination of multiples of τ^2Mg and d_0. But Table 7 shows that $d_4(\tau^2M\Delta h_2^2g)$ equals $MP\Delta h_0^2e_0$, which is not such a linear combination in the Adams E_4-page. □

Lemma **5.24.** — **(83, 5, 43)** $d_3(\tau h_6g + \tau h_2e_2) = 0$.

Proof. — In the Adams E_3-page, we have the matric Massey product

$$\tau h_6 g + \tau h_2 e_2 = \left\langle \begin{bmatrix} \tau g & \tau h_2 \end{bmatrix}, \begin{bmatrix} h_5^2 \\ x_1 \end{bmatrix}, h_0 \right\rangle$$

because of the Adams differentials $d_2(h_6) = h_0h_5^2$ and $d_2(e_2) = h_0x_1$, as well as the relation $\tau g \cdot h_5^2 + \tau h_2x_1$ in the Adams E_2-page. Moss's higher Leibniz rule 2.27 implies that $d_3(\tau h_6g + \tau h_2e_2)$ belongs to

$$\left\langle [0\ \ 0], \begin{bmatrix} h_5^2 \\ x_1 \end{bmatrix}, h_0 \right\rangle + \left\langle [\tau g\ \ \tau h_2], \begin{bmatrix} 0 \\ \tau h_1 m_1 \end{bmatrix}, h_0 \right\rangle + \left\langle [\tau g\ \ \tau h_2], \begin{bmatrix} h_5^2 \\ x_1 \end{bmatrix}, 0 \right\rangle$$

since $d_3(x_1) = \tau h_1m_1$, where the Massey products are formed in the Adams E_3-page using the d_2 differential. This expression simplifies to $\left\langle [\tau g\ \ \tau h_2], \begin{bmatrix} 0 \\ \tau h_1 m_1 \end{bmatrix}, h_0 \right\rangle$, which equals $\{0, \tau h_0^2h_4Q_3\}$.

Table 21 shows that there is a hidden ν extension from $h_0^2h_4Q_3$ to $Ph_1x_{76,6}$. The element $\tau Ph_1x_{76,6}$ is non-zero in the Adams E_∞-page. Therefore, $h_0^2h_4Q_3$ supports a (hidden or not hidden) τ extension whose target is in Adams filtration at most 10. The only possibility is that $\tau h_0^2h_4Q_3$ is non-zero in the Adams E_∞-page. □

Lemma **5.25.** — **(84, 4, 44)** $d_3(f_2) = \tau h_1h_4Q_3$.

Proof. — Table 21 shows that τh_1Q_3 detects $\nu^2\theta_5$, and Table 10 shows that $\tau h_1h_4Q_3$ detects $\langle \nu^2\theta_5, 2, \sigma^2\rangle$, with indeterminacy in strictly higher Adams filtration. This bracket contains $\nu^2\langle\theta_5, 2, \sigma\rangle$, so $\tau h_1 h_4 Q_3$ detects a multiple of ν^2. The only possibility is that $\tau h_1 h_4 Q_3$ is a multiple of h_2^2 in the E_∞-page. This implies that $d_3(f_2)$ equals $\tau h_1 h_4 Q_3$ or $\tau h_1 h_4 Q_3 + h_0^2 h_6 g$. The latter possibility is ruled out by comparison to $C\tau$. □

Lemma **5.26.** — **(85, 6, 45)** $d_3(\tau x_{85,6} + h_0^3 c_3) = 0$.

Proof. — Let α be an element of $\pi_{66,35}$ that is detected by $\tau h_2 C'$. Then $\nu\alpha$ is detected by $\tau h_2^2 C'$, and $\tau\nu\alpha$ is zero.

Let $\overline{\alpha}$ be an element of $\pi_{70,36}C\tau$ that is detected by $h_0^2 h_3 h_6$. Projection from $C\tau$ to the top cell takes $\overline{\alpha}$ to $\nu\alpha$. Moreover, in the homotopy of $C\tau$, the Toda bracket $\langle 2, \sigma^2, \overline{\alpha}\rangle$ is detected by $h_0^3 c_3$.

Now projection from $C\tau$ to the top cell takes $\langle 2, \sigma^2, \overline{\alpha}\rangle$ to $\langle 2, \sigma^2, \nu\alpha\rangle$, which equals zero by Lemma 6.28. Therefore, $h_0^3 c_3$ maps to zero under projection to the top cell of $C\tau$, so it must be in the image of inclusion of the bottom cell.

There are two possibilities. First, $\tau x_{85,6} + h_0^3 c_3$ could survive, and it could map to $h_0^3 c_3$ under inclusion of the bottom cell of $C\tau$. Second, $\tau h_1 f_2$ could map to $h_0^3 c_3$ under inclusion of the bottom cell. This could only occur if $d_{10}(h_1 f_2)$ equaled $M\Delta h_1 d_0$ and $d_9(\tau x_{85,6} + h_0^3 c_3)$ equaled $\tau M\Delta h_1 d_0$.

In either case, $d_3(\tau x_{85,6} + h_0^3 c_3)$ is zero. □

Remark **5.27.** — In the proof of Lemma 5.26, we have used that $d_5(\tau p_1 + h_0^2 h_3 h_6)$ equals $\tau^2 h_2^2 C'$ in order to conclude that $\tau\nu\alpha$ is zero. This differential depends on work in preparation [16].

However, we can also prove Lemma 5.26 independently of [16]. Lemma 5.61 shows that the other possible value of $d_5(\tau p_1 + h_0^2 h_3 h_6)$ is $\tau^2 h_2^2 C' + \tau h_3(\Delta e_1 + C_0)$. In this case, let β be an element of $\pi_{62,33}$ that is detected by $\Delta e_1 + C_0$. Then $\nu\alpha + \sigma\beta$ is detected by $\tau h_2^2 C' + h_3(\Delta e_1 + C_0)$, and $\tau(\nu\alpha + \sigma\beta)$ is zero.

Projection from $C\tau$ to the top cell takes $\overline{\alpha}$ to $\nu\alpha + \sigma\beta$, and takes $\langle 2, \sigma^2, \overline{\alpha}\rangle$ to $\langle 2, \sigma^2, \nu\alpha + \sigma\beta\rangle$, which equals zero by Lemmas 6.28 and 6.29. As in the proof of Lemma 5.26, $h_0^3 c_3$ maps to zero under projection to the top cell of $C\tau$, so it must be in the image of inclusion of the bottom cell.

Lemma **5.28.** — **(88, 18, 46)** $d_3(\tau^2 M h_0 d_0 k) = P\Delta^2 h_0 d_0 e_0 + \tau^3 \Delta h_1 d_0^2 e_0^2$.

Proof. — Table 7 shows that $d_4(\tau^2 M h_1 e_0) = MP^2 h_1$. Multiply by τd_0^2 to see that $d_4(\tau^3 M h_1 d_0^2 e_0) = \tau MP^2 h_1 d_0^2$. We have the relation $h_2 \cdot \tau^2 M h_0 d_0 k = \tau^3 M h_1 d_0^2 e_0$, but $\tau MP^2 h_1 d_0^2$ is not divisible by h_2. Therefore, $\tau^2 M h_0 d_0 k$ cannot survive to the E_4-page. By comparison to *mmf*, there is only one possible value for $d_3(\tau^2 M h_0 d_0 k)$. □

Remark **5.29.** — We are grateful to Dexter Chua for pointing out a small error in a previous version of Lemma 5.28.

Lemma **5.30.** — **(89, 15, 50)** $d_3(h_2 B_5 g) = M h_1 c_0 e_0^2$.

Proof. — We will first establish the relation $d_0 \cdot h_2 B_5 g = \tau M h_1 e_0 g^2$. We use the map p_* of [32, Theorem 1.1]. We have that $p_*(d_0 \cdot h_2 B_5 g) = p_*(\tau M h_1 e_0 g^2)$. Therefore, $d_0 \cdot h_2 B_5 g$ equals $\tau M h_1 e_0 g^2$, modulo a possible error term $P h_1^7 h_6 c_0 e_0$ in the kernel of p_*. However, multiplication by h_1 eliminates the error term.

Table 6 shows that $d_3(\tau e_0 g^2) = c_0 d_0 e_0^2$. Therefore, $d_3(\tau M h_1 e_0 g^2)$ equals $M h_1 c_0 d_0 e_0^2$. Observing that $M h_1 c_0 d_0 e_0^2$ is in fact non-zero in the Adams E_3-page, we conclude that $d_3(h_2 B_5 g)$ must equal $M h_1 c_0 e_0^2$. □

Lemma **5.31.** — **(90, 8, 48)** *The element* M^2 *is a permanent cycle.*

Proof. — Table 3 shows that the Massey product $\langle M h_1, h_0, h_0^2 g_2 \rangle$ equals $M^2 h_1$. Therefore, $M^2 h_1$ detects the Toda bracket $\langle \eta\theta_{4.5}, 2, \sigma^2\theta_4 \rangle$. The indeterminacy consists entirely of multiples of $\eta\theta_{4.5}$. The Toda bracket contains $\theta_4 \langle \eta\theta_{4.5}, 2, \sigma^2 \rangle$. Now $\langle \eta\theta_{4.5}, 2, \sigma^2 \rangle$ is zero because $\pi_{61,33}$ is zero.

We have now shown that $M^2 h_1$ detects a multiple of η. In fact, it detects a non-zero multiple of η because $M^2 h_1$ cannot be hit by a differential by comparison to the Adams spectral sequence for $C\tau$.

Therefore, there exists a non-zero element of $\pi_{90,48}$ that is detected in Adams filtration at most 12. The only possibility is that M^2 survives. □

Lemma **5.32.** — **(93, 7, 48)** $d_3(\Delta h_2^2 h_6) = \tau h_1 h_6 d_0^2$.

Proof. — In the Adams E_3-page, $\Delta h_2^2 h_6$ equals $\langle \Delta h_2^2, h_5^2, h_0 \rangle$, with no indeterminacy, because of the Adams differential $d_2(h_6) = h_0 h_5^2$. Using that $d_3(\Delta h_2^2) = \tau h_1 d_0^2$, Moss's higher Leibniz rule 2.27 implies that $d_3(\Delta h_2^2 h_6)$ is contained in

$$\langle \tau h_1 d_0^2, h_5^2, h_0 \rangle + \langle \Delta h_2^2, 0, h_0 \rangle + \langle \Delta h_2^2, h_5^2, 0 \rangle.$$

All of these brackets have no indeterminacy, and the last two equal zero. The first bracket equals $\tau h_1 h_6 d_0^2$, using the Adams differential $d_2(h_6) = h_0 h_5^2$. □

Lemma **5.33.** — **(93, 13, 48)** $d_3(P^2 h_6 d_0) = 0$.

Proof. — In the Adams E_3-page, the element $P^2 h_6 d_0$ equals the Massey product $\langle P^2 d_0, h_5^2, h_0 \rangle$, with no indeterminacy, because of the Adams differential $d_2(h_6) = h_0 h_5^2$. Moss's higher Leibniz rule 2.27 implies that $d_3(P^2 h_6 d_0)$ is a linear combination of multiples of h_0 and of $P^2 d_0$. The only possibility is that $d_3(P^2 h_6 d_0)$ is zero. □

Lemma **5.34.** — **(93, 22, 48)** $d_3(\tau^2 M P h_0 d_0 j) = P^2 \Delta^2 h_0 d_0^2 + \tau^3 P \Delta h_1 d_0^3 e_0$.

Proof. — Table 7 shows that $d_4(\tau^2 \mathrm{P} d_0 e_0) = \mathrm{P}^3 d_0$. Multiplication by $\tau \mathrm{MP} h_1$ shows that $d_4(\tau^3 \mathrm{MP}^2 h_1 d_0 e_0)$ equals $\tau \mathrm{MP}^4 h_1 d_0$. But $\tau^3 \mathrm{MP}^2 h_1 d_0 e_0$ equals $h_0 \cdot \tau^2 \mathrm{MP} h_0 d_0 j$, while $\tau \mathrm{MP}^4 h_1 d_0$ is not divisible by h_0. Therefore, $\tau^2 \mathrm{MP} h_0 d_0 j$ cannot survive to the E_4-page.

The possible values for $d_3(\tau^2 \mathrm{MP} h_0 d_0 j)$ are the non-zero linear combinations of $\mathrm{P}^2 \Delta^2 h_0 d_0^2$ and $\tau^3 \mathrm{P} \Delta h_1 d_0^3 e_0$. The map to the Adams spectral sequence for *tmf* takes both $\tau^2 \mathrm{MP} h_0 d_0 j$ and $\mathrm{P}^2 \Delta^2 h_0 d_0^2 + \tau^3 \mathrm{P} \Delta h_1 d_0^3 e_0$ to zero, but it takes each of $\mathrm{P}^2 \Delta^2 h_0 d_0^2$ and $\tau^3 \mathrm{P} \Delta h_1 d_0^3 e_0$ to the unique non-zero element in the appropriate degree. Therefore, $d_3(\tau^2 \mathrm{MP} h_0 d_0 j)$ cannot equal either $\mathrm{P}^2 \Delta^2 h_0 d_0^2$ or $\tau^3 \mathrm{P} \Delta h_1 d_0^3 e_0$. □

Lemma **5.35.** — **(95, 16, 49)** *The element* $\mathrm{P}^3 h_6 c_0$ *is a permanent cycle.*

Proof. — Table 18 shows that $\mathrm{P}^3 c_0$ detects the product $\eta \rho_{31}$. Using the Moss Convergence Theorem 2.16 and the Adams differential $d_2(h_6) = h_0 h_5^2$, the element $\mathrm{P}^3 h_6 c_0$ must survive to detect the Toda bracket $\langle \eta \rho_{31}, 2, \theta_5 \rangle$. □

Remark **5.36.** — We suspect that $\mathrm{P}^3 h_6 c_0$ detects the product $\eta_6 \rho_{31}$. However, the argument of Lemma 7.148 cannot be completed because the Toda bracket $\langle \eta \rho_{31}, 2, \theta_5 \rangle$ might have indeterminacy in lower Adams filtration.

5.3. *The Adams d_4 differential*

Table 7 lists the multiplicative generators of the Adams E_4-page through the 95-stem whose d_4 differentials are non-zero, or whose d_4 differentials are zero for non-obvious reasons.

Theorem **5.37.** — *Table 7 lists some values of the Adams d_4 differential on multiplicative generators. Through the 95-stem, the Adams d_4 differential is zero on all multiplicative generators not listed in the table.*

Proof. — The d_4 differential on many multiplicative generators is zero. A few of these multiplicative generators appear in Table 7 because their proofs require further explanation. For the remaining majority of such multiplicative generators, the d_4 differential is zero because there are no possible non-zero values, or because of comparison to the Adams spectral sequences for $\mathrm{C}\tau$, *tmf*, or *mmf*. In a few cases, the multiplicative generator is already known to be a permanent cycle as shown in Table 5. These cases do not appear in Table 7.

The last column of Table 7 gives information on the proof of each differential. Most follow immediately by comparison to the Adams spectral sequence for $\mathrm{C}\tau$, or by comparison to the classical Adams spectral sequence for *tmf*, or by comparison to the **C**-motivic Adams spectral sequence for *mmf*.

If an element is listed in the last column of Table 7, then the corresponding differential can be deduced from a straightforward argument using a multiplicative relation. For example,

$$d_4(d_0 \cdot \tau^2 e_0 g^2) = d_4(e_0^2 \cdot \tau^2 e_0 g) = e_0^2 \cdot \mathrm{P} d_0^2 = d_0^5,$$

so $d_4(\tau^2 e_0 g^2)$ must equal d_0^4.

The remaining more difficult computations are carried out in the following lemmas. □

Lemma **5.38.** — **(62, 10, 32)** $d_4(\tau h_1 \cdot \Delta x) = \tau^2 \Delta h_2^2 d_0 e_0$.

This differential was previously proved in [60, Remark 11.2]. We repeat the argument here for completeness.

Proof. — Table 7 shows that $\tau^3 \Delta h_2^2 g^2$ supports a d_4 differential, and Table 5 shows that $\tau \Delta^2 h_1^2 g + \tau^3 \Delta h_2^2 g^2$ is a permanent cycle. Therefore, $\tau \Delta^2 h_1^2 g$ also supports a d_4 differential.

On the other hand, we have

$$h_1 \cdot \tau \Delta^2 h_1 g = \mathrm{P} h_1 \cdot \Delta x = \Delta x \langle h_1, h_0^3 h_3, h_0 \rangle.$$

This expression equals $\langle h_1 \cdot \Delta x, h_0^3 h_3, h_0 \rangle$ by inspection of indeterminacies. Therefore, the Toda bracket $\langle \{\tau h_1 \cdot \Delta x\}, 8\sigma, 2 \rangle$ cannot be well-formed, since otherwise it would be detected by $\tau \Delta^2 h_1^2 g$. The only possibility is that $\tau h_1 \cdot \Delta x$ is not a permanent cycle, and the only possible differential is that $d_4(\tau h_1 \cdot \Delta x)$ equals $\tau^2 \Delta h_2^2 d_0 e_0$. □

Lemma **5.39.** — **(62, 10, 32)** $d_4(\Delta^2 h_3^2) = 0$.

Proof. — Table 8 shows that $d_5(\tau h_1^2 \cdot \Delta x)$ equals $\tau^3 d_0^2 e_0^2$. The element $\tau^3 d_0^2 e_0^2$ is not divisible by h_1 in the E_5-page, so $\tau h_1^2 \cdot \Delta x$ cannot be divisible by h_1 in the E_4-page.

If $d_4(\Delta^2 h_3^2)$ equaled $\tau^2 \Delta h_2^2 d_0 e_0$, then $\Delta^2 h_3^2 + \tau h_1 \cdot \Delta x$ would survive to the E_5-page, and $\tau^2 h_1^2 \cdot \Delta x$ would be divisible by h_1 in the E_5-page. □

Lemma **5.40.** — **(63, 7, 33)** $d_4(\tau \mathrm{X}_2) = \tau \mathrm{M} h_2 d_0$.

Proof. — Table 7 shows that $d_4(\mathrm{C}')$ equals $\mathrm{M} h_2 d_0$. Therefore, either $\tau \mathrm{X}_2$ or $\tau \mathrm{X}_2 + \tau \mathrm{C}'$ is non-zero on the E_∞-page. The inclusion of the bottom cell into $\mathrm{C}\tau$ takes this element to $\overline{h_5 d_0 e_0}$.

In the homotopy of $\mathrm{C}\tau$, there is a ν extension from $\overline{h_5 d_0 e_0}$ to $\tau \mathrm{B}_5$, and inclusion of the bottom cell into $\mathrm{C}\tau$ takes $\tau h_2 \mathrm{C}'$ to $\tau \mathrm{B}_5$.

It follows that there must be a ν extension with target $\tau h_2 \mathrm{C}'$. The only possibility is that $\tau \mathrm{X}_2 + \tau \mathrm{C}'$ is non-zero on the E_∞-page, and therefore $d_4(\tau \mathrm{X}_2)$ equals $d_4(\tau \mathrm{C}')$. □

Lemma **5.41.** — **(68, 5, 36)** $d_4(h_0 d_2) = X_3$.

Proof. — The element X_3 is a permanent cycle. The only possible target for a differential is $\tau^2 d_0 e_0 m$, but this is ruled out by comparison to *tmf*.

In the Adams E_∞-page for $C\tau$, there are several elements in stem 67 and weight 36. However, they all have filtration lower than 9. Since X_3 has filtration 9, it must map to zero under inclusion of the bottom cell into $C\tau$. Therefore, X_3 is the target of a hidden τ extension, or it is hit by a differential.

The only possible hidden τ extension would have source $h_1 \cdot \Delta_1 h_3^2$. In $C\tau$, there is an η extension from $h_0 d_2$ to $\overline{h_1^2 \cdot \Delta_1 h_3^2}$. Since $\overline{h_1^2 \cdot \Delta_1 h_3^2}$ maps non-trivially (to $h_1^2 \cdot \Delta_1 h_3^2$) under projection to the top cell of $C\tau$, it follows that $h_0 d_2$ also maps non-trivially under projection. For degree reasons, the only possibility is that $h_0 d_2$ maps to $h_1 \cdot \Delta_1 h_3^2$, and therefore $h_1 \cdot \Delta_1 h_3^2$ does not support a hidden τ extension.

Therefore, X_3 must be hit by a differential, and there is just one possibility. □

Lemma **5.42.** — **(68, 11, 38)** $d_4(M h_2 g) = 0$.

Proof. — Table 3 shows that the Massey product $\langle h_2 g, h_0^3, g_2 \rangle$ equals $M h_2 g$. The Moss Convergence Theorem 2.16 shows that $M h_2 g$ must survive to detect the Toda bracket $\langle \{h_2 g\}, 8, \overline{\kappa}_2 \rangle$. □

Lemma **5.43.** — **(72, 9, 40)** $d_4(h_2^2 G_0) = \tau g^2 n$.

Proof. — Table 15 shows that there is a hidden 2 extension from $h_0 h_3 g_2$ to $\tau g n$. Therefore, $\tau g n$ detects $4\sigma\overline{\kappa}_2$.

Table 3 shows that $\langle h_1^3 h_4, h_1, \tau g n \rangle$ consists of the two elements $\tau g^2 n$ and $\tau g^2 n + M h_2 g \cdot h_0 h_2$. Then the Toda bracket $\langle \eta^2 \eta_4, \eta, 4\sigma\overline{\kappa}_2 \rangle$ is detected by either $\tau g^2 n$ or $\tau g^2 n + M h_2 g \cdot h_0 h_2$. But $M h_2 g \cdot h_0 h_2$ is hit by an Adams d_2 differential, so $\tau g^2 n$ detects the Toda bracket.

The Toda bracket has no indeterminacy, so it equals $\langle \eta^2 \eta_4, \eta, 2 \rangle 2\sigma\overline{\kappa}_2$. This last expression must be zero.

We have shown that $\tau g^2 n$ must be hit by some differential. The only possibility is that $d_4(h_2^2 G_0) = \tau g^2 n$. □

Lemma **5.44.** — **(75, 11, 40)** $d_4(\Delta h_0^2 h_3 g_2) = \tau M h_1 d_0^2$.

Proof. — Table 8 shows that $d_5(A') = \tau M h_1 d_0$. Now $d_0 A'$ is zero in the E_5-page, so $\tau M h_1 d_0^2$ must also be zero in the E_5-page. □

Lemma **5.45.** — **(76, 14, 41)** $d_4(\Delta^2 h_1 h_3 g) = \tau \Delta h_2^2 d_0^2 e_0$.

Proof. — Table 10 shows that the element $\Delta h_2^2 d_0^2 e_0$ detects the Toda bracket $\langle \tau\eta\kappa\overline{\kappa}^2, \eta, \eta^2\eta_4\rangle$. Now shuffle to obtain

$$\tau\langle \tau\eta\kappa\overline{\kappa}^2, \eta, \eta^2\eta_4\rangle = \langle \tau, \tau\eta\kappa\overline{\kappa}^2, \eta\rangle\eta^2\eta_4.$$

Table 10 shows that $\langle \tau, \tau\eta\kappa\overline{\kappa}^2, \eta\rangle$ is detected by $h_0h_2h_5i$. It follows that the expression $\langle \tau, \tau\eta\kappa\overline{\kappa}^2, \eta\rangle\eta^2\eta_4$ is zero, so $\tau\Delta h_2^2 d_0^2 e_0$ must be hit by some differential. The only possibility is that $d_4(\Delta^2 h_1 h_3 g)$ equals $\tau\Delta h_2^2 d_0^2 e_0$. □

Lemma **5.46.** — **(80, 5, 42)** $d_4(h_0 e_2) = \tau h_1^3 x_{76,6}$.

Proof. — Table 24 shows that $\sigma^2\theta_5$ is detected by $h_0h_4\mathrm{A}$ or $h_0h_4\mathrm{A} + \tau^2 d_1 g_2$. Note that both $h_2 \cdot h_0h_4\mathrm{A}$ and $h_2(h_0h_4\mathrm{A} + \tau^2 d_1 g_2)$ equal $\tau h_1^3 x_{76,6}$.

Since $\nu\sigma = 0$, the element $\tau h_1^3 x_{76,6}$ must be hit by a differential. The only possibility is that $d_4(h_0 e_2)$ equals $\tau h_1^3 x_{76,6}$. □

Lemma **5.47.** — **(81, 15, 42)** $d_4(\Delta^3 h_1^2 h_3) = \tau^4 d_0 e_0^2 l$.

Proof. — Table 18 shows that there is a hidden η extension from $\tau^2\Delta h_1 g^2$ to $\tau^2 d_0 e_0 m$. Multiply by d_0 to see that there is also a hidden η extension from $\tau^2\Delta h_1 e_0^2 g$ to $\tau^2 d_0 e_0^2 l$.

Also, $\tau^2\Delta h_1 e_0^2 g$ detects an element in $\pi_{79,43}$ that is annihilated by τ^2. Therefore, $\tau^4 d_0 e_0^2 l$ must be hit by some differential. Moreover, comparison to *mmf* shows that $\tau^3 d_0 e_0^2 l$ is not hit by a differential.

The hidden η extension from $\tau^3\Delta h_1 e_0^2 g$ to $\tau^3 d_0 e_0^2 l$ is detected by projection from $\mathrm{C}\tau$ to the top cell. The only possibility is that this hidden η extension is the image of the h_1 extension from $\Delta^3 h_1 h_3$ to $\Delta^3 h_1^2 h_3$ in the Adams E_∞-page for $\mathrm{C}\tau$.

Therefore, $\Delta^3 h_1^2 h_3$ maps non-trivially under projection from $\mathrm{C}\tau$ to the top cell. Consequently, $\Delta^3 h_1^2 h_3$ cannot be a permanent cycle in the Adams spectral sequence for the sphere. □

Lemma **5.48.** — **(83, 11, 45)** $d_4(\Delta j_1) = \tau \mathrm{M} h_0 e_0 g$.

Proof. — Otherwise, both Δj_1 and $\tau g\mathrm{C}'$ would survive to the E_∞-page, and neither could be the target of a hidden τ extension. They would both map non-trivially under inclusion of the bottom cell into $\mathrm{C}\tau$. But there are not enough elements in $\pi_{83,45}\mathrm{C}\tau$ for this to occur. □

Lemma **5.49.** — **(85, 5, 45)** $d_4(h_1 f_2) = 0$.

Proof. — Table 21 shows that there is a hidden ν extension from $h_2 g\mathrm{D}_3$ to $\mathrm{B}_6 d_1$. If $d_4(h_1 f_2)$ equaled $\tau h_2 g \mathrm{D}_3$, then this ν extension would be in the image of projection from

$C\tau$ to the top cell, since $h_2 g D_3$ cannot support a hidden τ extension. However, there is no such ν extension in the homotopy of $C\tau$. □

Lemma **5.50.** — **(85, 6, 44)** $d_4(\tau x_{85,6} + h_0^3 c_3) = 0$.

Proof. — We showed in Lemma 5.26 that $h_0^3 c_3$ is in the image of inclusion of the bottom cell of $C\tau$. Therefore, $Px_{76,6}$ cannot be in the image of projection from $C\tau$ to the top cell. Since $Px_{76,6}$ cannot support a hidden τ extension, there can be no differential whose value is $\tau P x_{76,6}$. □

Lemma **5.51.** — **(86, 4, 45)** $d_4(h_1 c_3) = \tau h_0 h_2 h_4 Q_3$.

Proof. — Lemma 7.152 shows that there exists an element α in $\pi_{67,36}$ that is detected by $h_0 Q_3 + h_0 n_1$ such that $\tau\nu\alpha$ equals $(\eta\sigma + \epsilon)\theta_5$.

Table 10 shows that the Toda bracket $\langle \nu, \sigma, 2\sigma \rangle$ is detected by $h_2 h_4$, so the element $\tau h_0 h_2 h_4 Q_3$ detects $\tau\alpha\langle \nu, \sigma, 2\sigma\rangle$, which is contained in $\langle \tau\nu\alpha, \sigma, 2\sigma\rangle$. The indeterminacy in these expressions is zero because $\tau\nu\alpha \cdot \pi_{15,8}$ and $2\sigma \cdot \pi_{78,41}$ are both zero.

We now know that $\tau h_0 h_2 h_4 Q_3$ detects the Toda bracket $\langle (\epsilon + \eta\sigma)\theta_5, \sigma, 2\sigma\rangle$. This bracket contains $\theta_5\langle \epsilon + \eta\sigma, \sigma, 2\sigma\rangle$. Lemma 6.6 shows that the bracket $\langle \epsilon + \eta\sigma, \sigma, 2\sigma\rangle$ contains 0, so $\theta_5\langle \epsilon + \eta\sigma, \sigma, 2\sigma\rangle$ equals zero.

Finally, we have shown that $\tau h_0 h_2 h_4 Q_3$ detects zero, so it must be hit by some differential. □

Lemma **5.52.** — **(87, 7, 45)** $d_4(x_{87,7}) = 0$.

Proof. — Consider the exact sequence

$$\pi_{87,45} \to \pi_{87,45} C\tau \to \pi_{86,46}.$$

The middle term $\pi_{87,45} C\tau$ is isomorphic to $(\mathbf{Z}/2)^4$. The elements of $\pi_{87,45}$ that are not divisible by τ are detected by $P^2 h_6 c_0$, and possibly $x_{87,7}$ and $\tau \Delta h_1 H_1$. On the other hand, the elements of $\pi_{86,46}$ that are annihilated by τ are detected by $\tau^3 \Delta c_0 e_0^2 g$ and possibly $M\Delta h_0^2 e_0$.

In order for the possibility $M\Delta h_0^2 e_0$ to occur, either $x_{87,7}$ or $\tau\Delta h_1 H_1$ would have to support a differential hitting $\tau M \Delta h_0^2 e_0$, in which case one of those possibilities could not occur.

If $d_4(x_{87,7})$ equaled $\tau^3 g G_0$, then there would not be enough elements to make the above sequence exact. □

Lemma **5.53.** — **(87, 1045)** $d_4(\tau\Delta h_1 H_1) = 0$.

Proof. — The element $\Delta^2 h_2^2 d_1$ is a permanent cycle that cannot be hit by any differential because $h_2 \cdot \Delta^2 h_2^2 d_1$ cannot be hit by a differential. The element $\Delta^2 h_2^2 d_1$ cannot be in the image of projection from $C\tau$ to the top cell, and it cannot support a hidden τ extension. Therefore, $\tau \Delta^2 h_2^2 d_1$ cannot be hit by a differential. □

Lemma **5.54.** — **(89, 15, 49)** $d_4(\tau h_2 B_5 g) = M h_1 d_0^3$.

Proof. — Table 24 shows that Md_0 detects $\kappa\theta_{4.5}$. Therefore, Md_0^3 detects $\kappa^3\theta_{4.5}$, which equals $\eta^2\overline{\kappa}^2\theta_{4.5}$ because Table 18 shows that there is a hidden η extension from $\tau^2 h_1 g^2$ to d_0^3.

Now $\eta^2\overline{\kappa}^2\theta_{4.5}$ is zero because $\eta^2\overline{\kappa}\theta_{4.5}$ is zero. Therefore, Md_0^3 and $Mh_1 d_0^3$ must both be hit by differentials.

There are several possible differentials that can hit $Mh_1 d_0^3$. The element $h_1 x_{88,10}$ cannot be the source of this differential because Table 5 shows that $x_{88,10}$ is a permanent cycle. The element $\tau h_2^2 g C'$ cannot be the source of the differential because $h_2^2 g C'$ is a permanent cycle by comparison to *mmf*. The element $\Delta h_1 g_2 g$ cannot be the source because it equals $h_3(\Delta e_1 + C_0)g$. The only remaining possibility is that $d_4(\tau h_2 B_5 g)$ equals $Mh_1 d_0^3$. □

Lemma **5.55.** — **(91, 12, 48)** $d_4(\Delta h_2^2 A') = 0$.

Proof. — In the Adams E_4-page, the element $\Delta h_2^2 A'$ equals the Massey product $\langle A', h_1, \tau d_0^2 \rangle$, with no indeterminacy because of the Adams differential $d_3(\Delta h_2^2) = \tau h_1 d_0^2$. Moss's higher Leibniz rule 2.27 implies that $d_4(\Delta h_2^2 A')$ is contained in

$$\langle 0, h_1, \tau d_0^2 \rangle + \langle A', 0, \tau d_0^2 \rangle + \langle A', h_1, 0 \rangle,$$

so it is a linear combination of multiples of A' and τd_0^2. The only possibility is that $d_4(\Delta h_2^2 A')$ is zero. □

Lemma **5.56.** — **(93, 3, 48)** $d_4(h_4^2 h_6) = h_0^3 g_3$.

Proof. — By comparison to the Adams spectral sequence for $C\tau$, the value of $d_4(h_4^2 h_6)$ is either $h_0^3 g_3$ or $h_0^3 g_3 + \tau h_1 h_4^2 D_3$.

Table 24 shows that $h_0^2 g_3$ detects the product $\theta_4\theta_5$. Since $2\theta_4\theta_5$ equals zero, $h_0^3 g_3$ must be hit by a differential. □

Lemma **5.57.** — **(95, 16, 50)** $d_4(M\Delta^2 h_1^2) = MP\Delta h_0^2 e_0$.

Proof. — Table 10 shows that $M\Delta^2 h_1^2 + \tau^2 M\Delta h_2^2 g$ detects the Toda bracket $\langle \eta, \tau\kappa^2, \tau\theta_{4.5}\overline{\kappa} \rangle$. Therefore, $d_4(M\Delta^2 h_1^2)$ equals $d_4(\tau^2 M\Delta h_2^2 g)$. □

5.4. *The Adams d_5 differential*

Table 8 lists the multiplicative generators of the Adams E_5-page through the 92-stem whose d_5 differentials are non-zero, or whose d_5 differentials are zero for non-obvious reasons.

Theorem **5.58.** — *Table 8 lists some values of the Adams d_5 differential on multiplicative generators. Through the 92-stem, the Adams d_5 differential is zero on all multiplicative generators not listed in the table.*

Proof. — The d_5 differential on many multiplicative generators is zero. For the majority of such multiplicative generators, the d_5 differential is zero because there are no possible non-zero values, or by comparison to the Adams spectral sequence for $C\tau$, or by comparison to *tmf* or *mmf*. In a few cases, the multiplicative generator is already known to be a permanent cycle; $h_1 h_6$ is one such example. A few additional cases appear in Table 8 because their proofs require further explanation.

The last column of Table 8 gives information on the proof of each differential. Many computations follow immediately by comparison to the Adams spectral sequence for $C\tau$.

If an element is listed in the last column of Table 8, then the corresponding differential can be deduced from a straightforward argument using a multiplicative relation. For example,

$$d_5(\tau \cdot gA') = d_5(\tau g \cdot A') = \tau g \cdot \tau M h_1 d_0 = \tau^2 M h_1 e_0^2,$$

so $d_5(gA')$ must equal $\tau M h_1 e_0^2$.

A few of the more difficult computations appear in [16]. The remaining more difficult computations are carried out in the following lemmas. □

Lemma **5.59.** — **(63, 11, 33)** $d_5(\tau h_1^2 \cdot \Delta x) = \tau^3 d_0^2 e_0^2$.

Proof. — The element $\tau^2 d_0^2 e_0^2$ cannot be hit by a differential. There is a hidden η extension from $\tau \Delta h_2^2 d_0 e_0$ to $\tau^2 d_0^2 e_0^2$ because of the hidden τ extensions from $\tau h_1 g^3 + h_1^5 h_5 c_0 e_0$ to $\Delta h_2^2 d_0 e_0$ and from $h_1^6 h_5 c_0 e_0$ to $d_0^2 e_0^2$. This shows that $\tau^3 d_0^2 e_0^2$ must be hit by some differential.

This hidden η extension is detected by projection from $C\tau$ to the top cell. Since $\overline{Ph_5 c_0 d_0}$ in $C\tau$ maps to $\tau \Delta h_2^2 d_0 e_0$ under projection to the top cell, it follows that $\overline{Ph_1 h_5 c_0 d_0}$ in $C\tau$ maps to $\tau^2 d_0^2 e_0^2$ under projection to the top cell.

If $\tau h_1^2 \cdot \Delta x$ survived, then it could not be the target of a hidden τ extension and it could not be hit by a differential. Also, it could not map non-trivially under inclusion of the bottom cell into $C\tau$, since the only possible value $\overline{Ph_1 h_5 c_0 d_0}$ has already been accounted for in the previous paragraph. □

Lemma **5.60.** — **(68, 12, 36)** $d_5(h_5 d_0 i) = \tau \Delta h_1 d_0^3$.

Proof. — We showed in Lemma 5.12 that Ph_2h_5j cannot be divisible by h_0 in the E_∞-page. Therefore, $h_5 d_0 i$ must support a differential. □

Lemma **5.61.** — **(70, 4, 36)** *$d_5(\tau p_1 + h_0^2 h_3 h_6)$ equals either $\tau^2 h_2^2 C'$ or $\tau^2 h_2^2 C' + \tau h_3(\Delta e_1 + C_0)$.*

Proof. — Projection to the top cell of $C\tau$ takes $h_4 D_2$ to $\tau^3 d_1 g^2$. Moreover, there is a ν extension in the homotopy of $C\tau$ from $h_0^2 h_3 h_6$ to $h_4 D_2$. Therefore, this ν extension must be in the image of projection to the top cell.

Table 21 shows that there is a hidden ν extension from $\tau h_2^2 C'$ to $\tau^3 d_1 g^2$. Therefore, either $\tau h_2^2 C'$ or $\tau h_2^2 C' + h_3(\Delta e_1 + C_0)$ is in the image of projection to the top cell, so $\tau^2 h_2^2 C'$ or $\tau^2 h_2^2 C' + \tau h_2(\Delta e_1 + C_0)$ is hit by a differential. The element $\tau p_1 + h_0^2 h_3 h_6$ is the only possible source for this differential. □

Lemma **5.62.** — **(72, 7, 39)** $d_5(h_1 x_{71,6}) = 0$.

Proof. — Table 14 shows that there is a hidden τ extension from $Mh_1^2 h_3 g$ to $Mh_1 d_0^2$. Therefore, $Mh_2^2 g$ must also support a τ extension. This shows that $\tau M h_2^2 g$ cannot be the target of a differential. □

Lemma **5.63.** — **(73, 7, 38)** $d_5(h_4 D_2) = \tau^4 d_1 g^2$.

Proof. — Suppose for sake of contradiction that $h_4 D_2$ survived, and let α be an element of $\pi_{73,38}$ that is detected by it. Table 14 shows that there is a hidden τ extension from $h_1^2 h_6 c_0$ to $h_0 h_4 D_2$. Therefore, $h_0 h_4 D_2$ detects both 2α and $\tau\eta\epsilon\eta_6$. However, it is possible that the difference between these two elements is detected by $\tau^2 M d_0^2$ or by $\tau^3 \Delta h_1 d_0 e_0^2$. We will handle of each of these cases.

First, suppose that 2α equals $\tau\eta\epsilon\eta_6$. Then the Toda bracket

$$\left\langle \eta, \begin{bmatrix} 2 & \tau\eta\epsilon \end{bmatrix}, \begin{bmatrix} \alpha \\ \eta_6 \end{bmatrix} \right\rangle$$

is well-defined. Inclusion of the bottom cell into $C\tau$ takes this bracket to

$$\left\langle \eta, \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} \alpha \\ \eta_6 \end{bmatrix} \right\rangle = \langle \eta, 2, \alpha \rangle,$$

so $\langle \eta, 2, \alpha \rangle$ is in the image of inclusion of the bottom cell.

On the other hand, in the homotopy of $C\tau$, the bracket $\langle \eta, 2, \alpha \rangle$ is detected by $\overline{h_1^3 h_6 c_0}$, with indeterminacy generated by $\overline{h_1^2 h_4 Q_2}$. These elements map non-trivially under projection to the top cell, which contradicts that they are in the image of inclusion of the bottom cell.

Next, suppose that $2\alpha + \tau\eta\epsilon\eta_6$ is detected by $\tau^3 \Delta h_1 d_0 e_0^2$. Then the Toda bracket

$$\left\langle \eta, \begin{bmatrix} 2 & \tau\eta\epsilon & \tau^2\beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \eta_6 \\ \overline{\kappa} \end{bmatrix} \right\rangle$$

is well-defined, where β is an element of $\pi_{53,29}$ that is detected by $\Delta h_1 d_0^2$. The same argument involving inclusion of the bottom cell into $C\tau$ applies to this Toda bracket.

Finally, assume that $2\alpha + \tau\eta\epsilon\eta_6$ is detected by $\tau^2 M d_0^2$. Table 24 shows that $M d_0$ detects $\kappa\theta_{4.5}$, so $\tau^2 M d_0^2$ detects $\tau^2\kappa^2\theta_{4.5}$. Then $2\alpha + \tau\eta\epsilon\eta_6$ equals either $\tau^2\kappa^2\theta_{4.5}$ or $\tau^2\kappa^2\theta_{4.5} + \tau^2\beta\overline{\kappa}$. We can apply the same argument to the Toda bracket

$$\left\langle \eta, \begin{bmatrix} 2 & \tau\eta\epsilon & \tau^2\kappa\theta_{4.5} \end{bmatrix}, \begin{bmatrix} \alpha \\ \eta_6 \\ \kappa \end{bmatrix} \right\rangle,$$

or to the Toda bracket

$$\left\langle \eta, \begin{bmatrix} 2 & \tau\eta\epsilon & \tau^2\kappa\theta_{4.5} & \tau^2\beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \eta_6 \\ \kappa \\ \overline{\kappa} \end{bmatrix} \right\rangle.$$

We have now shown by contradiction that $h_4 D_2$ does not survive. After ruling out other possibilities by comparison to $C\tau$ and to *mmf*, the only remaining possibility is that $d_5(h_4 D_2)$ equals $\tau^4 d_1 g^2$. □

Lemma **5.64.** — **(86, 11, 45)** $d_5(\tau^3 g G_0) = \tau M \Delta h_1^2 d_0$.

Proof. — Suppose for sake of contradiction that the element $\tau^3 g G_0$ survived. It cannot be the target of a hidden τ extension, and it cannot be hit by a differential. Therefore, it maps non-trivially under inclusion of the bottom cell into $C\tau$, and the only possible image is $\Delta^2 e_1 + \overline{\tau \Delta h_2 e_1 g}$.

Let α be an element of $\pi_{86,45}$ that is detected by $\tau^3 g G_0$. Consider the Toda bracket $\langle \alpha, 2\nu, \nu \rangle$. Lemma 4.15 implies that this Toda bracket is detected by $e_0 x_{76,9}$, or is detected in higher Adams filtration.

On the other hand, under inclusion of the bottom cell into $C\tau$, the Toda bracket is detected by $\Delta^2 h_1 g_2$. This is inconsistent with the conclusion of the previous paragraph, since inclusion of the bottom cell can only increase Adams filtrations.

We now know that $\tau^3 g G_0$ does not survive. After eliminating other possibilities by comparison to *mmf*, the only remaining possibility is that $d_5(\tau^3 g G_0)$ equals $\tau M \Delta h_1^2 d_0$. □

Lemma **5.65.** — **(92, 4, 48)** $d_5(g_3) = h_6 d_0^2$.

Proof. — Table 10 shows that $h_1 h_6$ detects the Toda bracket $\langle \eta, 2, \theta_5 \rangle$. Therefore, $h_1 h_6 d_0^2$ detects $\kappa^2 \langle \eta, 2, \theta_5 \rangle$. Now consider the shuffle

$$\tau\kappa^2 \langle \eta, 2, \theta_5 \rangle = \langle \tau\kappa^2, \eta, 2 \rangle \theta_5.$$

Lemma 6.7 shows that the last bracket is zero. Therefore, $h_1 h_6 d_0^2$ does not support a hidden τ extension, so it is either hit by a differential or in the image of projection from $C\tau$ to the top cell.

In the Adams spectral sequence for $C\tau$, the element $h_0^3 h_4^2 h_6$ detects the Toda bracket $\langle \theta_4, 2, \theta_5 \rangle$. Therefore, $h_0^3 h_4^2 h_6$ must be in the image of inclusion of the bottom cell into $C\tau$. In particular, $h_0^3 h_4^2 h_6$ cannot map to $h_1 h_6 d_0^2$ under projection from $C\tau$ to the top cell.

Now $h_1 h_6 d_0^2$ cannot be in the image of projection from $C\tau$ to the top cell, so it must be hit by some differential. The only possibility is that $d_5(h_1 g_3)$ equals $h_1 h_6 d_0^2$. □

Lemma **5.66.** — **(92, 12, 48)** $d_5(\Delta^2 g_2) = 0$.

Proof. — The only possible values for $d_2(\Delta^2 g_2)$ are the linear combinations of $\tau M\Delta c_0 d_0 + \tau^2 M d_0 l$ and $\tau^2 \Delta^2 h_2 g^2$. The possibilities $\tau M\Delta c_0 d_0 + \tau^2 M d_0 l$ and $\tau M\Delta c_0 d_0 + \tau^2 M d_0 l + \tau^2 \Delta^2 h_2 g^2$ are ruled out by d_0 extensions. More specifically, $d_0(\tau M\Delta c_0 d_0 + \tau^2 M d_0 l)$ and $d_0(\tau M\Delta c_0 d_0 + \tau^2 M d_0 l + \tau^2 \Delta^2 h_2 g^2)$ equal the non-zero element $\tau^2 M d_0^2 l$ in the E_5-page, while $d_0 \cdot \Delta^2 g_2$ is zero already in the E_2-page.

If $\tau^2 \Delta^2 h_2 g^2$ were the value of a differential, then the 2 extension from $\tau\Delta^2 h_2 g^2$ to $\tau\Delta^2 h_0 h_2 g^2$ would be detected by the top cell of $C\tau$. However, there is no such 2 extension in the homotopy of $C\tau$. □

Lemma **5.67.** — **(93, 13, 50)** $d_5(e_0 x_{76,9}) = M\Delta h_1 c_0 d_0$.

Proof. — If $M\Delta h_1 c_0 d_0$ were a permanent non-zero cycle, then it could not support a hidden τ extension because Lemma 5.87 shows that $MP\Delta h_1 d_0$ is hit by some differential. Therefore, it would lie in the image of projection from $C\tau$ to the top cell, and the only possible pre-image is the element $\Delta^2 h_1 g_2$ in the E_∞-page of the Adams spectral sequence for $C\tau$.

There is a σ extension from $\Delta^2 e_1 + \overline{\tau\Delta h_2 e_1 g}$ to $\Delta^2 h_1 g_2$ in the Adams spectral sequence for $C\tau$. Then $M\Delta h_1 c_0 d_0$ would also have to be the target of a σ extension. The only possible source for this extension would be $M\Delta h_1^2 d_0$.

Table 18 shows that Mh_1 detects $\eta\theta_{4.5}$, so $M\Delta h_1^2 d_0$ detects $\eta\theta_{4.5}\{\Delta h_1 d_0\}$. The product $\eta\sigma\theta_{4.5}\{\Delta h_1 d_0\}$ equals zero because $\sigma\{\Delta h_1 d_0\}$ is zero. Therefore, $M\Delta h_1^2 d_0$ cannot support a hidden σ extension to $M\Delta h_1 c_0 d_0$.

We have now shown that $M\Delta h_1 c_0 d_0$ must be hit by some differential, and the only possibility is that equals $d_5(e_0 x_{76,9})$. □

5.5. *Higher differentials*

Table 9 lists the multiplicative generators of the Adams E_r-page, for $r \geq 6$, through the 90-stem whose d_r differentials are non-zero, or whose d_r differentials are zero for non-obvious reasons.

Theorem **5.68.** — *Table 9 lists some values of the Adams d_r differential on multiplicative generators of the E_r-page, for $r \geq 6$. For $r \geq 6$, the Adams d_r differential is zero on all multiplicative generators of the E_r-page not listed in the table. The list is complete through the 90-stem, except that:*

(1) *$d_{10}(h_1 f_2)$ might equal $M\Delta h_1 d_0$.*
(2) *$d_9(\tau x_{85,6} + h_0^3 c_3)$ might equal $\tau M\Delta h_1 d_0$.*
(3) *$d_9(h_4^2 D_3)$ might equal $\tau M\Delta h_1 g$.*

Proof. — The d_r differential on many multiplicative generators is zero. For the majority of such multiplicative generators, the d_r differential is zero because there are no possible non-zero values, or by comparison to the Adams spectral sequence for $C\tau$, or by comparison to *tmf* or *mmf*. In a few cases, the multiplicative generator is already known to be a permanent cycle, as shown in Table 5. A few additional cases appear in Table 9 because their proofs require further explanation.

Some of the more difficult computations appear in [16]. The remaining more difficult computations are carried out in the following lemmas. □

Lemma **5.69.**

(1) $\mathbf{(67, 5, 35)}$ $d_6(\tau Q_3 + \tau n_1) = 0$.
(2) $\mathbf{(87, 9, 48)}$ $d_6(g Q_3) = 0$.

Proof. — Several possible differentials on these elements are eliminated by comparison to the Adams spectral sequences for $C\tau$ and for *tmf*. The only remaining possibility is that $d_6(\tau Q_3 + \tau n_1)$ might equal $\tau^2 M h_1 g$, and that $d_6(g Q_3)$ might equal $\tau M h_1 g^2$.

The element $M\Delta h_0^2 e_0$ is not hit by any differential because Table 5 shows that $h_1^2 c_3$ is a permanent cycle, and Table 10 shows that $\tau^2 g Q_3 = h_4^2 Q_2$ must survive to detect the Toda bracket $\langle \theta_4, \tau\overline{\kappa}, \{t\}\rangle$.

Lemma 6.30 shows that $M\Delta h_0^2 e_0$ detects the Toda bracket $\langle \tau\eta\overline{\kappa}^2, 2, 4\overline{\kappa}_2\rangle$, which contains $\tau\overline{\kappa}^2\langle \eta, 2, 4\overline{\kappa}_2\rangle$. Lemma 6.12 shows that this expression contains zero. We now know that $M\Delta h_0^2 e_0$ detects an element in the indeterminacy of the bracket $\langle \tau\eta\overline{\kappa}^2, 4, 2\overline{\kappa}_2\rangle$. In fact, it must detect a multiple of $\tau\eta\overline{\kappa}^2$ since $2\overline{\kappa}_2 \cdot \pi_{42,22}$ is zero.

The only possibility is that $M\Delta h_0^2 e_0$ detects $\overline{\kappa}$ times an element detected by $\tau^2 M h_1 g$. Therefore, $\tau^2 M h_1 g$ cannot be hit by a differential. This shows that $\tau Q_3 + \tau n_1$ is a permanent cycle.

We also know that $M\Delta h_0^2 e_0$ is the target of a hidden τ extension, since it detects a multiple of τ. The element $\tau^2 M h_1 g^2$ is the only possible source of this hidden τ extension, so it cannot be hit by a differential. This shows that $d_6(g Q_3)$ cannot equal $\tau M h_1 g^2$. □

Lemma **5.70.**

(1) **(68, 7, 37)** $d_6(h_2^2 \mathrm{H}_1) = \mathrm{M}c_0 d_0$.
(2) **(68, 7, 36)** $d_7(\tau h_2^2 \mathrm{H}_1) = \mathrm{MP}d_0$.

Proof. — Table 3 shows that $\mathrm{MP}d_0$ equals the Massey product $\langle \mathrm{P}d_0, h_0^3, g_2 \rangle$. This implies that $\mathrm{MP}d_0$ detects the Toda bracket $\langle \tau\eta^2\overline{\kappa}, 8, \overline{\kappa}_2 \rangle$. Lemma 6.19 shows that this Toda bracket consists entirely of multiples of $\tau\eta^2\overline{\kappa}$.

We now know that $\mathrm{MP}d_0$ detects a multiple of $\tau\eta^2\overline{\kappa}$. The only possibility is that $\mathrm{MP}d_0$ detects η times an element detected by $\tau^2 \mathrm{M}h_1 g$.

We will show in Lemma 7.105 that $\tau^2 \mathrm{M}h_1 g$ is the target of a ν extension, so $\tau^2 \mathrm{M}h_1 g$ cannot support a hidden η extension. Therefore, $\mathrm{MP}d_0$ must be hit by some differential. The only possibility is that $d_7(\tau h_2^2 \mathrm{H}_1)$ equals $\mathrm{MP}d_0$. Then $h_2^2 \mathrm{H}_1$ cannot survive to the E_7-page, so $d_6(h_2^2 \mathrm{H}_1)$ equals $\mathrm{M}c_0 d_0$. □

Lemma **5.71.** — **(71, 5, 37)** *The element $\tau h_1 p_1$ is a permanent cycle.*

Proof. — Lemma 5.61, together with results of [16], show that $\tau h_1 p_1$ survives to the E_6-page. We must eliminate possible higher differentials.

Table 14 shows that there is a hidden τ extension from $\tau h_2^2 \mathrm{C}''$ to $\Delta^2 h_1^2 h_4 c_0$. This means that $\tau h_2 \mathrm{C}'' + h_1 h_3(\Delta e_1 + \mathrm{C}_0)$ must also support a hidden τ extension.

The two possible targets for this hidden τ extension are $\Delta^2 h_2 c_1$ and $\tau\Delta^2 h_1^2 g + \tau^3 \Delta h_2^2 g^2$. The second possibility is ruled out by comparison to *tmf*, so $\Delta^2 h_2 c_1$ cannot be hit by a differential. □

Lemma **5.72.** — **(74, 7, 38)** *The element $\mathrm{P}h_0 h_2 h_6$ is a permanent cycle.*

Proof. — First note that projection from $\mathrm{C}\tau$ to the top cell takes $\mathrm{P}h_2 h_6$ to a non-zero element. If $\mathrm{P}h_0 h_2 h_6$ were not a permanent cycle in the Adams spectral sequence for the sphere, then projection from $\mathrm{C}\tau$ to the top cell would also take $\mathrm{P}h_0 h_2 h_6$ to a non-zero element. Then the 2 extension from $\mathrm{P}h_2 h_6$ to $\mathrm{P}h_0 h_2 h_6$ in $\pi_{74,38}\mathrm{C}\tau$ would project to a 2 extension in $\pi_{73,39}$. However, there are no possible 2 extensions in $\pi_{73,39}$. □

Lemma **5.73.** — **(77, 7, 42)** $d_7(m_1) = 0$.

Proof. — The only other possibility is that $d_7(m_1)$ equals $\tau^2 g^2 t$. If that were the case, then the ν extension from $\tau g^2 t$ to $\tau^2 c_1 g^3$ would be detected by projection from $\mathrm{C}\tau$ to the top cell. However, the homotopy groups of $\mathrm{C}\tau$ have no such ν extension. □

Lemma **5.74.** — **(80, 6, 43)** $d_8(h_1 x_1) = 0$.

Proof. — Table 5 shows that $\tau h_1 x_1$ is a permanent cycle. Then $d_8(\tau h_1 x_1)$ cannot equal $\tau^2 \mathrm{M}e_0^2$, and $d_8(h_1 x_1)$ cannot equal $\tau \mathrm{M}e_0^2$. □

Lemma **5.75.** — **(81, 3, 42)** $d_6(h_2h_4h_6) = 0$.

Proof. — Table 5 shows that $h_2^2h_4h_6$ is a permanent cycle. Therefore, the Adams differential $d_6(h_2^2h_4h_6)$ does not equal $\tau h_2c_1\mathrm{A}'$, and $d_6(h_2h_4h_6)$ does not equal $\tau c_1\mathrm{A}'$. □

Lemma **5.76.** — **(86, 6, 46)** $d_{10}(h_2h_6g + h_1^2f_2) = 0$.

Proof. — If $d_{10}(h_2h_6g + h_1^2f_2)$ equaled $\mathrm{M}\Delta h_1^2d_0$, then the ν extension from $\tau h_6g + \tau h_2e_2$ to $\tau h_2h_6g + \tau h_1^2f_2$ would be detected by the bottom cell of $\mathrm{C}\tau$. However, there is no such ν extension in the homotopy of $\mathrm{C}\tau$. □

Lemma **5.77.** — **(87, 7, 45)** $d_7(x_{87,7}) = 0$.

Proof. — If $\tau\Delta^2h_2^2d_1$ were hit by a differential, then the ν extension from $\Delta^2h_2^2d_1$ to $\Delta^2h_1^2h_3d_1$ would be detected by projection from $\mathrm{C}\tau$ to the top cell. But the homotopy of $\mathrm{C}\tau$ has no such ν extension. □

Lemma **5.78.** — **(87, 10, 45)** $d_6(\tau\Delta h_1\mathrm{H}_1) = \tau\mathrm{M}\Delta h_0^2e_0$.

Proof. — Suppose for sake of contradiction that $\tau\Delta h_1\mathrm{H}_1$ survives. This element cannot be hit, nor can it be the target of a hidden τ extension. Therefore, it would have non-zero image under inclusion of the bottom cell into $\mathrm{C}\tau$, and it would map to $\overline{\Delta h_1\mathrm{B}_7}$.

In the homotopy of $\mathrm{C}\tau$, the Toda bracket $\langle\overline{\Delta h_1\mathrm{B}_7}, h_0, h_2^2\rangle$ is detected by the element $\underline{\mathrm{M}\Delta^2h_1}$. Beware that this bracket has indeterminacy in lower filtration since $h_3 \cdot \overline{\Delta h_1\mathrm{B}_7} = \underline{h_1^2x_{91,11}}$.

This implies that the Toda bracket $\langle\{\tau\Delta h_1\mathrm{H}_1\}, 2, \nu^2\rangle$ would be non-zero in $\pi_{94,49}$, and all of its elements would be detected in Adams filtration at most 15. (Beware that this Toda bracket would have indeterminacy detected by $\tau\Delta h_1h_3\mathrm{H}_1$.)

On the other hand, the Moss Convergence Theorem 2.16 would imply that the Toda bracket is detected in filtration at least 12. However, there are no possible elements in filtrations 12 through 15.

We have now shown that $\tau\Delta h_1\mathrm{H}_1$ cannot survive. There is only one possible value for a differential on $\tau\Delta h_1\mathrm{H}_1$.

The previous argument assumed that $2\{\tau\Delta h_1\mathrm{H}_1\}$ is zero in order to form the Toda bracket $\langle\{\tau\Delta h_1\mathrm{H}_1\}, 2, \nu^2\rangle$. However, it is possible that $\tau\Delta h_1\mathrm{H}_1$ supports a hidden 2 extension to $\tau\Delta^2h_3d_1$ or to $\tau^2\Delta^2c_1g$. Therefore, $2\{\tau\Delta h_1\mathrm{H}_1\}$ might equal $\tau\sigma\{\Delta^2d_1\}$, $\tau\nu\{\Delta^2t\}$, or their sum. In those cases, we would need to consider the matric Toda brackets

$$\left\langle \begin{bmatrix} \{\tau\Delta h_1\mathrm{H}_1\} & \tau\{\Delta^2d_1\} \end{bmatrix}, \begin{bmatrix} 2 \\ \sigma \end{bmatrix}, \nu^2 \right\rangle,$$

$$\left\langle \begin{bmatrix} \{\tau \Delta h_1 H_1\} & \nu \end{bmatrix}, \begin{bmatrix} 2 \\ \tau\{\Delta^2 t\} \end{bmatrix}, \nu^2 \right\rangle,$$

or

$$\left\langle \begin{bmatrix} \{\tau \Delta h_1 H_1\} & \tau\{\Delta^2 d_1\} & \nu \end{bmatrix}, \begin{bmatrix} 2 \\ \sigma \\ \tau\{\Delta^2 t\} \end{bmatrix}, \nu^2 \right\rangle$$

respectively. Under inclusion of the bottom cell into $C\tau$, these three brackets would map to

$$\left\langle \begin{bmatrix} \overline{\Delta h_1 B_7} & 0 \end{bmatrix}, \begin{bmatrix} h_0 \\ h_3 \end{bmatrix}, h_2^2 \right\rangle,$$

$$\left\langle \begin{bmatrix} \overline{\Delta h_1 B_7} & h_2 \end{bmatrix}, \begin{bmatrix} h_0 \\ 0 \end{bmatrix}, h_2^2 \right\rangle,$$

or

$$\left\langle \begin{bmatrix} \overline{\Delta h_1 B_7} & 0 & h_2 \end{bmatrix}, \begin{bmatrix} h_0 \\ h_3 \\ 0 \end{bmatrix}, h_2^2 \right\rangle$$

respectively. All three of these brackets in $C\tau$ equal $\langle \overline{\Delta h_1 B_7}, h_0, h_2^2 \rangle$. Beware that the last two could have larger indeterminacy, but in fact do not. □

Lemma **5.79.** — **(88, 10, 48)** *The element $x_{88,10}$ is a permanent cycle.*

Proof. — In the Adams spectral sequence for $C\tau$, there is a hidden η extension from $h_1^2 x_{85,6}$ to $x_{88,10}$. Therefore, $x_{88,10}$ lies in the image of inclusion of the bottom cell into $C\tau$. The only possible pre-image is the element $x_{88,10}$ in the Adams spectral sequence in the sphere, so $x_{88,10}$ must survive. □

Lemma **5.80.**

(1) **(88, 11, 49)** $d_6(h_2^2 g H_1) = M c_0 e_0^2$.
(2) **(88, 11, 48)** $d_7(\tau h_2^2 g H_1) = 0$.

Proof. — If $M c_0 e_0^2$ is non-zero in the E_∞-page, then it detects an element that is annihilated by τ because Lemma 5.81 shows that the only possible target of such an extension is hit by a differential. Then $M c_0 e_0^2$ would be in the image of projection from $C\tau$ to the top cell. The only possible pre-image would be the element $\Delta g_2 g$ of the Adams spectral sequence for $C\tau$.

In the Adams spectral sequence for $C\tau$, there is a σ extension from gA' to $\Delta g_2 g$. Projection from $C\tau$ to the top cell would imply that there is a hidden σ extension in the homotopy groups of the sphere, from $Mh_1 e_0^2$ to $Mc_0 e_0^2$, because gA' maps to $Mh_1 e_0^2$ under projection from $C\tau$ to the top cell.

But $Mh_1 e_0^2$ detects $\eta\theta_{4.5}\{e_0^2\}$, which cannot support a σ extension. This establishes the first formula.

For the second formula, if $d_7(\tau h_2^2 g H_1)$ were equal to $\tau^2 \Delta h_2^2 e_0 g^2$, then the same argument would apply, with $\tau \Delta h_2^2 e_0 g^2$ substituted for $Mc_0 e_0^2$. □

Lemma **5.81.** — **(88, 12, 48)** $d_6(\Delta g_2 g) = Md_0^3$.

Proof. — The proof of Lemma 5.54 shows that Md_0^3 must be hit by a differential. The only possibility is that $d_6(\Delta g_2 g)$ equals Md_0^3.

Alternatively, Lemma 5.70 shows that $d_7(\tau h_2^2 H_1) = MPd_0$. Note that $\tau g \cdot \tau h_2^2 H_1 = 0$ in the E_7-page. Therefore, $\tau M d_0^3 = \tau g \cdot MPd_0$ must already be zero in the E_7-page. The only possibility is that $d_6(\tau \Delta g_2 g) = \tau M d_0^3$, and then $d_6(\Delta g_2 g) = Md_0^3$. □

Remark **5.82.** — Table 9 shows that $d_6(\Delta^2 f_1)$ equals $\tau^2 M d_0^3$. The proof relies on $d_6(\tau \Delta h_1 H_1) = \tau M \Delta h_0^2 e_0$ and uses techniques similar to the ones in [17].

Lemma **5.83.** — **(92, 5, 48)** *The element* $h_0 g_3$ *is a permanent cycle.*

Proof. — In the homotopy of $C\tau$, the product $\theta_4\theta_5$ is detected by $h_0^2 g_3$. In the sphere, the product $\theta_4\theta_5$ is therefore non-zero and detected in Adams filtration at most 6.

Table 10 shows that the Toda bracket $\langle 2, \theta_4, \theta_4, 2\rangle$ contains θ_5. Therefore, the product $\theta_4\theta_5$ is contained in

$$\theta_4 \langle 2, \theta_4, \theta_4, 2\rangle = \langle \theta_4, 2, \theta_4, \theta_4\rangle 2.$$

(Note that the sub-bracket $\langle \theta_4, \theta_4, 2\rangle$ is zero because $\pi_{61,32}$ is zero.) Therefore, $\theta_4\theta_5$ is divisible by 2. It follows that $\theta_4\theta_5$ is detected by $h_0^2 g_3$, and $h_0 g_3$ is a permanent cycle that detects $\langle \theta_4, 2, \theta_4, \theta_4\rangle$. □

Lemma **5.84.** — **(92, 10, 51)** $d_6(\Delta_1 h_1^2 e_1) = 0$.

Proof. — Consider the element $\overline{\tau M h_2^2 g^2}$ in the Adams spectral sequence for $C\tau$. This element cannot be in the image of inclusion of the bottom cell into $C\tau$. Therefore, it must map non-trivially under projection from $C\tau$ to the top cell. The only possibility is that $\tau M h_2^2 g^2$ is the image. Therefore, $\tau M h_2^2 g^2$ cannot be the target of a differential. □

Lemma **5.85.** — **(92, 10, 48)** $d_7(x_{92,10})$ *does not equal* $\tau^2 \Delta^2 h_2 g^2$.

Proof. — If $\tau^2\Delta^2 h_2 g^2$ were hit by a differential, then the 2 extension from $\tau\Delta^2 h_2 g^2$ to $\tau\Delta^2 h_0 h_2 g^2$ would be detected by projection from $C\tau$ to the top cell. But the homotopy of $C\tau$ has no such 2 extension. □

Lemma **5.86.** — **(92, 10, 51)** $d_8(\Delta_1 h_2 e_1) = 0$.

Proof. — Consider the element $e_0 x_{76,9}$ in the Adams E_∞-page for $C\tau$. It cannot be in the image of inclusion of the bottom cell into $C\tau$, so it must project to a non-zero element in the top cell. The only possible image is $M\Delta h_1^3 g$. Therefore, $M\Delta h_1^3 g$ cannot be the target of a differential. □

Lemma **5.87.** — *The element* $MP\Delta h_1 d_0$ *is hit by some differential.*

Proof. — Table 14 shows that there is a hidden τ extension from $\Delta h_1 c_0 d_0$ to $P\Delta h_1 d_0$. Therefore, $P\Delta h_1 d_0$ detects $\tau\epsilon\{\Delta h_1 d_0\}$. On the other hand, Tables 18 and 24 show that $P\Delta h_1 d_0$ also detects $\tau\eta\kappa\{\Delta h_1 h_3\}$. Since there are no elements in higher Adams filtration, we have that $\tau\epsilon\{\Delta h_1 d_0\}$ equals $\tau\eta\kappa\{\Delta h_1 h_3\}$.

Table 24 shows that MP detects $\tau\epsilon\theta_{4.5}$, so $MP\Delta h_1 d_0$ detects $\tau\epsilon\{\Delta h_1 d_0\}\theta_{4.5}$, which equals $\tau\eta\kappa\{\Delta h_1 h_3\}\theta_{4.5}$. But $\tau\eta\kappa\theta_{4.5}$ is zero because all elements of $\pi_{60,32}$ are detected by *tmf*. This shows that $MP\Delta h_1 d_0$ detects zero, so it must be hit by a differential. □

Remark **5.88.** — Lemma 5.87 does not specify the differential that hits the element $MP\Delta h_1 d_0$. In fact, $d_6(\tau e_0 x_{76,9})$ equals $MP\Delta h_1 d_0$ [16].

6. Toda brackets

The purpose of this section is to establish various Toda brackets that are used elsewhere in this manuscript. Tables 10 and 11 collect all of this information in one place. Many Toda brackets can be easily computed from the Moss Convergence Theorem 2.16. These are summarized in the tables without further discussion. However, some brackets require more complicated arguments. Those arguments are collected in this section. For easy reference, the lemmas in this section are labelled with degrees that match the degrees given in the tables.

We will need the following **C**-motivic version of a theorem of Toda [55, Theorem 3.6] that applies to symmetric Toda brackets.

Theorem **6.1.** — *Let* α *be an element of* $\pi_{s,w}$*, with* s *even. There exists an element* α^* *in* $\pi_{2s+1,2w}$ *such that* $\langle\alpha, \beta, \alpha\rangle$ *contains the product* $\beta\alpha^*$ *for all* β *such that* $\alpha\beta$ *equals zero.*

Corollary **6.2.** — *If* $2\beta = 0$*, then* $\langle 2, \beta, 2\rangle$ *contains* $\tau\eta\beta$*.*

Proof. — Apply Theorem 6.1 to $\alpha = 2$. We need to find the value of α^*. Table 3 shows that the Massey product $\langle h_0, h_1, h_0\rangle$ equals τh_1^2. The Moss Convergence Theorem 2.16 then shows that $\langle 2, \eta, 2\rangle$ equals $\tau\eta^2$. It follows that α^* equals $\tau\eta$. □

Theorem **6.3.** — *Tables 10 and 11 list some Toda brackets in the* **C**-*motivic stable homotopy groups.*

Proof. — The fourth column of the table gives information about the proof of each Toda bracket.

If the fourth column shows a Massey product, then the Toda bracket follows from the Moss Convergence Theorem 2.16. If the fourth column shows an Adams differential, then the Toda bracket follows from the Moss Convergence Theorem 2.16, using the mentioned differential.

A few Toda brackets are established elsewhere in the literature; specific citations are given in these cases.

Additional more difficult cases are established in the following lemmas. □

Tables 10 lists information about some Toda brackets that do not contain zero, while Table 11 lists information about some Toda brackets that do contain zero. The third columns of the tables give elements of the Adams E_∞-page that detect elements of the Toda brackets. The fourth columns of the tables give partial information about indeterminacies, again by giving detecting elements of the Adams E_∞-page. We have not completely analyzed the indeterminacies of all brackets when the details are inconsequential for our purposes. The fifth columns indicate the proofs of the Toda brackets, and the sixth columns shows where each specific Toda bracket is used in the manuscript.

Lemma **6.4.** — **(16, 9)** *The Toda bracket $\langle \kappa, 2, \eta\rangle$ contains zero, with indeterminacy generated by $\eta\rho_{15}$.*

Proof. — Using the Adams differential $d_3(h_0h_4) = h_0d_0$, the Moss Convergence Theorem 2.16 shows that the Toda bracket is detected in filtration at least 3. The only element in sufficiently high filtration is Pc_0, which detects the product $\eta\rho_{15}$. This product lies in the indeterminacy, so the bracket must contain zero. □

Lemma **6.5.** — **(20, 11)** *The Toda bracket $\langle \kappa, 2, \eta, \nu\rangle$ is detected by τg.*

Proof. — The subbracket $\langle 2, \eta, \nu\rangle$ is strictly zero, since $\pi_{5,3}$ is zero. The subbracket $\langle \kappa, 2, \eta\rangle$ contains zero by Lemma 6.4. Therefore, the fourfold bracket $\langle \kappa, 2, \eta, \nu\rangle$ is well-defined.

Shuffle to obtain

$$\langle \kappa, 2, \eta, \nu\rangle\eta^2 = \kappa\langle 2, \eta, \nu, \eta^2\rangle.$$

Table 10 shows that ϵ is contained in the Toda bracket $\langle \eta^2, \nu, \eta, 2\rangle$, so the latter expression equals $\epsilon\kappa$, which is detected by $c_0 d_0$. It follows that $\langle \kappa, 2, \eta, \nu\rangle$ must be detected by τg. □

Lemma **6.6.** — **(23, 13)** *The Toda bracket* $\langle \epsilon + \eta\sigma, \sigma, 2\sigma\rangle$ *contains zero, with indeterminacy generated by* $4\nu\overline{\kappa}$ *in* $\{Ph_1 d_0\}$.

Proof. — Consider the shuffle

$$\langle \epsilon + \eta\sigma, \sigma, 2\sigma\rangle\eta = (\epsilon + \eta\sigma)\langle \sigma, 2\sigma, \eta\rangle.$$

Table 10 shows that $h_1 h_4$ detects $\langle \sigma, 2\sigma, \eta\rangle$, so $h_1 h_4 c_0$ detects the product $\epsilon\langle \sigma, 2\sigma, \eta\rangle$. On the other hand, Table 24 shows that there is a hidden σ extension from $h_1 h_4$ to $h_4 c_0$. Therefore, $h_1 h_4 c_0$ also detects $\eta\sigma\langle \sigma, 2\sigma, \eta\rangle$. It follows that $(\epsilon + \eta\sigma)\langle \sigma, 2\sigma, \eta\rangle$ is detected in filtration higher than 5.

Consider the shuffle

$$2\langle \epsilon + \eta\sigma, \sigma, 2\sigma\rangle = \langle 2, \epsilon + \eta\sigma, \sigma\rangle 2\sigma.$$

The latter expression is zero since 2σ annihilates all elements of $\pi_{16,9}$.

This shows that no elements of the Toda bracket can be detected by $\tau h_2 g$ or $\tau h_0 h_2 g$.

The element $4\nu\overline{\kappa}$ generates the indeterminacy because it equals $\tau\eta\kappa(\epsilon + \eta\sigma)$. □

Lemma **6.7.** — **(30, 16)** *The Toda bracket* $\langle \tau\kappa^2, \eta, 2\rangle$ *equals zero, with no indeterminacy.*

Proof. — The Adams differential $d_3(\Delta h_2^2) = \tau h_1 d_0^2$ implies that the bracket is detected by $h_0 \cdot \Delta h_2^2$, which equals zero in the E_∞-page. Therefore, the Toda bracket is detected in Adams filtration at least 7, but there are no elements in the Adams E_∞-page in sufficiently high filtration.

The indeterminacy can be computed by inspection. □

Lemma **6.8.** — **(35, 20)** *The Toda bracket* $\langle \eta^2, \theta_4, \eta^2\rangle$ *contains zero, with indeterminacy generated by* $\eta^3\eta_5$.

Proof. — If the bracket were detected by $h_2 d_1$, then

$$\nu\langle \eta^2, \theta_4, \eta^2\rangle = \langle \nu, \eta^2, \theta_4\rangle\eta^2$$

would be detected by $h_2^2 d_1$. However, $h_2^2 d_1$ does not detect a multiple of η^2.

The bracket cannot be detected by $\tau h_1 e_0^2$ by comparison to *tmf*.

By inspection, the only remaining possibility is that the bracket contains zero. The indeterminacy can be computed by inspection. □

Lemma **6.9.** — **(36, 20)** *The Toda bracket* $\langle \tau, \eta^2\kappa_1, \eta\rangle$ *is detected by* t*, with indeterminacy generated by* $\eta^3\mu_{33}$.

Proof. — There is a relation $h_1 \cdot \overline{h_1^2 d_1} = t$ in the homotopy of $C\tau$. Using the connection between Toda brackets and cofibers as described in [30, Section 3.1.1], this shows that t detects the Toda bracket.

The indeterminacy is computed by inspection. □

The third author presents the following Lemma 6.10 as a correction to [62, Theorem 2.1], where it states that the said Toda bracket contains 0.

Lemma **6.10.** — **(45)** *The classical Toda bracket* $\langle \theta_4, 2, \sigma^2 + \kappa\rangle$ *contains* 0 *or* $\eta\overline{\kappa}_2$*. Its indeterminacy is generated by* $\rho_{15}\theta_4$*, which is detected by* $h_0^2 h_5 d_0$.

Proof. — The gap originated in [62, Remark 3.3], where it was claimed that $\langle \theta_4, 2, \sigma^2\rangle$ contains an order 2 element of the form $2\alpha + \beta$, where α is detected by h_4^3 and β is detected by $h_5 d_0$. In fact, since $h_5 d_0$ and $h_1 g_2$ are in the same filtration, we can only conclude that β is detected by $h_5 d_0$ or $h_5 d_0 + h_1 g_2$, therefore the missed possibility in the statement of the lemma. □

Remark **6.11.** — In fact, we have evidence that this classical Toda bracket $\langle \theta_4, 2, \sigma^2 + \kappa\rangle$ contains $\eta\overline{\kappa}_2$. However, the argument depends on computations as far as the 110-stem.

Lemma **6.12.** — **(46, 25)** *The Toda bracket* $\langle \eta, 2, 4\overline{\kappa}_2\rangle$ *contains zero.*

Proof. — The Massey product $Mh_1 = \langle h_1, h_0, h_0^2 g_2\rangle$ shows that Mh_1 detects the Toda bracket. Table 18 shows that Mh_1, $\Delta h_2 c_1$, and $\tau d_0 l + \Delta c_0 d_0$ are all targets of hidden η extensions. (Beware that the hidden η extension from $h_3^2 h_5$ to Mh_1 is a crossing extension in the sense of Section 2.1, but that does not matter.) Therefore, Mh_1 detects only multiples of η, so the Toda bracket contains a multiple of η. This implies that it contains zero, since multiples of η belong to the indeterminacy. □

Lemma **6.13.** — **(59, 31)** *The Toda bracket* $\langle \tau\overline{\kappa}_2, \sigma^2, 2\rangle$ *equals zero.*

Proof. — No elements of the bracket can be detected by $\tau^2 \Delta h_1 d_0 g$ by comparison to *tmf*.

Consider the shuffle

$$\langle \tau\overline{\kappa}_2, \sigma^2, 2\rangle\kappa = \tau\overline{\kappa}_2\langle \sigma^2, 2, \kappa\rangle.$$

The bracket $\langle \sigma^2, 2, \kappa\rangle$ is zero because it is contained in $\pi_{29,16} = 0$. On the other hand, $\{\tau M d_0\}\kappa$ is non-zero and detected by $\tau M d_0^2$. Therefore, no elements of $\langle \tau\overline{\kappa}_2, \sigma^2, 2\rangle$ can be can be detected by $\tau M d_0$. □

The third author presents the following Lemma 6.14, which is needed in the proof of Lemma 7.137.

Lemma **6.14.** — **(60)** *The classical Toda bracket* $\langle \eta\overline{\kappa}_2, 2\sigma, \sigma\rangle$ *equals zero.*

Proof. — Due to $d_3(e_1) = h_2^2 n$ and that $h_1 g_2 = h_3 e_1$, we have the following Massey product in the E_4-page

$$h_1 g_2 = \langle h_2, h_2 n, h_3\rangle$$

with zero indeterminacy. Since there are no crossing differentials, we conclude that $h_1 g_2$ detects a homotopy class in the Toda bracket $\langle \nu, \nu\{n\}, \sigma\rangle$. We claim that $\eta\overline{\kappa}_2$ is contained in this bracket. In fact, they might differ by classes detected in filtration 6 or higher: $h_0 h_5 d_0,\ h_0^2 h_5 d_0,\ w$. The first two detect σ-multiples so they are in the indeterminacy. The homotopy class $\{w\}$ is detected by tmf, but $\overline{\kappa}_2$ and $\{n\}$ are not, so w can be ruled out too.

Therefore, we have

$$\langle \eta\overline{\kappa}_2, 2\sigma, \sigma\rangle \subseteq \langle\langle \nu, \nu\{n\}, \sigma\rangle, 2\sigma, \sigma\rangle \supseteq \nu\langle \nu\{n\}, \sigma, 2\sigma, \sigma\rangle = 0.$$

Here $\langle \sigma, 2\sigma, \sigma\rangle = 0$, and $\langle \nu\{n\}, \sigma, 2\sigma\rangle$ contains 0 since coker J in π_{49} is 0. So the 4-fold bracket $\langle \nu\{n\}, \sigma, 2\sigma, \sigma\rangle$ in π_{57} is well-defined. By comparison with $\pi_{57}tmf$, we know it is 0.

We remain to show the indeterminacy of $\langle\langle \nu, \nu\{n\}, \sigma\rangle, 2\sigma, \sigma\rangle$ is 0. In fact,

- $\pi_{15} \cdot \langle \nu, \nu\{n\}, \sigma\rangle = \langle \pi_{15}, \nu, \nu\{n\}\rangle\sigma \subseteq \sigma\pi_{53} = 0$. (Lemma 2.3 in [62].)
- $\langle \nu \cdot \pi_{42}, 2\sigma, \sigma\rangle = \langle 0, 2\sigma, \sigma\rangle + \langle \rho_{15}\theta_4, 2\sigma, \sigma\rangle = 0$. (Lemmas 2.3 and 2.4 in [62].)
- $\langle \sigma \cdot \pi_{38}, 2\sigma, \sigma\rangle \supseteq \pi_{38} \cdot \langle \sigma, 2\sigma, \sigma\rangle = 0$.

This completes the proof. □

Lemma **6.15.** — **(60, 32)** *For every* α *that is detected by* $h_3^2 h_5$*, the Toda bracket* $\langle 2, \sigma^2, \alpha\rangle$ *contains zero. The indeterminacy is generated by* $2\tau\overline{\kappa}^3$*, which is detected by* $\tau^2 d_0^2 l$*.*

Proof. — Let α be detected by $h_3^2 h_5$. For degree reasons, the only elements that could detect $\sigma^2\alpha$ either support η extensions or are detected by *tmf*. Therefore, $\sigma^2\alpha$ is zero. Hence the bracket is defined.

By comparison to *tmf*, the bracket cannot be detected by $\tau^4 g^3$. Table 15 shows that $\tau^2 d_0^2 l$ is the target of a hidden 2 extension, so it detects an element in the indeterminacy. Since there are no other possibilities, the bracket must contain zero. □

Remark **6.16.** — This result is consistent with Table 23 of [30], which claims that the bracket $\langle 2, \sigma^2, \theta_{4.5}\rangle$ contains an element that is detected by B_3. The element B_3 is now known to be zero in the Adams E_∞-page, so this just means that the bracket contains an element detected in Adams filtration strictly higher than the filtration of B_3.

Lemma **6.17.** — *deg63, 34 The Toda bracket* $\langle \theta_4, \eta^2, \theta_4 \rangle$ *equals zero.*

Proof. — Theorem 6.1 says that there exists an element θ_4^* in $\pi_{61,32}$ such that $\langle \theta_4, \eta^2, \theta_4 \rangle$ contains $\eta^2\theta_4^*$. The group $\pi_{61,32}$ is zero, so θ_4^* must be zero, and the bracket must contain zero.

In order to compute the indeterminacy of $\langle \theta_4, \eta^2, \theta_4 \rangle$, we must consider the product of θ_4 with elements of $\pi_{33,18}$. There are several cases to consider.

First consider $\{\Delta h_1^2 h_3\}$. The product $\theta_4\{\Delta h_1^2 h_3\}$ is detected in Adams filtration at least 10, but there are no elements in sufficiently high filtration.

Next consider $\nu\theta_4$ detected by p. The product θ_4^2 is zero [62], so $\nu\theta_4^2$ is also zero.

Finally, consider $\eta\eta_5$ detected by $h_1^2 h_5$. Table 10 shows that $\langle \eta, 2, \theta_4 \rangle$ detects η_5. Shuffle to obtain

$$\eta\eta_5\theta_4 = \eta\langle \eta, 2, \theta_4 \rangle\theta_4 = \eta^2\langle 2, \theta_4, \theta_4 \rangle.$$

The bracket $\langle 2, \theta_4, \theta_4 \rangle$ is zero because it is contained in $\pi_{61,32} = 0$. □

Lemma **6.18.** — **(66, 36)** *The Toda bracket* $\langle \eta^2, \theta_4, \eta^2, \theta_4 \rangle$ *is detected by* $\Delta_1 h_3^2$.

Proof. — Table 3 shows that $\Delta_1 h_3^2$ equals $\langle h_1^2, h_4^2, h_1^2, h_4^2 \rangle$. Therefore, $\Delta_1 h_3^2$ detects $\langle \eta^2, \theta_4, \eta^2, \theta_4 \rangle$, if the Toda bracket is well-defined.

In order to show that the Toda bracket is well-defined, we need to know that the subbrackets $\langle \eta^2, \theta_4, \eta^2 \rangle$ and $\langle \theta_4, \eta^2, \theta_4 \rangle$ contain zero. These are handled by Lemmas 6.8 and 6.17. □

Lemma **6.19.** — **(67, 36)** *The Toda bracket* $\langle \tau\eta^2\overline{\kappa}, 8, \overline{\kappa}_2 \rangle$ *contains zero, and its indeterminacy is generated by multiples of* $\tau\eta^2\overline{\kappa}$.

Proof. — The bracket $\langle \tau\eta^2\overline{\kappa}, 8, \overline{\kappa}_2 \rangle$ contains $\tau\eta\overline{\kappa}\langle \eta, 2, 4\overline{\kappa}_2 \rangle$. Lemma 6.12 shows that this expression contains zero.

It remains to show that $\overline{\kappa}_2 \cdot \pi_{23,12}$ equals zero. There are several cases to consider.

First, the product $\tau\sigma\eta_4\overline{\kappa}_2$ in $\pi_{60,32}$ could only be detected by $\tau^4 g^3$ or $\tau^2 d_0^2 l$. Comparison to *tmf* rules out both possibilities. Therefore, $\tau\sigma\eta_4\overline{\kappa}_2$ is zero.

Second, the product $\overline{\kappa}\overline{\kappa}_2$ in $\pi_{64,35}$ must be detected in filtration at least 9, since $\tau g g_2$ equals zero, so it could only be detected by $h_1^2(\Delta e_1 + C_0)$. This implies that $\tau\nu\overline{\kappa}\overline{\kappa}_2$ is zero.

Third, we must consider the product $\rho_{23}\overline{\kappa}_2$. Table 10 shows that the Toda bracket $\langle \sigma, 16, 2\rho_{15} \rangle$ detects ρ_{23}. Then $\rho_{23}\overline{\kappa}_2$ is contained in

$$\langle \sigma, 16, 2\rho_{15} \rangle\overline{\kappa}_2 = \sigma\langle 16, 2\rho_{15}, \overline{\kappa}_2 \rangle.$$

The latter bracket is contained in $\pi_{60,32}$. As above, comparison to *tmf* shows that the expression is zero. □

Lemma **6.20.** — **(70, 37)** *The Toda bracket* $\langle \eta, \nu, \tau\theta_{4.5}\overline{\kappa} \rangle$ *is detected by* $\tau h_1 D_3'$.

Proof. — Table 6 shows that $d_3(\tau D_3')$ equals $\tau^2 M h_2 g$. The Moss Convergence Theorem 2.16 implies that $\tau h_1 D_3'$ detects the Toda bracket. □

Lemma **6.21.** — **(71, 37)** *There exists an element* α *in* $\pi_{66,34}$ *detected by* $\tau^2 h_2 C'$ *such that the Toda bracket* $\langle \eta, \nu, \alpha \rangle$ *is defined and detected by* $\tau h_1 p_1$.

Proof. — The differential $d_5(\tau p_1 + h_0^2 h_3 h_6) = \tau^2 h_2^2 C'$ and the Moss Convergence Theorem 2.16 establish that the Toda bracket is detected by $\tau h_1 p_1$, provided that the Toda bracket is well-defined.

Let α be an element of $\pi_{66,34}$ that is detected by $\tau^2 h_2 C'$. Then $\nu\alpha$ does not necessarily equal zero; it could be detected in higher filtration by $\tau^2 h_2 B_5 + h_2 D_2'$. Then we can adjust our choice of α by an element detected by $\tau^2 B_5 + D_2'$ to ensure that $\nu\alpha$ is zero. □

Lemma **6.22.** — **(72, 38)** *The Toda bracket* $\langle \sigma^2, 2, \{t\}, \tau\overline{\kappa} \rangle$ *is detected by* $h_4 Q_2 + h_3^2 D_2$.

Proof. — The subbracket $\langle \sigma^2, 2, \{t\} \rangle$ contains zero by comparison to $C\tau$, and its indeterminacy is generated by $\sigma^3\theta_4 = 4\sigma\overline{\kappa}_2$ detected by $\tau g n$. The subbracket $\langle 2, \{t\}, \tau\overline{\kappa} \rangle$ is strictly zero because it cannot be detected by $h_0 h_2 h_5 i$ by comparison to *tmf*. This shows that the desired four-fold Toda bracket is well-defined.

Consider the relation

$$\eta \langle \sigma^2, 2, \{t\}, \tau\overline{\kappa} \rangle \subseteq \langle \langle \eta, \sigma^2, 2 \rangle, \{t\}, \tau\overline{\kappa} \rangle.$$

Let α be any element of $\langle \eta, \sigma^2, 2 \rangle$. Table 10 shows that α is detected by $h_1 h_4$ and equals either η_4 or $\eta_4 + \eta\rho_{15}$. By inspection, the indeterminacy of $\langle \alpha, \{t\}, \tau\overline{\kappa} \rangle$ equals $\tau\overline{\kappa} \cdot \pi_{53,29}$, which is detected in Adams filtration at least 14. (In fact, the indeterminacy is non-zero, since it contains both $\tau\overline{\kappa} \cdot \{Mc_0\}$ detected by $\tau M d_0^2$ and also $\tau\overline{\kappa} \cdot \{\Delta h_1 d_0^2\}$ detected by $\tau^2 \Delta h_1 d_0 e_0^2$.)

Table 10 shows that $\langle \alpha, \{t\}, \tau\overline{\kappa} \rangle$ is detected by $h_1 h_4 Q_2$. Together with the partial analysis of the indeterminacy in the previous paragraph, this shows that $\langle \alpha, \{t\}, \tau\overline{\kappa} \rangle$ does not contain zero.

Then $\eta \langle \sigma^2, 2, \{t\}, \tau\overline{\kappa} \rangle$ also does not contain zero, and the only possibility is that $\langle \sigma^2, 2, \{t\}, \tau\overline{\kappa} \rangle$ is detected by $h_4 Q_2 + h_3^2 D_2$. □

Lemma **6.23.** — **(75, 40)** *The Toda bracket* $\langle \theta_4, \theta_4, \kappa \rangle$ *equals zero.*

Proof. — The Massey product $\langle h_4^2, h_4^2, d_0 \rangle$ equals zero, since

$$h_1^2 \langle h_4^2, h_4^2, d_0 \rangle = \langle h_1^2, h_4^2, h_4^2 \rangle d_0 = 0,$$

while $h_1^2 x_{75,7}$ is not zero. The Moss Convergence Theorem 2.16 then implies that $\langle \theta_4, \theta_4, \kappa \rangle$ is detected in Adams filtration at least 8.

The only element in sufficiently high filtration is $Ph_1^4 h_6$. However,

$$\eta^2 \langle \theta_4, \theta_4, \kappa \rangle = \langle \eta^2, \theta_4, \theta_4 \rangle \kappa = 0,$$

while $h_1^2 \cdot Ph_1^4 h_6$ is not zero. Then $\langle \theta_4, \theta_4, \kappa \rangle$ must contain zero because there are no remaining possibilities.

The indeterminacy can be computed by inspection, using that $\theta_4 \theta_{4.5}$ is zero by comparison to $C\tau$. □

Lemma **6.24.** — **(77, 40)** *The Toda bracket* $\langle \kappa, 2, \theta_5 \rangle$ *is detected by* $h_6 d_0$.

Proof. — The differential $d_3(h_0 h_4) = h_0 d_0$ implies that $\langle \kappa, 2, \theta_5 \rangle$ is detected by $h_0 h_4 \cdot h_5^2 = 0$ in filtration 4. In other words, the Toda bracket is detected in Adams filtration at least 5.

The element $h_1 h_6 d_0$ detects $\langle \eta\kappa, 2, \theta_5 \rangle$, using the Adams differential $d_2(h_6) = h_0 h_5^2$. This expression contains $\eta \langle \kappa, 2, \theta_5 \rangle$, which shows that $\langle \kappa, 2, \theta_5 \rangle$ is detected in filtration at most 5.

The only possibility is that the Toda bracket is detected by $h_6 d_0$. □

Lemma **6.25.** — **(79, 42)** *There exists an element* μ *in* $\pi_{77,41}$ *that is detected by* τm_1 *such that* $\eta\mu$ *is zero and* μ *is not divisible by* τ. *Moreover, the Toda bracket* $\langle \mu, \eta, 2 \rangle$ *contains zero or is detected by* $\tau^2 Me_0^2$, *and its indeterminacy is detected by* $h_0 h_2 x_{76,6}$.

Proof. — Let μ' be an element of $\pi_{77,42}$ that is detected by m_1. Then $\tau\mu'$ is detected by τm_1, and $\eta\mu'$ is detected by $h_1 m_1$. Table 14 shows that there is a hidden τ extension from $h_1 m_1$ to $M\Delta h_1^2 h_3$. Therefore, $\tau\eta\mu'$ is detected by $M\Delta h_1^2 h_3$.

Now let μ'' be an element of $\pi_{77,41}$ that is detected by $M\Delta h_1 h_3$. Then $\eta\mu''$ is also detected by $M\Delta h_1^2 h_3$. This shows that $\eta(\tau\mu' + \mu'')$ is zero because there are no possible detecting elements in higher filtration.

Choose μ to be $\tau\mu' + \mu''$. Note that μ'' is not divisible by τ because inclusion of the bottom cell of $C\tau$ takes $M\Delta h_1 h_3$ to a non-zero element. Therefore, μ is also not divisible by τ.

Now that μ is defined, it remains to study the Toda bracket. We begin with an analysis of its indeterminacy, which is generated by $\tau\eta^2 \cdot \mu$ and the multiples of 2 in $\pi_{79,42}$. The first expression is zero by the construction of μ. Let α be an element of $\pi_{79,42}$ that is detected by $h_2 x_{76,6}$, so 2α is detected by $h_0 h_2 x_{76,6}$. Tables 15 and 17 show that there are no hidden 2 extensions in the 79-stem with weight 42. Therefore, the indeterminacy is generated by 2α.

Inclusion of the bottom cell of $C\tau$ takes the bracket to $\langle M\Delta h_1 h_3, h_1, h_0 \rangle$. Machine-generated data [58] shows that this bracket equals $\{0, h_0 h_2 x_{76,6}\}$ in $C\tau$.

Let β be any element of $\langle \mu, \eta, 2\rangle$. It is possible that β maps to $h_0 h_2 x_{76,6}$ under inclusion of the bottom cell of $C\tau$. In that case, $\beta + 2\alpha$ also belongs to $\langle \mu, \eta, 2\rangle$ and must map to zero under inclusion of the bottom cell of $C\tau$.

In either case, the original Toda bracket contains an element that maps to zero under inclusion of the bottom cell of $C\tau$, and that element is therefore divisible by τ. By inspection, the only possible detecting elements are $\tau^2 Me_0^2$ and $\tau^3 \Delta h_1 e_0^2 g$. The latter option is ruled out by comparison to *mmf*. □

Lemma **6.26.** — **(80, 42)** *The Toda bracket* $\langle 2, \eta, \tau\eta\{h_1 x_{76,6}\}\rangle$ *is detected by* $\tau h_1 x_1$.

Proof. — Let α be an element of $\pi_{77,41}$ that is detected by $h_1 x_{76,6}$. First we must show that the Toda bracket is well-defined.

Note that 2α is zero because there are no 2 extensions in $\pi_{77,41}$ in sufficiently high Adams filtration. Now consider the shuffle

$$\tau\eta^2\alpha = \langle 2, \eta, 2\rangle\alpha = 2\langle \eta, 2, \alpha\rangle.$$

Table 10 shows that $\langle \eta, 2, \alpha\rangle$ is detected by $h_0 h_2 x_{76,6}$, but this element does not support a hidden 2 extension. This shows that $\tau\eta^2\alpha$ is zero and that the Toda bracket is well-defined.

Finally, use the Adams differential $d_4(h_0 e_2) = \tau h_1^3 x_{76,6}$ and the relation $h_0 \cdot h_0 e_2 = \tau h_1 x_1$ to compute the Toda bracket. □

Lemma **6.27.** — **(81, 43)** *There exists an element* α *in* $\pi_{79,42}$ *that is detected by* $h_2 x_{76,6}$ *such that* $\eta\alpha$ *is zero. Moreover, the Toda bracket* $\langle 2, \eta, \alpha\rangle$ *is zero, with no indeterminacy.*

Proof. — There is no hidden η extension on $h_2 x_{76,6}$ because the possible targets $\tau^3 d_0 e_0^2 l$ and $\tau^5 g^4$ are ruled out by comparison to *mmf*. Therefore, α exists.

The Massey product $\langle h_0, h_1, h_2 x_{76,6}\rangle$ has no indeterminacy by inspection. Consequently,

$$\langle h_0, h_1, h_2 x_{76,6}\rangle = \langle h_0, h_1, h_2\rangle x_{76,6} = 0.$$

The Moss Convergence Theorem 2.16 implies that the Toda bracket $\langle 2, \eta, \alpha\rangle$ is detected in filtration 9 or higher. The possible detecting elements are $Ph_1^2 h_6 c_0$ and $\Delta^2 h_1 d_1$. In either of these cases, the Toda bracket would be detected by inclusion of the bottom cell of $C\tau$, but the corresponding bracket is zero in $C\tau$.

The indeterminacy is generated by $\tau\eta^2\alpha$ and the multiples of 2 in $\pi_{81,43}$. The first expression is zero by the choice of α. Tables 15 and 17 show that there are no multiples of 2 in $\pi_{81,43}$. □

Lemma **6.28.** — **(84, 45)** *The Toda bracket* $\langle 2, \sigma^2, \{\tau h_2^2 C'\}\rangle$ *equals zero, with no indeterminacy.*

Proof. — Let α be an element of $\pi_{66,35}$ that is detected by $\tau h_2 C'$, so $\nu\alpha$ is the unique element that is detected by $\tau h_2^2 C'$. We consider the Toda bracket $\langle 2, \sigma^2, \nu\alpha \rangle$. By inspection, the indeterminacy is zero, so the bracket equals $\langle 2, \sigma^2, \nu \rangle \alpha$, which equals $\langle \alpha, 2, \sigma^2 \rangle \nu$.

Apply the Moss Convergence Theorem 2.16 with the Adams d_2 differential to see that the Toda bracket $\langle \alpha, 2, \sigma^2 \rangle$ is detected by 0 in Adams filtration 9, but it could be detected by a non-zero element in higher filtration. However, this shows that $\langle \alpha, 2, \sigma^2 \rangle \nu$ is zero by inspection. □

Lemma **6.29.** — **(84, 45)** *The Toda bracket* $\langle 2, \sigma^2, \{h_3(\Delta e_1 + C_0)\}\rangle$ *equals zero, with no indeterminacy.*

Proof. — Let β be an element of $\pi_{62,33}$ that is detected by $\Delta e_1 + C_0$, so $\sigma\beta$ is the unique element that is detected by $h_3(\Delta e_1 + C_0)$. We consider the Toda bracket $\langle 2, \sigma^2, \sigma\beta \rangle$. By inspection, the indeterminacy is zero, so the bracket equals $\langle 2, \sigma^2, \beta \rangle \sigma$.

Apply the Moss Convergence Theorem 2.16 with the Adams d_2 differential to see that the Toda bracket $\langle 2, \sigma^2, \beta \rangle$ is detected by 0 in Adams filtration 9, but it could be detected by a non-zero element in higher filtration. Then the only possible non-zero value for $\langle 2, \sigma^2, \beta \rangle \sigma$ is $\{M\Delta h_1 h_3\}\sigma$. Table 24 shows that $M\Delta h_1 h_3$ detects $\{\Delta h_1 h_3\}\theta_{4.5}$, so $\sigma\{M\Delta h_1 h_3\}$ equals $\sigma\{\Delta h_1 h_3\}\theta_{4.5}$, which equals zero. □

Lemma **6.30.** — **(86, 46)** *The Toda bracket* $\langle \tau\eta\overline{\kappa}^2, 2, 4\overline{\kappa}_2 \rangle$ *is detected by* $M\Delta h_0^2 e_0$.

Proof. — Table 3 shows that the Massey product $\langle \Delta h_0^2 e_0, h_0^2, h_0 g_2 \rangle$ equals the element $M\Delta h_0^2 e_0$. Now apply the Moss Convergence Theorem 2.16, using that Table 18 shows that $\Delta h_0^2 e_0$ detects $\tau\eta\overline{\kappa}^2$. □

Lemma **6.31.** — **(87, 46)** *There exists an element* α *in* $\pi_{67,36}$ *that is detected by* $h_0 Q_3 + h_0 n_1$ *such that* $h_1^2 c_3$ *detects the Toda bracket* $\langle \tau\alpha, \nu_4, \eta \rangle$.

Proof. — A consequence of the proof of Lemma 5.51 is that there exists α in $\pi_{67,36}$ that is detected by $h_0 Q_3 + h_0 n_1$ such that the product $\tau\nu_4\alpha$ is zero. Therefore, $h_1^2 c_3$ detects the Toda bracket $\langle \tau\alpha, \nu_4, \eta \rangle$ because of the Adams differential $d_4(h_1 c_3) = \tau h_0 h_2 h_4 Q_3$. □

7. Hidden extensions

In this section, we will discuss hidden extensions in the E_∞-page of the Adams spectral sequence. We methodically explore hidden extensions by τ, 2, η, and ν, and we study other miscellaneous hidden extensions that are relevant for specific purposes. For easy reference, the lemmas in this section are labelled with degrees that match the degrees given in the tables.

7.1. *Hidden τ extensions*

In order to study hidden τ extensions, we will use the long exact sequence

$$\cdots \longrightarrow \pi_{p,q+1} \xrightarrow{\tau} \pi_{p,q} \longrightarrow \pi_{p,q}C\tau \longrightarrow \pi_{p-1,q+1} \xrightarrow{\tau} \pi_{p-1,q} \longrightarrow \cdots \tag{7.1}$$

extensively. This sequence governs hidden τ extensions in the following sense. An element α in $\pi_{p,q}$ is divisible by τ if and only if it maps to zero in $\pi_{p,q}C\tau$, and an element α in $\pi_{p-1,q+1}$ supports a τ extension if and only if it is not in the image of $\pi_{p,q}C\tau$. Therefore, we need to study the maps $\pi_{*,*} \to \pi_{*,*}C\tau$ and $\pi_{*,*}C\tau \to \pi_{*-1,*+1}$ induced by inclusion of the bottom cell into $C\tau$ and by projection from $C\tau$ to the top cell.

The E_∞-pages of the Adams spectral sequences for $S^{0,0}$ and $C\tau$ give associated graded objects for the homotopy groups that are the sources and targets of these maps. Naturality of the Adams spectral sequence induces maps on associated graded objects.

These maps on associated graded objects often detect the values of the maps on homotopy groups. For example, the element h_0 in the Adams spectral sequence for the sphere is mapped to the element h_0 in the Adams spectral sequence for $C\tau$. In homotopy groups, this means that inclusion of the bottom cell into $C\tau$ takes the element 2 in $\pi_{0,0}$ to the element 2 in $\pi_{0,0}C\tau$.

On the other side, the element $\overline{h_1^4}$ in the Adams spectral sequence for $C\tau$ is mapped to the element h_1^4 in Adams spectral sequence for the sphere. In homotopy groups, this means that projection from $C\tau$ to the top cell takes the element $\{\overline{h_1^4}\}$ in $\pi_{5,3}C\tau$ to the element η^4 in $\pi_{4,4}$.

However, some values of the maps on homotopy groups can be hidden in the map of associated graded objects. This situation is rare in low stems but becomes more and more common in higher stems. The first such example occurs in the 30-stem. The element Δh_2^2 is a permanent cycle in the Adams spectral sequence for $C\tau$, so $\{\Delta h_2^2\}$ is an element in $\pi_{30,16}C\tau$. Now Δh_2^2 maps to zero in the E_∞-page of the Adams spectral sequence for the sphere, but $\{\Delta h_2^2\}$ does not map to zero in $\pi_{29,17}$. In fact $\{\Delta h_2^2\}$ maps to $\eta\kappa^2$, which is detected by $h_1 d_0^2$. This demonstrates that projection from $C\tau$ to the top cell has a hidden value.

We refer the reader to Section 2.1 for a precise discussion of these issues.

Theorem **7.1.**

(1) *Through the 90-stem, Table 12 lists all hidden values of inclusion of the bottom cell into* $C\tau$*, except that:*
  (a) *If $h_1 f_2$ does not survive but $\tau h_1 f_2$ does survive, then $\tau h_1 f_2$ maps to $h_0^3 c_3$.*
  (b) *If $h_1^2 f_2$ does not survive, then $\tau h_1^2 f_2$ maps to $\overline{\tau h_1^3 h_4 Q_3}$ or $\Delta^2 e_1 + \overline{\tau \Delta h_2 e_1 g}$.*
  (c) *$\tau h_1 x_{85,6}$ maps to $\overline{\tau h_1^3 h_4 Q_3}$ or $\Delta^2 e_1 + \overline{\tau \Delta h_2 e_1 g}$.*

(2) *Through the 90-stem, Table 13 lists all hidden values of projection from* $C\tau$ *to the top cell, except that:*

(a) *If* $h_1 f_2$ *does not survive, then* $h_1 f_2$ *maps to* $h_1^2 h_4 Q_3$ *or* $\Delta h_1 j_1$.
(b) $x_{85,6}$ *maps to* $h_1^2 h_4 Q_3$ *or* $\Delta h_1 j_1$.
(c) *If* $h_1^2 f_2$ *does not survive, then* $h_1^2 f_2$ *maps to* $h_1^3 h_4 Q_3$ *or* $\tau M h_0 g^2$.
(d) $h_1 x_{85,6}$ *maps to* $h_1^3 h_4 Q_3$ *or* $\tau M h_0 g^2$.
(e) *If* $h_1^2 f_2$ *survives, then* $\overline{\tau h_1^3 h_4 Q_3}$ *or* $\Delta^2 e_1 + \overline{\tau \Delta h_2 e_1 g}$ *maps to* $M\Delta h_1^2 d_0$.

Proof. — The values of inclusion of the bottom cell and projection to the top cell are almost entirely determined by inspection of Adams E_∞-pages. Taking into account the multiplicative structure, there are no other combinatorial possibilities. For example, consider the exact sequence

$$\pi_{30,16} \to \pi_{30,16} C\tau \to \pi_{29,17}.$$

In the Adams E_∞-page for $C\tau$, h_4^2 and Δh_2^2 are the only two elements in the 30-stem with weight 16. In the Adams E_∞-page for the sphere, h_4^2 is the only element in the 30-stem with weight 16, and $h_1 d_0^2$ is the only element in the 29-stem with weight 17. The only possibility is that h_4^2 maps to h_4^2 under inclusion of the bottom cell, and Δh_2^2 maps to $h_1 d_0^2$ under projection to the top cell.

One case, given below in Lemma 7.6, requires a more complicated argument. □

Remark **7.2.** — Through the 90-stem, inclusion of the bottom cell into $C\tau$ has only one hidden value with target indeterminacy. Namely, $h_2 c_1 A'$ is the hidden value of $\overline{h_1 g B_7}$, with target indeterminacy generated by Δj_1. Through the 90-stem, projection from $C\tau$ to the top cell has no hidden values with target indeterminacy.

Remark **7.3.** — Through the 90-stem, inclusion of the bottom cell into $C\tau$ has no crossing values. On the other hand, projection from $C\tau$ to the top cell does have crossing values in this range. These occurrences are described in the fourth column of Table 13. Each can be verified by direct inspection.

Theorem **7.4.** — *Through the 90-stem, Table 14 lists all hidden* τ *extensions in* $\mathbb{C}$*-motivic stable homotopy groups, except that:*

(1) *if* $M\Delta h_1 d_0$ *is not hit by a differential, then there is a hidden* τ *extension from* $\Delta h_1 j_1$ *to* $M\Delta h_1 d_0$.
(2) *if* $M\Delta h_1^2 d_0$ *is not hit by a differential, then there is a hidden* τ *extension from* $\tau M h_0 g^2$ *to* $M\Delta h_1^2 d_0$.

In this range, the only crossing extension is:

(1) *the hidden* τ *extension from* $h_1^2 h_6 c_0$ *to* $h_0 h_4 D_2$*, and the not hidden* τ *extension on* $\tau h_2^2 Q_3$.

Proof. — Almost all of these hidden τ extensions follow immediately from the values of the maps in the long exact sequence (7.1) given in Tables 12 and 13.

For example, consider the element $\mathrm{P}d_0$ in the Adams E_∞-page for the sphere, which belongs to the 22-stem with weight 12. Now $\pi_{22,12}\mathrm{C}\tau$ is zero because there are no elements in that degree in the Adams E_∞-page for $\mathrm{C}\tau$, so inclusion of the bottom cell takes $\{\mathrm{P}d_0\}$ to zero. Therefore, $\{\mathrm{P}d_0\}$ must be in the image of multiplication by τ. The only possibility is that there is a hidden τ extension from $c_0 d_0$ to $\mathrm{P}d_0$. □

Remark **7.5.** — If $\mathrm{M}\Delta h_1 d_0$ and $\mathrm{M}\Delta h_1^2 d_0$ are not hit by differentials, then a straightforward analysis of the sequence (7.1) shows that the possible τ extensions on $\Delta h_1 j_1$ and $\tau \mathrm{M} h_0 g^2$ must occur. Thus these uncertainties are entirely determined by corresponding uncertainties in values of the Adams differentials.

Lemma **7.6.**

(1) **(70, 10, 38)** *The element* $h_1 h_3(\Delta e_1 + \mathrm{C}_0) + \tau h_2 \mathrm{C}''$ *maps to* $\overline{h_1^4 c_0 \mathrm{Q}_2}$ *under inclusion of the bottom cell into* $\mathrm{C}\tau$.
(2) **(70, 8, 39)** *There is a hidden* τ *extension from* $d_1 e_1$ *to* $h_1 h_3(\Delta e_1 + \mathrm{C}_0)$.

Proof. — Consider the exact sequence $\pi_{70,38} \to \pi_{70,38}\mathrm{C}\tau \to \pi_{69,39}$. For combinatorial reasons, one of the following two possibilities must occur:

(a) the element $h_1 h_3(\Delta e_1 + \mathrm{C}_0) + \tau h_2 \mathrm{C}''$ maps to $\overline{h_1^4 c_0 \mathrm{Q}_2}$ under inclusion of the bottom cell into $\mathrm{C}\tau$, and there is a hidden τ extension from $d_1 e_1$ to $h_1 h_3(\Delta e_1 + \mathrm{C}_0)$.
(b) the element $h_1 h_3(\Delta e_1 + \mathrm{C}_0)$ maps to $\overline{h_1^4 c_0 \mathrm{Q}_2}$ under inclusion of the bottom cell into $\mathrm{C}\tau$, and there is a hidden τ extension from $d_1 e_1$ to $h_1 h_3(\Delta e_1 + \mathrm{C}_0) + \tau h_2 \mathrm{C}''$.

We will show that there cannot be a hidden τ extension from $d_1 e_1$ to $h_1 h_3(\Delta e_1 + \mathrm{C}_0) + \tau h_2 \mathrm{C}''$.

Lemma 7.154 shows that $\tau\nu\{d_1 e_1\}$ equals $\tau\eta\sigma\{k_1\}$. Since there is no hidden τ extension on $h_1 k_1$, there must exist an element α in $\{k_1\}$ such that $\tau\eta\alpha = 0$. Therefore, $\tau\nu\{d_1 e_1\}$ must be zero.

If there were a τ extension from $d_1 e_1$ to $h_1 h_3(\Delta e_1 + \mathrm{C}_0) + \tau h_2 \mathrm{C}''$, then $\tau\nu\{d_1 e_1\}$ would be detected by

$$h_2 \cdot (h_1 h_3(\Delta e_1 + \mathrm{C}_0) + \tau h_2 \mathrm{C}'') = \tau h_2^2 \mathrm{C}'',$$

and in particular would be non-zero. □

7.2. *Hidden* 2 *extensions*

Theorem **7.7.** — *Tables 15 and 16 list some hidden extensions by* 2.

Proof. — Many of the hidden extensions follow by comparison to $C\tau$. For example, there is a hidden 2 extension from $h_0 h_2 g$ to $h_1 c_0 d_0$ in the Adams spectral sequence for $C\tau$. Pulling back along inclusion of the bottom cell into $C\tau$, there must also be a hidden 2 extension from $h_0 h_2 g$ to $h_1 c_0 d_0$ in the Adams spectral sequence for the sphere. This type of argument is indicated by the notation $C\tau$ in the fourth column of Table 15.

Next, Table 14 shows a hidden τ extension from $h_1 c_0 d_0$ to $Ph_1 d_0$. Therefore, there is also a hidden 2 extension from $\tau h_0 h_2 g$ to $Ph_1 d_0$. This type of argument is indicated by the notation τ in the fourth column of Table 15.

Many cases require more complicated arguments. In stems up to approximately dimension 62, see [30, Section 4.2.2 and Tables 27–28] [61], and [62]. The higher-dimensional cases are handled in the following lemmas. □

Remark **7.8.** — Through the 90-stem, there are no crossing 2 extensions.

Remark **7.9.** — The hidden 2 extension from $h_0 h_3 g_2$ to τgn is proved in [61], which uses on the "$\mathbf{R}P^\infty$-method" to establish a hidden σ extension from $\tau h_3 d_1$ to $\Delta h_2 c_1$ and a hidden η extension from $\tau h_1 g_2$ to $\Delta h_2 c_1$. We now have easier proofs for these η and σ extensions, using the hidden τ extension from $h_1^2 g_2$ to $\Delta h_2 c_1$ given in Table 14, as well as the relation $h_3^2 d_1 = h_1^2 g_2$.

Remark **7.10.** — Comparison to synthetic homotopy gives additional information about some possible hidden 2 extensions, including:

(1) there is a hidden 2 extension from $h_0 h_5 i$ to $\tau^4 e_0^2 g$.
(2) there is no hidden 2 extension from $Px_{76,6}$ to $M\Delta h_1 d_0$.

See [15] and [16] for more details. We are grateful to John Rognes for pointing out a mistake in [30, Lemma 4.56 and Table 27] concerning the hidden 2 extension on $h_0 h_5 i$. Lemma 7.18 shows that the extension occurs but does not determine its target precisely.

Remark **7.11.** — The first correct proof of the relation $2\theta_5 = 0$ appeared in [62]. Earlier claims in [44] and [38] were based upon a mistaken understanding of the Toda bracket $\langle \sigma^2, 2, \theta_4 \rangle$. See [30, Table 23] for the correct value of this bracket.

Remark **7.12.** — If $M\Delta h_1^2 d_0$ is non-zero in the E_∞-page, then there is a hidden τ extension from $\tau M h_0 g^2$ to $M\Delta h_1^2 d_0$. This implies that there must be a hidden 2 extension from $\tau^2 M g^2$ to $M\Delta h_1^2 d_0$.

Remark **7.13.** — Table 15 shows that there is a hidden 2 extension from $x_{87,7}$ to $\tau^3 g Q_3$. This follows from data recently produced by Dexter Chua on the d_2 differentials in the Adams spectral sequence for the cofiber of 2.

Theorem **7.14.** — *Table 17 lists all unknown hidden 2 extensions, through the 90-stem.*

Proof. — Many possibilities are eliminated by comparison to $C\tau$, to *tmf*, or to *mmf*. For example, there cannot be a hidden 2 extension from $h_2^2 h_4$ to $\tau h_1 g$ by comparison to $C\tau$.

Many additional possibilities are eliminated by consideration of other parts of the multiplicative structure. For example, there cannot be a hidden 2 extension from $Ph_1 h_5$ to $\tau^3 g^2$ because $\tau^3 g^2$ supports an h_1 extension and 2η equals zero.

Several cases are a direct consequence of Proposition 7.16.

Some possibilities are eliminated by more complicated arguments. These cases are handled in the following lemmas. □

Remark **7.15.** — If $M\Delta h_1^2 d_0$ is not zero in the E_∞-page, then $M\Delta h_1 d_0$ supports an h_1 multiplication, and there cannot be a hidden 2 extension from $Px_{76,6}$ to $M\Delta h_1 d_0$.

Proposition **7.16.** — *Suppose that* 2α *and* $\tau\eta\alpha$ *are both zero. Then* $2\langle \alpha, 2, \theta_5 \rangle$ *is zero.*

Proof. — Consider the shuffle

$$2\langle \alpha, 2, \theta_5 \rangle = \langle 2, \alpha, 2 \rangle \theta_5.$$

Since $2\theta_5$ is zero, this expression has no indeterminacy. Corollary 6.2 implies that it equals $\tau\eta\alpha\theta_5$, which is zero by assumption. □

Remark **7.17.** — Proposition 7.16 eliminates possible hidden 2 extensions on several elements, including $h_2^2 h_6$, $h_0^3 h_3 h_6$, $h_3^2 h_6$, $h_6 c_1$, $h_2^2 h_4 h_6$, $h_0^5 h_6 i$, and $h_2^2 h_6 g$.

Lemma **7.18.** — **(54, 9, 28)** *There is a hidden* 2 *extension from* $h_0 h_5 i$ *to either* τMPh_1 *or to* $\tau^4 e_0^2 g$.

Proof. — Table 2 shows that $h_0 h_5 i$ maps to $\Delta^2 h_2^2$ in the homotopy of *tmf*. The element $\Delta^2 h_2^2$ supports a hidden 2 extension, so $h_0 h_5 i$ must support a hidden 2 extension as well. □

Lemma **7.19.**

(1) **(63, 6, 33)** *There is a hidden* 2 *extension from* $\tau h_1 H_1$ *to* $\tau h_1(\Delta e_1 + C_0)$.
(2) **(63, 7, 33)** *There is no hidden* 2 *extension on* $\tau X_2 + \tau C'$.
(3) **(70, 7, 37)** *There is a hidden* 2 *extension from* $\tau h_1 h_3 H_1$ *to* $\tau h_1 h_3(\Delta e_1 + C_0)$.

Proof. — Table 18 shows that there is an η extension from $\tau h_1 H_1$ to $h_3 Q_2$. Let α be any element of $\pi_{63,33}$ that is detected by $\tau h_1 H_1$. Then $\tau\eta^2\alpha$ is non-zero and detected by $\tau h_1 h_3 Q_2$. Note that $\tau h_1 h_3 Q_2$ cannot be the target of a hidden 2 extension because there are no possibilities.

If 2α were zero, then we would have the shuffling relation

$$\tau\eta^2\alpha = \langle 2, \eta, 2\rangle\alpha = 2\langle \eta, 2, \alpha\rangle.$$

But this would contradict the previous paragraph.

We now know that 2α must be non-zero for every possible choice of α. The only possibility is that there is a hidden 2 extension from $\tau h_1 \mathrm{H}_1$ to $\tau h_1(\Delta e_1 + \mathrm{C}_0)$, and that there is no hidden 2 extension on $\tau \mathrm{X}_2 + \tau \mathrm{C}'$. This establishes the first two parts.

The third part follows immediately from the first part by multiplication by h_3. □

Lemma **7.20.** — **(64, 2, 33)** *There is a hidden 2 extension from h_1h_6 to $\tau h_1^2 h_5^2$.*

Proof. — Table 10 shows that h_1h_6 detects $\langle \eta, 2, \theta_5\rangle$. Now shuffle to obtain

$$2\langle \eta, 2, \theta_5\rangle = \langle 2, \eta, 2\rangle\theta_5 = \tau\eta^2\theta_5.$$ □

Lemma **7.21.** — **(66, 6, 36)** *There is no hidden 2 extension on $\Delta_1 h_3^2$.*

Proof. — Table 10 shows that $\Delta_1 h_3^2$ detects the Toda bracket $\langle \eta^2, \theta_4, \eta^2, \theta_4\rangle$. We have

$$2\langle \eta^2, \theta_4, \eta^2, \theta_4\rangle \subseteq \langle\langle 2, \eta^2, \theta_4\rangle, \eta^2, \theta_4\rangle.$$

Table 10 shows that

$$\nu\theta_4 = \langle 2, \eta, \eta\theta_4\rangle = \langle 2, \eta^2, \theta_4\rangle,$$

so we must compute $\langle \nu\theta_4, \eta^2, \theta_4\rangle$.

This bracket contains $\nu\langle \theta_4, \eta^2, \theta_4\rangle$, which equals zero by Lemma 6.17. Therefore, we only need to compute the indeterminacy of $\langle \nu\theta_4, \eta^2, \theta_4\rangle$.

The only possible non-zero element in the indeterminacy is the product $\theta_4\{t\}$. Table 10 shows that $\{t\} = \langle \nu, \eta, \eta\theta_4\rangle$. Now

$$\theta_4\{t\} = \langle \nu, \eta, \eta\theta_4\rangle\theta_4 = \nu\langle \eta, \eta\theta_4, \theta_4\rangle.$$

This last expression is well-defined because θ_4^2 is zero [62], and it must be zero because $\pi_{63,34}$ consists entirely of multiples of η. □

Lemma **7.22.** — **(67, 6, 36)** *There is no hidden 2 extension on $h_0\mathrm{Q}_3 + h_2^2\mathrm{D}_3$.*

Proof. — By comparison to the homotopy of $C\tau$, there is no hidden extension with value $h_2^2\mathrm{A}'$. Table 18 shows that $\tau^2\Delta h_2^2 e_0 g$ supports a hidden η extension. Therefore, it cannot be the target of a 2 extension. □

Lemma **7.23.** — **(68, 7, 36)** *There is no hidden* 2 *extension on* $h_3 \mathrm{A}'$.

Proof. — Table 10 shows that $h_3\mathrm{A}'$ detects the Toda bracket $\langle \sigma, \kappa, \tau\eta\theta_{4.5} \rangle$. Shuffle to obtain

$$\langle \sigma, \kappa, \tau\eta\theta_{4.5} \rangle 2 = \sigma \langle \kappa, \tau\eta\theta_{4.5}, 2 \rangle.$$

The bracket $\langle \kappa, \tau\eta\theta_{4.5}, 2 \rangle$ is zero because it is contained in $\pi_{61,32} = 0$. □

Lemma **7.24.** — **(69, 4, 36)** *There is no hidden* 2 *extension on* p'.

Proof. — Table 24 shows that p' detects the product $\sigma\theta_5$, and $2\theta_5$ is already known to be zero [62]. □

Lemma **7.25.** — **(70, 9, 37)** *There is no hidden* 2 *extension on* $\tau h_1 \mathrm{D}'_3$.

Proof. — Table 10 shows that $\tau h_1 \mathrm{D}'_3$ detects the Toda bracket $\langle \eta, \nu, \tau\theta_{4.5}\overline{\kappa} \rangle$. Now shuffle to obtain

$$2\langle \eta, \nu, \tau\theta_{4.5}\overline{\kappa} \rangle = \langle 2, \eta, \nu \rangle \tau\theta_{4.5}\overline{\kappa},$$

which equals zero because $\langle 2, \eta, \nu \rangle$ is contained in $\pi_{5,3} = 0$. □

Lemma **7.26.** — **(71, 3, 37)** *There is no hidden* 2 *extension on* $h_1 h_3 h_6$.

Proof. — Table 10 shows that $h_1 h_6$ detects the Toda bracket $\langle \eta, 2, \theta_5 \rangle$. Let α be an element of this bracket. Then $h_1 h_3 h_6$ detects $\sigma\alpha$, and

$$2\sigma\alpha = 2\sigma\langle \eta, 2, \theta_5 \rangle = \sigma\langle 2, \eta, 2 \rangle \theta_5 = \tau\eta^2\sigma\theta_5.$$

Table 10 also shows that $h_6 c_0$ detects the Toda bracket $\langle \epsilon, 2, \theta_5 \rangle$. Let β be an element of this bracket. As in the proof of Lemma 7.27, we compute that 2β equals $\tau\eta\epsilon\theta_5$.

Now consider the element $\sigma\alpha + \beta$, which is also detected by $h_1 h_3 h_6$. Then

$$2(\sigma\alpha + \beta) = \tau\eta^2\sigma\theta_5 + \tau\eta\epsilon\theta_5 = \tau\nu^3\theta_5,$$

using Toda's relation $\eta^2\sigma + \nu^3 = \eta\epsilon$ [55].

Table 21 shows that there is a hidden ν extension from $h_2 h_5^2$ to $\tau h_1 \mathrm{Q}_3$. Therefore, $\tau h_1 \mathrm{Q}_3$ detects $\nu^2\theta_5$.

This does not yet imply that $\nu^3\theta_5$ is zero, because $\nu^2\theta_5 + \eta\{\tau \mathrm{Q}_3 + \tau n_1\}$ might be detected $h_3\mathrm{A}'$ or $\mathrm{P}h_2 h_5 j$ in higher filtration. However, $h_3\mathrm{A}'$ does not support a hidden ν extension by Lemma 7.110. Also, Table 2 shows that $\mathrm{P}h_2 h_5 j$ maps non-trivially to *tmf*, while $\nu^2\theta_5 + \eta\{\tau \mathrm{Q}_3 + \tau n_1\}$ maps to zero. This is enough to conclude that $\nu^3\theta_5$ is zero.

We have now shown that $2(\sigma\alpha + \beta)$ is zero in $\pi_{71,37}$. Since $h_1 h_3 h_6$ detects $\sigma\alpha + \beta$, it follows that $h_1 h_3 h_6$ does not support a hidden 2 extension. □

Lemma **7.27.** — **(71, 4, 37)** *There is a hidden 2 extension from $h_6 c_0$ to $\tau h_1^2 p'$.*

Proof. — Table 10 shows that $h_6 c_0$ detects the Toda bracket $\langle \epsilon, 2, \theta_5 \rangle$. Now shuffle to obtain

$$2\langle \epsilon, 2, \theta_5 \rangle = \langle 2, \epsilon, 2 \rangle \theta_5 = \tau \eta \epsilon \theta_5.$$

Finally, $\tau \eta \epsilon \theta_5$ is detected by $\tau h_1^2 p'$ because of the relation $h_1^2 p' = h_1 h_5^2 c_0$. □

Lemma **7.28.** — **(71, 5, 37)** *There is no hidden 2 extension on $\tau h_1 p_1$.*

Proof. — Lemma 6.21 shows that $\tau h_1 p_1$ detects $\langle \eta, \nu, \alpha \rangle$ for some α detected by $\tau^2 h_2 \mathrm{C}'$. Now shuffle to obtain

$$2\langle \eta, \nu, \alpha \rangle = \langle 2, \eta, \nu \rangle \alpha,$$

which is zero because $\langle 2, \eta, \nu \rangle$ is contained in $\pi_{5,3} = 0$. □

Lemma **7.29.** — **(71, 8, 39)** *There is a hidden 2 extension from $h_2^3 \mathrm{H}_1$ to $\tau \mathrm{M} h_2^2 g$.*

Proof. — Table 21 shows that there are hidden ν extensions from $h_2^3 \mathrm{H}_1$ to $h_3 \mathrm{C}''$, and from $\tau \mathrm{M} h_2^2 g$ to $\mathrm{M} h_1 d_0^2$. Table 15 shows that there is also a hidden 2 extension from $h_3 \mathrm{C}''$ to $\mathrm{M} h_1 d_0^2$. The only possibility is that there must also be a hidden 2 extension on $h_1^2 h_3 \mathrm{H}_1$. □

Lemma **7.30.** — **(72, 6, 37)** *There is no hidden 2 extension on $\mathrm{P} h_1 h_6$.*

Proof. — Table 10 shows that $\mathrm{P} h_1 h_6$ detects the Toda bracket $\langle \mu_9, 2, \theta_5 \rangle$. Shuffle to obtain

$$2\langle \mu_9, 2, \theta_5 \rangle = \langle 2, \mu_9, 2 \rangle \theta_5 = \tau \eta \mu_9 \theta_5.$$

Table 10 also shows that μ_9 is contained in the Toda bracket $\langle \eta, 2, 8\sigma \rangle$. Shuffle again to obtain

$$\tau \eta \mu_9 \theta_5 = \langle \eta, 2, 8\sigma \rangle \tau \eta \theta_5 = \tau \eta^2 \langle 2, 8\sigma, \theta_5 \rangle.$$

Table 10 shows that $h_0^3 h_3 h_6$ detects $\langle 2, 8\sigma, \theta_5 \rangle$.

By inspection, the product $\eta^2 \{h_0^3 h_3 h_6\}$ can only be detected by $\Delta^2 h_1 h_4 c_0$. However, this cannot occur by comparison to $C\tau$. Therefore, $\eta^2 \{h_0^3 h_3 h_6\}$, and also $\tau \eta^2 \{h_0^3 h_3 h_6\}$, must be zero. □

Lemma **7.31.** — **(72, 8, 38)** *There is no hidden 2 extension on $h_4 \mathrm{Q}_2 + h_3^2 \mathrm{D}_2$.*

Proof. — Table 10 shows that the element $h_4Q_2 + h_3^2D_2$ detects the Toda bracket $\langle\sigma^2, 2, \{t\}, \tau\overline{\kappa}\rangle$. Consider the relation

$$2\langle\sigma^2, 2, \{t\}, \tau\overline{\kappa}\rangle \subseteq \langle\langle 2, \sigma^2, 2\rangle, \{t\}, \tau\overline{\kappa}\rangle.$$

Corollary 6.2 shows that the Toda bracket $\langle 2, \sigma^2, 2\rangle$ contains zero since $\eta\sigma^2$ is zero. Therefore, it consists of even multiples of ρ_{15}; let $2k\rho_{15}$ be any such element in $\pi_{15,8}$.

The Toda bracket $\langle 2k\rho_{15}, \{t\}, \tau\overline{\kappa}\rangle$ contains $k\rho_{15}\langle 2, \{t\}, \tau\overline{\kappa}\rangle$, which equals zero as discussed in the proof of Lemma 6.22. Moreover, its indeterminacy is equal to $\tau\overline{\kappa}\cdot\pi_{52,28}$, which is detected in Adams filtration at least 12. This implies that $\langle 2k\rho_{15}, \{t\}, \tau\overline{\kappa}\rangle$ is detected in Adams filtration at least 12, and that the target of a hidden 2 extension on $h_4Q_2 + h_3^2D_2$ must have Adams filtration at least 12.

The remaining possible targets with Adams filtration at least 12 are eliminated by comparison to $C\tau$ or to *mmf*. □

Remark **7.32.** — The proof of Lemma 7.31 might be simplified by considering the shuffle

$$2\langle\sigma^2, 2, \{t\}, \tau\overline{\kappa}\rangle = \langle 2, \sigma^2, 2, \{t\}\rangle\tau\overline{\kappa}.$$

However, the latter four-fold bracket may not exist, since both three-fold subbrackets have indeterminacy. See [29] for a discussion of the analogous difficulty with Massey products.

Lemma **7.33.** — **(73, 7, 40)** *There is no hidden 2 extension on $h_2^2Q_3$.*

Proof. — The element $\tau h_2^2Q_3$ detects $\nu^2\{\tau Q_3 + \tau n_1\}$, so it cannot support a hidden 2 extension. This rules out all possible 2 extensions on $h_2^2Q_3$. □

Lemma **7.34.** — **(73, 8, 38)** *There is no hidden 2 extension on $h_0h_4D_2$.*

Proof. — Table 14 shows that there is a hidden τ extension from $h_1^2h_6c_0$ to $h_0h_4D_2$. Therefore, $h_0h_4D_2$ detects either $\tau\eta\epsilon\eta_6$ or $\tau\eta\epsilon\eta_6 + \nu^2\{\tau Q_3 + \tau n_1\}$, because of the presence of $\tau h_2^2Q_3$ in higher filtration. In either case, $h_0h_4D_2$ cannot support a hidden 2 extension. □

Lemma **7.35.** — **(74, 6, 39)** *There is a hidden 2 extension from $h_3(\tau Q_3 + \tau n_1)$ to $\tau x_{74,8}$.*

Proof. — Table 24 shows that $\tau x_{74,8}$ detects $\tau\overline{\kappa}_2\theta_4$. Table 10 shows that θ_4 equals the Toda bracket $\langle\sigma^2, 2, \sigma^2, 2\rangle$.

Now consider the shuffle

$$\tau\overline{\kappa}_2\theta_4 = \tau\overline{\kappa}_2\langle\sigma^2, 2, \sigma^2, 2\rangle = \langle\tau\overline{\kappa}_2, \sigma^2, 2, \sigma^2\rangle 2.$$

Lemma 6.13 shows that the latter bracket is well-defined. This implies that $\tau x_{74,8}$ is the target of a hidden 2 extension, and $h_3(\tau Q_3 + \tau n_1)$ is the only possible source. □

Lemma **7.36.** — **(77, 5, 40)** *There is no hidden* 2 *extension on* $h_6 d_0$.

Proof. — Table 10 shows that $h_6 d_0$ detects the Toda bracket $\langle \kappa, 2, \theta_5 \rangle$. Now shuffle to obtain

$$2\langle \kappa, 2, \theta_5 \rangle = \langle 2, \kappa, 2 \rangle \theta_5 = \tau \eta \kappa \theta_5.$$

Lemma 7.153 shows that this product equals either $\tau \eta \sigma^2 \theta_5$ or $\tau \eta \sigma^2 \theta_5 + \tau^3 \eta \kappa_1 \overline{\kappa}_2$. Both possibilities are zero because $\eta \sigma^2$ and $\tau \eta \kappa_1$ are zero. □

Lemma **7.37.** — **(78, 10, 42)** *There is a hidden* 2 *extension from* $e_0 \mathrm{A}'$ *to* $\mathrm{M}\Delta h_1^2 h_3$.

Proof. — Let α be an element of $\pi_{76,40}$ that is detected by $x_{76,9}$. Table 18 shows that there is a hidden η extension from $x_{76,9}$ to $\mathrm{M}\Delta h_1 h_3$, so $\tau \eta^2 \alpha$ is detected by $\tau \mathrm{M}\Delta h_1^2 h_3$. Now shuffle to obtain

$$\tau \eta^2 \alpha = \langle 2, \eta, 2 \rangle \alpha = 2\langle \eta, 2, \alpha \rangle.$$

This shows that $\tau \mathrm{M}\Delta h_1^2 h_3$ must be the target of a hidden 2 extension.

Moreover, the source of this hidden 2 extension must be in Adams filtration at least 10, since the Adams differential $d_2(\tau x_{77,8}) = h_0 x_{76,9}$ implies that $\langle \eta, 2, \alpha \rangle$ is detected by $h_1 x_{77,8} = 0$ in filtration 9. The only possible source is $e_0 \mathrm{A}'$. □

Lemma **7.38.** — **(79, 3, 41)** *There is no hidden* 2 *extension on* $h_1 h_4 h_6$.

Proof. — Table 10 shows that $h_1 h_4 h_6$ detects the Toda bracket $\langle \eta_4, 2, \theta_5 \rangle$. Now shuffle to obtain

$$2\langle \eta_4, 2, \theta_5 \rangle = \langle 2, \eta_4, 2 \rangle \theta_5,$$

which equals $\tau \eta \eta_4 \theta_5$ by Table 10. We will show that this product is zero.

There are several elements in the Adams E_∞-page that might detect $\eta_4 \theta_5$. The possibilities $h_1 h_6 d_0$ and $x_{78,9}$ are ruled out by comparison to $\mathrm{C}\tau$. The possibility $\tau e_0 \mathrm{A}'$ is ruled out because Table 15 shows that $e_0 \mathrm{A}'$ supports a hidden 2 extension.

Two possibilities remain. If $\eta_4 \theta_5$ is detected by $\tau \mathrm{M}\Delta h_1^2 h_3$, then $\tau \eta \eta_4 \theta_5$ must be zero because there are no elements in sufficiently high Adams filtration.

Finally, suppose that $\eta_4 \theta_5$ is detected by $\tau h_1^2 x_{76,6}$. Let α be an element of $\pi_{77,41}$ that is detected by $h_1 x_{76,6}$. If $\eta_4 \theta_5 + \tau \eta \alpha$ is not zero, then it is detected in higher filtration. It cannot be detected by $x_{78,9}$ by comparison to $\mathrm{C}\tau$, and it cannot be detected by $\tau e_0 \mathrm{A}'$ because of the hidden 2 extension on $e_0 \mathrm{A}'$. If it is detected by $\tau \mathrm{M}\Delta h_1^2 h_3$, then we may change the choice of α to ensure that $\eta_4 \theta_5 + \tau \eta \alpha$ is zero.

We have now shown that $\tau \eta \eta_4 \theta_5$ equals $\tau^2 \eta^2 \alpha$. Shuffle to obtain

$$\tau^2 \eta^2 \alpha = \tau \alpha \langle 2, \eta, 2 \rangle = \langle \alpha, 2, \eta \rangle 2\tau.$$

Here we are using that 2α is zero; the possible 2 extensions on $h_1x_{76,6}$ are easily eliminated by the presence of η extensions and by comparison to *mmf*.

Table 10 shows that $\langle\alpha, 2, \eta\rangle$ is detected by $h_0h_2x_{76,6}$, and Lemma 7.39 shows that this element does not support a hidden 2 extension. Therefore, $\langle\alpha, 2, \eta\rangle 2\tau$ is zero. □

Lemma **7.39.** — **(79, 8, 42)** *There is no hidden* 2 *extension on* $h_0h_2x_{76,6}$.

Proof. — Let α be an element of $\pi_{76,40}$ that is detected by h_4A. Then $\nu\alpha$ is detected by $h_0h_2x_{76,6}$, and 2α is detected by h_0h_4A. We will show that $2\nu\alpha$ is zero.

Table 24 shows that h_0h_4A also detects either $\sigma^2\theta_5$ or $\sigma^2\theta_5 + \tau^2\kappa_1\overline{\kappa}_2$. Then $2\alpha + \sigma^2\theta_5$ or $2\alpha + \sigma^2\theta_5 + \tau^2\kappa_1\overline{\kappa}_2$ could be detected in higher filtration. However, only $x_{76,9}$ could detect this error term, and inclusion of the bottom cell into Cτ rules it out.

We now know that $\sigma^2\theta_5 + 2\alpha$ is either zero or $\tau^2\kappa_1\overline{\kappa}_2$. Multiply by ν to conclude that $2\nu\alpha$ is either zero or $\tau^2\nu\kappa_1\overline{\kappa}_2$. As in the proof of Lemma 7.80, this last expression is also zero. □

Lemma **7.40.** — **(79, 8, 41)** *There is no hidden* 2 *extension on* Ph_6c_0.

Proof. — Table 24 shows that Ph_6c_0 detects the product $\rho_{15}\eta_6$. Table 10 shows that η_6 is contained in the Toda bracket $\langle\eta, 2, \theta_5\rangle$. Now shuffle to obtain

$$2\rho_{15}\eta_6 = 2\rho_{15}\langle\eta, 2, \theta_5\rangle = \rho_{15}\theta_5\langle 2, \eta, 2\rangle,$$

which equals $\tau\eta^2\rho_{15}\theta_5$ by Table 10.

Table 24 shows that $\rho_{15}\theta_5$ is detected by either $h_0x_{77,7}$ or τ^2m_1. First suppose that it is detected by $h_0x_{77,7}$. Table 15 shows that $h_0x_{77,7}$ is the target of a 2 extension. Then $\rho_{15}\theta_5$ equals 2α modulo higher filtration. In any case, $\tau\eta^2\rho_{15}\theta_5$ is zero.

Next suppose that $\rho_{15}\theta_5$ is detected by τ^2m_1. Then $\rho_{15}\theta_5$ equals $\tau^2\alpha$ modulo higher filtration for some element α detected by m_1. Table 14 shows that there is a hidden τ extension from h_1m_1 to M$\Delta h_1^2h_3$. This implies that $\tau\eta\alpha$ is detected by M$\Delta h_1^2h_3$. Finally, $\tau^3\eta^2\alpha = \tau\eta^2\rho_{15}\theta_5$ must be zero. □

Lemma **7.41.** — **(79, 11, 42)** *There is no hidden* 2 *extension on* ΔB$_6$.

Proof. — Table 18 shows that there is a hidden η extension from $h_0^6h_4h_6$ to $\tau\Delta$B$_6$. Therefore, $\tau\Delta$B$_6$ cannot be the source of a hidden 2 extension, so there cannot be a hidden 2 extension from ΔB$_6$ to τ^2Me_0^2. □

Lemma **7.42.** — **(82, 6, 44)** *There is no hidden* 2 *extension on* h_5^2g.

Proof. — The element τh_5^2g detects the product $\overline{\kappa}\theta_5$, so it cannot support a hidden 2 extension since $2\theta_5$ is zero.

If there were a hidden 2 extension from $h_5^2 g$ to $\tau(\Delta e_1 + C_0)g$, then the hidden τ extension from $\tau(\Delta e_1 + C_0)g$ to $\Delta^2 h_2 n$ would imply that there is a hidden 2 extension from $\tau h_5^2 g$ to $\Delta^2 h_2 n$. □

Lemma **7.43.** — **(82, 8, 44)** *There is no hidden* 2 *extension on* $h_2^2 x_{76,6}$.

Proof. — As in the proof of Lemma 6.27, let α be an element in $\pi_{79,42}$ that is detected by $h_2 x_{76,6}$ such that $\eta\alpha$ is zero. Then $\nu\alpha$ is detected by $h_2^2 x_{76,6}$, and we wish to show that $2\nu\alpha$ is zero.

Table 10 shows that 2ν is contained in $\langle \eta, 2, \eta \rangle$. Consider the shuffle

$$2\nu\alpha = \langle \eta, 2, \eta \rangle \alpha = \eta \langle 2, \eta, \alpha \rangle.$$

Table 10 shows that the last Toda bracket is zero. □

Lemma **7.44.** — **(83, 7, 44)** *There is no hidden* 2 *extension on* $h_0^2 h_6 g$.

Proof. — The element $h_0^2 h_6 g$ equals $h_2^2 h_6 d_0$, so it detects $\nu^2\{h_6 d_0\}$. □

Lemma **7.45.** — **(85, 7, 45)** *There is no hidden* 2 *extension on* $\tau h_2 h_4 Q_3$.

Proof. — There cannot be a hidden 2 extension from $\tau h_2 h_4 Q_3$ to $\tau P h_1 x_{76,6}$ because there is no hidden τ extension from $h_0 h_2 h_4 Q_3$ to $P h_1 x_{76,6}$.

Table 18 shows that $\tau^3 M g^2$ supports a hidden η extension. Therefore, it cannot be the target of a hidden 2 extension. □

Lemma **7.46.**

(1) **(85, 8, 45)** *There is no hidden* 2 *extension on* $h_6 c_0 d_0$.
(2) **(85, 9, 44)** *There is no hidden* 2 *extension on* $P h_6 d_0$.

Proof. — Table 18 shows that both elements are targets of hidden η extensions. □

Lemma **7.47.** — **(86, 5, 45)** *There is no hidden* 2 *extension on* $h_4 h_6 c_0$.

Proof. — Table 24 shows that $h_4 h_6 c_0$ detects the product $\sigma\{h_1 h_4 h_6\}$, and the element $h_1 h_4 h_6$ does not support a hidden 2 extension by Lemma 7.38. □

Lemma **7.48.** — **(86, 12, 47)** *There is no hidden* 2 *extension on* $\tau h_2 g C'$.

Proof. — The possible target $\tau^3 e_0^3 m$ is ruled out by comparison to *mmf*. The possible target $P h_1^7 h_6$ is ruled out by comparison to $C\tau$.

It remains to eliminate the possible target $\tau^2 \mathrm{M} h_1 g^2$. Table 14 shows that there are hidden τ extensions from $\tau h_2 g \mathrm{C}'$ and $\tau^2 \mathrm{M} h_1 g^2$ to $\Delta^2 h_2^2 d_1$ and $\mathrm{M}\Delta h_0^2 e_0$ respectively. However, there is no hidden 2 extension from $\Delta^2 h_2^2 d_1$ to $\mathrm{M}\Delta h_0^2 e_0$, so there cannot be a 2 extension from $\tau h_2 g \mathrm{C}'$ to $\tau^2 \mathrm{M} h_1 g^2$. □

Lemma **7.49.** — **(87, 5, 46)** *There is no hidden* 2 *extension on* $h_1^2 c_3$.

Proof. — Table 10 shows that the Toda bracket $\langle \tau\{h_0 \mathrm{Q}_3 + h_0 n_1\}, \nu_4, \eta \rangle$ is detected by $h_1^2 c_3$. Shuffle to obtain

$$\langle \tau\{h_0 \mathrm{Q}_3 + h_0 n_1\}, \nu_4, \eta \rangle 2 = \tau\{h_0 \mathrm{Q}_3 + h_0 n_1\} \langle \nu_4, \eta, 2 \rangle.$$

These expressions have no indeterminacy because $\tau\{h_0 \mathrm{Q}_3 + h_0 n_1\}$ does not support a 2 extension. Finally, the bracket $\langle \nu_4, \eta, 2 \rangle$ contains zero by comparison to *tmf*. □

Lemma **7.50.** — **(87, 12, 45)** *There is no hidden* 2 *extension on* $\mathrm{P}^2 h_6 c_0$.

Proof. — Table 24 shows that $\mathrm{P}^2 h_6 c_0$ detects the product $\rho_{23}\eta_6$. Table 10 shows that η_6 is contained in the Toda bracket $\langle \eta, 2, \theta_5 \rangle$. Shuffle to obtain

$$2\rho_{23}\eta_6 = 2\rho_{23}\langle \eta, 2, \theta_5 \rangle = \langle 2, \eta, 2 \rangle \rho_{23}\theta_5 = \tau\eta^2 \rho_{23}\theta_5.$$

The product $\rho_{23}\theta_5$ is detected in Adams filtration at least 13, and then $\tau\eta^2\rho_{23}\theta_5$ is detected in filtration at least 16. This rules out all possible targets for a hidden 2 extension on $\mathrm{P}^2 h_6 c_0$. □

Lemma **7.51.** — **(90, 12, 48)** *There is no hidden* 2 *extension on* M^2.

Proof. — Table 24 shows that M^2 detects $\theta_{4.5}^2$. Graded commutativity implies that $2\theta_{4.5}^2$ is zero. □

7.3. *Hidden η extensions*

Theorem **7.52.** — *Tables 18 and 19 list some hidden extensions by* η.

Proof. — Many of the hidden extensions follow by comparison to $\mathrm{C}\tau$. For example, there is a hidden η extension from $\tau h_1 g$ to $c_0 d_0$ in the Adams spectral sequence for $\mathrm{C}\tau$. Pulling back along inclusion of the bottom cell into $\mathrm{C}\tau$, there must also be a hidden η extension from $\tau h_1 g$ to $c_0 d_0$ in the Adams spectral sequence for the sphere. This type of argument is indicated by the notation $\mathrm{C}\tau$ in the fourth column of Table 18.

Next, Table 14 shows a hidden τ extension from $c_0 d_0$ to $\mathrm{P} d_0$. Therefore, there is also a hidden η extension from $\tau^2 h_1 g$ to $\mathrm{P} d_0$. This type of argument is indicated by the notation τ in the fourth column of Table 18.

The proofs of several of the extensions in Table 18 rely on analogous extensions in *mmf*. Extensions in *mmf* have not been rigorously analyzed [31]. However, the specific extensions from *mmf* that we need are easily deduced from extensions in *tmf*, together with the multiplicative structure. For example, there is a hidden η extension in *tmf* from *an* to τd_0^2. Therefore, there is a hidden η extension in *mmf* from *ang* to $\tau d_0^2 g$, and also a hidden η extension from $\Delta h_2^2 e_0$ to $\tau d_0 e_0^2$ in the homotopy groups of the sphere spectrum. Note that *mmf* really is required here, since *ang* and $d_0^2 g$ equal zero in the homotopy of *tmf*.

Many cases require more complicated arguments. In stems up to approximately dimension 62, see [30, Section 4.2.3 and Tables 29–30] and [61]. The higher-dimensional cases are handled in the following lemmas. □

Remark **7.53.** — The hidden η extension from τC to $\tau^2 gn$ is proved in [61], which uses on the "$\mathbf{R}\mathrm{P}^\infty$-method" to establish a hidden σ extension from $\tau h_3 d_1$ to $\Delta h_2 c_1$ and a hidden η extension from $\tau h_1 g_2$ to $\Delta h_2 c_1$. We now have easier proofs for these η and σ extensions, using the hidden τ extension from $h_1^2 g_2$ to $\Delta h_2 c_1$ given in Table 14, as well as the relation $h_3^2 d_1 = h_1^2 g_2$.

Remark **7.54.** — If $h_1 f_2$ survives, then there is a hidden τ extension from $\Delta h_1 j_1$ to $M\Delta h_1 d_0$. It follows that there must be a hidden η extension from $\tau \Delta j_1 + \tau^2 g C'$ to $M\Delta h_1 d_0$.

Remark **7.55.** — The last column of Table 18 indicates the crossing η extensions.

Theorem **7.56.** — *Table 20 lists all unknown hidden η extensions, through the* 90*-stem.*

Proof. — Many possible extensions can be eliminated by comparison to $C\tau$, to *tmf*, or to *mmf*. For example, there cannot be a hidden η extension from $\tau M d_0$ to $\tau^4 g^3$ because $\tau^4 g^3$ maps to a non-zero element in $\pi_{60} tmf$ that is not divisible by η.

Other possibilities are eliminated by consideration of other parts of the multiplicative structure. For example, there cannot be a hidden η extension whose target supports a multiplication by 2, since 2η equals zero.

Many cases are eliminated by more complicated arguments. These are handled in the following lemmas. □

Remark **7.57.** — There is no hidden η extension on $h_2 D_3$. The possible target τk_1 is eliminated by computer data recently produced by Dexter Chua on d_2 differentials the Adams spectral sequence for the cofiber of η.

Remark **7.58.** — There is a hidden τ extension from $\tau(\Delta e_1 + C_0)g$ to $\Delta^2 h_2 n$. The possible extension from $\tau g D_3$ to $\tau(\Delta e_1 + C_0)g$ occurs if and only if the possible extension from $\tau^2 g D_3$ to $\Delta^2 h_2 n$ occurs.

Remark **7.59.** — Computer data recently produced by Dexter Chua on d_2 differentials in the Adams spectral sequence for the cofiber of η shows that the classical element gQ_3 must be the target of a hidden η extension. Therefore, there is either a hidden η extension from $h_1^2 f_2$ to $\tau g Q_3$, or from $\tau h_1 x_{85,6}$ to $\tau^2 g Q_3$.

Lemma **7.60.** — **(58, 8, 30)** *There is no hidden η extension on $\tau h_1 Q_2$.*

Proof. — There cannot be a hidden η extension from $\tau h_1 Q_2$ to $\tau^2 \Delta h_1 d_0 g$ by comparison to *tmf*. It remains to show that there cannot be a hidden η extension from $\tau h_1 Q_2$ to $\tau M d_0$.

Note that $h_1 d_0 Q_2 = \tau^3 d_1 g^2$, so $\kappa\{h_1 Q_2\}$ is detected by $\tau^3 d_1 g^2$. Therefore, $\kappa\{h_1 Q_2\} + \tau\kappa_1\overline{\kappa}^2$ is detected in higher filtration. The only possibility is $\tau^3 e_0 g m$, but that cannot occur by comparison to *mmf*. Therefore, $\kappa\{h_1 Q_2\} + \tau\kappa_1\overline{\kappa}^2$ is zero.

Now $\tau\eta\kappa_1\overline{\kappa}^2$ is zero because $\tau\eta\kappa_1\overline{\kappa}$ cannot be detected by $\Delta h_1 d_0^2$ by comparison to *tmf*. Therefore, $\eta\kappa\{h_1 Q_2\}$ is zero, so $\tau\eta\kappa\{h_1 Q_2\}$ is also zero.

On the other hand, $\tau\kappa\{M d_0\}$ is non-zero because it is detected by $\tau M d_0^2$. Therefore $\tau\eta\{h_1 Q_2\}$ cannot be detected by $\tau M d_0$. □

Lemma **7.61.** — **(64, 4, 33)** *There is no hidden η extension on $\tau h_1^2 h_5^2$.*

Proof. — Table 15 shows that $\tau h_1^2 h_5^2$ is the target of a hidden 2 extension. □

Lemma **7.62.** — **(64, 8, 33)** *There is a hidden η extension from $\tau^2 h_1 X_2$ to $\tau^2 M h_0 g$.*

Proof. — Table 18 shows that there is a hidden η extension from $\tau h_1 X_2$ to $c_0 Q_2$. Since $c_0 Q_2$ does not support a hidden τ extension, there exists an element β in $\pi_{65,35}$ that is detected by $c_0 Q_2$ such that $\tau\beta = 0$.

Projection from $C\tau$ to the top cell takes $\overline{c_0 Q_2}$ and $P(A + A')$ to $c_0 Q_2$ and $\tau M h_0 h_2 g$ respectively. Since $h_2 \cdot \overline{c_0 Q_2} = P(A + A')$ in the Adams spectral sequence for $C\tau$, it follows that $\nu\beta$ is non-zero and detected by $\tau M h_0 h_2 g$.

Let α be an element of $\pi_{63,33}$ that is detected by $\tau X_2 + \tau C'$, and consider the sum $\eta^2\alpha + \beta$. Both terms are detected by $c_0 Q_2$, but the sum could be detected in higher filtration. In fact, the sum is non-zero because $\nu(\eta^2\alpha + \beta)$ is non-zero.

It follows that $\eta^2\alpha + \beta$ is detected by $\tau M h_0 g$, and that $\tau\eta^2\alpha$ is detected by $\tau^2 M h_0 g$. □

Lemma **7.63.** — **(66, 4, 34)** *There is no hidden η extension on $\tau h_1^3 h_6$.*

Proof. — The element $\tau\eta^2\eta_6$ is detected by $\tau h_1^3 h_6$. Table 10 shows that η_6 is contained in the Toda bracket $\langle \eta, 2, \theta_5 \rangle$ Now shuffle to obtain

$$\eta \cdot \tau\eta^2\eta_6 = 4\nu\eta_6 = 4\nu\langle \eta, 2, \theta_5 \rangle = 4\langle \nu, \eta, 2 \rangle \theta_5,$$

which equals zero because $2\theta_5$ is zero. □

Lemma **7.64.** — $(\mathbf{66}, \mathbf{6}, \mathbf{35})$ *There is no hidden* η *extension from* $\tau \Delta_1 h_3^2$ *to* $h_2^2 A'$.

Proof. — Table 21 shows that $h_2^2 A'$ supports a hidden ν extension, so it cannot be the target of a hidden η extension. □

Lemma **7.65.** — $(\mathbf{68}, \mathbf{6}, \mathbf{36})$ *There is no hidden* η *extension on* $\tau h_1 Q_3$.

Proof. — Table 21 shows that $\tau h_1 Q_3$ is the target of a hidden ν extension. Therefore, it cannot be the source of a hidden η extension. □

Lemma **7.66.** — $(\mathbf{68}, \mathbf{7}, \mathbf{36})$ *There is a hidden* η *extension from* $h_3 A'$ *to* $h_3(\Delta e_1 + C_0)$.

Proof. — Comparison to $C\tau$ shows that there is a hidden η extension from $h_3 A'$ to either $\tau h_2^2 C' + h_3(\Delta e_1 + C_0)$ or $h_3(\Delta e_1 + C_0)$. Table 21 shows that $\tau h_2^2 C' + h_3(\Delta e_1 + C_0)$ supports a hidden ν extension. Therefore, it cannot be the target of a hidden η extension. □

Lemma **7.67.** — $(\mathbf{69}, \mathbf{3}, \mathbf{36})$ *There is a hidden* η *extension from* $h_2^2 h_6$ *to* $\tau h_0 h_2 Q_3$.

Proof. — Table 3 gives the Massey product $h_0 h_2 = \langle h_1, h_0, h_1 \rangle$. Therefore,

$$\langle \tau h_1 Q_3, h_0, h_1 \rangle = \{\tau h_0 h_2 Q_3, \tau h_0 h_2 Q_3 + \tau h_1 h_3 H_1\}.$$

Table 21 shows that there is a hidden ν extension from $h_2 h_5^2$ to $\tau h_1 Q_3$, so $\nu^2 \theta_5$ is detected by $\tau h_1 Q_3$. Therefore, the Toda bracket $\langle \nu^2 \theta_5, 2, \eta \rangle$ is detected by $\tau h_0 h_2 Q_3$ or by $\tau h_0 h_2 Q_3 + \tau h_1 h_3 H_1$.

Now $\langle \nu^2 \theta_5, 2, \eta \rangle$ contains $\nu^2 \langle \theta_5, 2, \eta \rangle$. This expression equals $\nu \theta_5 \langle 2, \eta, \nu \rangle$, which equals zero because $\langle 2, \eta, \nu \rangle$ is contained in $\pi_{5,3} = 0$.

We now know that $\langle \nu^2 \theta_5, 2, \eta \rangle$ equals its own determinacy, so $\tau h_0 h_2 Q_3$ or $\tau h_0 h_2 Q_3 + \tau h_1 h_3 H_1$ detects a multiple of η. The only possibility is that there is a hidden η extension on $h_2^2 h_6$.

The target of this extension cannot be $\tau h_0 h_2 Q_3 + \tau h_1 h_3 H_1$ by comparison to $C\tau$. □

Lemma **7.68.** — $(\mathbf{70}, \mathbf{5}, \mathbf{36})$ *There is no hidden* η *extension on* $h_0^3 h_3 h_6$.

Proof. — There are several possible targets for a hidden η extension on $h_0^3 h_3 h_6$. The element $\tau \Delta^2 h_2 g$ is ruled out because it supports an h_2 extension. The element $\Delta^2 h_4 c_0$ is ruled out by comparison to $C\tau$. The elements $\tau h_3^2 Q_2$ and $\tau d_0 Q_2$ are ruled out because Table 18 shows that they are targets of hidden η extensions from $\tau^2 h_1 h_3 H_1$ and $\tau^2 h_1 D_3'$ respectively.

The only remaining possibility is $\tau^2 l_1$. This case is more complicated.

Table 10 shows that $h_0^3 h_3 h_6$ detects the Toda bracket $\langle 8\sigma, 2, \theta_5 \rangle$. Now shuffle to obtain

$$\eta \langle 8\sigma, 2, \theta_5 \rangle = \langle \eta, 8\sigma, 2 \rangle \theta_5.$$

Table 10 shows that $\langle \eta, 8\sigma, 2 \rangle$ contains μ_9 and has indeterminacy generated by $\tau\eta^2\sigma$ and $\tau\eta\epsilon$. Thus the expression $\langle \eta, 8\sigma, 2 \rangle \theta_5$ contains at most four elements.

The product $\mu_9\theta_5$ is detected in filtration at least 8, so it is not detected by $\tau^2 l_1$. The product $(\mu_9 + \tau\eta^2\sigma)\theta_5$ is detected by $\tau h_1^2 p'$ because Table 24 shows that there is a hidden σ extension from h_5^2 to p'. The product $(\mu_9 + \tau\eta\epsilon)\theta_5$ is also detected by $\tau h_1^2 p' = \tau h_1 h_5^2 c_0$. Finally, the product $(\mu_9 + +\tau\eta^2\sigma + \tau\eta\epsilon)\theta_5$ equals $(\mu_9 + \tau\nu^3)\theta_5$, which also must be detected in filtration at least 8. □

Lemma **7.69.** — **(70, 6, 38)** *There is no hidden η extension on $h_2 Q_3$.*

Proof. — There cannot be a hidden η extension on $\tau h_2 Q_3$ because it is a multiple of h_2. Therefore, the possible targets for an η extension on $h_2 Q_3$ must be annihilated by τ.

The element $h_1^3 h_3 H_1$ cannot be the target because Table 14 shows that it supports a hidden τ extension. The element $\tau M h_2^2 g$ cannot be the target because Table 21 shows that it supports a hidden ν extension to $M h_1 d_0^2$. □

Lemma **7.70.** — **(70, 7, 37)** *There is a hidden η extension from $\tau h_1 h_3 H_1$ to $h_3^2 Q_2$.*

Proof. — Table 18 shows that there is a hidden η extension from $\tau h_1 H_1$ to $h_3 Q_2$. Now multiply by h_3. □

Lemma **7.71.**

(1) **(70, 10, 38)** *There is no hidden η extension on $h_1 h_3(\Delta e_1 + C_0)$.*
(2) **(70, 10, 38)** *There is no hidden η extension on $\tau h_2 C'' + h_1 h_3(\Delta e_1 + C_0)$.*

Proof. — The element $\tau M h_2^2 g$ is the only possible target for such hidden η extensions. However, Table 21 shows that there is a hidden ν extension from $\tau M h_2^2 g$ to $M h_1 d_0^2$. □

Lemma **7.72.** — **(71, 6, 37)** *There is no hidden η extension on $\tau h_1^2 p'$.*

Proof. — The element $\tau h_1^2 p'$ detects $\tau\eta^2\sigma\theta_5$ because Table 24 shows that there is a hidden σ extension from h_5^2 to p'. Then $\tau\eta^3\sigma\theta_5$ is zero since $\tau\eta^3\sigma$ is zero. □

Lemma **7.73.** — **(71, 8, 39)** *There is no hidden η extension on $h_2^3 H_1$.*

Proof. — Table 24 shows that $\mathrm{M}d_0$ detects the product $\kappa\theta_{4.5}$. Then Table 10 shows that $h_2^3\mathrm{H}_1$ detects the Toda bracket $\langle \nu, \epsilon, \kappa\theta_{4.5}\rangle$. Now shuffle to obtain

$$\eta\langle \nu, \epsilon, \kappa\theta_{4.5}\rangle = \langle \eta, \nu, \epsilon\rangle\kappa\theta_{4.5},$$

which is zero because $\langle \eta, \nu, \epsilon\rangle$ is contained in $\pi_{13,8} = 0$. □

Lemma **7.74.** — **(72, 5, 37)** *There is a hidden η extension from $\tau h_1 h_6 c_0$ to $\tau^2 h_2^2 \mathrm{Q}_3$.*

Proof. — The hidden τ extension from $h_1^2 h_6 c_0$ to $h_0 d_0 \mathrm{D}_2$ implies that $\tau h_1 h_6 c_0$ must support a hidden η extension. However, this hidden τ extension crosses the τ extension from $\tau h_2^2 \mathrm{Q}_3$ to $\tau^2 h_2^2 \mathrm{Q}_3$. Therefore, the target of the hidden η extension is either $\tau^2 h_2^2 \mathrm{Q}_3$ or $h_0 d_0 \mathrm{D}_2$.

The element $\tau h_1 h_6 c_0$ detects the product $\tau\eta_6\epsilon$, so we want to compute $\tau\eta\eta_6\epsilon$. Table 10 shows that η_6 belongs to $\langle \theta_5, 2, \eta\rangle$. Shuffle to obtain

$$\tau\eta\eta_6\epsilon = \langle \theta_5, 2, \eta\rangle\tau\eta\epsilon = \theta_5\langle 2, \eta, \tau\eta\epsilon\rangle.$$

Table 10 shows that $\langle 2, \eta, \tau\eta\epsilon\rangle$ contains ζ_{11}. Finally, $\theta_5\zeta_{11}$ is detected by $\tau^2 h_2^2 \mathrm{Q}_3 = h_5^2 \cdot \mathrm{P}h_2$. □

Lemma **7.75.** — **(72, 7, 39)** *There is no hidden η extension on $h_1^3 p'$.*

Proof. — The element $h_1^3 p'$ does not support a hidden τ extension, while Table 14 shows that there is a hidden τ extension from $\tau h_2^2 \mathrm{C}''$ to $\Delta^2 h_1^2 h_4 c_0$. Therefore, there cannot be a hidden η extension from $h_1^3 p'$ to $\tau h_2^2 \mathrm{C}''$. □

Lemma **7.76.** — **(72, 11, 38)** *There is a hidden η extension from $h_0 d_0 \mathrm{D}_2$ to $\tau \mathrm{M} d_0^2$.*

Proof. — Table 10 shows that $h_0 d_0 \mathrm{D}_2$ detects the Toda bracket $\langle \tau\overline{\kappa}\theta_{4.5}, 2\nu, \nu\rangle$. Now shuffle to obtain

$$\langle \tau\overline{\kappa}\theta_{4.5}, 2\nu, \nu\rangle\eta = \tau\overline{\kappa}\theta_{4.5}\langle 2\nu, \nu, \eta\rangle.$$

Table 10 shows that the Toda bracket $\langle 2\nu, \nu, \eta\rangle$ contains ϵ. Finally, $\tau\overline{\kappa}\theta_{4.5}\epsilon$ is detected by $\tau \mathrm{M} d_0^2$ because Table 24 shows that there is a hidden ϵ extension from $\tau \mathrm{M} g$ to $\mathrm{M} d_0^2$. □

Lemma **7.77.** — **(75, 6, 40)** *There is a hidden η extension from $h_0 h_3 d_2$ to $\tau d_1 g_2$.*

Proof. — Table 10 shows that the Toda bracket $\langle \eta, \sigma^2, \eta, \sigma^2\rangle$ equals κ_1. We would like to consider the shuffle

$$\langle \eta, \sigma^2, \eta, \sigma^2\rangle\tau\overline{\kappa}_2 = \eta\langle \sigma^2, \eta, \sigma^2, \tau\overline{\kappa}_2\rangle,$$

but we must show that the Toda bracket $\langle \eta, \sigma^2, \tau\overline{\kappa}_2 \rangle$ is well-defined and contains zero. It is well-defined because $\sigma^2\overline{\kappa}_2$ is detected by $h_3^2 g_2$ in $\pi_{58,32}$, and there are no τ extensions on this group. The bracket contains zero by comparison to *tmf*, since all non-zero elements of $\pi_{60,32}$ are detected by *tmf*.

We have now shown that $\tau\kappa_1\overline{\kappa}_2$ is divisible by η. The only possibility is that there is a hidden η extension from $h_0 h_3 d_2$ to $\tau d_1 g_2$. □

Lemma **7.78.**

(1) **(77, 3, 40)** *There is no hidden η extension on $h_3^2 h_6$.*
(2) **(77, 7, 41)** *There is no hidden η extension on τm_1.*

Proof. — Table 15 shows that $e_0 A'$ and $\tau e_0 A'$ support hidden 2 extensions, so they cannot be the targets of hidden η extensions. □

Lemma **7.79.** — **(77, 8, 40)** *There is no hidden η extension on $h_0 x_{77,7}$.*

Proof. — Table 15 shows that $h_0 x_{77,7}$ is the target of a hidden 2 extension. □

Lemma **7.80.** — **(78, 6, 41)** *There is no hidden η extension on $h_1 h_6 d_0$.*

Proof. — Table 10 shows that $h_1 h_6$ detects the Toda bracket $\langle \theta_5, 2, \eta \rangle$, so $h_1 h_6 d_0$ detects $\langle \theta_5, 2, \eta \rangle \kappa$. Now shuffle to obtain

$$\langle \theta_5, 2, \eta \rangle \eta\kappa = \theta_5 \langle 2, \eta, \eta\kappa \rangle.$$

Table 10 shows that the Toda bracket $\langle 2, \eta, \eta\kappa \rangle$ equals $\nu\kappa$. Thus we need to compute the product $\nu\kappa\theta_5$. Lemma 7.153 shows that this product equals either $\nu\sigma^2\theta_5$ or $\nu(\sigma^2\theta_5 + \tau^2\kappa_1\overline{\kappa}_2)$. These expressions equal 0 and $\tau^2\nu\kappa_1\overline{\kappa}_2$ respectively since $\nu\sigma = 0$.

It remains to show that $\tau^2\nu\kappa_1\overline{\kappa}_2$ is zero. The proof of Lemma 7.77 shows that $\tau\kappa_1\overline{\kappa}_2$ is divisible by η. Therefore, $\tau^2\nu\kappa_1\overline{\kappa}_2$ is zero since $\eta\nu = 0$. □

Lemma **7.81.** — **(78, 8, 41)** *There is no hidden η extension on $\tau h_1^2 x_{76,6}$.*

Proof. — Let α be an element of $\pi_{77,41}$ that is detected by $h_1 x_{76,6}$. Then $\tau h_1^2 x_{76,6}$ detects $\tau\eta\alpha$. Now consider the shuffle

$$\tau\eta^2\alpha = \langle 2, \eta, 2 \rangle \alpha = 2\langle \eta, 2, \alpha \rangle.$$

Note that 2α is zero because there are no 2 extensions in $\pi_{77,41}$, so the second bracket is well-defined.

Finally, $2\langle \eta, 2, \alpha \rangle$ must be zero because there are no 2 extensions in $\pi_{79,42}$ in sufficiently high filtration. □

Lemma **7.82.** — $(\mathbf{78}, \mathbf{8}, \mathbf{40})$ *There is a hidden* η *extension from* $h_0^6 h_4 h_6$ *to* $\tau \Delta B_6$.

Proof. — In the homotopy of $C\tau$, there is a hidden η extension from $h_0^6 h_4 h_6$ to $\Delta^2 n$. Therefore, $h_0^6 h_4 h_6$ must support a hidden η extension whose target lies in Adams filtration 13 or lower. However, $\Delta^2 n$ is not the target because it supports an h_2 extension. The only remaining possible target is $\tau \Delta B_6$. □

Lemma **7.83.** — $(\mathbf{78}, \mathbf{10}, \mathbf{42})$ *There is a hidden* η *extension from* $e_0 A'$ *to* $\tau M e_0^2$.

Proof. — The classical relation $gC' = e_0 G_0$ implies the $\mathbf{C}$-motivic relation $\tau g \cdot C' = e_0 \cdot \tau G_0$ modulo the possible error term Δj_1. The error term does not appear because of h_1^2 extensions.

Table 3 shows that τG_0 equals $\langle A', h_1, h_2 \rangle$. Therefore, we have

$$\tau g C' = e_0 \langle A', h_1, h_2 \rangle = \langle e_0 A', h_1, h_2 \rangle.$$

The second equality holds because there is no indeterminacy by inspection.

Let α be an element of $\pi_{78,44}$ that is detected by $e_0 A'$. If the product $\eta\alpha$ were zero, then the Moss Convergence Theorem would imply that $\tau g C'$ is a permanent cycle that detects the Toda bracket $\langle \alpha, \eta, \nu \rangle$. However, $\tau g C'$ supports a d_4 differential and does not survive.

We now know that $e_0 A'$ supports a hidden η extension. After ruling out $\tau^2 \Delta h_1 e_0^2 g$ by comparison to *mmf*, the only remaining possible target is $\tau M e_0^2$. □

Lemma **7.84.** — $(\mathbf{81}, \mathbf{3}, \mathbf{42})$ *If* $h_2 h_4 h_6$ *supports a hidden* η *extension, then its target is not* $\tau h_2^2 x_{76,6}$.

Proof. — Table 21 shows that $\tau h_2^2 x_{76,6}$ supports a hidden ν extension, so it cannot be the target of a hidden η extension. □

Lemma **7.85.** — $(\mathbf{81}, \mathbf{5}, \mathbf{43})$ *There is no hidden* η *extension on* $h_1^3 h_4 h_6$.

Proof. — The element $\tau h_1^3 h_4 h_6$ is a multiple of h_0, so it cannot support a hidden η extension. This eliminates all possible targets except for $\tau(\Delta e_1 + C_0) g$.

However, $\tau(\Delta e_1 + C_0) g$ supports a hidden τ extension. As in the previous paragraph, this eliminates $\tau(\Delta e_1 + C_0) g$ as a possible target. □

Lemma **7.86.** — $(\mathbf{81}, \mathbf{7}, \mathbf{44})$ *There is no hidden* η *extension on* $h_3^2 n_1$.

Proof. — The element $\tau h_3^2 n_1 = h_3^2(\tau Q_3 + \tau n_1)$ detects $\sigma^2\{\tau Q_3 + \tau n_1\}$. Then $\eta\sigma^2\{\tau Q_3 + \tau n_1\}$ is zero because $\eta\sigma^2$ is zero. □

Lemma **7.87.** — **(81, 12, 42)** *There is no hidden* η *extension on* $\Delta^2 p$.

Proof. — Table 21 shows that $\Delta^2 p$ is the target of a hidden ν extension, so it cannot be the source of an η extension. □

Lemma **7.88.** — **(82, 4, 43)** *There is no hidden* η *extension on* $h_6 c_1$.

Proof. — Table 10 shows that $h_6 c_1$ detects the Toda bracket $\langle \overline{\sigma}, 2, \theta_5 \rangle$. By inspection, all possible indeterminacy is in higher Adams filtration, so $h_6 c_1$ detects every element of the Toda bracket.

Shuffle to obtain

$$\eta \langle \overline{\sigma}, 2, \theta_5 \rangle = \langle \eta, \overline{\sigma}, 2 \rangle \theta_5.$$

The Toda bracket $\langle \eta, \overline{\sigma}, 2 \rangle$ is detected in filtration at least 5 since the Massey product $\langle h_1, c_1, h_0 \rangle$ is zero. Therefore, the Toda bracket equals $\{0, \eta\overline{\kappa}\}$.

We now know that $\eta \langle \overline{\sigma}, 2, \theta_5 \rangle$ contains zero, and therefore $h_6 c_1$ does not support a hidden η extension. □

Lemma **7.89.** — **(83, 6, 44)** *There is no hidden* η *extension on* $h_0 h_6 g$.

Proof. — Table 10 shows that $h_0 h_6 g$ detects the Toda bracket $\langle \nu, \eta, \eta_6 \kappa \rangle$. Shuffle to obtain

$$\eta \langle \nu, \eta, \eta_6 \kappa \rangle = \langle \eta, \nu, \eta \rangle \eta_6 \kappa.$$

Table 10 shows that $\langle \eta, \nu, \eta \rangle$ equals ν^2. Finally,

$$\nu^2 \eta_6 \kappa = \nu^2 \kappa \langle \eta, 2, \theta_5 \rangle = \nu \theta_5 \kappa \langle \nu, \eta, 2 \rangle,$$

which equals zero because $\langle \nu, \eta, 2 \rangle$ is contained in $\pi_{5,3} = 0$. □

Lemma **7.90.** — **(85, 7, 46)** *There is no hidden* η *extension on* $h_2 h_4 Q_3$.

Proof. — We must eliminate $\tau h_2 g C'$ as a possible target. One might hope to use the homotopy of $C\tau$ in order to do this, but the homotopy of $C\tau$ has an η extension in the relevant degree that could possibly detect a hidden extension from $h_2 h_4 Q_3$ to $\tau h_2 g C'$.

If there were a hidden η extension from $h_2 h_4 Q_3$ to $\tau h_2 g C'$, then the hidden τ extension from $\tau h_2 g C'$ to $\Delta^2 h_2^2 d_1$ would imply that there is a hidden η extension from $\tau h_2 h_4 Q_3$ to $\Delta^2 h_2^2 d_1$. However, $\tau h_2 h_4 Q_3$ detects the product $\nu_4 \{\tau Q_3 + \tau n_1\}$, and $\eta \nu_4$ is zero. Therefore, $\tau h_2 h_4 Q_3$ cannot support a hidden η extension. □

Lemma **7.91.**

(1) **(86, 9, 46)** *There is no hidden η extension on $h_1 h_6 c_0 d_0$.*
(2) **(86, 10, 45)** *There is no hidden η extension on $\mathrm{P}h_1 h_6 d_0$.*

Proof. — Table 15 shows that $h_1 h_6 c_0 d_0$ and $\mathrm{P}h_1 h_6 d_0$ are targets of hidden 2 extensions, so they cannot be the sources of hidden η extensions. □

Lemma **7.92.** — **(86, 11, 44)** *There is a hidden η extension from $h_0^3 h_6 i$ to $\tau^2 \Delta^2 c_1 g$.*

Proof. — The Adams differential $d_2(\Delta^3 h_3^2) = \Delta^2 h_0^3 x$ implies that $\tau^2 \Delta^2 c_1 g = h_1 \cdot \Delta^3 h_3^2$ detects the Toda bracket $\langle \eta, 2, \{\Delta^2 h_0^2 x\} \rangle$. However, the later Adams differential $d_5(h_0^2 h_6 i) = \Delta^2 h_0^2 x$ implies that 0 belongs to $\{\Delta^2 h_0^2 x\}$. Therefore, $\tau^2 \Delta^2 c_1 g$ detects $\langle \eta, 2, 0 \rangle$, so $\tau^2 \Delta^2 c_1 g$ detects a multiple of η. The only possibility is that there is a hidden η extension from $h_0^3 h_6 i$ to $\tau^2 \Delta^2 c_1 g$. □

Lemma **7.93.** — **(87, 11, 48)** *There is no hidden η extension on $\mathrm{B}_6 d_1$.*

Proof. — Table 15 shows that $\mathrm{B}_6 d_1$ is the target of a hidden 2 extension, so it cannot be the source of a hidden η extension. □

Lemma **7.94.** — **(88, 7, 47)** *There is no hidden η extension on $h_1^2 h_4 h_6 c_0$.*

Proof. — Table 21 shows that $h_1^2 h_6 h_6 c_0$ is the target of a hidden ν extension, so it cannot support a hidden η extension. □

Lemma **7.95.** — **(89, 13, 47)** *There is a hidden η extension from $\Delta^2 h_1 f_1$ to the element $\tau \Delta^2 h_2 c_1 g$.*

Proof. — The element $\tau \Delta^2 h_2 c_1 g$ detects the product $\nu^2 \{\Delta^2 t\}$. Table 10 shows that ν^2 equals the Toda bracket $\langle \eta, \nu, \eta \rangle$. Shuffle to obtain

$$\langle \eta, \nu, \eta \rangle \{\Delta^2 t\} = \eta \langle \nu, \eta, \{\Delta^2 t\} \rangle.$$

This shows that $\tau \Delta^2 h_2 c_1 g$ is the target of a hidden η extension. The only possible source for this extension is $\Delta^2 h_1 f_1$. □

7.4. *Hidden ν extensions*

Theorem **7.96.** — *Tables 21 and 22 list some hidden extensions by ν.*

Proof. — Many of the hidden extensions follow by comparison to $C\tau$. For example, there is a hidden ν extension from $h_0^2 g$ to $h_1 c_0 d_0$ in the Adams spectral sequence for $C\tau$. Pulling back along inclusion of the bottom cell into $C\tau$, there must also be a hidden ν extension from $h_0^2 g$ to $h_1 c_0 d_0$ in the Adams spectral sequence for the sphere. This type of argument is indicated by the notation $C\tau$ in the fourth column of Table 18.

Next, Table 14 shows a hidden τ extension from $h_1 c_0 d_0$ to $Ph_1 d_0$. Therefore, there is also a hidden ν extension from $\tau h_0^2 g$ to $Ph_1 d_0$. This type of argument is indicated by the notation τ in the fourth column of Table 18.

Some extensions can be resolved by comparison to *tmf* or to *mmf*. For example, Table 2 shows that the classical unit map $S \to tmf$ takes $\{\Delta h_1 h_3\}$ in π_{32} to a non-zero element α of $\pi_{32} tmf$ such that $\nu\alpha = \eta\kappa\overline{\kappa}$ in $\pi_{35} tmf$. Therefore, there must be a hidden ν extension from $\Delta h_1 h_3$ to $\tau h_1 e_0^2$.

The proofs of several of the extensions in Table 21 rely on analogous extensions in *mmf*. Extensions in *mmf* have not been rigorously analyzed [31]. However, the specific extensions from *mmf* that we need are easily deduced from extensions in *tmf*, together with the multiplicative structure. For example, there is a hidden ν extension in *tmf* from Δh_1 to τd_0^2. Therefore, there is a hidden ν extension in *mmf* from $\Delta h_1 g$ to $\tau d_0^2 g$, and also a hidden ν extension from $\tau\Delta h_1 g$ to $\tau^2 d_0 e_0^2$ in the homotopy groups of the sphere spectrum. Note that *mmf* really is required here, since $d_0^2 g$ equals zero in the homotopy of *tmf*.

Many cases require more complicated arguments. In stems up to approximately dimension 62, see [30, Section 4.2.4 and Tables 31–32] and [61]. The higher-dimensional cases are handled in the following lemmas. □

Remark **7.97.** — The last column of Table 21 indicates which ν extensions are crossing, as well as which extensions have indeterminacy in the sense of Section 2.1.1.

Remark **7.98.** — The hidden ν extension from $h_2 h_5 d_0$ to τgn is proved in [61], which relies on the "$\mathbf{R}P^\infty$-method" to establish a hidden σ extension from $\tau h_3 d_1$ to $\Delta h_2 c_1$ and a hidden η extension from $\tau h_1 g_2$ to $\Delta h_2 c_1$. We now have easier proofs for these η and σ extensions, using the hidden τ extension from $h_1^2 g_2$ to $\Delta h_2 c_1$ given in Table 14, as well as the relation $h_3^2 d_1 = h_1^2 g_2$.

Remark **7.99.** — If $M\Delta h_1^2 d_0$ is not hit by a differential, then there is a hidden τ extension from $\tau M h_0 g^2$ from $M\Delta h_1^2 d_0$. This implies that there must be a hidden ν extension from $\tau(\Delta e_1 + C_0)g$ to $M\Delta h_1^2 d_0$.

Theorem **7.100.** — *Table 23 lists all unknown hidden ν extensions, through the 90-stem.*

Proof. — Many possible extensions can be eliminated by comparison to $C\tau$, to *tmf*, or to *mmf*. For example, there cannot be a hidden ν extension from $h_0 h_2 h_4$ to $\tau h_1 g$ by comparison to $C\tau$.

Other possibilities are eliminated by consideration of other parts of the multiplicative structure. For example, there cannot be a hidden ν extension whose target supports a multiplication by η, since $\eta\nu$ equals zero.

Many cases are eliminated by more complicated arguments. These are handled in the following lemmas. □

Remark **7.101.** — Comparison to synthetic homotopy eliminates several possible hidden ν extensions, including:

(1) from $\tau h_1 p_1$ to $\tau x_{74,8}$.
(2) from $\Delta^2 p$ to $\tau \mathrm{M}\Delta h_1 d_0$.

See [16] for more details.

Remark **7.102.** — If $\mathrm{M}\Delta h_1^2 d_0$ is not hit by a differential, then $\mathrm{M}\Delta h_1 d_0$ supports an h_1 extension, and there cannot be a hidden ν extension from $h_0 h_2 h_4 h_6$ to $\mathrm{M}\Delta h_1 d_0$.

Lemma **7.103.** — **(62, 8, 33)** *There is a hidden ν extension from $\Delta e_1 + \mathrm{C}_0$ to $\tau \mathrm{M} h_0 g$.*

Proof. — Table 10 shows that $2\overline{\kappa}$ is contained in $\tau\langle \nu, \eta, \eta\kappa\rangle$. Shuffle to obtain that

$$\nu\langle \eta, \eta\kappa, \tau\theta_{4.5}\rangle = \langle \nu, \eta, \eta\kappa\rangle \tau\theta_{4.5},$$

so $2\overline{\kappa}\theta_{4.5}$ is divisible by ν.

Table 24 shows that $\tau \mathrm{M} g$ detects $\overline{\kappa}\theta_{4.5}$, so $\tau \mathrm{M} h_0 g$ detects $2\overline{\kappa}\theta_{4.5}$. Now we know that there is a hidden ν extension whose target is $\tau \mathrm{M} h_0 g$, and the only possible source is $\Delta e_1 + \mathrm{C}_0$. □

Remark **7.104.** — One consequence of the proof of Lemma 7.103 is that $\Delta e_1 + \mathrm{C}_0$ detects the Toda bracket $\langle \eta, \eta\kappa, \tau\theta_{4.5}\rangle$.

Lemma **7.105.** — **(63, 6, 33)** *There is a hidden ν extension from $\tau h_1 \mathrm{H}_1$ to $\tau^2 \mathrm{M} h_1 g$.*

Proof. — Lemma 6.4 shows that the bracket $\langle \kappa, 2, \eta\rangle$ contains zero with indeterminacy generated by $\eta\rho_{15}$. The bracket $\langle \tau\eta\theta_{4.5}, \kappa, 2\rangle$ equals zero since $\pi_{61,32}$ is zero. Therefore, the Toda bracket $\langle \tau\eta\theta_{4.5}, \kappa, 2, \eta\rangle$ is well-defined.

Table 10 shows that τg detects $\langle \kappa, 2, \eta, \nu\rangle$. Therefore, $\tau^2 \mathrm{M} h_1 g$ detects

$$\tau\eta\theta_{4.5}\langle \kappa, 2, \eta, \nu\rangle = \langle \tau\eta\theta_{4.5}, \kappa, 2, \eta\rangle \nu.$$

This shows that $\tau^2 \mathrm{M} h_1 g$ is the target of a ν extension, and the only possible source is $\tau h_1 \mathrm{H}_1$. □

Remark **7.106.** — The proof of Lemma 7.105 shows that $\tau h_1 H_1$ detects the Toda bracket $\langle \tau\eta\theta_{4.5}, \kappa, 2, \eta\rangle$.

Lemma **7.107.** — **(64, 2, 33)** *There is no hidden ν extension on $h_1 h_6$.*

Proof. — Table 10 shows that $h_1 h_6$ detects the Toda bracket $\langle \eta, 2, \theta_5\rangle$. Shuffle to obtain

$$\nu\langle \eta, 2, \theta_5\rangle = \langle \nu, \eta, 2\rangle\theta_5 = 0,$$

since $\langle \nu, \eta, 2\rangle$ is contained in $\pi_{5,3} = 0$. □

Lemma **7.108.** — **(64, 8, 34)** *There is no hidden ν extension on $h_3 Q_2$.*

Proof. — Table 18 shows that $\tau^2 \Delta h_2^2 e_0 g$ supports a hidden η extension. Therefore, it cannot be the target of a ν extension. □

Lemma **7.109.** — **(67, 8, 36)** *There is a hidden ν extension from the element $h_2^2 A'$ to $h_1 h_3(\Delta e_1 + C_0)$.*

Proof. — By comparison to $C\tau$, There cannot be a hidden ν extension from $h_2^2 A'$ to $\tau h_2 C'' + h_1 h_3(\Delta e_1 + C_0)$

Table 10 shows that $\Delta e_1 + C_0$ detects the Toda bracket $\langle \eta, \eta\kappa, \tau\theta_{4.5}\rangle$, and $h_2 A'$ detects the Toda bracket $\langle \nu, \eta, \tau\kappa\theta_{4.5}\rangle$. Note that $h_2 A'$ also detects $\langle \nu, \eta\kappa, \tau\theta_{4.5}\rangle$.

Now shuffle to obtain

$$\begin{aligned}&(\eta\sigma + \epsilon)\langle \eta, \eta\kappa, \tau\theta_{4.5}\rangle + \nu^2\langle \nu, \eta\kappa, \tau\theta_{4.5}\rangle\\ &\quad = \left\langle \begin{bmatrix} \eta\sigma + \epsilon & \nu^2 \end{bmatrix}, \begin{bmatrix} \eta \\ \nu \end{bmatrix}, \eta\kappa \right\rangle \tau\theta_{4.5}.\end{aligned}$$

The matric Toda bracket $\left\langle \begin{bmatrix} \eta\sigma + \epsilon & \nu^2 \end{bmatrix}, \begin{bmatrix} \eta \\ \nu \end{bmatrix}, \eta\kappa \right\rangle$ must equal $\{0, \nu^2\overline{\sigma}\}$, since $\nu^2\overline{\sigma} = \{h_1^2 h_4 c_0\}$ is the only non-zero element of $\pi_{25,15}$, and that element belongs to the indeterminacy because it is a multiple of ν^2.

Next observe that $\tau\nu^2\overline{\sigma}\theta_{4.5}$ is zero because all possible values of $\overline{\sigma}\theta_{4.5}$ are multiples of η. This shows that

$$(\eta\sigma + \epsilon)\alpha + \nu^2\beta = 0,$$

for some α and β detected by $\Delta e_1 + C_0$ and $h_2 A'$ respectively. The product $(\eta\sigma + \epsilon)\alpha$ is detected by $h_1 h_3(\Delta e_1 + C_0)$, so there must be a hidden ν extension from $h_2^2 A'$ to $h_1 h_3(\Delta e_1 + C_0)$. □

Lemma **7.110.** — **(68, 7, 36)** *There is no hidden ν extension on $h_3 A'$.*

Proof. — Table 10 shows that $h_3 A'$ detects the Toda bracket $\langle \sigma, \kappa, \tau\eta\theta_{4.5} \rangle$. Now shuffle to obtain

$$\nu\langle \sigma, \kappa, \tau\eta\theta_{4.5} \rangle = \langle \nu, \sigma, \kappa \rangle \tau\eta\theta_{4.5} = \langle \eta, \nu, \sigma \rangle \tau\kappa\theta_{4.5}.$$

The Toda bracket $\langle \eta, \nu, \sigma \rangle$ is zero because it is contained in $\pi_{12,7} = 0$. □

Lemma **7.111.** — **(69, 4, 36)** *There is no hidden ν extension on p'.*

Proof. — Table 24 shows that p' detects the product $\sigma\theta_5$. Therefore, it cannot support a hidden ν extension. □

Lemma **7.112.** — **(69, 9, 38)** *There is a hidden ν extension from $h_2^2 C'$ to $\tau^2 d_1 g^2$.*

Proof. — Let α be an element of $\pi_{63,33}$ that is detected by $\tau X_2 + \tau C'$. Table 24 shows that $\epsilon\alpha$ is detected by $d_0 Q_2$, so $\eta\epsilon\alpha$ is detected by $\tau^3 d_1 g^2$. On the other hand, $\eta\sigma\alpha$ is zero by comparison to $C\tau$.

Now consider the relation $\eta^2\sigma + \nu^3 = \eta\epsilon$. This shows that $\nu^3\alpha$ is detected by $\tau^3 d_1 g^2$. Since $\nu^2\alpha$ is detected by $\tau h_2^2 C'$, there must be a hidden ν extension from $h_2^2 C'$ to $\tau^2 d_1 g^2$. □

Lemma **7.113.** — **(70, 9, 37)** *There is a hidden ν extension from $\tau h_1 D_3'$ to $\tau M d_0^2$.*

Proof. — Table 10 shows that $\tau h_1 D_3'$ detects the Toda bracket $\langle \eta, \nu, \tau\overline{\kappa}\theta_{4.5} \rangle$. Now shuffle to obtain

$$\nu\langle \eta, \nu, \tau\overline{\kappa}\theta_{4.5} \rangle = \langle \nu, \eta, \nu \rangle \tau\overline{\kappa}\theta_{4.5}.$$

The bracket $\langle \nu, \eta, \nu \rangle$ equals $\eta\sigma + \epsilon$ [55].

Now we must compute $(\eta\sigma + \epsilon)\tau\overline{\kappa}\theta_{4.5}$. The product $\sigma\overline{\kappa}$ is zero, and Table 24 shows that $\epsilon\overline{\kappa}\theta_{4.5}$ is detected by $M d_0^2$. These two observations imply that $(\eta\sigma + \epsilon)\tau\overline{\kappa}\theta_{4.5}$ is detected by $\tau M d_0^2$. □

Lemma **7.114.** — **(71, 4, 37)** *There is no hidden ν extension on $h_6 c_0$.*

Proof. — Table 10 shows that $h_6 c_0$ detects the Toda bracket $\langle \epsilon, 2, \theta_5 \rangle$. Now shuffle to obtain

$$\nu\langle \epsilon, 2, \theta_5 \rangle = \langle \nu, \epsilon, 2 \rangle \theta_5.$$

Finally, the Toda bracket $\langle \nu, \epsilon, 2 \rangle$ is zero because it is contained in $\pi_{12,7} = 0$. □

Lemma **7.115.** — **(73, 11, 41)** *There is a hidden ν extension from h_2^2C'' to $\tau g^2 t$.*

Proof. — Let α be an element of $\pi_{53,30}$ that is detected by i_1. Table 21 shows gt detects $\nu\alpha$. Therefore $\tau g^2 t$ detects $\nu\kappa\alpha$, so $\tau g^2 t$ must be the target of a hidden ν extension. The element h_2^2C'' is the only possible source for this extension. □

Lemma **7.116.** — **(73, 12, 41)** *There is no hidden ν extension on Mh_1h_3g.*

Proof. — If there were a hidden ν extension from Mh_1h_3g to $\tau g^2 t$, then there would also be a hidden ν extension with target $\tau^2 g^2 t$. But there is no possible source for such an extension. □

Lemma **7.117.** — **(75, 6, 40)** *If there is a hidden ν extension on $h_0h_3d_2$, then its target is $M\Delta h_1^2 h_3$.*

Proof. — The only other possible target is e_0A'. However, Table 15 shows that e_0A' supports a hidden 2 extension, while $h_0h_3d_2$ does not. □

Lemma **7.118.** — **(76, 8, 41)** *There is no hidden ν extension on $\tau d_1 g_2$.*

Proof. — Table 18 shows that $\tau d_1 g_2$ is the target of a hidden η extension. Therefore, it cannot be the source of a hidden ν extension. □

Lemma **7.119.** — **(76, 8, 40)** *There is no hidden ν extension on h_0h_4A.*

Proof. — Table 24 shows that h_0h_4A detects either $\sigma^2\theta_5$ or $\sigma^2\theta_5 + \tau^2\kappa_1\overline{\kappa}_2$. As in the proof of Lemma 7.80, both possibilities are annihilated by ν. □

Lemma **7.120.** — **(77, 3, 40)** *There is a hidden ν extension from $h_3^2h_6$ to $\tau h_1 x_1$.*

Proof. — Table 10 shows that $h_3^2h_6$ detects $\langle\theta_5, 2, \sigma^2\rangle$. Let α be an element of $\pi_{77,40}$ that is contained in this Toda bracket. Then $\nu\alpha$ is an element of

$$\langle\theta_5, 2, \sigma^2\rangle\nu = \langle\theta_5, 2\sigma, \sigma\rangle\nu = \theta_5\langle 2\sigma, \sigma, \nu\rangle \subseteq \langle 2\sigma, \sigma\theta_5, \nu\rangle.$$

Table 24 shows that p' detects $\sigma\theta_5$. Therefore, the Toda bracket $\langle 2\sigma, \sigma\theta_5, \nu\rangle$ is detected by an element of the Massey product $\langle h_0h_3, p', h_2\rangle$. Table 3 shows that h_0e_2 equals the Massey product $\langle h_3, p', h_2\rangle$. By inspection of indeterminacy, the Massey product $\langle h_0h_3, p', h_2\rangle$ contains $h_0^2e_2 = \tau h_1 x_1$ with indeterminacy generated by $h_0h_6e_0$.

We have now shown that $\nu\alpha$ is detected by either $\tau h_1 x_1$ or $\tau h_1 x_1 + h_0h_6e_0$. But $h_0h_6e_0 = h_2h_6d_0$ is a multiple of h_2, so we may add an element in higher Adams filtration to α, if necessary, to conclude that $\nu\alpha$ is detected by $\tau h_1 x_1$. □

Lemma **7.121.** — **(78, 9, 40)** *There is a hidden ν extension from the element $h_0^7 h_4 h_6$ to $\tau \Delta^2 h_1 d_1$.*

Proof. — Table 21 shows that there is a hidden ν extension from $h_0^6 h_4 h_6$ to $\Delta^2 p$. Therefore, there is also a hidden ν extension from $h_0^7 h_4 h_6$ to $h_0 \cdot \Delta^2 p = \tau \Delta^2 h_1 d_1$. □

Lemma **7.122.** — **(78, 10, 42)** *There is no hidden ν extension on $e_0 A'$.*

Proof. — A possible hidden ν extension from $e_0 A'$ to $\Delta h_1^2 B_6$ would be detected by $C\tau$, but we have to be careful with the analysis of the homotopy of $C\tau$ because of the h_2 extension from $\overline{\Delta h_1 d_1 g}$ to $\Delta h_1^2 B_6$ in the Adams E_∞-page for $C\tau$.

Let α be an element of $\pi_{75,40} C\tau$ that is detected by $\overline{h_3 C'}$. Then $\nu\alpha$ is detected by $e_0 A'$, and $\nu\alpha$ maps to zero under projection to the top cell because $h_3 C'$ does not support a ν extension in the homotopy of the sphere.

Therefore, $\nu\alpha$ lies in the image of $e_0 A'$ under inclusion of the bottom cell. Since $\nu^2\alpha$ is zero, $e_0 A'$ cannot support a hidden ν extension to $\Delta h_1^2 B_6$. □

Lemma **7.123.** — **(79, 3, 41)** *There is no hidden ν extension on $h_1 h_4 h_6$.*

Proof. — Table 10 shows that $h_1 h_4 h_6$ detects the Toda bracket $\langle \eta_4, 2, \theta_5 \rangle$. Shuffle to obtain

$$\nu \langle \eta_4, 2, \theta_5 \rangle = \langle \nu, \eta_4, 2 \rangle \theta_5.$$

Finally, $\langle \nu, \eta_4, 2 \rangle$ must contain zero in $\pi_{20,11}$ because *tmf* detects every element of $\pi_{20,11}$. □

Lemma **7.124.** — **(81, 7, 44)** *There is no hidden ν extension on $h_3^2 n_1$.*

Proof. — The element $h_2 g D_3$ cannot be the target of a hidden ν extension by comparison to $C\tau$.

The element $\tau h_3^2 n_1 = h_3 \cdot h_3(\tau Q_3 + \tau n_1)$ detects a multiple of σ, so it cannot support a hidden ν extension. This rules out $h_2 g A'$ as a possible target. □

Lemma **7.125.** — **(82, 8, 44)** *There is no hidden ν extension on $\tau e_1 g_2$.*

Proof. — After eliminating other possibilities by comparison to *tmf*, comparison to *mmf*, and by inspection of h_1 multiplications, the only possible target for a hidden ν extension is $P h_1 x_{76,6}$.

Let α be an element of $\pi_{82,45}$ that is detected by $e_1 g_2$. Then $\nu\alpha$ is detected by $h_2 e_1 g_2 = h_1^3 h_4 Q_3$. Choose an element β of $\pi_{83,45}$ that is detected by $h_1 h_4 Q_3$ such that $\tau\beta$ is zero. Then $\eta^2\beta$ is also detected by $h_1^3 h_4 Q_3$. However, $\nu\alpha + \eta^2\beta$ is not necessarily zero;

it could be detected in Adams filtration at least 13. In any case, $\tau\nu\alpha$ equals $\tau\eta^2\beta = 0$ modulo filtration 13. In particular, $\tau\nu\alpha$ cannot be detected by $\mathrm{P}h_1x_{76,6}$ in filtration 11. □

Lemma **7.126.** — **(82, 11, 42)** *There is no hidden ν extension on* $\mathrm{P}^2h_0h_2h_6$.

Proof. — Table 21 shows that there is a hidden ν extension from $\mathrm{P}^2h_2h_6$ to Δ^2h_0x. The target of a hidden ν extension on $\mathrm{P}^2h_0h_2h_6$ must have Adams filtration higher than the filtration of Δ^2h_0x. The only possibilities are ruled out by comparison to *tmf*. □

Lemma **7.127.** — **(82, 12, 45)** *There is a hidden ν extension from* $(\Delta e_1 + \mathrm{C}_0)g$ *to* $\tau\mathrm{M}h_0g^2$.

Proof. — Let α be an element of $\pi_{62,33}$ that is detected by $\Delta e_1 + \mathrm{C}_0$. Table 10 shows that $(\Delta e_1 + \mathrm{C}_0)g$ detects $\langle \alpha, \eta^3, \eta_4\rangle$. Then

$$\nu\langle \alpha, \eta^3, \eta_4\rangle = \langle \nu\alpha, \eta^3, \eta_4\rangle$$

by inspection of indeterminacies. Table 21 shows that $\tau\mathrm{M}h_0g$ detects $\nu\alpha$. The Toda bracket $\langle \nu\alpha, \eta^3, \eta_4\rangle$ is detected by the Massey product

$$\langle \tau\mathrm{M}h_0g, h_1^3, h_1h_4\rangle = \langle \tau\mathrm{M}h_0g, h_1^4, h_4\rangle = \mathrm{M}h_0g\langle \tau, h_1^4, h_4\rangle = \tau\mathrm{M}h_0g^2.$$ □

Lemma **7.128.** — **(83, 10, 45)** *There is no hidden ν extension on* $h_2c_1\mathrm{A}'$.

Proof. — Table 10 shows that $\tau h_2c_1\mathrm{A}'$ detects $\langle \tau\theta_{4.5}\kappa, \eta, \nu\rangle\tau\overline{\sigma}$. Shuffle to obtain

$$\langle \tau\theta_{4.5}\kappa, \eta, \nu\rangle\tau\overline{\sigma}\nu = \tau\theta_{4.5}\kappa\langle \eta, \nu, \tau\nu\overline{\sigma}\rangle.$$

The Toda bracket $\langle \eta, \nu, \tau\nu\overline{\sigma}\rangle$ is zero because $\pi_{27,15}$ contains only a v_1-periodic element detected by $\mathrm{P}^3h_1^3$.

We now know that $\tau h_2c_1\mathrm{A}'$ does not support a hidden ν extension. In particular, there cannot be a hidden ν extension from $\tau h_2c_1\mathrm{A}'$ to $\mathrm{M}\Delta h_0^2e_0$. The hidden τ extension from $\tau^2\mathrm{M}h_1g^2$ to $\mathrm{M}\Delta h_0^2e_0$ implies that there cannot be a hidden ν extension from $h_2c_1\mathrm{A}'$ to $\tau^2\mathrm{M}h_1g^2$.

Additional cases are ruled out by comparison to $\mathrm{C}\tau$ and to *mmf*. □

Lemma **7.129.**

(1) **(83, 11, 45)** *There is a hidden ν extension from* $\Delta j_1 + \tau g\mathrm{C}'$ *to* $\tau^2\mathrm{M}h_1g^2$.
(2) **(83, 11, 44)** *There is a hidden ν extension from* $\tau^2g\mathrm{C}'$ *to* $\mathrm{M}\Delta h_0^2e_0$.

Proof. — Table 21 shows that there exists an element α in $\pi_{63,33}$ detected by $\tau h_1 H_1$ such that ν is detected by $\tau^2 M h_1 g$. (Beware that there is a crossing extension here, so not every element detected by $\tau h_1 H_1$ has the desired property.) Table 24 shows that $\tau^2 M h_1 g$ also detects $\tau\theta_{4.5}\eta\overline{\kappa}$. However, $\nu\alpha$ does not necessarily equal $\tau\theta_{4.5}\eta\overline{\kappa}$ because the difference could be detected in higher filtration by $\Delta^2 h_1^3 h_4$. In any case, $\nu\overline{\kappa}\alpha$ equals $\tau\theta_{4.5}\eta\overline{\kappa}^2$.

The product $\theta_{4.5}\eta\overline{\kappa}^2$ is detected by $\tau^2 M h_1 g^2$. The hidden τ extension from $\tau^2 M h_1 g^2$ to $M\Delta h_0^2 e_0$ then implies that $\nu\overline{\kappa}\alpha = \tau\theta_{4.5}\eta\overline{\kappa}^2$ is detected by $M\Delta h_0^2 e_0$.

We now know that $M\Delta h_0^2 e_0$ is the target of a hidden ν extension. The only possible source is $\tau^2 g C'$. (Lemma 7.128 eliminates another possible source.) This establishes the second extension. The first extension follows from onsideration of τ extensions. □

Remark **7.130.** — The proof of Lemma 7.129 shows that $\nu\overline{\kappa}\alpha$ is detected by $M\Delta h_0^2 e_0$, where α is detected by $\tau h_1 H_1$. Note that $\overline{\kappa}\alpha$ is detected by $\tau^2 h_1 H_1 g = \tau h_2 c_1 A'$. But this does not show that $\tau h_2 c_1 A'$ supports a hidden ν extension. Rather, it shows that the source of the hidden ν extension is either $\tau h_2 c_1 A'$, or a non-zero element in higher filtration.

Lemma **7.131.** — **(84, 4, 44)** *There is no hidden ν extension on $h_2^2 h_4 h_6$.*

Proof. — Table 10 shows that $h_2^2 h_4 h_6$ detects the Toda bracket $\langle \nu\nu_4, 2, \theta_5\rangle$. Shuffle to obtain

$$\nu\langle \nu\nu_4, 2, \theta_5\rangle = \langle \nu, \nu\nu_4, 2\rangle\theta_5.$$

The Toda bracket $\langle \nu, \nu\nu_4, 2\rangle$ is zero because $\pi_{25,14}$ consists only of a v_1-periodic element detected by $P^2 h_1 c_0$. □

Lemma **7.132.** — **(85, 6, 44)** *If $\tau x_{85,6} + h_0^3 c_3$ survives, then it supports a hidden ν extension to $h_1 x_{87,7} + \tau^2 g_2^2$.*

Proof. — By comparison to $C\tau$, there must be a hidden ν extension whose target is either $h_1 x_{87,7}$ or $h_1 x_{87,7} + \tau^2 g_2^2$.

Table 21 shows that there is a hidden ν extension from $\tau^2 h_2 h_4 Q_3$ to $\tau^2 h_0 g_2^2$. This implies that the target of the ν extension on $\tau x_{85,6} + h_0^3 c_3$ must be $h_1 x_{87,7} + \tau^2 g_2^2$. □

Lemma **7.133.** — **(87, 12, 45)** *There is no hidden ν extension on $P^2 h_6 c_0$.*

Proof. — Table 24 shows that $P^2 h_6 c_0$ detects the product $\rho_{23}\eta_6$, and $\nu\rho_{23}\eta_6$ is zero. □

Lemma **7.134.** — **(87, 12, 48)** *There is a hidden ν extension from $h_2^2 g A'$ to $\Delta h_1^2 g_2 g$.*

Proof. — Comparison to $C\tau$ shows that $h_2^2 g A'$ supports a hidden ν extension whose target is either $\Delta h_1^2 g_2 g$ or $\Delta h_1^2 g_2 g + \tau h_2 g C''$.

Let α be an element of $\pi_{84,46}$ that is detected by $h_2 g A'$. Since $h_2 g A'$ does not support a hidden η extension, we may choose α such that $\eta\alpha$ is zero. Note that $h_2^2 g A'$ detects $\nu\alpha$.

Shuffle to obtain

$$\nu^2\alpha = \langle \eta, \nu, \eta \rangle \alpha = \eta \langle \nu, \eta, \alpha \rangle.$$

This shows that $\nu^2\alpha$ must be divisible by η. Consequently, the hidden ν extension on $h_2^2 g A'$ must have target $\Delta h_1^2 g_2 g$. □

Remark **7.135.** — The proof of Lemma 7.134 shows that $\Delta h_1 g_2 g$ detects the Toda bracket $\langle \nu, \eta, \{h_2 g A'\} \rangle$.

7.5. *Miscellaneous hidden extensions*

Theorem **7.136.** — *Tables 24 and 25 list some miscellaneous hidden extensions.*

Proof. — Similarly to Theorems 7.7, 7.52, and 7.96, some of the extensions follow by comparison to $C\tau$ or to *tmf*. The more difficult cases are handled in the following lemmas. □

Based on the corrected statement of Lemma 6.10 ([62, Theorem 2.1]) and Lemma 6.14, the third author presents the proof of the following Lemma 7.137 ([62, Theorem 1.2]), fixing a gap in its original proof. Note that as in the original proof, it only uses classical knowledge back then up to the 60-stem.

Lemma **7.137.** — **(30, 2)** *Classically, there is no hidden θ_4 extension on h_4^2. In other words θ_4^2 is zero in π_{60}.*

Proof. — We have

$$\theta_4^2 = \theta_4 \langle 2, \sigma^2 + \kappa, 2\sigma, \sigma \rangle \subseteq \langle \langle \theta_4, 2, \sigma^2 + \kappa \rangle, 2\sigma, \sigma \rangle.$$

Using [62, Theorem 2.2] and Lemma 6.10, the last expression is contained in the union of

$$\langle 0, 2\sigma, \sigma \rangle, \ \langle \eta\overline{\kappa}_2, 2\sigma, \sigma \rangle, \ \langle \rho_{15}\theta_4, 2\sigma, \sigma \rangle.$$

By [62, Lemmas 2.3 and 2.4] and Lemma 6.14, all three brackets contain a single element 0. Therefore, $\theta_4^2 = 0$. □

Lemma **7.138.**

(1) **(45, 3, 24)** *There is a hidden* ϵ *extension from* $h_3^2 h_5$ *to* $\mathrm{M}c_0$.
(2) **(45, 3, 23)** *There is a hidden* ϵ *extension from* $\tau h_3^2 h_5$ *to* MP.

Proof. — Table 18 shows that $\mathrm{M}h_1$ detects the product $\eta\theta_{4.5}$. Then $\mathrm{M}h_1 c_0$ detects $\eta\epsilon\theta_{4.5}$. This implies that $\mathrm{M}c_0$ detects $\epsilon\theta_{4.5}$.

This only shows that $\mathrm{M}c_0$ is the target of a hidden ϵ extension, whose source could be $h_3^2 h_5$ or $h_5 d_0$. However, Lemma 7.145 rules out the latter case. This establishes the first hidden extension.

Table 14 shows that there is a hidden τ extension from $\mathrm{M}c_0$ to MP. Then the first hidden extension implies the second one. □

Remark **7.139.** — We claimed in [30, Table 33] that there is a hidden ϵ extension from $h_3^2 h_5$ to $\mathrm{M}c_0$. However, the argument given in [30, Lemma 4.108] only implies that $\mathrm{M}c_0$ is the target of a hidden extension from either $h_3^2 h_5$ or $h_5 d_0$.

Lemma **7.140.** — **(45, 3, 24)** *There is a hidden* κ *extension from* $h_3^2 h_5$ *to* $\mathrm{M}d_0$.

Proof. — Table 18 shows that $\mathrm{M}h_1$ detects the product $\eta\theta_{4.5}$. Then $\mathrm{M}h_1 d_0$ detects the product $\eta\kappa\theta_{4.5}$, so $\mathrm{M}d_0$ must detect the product $\kappa\theta_{4.5}$. This shows that $\mathrm{M}d_0$ is the target of a hidden κ extension whose source is either $h_3^2 h_5$ or $h_5 d_0$.

We showed in Lemma 7.145 that $\epsilon\alpha$ is zero for some element α of $\pi_{45,24}$ that is detected by $h_5 d_0$. Then $\epsilon\overline{\kappa}\alpha$ is also zero. Table 24 shows that $\epsilon\overline{\kappa}$ equals κ^2. Therefore, $\kappa^2\alpha$ is zero. If $\kappa\alpha$ were detected by $\mathrm{M}d_0$, then $\kappa^2\alpha$ would be detected by $\mathrm{M}d_0^2$. It follows that there is no hidden κ extension from $h_5 d_0$ to $\mathrm{M}d_0$. □

Remark **7.141.** — We showed in [30, Table 33] that there is a hidden κ extension from either $h_3^2 h_5$ or $h_5 d_0$ to $\mathrm{M}d_0$. Lemma 7.140 settles this uncertainty.

Lemma **7.142.** — **(45, 3, 24)** *There is a hidden* $\overline{\kappa}$ *extension from* $h_3^2 h_5$ *to* $\tau\mathrm{M}g$.

Proof. — Table 18 shows that $\mathrm{M}h_1$ detects the product $\eta\theta_{4.5}$. Then $\tau\mathrm{M}h_1 g$ detects the product $\eta\overline{\kappa}\theta_{4.5}$, so $\tau\mathrm{M}g$ must detect the product $\overline{\kappa}\theta_{4.5}$. This shows that $\tau\mathrm{M}g$ is the target of a hidden $\overline{\kappa}$ extension whose source is either $h_3^2 h_5$ or $h_5 d_0$.

We showed in Lemma 7.145 that $\epsilon\alpha$ is zero for some element α of $\pi_{45,24}$ that is detected by $h_5 d_0$. If $\overline{\kappa}\alpha$ were detected by $\tau\mathrm{M}g$, then $\epsilon\overline{\kappa}\alpha$ would be detected by $\mathrm{M}d_0^2$ because Table 24 shows that there is a hidden ϵ extension from $\tau\mathrm{M}g$ to $\mathrm{M}d_0^2$. Therefore, there is no hidden $\overline{\kappa}$ extension from $h_5 d_0$ to $\tau\mathrm{M}g$. □

Lemma **7.143.** — **(45, 3, 24)** *There is a hidden* $\{\Delta h_1 h_3\}$ *extension from* $h_3^2 h_5$ *to* $\mathrm{M}\Delta h_1 h_3$.

Proof. — Table 18 shows that $\mathrm{M}h_1$ detects the product $\eta\theta_{4.5}$. Therefore, the element $\mathrm{M}\Delta h_1^2 h_3$ detects $\eta\{\Delta h_1 h_3\}\theta_{4.5}$. This shows that $\mathrm{M}\Delta h_1 h_3$ is the target of a hidden $\{\Delta h_1 h_3\}$ extension. Lemma 7.146 rules out $h_5 d_0$ as a possible source. The only remaining possible source is $h_3^2 h_5$. □

Lemma **7.144.** — **(45, 3, 24)** *There is a hidden $\theta_{4.5}$ extension from $h_3^2 h_5$ to* M^2.

Proof. — The proof of Lemma 5.31 shows that $\mathrm{M}^2 h_1$ detects a multiple of $\eta\theta_{4.5}$. Therefore, it detects either $\eta\theta_{4.5}^2$ or $\eta\theta_{4.5}\{h_5 d_0\}$.

Now $h_1 h_5 d_0$ detects $\eta\{h_5 d_0\}$, which also detects $\eta_4\theta_4$ by Table 24. In fact, the proof of [30, Lemma 4.112] shows that these two products are equal. Then $\eta\theta_{4.5}\{h_5 d_0\}$ equals $\eta_4\theta_4\theta_{4.5}$. Next, $\eta_4\theta_{4.5}$ lies in $\pi_{61,33}$. The only non-zero element of $\pi_{61,33}$ is detected by *mmf*, so the product $\eta_4\theta_{4.5}$ must be zero.

We have now shown that $\mathrm{M}^2 h_1$ detects $\eta\theta_{4.5}^2$. This implies that M^2 detects $\theta_{4.5}^2$. □

Lemma **7.145.** — **(45, 5, 24)** *There is no hidden ϵ extension on $h_5 d_0$.*

Proof. — Table 10 shows that $h_5 d_0$ detects the Toda bracket $\langle 2, \theta_4, \kappa\rangle$. Now shuffle to obtain

$$\epsilon\langle 2, \theta_4, \kappa\rangle = \langle \epsilon, 2, \theta_4\rangle\kappa.$$

Table 10 shows that $h_5 c_0$ detects the Toda bracket $\langle \epsilon, 2, \theta_4\rangle$, and there is no indeterminacy. Let α in $\pi_{39,21}$ be the unique element of this Toda bracket. We wish to compute $\alpha\kappa$.

Table 10 shows that $h_5 c_0$ also detects the Toda bracket $\langle \eta_5, \nu, 2\nu\rangle$, with indeterminacy generated by $\sigma\eta_5$. Let β in $\pi_{39,21}$ be an element of this Toda bracket. Then α and β are equal, modulo $\sigma\eta_5$ and modulo elements in higher filtration. Both $\tau h_3 d_1$ and $\tau^2 c_1 g$ detect multiples of σ. Also, the difference between α and β cannot be detected by $\Delta h_1 d_0$ by comparison to *tmf*.

This implies that α equals $\beta + \sigma\gamma$ for some element γ in $\pi_{31,17}$. Then

$$\alpha\kappa = (\beta + \sigma\gamma)\kappa = \beta\kappa$$

because $\sigma\kappa$ is zero.

Now shuffle to obtain

$$\beta\kappa = \langle \eta_5, \nu, 2\nu\rangle\kappa = \eta_5\langle \nu, 2\nu, \kappa\rangle.$$

Table 10 shows that $\langle \nu, 2\nu, \kappa\rangle$ contains $\eta\overline{\kappa}$, and its indeterminacy is generated by $\nu\nu_4$. We now need to compute $\eta_5\eta\overline{\kappa}$.

The product $\eta_5\overline{\kappa}$ is detected by $\tau h_1 h_3 g_2 = \tau h_1 h_5 g$, so $\eta_5\overline{\kappa}$ equals $\tau\eta\sigma\overline{\kappa}_2$, modulo elements of higher filtration. But these elements of higher filtration are either annihilated by η or detected by *tmf*, so $\eta_5\eta\overline{\kappa}$ equals $\tau\eta^2\sigma\overline{\kappa}_2$. By comparison to *tmf*, this latter expression must be zero. □

Lemma **7.146.** — **(45, 5, 24)** *There is no hidden* $\{\Delta h_1 h_3\}$ *extension on* $h_5 d_0$.

Proof. — Table 10 shows that $h_5 d_0$ detects the Toda bracket $\langle \kappa, \theta_4, 2\rangle$. By inspection of indeterminacies, we have

$$\{\Delta h_1 h_3\}\langle \kappa, \theta_4, 2\rangle = \langle \{\Delta h_1 h_3\}\kappa, \theta_4, 2\rangle.$$

Table 24 shows that $\tau d_0 l + \Delta c_0 d_0$ detects the product $\{\Delta h_1 h_3\}\kappa$. Now apply the Moss Convergence Theorem 2.16 with the Adams differential $d_2(h_5) = h_0 h_4^2$ to determine that the Toda bracket $\langle \{\Delta h_1 h_3\}\kappa, \theta_4, 2\rangle$ is detected in Adams filtration at least 13.

The only element in sufficiently high filtration is $\tau^5 e_0 g^3$, but comparison to *mmf* rules this out. Thus the Toda bracket $\langle \{\Delta h_1 h_3\}\kappa, \theta_4, 2\rangle$ contains zero. □

Lemma **7.147.** — **(62, 2, 32)** *There is a hidden* ρ_{15} *extension from* h_5^2 *to either* $h_0 x_{77,7}$ *or* $\tau^2 m_1$.

Proof. — Table 10 shows that the Toda bracket $\langle 8, 2\sigma, \sigma\rangle$ contains ρ_{15}. Then ρ_{15} is also contained in $\langle 2, 8\sigma, \sigma\rangle$, although the indeterminacy increases.

Now shuffle to obtain

$$\rho_{15}\theta_5 = \theta_5\langle 2, 8\sigma, \sigma\rangle = \langle \theta_5, 2, 8\sigma\rangle\sigma.$$

Table 10 shows that $h_0^3 h_3 h_6$ detects $\langle \theta_5, 2, 8\sigma\rangle$. Also, there is a σ extension from $h_0^3 h_3 h_6$ to $h_0 x_{77,7}$ in the homotopy of $C\tau$.

This implies that $\rho_{15}\theta_5$ is non-zero in $\pi_{77,40}$, and that it is detected in filtration at most 8. Moreover, it is detected in filtration at least 7, since ρ_{15} and θ_5 are detected in filtrations 4 and 2 respectively.

There are several elements in filtration 7 that could detect $\rho_{15}\theta_5$. The element $x_{77,7}$ (if it survives to the E_∞-page) is ruled out by comparison to $C\tau$. The element $\tau h_1 x_{76,6}$ is ruled out because $\eta\rho_{15}\theta_5$ is detected in filtration at least 10, since $\eta\rho_{15}$ is detected in filtration 7.

The only remaining possibilities are $h_0 x_{77,7}$ and $\tau^2 m_1$. □

Lemma **7.148.**

(1) **(64, 2, 33)** *There is a hidden* ρ_{15} *extension from* $h_1 h_6$ *to* $Ph_6 c_0$.
(2) **(64, 2, 33)** *There is a hidden* ρ_{23} *extension from* $h_1 h_6$ *to* $P^2 h_6 c_0$.

Proof. — Table 10 shows that $h_1 h_6$ detects $\langle \eta, 2, \theta_5\rangle$. Then

$$\rho_{15}\langle \eta, 2, \theta_5\rangle \subseteq \langle \eta\rho_{15}, 2, \theta_5\rangle.$$

The last bracket is detected by $Ph_6 c_0$ because $d_2(h_6) = h_0 h_5^2$ and because Pc_0 detects $\eta\rho_{15}$. Also, its indeterminacy is in Adams filtration higher than 8. This establishes the first hidden extension.

The proof for the second extension is essentially the same, using that $P^2 c_0$ detects $\eta\rho_{23}$ and that the indeterminacy of $\langle \eta\rho_{23}, 2, \theta_5\rangle$ is in Adams filtration higher than 12. ☐

Lemma **7.149.** — **(65, 10, 35)** *There is a hidden* ϵ *extension from* τMg *to* Md_0^2.

Proof. — First, we have the relation $c_0 \cdot h_1^2 X_2 = Mh_1h_3g$ in the Adams E_2-page, which is detected in the homotopy of $C\tau$. Table 14 shows that there are hidden τ extensions from $h_1^2X_2$ and Mh_1h_3g to τMg and Md_0^2 respectively. ☐

Lemma **7.150.** — **(77, 12, 41)** *If* $M\Delta h_1^2 d_0$ *is non-zero in the* E_∞*-page, then there is a hidden* ϵ *extension from* $M\Delta h_1 h_3$ *to* $M\Delta h_1^2 d_0$.

Proof. — Table 18 shows that Mh_1 detects the product $\eta\theta_{4.5}$. Since $M\Delta h_1^2 d_0$ equals $\Delta h_1 d_0 \cdot Mh_1$, it detects $\eta\{\Delta h_1 d_0\}\theta_{4.5}$.

Table 24 shows that $\eta\{\Delta h_1 d_0\}$ equals $\epsilon\{\Delta h_1 h_3\}$, since they are both detected by $\Delta h_1^2 d_0$ and there are no elements in higher Adams filtration. Therefore, the product $\epsilon\{\Delta h_1 h_3\}\theta_{4.5}$ is detected by $M\Delta h_1^2 d_0$. In particular, $\{\Delta h_1 h_3\}\theta_{4.5}$ is non-zero, and it can only be detected by $M\Delta h_1 h_3$. ☐

7.6. *Additional relations*

Lemma **7.151.** — *The product* $(\eta\sigma + \epsilon)\theta_5$ *is detected by* $\tau h_0 h_2 Q_3$.

Proof. — Table 10 indicates a hidden η extension from $h_2^2 h_6$ to $\tau h_0 h_2 Q_3$. Therefore, there exists an element α in $\pi_{69,36}$ such that $\tau h_0 h_2 Q_3$ detects $\eta\alpha$. (Beware of the crossing extension from p' to $h_1 p'$. This means that it is possible to choose such an α, but not any element detected by $h_2^2 h_6$ will suffice.)

Table 10 shows that $h_2^2 h_6$ detects the Toda bracket $\langle \nu^2, 2, \theta_5\rangle$. Let β be an element of this Toda bracket. Since α and β are both detected by $h_2^2 h_6$, the difference $\alpha - \beta$ is detected in Adams filtration at least 4.

Table 24 shows that p' detects $\sigma\theta_5$, which belongs to the indeterminacy of $\langle \nu^2, 2, \theta_5\rangle$. Therefore, we may choose β such that the difference $\alpha - \beta$ is detected in filtration at least 9. Since $\eta\alpha$ is detected by $\tau h_0 h_2 Q_3$ in filtration 7, it follows that $\eta\beta$ is also detected by $\tau h_0 h_2 Q_3$.

We now have an element β contained in $\langle \nu^2, 2, \theta_5\rangle$ such that $\eta\beta$ is detected by $\tau h_0 h_2 Q_3$. Now consider the shuffle

$$\eta\langle \nu^2, 2, \theta_5\rangle = \langle \eta, \nu^2, 2\rangle\theta_5.$$

Table 10 shows that the last bracket equals $\{\epsilon, \epsilon + \eta\sigma\}$. Therefore, either $\epsilon\theta_5$ or $(\epsilon + \eta\sigma)\theta_5$ is detected by $\tau h_0 h_2 Q_3$. But $\epsilon\theta_5$ is detected by $h_1 p' = h_5^2 c_0$. ☐

Lemma **7.152.** — *There exists an element* α *of* $\pi_{67,36}$ *that is detected by* $h_0 Q_3 + h_0 n_1$ *such that* $\tau\nu\alpha$ *equals* $(\eta\sigma + \epsilon)\theta_5$.

Proof. — Lemma 7.151 shows that $\tau h_0 h_2 Q_3$ detects $(\epsilon + \eta\sigma)\theta_5$. The element $\tau h_0 h_2 Q_3$ also detects $\tau\nu\alpha$. Let β be the difference $\tau\nu\alpha - (\epsilon + \eta\sigma)\theta_5$, which is detected in higher Adams filtration. We will show that β must equal zero.

First, $\tau h_1 D_3'$ cannot detect β because $\eta^2\beta$ is zero, while Table 18 shows that $\tau^3 d_1 g^2$ detects $\eta^2\{\tau h_1 D_3'\}$. Second, Table 21 shows that $\tau h_1 h_3(\Delta e_1 + C_0)$ is the target of a hidden ν extension. Therefore, we may alter the choice of α to ensure that β is not detected by $\tau h_1 h_3(\Delta e_1 + C_0)$. Third, $\Delta^2 h_2 c_1$ is also the target of a ν extension. Therefore, we may alter the choice of α to ensure that β is not detected by $\Delta^2 h_2 c_1$. Finally, comparison to *tmf* implies that β is not detected by $\tau\Delta^2 h_1^2 g + \tau^3 \Delta h_2^2 g^2$. □

Lemma **7.153.** — *The product* $(\sigma^2 + \kappa)\theta_5$ *is zero, or it is equal to* $\tau^2\kappa_1\overline{\kappa}_2$ *detected by* $\tau^2 d_1 g_2$.

Proof. — The product $(\sigma^2 + \kappa)\theta_5$ maps to zero under inclusion of the bottom cell of $C\tau$. Therefore, $(\sigma^2 + \kappa)\theta_5$ is divisible by τ. The only two possibilities are 0 and $\tau^2\kappa_1\overline{\kappa}_2$. □

Lemma **7.154.** — $\eta\sigma\{k_1\} + \nu\{d_1 e_1\} = 0$ *in* $\pi_{73,41}$.

Proof. — We have the relation $h_1 h_3 k_1 + h_2 d_1 e_1 = 0$ in the Adams E_∞-page, but $\eta\sigma\{k_1\} + \nu\{d_1 e_1\}$ could possibly be detected in higher Adams filtration. However, it cannot be detected by $h_2^2 C''$ or $M h_1 h_3 g$ by comparison to $C\tau$. Also, it cannot be detected by $\Delta h_1 d_0 e_0^2$ by comparison to *mmf*. □

8. Tables

Table 1 gives some notation for elements in $\pi_{*,*}$. The fourth column gives partial information that reduces the indeterminacies in the definitions, but does not completely specify a unique element in all cases. See Section 1.5 for further discussion.

Table 2 gives hidden values of the unit map $\pi_{*,*} \to \pi_{*,*}mmf$. The elements in the third column belong to the Adams E_∞-page for *mmf* [28, 31]. See Section 2.2 for further discussion.

Table 3 lists information about some Massey products. The fifth column indicates the proof. When a differential appears in this column, it indicates the May differential that can be used with the May Convergence Theorem (see Remark 2.26) to compute the bracket. The sixth column shows where each specific Massey product is used in the manuscript. See Section 4 for more discussion.

Table 4 lists all of the multiplicative generators of the Adams E_2-page through the 95-stem. The third column indicates the value of the d_2 differential, if it is non-zero.

A blank entry in the third column indicates that the d_2 differential is zero. The fourth column indicates the proof. A blank entry in the fourth column indicates that there are no possible values for the differential. The fifth column gives alternative names for the element, as used in [11, 30], or [54]. See Sections 1.5 and 5.1 for further discussion.

Table 5 lists some elements in the Adams spectral sequence that are known to be permanent cycles. The third column indicates the proof. When a Toda bracket appears in the third column, the Moss Convergence Theorem 2.16 applied to that Toda bracket implies that the element is a permanent cycle (see Table 10 for more information). When a product appears in the third column, the element must survive to detect that product.

Table 6 lists the multiplicative generators of the Adams E_3-page through the 95-stem whose d_3 differentials are non-zero, or whose d_3 differentials are zero for non-obvious reasons. See Section 5.2 for further discussion.

Table 7 lists the multiplicative generators of the Adams E_4-page through the 95-stem whose d_4 differentials are non-zero, or whose d_4 differentials are zero for non-obvious reasons. See Section 5.3 for further discussion.

Table 8 lists the multiplicative generators of the Adams E_5-page through the 95-stem whose d_5 differentials are non-zero, or whose d_5 differentials are zero for non-obvious reasons. See Section 5.4 for further discussion.

Table 9 lists the multiplicative generators of the Adams E_r-page, for $r \geq 6$, through the 90-stem whose d_r differentials are non-zero, or whose d_r differentials are zero for non-obvious reasons. See Section 5.5 for further discussion.

Table 10 lists information about some Toda brackets. Whenever possible, we use Greek letter names to refer to specific homotopy elements. An expression of the form $\{x\}$ means that the Toda bracket computation applies to any homotopy element detected by the element x of the Adams E_∞-page. An expression of the form $[x]$ means that the Toda bracket computation applies to at least one homotopy element that is detected by x. The third column of Table 10 gives an element of the Adams E_∞-page that detects an element of the Toda bracket. The fourth column of Table 10 gives partial information about indeterminacies, again by giving detecting elements of the Adams E_∞-page. We have not completely analyzed the indeterminacies of some brackets when the details are inconsequential for our purposes; this is indicated by a blank entry in the fourth column. The fifth column indicates the proof of the Toda bracket, and the sixth column shows where each specific Toda bracket is used in the manuscript. See Section 6 for further discussion.

Tables 12 and 13 gives hidden values of the inclusion $\pi_{*,*} \to \pi_{*,*}C\tau$ of the bottom cell, and of the projection $\pi_{*,*}C\tau \to \pi_{*-1,*+1}$ to the top cell. See Section 7.1 for further discussion.

Table 14 lists hidden τ extensions in the E_∞-page of the $\mathbf{C}$-motivic Adams spectral sequence. See Section 7.1 for further discussion.

Tables 15, 18, and 21 list hidden extensions by 2, η, and ν. The fourth column indicates the proof of each extension. The fifth column gives additional information about each extension, including whether it is a crossing extension and whether it has indeterminacy in the sense of Section 2.1.1. See Sections 7.2, 7.3, and 7.4 for further discussion.

Tables 17, 20, and 23 list possible hidden extensions by 2, η, and ν that we have not yet resolved.

Finally, Table 24 gives some various hidden extensions by elements other than 2, η, and ν. See Section 7.5 for further discussion.

TABLE 1. — Notation for $\pi_{*,*}$

(s, w)	Element	Detected by	Definition
$(0, -1)$	τ	τ	
$(0, 0)$	2	h_0	
$(1, 1)$	η	h_1	
$(3, 2)$	ν	h_2	
$(7, 4)$	σ	h_3	
$(8, 5)$	ϵ	c_0	
$(9, 5)$	μ_9	$\mathrm{P}h_1$	
$(14, 8)$	κ	d_0	
$(15, 8)$	ρ_{15}	$h_0^3 h_4$	
$(16, 9)$	η_4	$h_1 h_4$	
$(17, 9)$	μ_{17}	$\mathrm{P}^2 h_1$	
$(19, 11)$	$\overline{\sigma}$	c_1	
$(20, 11)$	$\overline{\kappa}$	τg	$\langle \kappa, 2, \eta, \nu \rangle$
$(23, 12)$	ρ_{23}	$h_0^2 i + \tau \mathrm{P} h_1 d_0$	
$(25, 13)$	μ_{25}	$\mathrm{P}^3 h_1$	
$(30, 16)$	θ_4	h_4^2	
$(32, 17)$	η_5	$h_1 h_5$	in $\langle \eta, 2, \theta_4 \rangle$
$(32, 18)$	κ_1	d_1	
$(44, 24)$	$\overline{\kappa}_2$	g_2	
$(45, 24)$	$\theta_{4.5}$	h_4^3	$\eta \theta_{4.5} \in \{\mathrm{M} h_1\}$
$(62, 32)$	θ_5	h_5^2	
$(63, 32)$	η_6	$h_1 h_6$	in $\langle \eta, 2, \theta_5 \rangle$

TABLE 2. — Some hidden values of the unit map of *mmf*

(s, f, w)	Element	Image
$(28, 6, 17)$	$h_1 h_3 g$	cg
$(29, 7, 18)$	$h_1^2 h_3 g$	$h_1 cg$
$(32, 6, 17)$	$\Delta h_1 h_3$	$\Delta c + \tau ag$
$(33, 7, 18)$	$\Delta h_1^2 h_3$	$\Delta h_1 c$
$(35, 8, 18)$	$\tau^3 h_1 e_0^2$	$\mathrm{P}an$
$(40, 10, 21)$	$\tau \Delta h_1^2 d_0$	$\mathrm{P}(\Delta c + \tau ag)$
$(48, 10, 29)$	$h_1 h_3 g^2$	cg^2
$(49, 11, 30)$	$h_1^2 h_3 g^2$	$h_1 cg^2$
$(52, 10, 29)$	$\Delta h_1 h_3 g$	$(\Delta c + \tau ag)g$
$(53, 11, 30)$	$\Delta h_1^2 h_3 g$	$\Delta h_1 cg$
$(54, 9, 28)$	$h_0 h_5 i$	$\Delta^2 h_2^2$
$(54, 11, 32)$	$h_1^6 h_5 e_0$	dg^2
$(55, 12, 33)$	$h_1^7 h_5 e_0$	$h_1 dg^2$

TABLE 2. — (*Continued*)

(s, f, w)	Element	Image
(57, 10, 30)	$h_0 h_2 h_5 i$	$\Delta h_1(\Delta c + \tau a g)$
(59, 12, 33)	$\mathrm{P} h_1^3 h_5 e_0$	$\Delta h_1 d g$
(60, 13, 34)	$\tau^2 h_0 g^3$	$\Delta h_1^2 d g$
(62, 14, 37)	$h_1^6 h_5 c_0 e_0$	$c d g^2$
(63, 15, 38)	$h_1^7 h_5 c_0 e_0$	$h_1 c d g^2$
(65, 12, 34)	$\mathrm{P} h_5 j$	$\Delta^2 h_2 d$
(66, 14, 37)	$\mathrm{P} h_1^2 h_5 c_0 e_0$	$(\Delta c + \tau a g) d g$
(67, 15, 38)	$\mathrm{P} h_1^3 h_5 c_0 e_0$	$\Delta h_1 c d g$
(68, 13, 37)	$\mathrm{P} h_2 h_5 j$	$\Delta^2 h_2^2 d$
(68, 14, 41)	$h_1 h_3 g^3$	$c g^3$
(69, 15, 42)	$h_1^2 h_3 g^3$	$h_1 c g^3$
(71, 15, 38)	$\Delta^2 h_0^2 h_2 g$	$\Delta h_1(\Delta c + \tau a g) d$
(72, 14, 41)	$\Delta h_1 h_3 g^2$	$(\Delta c + \tau a g) g^2$
(73, 15, 42)	$\Delta h_1^2 h_3 g^2$	$\Delta h_1 c g^2$
(77, 15, 42)	$\Delta^2 h_2^3 g$	$\Delta h_1(\Delta c + \tau a g) g$
(88, 18, 53)	$h_1 h_3 g^4$	$c g^4$
(89, 19, 54)	$h_1^2 h_3 g^4$	$h_1 c g^4$

TABLE 3. — Some Massey products in $\mathrm{Ext}_{\mathbf{C}}$

(s, f, w)	Bracket	Contains	Indeterminacy	Proof	Used for
(2, 2, 1)	$\langle h_0, h_1, h_0 \rangle$	τh_1^2	0	Theorem 4.1	$\langle 2, \eta, 2 \rangle$
(3, 2, 2)	$\langle h_1, h_0, h_1 \rangle$	$h_0 h_2$	0	Theorem 4.1	$\langle \eta, 2, \eta \rangle$, 7.67 $\langle \{h_1 x_{76,6}\}, 2, \eta \rangle$
(6, 2, 4)	$\langle h_1, h_2, h_1 \rangle$	h_2^2	0	Theorem 4.1	$\langle \eta, \nu, \eta \rangle$
(8, 2, 5)	$\langle h_2, h_1, h_2 \rangle$	$h_1 h_3$	0	Theorem 4.1	$\langle \nu, \eta, \nu \rangle$
(8, 3, 5)	$\langle h_1^2, h_0, h_1, h_2 \rangle$	c_0	0	$d_1(h_{20}) = h_0 h_1$ $d_1(h_{21}) = h_1 h_2$	$\langle \eta^2, 2, \eta, \nu \rangle$
(8, 3, 5)	$\langle h_1, h_2, h_0 h_2 \rangle$	c_0	0	$d_2(h_0(1) = h_0 h_2^2$	$\langle \eta, \nu, 2\nu \rangle$, $\langle \eta_5, \nu, 2\nu \rangle$
(9, 5, 5)	$\langle h_1, h_0, h_0^3 h_3 \rangle$	$\mathrm{P} h_1$	0	$d_4(b_{20}^2) = h_0^4 h_3$	$\langle \eta, 2, 8\sigma \rangle$
(11, 5, 6)	$\langle h_0, h_1, \tau h_1 c_0 \rangle$	$\mathrm{P} h_2$	0	$d_2(b_{20} h_0(1)) = \tau h_1^2 c_0$	$\langle 2, \eta, \tau \eta \epsilon \rangle$
(20, 4, 11)	$\langle \tau, h_1^4, h_4 \rangle$	τg	0	$d_4(g) = h_1^4 h_4$	7.127
(23, 5, 14)	$\langle h_2, h_1, h_1^3 h_4 \rangle$	$h_2 g$	0	$d_4(g) = h_1^4 h_4$	4.11
(23, 9, 12)	$\langle h_3, h_0^4, h_0^4 h_4 \rangle$	$h_0^2 i + \tau \mathrm{P} h_1 d_0$	0	$d_8(b_{20}^4) = h_0^8 h_4$	$\langle \sigma, 16, 2\rho_{15} \rangle$
(30, 6, 16)	$\langle h_0^2, h_3^2, h_0^2, h_3^2 \rangle$	Δh_2^2	0	$d_4(h_2 b_{30}) = h_0^2 h_3^2$	4.9
(32, 4, 18)	$\langle h_1, h_3^2, h_1, h_3^2 \rangle$	d_1	0	$d_2(h_1(1)) = h_1 h_3^2$	$\langle \eta, \sigma^2, \eta, \sigma^2 \rangle$
(33, 4, 18)	$\langle h_1 h_4^2, h_1, h_0 \rangle$	p	0	$d_4(h_3 b_{31}) = h_1^2 h_4^2$	$\langle \eta \theta_4, \eta, 2 \rangle$
(41, 3, 22)	$\langle h_2 h_4, h_4, h_3 \rangle$	c_2	0	$d_2(h_2(1)) = h_2 h_4^2$	4.12
(46, 7, 25)	$\langle h_1, h_0, h_0^2 g_2 \rangle$	$\mathrm{M} h_1$	0	M	6.12
(54, 6, 29)	$\langle h_1, h_0, \mathrm{D}_1 \rangle$	$\tau \Delta_1 h_1^2$	0	[35]	Example 2.31
(66, 6, 36)	$\langle h_1^2, h_4^2, h_1^2, h_4^2 \rangle$	$\Delta_1 h_3^2$	0	Lemma 4.9	6.18
(66, 7, 35)	$\langle \mathrm{A}', h_1, h_2 \rangle$	$\tau \mathrm{G}_0$	0	Lemma 4.10	7.83
(67, 14, 36)	$\langle \mathrm{P} d_0, h_0^3, g_2 \rangle$	$\mathrm{MP} d_0$	0	M	5.70
(68, 11, 38)	$\langle h_2 g, h_0^3, g_2 \rangle$	$\mathrm{M} h_2 g$	0	M	5.42
(71, 13, 40)	$\langle h_1^3 h_4, h_1, \tau g n \rangle$	$\tau g^2 n$	$\mathrm{M} h_0 h_2^2 g$	Lemma 4.11	5.43
(75, 18, 42)	$\langle \Delta h_0^2 d_0 e_0, h_1, h_1^3 h_4 \rangle$	$\Delta h_2^2 d_0^2 e_0$	0	Proposition 4.3	$\langle \tau \eta \kappa \overline{\kappa}^2, \eta, \eta^2 \eta_4 \rangle$

Table 3. — (*Continued*)

(s, f, w)	Bracket	Contains	Indeterminacy	Proof	Used for
(80, 5, 42)	$\langle h_3, p', h_2 \rangle$	$h_0 e_2$	0	Lemma 4.12	7.120
(82, 12, 45)	$\langle \Delta e_1 + C_0, h_1^3, h_1 h_4 \rangle$	$(\Delta e_1 + C_0) g$	0	Lemma 4.14	$\langle \{\Delta e_1 + C_0\}, \eta^3, \eta_4 \rangle$
(86, 16, 46)	$\langle \Delta h_0^2 e_0, h_0^2, h_0 g_2 \rangle$	$M \Delta h_0^2 e_0$	0	M	6.30
(91, 13, 49)	$\langle M h_1, h_0, h_0^2 g_2 \rangle$	$M^2 h_1$	0	M	5.31
(93, 13, 49)	$\langle \tau^3 g G_0, h_0 h_2, h_2 \rangle$	$? \tau e_0 x_{76,9}$	$\tau M^2 h_2$	Lemma 4.15	5.64

Table 4. — Generators of the **C**-motivic Adams E_2-page

(s, f, w)	Element	d_2	Proof	Other names
(0, 1, 0)	h_0			
(1, 1, 1)	h_1			
(3, 1, 2)	h_2			
(7, 1, 4)	h_3			
(8, 3, 5)	c_0			
(9, 5, 5)	Ph_1			
(11, 5, 6)	Ph_2			
(14, 4, 8)	d_0			
(15, 1, 8)	h_4	$h_0 h_3^2$	$C\tau$	
(16, 7, 9)	Pc_0			
(17, 4, 10)	e_0	$h_1^2 d_0$	$C\tau$	
(17, 9, 9)	$P^2 h_1$			
(18, 4, 10)	f_0	$h_0^2 e_0$	$C\tau$	
(19, 3, 11)	c_1		$C\tau$	
(19, 9, 10)	$P^2 h_2$			
(20, 4, 11)	τg			
(22, 8, 12)	Pd_0			
(23, 5, 14)	$h_2 g$			
(23, 7, 12)	i	$Ph_0 d_0$	$C\tau$	
(24, 11, 13)	$P^2 c_0$			
(25, 8, 14)	Pe_0	$Ph_1^2 d_0$	$C\tau$	
(25, 13, 13)	$P^3 h_1$			
(26, 7, 14)	j	$Ph_0 e_0$	$C\tau$	
(27, 5, 16)	$h_3 g$	$h_0 h_2^2 g$	$C\tau$	
(27, 13, 14)	$P^3 h_2$			
(29, 7, 16)	k	$h_0 d_0^2$	$C\tau$	
(30, 6, 16)	Δh_2^2		$C\tau$	r
(30, 12, 16)	$P^2 d_0$			
(31, 1, 16)	h_5	$h_0 h_4^2$	$C\tau$	
(31, 5, 17)	n			
(32, 4, 18)	d_1			
(32, 6, 17)	$\Delta h_1 h_3$		h_0	q
(32, 7, 18)	l	$h_0 d_0 e_0$	$C\tau$	
(32, 15, 17)	$P^3 c_0$			
(33, 4, 18)	p			
(33, 12, 18)	$P^2 e_0$	$P^2 h_1^2 d_0$	$C\tau$	
(33, 17, 17)	$P^4 h_1$			
(34, 11, 18)	Pj	$P^2 h_0 e_0$	$C\tau$	
(35, 7, 20)	m	$h_0 e_0^2$	$C\tau$	

TABLE 4. — (*Continued*)

(s, f, w)	Element	d_2	Proof	Other names
(35, 17, 18)	P^4h_2			
(36, 6, 20)	t		$C\tau$	
(37, 5, 20)	x			
(37, 8, 22)	e_0g	$h_1^2e_0^2$	$C\tau$	
(38, 4, 21)	e_1			
(38, 6, 20)	Δh_3^2	h_0^3x	$C\tau$	$y+?h_2^2d_1$
(38, 16, 20)	P^3d_0			
(39, 7, 23)	c_1g		$C\tau$	
(39, 9, 21)	Δh_1d_0			u
(39, 15, 20)	P^2i	$P^3h_0d_0$	$C\tau$	
(40, 4, 22)	f_1			
(40, 8, 23)	τg^2			
(40, 19, 21)	P^4c_0			
(41, 3, 22)	c_2	h_0f_1	$C\tau$	
(41, 10, 22)	$\Delta h_0^2e_0$		$C\tau$	z
(41, 16, 22)	P^3e_0	$P^3h_1^2d_0$	$C\tau$	
(41, 21, 21)	P^5h_1			
(42, 9, 23)	Δh_1e_0	$\Delta h_1^3d_0$	$C\tau$	v
(42, 15, 22)	P^2j	$P^3h_0e_0$	$C\tau$	
(43, 9, 26)	h_2g^2			
(43, 21, 22)	P^5h_2			
(44, 4, 24)	g_2			
(45, 9, 24)	$\tau\Delta h_1g$			w
(46, 7, 25)	Mh_1			B_1
(46, 8, 25)	Δh_2c_1			N
(46, 11, 25)	Δc_0d_0	$\tau h_0d_0^2e_0$	$C\tau$	u'
(46, 20, 24)	P^4d_0			
(47, 9, 28)	h_3g^2	$h_0h_2^2g^2$	$C\tau$	
(47, 13, 24)	Δh_0^2i	h_0i^2	$C\tau$	Q', $Q+Pu$
(47, 13, 25)	$P\Delta h_1d_0$			
(48, 7, 26)	Mh_2			$B_2+?h_0^2h_5e_0$
(48, 23, 25)	P^5c_0			
(49, 11, 27)	Δc_0e_0	$\Delta h_1^2c_0d_0+\tau h_0d_0e_0^2$	$C\tau$	v'
(49, 20, 26)	P^4e_0	$P^4h_1^2d_0$	$C\tau$	
(49, 25, 25)	P^6h_1			
(50, 6, 27)	C			
(50, 10, 28)	Δh_2^2g		$C\tau$	
(50, 13, 27)	$P\Delta h_1e_0$	$P\Delta h_1^3d_0$	$C\tau$	
(50, 19, 26)	P^3j	$P^4h_0e_0$	$C\tau$	
(51, 9, 28)	Δh_3g	$\Delta h_0h_2^2g$	$C\tau$	G_3
(51, 9, 29)	gn			
(51, 25, 26)	P^6h_2			
(52, 5, 28)	D_1	$h_0^2h_3g_2$	$C\tau$	
(52, 8, 30)	d_1g			
(53, 7, 30)	i_1		$C\tau$	
(53, 9, 29)	Mc_0		h_0	B_8, Ph_5d_0
(53, 10, 28)	MP			x'
(54, 6, 29)	$\tau\Delta_1h_1^2$	Mh_1h_3	$C\tau$	G
(54, 10, 28)	$\Delta^2h_2^2$	MPh_0^2	$C\tau$	$R_1+?h_0^2h_5i$
(54, 15, 29)	$P\Delta c_0d_0$	$\tau Ph_0d_0^2e_0$	$C\tau$	

TABLE 4. — (*Continued*)

(s, f, w)	Element	d_2	Proof	Other names
(54, 24, 28)	$P^5 d_0$			
(55, 7, 30)	B_6		$C\tau$	
(55, 11, 32)	gm	$h_0 e_0^2 g$	$C\tau$	
(55, 17, 29)	$P^2 \Delta h_1 d_0$			
(55, 23, 28)	$P^4 i$	$P^5 h_0 d_0$	$C\tau$	
(56, 10, 29)	$\Delta^2 h_1 h_3$	$\tau M P h_1^2$	h_1	$Q_1 + ?gt$
(56, 10, 32)	gt		$C\tau$	
(56, 27, 29)	$P^6 c_0$			
(57, 6, 31)	D_4	$h_1 B_6$	$C\tau$	
(57, 7, 30)	Q_2		$C\tau$	
(57, 9, 31)	$\Delta h_1 d_1$			D_{11}
(57, 15, 31)	$P \Delta c_0 e_0$	$P \Delta h_1^2 c_0 d_0 + \tau h_0 d_0^4$	$C\tau$	
(57, 24, 30)	$P^5 e_0$	$P^5 h_1^2 d_0$	$C\tau$	
(57, 29, 29)	$P^7 h_1$			
(58, 6, 30)	D_2	$h_0 Q_2$	$C\tau$	
(58, 8, 33)	$e_1 g$			
(58, 17, 31)	$P^2 \Delta h_1 e_0$	$P^2 \Delta h_1^3 d_0$	$C\tau$	
(58, 23, 30)	$P^4 j$	$P^5 h_0 e_0$	$C\tau$	
(59, 7, 33)	j_1			
(59, 10, 32)	$M d_0$			B_{21}
(59, 11, 35)	$c_1 g^2$			
(59, 29, 30)	$P^7 h_2$			
(60, 7, 32)	$M h_4$		$C\tau$	B_3
(60, 9, 32)	B_4	$M h_0 d_0$	$C\tau$	
(60, 12, 35)	τg^3			
(60, 13, 36)	$h_0 g^3$			
(61, 4, 32)	D_3			
(61, 6, 32)	A'		$C\tau$	
(61, 6, 32)	$A + A'$	$M h_0 h_4$	$C\tau$	
(61, 7, 33)	B_7			
(61, 9, 32)	Δx	$h_0^2 B_4 + \tau M h_1 d_0$	Lemma 5.3 or $C\tau$, h_1^2	X_1
(62, 5, 33)	H_1	B_7	$C\tau$	
(62, 8, 33)	C_0			$x_{8,33} + h_0^6 h_5^2$
(62, 8, 33)	Δe_1			E_1, $x_{8,32} + x_{8,33}$
(62, 10, 32)	$\Delta^2 h_3^2$		$C\tau$	$x_{10,27} + x_{10,28} +$ $+ h_1 X_1$, R
(62, 10, 34)	$M e_0$	$M h_1^2 d_0$	$C\tau$	B_{22}, $x_{10,28}$
(62, 19, 33)	$P^2 \Delta c_0 d_0$	$\tau P^2 h_0 d_0^2 e_0$	$C\tau$	
(62, 28, 32)	$P^6 d_0$			
(63, 1, 32)	h_6	$h_0 h_5^2$	$C\tau$	
(63, 7, 34)	C'		$C\tau$	$x_{7,33} + x_{7,34}$
(63, 7, 34)	X_2	$M h_1^2 h_4$	$C\tau$	$x_{7,33}$
(63, 13, 38)	$h_2 g^3$			
(63, 21, 33)	$P^3 \Delta h_1 d_0$			
(64, 6, 34)	A''	$h_0 X_2$	$C\tau$	
(64, 10, 33)	$\Delta^2 h_1 h_4$			$x_{10,32} + ?h_0^2 h_3 Q_2$, q_1
(64, 14, 34)	$\Delta^2 h_1^2 d_0$	$M P^2 h_1^2$	$C\tau$	U, $PQ_1 + ?km$
(64, 31, 33)	$P^7 c_0$			

TABLE 4. — (*Continued*)

(s, f, w)	Element	d_2	Proof	Other names
(65, 7, 36)	k_1			
(65, 10, 35)	τMg		$C\tau$	B_{23}
(65, 13, 34)	$\Delta^2 h_0 e_0$	$h_0 \cdot \Delta^2 h_1^2 d_0$	$C\tau$	$R_2+?gw$
(65, 19, 35)	$P^2 \Delta c_0 e_0$	$P^2 \Delta h_1^2 c_0 d_0 +$ $+ \tau P h_0 d_0^4$	$C\tau$	
(65, 28, 34)	$P^6 e_0$	$P^6 h_1^2 d_0$	$C\tau$	
(65, 33, 33)	$P^8 h_1$			
(66, 6, 36)	$\Delta_1 h_3^2$			r_1
(66, 7, 35)	τG_0	$h_2 C_0 + h_1 h_3 Q_2$	$C\tau$	$x_{7,40}+?h_0 r_1$
(66, 10, 34)	D_2'	$\tau^2 M h_0^2 g$	$C\tau$, i	PD_2
(66, 10, 35)	τB_5	$\tau M h_0^2 g$	i	
(66, 21, 35)	$P^3 \Delta h_1 e_0$	$P^3 \Delta h_1^3 d_0$	$C\tau$	
(66, 27, 34)	$P^5 j$	$P^6 h_0 e_0$	$C\tau$	
(67, 5, 35)	τQ_3	$\tau \Delta_1 h_0 h_3^2$	g_2	
(67, 5, 36)	n_1	$\Delta_1 h_0 h_3^2$	$C\tau$	
(67, 6, 36)	$h_0 Q_3 + h_2^2 D_3$		$C\tau$	
(67, 9, 36)	X_3		$C\tau$	$x_{9,40}$
(67, 9, 37)	C''			$x_{9,39}$
(67, 11, 35)	$\Delta^2 c_1$			$x_{11,35} + h_0^2 x_{9,40}$
(67, 13, 40)	$h_3 g^3$	$h_0 h_2^2 g^3$	$C\tau$	
(67, 33, 34)	$P^8 h_2$			
(68, 4, 36)	d_2		$C\tau$	
(68, 8, 36)	Δg_2	$h_0 X_3$	$C\tau$	$G_{21}+?h_0 h_3 A'$
(68, 11, 38)	$M h_2 g$			
(68, 13, 36)	$\Delta^2 h_0 g$	$MP h_0 d_0$	$C\tau$	$P^2 D_1+?h_0^5 G_{21}$, G_{11}
(69, 4, 36)	p'		$C\tau$	
(69, 8, 37)	D_3'	$h_1 X_3$	$C\tau$, [17]	PD_3
(69, 8, 38)	$h_2 G_0$	$h_1 C''$	$C\tau$	
(69, 10, 36)	$P(A + A')$	$\tau^2 M h_0 h_2 g$	$C\tau$, i	
(69, 11, 38)	$h_2 B_5$		$C\tau$	
(69, 13, 36)	$\tau \Delta^2 h_1 g$		d_0	W_1
(69, 18, 36)	MP^3			$x_{18,20}+?d_0 il$
(70, 4, 37)	p_1		$C\tau$	
(70, 6, 38)	$h_2 Q_3$		$C\tau$	
(70, 17, 36)	$P\Delta^2 h_0 d_0$	$MP^3 h_0$	$C\tau$	R_1', $R_1+?d_0^2 v$
(70, 23, 37)	$P^3 \Delta c_0 d_0$	$\tau P^3 h_0 d_0^2 e_0$	$C\tau$	
(70, 32, 36)	$P^7 d_0$			
(71, 6, 38)	$x_{71,6}$	$\tau d_1 e_1$	h_3	$x_{6,47}+?h_1^2 p'$
(71, 7, 39)	l_1			
(71, 12, 37)	$\Delta^2 h_4 c_0$		$C\tau$, *tmf*, h_0	$x_{12,37}+?h_0 d_0 Q_2$
(71, 13, 38)	$\Delta^2 h_2 g$		$C\tau$	$x_{13,34}$
(71, 13, 38)	Mj	$MP h_0 e_0$	$C\tau$	$x_{13,35}$
(71, 13, 40)	$\Delta h_3 g^2$	$h_0 m^2$	$C\tau$	
(71, 13, 41)	$g^2 n$			
(71, 25, 37)	$P^4 \Delta h_1 d_0$			
(71, 31, 36)	$P^6 i$	$P^7 h_0 d_0$	$C\tau$	
(72, 12, 42)	$d_1 g^2$			
(72, 17, 39)	$\Delta^2 h_1^2 c_0 d_0$	$MP^2 h_1^2 c_0$	$C\tau$	
(72, 18, 38)	$P\Delta^2 h_1^2 d_0$	$MP^3 h_1^2$	$C\tau$	

TABLE 4. — (*Continued*)

(s, f, w)	Element	d_2	Proof	Other names
(72, 35, 37)	$P^8 c_0$			
(73, 17, 38)	$P\Delta^2 h_0 e_0$	$MP^3 h_2$	$C\tau$	
(73, 23, 39)	$P^3 \Delta c_0 e_0$	$P^3 \Delta h_1^2 c_0 d_0 +$ $+ \tau P^2 h_0 d_0^4$	$C\tau$	
(73, 32, 38)	$P^7 e_0$	$P^7 h_1^2 d_0$	$C\tau$	
(73, 37, 37)	$P^9 h_1$			
(74, 8, 40)	$x_{74,8}$			$x_{8,51} + ?Ph_0^2 h_2 h_6$
(74, 25, 39)	$P^4 \Delta h_1 e_0$	$P^4 \Delta h_1^3 d_0$	$C\tau$	
(74, 31, 38)	$P^6 j$	$P^7 h_0 e_0$	$C\tau$	
(75, 7, 40)	$x_{75,7}$		$C\tau$	$x_{7,53}$
(75, 11, 42)	gB_6		$C\tau$	
(75, 13, 40)	$\Delta^2 h_3 g$	$\Delta^2 h_0 h_2^2 g$	$C\tau$	
(75, 15, 44)	$g^2 m$	$h_0 e_0^2 g^2$	$C\tau$	
(75, 37, 38)	$P^9 h_2$			
(76, 6, 40)	$x_{76,6}$	$h_0 x_{75,7}$	$C\tau$	$x_{6,53}$
(76, 9, 40)	$x_{76,9}$		$C\tau$	$x_{9,51} + ?h_1 h_4 B_3$
(76, 14, 44)	$g^2 t$		$C\tau$	
(76, 16, 40)	$\Delta^2 d_0^2$	$\tau^2 d_0 jm$	$C\tau$, *tmf*	$x_{16,32}$
(77, 7, 40)	$x_{77,7}$	$\tau M h_1 h_4^2$	Lemma 5.5	$x_{7,57} + m_1$
(77, 7, 42)	m_1		$C\tau$	
(77, 8, 41)	$x_{77,8}$	$\Delta h_1 h_3 g_2$	$C\tau$	$x_{8,57}$
(77, 12, 41)	$M\Delta h_1 h_3$		$C\tau$, h_0, τg	$P^2 D_3$
(77, 13, 43)	$\Delta h_1 d_1 g$			
(77, 16, 40)	$\Delta^2 h_0 k$	$\Delta^2 h_0^2 d_0^2$	$C\tau$	$x_{16,33} + ?e_0 g^3$
(77, 16, 46)	$e_0 g^3$	$h_1^2 e_0^2 g^2$	$C\tau$	
(78, 6, 42)	t_1	$h_0 m_1$	$C\tau$	
(78, 9, 41)	$x_{78,9}$		$C\tau$	$x_{9,55} + ?h_0^7 h_4 h_6$
(78, 10, 40)	$x_{78,10}$	$h_0^5 x_{77,7}$	$C\tau$, h_1	$P^2 h_5^2$
(78, 12, 45)	$e_1 g^2$			
(78, 27, 41)	$P^4 \Delta c_0 d_0$	$\tau P^4 h_0 d_0^2 e_0$	$C\tau$	
(78, 36, 40)	$P^8 d_0$			
(79, 5, 42)	x_1		$C\tau$	
(79, 11, 42)	ΔB_6		$C\tau$	
(79, 11, 45)	gj_1			
(79, 13, 41)	$\Delta^2 n$			$x_{13,42}$
(79, 15, 47)	$c_1 g^3$		$C\tau$	
(79, 16, 42)	$\Delta^2 d_0 e_0$	$\Delta^2 h_1^2 d_0^2 + \tau^2 d_0 km$	$C\tau$, *tmf*	$x_{16,35}$
(79, 29, 40)	$P^4 \Delta h_0^2 i$	$P^4 h_0 i^2$	$C\tau$	
(79, 29, 41)	$P^5 \Delta h_1 d_0$		$C\tau$	
(80, 4, 42)	e_2	$h_0 x_1$	$C\tau$	
(80, 12, 42)	$\Delta^2 d_1$		h_0	$x_{12,44}$
(80, 13, 44)	gB_4	$M h_0 e_0^2$	$C\tau$	
(80, 14, 41)	$\Delta^3 h_1 h_3$		h_0	$x_{14,42}$
(80, 16, 42)	$\Delta^2 h_0 l$	$\Delta^2 h_0^2 d_0 e_0$	$C\tau$	$x_{16,37} + ?g^4$
(80, 16, 47)	τg^4			
(80, 22, 42)	$P^2 \Delta^2 h_1^2 d_0$	$MP^4 h_1^2$	$C\tau$	
(80, 39, 41)	$P^9 c_0$			
(81, 10, 44)	gA'			
(81, 11, 45)	gB_7			
(81, 12, 42)	$\Delta^2 p$		h_0	$x_{12,45}$

TABLE 4. — (*Continued*)

(s, f, w)	Element	d_2	Proof	Other names
(81, 21, 42)	$P^2\Delta^2 h_0 e_0$	$h_0 \cdot P^2\Delta^2 h_1^2 d_0$	$C\tau$	
(81, 27, 43)	$P^4\Delta c_0 e_0$	$P^4\Delta h_1^2 c_0 d_0 +$ $+\tau P^3 h_0 d_0^4$	$C\tau$	
(81, 36, 42)	$P^8 e_0$	$P^8 h_1^2 d_0$	$C\tau$	
(81, 41, 41)	$P^{10} h_1$			
(82, 6, 44)	$h_4 Q_3$			
(82, 9, 45)	gH_1	gB_7	$C\tau$	
(82, 12, 45)	$(\Delta e_1 + C_0)g$			
(82, 12, 45)	gC_0			
(82, 14, 46)	$Me_0 g$	$Mh_1^2 e_0^2$	$C\tau$	
(82, 16, 44)	$\Delta^2 e_0^2$	$\tau^2\Delta h_2^2 e_0^3$	$C\tau$, *tmf*	$x_{16,38}$
(82, 17, 47)	$\Delta h_1 e_0 g^2$	$\Delta h_1^3 e_0^2 g + Mh_1^5 d_0 e_0$	$C\tau$	
(82, 29, 43)	$P^5\Delta h_1 e_0$	$P^5\Delta h_1^3 d_0$	$C\tau$	
(82, 35, 42)	$P^7 j$	$P^8 h_0 e_0$	$C\tau$	
(83, 11, 45)	Δj_1			
(83, 11, 46)	gC'			
(83, 15, 44)	$\Delta^2 m$	$\Delta^2 h_0 e_0^2 +$ $+\tau^3\Delta h_1 e_0 g^2$	$C\tau$, *tmf*	$x_{15,41}$
(83, 17, 50)	$h_2 g^4$			
(83, 41, 42)	$P^{10} h_2$			
(84, 4, 44)	f_2		$C\tau$	
(84, 10, 45)	$Px_{76,6}$		$C\tau$	
(84, 14, 44)	$\Delta^2 t$		$C\tau$	$x_{14,46}$
(84, 15, 44)	$\Delta h_0^2 B_4$	$\Delta^2 h_0^2 m + \tau\Delta^2 h_1 e_0^2$	$C\tau$, *mmf*	$x_{15,42} + x_{15,43}$
(84, 15, 45)	$M\Delta h_1 d_0$		$C\tau$	$x_{15,43}$
(85, 3, 44)	c_3	$h_0 f_2$	$C\tau$	
(85, 6, 45)	$x_{85,6}$	$\tau h_1^2 h_4 Q_3$	h_1	$x_{6,68} + h_0^3 c_3$
(85, 13, 44)	$\Delta^2 x$		$C\tau$, h_1^2	$x_{13,46}$
(85, 14, 48)	Mg^2		$C\tau$	
(85, 16, 46)	$\Delta^2 e_0 g$	$\Delta^2 h_1^2 e_0^2 + \tau^2 d_0 m^2$	*tmf*	$x_{16,42} + h_0^3 x_{13,46}$
(85, 26, 44)	MP^5			
(86, 11, 47)	τgG_0	$h_2(\Delta e_1 + C_0)g$	$C\tau$	
(86, 12, 45)	$\Delta^2 e_1$		d_0	$x_{12,48}$
(86, 12, 46)	$\Delta h_1 B_7$			
(86, 14, 44)	$\Delta^3 h_3^2$	$\Delta^2 h_0^3 x$	$C\tau$, h_1	$P^3 h_5^2 + ?gB_5$
(86, 14, 47)	$\tau B_5 g$	$\tau Mh_0^2 g^2$	Lemma 5.6	
(86, 25, 44)	$\Delta^2 P^3 h_0 d_0$	$MP^5 h_0$	$C\tau$	$x_{25,24} + ?P^2 d_0^2 v$
(86, 31, 45)	$P^5\Delta c_0 d_0$	$\tau P^5 h_0 d_0^2 e_0$	h_0	
(86, 40, 44)	$P^9 d_0$			
(87, 7, 45)	$x_{87,7}$			$x_{7,74}$
(87, 9, 48)	gQ_3		$C\tau$	
(87, 10, 46)	$\Delta h_1 H_1$	$\Delta h_1 B_7$	$C\tau$	$x_{10,60}$
(87, 13, 49)	gC''		$C\tau$	
(87, 15, 47)	$\Delta^2 c_1 g$		$C\tau$	
(87, 15, 47)	$M\Delta h_1 e_0$	$M\Delta h_1^3 d_0$	$C\tau$	$x_{15,47} + ?h_2 x_{14,46}$
(87, 17, 45)	$\Delta^3 h_1 d_0$	$\tau^5 e_0^3 m$	*tmf*	$x_{17,50}$
(87, 17, 52)	$h_3 g^4$	$h_0 h_2^2 g^4$	$C\tau$	
(87, 20, 46)	$P\Delta^2 d_0 e_0$	$P\Delta^2 h_1^2 d_0^2 +$ $+\tau^2\Delta h_2^2 d_0^4$	$C\tau$, *tmf*	
(87, 33, 45)	$P^6\Delta h_1 d_0$		$C\tau$	

TABLE 4. — (*Continued*)

(s,f,w)	Element	d_2	Proof	Other names
(87, 39, 36)	$P^8 i$	$P^9 h_0 d_0$	$C\tau$	
(88, 10, 48)	$x_{88,10}$			
(88, 12, 46)	$\Delta^2 f_1$			$x_{12,51}$+?$g G_{21}$+ +?$P h_0^3 h_6 e_0$
(88, 12, 48)	$\Delta g_2 g$			
(88, 16, 47)	$\tau\Delta^2 g^2$	$\tau^3 \Delta h_2^2 e_0 g^2$	*tmf*, h_0	$x_{16,48}$
(88, 26, 46)	$P^3 \Delta^2 h_1^2 d_0$	$MP^5 h_1^2$	$C\tau$	
(88, 43, 45)	$P^{10} c_0$			
(89, 11, 46)	$\Delta^2 c_2$	$\Delta^2 h_0 f_1$	$C\tau$	$x_{11,59}$
(89, 12, 50)	$h_2 g G_0$	$h_1 g C''$	$C\tau$	
(89, 15, 50)	$h_2 B_5 g$		$C\tau$	
(89, 18, 46)	$\Delta^3 h_0^2 e_0$	$\tau^6 e_0^4 g$	$C\tau$, *tmf*	$x_{18,50}$
(89, 19, 51)	$\Delta c_0 e_0 g^2$	$\Delta h_1^2 c_0 e_0^2 g + \tau h_0 e_0^4 g$ $+M h_1^4 c_0 d_0 e_0$	$C\tau$, h_0	
(89, 25, 46)	$P^3 \Delta^2 h_0 e_0$	$MP^5 h_2$	$C\tau$	
(89, 31, 47)	$P^5 \Delta c_0 e_0$	$P^5 \Delta h_1^2 c_0 d_0$+ $+\tau P^4 h_0 d_0^4$	$C\tau$	
(89, 40, 46)	$P^9 e_0$	$P^9 h_1^2 d_0$	$C\tau$	
(89, 45, 45)	$P^{11} h_1$			
(90, 10, 48)	$x_{90,10}$			$x_{10,63}$
(90, 12, 48)	M^2			$x_{12,55}$
(90, 15, 49)	$M\Delta h_1 g$		$C\tau$	$r B_4$
(90, 17, 47)	$\Delta^3 h_1 e_0$	$\Delta^3 h_1^3 d_0 + \tau^5 e_0^2 g m$	$C\tau$, *tmf*	$x_{17,52}$
(90, 33, 47)	$P^6 \Delta h_1 e_0$	$P^6 \Delta h_1^3 d_0$	$C\tau$	
(90, 39, 46)	$P^8 j$	$P^9 h_0 e_0$	$C\tau$	
(91, 8, 48)	$x_{91,8}$	$x_{90,10}$	$C\tau$	$x_{8,75}$
(91, 11, 48)	$x_{91,11}$	$M^2 h_0$	$C\tau$	$x_{11,61}$+?$h_0^2 h_6 d_0^2$
(91, 17, 49)	$M\Delta c_0 d_0$	$\tau M h_0 d_0^2 e_0$	$C\tau$, h_0	
(91, 17, 50)	$\Delta^2 h_2 g^2$			
(91, 17, 52)	$\Delta h_3 g^3$	$h_0 g m^2$	$C\tau$	
(91, 17, 53)	$g^3 n$			
(91, 45, 46)	$P^{11} h_2$			
(92, 4, 48)	g_3		$C\tau$	
(92, 10, 48)	$x_{92,10}$		$C\tau$	$x_{10,65}$+?$h_0^2 h_6 k$
(92, 10, 51)	$\Delta_1 h_1^2 e_1$		$C\tau$	
(92, 12, 48)	$\Delta^2 g_2$		$C\tau$	$x_{12,58}$+?$h_0^2 x_{10,65}$
(92, 16, 54)	$d_1 g^3$			
(92, 18, 48)	$\Delta^3 h_2^2 d_0$	$\tau^6 d_0 e_0 g^3$	$C\tau$, *tmf*	$x_{18,55}$
(92, 24, 48)	$P^2 \Delta^2 d_0^2$	$\tau^2 d_0^3 \ddot{y}$	$C\tau$, *tmf*	
(93, 8, 49)	$x_{93,8}$		$C\tau$	$x_{8,78} + h_0 h_6 r$
(93, 9, 51)	$\Delta_1 h_2 e_1$			
(93, 10, 49)	$\Delta h_3 H_1$	$h_1 x_{91,11}$	$C\tau$	$x_{10,67} + h_0^4 h_6 r$
(93, 12, 50)	$\Delta\Delta_1 e_0$	$M^2 h_1^2$	$C\tau$	$x_{12,60}$+?$P h_6 c_0 d_0$
(93, 15, 54)	$g^2 i_1$		$C\tau$	
(93, 17, 48)	$\tau\Delta^3 h_1 g$	$\tau^6 e_0^2 g m$	*tmf*, h_1	$x_{17,57}$
(94, 8, 49)	$x_{94,8}$		[17]	$x_{8,80}$
(94, 9, 50)	$x_{94,9}$		$C\tau$	
(94, 10, 50)	$y_{94,10}$		$C\tau$	$x_{10,70}$+?$h_0^2 x_{8,80}$
(94, 10, 51)	$x_{94,10}$			
(94, 15, 49)	$M\Delta^2 h_1$			$x_{15,56}$

Table 4. — (*Continued*)

(s, f, w)	Element	d_2	Proof	Other names
(94, 16, 49)	$\Delta^3 h_2 c_1$		d_0	$x_{16,54}$
(94, 17, 51)	$\mathrm{M}\Delta c_0 e_0$	$\mathrm{M}\Delta h_1^2 c_0 d_0 + {} + \tau \mathrm{M} h_0 d_0 e_0^2$	$\mathrm{C}\tau$, h_0	
(94, 19, 49)	$\Delta^3 c_0 d_0$	$\tau \Delta^2 h_0 d_0^2 e_0$	h_0	
(94, 19, 50)	$\Delta^2 d_0 l$	$\Delta^2 h_0 d_0^2 e_0 + {} + \tau^3 \Delta h_1 e_0^4$	$\mathrm{C}\tau$, *tmf*	$x_{19,49}$
(94, 22, 48)	$\Delta^2 i^2$	$\tau^6 d_0^3 e_0^3$	*tmf*	$x_{22,39}$
(94, 35, 49)	$\mathrm{P}^6 \Delta c_0 d_0$	$\tau \mathrm{P}^6 h_0 d_0^2 e_0$	$\mathrm{C}\tau$	
(94, 44, 48)	$\mathrm{P}^{10} d_0$			
(95, 7, 50)	$x_{95,7}$	$x_{94,9}$	$\mathrm{C}\tau$, [4]	$x_{7,79}$
(95, 8, 51)	$x_{95,8}$	$x_{94,10}$	$\mathrm{C}\tau$	
(95, 10, 50)	$\Delta x_{71,6}$	$\tau \Delta d_1 e_1$	$\mathrm{C}\tau$, h_3	$x_{10,73} + ?h_0^2 h_6 l$
(95, 11, 51)	$x_{95,11}$		$\mathrm{C}\tau$	
(95, 15, 54)	$g^2 \mathrm{B}_6$			
(95, 16, 52)	$\mathrm{M}\Delta h_2^2 g$		$\mathrm{C}\tau$	
(95, 17, 52)	$\Delta^2 h_3 g^2$	$\mathrm{P}^2 h_1^7 h_6 c_0$	$\mathrm{C}\tau$	
(95, 18, 50)	$\Delta^3 h_2^2 e_0$	$\tau^6 e_0^2 g^3$	$\mathrm{C}\tau$, *tmf*	$x_{18,57}$
(95, 19, 56)	$g^3 m$	$h_0 e_0^2 g^3$	$\mathrm{C}\tau$	
(95, 21, 48)	$\Delta^3 h_0^2 i$	$\Delta^2 h_0 i^2 + \tau^6 d_0^3 e_0 m$	$\mathrm{C}\tau$, *tmf*	$x_{21,43} + ?\mathrm{P}x_{17,50}$
(95, 21, 49)	$\mathrm{P}\Delta^3 h_1 d_0$	$\tau^5 d_0^3 e_0 m$	*tmf*	
(95, 24, 50)	$\mathrm{P}^2 \Delta^2 d_0 e_0$	$\mathrm{P}^2 \Delta^2 h_1^2 d_0^2 + \tau^2 d_0^3 ik$	$\mathrm{C}\tau$, *tmf*	
(95, 37, 49)	$\mathrm{P}^7 \Delta h_1 d_0$		$\mathrm{C}\tau$	

Table 5. — Some **C**-motivic permanent cycles

(s, f, w)	Element	Proof
(36, 6, 20)	t	$\langle \tau, \eta^2 \kappa_1, \eta \rangle$
(64, 2, 33)	$h_1 h_6$	$\langle \eta, 2, \theta_5 \rangle$
(68, 7, 36)	$h_3 \mathrm{A}'$	$\langle \sigma, \kappa, \tau \eta \theta_{4.5} \rangle$
(69, 3, 36)	$h_2^2 h_6$	$\langle \nu^2, 2, \theta_5 \rangle$
(69, 4, 36)	p'	$\sigma \theta_5$
(70, 5, 36)	$h_0^3 h_3 h_6$	$\langle 8\sigma, 2, \theta_5 \rangle$
(70, 14, 37)	$\tau \Delta^2 h_1^2 g + \tau^3 \Delta h_2^2 g^2$	$\langle \eta, \tau \kappa^2, \tau \overline{\kappa}^2 \rangle$
(71, 4, 37)	$h_6 c_0$	$\langle \epsilon, 2, \theta_5 \rangle$
(71, 5, 37)	$\tau h_1 p_1$	Lemma 5.71
(72, 6, 37)	$\mathrm{P} h_1 h_6$	$\langle \mu_9, 2, \theta_5 \rangle$
(74, 7, 38)	$\mathrm{P} h_0 h_2 h_6$	Lemma 5.72
(74, 8, 40)	$x_{74,8}$	$\theta_4 \overline{\kappa}_2$
(77, 3, 40)	$h_3^2 h_6$	$\langle \sigma^2, 2, \theta_5 \rangle$
(77, 5, 40)	$h_6 d_0$	$\langle \kappa, 2, \theta_5 \rangle$
(79, 3, 41)	$h_1 h_4 h_6$	$\langle \eta_4, 2, \theta_5 \rangle$
(79, 8, 41)	$\mathrm{P} h_6 c_0$	$\rho_{15} \eta_6$
(80, 6, 42)	$\tau h_1 x_1$	$\langle 2, \eta, \tau \eta \{h_1 x_{76,6}\} \rangle$
(80, 10, 41)	$\mathrm{P}^2 h_1 h_6$	$\langle \mu_{17}, 2, \theta_5 \rangle$
(80, 12, 42)	$\Delta^2 d_1$	Lemma 5.17
(82, 4, 43)	$h_6 c_1$	$\langle \overline{\sigma}, 2, \theta_5 \rangle$
(83, 6, 44)	$h_0 h_6 g$	$\langle \kappa \eta_6, \eta, \nu \rangle$
(84, 4, 44)	$h_2^2 h_4 h_6$	$\langle \nu \nu_4, 2, \theta_5 \rangle$
(85, 9, 44)	$\mathrm{P} h_6 d_0$	$\langle \tau \eta^2 \overline{\kappa}, 2, \theta_5 \rangle$

TABLE 5. — (*Continued*)

(s, f, w)	Element	Proof
(86, 5, 45)	$h_4 h_6 c_0$	$\sigma\{h_1 h_4 h_6\}$
(87, 5, 46)	$h_1^2 c_3$	$\langle \tau\{h_0 Q_3 + h_0 n_1\}, \nu_4, \eta \rangle$
(87, 8, 47)	$h_1 h_4 x_{71,6}$	$\langle \{h_1 x_{71,6}\}, 2, \sigma^2 \rangle$
(87, 12, 45)	$P^2 h_6 c_0$	$\rho_{23}\eta_6$
(88, 10, 48)	$x_{88,10}$	Lemma 5.79
(88, 14, 45)	$P^3 h_1 h_6$	$\langle \mu_{25}, 2, \theta_5 \rangle$
(90, 12, 48)	M^2	Lemma 5.31
(91, 7, 49)	$h_1 h_3 h_6 g$	$\langle \{h_1 h_3 g\}, 2, \theta_5 \rangle$
(92, 5, 48)	$h_0 g_3$	Lemma 5.83
(93, 10, 50)	$h_1^2 x_{91,8}$	$\langle \kappa_1, \kappa, \tau\eta\theta_{4.5} \rangle$
(94, 6, 49)	$h_6 n$	$\langle \{n\}, 2, \theta_5 \rangle$
(95, 5, 50)	$h_6 d_1$	$\langle \kappa_1, 2, \theta_5 \rangle$
(95, 7, 49)	$\Delta h_1 h_3 h_6$	$\langle \{\Delta h_1 h_3\}, 2, \theta_5 \rangle$
(95, 16, 49)	$P^3 h_6 c_0$	Lemma 5.35

TABLE 6. — $\mathbb{C}$-motivic Adams d_3 differentials

(s, f, w)	Element	d_3	Proof
(15, 2, 8)	$h_0 h_4$	$h_0 d_0$	$C\tau$
(30, 6, 16)	Δh_2^2	$\tau h_1 d_0^2$	*tmf*
(31, 4, 16)	$h_0^3 h_5$	$h_0 \cdot \Delta h_2^2$	$C\tau$
(31, 8, 17)	$\tau d_0 e_0$	$P c_0 d_0$	$d_4(\tau^2 d_0 e_0 + h_0^7 h_5)$
(34, 2, 18)	$h_2 h_5$	$\tau h_1 d_1$	Lemma 5.10
(37, 8, 21)	$\tau e_0 g$	$c_0 d_0^2$	$d_4(\tau^2 e_0 g)$
(38, 4, 21)	e_1	$h_1 t$	$C\tau$
(39, 12, 21)	$\tau P d_0 e_0$	$P^2 c_0 d_0$	h_1
(40, 4, 22)	f_1	0	h_2
(46, 14, 24)	i^2	$\tau P^2 h_1 d_0^2$	*tmf*
(47, 16, 25)	$\tau P^2 d_0 e_0$	$P^3 c_0 d_0$	h_1
(47, 18, 24)	$h_0^5 Q'$	$P^4 h_0 d_0$	$C\tau$
(49, 6, 27)	$h_1 h_5 e_0$	$M h_1^3$	$C\tau$
(49, 11, 26)	$\tau^2 d_0 m$	$P \Delta h_1^2 d_0$	*mmf*
(50, 10, 28)	$\Delta h_2^2 g$	$\tau h_1 d_0 e_0^2$	τ
(54, 8, 28)	$h_5 i$	$M P h_0$	$C\tau$
(54, 6, 28)	$\tau^2 \Delta_1 h_1^2$	$\tau M c_0$	Lemma 5.11
(55, 11, 30)	$\tau^2 g m$	$\Delta h_1^2 d_0^2$	$d_4(\tau^3 g m)$
(55, 20, 29)	$\tau P^3 d_0 e_0$	$P^4 c_0 d_0$	h_1
(55, 7, 30)	B_6	$\tau h_2 g n$	$C\tau$
(56, 8, 31)	$h_5 c_0 e_0$	$M h_1^2 c_0$	$C\tau$
(56, 9, 30)	$P h_5 e_0$	$M P h_1^2$	$C\tau$
(56, 10, 32)	$g t$	0	τ
(56, 13, 30)	$\tau \Delta h_1 d_0 e_0$	$P \Delta h_1 c_0 d_0$	$d_4(\tau^2 \Delta h_1 d_0 e_0)$
(57, 12, 33)	$\tau e_0 g^2$	$c_0 d_0 e_0^2$	$C\tau$
(57, 8, 30)	$h_5 j$	$M P h_2$	$C\tau$
(57, 15, 30)	$\tau^2 P d_0 m$	$P^2 \Delta h_1^2 d_0$	*mmf*
(57, 7, 30)	Q_2	$\tau^2 g t$	$\tau \Delta h_1 g$
(58, 8, 33)	$e_1 g$	$h_1 g t$	$C\tau$
(60, 12, 35)	τg^3	$M h_1^6 c_0$	$C\tau$
(61, 4, 32)	D_3	$M h_4$	$C\tau$

TABLE 6. — (*Continued*)

(s, f, w)	Element	d_3	Proof
(62, 13, 34)	$\tau \Delta h_1 e_0 g$	$\Delta h_1 c_0 d_0^2$	$C\tau$
(62, 8, 33)	Δe_1	$\Delta h_2^2 n$	$C\tau$
(62, 8, 33)	C_0	$\Delta h_2^2 n$	$C\tau$
(62, 10, 33)	$h_1 \cdot \Delta x + \tau M e_0$	MPc_0	$C\tau$
(62, 10, 32)	$\tau h_1 \cdot \Delta x$	0	Δh_2^2
(62, 10, 32)	$\Delta^2 h_3^2$	0	Δh_2^2
(62, 22, 32)	$P^2 i^2$	$\tau P^4 h_1 d_0^2$	*tmf*
(63, 8, 32)	$h_0^7 h_6$	$\Delta^2 h_0 h_3^2$	$C\tau$, [17]
(63, 24, 33)	$\tau P^4 d_0 e_0$	$P^5 c_0 d_0$	h_1
(64, 17, 34)	$\tau P \Delta h_1 d_0 e_0$	$P^2 \Delta h_1 c_0 d_0$	$d_4(\tau^2 P \Delta h_1 d_0 e_0)$
(65, 13, 36)	$\Delta h_2^2 m$	$MPh_1^3 c_0$	$C\tau$
(65, 19, 34)	$\tau^2 P^2 d_0 m$	$P^3 \Delta h_1^2 d_0$	*mmf*
(67, 5, 35)	$\tau Q_3 + \tau n_1$	0	h_2
(67, 9, 37)	C''	nm	$C\tau$
(67, 9, 36)	X_3	0	τg
(68, 4, 36)	d_2	$h_0^2 Q_3$	$C\tau$
(68, 11, 35)	$\tau h_0^3 \cdot \Delta g_2$	$\tau^3 \Delta h_2^2 e_0 g$	Lemma 5.12
(68, 11, 38)	$M h_2 g$	0	τ
(69, 8, 36)	$\tau D_3'$	$\tau^2 M h_2 g$	Lemma 5.13
(69, 11, 38)	$h_2 B_5$	$M h_1 c_0 d_0$	$C\tau$
(69, 13, 36)	$\tau \Delta^2 h_1 g$	$\tau^4 e_0^4$	*tmf*
(70, 2, 36)	$h_3 h_6$	$h_0 p'$	$C\tau$
(70, 4, 37)	p_1	$\tau h_1^2 Q_3$	$C\tau$
(70, 14, 37)	$\tau MP e_0$	$MP^2 c_0$	$d_4(\tau^2 MP e_0)$
(70, 14, 40)	m^2	$\tau h_1 e_0^4$	τ
(71, 28, 37)	$\tau P^5 d_0 e_0$	$P^6 c_0 d_0$	h_1
(72, 21, 39)	$\tau P^2 \Delta h_1 d_0 e_0$	$P^3 \Delta h_1 c_0 d_0$	$d_4(\tau^2 P^2 \Delta h_1 d_0 e_0)$
(73, 23, 38)	$\tau^2 P^3 d_0 m$	$P^4 \Delta h_1^2 d_0$	*mmf*
(74, 6, 38)	$P h_2 h_6$	$\tau h_1 h_4 Q_2$	Lemma 5.10
(75, 7, 40)	$x_{75,7}$	$h_0^2 x_{74,8}$	$C\tau$
(75, 11, 42)	$g B_6$	$\tau h_2 g^2 n$	h_2
(75, 15, 42)	$\tau^2 g^2 m$	$\Delta h_1^2 d_0 e_0^2$	$d_4(\tau^3 g^2 m)$
(76, 5, 40)	$h_4 D_3$	$d_0 D_3$	$C\tau$
(76, 14, 41)	$\Delta^2 h_1 h_3 g$	0	h_1
(76, 14, 44)	$g^2 t$	0	τ
(77, 14, 40)	$\tau^2 M h_0 l$	$\Delta^2 h_0 d_0^2$	Lemma 5.14
(77, 16, 45)	$\tau e_0 g^3$	$c_0 e_0^4$	$d_4(\tau^2 e_0 g^3)$
(77, 17, 41)	$\Delta^2 h_1 d_0^2$	$\tau^3 d_0^3 e_0^2$	*tmf*
(78, 3, 40)	$h_0 h_4 h_6$	$h_0 h_6 d_0$	$C\tau$
(78, 12, 45)	$e_1 g^2$	$h_1 g^2 t$	$C\tau$
(78, 13, 40)	$h_0^3 x_{78,10}$	$\tau^6 e_0 g^3$	Lemma 5.15
(78, 18, 41)	$\tau MP^2 e_0$	$MP^3 c_0$	h_1
(78, 30, 40)	$P^4 i^2$	$\tau P^6 h_1 d_0^2$	*tmf*
(79, 5, 42)	x_1	$\tau h_1 m_1$	Lemma 5.16
(79, 32, 41)	$\tau P^6 d_0 e_0$	$P^7 c_0 d_0$	h_1
(79, 34, 40)	$P^4 h_0^5 Q'$	$P^8 h_0 d_0$	$C\tau$
(80, 14, 41)	$\Delta^3 h_1 h_3$	$\tau^4 \Delta h_1 e_0^2 g$	Lemma 5.18, [17]
(80, 14, 42)	$\tau^2 d_0 B_5$	$\Delta^2 h_0 d_0 e_0 + \tau^3 \Delta h_1 e_0^2 g$	Lemma 5.20, [17]
(80, 25, 42)	$\tau P^3 \Delta h_1 d_0 e_0$	$P^4 \Delta h_1 c_0 d_0$	$d_4(\tau^2 P^3 \Delta h_1 d_0 e_0)$
(81, 3, 42)	$h_2 h_4 h_6$	0	Lemma 5.22

TABLE 6. — (*Continued*)

(s, f, w)	Element	d_3	Proof
(81, 12, 42)	$\Delta^2 p$	0	Lemma 5.23
(81, 27, 42)	$\tau^2 P^4 d_0 m$	$P^5 \Delta h_1^2 d_0$	*mmf*
(82, 6, 44)	$h_4 Q_3$	$h_3 x_{74,8}$	$C\tau$
(82, 10, 42)	$P^2 h_2 h_6$	0	Lemma 5.22
(82, 12, 45)	$g C_0$	$\Delta h_2^2 g n$	$C\tau$
(82, 14, 45)	$\tau M e_0 g$	$M c_0 d_0^2$	h_1
(82, 17, 46)	$\tau \Delta h_1 e_0 g^2$	$\Delta h_1 c_0 d_0 e_0^2$	$d_4(\tau^2 \Delta h_1 e_0 g^2)$
(83, 5, 43)	$\tau h_6 g + \tau h_2 e_2$	0	Lemma 5.24
(83, 17, 45)	$\Delta^2 h_1 e_0^2$	$\tau^3 d_0 e_0^4$	τ
(84, 4, 44)	f_2	$\tau h_1 h_4 Q_3$	Lemma 5.25
(84, 19, 45)	$\Delta^2 c_0 d_0^2$	$\tau \Delta h_0^2 d_0^3 e_0$	*mmf*
(85, 6, 45)	$\tau x_{85,6} + h_0^3 c_3$	0	Lemma 5.26
(85, 21, 45)	$P \Delta^2 h_1 d_0^2$	$\tau^3 d_0^6$	*tmf*
(86, 4, 45)	$h_1 c_3$	0	h_0
(86, 11, 45)	$\tau^3 g G_0$	0	d_0
(86, 12, 45)	$\Delta^2 e_1$	$\Delta^2 h_1 t$	$C\tau$
(86, 17, 46)	$\tau \Delta^2 h_1 e_0 g$	$\Delta^2 h_1 c_0 d_0^2 + \tau^4 e_0^5$	*mmf*
(86, 22, 45)	$\tau M P^3 e_0$	$M P^4 c_0$	h_1
(87, 7, 45)	$x_{87,7}$	0	d_0
(87, 13, 49)	$g C''$	$g n m$	$C\tau$
(87, 36, 45)	$\tau P^7 d_0 e_0$	$P^8 c_0 d_0$	h_1
(88, 12, 46)	$\Delta^2 f_1$	0	h_2
(88, 18, 46)	$\tau^2 M h_0 d_0 k$	$P \Delta^2 h_0 d_0 e_0 + \tau^3 \Delta h_1 d_0^2 e_0^2$	Lemma 5.28
(88, 18, 46)	$\Delta^3 h_1^2 d_0$	$\tau^3 \Delta h_1 d_0^2 e_0^2$	*mmf*, [17]
(88, 29, 46)	$\tau P^4 \Delta h_1 d_0 e_0$	$P^5 \Delta h_1 c_0 d_0$	$d_4(\tau^2 P^4 \Delta h_1 d_0 e_0)$
(89, 14, 51)	$h_1^9 h_6 e_0$	$M h_1^3 g^2$	$C\tau$
(89, 15, 50)	$h_2 B_5 g$	$M h_1 c_0 e_0^2$	Lemma 5.30
(89, 17, 48)	$\tau \Delta^2 h_1 g^2$	$\tau^4 e_0^4 g$	τ
(89, 31, 46)	$\tau^2 P^5 d_0 m$	$P^6 \Delta h_1^2 d_0$	*mmf*
(90, 18, 52)	$g m^2$	$\tau h_1 e_0^4 g$	τ
(90, 19, 49)	$\Delta^2 c_0 e_0^2$	$\tau \Delta h_2^2 d_0^3 e_0$	*mmf*
(92, 10, 48)	$x_{92,10}$	0	d_0
(92, 14, 50)	$m Q_2$	$\tau^3 g^3 n$	$C\tau$, τ
(92, 23, 49)	$P \Delta^2 c_0 d_0^2$	$\tau P^2 \Delta h_2^2 d_0^2 e_0$	*mmf*
(93, 7, 48)	$\Delta h_2^2 h_6$	$\tau h_1 h_6 d_0^2$	Lemma 5.32
(93, 8, 49)	$x_{93,8}$	$\Delta h_2^2 H_1$	$C\tau$
(93, 13, 48)	$P^2 h_6 d_0$	0	Lemma 5.33
(93, 22, 48)	$\tau^2 M P h_0 d_0 j$	$P^2 \Delta^2 h_0 d_0^2 + \tau^3 P \Delta h_1 d_0^3 e_0$	Lemma 5.34
(93, 25, 49)	$P^2 \Delta^2 h_1 d_0^2$	$\tau^3 P d_0^6$	*tmf*
(94, 8, 49)	$x_{94,8}$	$h_1 x_{92,10}$	$C\tau$
(94, 9, 49)	$\tau h_6 d_0 e_0$	$P h_6 c_0 d_0$	h_1
(94, 15, 49)	$M \Delta^2 h_1$	$\tau^3 M d_0 e_0^2$	$P d_0^2$
(94, 15, 52)	$P h_1^6 h_6 e_0$	$M \Delta h_1^4 g$	$C\tau$
(94, 17, 50)	$\tau^2 M d_0 m$	$M P \Delta h_1^2 d_0$	d_0
(94, 21, 50)	$\tau \Delta^2 h_1 d_0^2 e_0$	$P \Delta^2 h_1 c_0 d_0^2 + \tau^4 d_0^3 e_0^3$	*mmf*
(94, 26, 49)	$\tau M P^4 e_0 +$ $+ \tau P^2 \Delta^2 h_1^2 d_0^2$	$M P^5 c_0$	h_1
(94, 38, 48)	$P^6 i^2$	$\tau P^8 h_1 d_0^2$	*tmf*
(95, 15, 54)	$g^2 B_6$	$\tau h_2 g^3 n$	$C\tau$
(95, 16, 52)	$M \Delta h_2^2 g$	$\tau M h_1 d_0 e_0^2$	τ

TABLE 6. — (*Continued*)

(s, f, w)	Element	d_3	Proof
(95, 19, 54)	$\tau^2 g^3 m$	$\Delta h_1^2 e_0^4 + \mathrm{M} h_1^4 d_0^2 e_0$	*mmf*
(95, 20, 50)	$\Delta^3 h_1 c_0 d_0$	$\tau^4 d_0^3 e_0 m$	*mmf*
(95, 36, 48)	$h_0^{15} \cdot \Delta^3 h_0^2 i$	$\mathrm{P}^6 h_0 i^2$	$\mathrm{C}\tau$
(95, 40, 49)	$\tau \mathrm{P}^8 d_0 e_0$	$\mathrm{P}^9 c_0 d_0$	h_1

TABLE 7. — **C**-motivic Adams d_4 differentials

(s, f, w)	Element	d_4	Proof
(31, 8, 16)	$\tau^2 d_0 e_0 + h_0^7 h_5$	$\mathrm{P}^2 d_0$	$\mathrm{C}\tau$
(37, 8, 20)	$\tau^2 e_0 g$	$\mathrm{P} d_0^2$	*tmf*
(38, 2, 20)	$h_3 h_5$	$h_0 x$	$\mathrm{C}\tau$
(39, 12, 20)	$\tau^2 \mathrm{P} d_0 e_0$	$\mathrm{P}^3 d_0$	*tmf*
(42, 6, 22)	$\mathrm{P} h_2 h_5$	0	d_0
(47, 16, 24)	$\tau^2 \mathrm{P}^2 d_0 e_0$	$\mathrm{P}^4 d_0$	*tmf*
(50, 6, 27)	C	$\mathrm{P} h_1^2 h_5 c_0$	$\mathrm{C}\tau$
(50, 10, 26)	$\tau^2 \Delta h_2^2 g$	$i j$	*tmf*
(55, 11, 29)	$\tau^3 g m$	$\mathrm{P} \Delta c_0 d_0 + \tau d_0^2 j$	*tmf*
(55, 20, 28)	$\tau^2 \mathrm{P}^3 d_0 e_0$	$\mathrm{P}^5 d_0$	*tmf*
(56, 13, 29)	$\tau^2 \Delta h_1 d_0 e_0$	$\mathrm{P}^2 \Delta h_1 d_0$	*tmf*
(57, 12, 32)	$\tau^2 e_0 g^2$	d_0^4	d_0
(58, 14, 30)	$\tau^2 \Delta h_2^2 d_0^2$	$\mathrm{P} i j$	*tmf*
(62, 10, 32)	$\tau h_1 \cdot \Delta x$	$\tau^2 \Delta h_2^2 d_0 e_0$	Lemma 5.38
(62, 10, 32)	$\Delta^2 h_3^2$	0	Lemma 5.39
(62, 13, 33)	$\tau^2 \Delta h_1 e_0 g$	$\mathrm{P} \Delta h_1 d_0^2$	*tmf*
(63, 7, 34)	C'	$\mathrm{M} h_2 d_0$	$\mathrm{C}\tau$
(63, 7, 33)	$\tau \mathrm{X}_2$	$\tau \mathrm{M} h_2 d_0$	Lemma 5.40
(63, 11, 33)	$\tau^2 \mathrm{M} h_1 e_0$	$\mathrm{M} \mathrm{P}^2 h_1$	d_0
(63, 15, 33)	$\tau^3 d_0^2 m$	$\mathrm{P}^2 \Delta c_0 d_0 + \tau \mathrm{P} d_0^2 j$	*tmf*
(63, 19, 32)	$h_0^{18} h_6$	$\mathrm{P}^2 h_0 i^2$	$\mathrm{C}\tau$
(64, 17, 33)	$\tau^2 \Delta h_2^2 d_0 g$	$\mathrm{P}^3 \Delta h_1 d_0$	*tmf*
(66, 18, 34)	$\tau^2 \mathrm{P} \Delta h_2^2 d_0^2$	$\mathrm{P}^2 i j$	*tmf*
(68, 5, 36)	$h_0 d_2$	X_3	Lemma 5.41
(68, 11, 38)	$\mathrm{M} h_2 g$	0	Lemma 5.42
(69, 11, 37)	$\tau h_2 \mathrm{B}_5$	$\mathrm{M} \mathrm{P} h_1 d_0$	d_0
(70, 10, 39)	$h_2 \mathrm{C}''$	$h_1^4 c_0 \mathrm{Q}_2$	$\mathrm{C}\tau$
(70, 14, 38)	$\tau^2 \Delta h_2^2 g^2$	$\Delta h_0^2 d_0^2 e_0$	τ
(70, 14, 36)	$\tau^2 \mathrm{M} \mathrm{P} e_0$	$\mathrm{M} \mathrm{P}^3$	d_0
(71, 19, 37)	$\tau^3 \mathrm{P} d_0^2 m$	$\mathrm{P}^3 \Delta c_0 d_0 + \tau \mathrm{P}^2 d_0^2 j$	*tmf*
(71, 28, 36)	$\tau^2 \mathrm{P}^5 d_0 e_0$	$\mathrm{P}^7 d_0$	*tmf*
(72, 9, 40)	$h_2^2 \mathrm{G}_0$	$\tau g^2 n$	Lemma 5.43
(72, 21, 37)	$\tau^2 \mathrm{P}^2 \Delta h_1 d_0 e_0$	$\mathrm{P}^4 \Delta h_1 d_0$	*tmf*
(74, 22, 38)	$\tau^2 \mathrm{P}^2 \Delta h_2^2 d_0^2$	$\mathrm{P}^3 i j$	*tmf*
(75, 5, 40)	$h_3 d_2$	$h_0 x_{74,8}$	$\mathrm{C}\tau$
(75, 11, 40)	$\Delta h_0^2 h_3 g_2$	$\tau \mathrm{M} h_1 d_0^2$	Lemma 5.44
(75, 15, 41)	$\tau^3 g^2 m$	$\Delta c_0 d_0^3 + \tau d_0^3 l$	*tmf*
(76, 14, 40)	$\tau^2 \mathrm{M} d_0 e_0$	$\mathrm{M} \mathrm{P}^2 d_0$	h_1
(76, 14, 41)	$\Delta^2 h_1 h_3 g$	$\tau \Delta h_2^2 d_0^2 e_0$	Lemma 5.45
(77, 16, 44)	$\tau^2 e_0 g^3$	$d_0^3 e_0^2$	τ

TABLE 7. — (*Continued*)

(s, f, w)	Element	d_4	Proof
(78, 5, 40)	$h_0^3h_4h_6$	$h_0^2x_{77,7}$	$C\tau$
(79, 23, 41)	$\tau^3P^2d_0^2m$	$P^4\Delta c_0d_0 + \tau P^3d_0^2j$	*tmf*
(79, 32, 40)	$\tau^2P^6d_0e_0$	P^8d_0	*tmf*
(80, 5, 42)	h_0e_2	$\tau h_1^3x_{76,6}$	Lemma 5.46
(80, 25, 41)	$\tau^2P^3\Delta h_1d_0e_0$	$P^5\Delta h_1d_0$	*tmf*
(81, 8, 43)	$\tau g D_3$	0	h_1
(81, 15, 42)	$\Delta^3h_1^2h_3$	$\tau^4d_0e_0^2l$	Lemma 5.47
(82, 14, 44)	τ^2Me_0g	MPd_0^2	h_1
(82, 17, 45)	$\tau^2\Delta h_1e_0g^2$	$\Delta h_1d_0^4$	τ
(82, 26, 42)	$\tau^2P^3\Delta h_2^2d_0^2$	P^4ij	*tmf*
(83, 11, 46)	gC'	Mh_0e_0g	$C\tau$
(83, 11, 45)	Δj_1	τMh_0e_0g	Lemma 5.48
(85, 5, 45)	h_1f_2	0	Lemma 5.49
(85, 6, 44)	$\tau x_{85,6} + h_0^3c_3$	0	Lemma 5.50
(86, 4, 45)	h_1c_3	$\tau h_0h_2h_4Q_3$	Lemma 5.51
(86, 10, 44)	$h_0^2h_6i$	0	d_0, $C\tau$
(86, 22, 44)	$\tau^2MP^3e_0$	MP^5	h_1
(87, 7, 45)	$x_{87,7}$	0	Lemma 5.52
(87, 10, 45)	$\tau\Delta h_1H_1$	0	Lemma 5.53
(87, 15, 47)	Δ^2c_1g	0	τ
(87, 27, 45)	$\tau^3P^3d_0^2m$	$P^5\Delta c_0d_0 + \tau P^4d_0^2j$	*tmf*
(87, 36, 44)	$\tau^2P^7d_0e_0$	P^9d_0	*tmf*
(88, 17, 48)	$\Delta^2h_0g^2$	$\tau\Delta h_1d_0^2e_0^2$	*mmf*
(88, 29, 45)	$\tau^2P^4\Delta h_1d_0e_0$	$P^6\Delta h_1d_0$	*tmf*
(89, 15, 49)	τh_2B_5g	$Mh_1d_0^3$	Lemma 5.54
(90, 14, 51)	h_2gC''	$Ph_1^{10}h_6c_0$	$C\tau$
(90, 18, 50)	τ^2gm^2	$\Delta h_2^2d_0^3e_0$	*mmf*
(90, 30, 46)	$\tau^2\Delta P^4h_2^2d_0^2$	P^5ij	*tmf*
(91, 12, 48)	$\Delta h_2^2A'$	0	Lemma 5.55
(91, 20, 50)	$\Delta^2h_1c_0e_0^2$	$\tau^2d_0^4e_0^2$	*mmf*
(92, 13, 52)	$h_2^2gG_0$	τg^3n	τ
(93, 3, 48)	$h_4^2h_6$	$h_0^3g_3$	Lemma 5.56
(95, 16, 50)	$M\Delta^2h_1^2$	$MP\Delta h_0^2e_0$	Lemma 5.57
(95, 16, 50)	$\tau^2M\Delta h_2^2g$	$MP\Delta h_0^2e_0$	d_0
(95, 19, 53)	τ^3g^3m	$\Delta c_0d_0^2e_0^2 + \tau d_0^3e_0m$	τ
(95, 31, 49)	$\tau^3P^4d_0^2m$	$P^6\Delta c_0d_0 + \tau P^5d_0^2j$	*tmf*
(95, 40, 48)	$P^6\Delta h_0^5i + \tau^2P^8d_0e_0$	$P^{10}d_0$	*tmf*

TABLE 8. — **C**-motivic Adams d_5 differentials

(s, f, w)	Element	d_5	Proof
(56, 9, 29)	τPh_5e_0	$\tau\Delta h_0^2d_0e_0$	[30, Lemma 3.92]
(61, 6, 32)	A'	τMh_1d_0	[60, Theorem 12.1]
(63, 11, 33)	$\tau h_1^2\cdot\Delta x$	$\tau^3d_0^2e_0^2$	Lemma 5.59
(63, 23, 32)	$h_0^{22}h_6$	P^6d_0	$C\tau$
(67, 6, 36)	$h_0Q_3 + h_2^2D_3$	0	$C\tau$, τ
(68, 12, 36)	h_5d_0i	$\tau\Delta h_1d_0^3$	Lemma 5.60
(70, 4, 36)	$\tau p_1 + h_0^2h_3h_6$	$\tau^2h_2^2C'$	Lemma 5.61, [16]
(72, 7, 39)	$h_1x_{71,6}$	0	Lemma 5.62

TABLE 8. — (*Continued*)

(s, f, w)	Element	d_5	Proof
(73, 7, 38)	$h_4 D_2$	$\tau^4 d_1 g^2$	Lemma 5.63
(81, 10, 44)	gA'	$\tau M h_1 e_0^2$	τ
(85, 6, 45)	$x_{85,6}$	0	h_2
(86, 10, 44)	$h_0^2 h_6 i$	$\Delta^2 h_0^2 x$	$C\tau$
(86, 11, 45)	$\tau^3 g G_0$	$\tau M \Delta h_1^2 d_0$	Lemma 5.64
(92, 4, 48)	g_3	$h_6 d_0^2$	Lemma 5.65
(92, 12, 48)	$\Delta^2 g_2$	0	Lemma 5.66, [17]
(93, 8, 48)	$h_0 \cdot \Delta h_2^2 h_6$	$\Delta^2 h_0 g_2$	$C\tau$
(93, 13, 50)	$e_0 x_{76,9}$	$M \Delta h_1 c_0 d_0$	Lemma 5.67

TABLE 9. — **C**-motivic higher Adams differentials

(s, f, w)	Element	r	d_r	Proof
(67, 5, 35)	$\tau Q_3 + \tau n_1$	6	0	Lemma 5.69
(68, 7, 37)	$h_2^2 H_1$	6	$M c_0 d_0$	Lemma 5.70
(68, 7, 36)	$\tau h_2^2 H_1$	7	$M P d_0$	Lemma 5.70
(77, 7, 42)	m_1	7	0	Lemma 5.73
(80, 6, 43)	$h_1 x_1$	8	0	Lemma 5.74
(81, 3, 42)	$h_2 h_4 h_6$	6	0	Lemma 5.75
(83, 5, 43)	$\tau h_6 g + \tau h_2 e_2$	9	0	[16]
(85, 5, 45)	$h_1 f_2$	10	$?M \Delta h_1 d_0$	
(85, 6, 44)	$\tau x_{85,6} + h_0^3 c_3$	9	$?\tau M \Delta h_1 d_0$	
(86, 6, 46)	$h_2 h_6 g + h_1^2 f_2$	10	0	Lemma 5.76
(87, 7, 45)	$x_{87,7}$	7	0	Lemma 5.77
(87, 9, 48)	$g Q_3$	6	0	Lemma 5.69
(87, 10, 45)	$\tau \Delta h_1 H_1$	6	$\tau M \Delta h_0^2 e_0$	Lemma 5.78
(88, 11, 49)	$h_2^2 g H_1$	6	$M c_0 e_0^2$	Lemma 5.80
(88, 11, 48)	$\tau h_2^2 g H_1$	7	0	Lemma 5.80
(88, 12, 46)	$\Delta^2 f_1$	6	$\tau^2 M d_0^3$	Remark 5.82
(88, 12, 48)	$\Delta g_2 g$	6	$M d_0^3$	Lemma 5.81
(91, 6, 48)	$h_4^2 D_3$	9	$?\tau M \Delta h_1 g$	
(92, 10, 51)	$\Delta_1 h_1^2 e_1$	6	0	Lemma 5.84
(92, 10, 48)	$x_{92,10}$	7	$?\tau M \Delta c_0 d_0 + \tau^2 M d_0 l$	Lemma 5.85
(93, 9, 51)	$\Delta_1 h_2 e_1$	8	0	Lemma 5.86
(93, 13, 49)	$\tau e_0 x_{76,9}$	6	$M P \Delta h_1 d_0$	Lemma 5.87, [16]

TABLE 10. — Some Toda brackets

(s, w)	Bracket	Contains	Indeterminacy	Proof	Used in
(2, 1)	$\langle 2, \eta, 2 \rangle$	τh_1^2	0	$\langle h_0, h_1, h_0 \rangle$	6.26, 7.19, 7.20, 7.26, 7.37 7.38, 7.40, 7.50, 7.81
(3, 2)	$\langle \eta, 2, \eta \rangle$	$h_0 h_2$	τh_1^3	$\langle h_1, h_0, h_1 \rangle$	7.43
(6, 4)	$\langle \eta, \nu, \eta \rangle$	h_2^2	0	$\langle h_1, h_2, h_1 \rangle$	7.89, 7.95
(8, 5)	$\langle \nu, \eta, \nu \rangle$	$h_1 h_3$	0	$\langle h_2, h_1, h_2 \rangle$	7.113
(8, 5)	$\langle \eta, \nu, 2\nu \rangle$	c_0	$h_1 h_3$	$\langle h_1, h_2, h_0 h_2 \rangle$	7.76, 7.151
(8, 5)	$\langle \eta^2, \nu, \eta, 2 \rangle$	c_0	0	$\langle h_1^2, h_2, h_1, h_0 \rangle$	6.5
(9, 5)	$\langle 2, \epsilon, 2 \rangle$	$\tau h_1 c_0$	0	Corollary 6.2	7.27
(9, 5)	$\langle \eta, 2, 8\sigma \rangle$	$P h_1$	$\tau h_1^2 h_3, \tau h_1 c_0$	$\langle h_1, h_0, h_0^3 h_3 \rangle$	7.30, 7.68
(10, 5)	$\langle 2, \mu_9, 2 \rangle$	$\tau P h_1^2$	0	Corollary 6.2	7.30

TABLE 10. — (*Continued*)

(s, w)	Bracket	Contains	Indeterminacy	Proof	Used in
$(11, 6)$	$\langle 2, \eta, \tau\eta\epsilon \rangle$	Ph_2	Ph_0h_2, τPh_1^3	$\langle h_0, h_1, \tau h_1 c_0 \rangle$	7.74
$(15, 8)$	$\langle 8, 2\sigma, \sigma \rangle$	$h_0^3 h_4$	$h_0^6 h_4$, $h_0^7 h_4$	$d_2(h_4) = h_0 h_3^2$	7.147
$(15, 8)$	$\langle 2, \kappa, 2 \rangle$	$\tau h_1 d_0$	$h_0^4 h_4$, $h_0^5 h_4$ $h_0^6 h_4$, $h_0^7 h_4$	Corollary 6.2	7.36, 7.153
$(16, 9)$	$\langle \eta, \sigma^2, 2 \rangle$	$h_1 h_4$	Pc_0	$d_2(h_4) = h_0 h_3^2$	6.22
$(16, 9)$	$\langle \eta, 2, \sigma^2 \rangle$	$h_1 h_4$	Pc_0	$d_2(h_4) = h_0 h_3^2$	6.6
$(17, 9)$	$\langle 2, \eta_4, 2 \rangle$	$\tau h_1^2 h_4$	0	Corollary 6.2	7.38
$(17, 10)$	$\langle 2, \eta, \eta\kappa \rangle$	$h_2 d_0$	0	$d_2(e_0) = h_1^2 d_0$	7.80
$(18, 10)$	$\langle \nu, \sigma, 2\sigma \rangle$	$h_2 h_4$	0	$d_2(h_4) = h_0 h_3^2$	5.51
$(20, 11)$	$\langle \kappa, 2, \eta, \nu \rangle$	τg	$\tau h_0^2 g$	Lemma 6.5	7.105
$(20, 12)$	$\langle \nu, \eta, \eta\kappa \rangle$	$h_0 g$	$h_0^2 g$	$d_2(e_0) = h_1^2 d_0$	7.103
$(21, 12)$	$\langle \nu, 2\nu, \kappa \rangle$	$\tau h_1 g$	$h_2^2 h_4$	$d_2(f0) = h_0^2 e_0$	7.145
$(23, 12)$	$\langle \sigma, 16, 2\rho_{15} \rangle$	$h_0^2 i + \tau P h_1 d_0$	$\tau h_4 c_0$	$\langle h_3, h_0^4, h_0^4 h_4 \rangle$	6.19
$(30, 16)$	$\langle \sigma^2, 2, \sigma^2, 2 \rangle$	h_4^2	0	$d_2(h_4) = h_0 h_3^2$	7.35
$(32, 17)$	$\langle \eta, 2, \theta_4 \rangle$	$h_1 h_5$	0	$d_2(h_5) = h_0 h_4^2$	6.17
$(32, 18)$	$\langle \eta, \sigma^2, \eta, \sigma^2 \rangle$	d_1	0	$\langle h_1, h_3^2, h_1, h_3^2 \rangle$	7.77
$(33, 18)$	$\langle \eta\theta_4, \eta, 2 \rangle$	p	0	$\langle h_1 h_4^2, h_1, h_0 \rangle$	7.21
$(36, 20)$	$\langle \tau, \eta^2\kappa_1, \eta \rangle$	t	Ph_1^4	Lemma 6.9	Table 5
$(36, 20)$	$\langle \nu, \eta, \eta\theta_4 \rangle$	t	0	[10, Corollary 4.3]	7.21
$(39, 21)$	$\langle \eta_5, \nu, 2\nu \rangle$	$h_5 c_0$	$h_1 h_3 h_5$	$\langle h_1 h_5, h_2, h_0 h_2 \rangle =$ $= h_5 \langle h_1, h_2, h_0 h_2 \rangle$	7.145
$(39, 21)$	$\langle \epsilon, 2, \theta_4 \rangle$	$h_5 c_0$	0	$d_2(h_5) = h_0 h_4^2$	7.145
$(45, 24)$	$\langle \theta_4, 2, \sigma^2 + \kappa \rangle$	0 or $\tau h_1 g_2$	$h_0^2 h_5 d_0$	Lemma 6.10	
$(45, 24)$	$\langle 2, \theta_4, \kappa \rangle$	$h_5 d_0$	$h_0 h_3^2 h_5$, $h_0 h_5 d_0$ $h_0^2 h_5 d_0$	$d_2(h_5) = h_0 h_4^2$	7.145, 7.146
$(57, 30)$	$\langle \tau, \tau\eta\kappa\overline{\kappa}^2, \eta \rangle$	$h_0 h_2 h_5 i$	$P^6 h_1 c_0$	$d_5(\tau P h_5 e_0) = \tau \Delta h_0^2 d_0 e_0$	5.45
$(62, 32)$	$\langle 2, \theta_4, \theta_4, 2 \rangle$	h_5^2	$h_5 n$	$d_2(h_5) = h_0 h_4^2$	5.83, 7.153
$(62, 33)$	$\langle \eta, \eta\kappa, \tau\theta_{4.5} \rangle$	$\Delta e_1 + C_0$	0	Remark 7.104	7.103
$(63, 33)$	$\langle \tau\eta\theta_{4.5}, \kappa, 2, \eta \rangle$	$\tau h_1 H_1$		Remark 7.106	7.105
$(64, 33)$	$\langle \eta, 2, \theta_5 \rangle$	$h_1 h_6$	$\tau h_1^2 h_5^2$, $\tau^2 h_1 X_2$ $\tau h_3 Q_2$, $P^7 c_0$	$d_2(h_6) = h_0 h_5^2$	5.65, 7.20, 7.26, 7.40, 7.50, 7.63 7.74, 7.80, 7.107, 7.148, Table 5
$(64, 34)$	$\langle \nu, \eta, \tau\kappa\theta_{4.5} \rangle$	$h_2 A'$	0	$d_5(A') = \tau M h_1 d_0$	7.109, 7.128
$(66, 36)$	$\langle \eta^2, \theta_4, \eta^2, \theta_4 \rangle$	$\Delta_1 h_3^2$		Lemma 6.18	7.21
$(68, 36)$	$\langle \sigma, \kappa, \tau\eta\theta_{4.5} \rangle$	$h_3 A'$	0	$d_5(A') = \tau M h_1 d_0$	7.23, 7.110, Table 5
$(69, 36)$	$\langle \nu^2, 2, \theta_5 \rangle$	$h_2^2 h_6$		$d_2(h_6) = h_0 h_5^2$	7.151, Table 5
$(70, 36)$	$\langle 8\sigma, 2, \theta_5 \rangle$	$h_0^3 h_3 h_6$	$\tau h_1 p'$	$d_2(h_6) = h_0 h_5^2$	7.30, 7.68, 7.147, Table 5
$(70, 37)$	$\langle \eta, \nu, \tau\theta_{4.5}\overline{\kappa} \rangle$	$\tau h_1 D_3'$		Lemma 6.20	7.25, 7.113
$(70, 37)$	$\langle \eta, \tau\kappa^2, \tau\overline{\kappa}^2 \rangle$	$\tau \Delta^2 h_1^2 g +$ $+ \tau^3 \Delta h_2^2 g^2$		$d_3(\Delta h_2^2) = \tau h_1 d_0^2$ $d_3(\tau \Delta^2 h_1 g) = \tau^4 e_0^4$	Table 5
$(71, 37)$	$\langle \epsilon, 2, \theta_5 \rangle$	$h_6 c_0$		$d_2(h_6) = h_0 h_5^2$	7.26, 7.27, 7.114, Table 5
$(71, 37)$	$\langle \eta, \nu, [\tau^2 h_2 C'] \rangle$	$\tau h_1 p_1$		Lemma 6.21	7.28
$(71, 39)$	$\langle \nu, \epsilon, \kappa\theta_{4.5} \rangle$	$h_2^3 H_1$	$\tau M h_2^2 g$	$d_6(h_2^2 H_1) = M c_0 d_0$	7.73
$(72, 37)$	$\langle \mu_9, 2, \theta_5 \rangle$	$Ph_1 h_6$		$d_2(h_6) = h_0 h_5^2$	7.30, Table 5
$(72, 38)$	$\langle \tau\overline{\kappa}\theta_{4.5}, 2\nu, \nu \rangle$	$h_0 d_0 D_2$	$h_1^2 h_3 h_6$	$d_2(P(A{+}A')) = \tau^2 M h_0 h_2 g$	7.76
$(72, 38)$	$\langle \sigma^2, 2, \{t\}, \tau\overline{\kappa} \rangle$	$h_4 Q_2 + h_3^2 D_2$		Lemma 6.22	7.31
$(75, 42)$	$\langle \tau\eta\kappa\overline{\kappa}^2, \eta, \eta^2\eta_4 \rangle$	$\Delta h_2^2 d_0^2 e_0$	0	$\langle \Delta h_0^2 d_0 e_0, h_1, h_1^3 h_4 \rangle$	5.45
$(77, 40)$	$\langle \sigma^2, 2, \theta_5 \rangle$	$h_3^2 h_6$		$d_2(h_6) = h_0 h_5^2$ $d_2(h_4) = h_0 h_3^2$	7.120, Table 5
$(77, 40)$	$\langle \kappa, 2, \theta_5 \rangle$	$h_6 d_0$		Lemma 6.24	7.36, Table 5
$(79, 41)$	$\langle \eta_4, 2, \theta_5 \rangle$	$h_1 h_4 h_6$		$d_2(h_6) = h_0 h_5^2$	7.38, 7.123, Table 5

TABLE 10. — (*Continued*)

(s, w)	Bracket	Contains	Indeterminacy	Proof	Used in
(79, 42)	$\langle\{\tau m_1\}, \eta, 2\rangle$	0 or $\tau^2 \mathrm{M} e_0^2$	$h_0 h_2 x_{76,6}$	Lemma 6.25	[16]
(79, 42)	$\langle\{h_1 x_{76,6}\}, 2, \eta\rangle$	$h_0 h_2 x_{76,6}$		$\langle h_1 x_{76,6}, h_0, h_1\rangle =$ $= x_{76,6}\langle h_1, h_0, h_1\rangle$	7.38
(80, 41)	$\langle\mu_{17}, 2, \theta_5\rangle$	$\mathrm{P}^2 h_1 h_6$		$d_2(h_6) = h_0 h_5^2$	Table 5
(80, 42)	$\langle 2, \eta, \tau\eta\{h_1 x_{76,6}\}\rangle$	$\tau h_1 x_1$		Lemma 6.26	Table 5
(82, 43)	$\langle\overline{\sigma}, 2, \theta_5\rangle$	$h_6 c_1$		$d_2(h_6) = h_0 h_5^2$	7.88, Table 5
(82, 45)	$\langle\{\Delta e_1 + \mathrm{C}_0\}, \eta^3, \eta_4\rangle$	$(\Delta e_1 + \mathrm{C}_0) g$	$h_1^3 x_1$	$\langle\Delta e_1 + \mathrm{C}_0, h_1^3, h_1 h_4\rangle$	7.127
(83, 44)	$\langle\eta_6 \kappa, \eta, \nu\rangle$	$h_0 h_6 g$		$d_2(h_6 e_0) = h_1^2 h_6 d_0$	Table 5, 7.89
(83, 44)	$\langle\nu^2 \theta_5, 2, \sigma^2\rangle$	$\tau h_1 h_4 \mathrm{Q}_3$		$d_2(h_4) = h_0 h_3^2$	5.25
(84, 44)	$\langle\nu\nu_4, 2, \theta_5\rangle$	$h_2^2 h_4 h_6$		$d_2(h_6) = h_0 h_5^2$	7.131, Table 5
(85, 44)	$\langle\tau\eta^2\overline{\kappa}, 2, \theta_5\rangle$	$\mathrm{P} h_6 d_0$		$d_2(h_6) = h_0 h_5^2$	Table 5
(86, 46)	$\langle\tau\eta\overline{\kappa}^2, 2, 4\overline{\kappa}_2\rangle$	$\mathrm{M}\Delta h_0^2 e_0$		Lemma 6.30	5.69
(87, 46)	$\langle\theta_4, \tau\overline{\kappa}, \{t\}\rangle$	$\tau^2 g \mathrm{Q}_3$	0	$d_3(\mathrm{Q}_2) = \tau^2 g t$	5.69
(87, 46)	$\langle\tau\{h_0 \mathrm{Q}_3 + h_0 n_1\}, \nu_4, \eta\rangle$	$h_1^2 c_3$		Lemma 6.31	7.49, Table 5
(87, 47)	$\langle\{h_1 x_{71,6}\}, 2, \sigma^2\rangle$	$h_1 h_4 x_{71,6}$		$d_2(h_4) = h_0 h_3^2$	Table 5
(88, 45)	$\langle\mu_{25}, 2, \theta_5\rangle$	$\mathrm{P}^3 h_1 h_6$		$d_2(h_6) = h_0 h_5^2$	Table 5
(89, 49)	$\langle\nu, \eta, \{h_2 g \mathrm{A}'\}\rangle$	$\Delta h_1 g_2 g$	$\tau h_2^2 g \mathrm{C}'$	Remark 7.135	7.134
(91, 49)	$\langle\{h_1 h_3 g\}, 2, \theta_5\rangle$	$h_1 h_3 h_6 g$		$d_2(h_6) = h_0 h_5^2$	Table 5
(92, 48)	$\langle\theta_4, \theta_4, 2, \theta_4\rangle$	$h_0 g_3$		Lemma 5.83	
(93, 50)	$\langle\kappa_1, \kappa, \tau\eta\theta_{4.5}\rangle$	$h_1^2 x_{91,8}$		$d_5(\mathrm{A}') = \tau \mathrm{M} h_1 d_0$	Table 5
(94, 49)	$\langle\{n\}, 2, \theta_5\rangle$	$h_6 n$		$d_2(h_6) = h_0 h_5^2$	Table 5
(95, 49)	$\langle\{\Delta h_1 h_3\}, 2, \theta_5\rangle$	$\Delta h_1 h_3 h_6$		$d_2(h_6) = h_0 h_5^2$	Table 5
(95, 50)	$\langle\kappa_1, 2, \theta_5\rangle$	$h_6 d_1$		$d_2(h_6) = h_0 h_5^2$	Table 5
(95, 50)	$\langle\eta, \tau\kappa^2, \tau\theta_{4.5}\overline{\kappa}\rangle$	$\mathrm{M}\Delta^2 h_1^2 +$ $+\tau^2 \mathrm{M}\Delta h_2^2 g$		$d_3(\Delta h_2^2) = \tau h_1 d_0^2$ $d_3(\mathrm{M}\Delta^2 h_1) = \tau^3 \mathrm{M} d_0 e_0^2$	5.57

TABLE 11. — Some null Toda brackets

(s, w)	Bracket	Contains	Indeterminacy	Proof	Used in
(16, 9)	$\langle\kappa, 2, \eta\rangle$	0	$\mathrm{P} c_0$	Lemma 6.4	6.5, 7.105
(23, 13)	$\langle\epsilon + \eta\sigma, \sigma, 2\sigma\rangle$	0	$\mathrm{P} h_1 d_0$	Lemma 6.6	5.51
(30, 16)	$\langle\tau\kappa^2, \eta, 2\rangle$	0	0	Lemma 6.7	5.65
(35, 20)	$\langle\eta^2, \theta_4, \eta^2\rangle$	0	$h_1^4 h_5$	Lemma 6.8	6.18
(46, 25)	$\langle\eta, 2, 4\overline{\kappa}_2\rangle$	0	$h_1 h_5 d_0$, $\mathrm{M} h_1$, $\Delta h_2 c_1$, $\tau d_0 l + \Delta c_0 d_0$	Lemma 6.12	5.69, 6.19
(59, 31)	$\langle\tau\overline{\kappa}_2, \sigma^2, 2\rangle$	0	0	Lemma 6.13	7.35
(60, 33)	$\langle\eta\overline{\kappa}_2, 2\sigma, \sigma\rangle$	0	0	Lemma 6.14	
(60, 32)	$\langle 2, \sigma^2, \{h_3^2 h_5\}\rangle$	0	$\tau^2 d_0^2 l$	Lemma 6.15	5.13
(63, 34)	$\langle\theta_4, \eta^2, \theta_4\rangle$	0	0	Lemma 6.17	6.18, 7.21
(67, 36)	$\langle\tau\eta^2\overline{\kappa}, 8, \overline{\kappa}_2\rangle$	0		Lemma 6.19	5.70
(81, 43)	$\langle 2, \eta, \{h_2 x_{76,6}\}\rangle$	0	0	Lemma 6.27	7.43
(84, 45)	$\langle 2, \sigma^2, \{\tau h_2^2 \mathrm{C}'\}\rangle$	0	0	Lemma 6.28	5.26
(84, 45)	$\langle 2, \sigma^2, \{h_3(\Delta e_1 + \mathrm{C}_0)\}\rangle$	0	0	Lemma 6.29	5.26

TABLE 12. — Hidden values of inclusion of the bottom cell into $C\tau$

(s, f, w)	Source	Value	Proof
(50, 6, 26)	$\tau \mathrm{C}$	$\overline{\mathrm{P} h_1^2 h_5 c_0}$	
(57, 10, 30)	$h_0 h_2 h_5 i$	$\overline{\Delta^2 h_1^2 h_3}$	
(63, 6, 33)	$\tau h_1 \mathrm{H}_1$	$\overline{h_1 \mathrm{B}_7}$	

TABLE 12. — (*Continued*)

(s, f, w)	Source	Value	Proof
$(63, 7, 33)$	$\tau X_2 + \tau C'$	$\overline{h_5 d_0 e_0}$	
$(64, 8, 34)$	$\tau h_1 X_2$	$\overline{h_1 h_5 d_0 e_0}$	
$(66, 8, 35)$	$\tau h_2 C'$	τB_5	
$(70, 7, 37)$	$\tau h_1 h_3 H_1$	$\overline{h_1 h_3 B_7}$	
$(70, 8, 37)$	$\tau h_1 D'_3$	$\overline{h_1^2 X_3}$	
$(70, 10, 38)$	$h_1 h_3 (\Delta e_1 + C_0) + \tau h_2 C''$	$\overline{h_1^4 c_0 Q_2}$	Lemma 7.6
$(71, 5, 37)$	$\tau h_1 p_1$	$\overline{h_0^2 h_2 Q_3}$	
$(74, 6, 39)$	$h_3(\tau Q_3 + \tau n_1)$	$\overline{h_1^4 p'}$	
$(75, 11, 41)$	$h_1^3 h_4 Q_2$	$\overline{\tau h_2 g^2 n}$	
$(77, 6, 40)$	$\tau h_1 h_4 D_3$	$x_{77,7}$	
$(81, 8, 43)$	$\tau g D_3$	$\overline{h_1^4 x_{76,6}}$	
$(83, 10, 45)$	$h_2 c_1 A'$	$\overline{h_1 g B_7}$	
$(85, 5, 44)$	$\tau h_1 f_2$	$?h_0^3 c_3$	
$(86, 6, 45)$	$\tau h_1^2 f_2$	$?\overline{\tau h_1^3 h_4 Q_3}$	
		$?\Delta^2 e_1 + \overline{\tau \Delta h_2 e_1 g}$	
$(86, 7, 45)$	$\tau h_1 x_{85,6}$	$?\overline{\tau h_1^3 h_4 Q_3}$	
		$?\Delta^2 e_1 + \overline{\tau \Delta h_2 e_1 g}$	
$(86, 12, 47)$	$\tau h_2 g C'$	$\tau B_5 g$	
$(88, 11, 48)$	$\tau h_2^2 g H_1$	$\Delta g_2 g$	
$(90, 14, 50)$	$\tau h_2 g C''$	$\overline{P h_1^{10} h_6 c_0}$	
$(90, 19, 49)$	$\tau^3 g m^2$	$\Delta^2 c_0 e_0^2$	

TABLE 13. — Hidden values of projection from $C\tau$ to the top cell

(s, f, w)	Source	Value	Crossing source
$(30, 6, 16)$	Δh_2^2	$h_1 d_0^2$	
$(34, 2, 18)$	$h_2 h_5$	$h_1 d_1$	
$(38, 7, 20)$	$h_0 y$	$\tau h_0 e_0 g$	
$(41, 4, 22)$	$h_0 c_2$	$h_1 h_3 d_1$	
$(44, 10, 24)$	$\Delta h_2^2 d_0$	$h_1 d_0^3$	
$(50, 10, 28)$	$\Delta h_2^2 g$	$h_1 d_0 e_0^2$	
$(55, 7, 30)$	B_6	$h_2 g n$	
$(56, 10, 29)$	$\Delta^2 h_1 h_3$	$\Delta h_0^2 d_0 e_0$	
$(57, 7, 30)$	Q_2	$\tau g t$	
$(58, 7, 30)$	$h_0 D_2$	$\Delta h_1 d_1$	
$(58, 11, 32)$	$P h_1^2 h_5 e_0$	$\tau h_2 e_0^2 g$	
$(59, 8, 33)$	$h_1^2 D_4$	$h_2^2 d_1 g$	
$(61, 6, 32)$	A'	$M h_1 d_0$	$\overline{h_1^3 Q_2}$
$(62, 11, 32)$	$\overline{P h_5 c_0 d_0}$	$\tau \Delta h_2^2 d_0 e_0$	
$(63, 12, 33)$	$\overline{P h_1 h_5 c_0 d_0}$	$\tau^2 d_0^2 e_0^2$	
$(64, 14, 36)$	km	$h_1 d_0^2 e_0^2$	
$(65, 6, 35)$	$h_2 H_1$	d_1^2	
$(65, 8, 34)$	$h_0 h_3 D_2$	$h_1^2 (\Delta e_1 + C_0)$	
$(66, 3, 34)$	$h_0 h_2 h_6$	$h_1^3 h_5^2$	
$(68, 5, 36)$	$h_0 d_2$	$h_1 \cdot \Delta_1 h_3^2$	
$(68, 7, 37)$	$h_2^2 H_1$	$h_1^4 X_2$	$\overline{h_1 h_3 j_1}$
$(68, 12, 35)$	$\overline{\tau M c_0 d_0}$	$\tau^2 \Delta h_2^2 e_0 g$	
$(68, 12, 36)$	$h_5 d_0 i$	$\Delta h_1 d_0^3$	

TABLE 13. — (*Continued*)

(s, f, w)	Source	Value	Crossing source
(69, 9, 36)	$\overline{h_1 X_3}$	$\tau M h_2 g$	
(69, 10, 36)	$P(A + A')$	$\tau M h_0 h_2 g$	
(69, 13, 36)	$\tau \Delta^2 h_1 g$	$\tau^3 e_0^4$	
(70, 4, 36)	$h_0^2 h_3 h_6$	$\tau h_2^2 C'$	
(70, 14, 40)	m^2	$h_1 e_0^4$	
(72, 9, 40)	$h_2^2 G_0$	$g^2 n$	
(72, 10, 38)	$d_0 D_2$	$\tau M h_2^2 g$	
(73, 7, 38)	$h_4 D_2$	$\tau^3 d_1 g^2$	
(74, 6, 38)	$P h_2 h_6$	$h_1 h_4 Q_2$	
(75, 11, 40)	$h_2 d_0 D_2$	$M h_1 d_0^2$	
(75, 11, 42)	$g B_6$	$h_2 g^2 n$	
(76, 14, 41)	$\Delta^2 h_1 h_3 g$	$\Delta h_2^2 d_0^2 e_0$	
(77, 11, 41)	$\tau g Q_2$	$\tau^2 g^2 t$	
(78, 7, 42)	$h_3 x_{71,6}$	$h_1 d_1 g_2$	
(78, 13, 40)	$h_0^3 x_{78,10}$	$\tau^5 e_0 g^3$	
(80, 5, 42)	$h_0 e_2$	$h_1^3 x_{76,6}$	
(80, 14, 41)	$\Delta^3 h_1 h_3$	$\tau^3 \Delta h_1 e_0^2 g$	
(81, 10, 44)	$g A'$	$M h_1 e_0^2$	$\overline{P h_1^9 h_6}$
(81, 15, 42)	$\Delta^3 h_1^2 h_3$	$\tau^3 d_0 e_0^2 l$	
(82, 16, 44)	$\Delta^2 e_0^2$	$\tau \Delta h_2^2 e_0^3$	
(83, 17, 45)	$\Delta^2 h_1 e_0^2$	$\tau^2 d_0 e_0^4$	
(84, 4, 44)	f_2	$h_1 h_4 Q_3$	
(84, 18, 48)	$d_0 m^2$	$h_1 d_0 e_0^4$	
(85, 5, 45)	$h_1 f_2$	$?h_1^2 h_4 Q_3$ $?\Delta h_1 j_1$	
(85, 6, 44)	$h_0^3 c_3$	$?M \Delta h_1 d_0$	
(85, 6, 45)	$x_{85,6}$	$?h_1^2 h_4 Q_3$ $?\Delta h_1 j_1$	
(85, 10, 47)	$h_2 g H_1$	$d_1^2 g$	
(86, 4, 45)	$h_1 c_3$	$h_0 h_2 h_4 Q_3$	
(86, 6, 46)	$h_1^2 f_2$	$?h_1^3 h_4 Q_3$ $?\tau M h_0 g^2$	
(86, 7, 46)	$h_1 x_{85,6}$	$?h_1^3 h_4 Q_3$ $?\tau M h_0 g^2$	
(86, 8, 45)	$\overline{\tau h_1^3 h_4 Q_3}$	$?M \Delta h_1^2 d_0$	$\overline{P^2 h_1^6 h_6}$
(86, 12, 45)	$\Delta^2 e_1 + \overline{\tau \Delta h_2 e_1 g}$	$?M \Delta h_1^2 d_0$	$\overline{P^2 h_1^6 h_6}$
(86, 15, 44)	$\overline{\Delta^3 h_0 h_3^2}$	$\tau \Delta^2 h_0 e_0 g$	
(87, 11, 45)	$\overline{\Delta h_1 B_7}$	$M \Delta h_0^2 e_0$	
(87, 17, 45)	$\Delta^3 h_1 d_0$	$\tau^4 e_0^3 m$	
(88, 11, 49)	$h_2^2 g H_1$	$M h_1^2 g^2$	
(88, 16, 47)	$\tau \Delta^2 g^2$	$\tau^2 \Delta h_2^2 e_0 g^2$	
(88, 17, 48)	$\Delta^2 h_0 g^2$	$\Delta h_1 d_0^2 e_0^2$	
(89, 12, 46)	$\Delta^2 h_0 c_2$	$\Delta^2 h_1 h_3 d_1$	
(89, 17, 48)	$\tau \Delta^2 h_1 g^2$	$\tau^3 e_0^4 g$	
(90, 18, 52)	$g m^2$	$h_1 e_0^4 g$	

TABLE 14. — Hidden τ extensions

(s, f, w)	From	To	Proof
$(22, 7, 13)$	$c_0 d_0$	$P d_0$	
$(23, 8, 14)$	$h_1 c_0 d_0$	$P h_1 d_0$	
$(28, 6, 17)$	$h_1 h_3 g$	d_0^2	
$(29, 7, 18)$	$h_1^2 h_3 g$	$h_1 d_0^2$	
$(40, 9, 23)$	$\tau h_0 g^2$	$\Delta h_1^2 d_0$	
$(41, 9, 23)$	$\tau^2 h_1 g^2$	$\Delta h_0^2 e_0$	
$(42, 11, 25)$	$c_0 e_0^2$	d_0^3	
$(43, 12, 26)$	$h_1 c_0 e_0^2$	$h_1 d_0^3$	
$(46, 6, 26)$	$h_1^2 g_2$	$\Delta h_2 c_1$	
$(47, 12, 26)$	$\Delta h_1 c_0 d_0$	$P \Delta h_1 d_0$	
$(48, 10, 29)$	$h_1 h_3 g^2$	$d_0 e_0^2$	
$(49, 11, 30)$	$h_1^2 h_3 g^2$	$h_1 d_0 e_0^2$	
$(52, 10, 29)$	$\Delta h_1 h_3 g$	$\tau^2 e_0 m$	
$(53, 9, 29)$	$M c_0$	MP	
$(53, 11, 30)$	$\Delta h_1^2 h_3 g$	$\Delta h_1 d_0^2$	
$(54, 8, 31)$	$h_1 i_1$	$M h_1 c_0$	
$(54, 10, 30)$	$M h_1 c_0$	$MP h_1$	
$(54, 11, 32)$	$h_1^6 h_5 e_0$	$\tau e_0^2 g$	
$(55, 12, 33)$	$h_1^7 h_5 e_0$	$\tau h_1 e_0^2 g$	
$(55, 13, 31)$	$\tau^2 h_1 e_0^2 g$	$\Delta h_0^2 d_0 e_0$	
$(59, 7, 33)$	j_1	$M d_0$	
$(59, 12, 33)$	$P h_1^3 h_5 e_0$	$\tau \Delta h_1 d_0 g$	
$(60, 9, 34)$	$h_1^3 D_4$	$M h_1 d_0$	
$(60, 13, 34)$	$\tau^2 h_0 g^3$	$\Delta c_0 d_0^2 + \tau d_0^2 l$	
$(61, 13, 35)$	$\tau^2 h_1 g^3$	$\Delta h_2^2 d_0 e_0$	
$(62, 14, 37)$	$h_1^6 h_5 c_0 e_0$	$d_0^2 e_0^2$	
$(63, 15, 38)$	$h_0^2 h_2 g^3$	$h_1 d_0^2 e_0^2$	
$(65, 9, 36)$	$h_1^2 X_2$	$\tau M g$	
$(66, 10, 37)$	$h_1^3 X_2$	$\tau M h_1 g$	
$(66, 14, 37)$	$P h_1^2 h_5 c_0 e_0$	$\tau^2 d_0 e_0 m$	
$(67, 15, 38)$	$P h_1^3 h_5 c_0 e_0$	$\Delta h_1 d_0^3$	
$(68, 14, 41)$	$h_1 h_3 g^3$	e_0^4	
$(69, 15, 42)$	$h_1^2 h_3 g^3$	$h_1 e_0^4$	
$(70, 8, 39)$	$d_1 e_1$	$h_1 h_3 (\Delta e_1 + C_0)$	Lemma 7.6
$(70, 10, 38)$	$\tau h_2 C'' + h_1 h_3 (\Delta e_1 + C_0)$	$\Delta^2 h_2 c_1$	
$(71, 8, 39)$	$h_2^3 H_1$	$h_3^2 Q_2$	
$(72, 7, 39)$	$h_1 x_{71,6}$	$h_0 d_0 D_2$	
$(72, 14, 41)$	$\Delta h_1 h_3 g^2$	$\tau^2 e_0 g m$	
$(73, 6, 39)$	$h_1^2 h_6 c_0$	$h_0 h_4 D_2$	
$(73, 11, 40)$	$\tau h_2^2 C''$	$\Delta^2 h_1^2 h_4 c_0$	
$(73, 12, 41)$	$M h_1 h_3 g$	$M d_0^2$	
$(73, 15, 42)$	$\Delta h_1^2 h_3 g^2$	$\Delta h_1 d_0 e_0^2$	
$(74, 13, 42)$	$M h_1^2 h_3 g$	$M h_1 d_0^2$	
$(75, 17, 43)$	$\tau^2 h_1 e_0^2 g^2$	$\Delta h_2^2 d_0^2 e_0$	
$(77, 15, 42)$	$\Delta^2 h_2^3 g$	$\tau^5 e_0 g^3$	
$(78, 8, 43)$	$h_1 m_1$	$M \Delta h_1^2 h_3$	
$(79, 11, 45)$	$g j_1$	$M e_0^2$	
$(80, 12, 46)$	$h_1 g j_1$	$M h_1 e_0^2$	
$(80, 17, 47)$	$\tau h_0 g^4$	$\Delta h_1^2 e_0^2 g$	
$(80, 18, 46)$	$\Delta h_1^2 e_0^2 g$	$\Delta c_0 d_0 e_0^2 + \tau d_0 e_0^2 l$	
$(81, 17, 47)$	$\tau^2 h_1 g^4$	$\Delta h_2^2 e_0^3$	

TABLE 14. — (*Continued*)

(s, f, w)	From	To	Proof
(82, 12, 44)	$\tau(\Delta e_1 + C_0)g$	$\Delta^2 h_2 n$	
(82, 19, 49)	$c_0 e_0^2 g^2$	$d_0 e_0^4$	
(83, 20, 50)	$h_1 c_0 e_0^2 g^2$	$h_1 d_0 e_0^4$	
(84, 12, 46)	$\Delta h_1 j_1$	$?M\Delta h_1 d_0$	
(85, 8, 45)	$h_6 c_0 d_0$	$Ph_6 d_0$	
(85, 15, 47)	$\tau M h_0 g^2$	$?M\Delta h_1^2 d_0$	
(86, 9, 46)	$h_1 h_6 c_0 d_0$	$Ph_1 h_6 d_0$	
(86, 12, 47)	$\tau h_2 g C'$	$\Delta^2 h_2^2 d_1$	
(86, 15, 47)	$\tau^2 M h_1 g^2$	$M\Delta h_0^2 e_0$	
(87, 20, 50)	$\Delta h_1 c_0 e_0^2 g$	$\Delta h_1 d_0^2 e_0^2$	
(88, 18, 53)	$h_1 h_3 g^4$	$e_0^4 g + M h_1^6 e_0 g$	
(89, 13, 49)	$\tau h_2^2 g C'$	$\Delta^2 h_1^2 h_3 d_1$	
(89, 19, 54)	$h_1^2 h_3 g^4$	$h_1 e_0^4 g + M h_1^7 e_0 g$	
(90, 14, 50)	$\tau h_2 g C''$	$\Delta^2 h_2 c_1 g$	

TABLE 15. — Hidden 2 extensions

(s, f, w)	Source	Target	Proof	Notes
(23, 6, 13)	$\tau h_0 h_2 g$	$Ph_1 d_0$	τ	
(23, 6, 14)	$h_0 h_2 g$	$h_1 c_0 d_0$	$C\tau$	
(40, 8, 22)	$\tau^2 g^2$	$\Delta h_1^2 d_0$	τ	
(43, 10, 25)	$\tau h_0 h_2 g^2$	$h_1 d_0^3$	τ	
(43, 10, 26)	$h_0 h_2 g^2$	$h_1 c_0 e_0^2$	$C\tau$	
(47, 10, 25)	$\tau \Delta h_2^2 e_0$	$P\Delta h_1 d_0$	τ	
(47, 10, 26)	$\Delta h_2^2 e_0$	$\Delta h_1 c_0 d_0$	$C\tau$	
(51, 6, 28)	$h_0 h_3 g_2$	$\tau g n$	[61]	
(54, 9, 28)	$h_0 h_5 i$	$\tau^4 e_0^2 g$	Lemma 7.18, [15]	
(60, 12, 33)	$\tau^3 g^3$	$\Delta c_0 d_0^2 + \tau d_0^2 l$	τ	
(63, 6, 33)	$\tau h_1 H_1$	$\tau h_1(\Delta e_1 + C_0)$	Lemma 7.19	
(63, 14, 37)	$\tau h_0 h_2 g^3$	$h_1 d_0^2 e_0^2$	τ	
(64, 2, 33)	$h_1 h_6$	$\tau h_1^2 h_5^2$	Lemma 7.20	
(65, 9, 36)	$h_1^2 X_2$	$M h_0 g$	τ	
(67, 14, 37)	$\tau \Delta h_2^2 e_0 g$	$\Delta h_1 d_0^3$	τ	
(70, 7, 37)	$\tau h_1 h_3 H_1$	$\tau h_1 h_3(\Delta e_1 + C_0)$	Lemma 7.19	
(71, 4, 37)	$h_6 c_0$	$\tau h_1^2 p'$	Lemma 7.27	
(71, 8, 39)	$h_2^3 H_1$	$\tau M h_2^2 g$	Lemma 7.29	
(74, 6, 39)	$h_3(\tau Q_3 + \tau n_1)$	$\tau x_{74,8}$	Lemma 7.35	indet
(74, 10, 41)	$h_3 C''$	$M h_1 d_0^2$	$C\tau$	
(74, 14, 40)	$\Delta^2 h_2^2 g$	$\tau^4 e_0^2 g^2$	*mmf*	
(77, 6, 41)	$\tau h_1 h_4 D_3$	$h_0 x_{77,7}$	$C\tau$	
(78, 10, 42)	$e_0 A'$	$M\Delta h_1^2 h_3$	Lemma 7.37	
(80, 16, 45)	$\tau^3 g^4$	$\Delta c_0 d_0 e_0^2 + \tau d_0 e_0^2 l$	τ	
(80, 16, 46)	$\tau^2 g^4$	$\Delta h_1^2 e_0^2 g + M h_1^3 d_0 e_0$	τ	
(83, 18, 49)	$\tau h_0 h_2 g^4$	$h_1 d_0 e_0^4$	τ	
(83, 18, 50)	$h_0 h_2 g^4$	$h_1 c_0 e_0^2 g^2$	$C\tau$	
(85, 14, 46)	$\tau^2 M g^2$	$?M\Delta h_1^2 d_0$	τ	
(86, 7, 45)	$\tau h_0 h_2 h_6 g$	$Ph_1 h_6 d_0$	τ	
(86, 7, 46)	$h_0 h_2 h_6 g$	$h_1 h_6 c_0 d_0$	$C\tau$	
(87, 7, 45)	$x_{87,7}$	$\tau^3 g Q_3$	Remark 7.13	
(87, 9, 48)	$g Q_3$	$B_6 d_1$	$C\tau$	
(87, 18, 49)	$\tau \Delta h_2^2 e_0 g^2$	$\Delta h_1 d_0^2 e_0^2$	$C\tau$	
(87, 18, 50)	$\Delta h_2^2 e_0 g^2$	$\Delta h_1 c_0 e_0^2 g + M h_1^3 c_0 d_0 e_0$	$C\tau$	
(90, 10, 50)	$h_2 g Q_3$	$\tau d_1 e_1 g$	$C\tau$	

Table 16. — Some null hidden 2 extensions

(s, f, w)	Source	Proof
(63, 7, 33)	$\tau X_2 + \tau C'$	Lemma 7.19
(66, 6, 36)	$\Delta_1 h_3^2$	Lemma 7.21
(67, 6, 36)	$h_0 Q_3 + h_2^2 D_3$	Lemma 7.22
(68, 7, 36)	$h_3 A'$	Lemma 7.23
(69, 4, 36)	p'	Lemma 7.24
(70, 9, 37)	$\tau h_1 D_3'$	Lemma 7.25
(71, 3, 37)	$h_1 h_3 h_6$	Lemma 7.26
(71, 5, 37)	$\tau h_1 p_1$	Lemma 7.28
(72, 6, 37)	$P h_1 h_6$	Lemma 7.30
(72, 8, 38)	$h_4 Q_2 + h_3^2 D_2$	Lemma 7.31
(73, 7, 40)	$h_2^2 Q_3$	Lemma 7.33
(73, 8, 38)	$h_0 h_4 D_2$	Lemma 7.34
(77, 5, 40)	$h_6 d_0$	Lemma 7.36
(79, 3, 41)	$h_1 h_4 h_6$	Lemma 7.38
(79, 8, 42)	$h_0 h_2 x_{76,6}$	Lemma 7.39
(79, 8, 41)	$P h_6 c_0$	Lemma 7.40
(79, 11, 42)	ΔB_6	Lemma 7.41
(82, 6, 44)	$h_5^2 g$	Lemma 7.42
(82, 8, 44)	$h_2^2 x_{76,6}$	Lemma 7.43
(83, 7, 44)	$h_0^2 h_6 g$	Lemma 7.44
(85, 7, 45)	$\tau h_2 h_4 Q_3$	Lemma 7.45
(85, 8, 45)	$h_6 c_0 d_0$	Lemma 7.46
(85, 9, 44)	$P h_6 d_0$	Lemma 7.46
(86, 5, 45)	$h_4 h_6 c_0$	Lemma 7.47
(86, 12, 47)	$\tau h_2 g C'$	Lemma 7.48
(87, 5, 46)	$h_1^2 c_3$	Lemma 7.49
(87, 12, 45)	$P^2 h_6 c_0$	Lemma 7.50
(90, 12, 48)	M^2	Lemma 7.51

Table 17. — Possible hidden 2 extensions

(s, f, w)	Source	Target
(59, 7, 33)	j_1	$?\tau^2 c_1 g^2$
(72, 7, 39)	$h_1 x_{71,6}$	$?\tau^3 d_1 g^2$
(79, 11, 45)	$g j_1$	$?\tau^2 c_1 g^3$
(85, 5, 45)	$h_1 f_2$	$?\tau h_2 h_4 Q_3$
		$?\tau P h_1 x_{76,6}$
(85, 5, 44)	$\tau h_1 f_2$	$?\tau^2 h_2 h_4 Q_3$
		$?\tau^2 P h_1 x_{76,6}$
		$?\tau^4 M g^2$
(86, 6, 46)	$h_1^2 f_2$	$?\Delta^2 h_2^2 d_1$
(86, 6, 45)	$\tau h_1^2 f_2$	$?\tau \Delta^2 h_2^2 d_1$
(86, 7, 45)	$\tau h_1 x_{85,6}$	$?\tau \Delta^2 h_2^2 d_1$
		$?P h_1 h_6 d_0$

TABLE 18. — Hidden η extensions

(s,f,w)	Source	Target	Proof	Notes
(15, 4, 8)	$h_0^3h_4$	Pc_0	$C\tau$	
(21, 5, 11)	τ^2h_1g	Pd_0	τ	
(21, 5, 12)	τh_1g	c_0d_0	$C\tau$	
(23, 9, 12)	h_0^2i	P^2c_0	$C\tau$	
(31, 11, 16)	$h_0^{10}h_5$	P^3c_0	$C\tau$	
(38, 4, 20)	$h_0^2h_3h_5$	τ^2c_1g	[30, Table 29]	
(39, 17, 20)	$P^2h_0^2i$	P^4c_0	$C\tau$	
(40, 8, 21)	τ^3g^2	$\Delta h_0^2e_0$	τ	
(41, 5, 23)	h_1f_1	τh_2c_1g	[30, Table 29]	crossing
(41, 9, 23)	$\tau^2h_1g^2$	d_0^3	τ	
(41, 9, 24)	τh_1g^2	$c_0e_0^2$	$C\tau$	
(41, 10, 22)	$\Delta h_0^2e_0$	τd_0^3	τ	
(45, 3, 24)	$h_3^2h_5$	Mh_1	[30, Table 29]	crossing
(45, 5, 24)	τh_1g_2	Δh_2c_1	τ	
(45, 9, 24)	$\tau\Delta h_1g$	$\tau d_0l+\Delta c_0d_0$	$C\tau$	
(46, 11, 24)	τ^2d_0l	$P\Delta h_1d_0$	τ	
(47, 10, 26)	$\Delta h_2^2e_0$	$\tau d_0e_0^2$	*mmf*	
(47, 20, 24)	h_0^7Q'	P^5c_0	$C\tau$	
(50, 6, 26)	τC	τ^2gn	[61]	
(52, 11, 28)	τ^2e_0m	$\Delta h_1d_0^2$	τ	
(54, 12, 29)	$\tau^3e_0^2g$	$\Delta h_0^2d_0e_0$	τ	
(55, 25, 28)	$P^4h_0^2i$	P^6c_0	$C\tau$	
(59, 13, 31)	$\tau\Delta h_1d_0g$	$\Delta c_0d_0^2+\tau d_0^2l$	τ	
(60, 12, 33)	τ^3g^3	$\Delta h_2^2d_0e_0$	τ	
(61, 9, 35)	$h_1^2j_1$	$\tau h_2c_1g^2$	$C\tau$	crossing
(61, 13, 35)	$\tau^2h_1g^3$	$d_0^2e_0^2$	τ	
(61, 14, 34)	$\Delta h_2^2d_0e_0$	$\tau d_0^2e_0^2$	τ	
(63, 6, 33)	τh_1H_1	h_3Q_2	$C\tau$	indet
(63, 26, 32)	$h_0^{25}h_6$	P^7c_0	$C\tau$	
(64, 8, 34)	τh_1X_2	c_0Q_2	$C\tau$	
(64, 8, 33)	$\tau^2h_1X_2$	τ^2Mh_0g	Lemma 7.62	indet
(65, 13, 35)	$\tau^2\Delta h_1g^2$	$\tau^2d_0e_0m$	τ	
(66, 15, 36)	$\tau^2d_0e_0m$	$\Delta h_1d_0^3$	τ	
(67, 14, 38)	$\Delta h_2^2e_0g$	τe_0^4	$C\tau$	
(68, 7, 36)	h_3A'	$h_3(\Delta e_1+C_0)$	Lemma 7.66	
(69, 3, 36)	$h_2^2h_6$	$\tau h_0h_2Q_3$	Lemma 7.67	crossing
(70, 7, 37)	$\tau h_1h_3H_1$	$h_3^2Q_2$	Lemma 7.70	
(70, 9, 37)	$\tau h_1D_3'$	d_0Q_2	$C\tau$	
(71, 5, 37)	τh_1p_1	h_4Q_2	$C\tau$	
(71, 13, 38)	Δ^2h_2g	τ^3e_0gm	*mmf*	
(71, 33, 36)	$P^6h_0^2i$	P^8c_0	$C\tau$	
(72, 5, 37)	$\tau h_1h_6c_0$	$\tau^2h_2^2Q_3$	Lemma 7.74	indet
(72, 11, 38)	$h_0d_0D_2$	τMd_0^2	Lemma 7.76	indet
(72, 15, 40)	τ^2e_0gm	$\Delta h_1d_0e_0^2$	τ	
(74, 16, 41)	$\tau^3e_0^2g^2$	$\Delta h_2^2d_0^2e_0$	τ	
(75, 6, 40)	$h_0h_3d_2$	τd_1g_2	Lemma 7.77	
(75, 10, 42)	$h_1^4x_{71,6}$	h_1gB_6	$C\tau$	
(75, 11, 41)	$h_1^3h_4Q_2$	τ^2g^2t	$C\tau$	
(76, 9, 40)	$x_{76,9}$	$M\Delta h_1h_3$	$C\tau$	
(77, 6, 40)	$\tau h_1h_4D_3$	$x_{78,9}$	$C\tau$	crossing

Table 18. — (*Continued*)

(s, f, w)	Source	Target	Proof	Notes
(78, 8, 40)	$h_0^6 h_4 h_6$	$\tau \Delta B_6$	Lemma 7.82	
(78, 10, 42)	$e_0 A'$	$\tau M e_0^2$	Lemma 7.83	
(79, 17, 44)	$\tau \Delta h_1 e_0^2 g$	$\Delta c_0 d_0 e_0^2 + \tau d_0 e_0^2 l$	τ	
(79, 36, 40)	$P^4 h_0^7 Q'$	$P^9 c_0$	$C\tau$	
(80, 16, 45)	$\tau^3 g^4$	$\Delta h_2^2 e_0^3$	τ	
(81, 13, 47)	$h_1^2 g j_1$	$\tau h_2 c_1 g^3$	$C\tau$	crossing
(81, 17, 47)	$\tau^2 h_1 g^4$	$d_0 e_0^4$	τ	
(81, 17, 48)	$\tau h_1 g^4$	$c_0 e_0^2 g^2$	$C\tau$	
(81, 18, 46)	$\Delta h_2^2 e_0^3$	$\tau d_0 e_0^4$	τ	
(83, 11, 44)	$\tau \Delta j_1 + \tau^2 g C'$	$?M\Delta h_1 d_0$	τ	crossing
(84, 6, 43)	$\tau^2 h_1 h_6 g$	$P h_6 d_0$	τ	
(84, 6, 44)	$\tau h_1 h_6 g$	$h_6 c_0 d_0$	$C\tau$	
(85, 14, 45)	$\tau^3 M g^2$	$M\Delta h_0^2 e_0$	τ	
(85, 17, 48)	$\tau \Delta h_1 g^3$	$\Delta c_0 e_0^2 g + M h_1^2 c_0 d_0 e_0$ $+ \tau e_0^3 m$	$C\tau$	
(86, 11, 44)	$h_0^3 h_6 i$	$\tau^2 \Delta^2 c_1 g$	Lemma 7.92	
(86, 16, 47)	$P^2 h_1^7 h_6$	$\tau^2 \Delta h_2^2 e_0 g^2$	$C\tau$	
(86, 19, 48)	$\tau^2 e_0^3 m$	$\Delta h_1 d_0^2 e_0^2$	τ	
(87, 8, 47)	$h_1 h_4 x_{71,6}$	$x_{88,10}$	$C\tau$	
(87, 18, 50)	$\Delta h_2^2 e_0 g^2$	$\tau e_0^4 g$	$C\tau$	
(87, 41, 44)	$P^8 h_0^2 i$	$P^{10} c_0$	$C\tau$	
(88, 11, 48)	$\tau h_2^2 g H_1$	$\Delta h_1 g_2 g + \tau h_2^2 g C'$	$C\tau$	
(89, 13, 47)	$\Delta^2 h_1 f_1$	$\tau \Delta^2 h_2 c_1 g$	Lemma 7.95	crossing

Table 19. — Some null hidden η extensions

(s, f, w)	Source	Proof
(58, 8, 30)	$\tau h_1 Q_2$	Lemma 7.60
(64, 4, 33)	$\tau h_1^2 h_5^2$	Lemma 7.61
(66, 4, 34)	$\tau h_1^3 h_6$	Lemma 7.63
(68, 6, 36)	$\tau h_1 Q_3$	Lemma 7.65
(70, 5, 36)	$h_0^3 h_3 h_6$	Lemma 7.68
(70, 6, 38)	$h_2 Q_3$	Lemma 7.69
(70, 10, 38)	$h_1 h_3 (\Delta e_1 + C_0)$	Lemma 7.71
(70, 10, 38)	$\tau h_2 C'' + h_1 h_3 (\Delta e_1 + C_0)$	Lemma 7.71
(71, 6, 37)	$\tau h_1^2 p'$	Lemma 7.72
(71, 8, 39)	$h_2^3 H_1$	Lemma 7.73
(72, 7, 39)	$h_1^3 p'$	Lemma 7.75
(77, 3, 40)	$h_3^2 h_6$	Lemma 7.78
(77, 7, 41)	τm_1	Lemma 7.78
(77, 8, 40)	$h_0 x_{77,7}$	Lemma 7.79
(78, 6, 41)	$h_1 h_6 d_0$	Lemma 7.80
(78, 8, 41)	$\tau h_1^2 x_{76,6}$	Lemma 7.81
(81, 5, 43)	$h_1^3 h_4 h_6$	Lemma 7.85
(81, 7, 44)	$h_3^2 n_1$	Lemma 7.86
(81, 12, 42)	$\Delta^2 p$	Lemma 7.87
(82, 4, 43)	$h_6 c_1$	Lemma 7.88
(83, 6, 44)	$h_0 h_6 g$	Lemma 7.89
(85, 7, 46)	$h_2 h_4 Q_3$	Lemma 7.90
(86, 9, 46)	$h_1 h_6 c_0 d_0$	Lemma 7.91
(86, 10, 45)	$P h_1 h_6 d_0$	Lemma 7.91
(87, 11, 48)	$B_6 d_1$	Lemma 7.93
(88, 7, 47)	$h_1^2 h_4 h_6 c_0$	Lemma 7.94

TABLE 20. — Possible hidden η extensions

(s, f, w)	Source	Target	Proof
(66, 6, 35)	$\tau\Delta_1 h_3^2$	$?\tau^2\Delta h_2^2 e_0 g$	Lemma 7.64
(66, 12, 35)	$\Delta^2 h_1^3 h_4$	$?\tau^2\Delta h_2^2 e_0 g$	
(67, 6, 36)	$h_0 Q_3 + h_2^2 D_3$	$?\tau M h_0 h_2 g$	
(81, 3, 42)	$h_2 h_4 h_6$	$?\tau h_2^2 g$ $?\tau^2 e_1 g_2$ $?\Delta^2 h_2 n$	Lemma 7.84
(81, 5, 39)	$h_1^3 h_4 h_6$	$?\tau(\Delta e_1 + C_0)g$	Lemma 7.85
(81, 8, 42)	$\tau^2 g D_3$	$?\Delta^2 h_2 n$	
(81, 8, 43)	$\tau g D_3$	$?\tau(\Delta e_1 + C_0)g$	
(86, 6, 45)	$\tau h_1^2 f_2$	$?\tau^2 g Q_3$ $?\Delta^2 h_3 d_1$	
(86, 6, 46)	$h_1^2 f_2$	$?h_1 h_4 x_{71,6}$ $?\tau g Q_3$ $?\tau h_2^2 g A'$	
(86, 6, 46)	$h_2 h_6 g + h_1^2 f_2$	$?h_1 h_4 x_{71,6}$ $?\tau h_2^2 g A'$	
(86, 7, 45)	$\tau h_1 x_{85,6}$	$?\tau^2 g Q_3$ $?\Delta^2 h_3 d_1$	
(87, 5, 46)	$h_1^2 c_3$	$?\tau h_0 g_2^2$	
(87, 6, 45)	$\tau h_1 h_4 h_6 c_0$	$?\tau^2 h_0 g_2^2$	
(87, 9, 48)	$g Q_3$	$?\tau M h_0 h_2 g^2$	
(88, 8, 48)	g_2^2	$?\tau h_2^2 g C'$ $?\Delta h_1 g_2 g$	

TABLE 21. — Hidden ν extensions

(s, f, w)	Source	Target	Proof	Notes
(20, 6, 11)	$\tau h_0^2 g$	$P h_1 d_0$	τ	
(20, 6, 12)	$h_0^2 g$	$h_1 c_0 d_0$	$C\tau$	
(22, 4, 13)	$h_2 c_1$	$h_1^2 h_4 c_0$	$C\tau$	
(26, 6, 15)	$\tau h_2^2 g$	$h_1 d_0^2$	τ	
(30, 2, 16)	h_4^2	p	$C\tau$	
(32, 6, 17)	$\Delta h_1 h_3$	$\tau^2 h_1 e_0^2$	*tmf*	
(39, 9, 21)	$\Delta h_1 d_0$	τd_0^3	*tmf*	
(40, 10, 23)	$\tau h_0^2 g^2$	$h_1 d_0^3$	τ	
(40, 10, 24)	$h_0^2 g^2$	$h_1 c_0 e_0^2$	$C\tau$	
(42, 8, 25)	$h_2 c_1 g$	$h_1^6 h_5 c_0$	$C\tau$	
(45, 3, 24)	$h_3^2 h_5$	$M h_2$	$C\tau$	crossing
(45, 4, 24)	$h_0 h_3^2 h_5$	$M h_0 h_2$	$C\tau$	crossing
(45, 9, 24)	$\tau\Delta h_1 g$	$\tau^2 d_0 e_0^2$	*mmf*	
(46, 10, 27)	$\tau h_2^2 g^2$	$h_1 d_0 e_0^2$	τ	
(48, 6, 26)	$h_2 h_5 d_0$	$\tau g n$	[61]	crossing
(51, 8, 27)	$\tau M h_2^2$	$M P h_1$	τ	
(51, 8, 28)	$M h_2^2$	$M h_1 c_0$	[30, Table 31]	
(52, 10, 29)	$\Delta h_1 h_3 g$	$\tau^2 h_1 e_0^2 g$	τ	
(52, 11, 28)	$\tau^2 e_0 m$	$\Delta h_0^2 d_0 e_0$	*mmf*	
(53, 7, 30)	i_1	$g t$	$C\tau$	
(54, 11, 32)	$h_1^6 h_5 e_0$	$h_2 e_0^2 g$	τ	
(57, 10, 30)	$h_0 h_2 h_5 i$	$\tau^2 d_0^2 l$	*tmf*	

TABLE 21. — (*Continued*)

(s, f, w)	Source	Target	Proof	Notes
(59, 12, 33)	$\mathrm{P}h_1^3 h_5 e_0$	$\tau d_0^2 e_0^2$	τ	
(59, 13, 32)	$\tau \Delta h_1 d_0 g$	$\tau^2 d_0^2 e_0^2$	*mmf*	
(60, 14, 35)	$\tau h_0^2 g^3$	$h_1 d_0^2 e_0^2$	τ	
(62, 8, 33)	$\Delta e_1 + \mathrm{C}_0$	$\tau \mathrm{M} h_0 g$	Lemma 7.103	
(62, 12, 37)	$h_2 c_1 g^2$	$h_1^8 \mathrm{D}_4$	$\mathrm{C}\tau$	
(63, 6, 33)	$\tau h_1 \mathrm{H}_1$	$\tau^2 \mathrm{M} h_1 g$	Lemma 7.105	crossing
(65, 3, 34)	$h_2 h_5^2$	$\tau h_1 \mathrm{Q}_3$	$\mathrm{C}\tau$	
(65, 9, 36)	$h_1^2 \mathrm{X}_2$	$\mathrm{M} h_2 g$	τ	
(65, 13, 36)	$\tau \Delta h_1 g^2 + \mathrm{P} h_1 h_5 c_0 e_0$	$\tau^2 e_0^4$	*mmf*	
(66, 6, 36)	$\Delta_1 h_3^2$	$h_2^2 \mathrm{C}'$	$\mathrm{C}\tau$	
(66, 14, 39)	$\tau h_2^2 g^3$	$h_1 e_0^4$	τ	
(67, 8, 36)	$h_2^2 \mathrm{A}'$	$h_1 h_3 (\Delta e_1 + \mathrm{C}_0)$	Lemma 7.109	
(68, 13, 36)	$\mathrm{P} h_2 h_5 j$	$\Delta^2 h_0^2 h_2 g$	$\mathrm{C}\tau$	
(69, 9, 38)	$h_2^2 \mathrm{C}'$	$\tau^2 d_1 g^2$	Lemma 7.112	
(70, 9, 37)	$\tau h_1 \mathrm{D}_3'$	$\tau \mathrm{M} d_0^2$	Lemma 7.113	indet
(70, 12, 37)	$\Delta^2 h_2 c_1$	$\Delta^2 h_1^2 h_4 c_0$	τ	
(70, 14, 37)	$\tau \Delta^2 h_1^2 g + \tau^3 m^2$	$\tau^2 \Delta h_1 d_0 e_0^2$	*mmf*	
(71, 8, 39)	$h_2^3 \mathrm{H}_1$	$h_3 \mathrm{C}''$	$\mathrm{C}\tau$	
(71, 12, 39)	$\tau \mathrm{M} h_2^2 g$	$\mathrm{M} h_1 d_0^2$	τ	
(71, 14, 38)	$\Delta^2 h_0 h_2 g$	$\tau^4 e_0^2 g^2$	*mmf*	
(72, 14, 41)	$\Delta h_1 h_3 g^2$	$\tau^2 h_1 e_0^2 g^2$	τ	
(72, 15, 40)	$\tau^2 e_0 g m$	$\Delta h_2^2 d_0^2 e_0$	*mmf*	
(73, 11, 41)	$h_2^2 \mathrm{C}''$	$\tau g^2 t$	Lemma 7.115	
(74, 14, 39)	$\tau \Delta^2 h_2^2 g$	$\tau^5 e_0 g^3$	τ	
(77, 3, 40)	$h_3^2 h_6$	$\tau h_1 x_1$	Lemma 7.120	indet
(77, 7, 41)	$h_1 x_{76,6}$	$c_1 \mathrm{A}'$	$\mathrm{C}\tau$	
(77, 15, 42)	$\Delta^2 h_2^3 g$	$\tau^2 d_0 e_0^2 l$	τ	
(77, 16, 41)	$\tau^5 e_0 g^3$	$\tau^3 d_0 e_0^2 l$	*mmf*	
(78, 8, 40)	$h_0^6 h_4 h_6$	$\Delta^2 p$	$\mathrm{C}\tau$	
(78, 9, 40)	$h_0^7 h_4 h_6$	$\tau \Delta^2 h_1 d_1$	Lemma 7.121	
(79, 17, 45)	$\Delta h_1 e_0^2 g + \mathrm{M} h_1^3 d_0 e_0$	$\tau d_0 e_0^4$	*mmf*	
(80, 18, 47)	$\tau h_0^2 g^4$	$h_1 d_0 e_0^4$	τ	
(80, 18, 48)	$h_0^2 g^4$	$h_1 c_0 e_0^2 g^2$	$\mathrm{C}\tau$	
(82, 6, 44)	$h_5^2 g$	$h_0 h_2 h_4 \mathrm{Q}_3$	$\mathrm{C}\tau$	
(82, 8, 44)	$h_2^2 x_{76,6}$	$\mathrm{P} h_1 x_{76,6}$	$\mathrm{C}\tau$	
(82, 10, 42)	$\mathrm{P}^2 h_2 h_6$	$\Delta^2 h_0 x$	$\mathrm{C}\tau$	
(82, 12, 44)	$\tau (\Delta e_1 + \mathrm{C}_0) g$	$?\mathrm{M} \Delta h_1^2 d_0$	τ	
(82, 12, 45)	$(\Delta e_1 + \mathrm{C}_0) g$	$\tau \mathrm{M} h_0 g^2$	Lemma 7.127	
(82, 16, 49)	$h_2 c_1 g^3$	$h_1^{14} h_6 c_0$	$\mathrm{C}\tau$	
(83, 7, 43)	$\tau h_0^2 h_6 g$	$\mathrm{P} h_1 h_6 d_0$	τ	
(83, 7, 44)	$h_0^2 h_6 g$	$h_1 h_6 c_0 d_0$	$\mathrm{C}\tau$	
(83, 11, 44)	$\tau^2 g \mathrm{C}'$	$\mathrm{M} \Delta h_0^2 e_0$	Lemma 7.129	crossing
(83, 11, 45)	$\Delta j_1 + \tau g \mathrm{C}'$	$\tau^2 \mathrm{M} h_1 g^2$	Lemma 7.129	
(84, 9, 46)	$h_2 g \mathrm{D}_3$	$\mathrm{B}_6 d_1$	$\mathrm{C}\tau$	
(85, 6, 44)	$\tau x_{85,6} + h_0^3 c_3$	$?h_1 x_{87,7} + \tau^2 g_2^2$	Lemma 7.132	
(85, 5, 45)	$h_2 h_6 c_1$	$h_1^2 h_4 h_6 c_0$	$\mathrm{C}\tau$	
(85, 7, 46)	$h_2 h_4 \mathrm{Q}_3$	$h_0 g_2^2$	$\mathrm{C}\tau$	
(85, 17, 48)	$\tau \Delta h_1 g^3$	$\tau^2 e_0^4 g$	*mmf*	
(86, 18, 51)	$\tau h_2^2 g^4$	$h_1 e_0^4 g + \mathrm{M} h_1^7 e_0 g$	τ	
(87, 12, 48)	$h_2^2 g \mathrm{A}'$	$\Delta h_1^2 g_2 g$	Lemma 7.134	

TABLE 22. — Some null hidden ν extensions

(s, f, w)	Source	Proof
(64, 2, 33)	$h_1 h_6$	Lemma 7.107
(64, 8, 34)	$h_3 Q_2$	Lemma 7.108
(68, 7, 36)	$h_3 A'$	Lemma 7.110
(69, 4, 36)	p'	Lemma 7.111
(71, 4, 37)	$h_6 c_0$	Lemma 7.114
(71, 5, 37)	$\tau h_1 p_1$	[16]
(73, 12, 41)	$M h_1 h_3 g$	Lemma 7.116
(76, 8, 41)	$\tau d_1 g_2$	Lemma 7.118
(76, 8, 40)	$h_0 h_4 A$	Lemma 7.119
(78, 10, 42)	$e_0 A'$	Lemma 7.122
(79, 3, 41)	$h_1 h_4 h_6$	Lemma 7.123
(81, 7, 44)	$h_3^2 n_1$	Lemma 7.124
(82, 8, 44)	$\tau e_1 g_2$	Lemma 7.125
(82, 11, 42)	$P^2 h_0 h_2 h_6$	Lemma 7.126
(83, 10, 45)	$h_2 c_1 A'$	Lemma 7.128
(84, 4, 44)	$h_2^2 h_4 h_6$	Lemma 7.131
(87, 12, 45)	$P^2 h_6 c_0$	Lemma 7.133

TABLE 23. — Possible hidden ν extensions

(s, f, w)	Source	Target	Proof
(70, 5, 36)	$h_0^3 h_3 h_6$	$?h_0 h_4 D_2$	
(75, 6, 40)	$h_0 h_3 d_2$	$?M\Delta h_1^2 h_3$	Lemma 7.117
(81, 12, 42)	$\Delta^2 p$	$?\tau M \Delta h_1 d_0$	
(85, 5, 45)	$h_1 f_2$	$?\tau g_2^2$	
(85, 5, 44)	$\tau h_1 f_2$	$?h_1 x_{87,7}$ or $?h_1 x_{87,7} + \tau^2 g_2^2$	
(86, 11, 44)	$h_0^3 h_6 i$	$?\tau \Delta^2 h_1 f_1$	
(87, 5, 46)	$h_1^2 c_3$	$?\tau M \Delta h_1 g$	
(87, 7, 45)	$x_{87,7}$	$?\tau^2 M \Delta h_1 g$	

TABLE 24. — Miscellaneous hidden extensions

(s, f, w)	Type	Source	Target	Proof
(16, 2, 9)	σ	$h_1 h_4$	$h_4 c_0$	$C\tau$
(20, 4, 11)	ϵ	τg	d_0^2	[30, Table 33]
(30, 2, 16)	σ	h_4^2	x	$C\tau$
(30, 2, 16)	η_4	h_4^2	$h_1 h_5 d_0$	[30, Table 33]
(32, 6, 17)	ϵ	$\Delta h_1 h_3$	$\Delta h_1^2 d_0$	*tmf*
(32, 6, 17)	κ	$\Delta h_1 h_3$	$\tau d_0 l + \Delta c_0 d_0$	*tmf*
(44, 4, 24)	θ_4	g_2	$x_{74,8}$	$C\tau$
(45, 3, 23)	ϵ	$\tau h_3^2 h_5$	MP	Lemma 7.138
(45, 3, 24)	ϵ	$h_3^2 h_5$	$M c_0$	Lemma 7.138
(45, 3, 24)	κ	$h_3^2 h_5$	$M d_0$	Lemma 7.140
(45, 3, 24)	$\overline{\kappa}$	$h_3^2 h_5$	$\tau M g$	Lemma 7.142
(45, 3, 24)	$\{\Delta h_1 h_3\}$	$h_3^2 h_5$	$M \Delta h_1 h_3$	Lemma 7.143
(45, 3, 24)	$\theta_{4.5}$	$h_3^2 h_5$	M^2	Lemma 7.144
(62, 2, 32)	σ	h_5^2	p'	$C\tau$
(62, 2, 32)	ρ_{15}	h_5^2	$?h_0 x_{77,7}$ $?\tau^2 m_1$	Lemma 7.147

TABLE 24. — (*Continued*)

(s, f, w)	Type	Source	Target	Proof
(62, 2, 32)	θ_4	h_5^2	$h_0^2 g_3$	$C\tau$
(63, 7, 33)	ϵ	$\tau X_2 + \tau C'$	$d_0 Q_2$	$C\tau$
(63, 7, 33)	κ	$\tau X_2 + \tau C'$	$M\Delta h_1 h_3$	$C\tau$
(63, 7, 33)	η_4	$\tau X_2 + \tau C'$	$h_1 x_{78,9}$	$C\tau$
(64, 2, 33)	ρ_{15}	$h_1 h_6$	$P h_6 c_0$	Lemma 7.148
(64, 2, 33)	ρ_{23}	$h_1 h_6$	$P^2 h_6 c_0$	Lemma 7.148
(65, 10, 35)	ϵ	$\tau M g$	$M d_0^2$	Lemma 7.149
(69, 4, 36)	σ	p'	$h_0 h_4 A$ or $h_0 h_4 A + \tau^2 d_1 g_2$	$C\tau$
(77, 12, 41)	ϵ	$M\Delta h_1 h_3$	$?M\Delta h_1^2 d_0$	Lemma 7.150
(79, 3, 41)	σ	$h_1 h_4 h_6$	$h_4 h_6 c_0$	$C\tau$

TABLE 25. — Some miscellaneous null hidden extensions

(s, f, w)	Type	Source	Proof
(30, 2, 16)	θ_4	h_4^2	Lemma 7.137
(45, 5, 24)	ϵ	$h_5 d_0$	Lemma 7.145
(45, 5, 24)	$\{\Delta h_1 h_3\}$	$h_5 d_0$	Lemma 7.146

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Competing Interests

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