

Diabatic representation of exotic hadrons in the dynamical diquark model

Richard F. Lebed^{*} and Steven R. Martinez[†]

Department of Physics, Arizona State University, Tempe, Arizona 85287, USA



(Received 19 July 2022; accepted 24 September 2022; published 12 October 2022)

We apply the diabatic formalism, an extension of the adiabatic approximation inherent to the Born-Oppenheimer (BO) approach of atomic physics, to the problem of mixing between exotic multiquark hadrons and their nearby di-hadron thresholds. The unperturbed BO eigenstates are obtained using the dynamical diquark model, while the diabatic calculation introduces a mixing potential between these states and the threshold states. We solve the resulting coupled Schrödinger equations numerically for hidden-charm tetraquarks of both open and closed strangeness to obtain physical mass eigenvalues, and we explore the di-hadron state content and spatial extent of the eigenstates. As an explicit example, $X(3872)$ emerges with a dominant $D^0\bar{D}^{*0}$ component, but also contains a considerable diquark-antidiquark component that can contribute significantly to its radiative decay widths, and this component also generates a full multiplet of other diquark-based exotic hadrons to be compared with experiment.

DOI: [10.1103/PhysRevD.106.074007](https://doi.org/10.1103/PhysRevD.106.074007)

I. INTRODUCTION

With the approach of the two-decade mark for experimental evidence of heavy-quark exotic hadrons [1], the field remains at a remarkable point: Well over 50 candidates have been observed with a high degree of statistical significance, and yet no single theoretical paradigm satisfactorily accounts for all of them [2–13]. Even so, several of these states lie exceptionally close to di-hadron thresholds, the most extraordinary example being the first one discovered, $X(3872)$,

$$m_{X(3872)} - m_{D^0} - m_{D^{*0}} = -0.04 \pm 0.09 \text{ MeV}, \quad (1)$$

using the average mass value for each hadron in Eq. (1) tabulated by the Particle Data Group (PDG) [14].

No other hadron lies in such close proximity to a decay threshold, suggesting a unique importance for the $D^0\bar{D}^{*0}$ component (the inclusion of charge conjugates being understood throughout) within the state. At minimum, a state so close to threshold exhibits a number of “universality” properties that depend only upon the large $D^0\bar{D}^{*0}$ scattering length [15]. More typically, $X(3872)$ has frequently been considered as a hadronic molecule analogous to the deuteron d , which is largely (but not entirely) bound

by π exchanges between its nucleon components. However, the 2.23 MeV d binding energy is many times larger than Eq. (1), and it corresponds to a typical hadronic size comparable to its root-mean-square (rms) charge radius $\sqrt{\langle r_d^2 \rangle} = 2.13 \text{ fm}$ [16]. By the same token, $X(3872)$ would be expected to be many femtometers across, larger than most nuclei, and its physical observables would be utterly dominated by long-distance $D^0\bar{D}^{*0}$ interactions.

Nevertheless, $X(3872)$ exhibits certain properties suggesting the significance of its short-distance wave-function components. Its known decays to conventional charmonium (J/ψ and χ_{c1}) account for more than 10% of $\Gamma_{X(3872)}$, and one of its observed radiative decay modes, $\gamma\psi(2S)$, has a branching fraction of at least a few percent [14]. Since charmonium rms charge radii are predicted from quark-potential models to be significantly less than 1 fm, the large expected separation of the charm quarks in the $D^0\bar{D}^{*0}$ pair would naively predict much smaller branching ratios for these processes. But $X(3872)$ also shares the J^{PC} quantum numbers 1^{++} of the yet-unseen conventional charmonium state $\chi_{c1}(2P)$ that quark-potential models predict (e.g., Ref. [17]) to lie several tens of mega-electron-volts (MeV) above $m_{X(3872)}$. As a result, $X(3872)$ has long been suggested to contain a substantial core of conventional $\chi_{c1}(2P)$ [18], in part as a mechanism to explain its surprising decay patterns.

Nevertheless, conventional charmonium is not the only short-distance component available to $X(3872)$. The valence quark content $c\bar{c}u\bar{u}$ of $D^0\bar{D}^{*0}$ allows for an alternative binding mechanism: that of a diquark-antidiquark $(cu)_3(\bar{c}\bar{u})_3$ pair, each one bound through the attractive color channel $3 \otimes 3 \rightarrow \bar{3}$.

^{*}Richard.Lebed@asu.edu

[†]srmart16@asu.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

A number of other exotic candidates lie quite close to (within a few MeV of) di-hadron thresholds, notably, $Z_c(3900)$ [$D\bar{D}^*$], $Z_c(4020)$ [$D^*\bar{D}^*$], $P_c(4312)$ [$\Sigma_c\bar{D}$], $P_c(4450)/P_c(4457)$ [$\Sigma_c\bar{D}^*$], $Z_b(10610)$ [$B\bar{B}^*$], $Z_b(10650)$ [$B^*\bar{B}^*$], and others. In fact, all of these Z meson states lie slightly *above* the corresponding thresholds, arguing against a traditional bound-state molecular picture. And yet, one cannot deny the significance of the proximity of the thresholds in these cases, suggesting a special importance of those particular hadron pairs for the exotic state. On the other hand, some exotic candidates [e.g., the Y states or $Z_c(4430)$] lack an obvious nearby threshold. A complete theoretical framework accommodating all of the heavy-quark exotic candidates must therefore recognize the physical significance of such nearby thresholds on their formation, mass, and decay modes.

One formalism that has predicted a specific spectrum of multi-quark heavy-quark exotic hadrons is the *dynamical diquark model* [19,20], in which the formation of a diquark-antidiquark (δ - $\bar{\delta}$) exotic meson requires the color-nonsinglet diquark quasiparticles $\delta, \bar{\delta}$ not to dissociate instantly into a di-meson pair, but rather to persist as components of a single multi-quark state connected by a color flux tube. Since each diquark is a color triplet containing a heavy quark, the same potentials that are computed on the lattice to describe heavy quarkonium and its hybrids can be imported into Schrödinger equations, which are solved numerically to obtain the spectrum of $\delta\bar{\delta}$ eigenstates. This procedure has been performed both for the multiplet average masses [21] and for the detailed spectrum once spin and isospin fine-structure effects are included [22–25]. Pentaquarks are handled similarly [26], by replacing the antidiquark $\bar{\delta}$ with a color-triplet triquark $\bar{\theta} \equiv [\bar{Q}_3(q_1q_2)_3]_3$ [27].

This treatment of obtaining eigenstates of heavy (quasi) particles at separation r connected by a static potential manifests the well-known Born-Oppenheimer (BO) approximation [28] from atomic physics. Intrinsic to the approximation is the assumption that the light degrees of freedom (d.o.f.) of the state adjust instantly to changes in the configuration Γ of the heavy sources (an *adiabatic* approximation), and that the eigenstates of the potential $V_\Gamma(r)$ thus derived from each such configuration Γ change gradually with $V_\Gamma(r)$. However, when the value of $V_\Gamma(r)$ crosses the energy of a di-hadron threshold, the physical eigenstates undergo a rapid level crossing between a predominantly $\delta\bar{\delta}$ state and a predominantly di-hadron state. The mixing of configurations induces *adiabatic* changes to the system, and solving for its new eigenstates requires a generalization beyond the strict BO limiting case.

Such a diabatic formalism has been extensively developed in atomic physics, and in recent years it has been standardized into textbook form [29]. This approach provides a specific, nonperturbative method for incorporating the mixing of coupled-channel contributions from di-hadron states into the calculations, and we briefly review

the relevant formalism below. The diabatic formalism was first applied to heavy-quark systems quite recently in Ref. [30]; in that case, the analysis examined mixing of di-hadron thresholds with conventional quarkonium, rather than with the four-quark $\delta\bar{\delta}$ states of the dynamical diquark model that are used here.

The initial study of Ref. [30] specifically considers states lying below or just above a di-hadron threshold. In this work we adopt the same restriction, specifically to study the lightest hidden-charm exotics of valence quark content $c\bar{c}q\bar{q}'$ (where $q^{(\prime)}$ are u or d quarks), $c\bar{c}s\bar{s}$, and $c\bar{c}q\bar{s}$. “Lightest” in this sense means the members of the (positive-parity) ground-state BO multiplet $\Sigma_g^+(1S)$; observed candidates with (presumed) corresponding flavor contents include $X(3872)$, $Y(4140)$, and $Z_{cs}(4000)$, respectively. Of course, the same methods can be applied as well to orbitally excited multiplets such as $\Sigma_g^+(1P)$ [containing, e.g., $Y(4220)$], hidden-bottom [e.g., $Z_b(10610)$] states, and fully charmed $c\bar{c}c\bar{c}$ [e.g., $X(6900)$] states, and can also be generalized to resonant states in order to study mass shifts and strong-decay widths [31].

Here we focus on identifying the δ - $\bar{\delta}$ and meson-meson content (including distinct contributing partial waves) of the lightest hidden-charm states, and we extract interesting features such as the expectation values $\langle r \rangle$ and $\langle r^2 \rangle^{1/2}$ of the δ - $\bar{\delta}$ separation r in each mass eigenstate. We show, for example, that the wave function of $X(3872)$ is indeed dominated by $D^0\bar{D}^{*0}$, but not overwhelmingly so, and find that the spectrum of states described by the original uncoupled dynamical diquark model is not greatly disrupted by the existence of di-hadron thresholds.

This paper is organized as follows. In Sec. II we review the state notation for $\delta\bar{\delta}$ systems, focusing for now only on the $\Sigma_g^+(1S)$ multiplet. Section III reviews the diabatic mixing formalism needed for the current set of calculations. Our numerical results appear in Sec. IV, and in Sec. V we summarize and indicate the next directions for future calculations.

II. STATES OF THE DYNAMICAL DIQUARK MODEL

The complete spectra of $c\bar{c}q\bar{q}'$, $c\bar{c}s\bar{s}$, $c\bar{c}q\bar{s}$, and $c\bar{c}c\bar{c}$ states as $\delta\bar{\delta}$ eigenstates of the dynamical diquark model are presented in Refs. [20,24,25,32], respectively. In this paper, all relevant states are accommodated by the lowest (Σ_g^+) BO potential, which consists of the (light) gluon field in its ground state connecting the heavy diquark [$\delta \equiv (Qq)_3$]-antidiquark [$\bar{\delta} \equiv (\bar{Q}\bar{q}')_3$] or diquark-triquark [$\bar{\theta} \equiv (\bar{Q}_3(q_1q_2)_3)_3$] quasiparticles. In all cases, $\delta, \bar{\delta}, \bar{\theta}$ are assumed to transform as color triplets (or antitriplets), and each quasiparticle contains no internal orbital angular momentum.

In this work, we consider $Q\bar{Q}q\bar{q}'$ states, in which $q, \bar{q}'$ may assume any of the flavors $\{u, d, s\}$. The classification

scheme, regardless of the combination, begins with six states, here grouped by J^{PC} quantum numbers. This spectrum,

$$\begin{aligned} J^{PC} = 0^{++}: X_0 &\equiv |0_{\delta}, 0_{\bar{\delta}}\rangle_0, & X'_0 &\equiv |1_{\delta}, 1_{\bar{\delta}}\rangle_0, \\ J^{PC} = 1^{++}: X_1 &\equiv \frac{1}{\sqrt{2}}(|1_{\delta}, 0_{\bar{\delta}}\rangle_1 + |0_{\delta}, 1_{\bar{\delta}}\rangle_1), \\ J^{PC} = 1^{+-}: Z &\equiv \frac{1}{\sqrt{2}}(|1_{\delta}, 0_{\bar{\delta}}\rangle_1 - |0_{\delta}, 1_{\bar{\delta}}\rangle_1), & Z' &\equiv |1_{\delta}, 1_{\bar{\delta}}\rangle_1, \\ J^{PC} = 2^{++}: X_2 &\equiv |1_{\delta}, 1_{\bar{\delta}}\rangle_2, \end{aligned} \quad (2)$$

which specifies the full multiplet of Σ_g^+ S -wave states, is written with the total $\delta(\bar{\delta})$ spin denoted by $s_{\delta}(s_{\bar{\delta}})$, and with the overall state total spin signified by an outer subscript. When needed, $9j$ angular momentum recoupling coefficients may be used to transform these states to another spin basis. For example, the transformation coefficients to the basis of good total heavy-quark ($Q\bar{Q}$) and light-quark ($q\bar{q}'$) spins read

$$\begin{aligned} &\langle (s_q s_{\bar{q}}) s_{q\bar{q}'}, (s_Q s_{\bar{Q}}) s_{Q\bar{Q}}, S | (s_q s_Q) s_{\delta}, (s_{\bar{q}} s_{\bar{Q}}) s_{\bar{\delta}}, S \rangle \\ &= ([s_{q\bar{q}'}][s_{Q\bar{Q}}][s_{\delta}][s_{\bar{\delta}}])^{1/2} \begin{Bmatrix} s_q & s_{\bar{q}} & s_{q\bar{q}'} \\ s_Q & s_{\bar{Q}} & s_{Q\bar{Q}} \\ s_{\delta} & s_{\bar{\delta}} & S \end{Bmatrix}, \end{aligned} \quad (3)$$

with $[s] \equiv 2s + 1$ denoting the multiplicity of a spin- s state. Using Eqs. (2) and (3), one may write

$$\begin{aligned} J^{PC} = 0^{++}: X_0 &= \frac{1}{2}|0_{q\bar{q}'}, 0_{Q\bar{Q}}\rangle_0 + \frac{\sqrt{3}}{2}|1_{q\bar{q}'}, 1_{Q\bar{Q}}\rangle_0, \\ X'_0 &= \frac{\sqrt{3}}{2}|0_{q\bar{q}'}, 0_{Q\bar{Q}}\rangle_0 - \frac{1}{2}|1_{q\bar{q}'}, 1_{Q\bar{Q}}\rangle_0, \\ J^{PC} = 1^{++}: X_1 &= |1_{q\bar{q}'}, 1_{Q\bar{Q}}\rangle_1, \\ J^{PC} = 1^{+-}: Z &= \frac{1}{\sqrt{2}}(|1_{q\bar{q}'}, 0_{Q\bar{Q}}\rangle_1 - |0_{q\bar{q}'}, 1_{Q\bar{Q}}\rangle_1), \\ Z' &= \frac{1}{\sqrt{2}}(|1_{q\bar{q}'}, 0_{Q\bar{Q}}\rangle_1 + |0_{q\bar{q}'}, 1_{Q\bar{Q}}\rangle_1), \\ J^{PC} = 2^{++}: X_2 &= |1_{q\bar{q}'}, 1_{Q\bar{Q}}\rangle_2. \end{aligned} \quad (4)$$

Further specifying the chosen combination of $\{u, d, s\}$ light-quark flavors enlarges this set. For example, considering combinations of $\{u, d\}$ alone expands the set to 12 states: 6 each with $I = 0$ and $I = 1$, but which nonetheless maintain spin structures in the forms of Eqs. (2) or (4). For the purposes of this work, we identify states solely based upon total J^{PC} , effectively ignoring the fine structure due to isospin. Additionally, we separately examine unique flavor combinations of light quarks: $c\bar{c}s\bar{s}$, $c\bar{c}q\bar{s}$, and $c\bar{c}q\bar{q}'$, where henceforth $q, q' \in \{u, d\}$. The only sources of $SU(3)_{\text{flavor}}$ dependence in these calculations arise through distinct explicit diquark and meson masses.

All states considered within this work are accommodated within the ground-state BO multiplet $\Sigma_g^+(1S)$, but it is worth noting that Ref. [20] provides a classification of states in higher multiplets such as $\Sigma_g^+(nP)$, as well as those with excited-gluon BO potentials such as Π_u^+ .

III. DIABATIC MIXING FORMALISM

In this work, we begin with the same construction as in the original dynamical diquark model [21]. That is, one separates the light d.o.f. from the heavy d.o.f. by writing the Hamiltonian as

$$H = K_{\text{heavy}} + H_{\text{light}} = \frac{\mathbf{p}^2}{2\mu_{\text{heavy}}} + H_{\text{light}}, \quad (5)$$

such that the Schrödinger equation now reads

$$\left(\frac{\mathbf{p}^2}{2\mu_{\text{heavy}}} + H_{\text{light}} - E \right) |\psi\rangle = 0. \quad (6)$$

Under the current analysis, “light field” refers to either just the glue fields (in the case of a $\delta\bar{\delta}$ state) or both glue and exchanged light-quark fields (in the case of the meson-meson states).

We now implement the Ansatz that the states defined in Sec. II may appreciably mix with nearby meson-meson thresholds sharing the same J^{PC} quantum numbers, but assume that the two types of states are clearly distinguishable away from the thresholds. Thus, one must determine and solve the multichannel Schrödinger equation connecting the $\delta\bar{\delta}$ states to such threshold states. We closely follow the work of Ref. [30], which carries out this process using conventional quarkonium rather than $\delta\bar{\delta}$ states. Applying the diabatic expansion to the eigenstates of said Schrödinger equation yields [30]

$$|\psi\rangle = \sum_i \int d\mathbf{r}' \tilde{\psi}_i(\mathbf{r}', \mathbf{r}_0) |\mathbf{r}'\rangle |\xi_i(\mathbf{r}_0)\rangle, \quad (7)$$

where $\mathbf{r}'$ denotes the separation of the heavy sources, $\mathbf{r}_0$ is a freely set fiducial parameter, and $|\xi_i\rangle$ are eigenstates of the light-field Hamiltonian. Inserting Eq. (7) into Eq. (6) and applying $\langle \xi_j(\mathbf{r}_0) |$ on the left-hand side produces

$$\sum_i \left[-\frac{\hbar^2}{2\mu_i} \delta_{ij} \nabla^2 + V_{ji}(\mathbf{r}, \mathbf{r}_0) - E \delta_{ji} \right] \tilde{\psi}_i(\mathbf{r}, \mathbf{r}_0) = 0, \quad (8)$$

with the *diabatic potential matrix* defined as

$$V_{ji}(\mathbf{r}, \mathbf{r}_0) \equiv \langle \xi_j(\mathbf{r}_0) | H_{\text{light}} | \xi_i(\mathbf{r}_0) \rangle. \quad (9)$$

We identify the $i = 0$ term with $\delta\bar{\delta}$ states, and $i > 0$ terms with meson-meson states.

This result may be written more compactly in matrix notation as

$$[\mathbf{K} + \mathbf{V}(\mathbf{r})]\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \quad (10)$$

with the parameter $\mathbf{r}_0$ implicit. Neglecting interactions between distinct meson-meson components, as is done in analogous lattice-QCD studies [33], the potential matrix then becomes

$$\mathbf{V} = \begin{pmatrix} V_{\delta\bar{\delta}}(\mathbf{r}) & V_{\text{mix}}^{(1)}(\mathbf{r}) & \cdots & V_{\text{mix}}^{(N)}(\mathbf{r}) \\ V_{\text{mix}}^{(1)}(\mathbf{r}) & V_{M_1\bar{M}_2}^{(1)}(\mathbf{r}) & & \\ \vdots & & \ddots & \\ V_{\text{mix}}^{(N)}(\mathbf{r}) & & & V_{M_1\bar{M}_2}^{(N)}(\mathbf{r}) \end{pmatrix}, \quad (11)$$

with the kinetic-energy operator expressed as

$$\mathbf{K} = \begin{pmatrix} -\frac{\hbar^2}{2\mu_{\delta\bar{\delta}}} & & & \\ & -\frac{\hbar^2}{2\mu_{M_1\bar{M}_2}^{(1)}} & & \\ & & \ddots & \\ & & & -\frac{\hbar^2}{2\mu_{M_1\bar{M}_2}^{(N)}} \end{pmatrix} \nabla^2, \quad (12)$$

where omitted elements are zeros. Inspection of the $i > 0$ diagonal elements leads to the identification of $V_{M_1\bar{M}_2}^{(i)}(\mathbf{r})$ as simply being the energy associated with that of the pure free i th meson-meson state. That is,

$$V_{M_1\bar{M}_2}^{(i)}(\mathbf{r}) = T_{M_1\bar{M}_2}, \quad (13)$$

where

$$T_{M_1\bar{M}_2} \equiv m_{M_1} + m_{\bar{M}_2}. \quad (14)$$

For S -wave $\delta\bar{\delta}$ states, we identify $V_{\delta\bar{\delta}}(\mathbf{r})$ as the uncoupled Σ_g^+ potential [20], with parameters that are calculated on the lattice [34]. Thus, we parametrize

$$V_{\delta\bar{\delta}}(r) = -\frac{\alpha}{r} + \sigma r + V_0 + m_\delta + m_{\bar{\delta}}, \quad (15)$$

where α , σ , and V_0 are 0.053 GeV · fm, 1.097 GeV/fm, and −0.180 GeV, respectively. For each flavor sector, one also requires values for the δ , $\bar{\delta}$ masses, and a corresponding list of di-hadron thresholds with matching J^{PC} quantum numbers must be identified. One then needs only to determine the appropriate form for the mixing potentials. Reference [30] argues for a Gaussian form, which we also adopt here. Explicitly,

$$|V_{\text{mix}}^{(i)}(r)| = \frac{\Delta}{2} \exp \left\{ -\frac{1}{2} \frac{[V_{\delta\bar{\delta}}(r) - T_{M_1\bar{M}_2}^{(i)}]^2}{(\rho\sigma)^2} \right\}, \quad (16)$$

where σ is the same string-tension parameter as in the Cornell-like potential of Eq. (15). Δ is a free parameter with units of energy indicating the strength of the mixing, and ρ is the radial scale for the level crossing [30].

It is a useful first exercise to consider this procedure for a single threshold. The diabatic-potential matrix then reads

$$\mathbf{V}(r) = \begin{pmatrix} V_{\delta\bar{\delta}}(r) & V_{\text{mix}}(r) \\ V_{\text{mix}}(r) & T_{M_1\bar{M}_2} \end{pmatrix}, \quad (17)$$

where we have replaced $\mathbf{r}$ with its magnitude, since all relevant potentials at this stage depend solely upon the diquark separation distance. The corresponding single-threshold kinetic-energy operator becomes

$$\mathbf{K} = \begin{pmatrix} -\frac{\hbar^2}{2\mu_{\delta\bar{\delta}}} & \\ & -\frac{\hbar^2}{2\mu_{M_1\bar{M}_2}} \end{pmatrix} \nabla^2. \quad (18)$$

The diabatic-potential matrix eigenvalues [denoted $V_-(r)$ and $V_+(r)$] and corresponding eigenvectors [$|\xi_-(r)\rangle$ and $|\xi_+(r)\rangle$] may be directly related to the $\delta\bar{\delta}$ and di-meson threshold light-field eigenstates via a generic transformation matrix:

$$\mathbf{R}(r) = \begin{pmatrix} \cos \theta(r) & \sin \theta(r) \\ -\sin \theta(r) & \cos \theta(r) \end{pmatrix}, \quad (19)$$

such that

$$\mathbf{R}(r)\mathbf{V}(r)\mathbf{R}^\dagger(r) = \text{diag}\{V_-(r), V_+(r)\}. \quad (20)$$

Here, θ is identified as the mixing angle between the $\delta\bar{\delta}$ and meson-meson terms. Since $\mathbf{R}(r)$ diagonalizes $\mathbf{V}(r)$, one can deduce an expression for θ in terms of its matrix elements [30]:

$$\theta(r) = \frac{1}{2} \arctan \left(\frac{2V_{\text{mix}}(r)}{T_{M_1\bar{M}_2} - V_{\delta\bar{\delta}}(r)} \right). \quad (21)$$

As an example, we plot θ for $\delta(\bar{\delta}) = c\bar{q}q'$ and $M_1\bar{M}_2 = D\bar{D}^*$ in Fig. 1 using phenomenologically viable values of Δ and ρ [see Eq. (22)], as obtained below.

IV. RESULTS

A. $c\bar{c}q\bar{q}'$ Sector

To obtain values for the Δ and ρ parameters in the mixing potentials, we fit them such that the lowest 1^{++} eigenvalue in the $c\bar{c}q\bar{q}'$ sector is near the measured value $m_{X(3872)} = 3871.7$ MeV. Of course, a range of Δ , ρ value

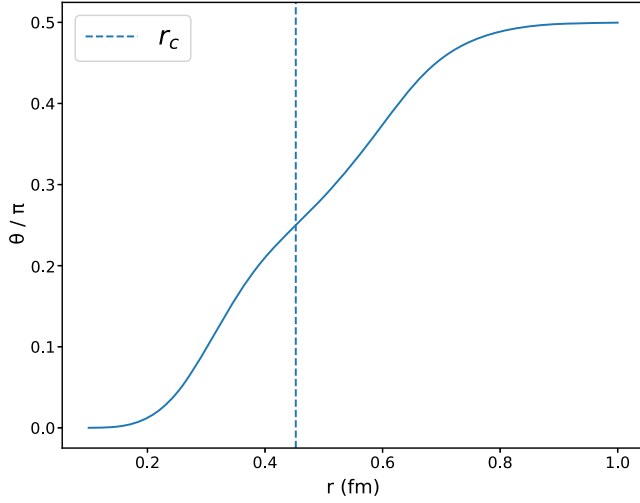


FIG. 1. The mixing angle $\theta(r)$ [Eq. (21), using Eqs. (16) and (22)] between $\delta\bar{\delta} = c\bar{c}qq'$ and $D\bar{D}^*$ in the 1^{++} channel, as a function of heavy-source radial separation r . The dashed line shows the critical radius r_c , i.e., the separation at which $V_{\delta\bar{\delta}}(r) = T_{D\bar{D}^*}$, using Eqs. (14), (15), and (23).

pairs achieves this result. Following the approach of Ref. [30], we select a pair that produces the correct asymptotic behavior for the single-threshold mixing angle $\theta(r)$, as modeled in Fig. 1. Specifically, as supported by unquenched lattice-QCD calculations [35], we require that $\theta \rightarrow \pi/2$ rapidly for $r > r_c$, the value at which the $\delta\bar{\delta}$ Cornell-like potential Eq. (15) crosses the $M_1\bar{M}_2$ threshold Eq. (14). According to Eq. (21), the Δ, ρ pair must also produce the nonzero value $\theta = \pi/4$ at $r = r_c$, indicating a nontrivial mixing of the $\delta\bar{\delta}$ and meson-meson components. The specific values used here are

$$\Delta_{\delta\bar{\delta}} = 0.300 \text{ GeV}, \quad \rho_{\delta\bar{\delta}} = 0.185 \text{ fm}. \quad (22)$$

Again, these values are not unique, but rather serve as proof of principle for the approach. The diquark mass is then the only remaining free parameter, with its value taken as reported in Ref. [25]:

$$m_{\delta=(cq)} = m_{\bar{\delta}=(\bar{c}\bar{q})} = 1.9271 \text{ GeV}. \quad (23)$$

One may then numerically solve Eq. (10) as a coupled set of equations. We follow the procedure described in Sec. IV. F of Ref. [30], solving for the lowest bound states. The resulting eigenvalues and their corresponding state mixtures are collected in Table I. Note the omission of a $J^{PC} = 1^{+-}$ entry in Table I as compared with Eqs. (2) or (4); while the diabatic formalism allows for the formation of $C = -1$ eigenstates, the di-hadron thresholds alone provide no mechanism to lift the degeneracy with the 1^{++} eigenstate.¹ Notably, we find that the 1^{++} eigenstate,

¹In the full dynamical diquark model, this degeneracy is lifted by operators that couple to individual diquark spins [22,24].

which we associate with $X(3872)$, consists primarily of $D\bar{D}^*$ content (an idea known for quite some time, e.g., Refs. [4,18,36]), consistent with expectations given its proximity to that threshold. In contrast, we find that the other eigenstates (0^{++} and 2^{++}) are primarily of $\delta\bar{\delta}$ content.

Using the results of this formalism, it is also possible to calculate certain transition rates, which, of course, provide numerous predictions for comparison with experimental results. Broadly accepted techniques for calculating the decays of $\delta\bar{\delta}$ to $Q\bar{Q}$ states do not yet exist in the literature, but until such methods are robustly developed, we can at least perform exploratory studies using analogs of known expressions. Here, we focus specifically on the radiative transition of the $J^{PC} = 1^{++}$ eigenstate to J/ψ and $\psi(2S)$. For E1 and M1 transitions of states within the dynamical diquark model, substantial work has already been performed in Ref. [37]. There, a standard equation for E1 partial widths for the process $i \rightarrow \gamma f$ is adapted to the case of exotic-to-exotic transitions, and it may be recast for the present case as

$$\Gamma_{\text{E1}}(n^{2s_{Q\bar{Q}}+1}(J_{q\bar{q}})_J \rightarrow n'^{2s'+1}(L')_{J'} + \gamma) = \frac{4}{3} C_{fi} \delta_{s_{Q\bar{Q}} s'_{Q\bar{Q}}} Q_\delta^2 \alpha |\langle \psi_f | r | \psi_i \rangle|^2 E_\gamma^3 \frac{E_f^{Q\bar{Q}}}{M_i^{Q\bar{Q}q\bar{q}}}, \quad (24)$$

where

$$C_{fi} \equiv \max(J_{q\bar{q}}, L')(2J' + 1) \left\{ \begin{matrix} L' & J' & s_{Q\bar{Q}} \\ J & J_{q\bar{q}} & 1 \end{matrix} \right\}^2. \quad (25)$$

In this expression, α is the QED fine-structure constant, E_γ is the photon energy (measured in the rest frame of the decaying exotic initial state $Q\bar{Q}q\bar{q}$ of mass $M_i^{Q\bar{Q}q\bar{q}}$, in which the conventional quarkonium final state $Q\bar{Q}$ recoils with energy $E_f^{Q\bar{Q}}$), and $Q_\delta = Q_Q + Q_q$ is the diquark charge in units of proton charge (meaning that the diquark is treated as a single quasiparticle coupling to the photon). Furthermore, $s_{Q\bar{Q}}^{(i)}$ denotes the total heavy-quark spin eigenvalue for the initial (final) state, $n^{(i)}$ indicates the principal quantum number for the initial (final) state, $J_{q\bar{q}} (L')$ represents the total angular momentum carried by the light d.o.f. in the initial (final) state, and $J (J')$ is the total angular momentum eigenvalue of the initial (final) state, respectively.

The original form of Eq. (24) refers solely to E1 transitions between two states with the same two fundamental charged constituents, which together generate an electric dipole moment. This expression is reasonable (and indeed is commonly used) for conventional quarkonium, albeit often including relativistic corrections that are outlined in Ref. [37]. However, several precarious assumptions enter into the use of Eq. (24) for the transition of a $\delta\bar{\delta}$ to a

TABLE I. Calculated eigenvalues and component-state admixtures for the $c\bar{c}q\bar{q}'$ sector obtained from solving Eq. (10) for specific J^{PC} numbers. Also presented are the expectation values $\langle r \rangle$ of the radial coordinate r (corresponding to $\delta\bar{\delta}$ separation) as well as $\langle r^2 \rangle^{1/2}$ for each state. Suppressed entries indicate contributions that are individually $<1\%$.

J^{PC}	E (MeV)	$\delta\bar{\delta}$	$D\bar{D}^*$	$D_s\bar{D}_s$	$D^*\bar{D}^*$	$D_s^*\bar{D}_s^*$	$\langle r \rangle$ (fm)	$\langle r^2 \rangle^{1/2}$ (fm)
0^{++}	3905.4	63.0%		27.4%	8.4%	1.2%	0.596	0.605
1^{++}	3871.5	8.6%	91.4%				4.974	5.459
2^{++}	3922.3	83.1%		1.5%	13.9%	1.5%	0.443	0.497

$Q\bar{Q}$ state. The first is the use of a diquark as a single charged quasiparticle to which a photon couples; this premise was explored in Ref. [37] along with alternative hypotheses, such as the incoherent coupling of the photon to the component quarks of the diquark, leading to different $O(1)$ factors taking the place of Q_δ^2 in the analog to Eq. (24). More significantly, however, Ref. [37] applied Eq. (24) only to the case of transitions between two $\delta\bar{\delta}$ states, again treating the diquarks as single compact quasiparticles. Applying it to $\delta\bar{\delta} \rightarrow Q\bar{Q}$ transitions is much more questionable since, among other possible objections, such a transition requires the annihilation of the $q\bar{q}$ pair. At minimum, Eq. (24) must at least be modified to accommodate the Okubo-Zweig-Iizuka (OZI) suppressed amplitude $q\bar{q} \rightarrow g$ in this process. While a proper treatment of the issue lies outside the scope of this work, we may at least note that the magnitude ϵ of the OZI suppression is expected to be significantly <1 , and so Γ_{El} of Eq. (24) is diminished by a factor ϵ^2 . One may expect ϵ to depend upon both the radial excitation number n of the initial state, as well as the $q\bar{q}$ spin state.

The most sensitive term to calculate in Eq. (24) is the overlap $|\langle \psi_f | r | \psi_i \rangle|^2$ of the initial and final radial wave functions weighted by r , the characteristic spatial separation of the heavy sources. Note especially that ψ_i (ψ_f) is a $\delta\bar{\delta}$ ($Q\bar{Q}$) state, so that computing this amplitude properly certainly requires more than the simple evaluation of wavefunction overlap performed here. Recall that the full exotic eigenstate in this analysis is taken to be a mixture of both $\delta\bar{\delta}$ and $M_1\bar{M}_2$ components. However, the expected relative separation of the mesons in the $D^0\bar{D}^{*0}$ component of $X(3872)$ is expected to be much greater than that of a pure $\delta\bar{\delta}$ state. Explicitly, using the results of Table I, we find that the $D^0\bar{D}^{*0}$ component alone should have $\langle r \rangle = 5.092$ fm, in agreement with the crude estimate provided by the Bohr-radius analog $r \sim \frac{1}{\sqrt{2\mu E_{\text{bind}}}}$.² Conversely, the value of $\langle r \rangle$ for the pure $\delta(\bar{\delta}) = c q(\bar{c}\bar{q}')$ component is only 0.361 fm. Since the decays $X(3872) \rightarrow \gamma\psi$ require the annihilation of the

light $q\bar{q}$ pair in $X(3872)$, one naively expects that the $\delta\bar{\delta}$ component should dominate in this process. Let us test this expectation.

Once the fractional $\delta\bar{\delta}$ content in the 1^{++} eigenstate is known, one may calculate the E1 partial width from Eq. (24). Using the potential of Eq. (15) and numerical inputs for its parameters given there, and adopting $m_c = 1840$ MeV [30] in place of m_δ , we compute the radial wave functions of J/ψ , $\psi(2S)$, and $\chi_{c1}(2P)$ treated as pure conventional charmonium states. We also solve for the $(1S)$ $\delta\bar{\delta} = c q(\bar{c}\bar{q})$ wave function, which was an essential ingredient in Ref. [37]. Introducing these values into Eq. (24), we find

$$\begin{aligned}\Gamma_{\text{El}}[X(3872) \rightarrow \gamma J/\psi] &= 469\epsilon^2 \text{ keV}, \\ \Gamma_{\text{El}}[X(3872) \rightarrow \gamma\psi(2S)] &= 1.56\epsilon^2 \text{ keV},\end{aligned}\quad (26)$$

while

$$\begin{aligned}\Gamma_{\text{El}}[\chi_{c1}(2P) \rightarrow \gamma J/\psi] &= 20.8 \text{ keV}, \\ \Gamma_{\text{El}}[\chi_{c1}(2P) \rightarrow \gamma\psi(2S)] &= 78.0 \text{ keV}.\end{aligned}\quad (27)$$

We note that the results of Eqs. (27) are comparable to the predictions of 71 keV and 95 keV, respectively, from Ref. [38] [which also uses Eq. (24), but with somewhat different numerical values for the matrix elements].

On the other hand, Eq. (26) predicts (setting $\epsilon \rightarrow 1$) wildly different $X(3872)$ radiative widths than the current PDG averages [14], 10.1 ± 4.7 keV for J/ψ and 54 ± 25 keV for $\psi(2S)$. The PDG separately gives an average for their ratio,

$$R \equiv \frac{\Gamma[X(3872) \rightarrow \gamma\psi(2S)]}{\Gamma[X(3872) \rightarrow \gamma J/\psi]} = 2.6 \pm 0.6, \quad (28)$$

while the corresponding ratio from Eqs. (26), taking the two values of ϵ equal, is only 3.3×10^{-3} . Interestingly, this number approximately equals an early estimate for R based upon a molecular-model calculation using vector-meson dominance [38]. However, a modern molecular-model calculation employing an effective Lagrangian supports values of R in a typical range of order several tenths [39], and these predictions are numerically stable when a small admixture of the conventional-charmonium state $\chi_{c1}(2P)$ is included [40].

²The specific $O(1)$ coefficient in this expression arises from using the literal Bohr radius for r . A different $O(1)$ coefficient arises when computing $\langle r^m \rangle^{1/m}$, but in any case one obtains a result for $X(3872)$ of multiple fm.

The origin of such a small predicted value for R in the dynamical diquark model can be traced to the fact that the $\delta\bar{\delta}$ component of $X(3872)$ is a ground-state ($n = 1$) radial mode like J/ψ , while $\chi_{c1}(2P)$ is a first-excited ($n = 2$) radial mode like $\psi(2S)$. Indeed, our calculated idealized $\delta\bar{\delta}$ wave function has almost exactly the same size as our calculated $c\bar{c}$ J/ψ wave function, leading to almost complete overlap and a large transition to J/ψ . In a more realistic treatment including a finite diquark (cq) size [22], one expects this overlap enhancement to be muted. Nevertheless, one still expects the phase-space advantage of J/ψ over $\psi(2S)$ to generate a rather small R value in this model. If a value of R substantially larger than 1 persists in the data, we conclude that one must have some underlying preferential coupling to $\psi(2S)$ over J/ψ , from a significant $\chi_{c1}(2P)$ component in $X(3872)$, from an enhanced ratio of $\psi(2S)$ to J/ψ effective-theory couplings as suggested in Ref. [39], or from an enhanced wave-function overlap of $\delta\bar{\delta}$ to the spatially larger $\psi(2S)$ state.

The overall size of the prediction for $\Gamma[X(3872) \rightarrow \gamma J/\psi]$ from the first of Eqs. (26) also points to the necessity of ϵ being no larger than about $1/6$, which is not an unreasonable expectation for an OZI-suppressed amplitude.

Last, in a model for $X(3872)$ with not just $\delta\bar{\delta}$ but also $D^0\bar{D}^{*0}$ components [and possibly $\chi_{c1}(2P)$ as well], the full value for its radiative decay widths can include contributions from more than one source. In that case, it becomes crucial to determine the relative phase of each contribution, since interference may be critical to obtaining a physically accurate result. This effect in a mixed $D^0\bar{D}^{*0}$ - $\chi_{c1}(2P)$ model was considered in Ref. [40].

B. $c\bar{c}s\bar{s}$ and $c\bar{c}q\bar{s}$ Sectors

In the $c\bar{c}s\bar{s}$ sector, one encounters an additional free parameter, the (cs) diquark mass. We fix this mass so that the 0^{++} energy eigenvalue matches the $X(3915)$ mass [14]. This choice produces the result

$$m_{\delta=(cs)} = m_{\bar{\delta}=(\bar{c}\bar{s})} = 1.9450 \text{ GeV}. \quad (29)$$

Note that this value only slightly exceeds $m_{\delta=(cq)}$ given in Eq. (23), despite containing a heavier s quark; in particular, it is substantially smaller than the value (2.080 GeV) obtained from the analysis of Ref. [25], because that work (unlike here) incorporates spin-splitting fine structure for the $\Sigma_q^+(1S)$ $\delta\bar{\delta}$ multiplet. The remaining eigenstates may then be calculated and are presented in Table II. Remarkably, the 1^{++} and 2^{++} states exhibit rather little $D_{(s)}^{(*)}\bar{D}_{(s)}^{*}$ content, while the 0^{++} state shows significant mixing with $D_s\bar{D}_s$.

The calculations for $c\bar{c}q\bar{s}$ states follow entirely from the previously determined parameters and are presented separately in Table III. This sector in particular contains a unique threshold structure, in that the $D^*\bar{D}_s$ and $D\bar{D}_s^*$ thresholds differ by less than 4 MeV, for each fixed value of total charge for the meson pair. Our calculations show that thresholds extremely close to each other tend to evenly share state content (assuming that the allowed ℓ quantum numbers are the same).

In all three sectors, whenever $\delta\bar{\delta}(\ell = 0)$ content dominates, we furthermore find a preferential coupling to S -wave meson-meson combinations over D -wave combinations, even if the latter threshold is closer to the mass eigenvalue. This result is exhibited by the $c\bar{c}q\bar{q}'2^{++}$, $c\bar{c}s\bar{s}1^{++}$, and $c\bar{c}q\bar{s}2^+$ states. The $c\bar{c}q\bar{q}'2^{++}$ state is the most extreme of these, preferring to couple more strongly to $D_s^*\bar{D}_s^*$ over $D_s\bar{D}_s$, despite the ~ 280 MeV difference. This behavior is further exhibited by the results for individual thresholds; for example, in the $2^{++}c\bar{c}q\bar{q}'$ state, the distinct $D^*\bar{D}^*$ channels with $\ell = 0$ and $\ell = 2$ contribute 11.6% and 1.2%, respectively. The expected suppression associated with the near-orthogonal overlap of an S -wave $\delta\bar{\delta}$ state and a D -wave meson-meson state provides a natural explanation.

We also note that the expectation values $\langle r \rangle$ (or $\langle r^2 \rangle^{1/2}$) for each state increase with increasing meson-meson

TABLE II. The same as in Table I, for the $c\bar{c}s\bar{s}$ sector.

J^{PC}	E (MeV)	$\delta\bar{\delta}$	$D_s\bar{D}_s$	$D^*\bar{D}^*$	$D_s\bar{D}_s^*$	$D_s^*\bar{D}_s$	$\langle r \rangle$ (fm)	$\langle r^2 \rangle^{1/2}$ (fm)
0^{++}	3922.0	50.6%	39.9%	8.3%		1.2%	0.627	0.757
1^{++}	3978.7	86.9%		1.5%	11.0%		0.421	0.470
2^{++}	3998.7	95.2%			1.5%	3.3%	0.394	0.437

TABLE III. The same as in Tables I and II, for the $c\bar{c}q\bar{s}$ sector.

J^P	E (MeV)	$\delta\bar{\delta}$	$D^*\bar{D}_s$	$D\bar{D}_s^*$	$D^*\bar{D}_s^*$	$\langle r \rangle$ (fm)	$\langle r^2 \rangle^{1/2}$ (fm)
0^+	3983.5	93.2%			6.8%	0.397	0.440
1^+	3914.5	66.7%	16.5%	15.9%		0.482	0.548
2^+	3962.1	90.4%	1.8%	1.8%	6.0%	0.411	0.460

content, consistent with predictions of $\langle r \rangle$ for hadron molecular states [41]. As discussed above, the $1^{++}c\bar{c}q\bar{q}'$ state exhibits a several-femtometer value for $\langle r \rangle$, as is expected for a pure $D\bar{D}^*$ molecular state with a binding energy of tenths of an MeV [41].

V. CONCLUSIONS

The introduction into a diquark-based model—in this paper, the variant known as the dynamical diquark model—of effects caused by the proximity of some of its eigenstates to di-hadron thresholds creates a model for multiquark exotic hadrons that combines the best features of both diquark and hadron-molecular models. One universal model can then incorporate exotic candidates that lie quite close to such thresholds, but which still maintain a close physical connection to other exotic candidates with no obvious di-hadron interpretation.

Coupled-channel calculations between bound states and thresholds have, of course, been carried out many times in the past, originally in the context of atomic and molecular physics, but more recently for exotic-hadron candidates as well. The so-called diabatic approach used here earns distinction as the natural (and rigorous) generalization of the Born-Oppenheimer approximation used to compute the unperturbed spectrum of heavy-heavy systems such as quarkonium and its hybrid excitations, and in this work for the first time to our knowledge, the diquark-antidiquark states of the dynamical diquark model.

The precise functional form for the mixing potential between unperturbed states and thresholds used here is phenomenological in origin, but it is motivated by lattice-QCD results involving the breaking of the color flux tube between the heavy sources. Clearly, advances in lattice simulations can be incorporated into improved future calculations.

We find, using parameters from the most recent analysis in the dynamical diquark model, that the famous $X(3872)$ can naturally contain a dominant component of $D^0\bar{D}^{*0}$ and yet originate as the unique isosinglet 1^{++} member of the lightest multiplet of hidden-charm diquark-antidiquark states. Masses of all the remaining members of the multiplet are then predicted, along with their di-hadron content generated by nearby thresholds. Since the original model has been extended to study hidden-charm, hidden-strangeness, and open-strange states as well, we also present calculations for those flavor sectors. General results include the observation that S -wave thresholds always produce larger effects than D -wave thresholds, even if the latter are substantially closer to the mass eigenvalue, and that coincident di-hadron thresholds with different arrangements of the light flavors in the hadrons have comparable effects.

We also use our results on state content to calculate the radiative decay widths for $X(3872) \rightarrow \gamma J/\psi$ and $\gamma\psi(2S)$,

which provides crucial information on the short-distance components of $X(3872)$, since its di-hadron component is spatially much larger than that of its diquark-antidiquark component. Interestingly, we obtain a $\gamma\psi(2S)$ width much smaller than the current measured value, due not just to the fact that the compact component amounts to only about 10% of the full state, but also that the overlap of the $1S$ diquark-antidiquark and $2S$ $c\bar{c}$ states is small. We conclude that obtaining the full radiative width may require including a comparable contribution from the diffuse di-hadron component, or even from a $\chi_{c1}(2P)$ component (which can also be included in the diabatic formalism).

These initial calculations, while quite encouraging, remain quite incomplete. First, no spin- or flavor-dependent effects (besides explicit differences in the diquark mass) have been incorporated into the unperturbed side of this calculation. The Hamiltonian for the full dynamical diquark model contains spin-spin and spin-isospin operators that, for example, distinguish masses of $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$. In this calculation, the only mass splittings arise from the explicit differences of $D_{(s)}^{(*)}$ masses. One major future research thrust is the inclusion of explicit fine-structure effects into the initial unperturbed states.

A second possible improvement involves the treatment of the threshold contribution. Di-hadron molecular models have been alluded to several times in this paper, but the actual treatment of the di-hadron state uncoupled from the diquark-antidiquark state, according to Eq. (14), is simply that of two free hadrons. Should it be desirable to regard the pair as forming a true di-hadron molecule, then directly replacing Eq. (14) with a mildly attractive (e.g., meson-exchange) potential would be straightforward in this formalism.

Finally, the diabatic formalism as presented here strictly applies only to states either below or not too far above significant di-hadron thresholds. Some of the exotic candidates [e.g., $Z_c(4430)$] lie rather far from relevant thresholds, and consequently have large decay widths. Such broad resonances should properly be treated as poles in di-hadron scattering amplitudes. However, the diabatic formalism has been developed, in the case of mixing with conventional quarkonium, to include the calculation of strong decay widths [31] and di-hadron scattering amplitudes [42]. These methods can be immediately adapted to the case of mixing with diquark-antidiquark states, and will also be incorporated into future work.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation (NSF) under Grants No. PHY-1803912 and No. PHY-2110278.

- [1] S. Choi *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **91**, 262001 (2003).
- [2] R. Lebed, R. Mitchell, and E. Swanson, *Prog. Part. Nucl. Phys.* **93**, 143 (2017).
- [3] H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, *Phys. Rep.* **639**, 1 (2016).
- [4] A. Hosaka, T. Iijima, K. Miyabayashi, Y. Sakai, and S. Yasui, *Prog. Theor. Exp. Phys.* **2016**, 062C01 (2016).
- [5] A. Esposito, A. Pilloni, and A. Polosa, *Phys. Rep.* **668**, 1 (2017).
- [6] F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, and B.-S. Zou, *Rev. Mod. Phys.* **90**, 015004 (2018).
- [7] A. Ali, J. Lange, and S. Stone, *Prog. Part. Nucl. Phys.* **97**, 123 (2017).
- [8] S. Olsen, T. Skwarnicki, and D. Zieminska, *Rev. Mod. Phys.* **90**, 015003 (2018).
- [9] M. Karliner, J. Rosner, and T. Skwarnicki, *Annu. Rev. Nucl. Part. Sci.* **68**, 17 (2018).
- [10] C.-Z. Yuan, *Int. J. Mod. Phys. A* **33**, 1830018 (2018).
- [11] Y.-R. Liu, H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, *Prog. Part. Nucl. Phys.* **107**, 237 (2019).
- [12] N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C.-P. Shen, C. Thomas, A. Vairo, and C.-Z. Yuan, *Phys. Rep.* **873**, 1 (2020).
- [13] H.-X. Chen, W. Chen, X. Liu, Y.-R. Liu, and S.-L. Zhu, *arXiv:2204.02649*.
- [14] P. Zyla *et al.* (Particle Data Group), *Prog. Theor. Exp. Phys.* **2020**, 083C01 (2020).
- [15] E. Braaten and M. Kusunoki, *Phys. Rev. D* **69**, 074005 (2004).
- [16] E. Tiesinga, P. Mohr, D. Newell, and B. Taylor, *Rev. Mod. Phys.* **93**, 025010 (2021).
- [17] T. Barnes, S. Godfrey, and E. Swanson, *Phys. Rev. D* **72**, 054026 (2005).
- [18] M. Suzuki, *Phys. Rev. D* **72**, 114013 (2005).
- [19] S. Brodsky, D. Hwang, and R. Lebed, *Phys. Rev. Lett.* **113**, 112001 (2014).
- [20] R. Lebed, *Phys. Rev. D* **96**, 116003 (2017).
- [21] J. Giron, R. Lebed, and C. Peterson, *J. High Energy Phys.* **05** (2019) 061.
- [22] J. Giron, R. Lebed, and C. Peterson, *J. High Energy Phys.* **01** (2020) 124.
- [23] J. Giron and R. Lebed, *Phys. Rev. D* **101**, 074032 (2020).
- [24] J. Giron and R. Lebed, *Phys. Rev. D* **102**, 014036 (2020).
- [25] J. Giron, R. Lebed, and S. Martinez, *Phys. Rev. D* **104**, 054001 (2021).
- [26] J. Giron and R. Lebed, *Phys. Rev. D* **104**, 114028 (2021).
- [27] R. Lebed, *Phys. Lett. B* **749**, 454 (2015).
- [28] M. Born and R. Oppenheimer, *Ann. Phys. (Berlin)* **389**, 457 (1927).
- [29] M. Baer, *Beyond Born-Oppenheimer: Electronic Nonadiabatic Coupling Terms and Conical Intersections* (Wiley, New Jersey, 2006).
- [30] R. Bruschini and P. González, *Phys. Rev. D* **102**, 074002 (2020).
- [31] R. Bruschini and P. González, *Phys. Rev. D* **103**, 074009 (2021).
- [32] J. Giron and R. Lebed, *Phys. Rev. D* **102**, 074003 (2020).
- [33] J. Bulava, B. Hörz, F. Knechtli, V. Koch, G. Moir, C. Morningstar, and M. Peardon, *Phys. Lett. B* **793**, 493 (2019).
- [34] M. Berwein, N. Brambilla, J. Tarrús Castellà, and A. Vairo, *Phys. Rev. D* **92**, 114019 (2015).
- [35] Z. Prkacin, G. Bali, T. Dussel, T. Lippert, H. Neff, and K. Schilling, *Proc. Sci. LAT2005* (**2006**) 308 [*arXiv:hep-lat/0510051*].
- [36] Y. Yamaguchi, A. Hosaka, S. Takeuchi, and M. Takizawa, *J. Phys. G* **47**, 053001 (2020).
- [37] J. Gens, J. Giron, and R. Lebed, *Phys. Rev. D* **103**, 094024 (2021).
- [38] E. Swanson, *Phys. Lett. B* **598**, 197 (2004).
- [39] F.-K. Guo, C. Hanhart, Y. S. Kalashnikova, U.-G. Meißner, and A. Nefediev, *Phys. Lett. B* **742**, 394 (2015).
- [40] E. Cincioglu and A. Ozpineci, *Phys. Lett. B* **797**, 134856 (2019).
- [41] Y.-R. Liu, X. Liu, W.-Z. Deng, and S.-L. Zhu, *Eur. Phys. J. C* **56**, 63 (2008).
- [42] R. Bruschini and P. González, *Phys. Rev. D* **104**, 074025 (2021).