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Mathematics and Computers in Simulation 202 (2022) 1–21

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original articles

A nonlinear discrete model for approximating a conservative multi-fractional Zakharov system: Analysis and computational simulations

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Received 17 March 2022; received in revised form 12 May 2022; accepted 22 May 2022
Available online 2 June 2022

Abstract

A system of two partial differential equations with fractional diffusion is considered in this study. The system extends the conventional Zakharov system with unknowns being nonlinearly coupled complex- and real-valued functions. The diffusion is understood in the Riesz sense, and suitable initial-boundary conditions are imposed on an open and bounded domain of the real numbers. It is shown that the mass and Higgs' free energy of the system are conserved. Moreover, the total energy is proven to be dissipated, and that both the free and the total energy are non-negative. As a corollary from the conservation of energy, we find that the solutions of the system are bounded throughout time. Motivated by these properties on the solutions of the system, we propose a numerical model to approximate the fractional Zakharov system via finite-difference approaches. Along with this numerical model for solving the continuous system, discrete analogues for the mass, the Higgs' free energy and the total energy are we provided. Furthermore, utilizing Browder's fixed-point theorem, we establish the solubility of the discrete model. It is shown that the discrete total mass and the discrete free energy are conserved, in agreement with the continuous case. The discrete energy functionals (both the discrete free energy and the discrete total energy) are proven to be non-negative functions of the discrete time thoroughly the boundedness of the numerical solutions. Properties of consistency, stability and convergence of the scheme are also studied rigorously. Numerical simulations illustrate some of the anticipated theoretical features of our finite-difference solution procedure.

Keywords: Fractional Zakharov systems; Conservation of Higgs' free energy; Riesz space-fractional equations; Energy-conserving method; Fractional-order central differences; Numerical efficiency analysis

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https://doi.org/10.1016/j.matcom.2022.05.026

1. Introduction

The design of discrete models for solving conservative systems from physics has been an interesting area of numerical analysis and simulations. Examples of well-known treatments include approximations of the unsaturated flow equation [4], two-phase flow equation in porous media [3], nonlinear Schrödinger equation with wave operators [45], classical Rosenau–regularized long-wave equation [32], coupled fractional Klein–Gordon–Schrödinger equation [15], Cahn–Hilliard equations in complex domains [37] and certain generalized hydrodynamic phase-field model equations with different densities [14,22]. It is worth pointing out that some of the first conservative media treated numerically under this perspective were the nonlinear Schrödinger equation [6], sine–Gordon equation [7] and nonlinear Klein–Gordon equation [12]. These efforts led to the development of the discrete variational derivative methods [8]. It is also important to mention that those continuous systems are usually derived from a conserved quantity which is related to an energy, either a Hamiltonian or a Higgs' free energy, and the governing equations may be derived from them using standard arguments from calculus of variations [13]. The derivation in the discrete case is usually carried out mimicking the continuous, and discrete forms of the formula for integration by parts are required to that end. More generally, it is necessary to consider employing discrete operators which are self adjoint and positive in order to guarantee the existence of some square-root operators [27]. Under these circumstances, a general form of the well-known formula for integration by parts may be readily applied. Standard variational arguments may guarantee the conservation of some energy-like functionals.

On the other hand, the recent development of fractional calculus has led to the design of discrete models for solving conservative systems with fractional-order operators [10,23]. In fact, it has been established that some fractional operators satisfy extensions of the formula of integration by parts and, thus, the use of variational arguments can be feasible [1]. The fact that these arguments are sometimes susceptible to be translated to the discrete case, has led to the development of conservative schemes for solving space-fractional partial differential equations. To illustrate these facts, there are numerous reports on numerical schemes for the nonlinear fractional Schrödinger equation [39], strongly coupled nonlinear fractional Schrödinger equations [35], fractional Klein–Gordon–Schrödinger equations [41], fractional sine–Gordon equation [42], fractional Klein–Gordon–Schrödinger system with generalized Yukawa interaction [2], two-component fractional Gross–Pitaevskii system [36], fractional Higgs' boson equation in the de Sitter space–time [28], fractional multidimensional Klein–Gordon–Zakharov equations [21], and fractional Kawarada equations [46]. It is worth pointing out that most of these reports consider nonlinear systems with fractional spatial partial derivatives of the Riesz or the Riemann–Liouville type [29] in view that they satisfy suitable properties which resemble the formula for integration by parts. As a consequence, variational arguments may be applied in their analysis, and conservation properties may be theoretically validated.

From the point of view of the discrete analysis, various discretizations for space-fractional operators of the Riesz type have been proposed. For example, fractional-order centered differences were introduced to approximate Riesz derivatives of fractional order with a quadratic order of consistency [30,31]. It is worthwhile mentioning that various numerical models that conserve some relevant physical quantities have been designed using this approach in order to solve space-fractional systems with Riesz derivatives. As examples, we can mention numerical methods to solve the nonlinear fractional Schrödinger equation [39], a double-fractional conservative Klein–Gordon–Zakharov system [24], a space-fractional Fermi–Pasta–Ulam–Tsingou medium [20], just to point out some examples. However, it is worth pointing out that there are other different discretizations for Riesz fractional operators which have been employed successfully in the literature, like weighted-shifted Grünwald–Letnikov differences, which have been employed to solve Riesz variable-order fractional diffusion equations [16] and Hamiltonian wave equations which extend the fractional nonlinear Klein–Gordon equation [11], among other systems. As the fractional centered differences, the weighted-shifted Grünwald–Letnikov differences have a second order of consistency. The advantage of the former approach over the latter is that they are relatively easy to implement computationally, while their disadvantage is that they require more regularity on the solutions to guarantee the order of consistency.

In the present study, we will investigate a multi-fractional form of the generalized Zakharov equations [38]. These equations have been used intensively to model plasmas with some quantum corrections [9]. In this manuscript, we will focus at a one-dimensional system defined over a closed and bounded interval of $\mathbb{R}$, considering spatial fractional derivatives of the Riesz type and homogeneous Neumann boundary conditions at the endpoints. As we will see, the system under investigation possesses a mass functional which is conserved throughout the time. Moreover, we will show that the Higgs' free energy of the system is conserved and non-negative. Furthermore, the total energy of the system is also non-negative and dissipated with respect to time. The boundedness property of the solutions of this model will be a straight-forward consequence of these results. Based on these facts, we will propose a finite-difference discretization of the system utilizing fractional-order central differences. Among the theoretical results reported in this manuscript, the existence of solutions using will be proven via a fixed-point theorem. Discrete forms of the mass, the free energy and the total energy of the system will be proposed. Discrete analogues of the results obtained for the continuous system will be established. More concretely, we will prove that the discrete mass and the discrete Higgs' free energy are conserved, and that the discrete free and total energies are non-negative functions under suitable parameter constraints. The consistency, stability and convergence of the finite difference scheme will be proven. Computational simulations will illustrate and validate some of our theoretical results.

2. Preliminaries

Let $I_q = \{1, \ldots, q\}$ and $\overline{I}_q = I_q \cup \{0\}$ for $q \in \mathbb{N}^+$. We use the symbol $\overline{X}$ to denote the closure of a set $X \subseteq \mathbb{R}^p$ under the usual topology of $\mathbb{R}^p$, where $p \in \mathbb{N}^+$ is a fixed natural number. For the remainder, we will suppose that $T > 0$ represents a fixed period of time, and $B = (x_L, x_R)$ is a nonempty interval in $\mathbb{R}$. Define $\Omega = B \times (0, T)$, and agree that all the functions of this study will be defined on the set $\overline{\Omega} \subseteq \mathbb{R}^2$. Moreover, we may extend the domain of definition of our functions to $\mathbb{R} \times [0, T]$ whenever needed, and by allowing them to be equal to zero on $(\mathbb{R} \setminus [x_L, x_R]) \times [0, T]$.

Definition 2.1 (*Podlubny [34]*)**.** Let Γ denote the usual Gamma function which extends the factorial function. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is any function, and assume that n is a non-negative integer and α is a real number, with the property that $n - 1 < \alpha \leq n$ is satisfied. Whenever it exists, the *Riesz fractional derivative* of f of order α at $x \in \mathbb{R}$ is given by

$$\frac{d^\alpha f(x)}{d|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{2.1}$$

When $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, the *Riesz fractional partial derivative* of u of order α with respect to x at $(x, t) \in \mathbb{R} \times [0, T]$ is given by (when it exists)

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} = \frac{-1}{2\cos(\frac{\pi\alpha}{2})\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\xi,t) d\xi}{|x-\xi|^{\alpha+1-n}}. \tag{2.2}$$

For any $z \in \mathbb{C}$, we will represent its complex conjugate using the standard notation $\overline{z}$. Let us define the set $L_{x,p}(\overline{\Omega}) = \{f : \overline{\Omega} \to F : f(\cdot, t) \in L_p(\overline{B}),\ \text{for each } t \in [0, T]\}$, where $p \in [1, \infty)$ and $F = \mathbb{R}, \mathbb{C}$. On the other hand, for any $f \in L_{x,p}(\overline{\Omega})$, we convey that

$$\|f\|_{x,p} = \left(\int_{\overline{B}} |f(x,t)|^p dx \right)^{1/p}, \quad \forall t \in [0, T], \tag{2.3}$$

which is a function of $t \in [0, T]$. Moreover, for each pair $f, g \in L_{x,2}(\overline{\Omega})$, define the following function of t:

$$\langle f, g \rangle_x = \int_{\overline{B}} f(x,t)\overline{g(x,t)} dx, \quad \forall t \in [0, T]. \tag{2.4}$$

For the remainder of this work, we fix $\alpha, \beta \in (1, 2]$. Assume that u and m are a complex- and a real-valued functions, respectively, whose domains are both equal to $\overline{\Omega}$. Moreover, let $u_0 : \overline{B} \to \mathbb{C}$ and $m_0, m_1 : \overline{B} \to \mathbb{R}$ be sufficiently smooth functions. Under these circumstances, the fractional extension of the Zakharov problem investigated in this work is given by the system

$$\begin{gathered}
i\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha} - u(x,t) - m(x,t)u(x,t) - |u(x,t)|^2 u(x,t) = 0, \quad \forall (x,t) \in \Omega, \\
\frac{\partial^2 m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial|x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial|x|^\beta} = 0, \quad \forall (x,t) \in \Omega, \\
\text{subject to } \begin{cases} u(x,0) = u_0(x), & m(x,0) = m_0(x), \quad \forall x \in \overline{B}, \\ \dfrac{\partial m(x,0)}{\partial t} = m_1(x), & \forall x \in B, \\ u(x_L,t) = u(x_R,t) = 0, & m(x_L,t) = m(x_R,t) = 0, \quad \forall t \in [0,T]. \end{cases}
\end{gathered} \tag{2.5}$$

Notice that the case $\alpha = \beta = 2$ is precisely the well-known Zakharov system [43,44]. For convenience, we define the function $v : \overline{\Omega} \to \mathbb{R}$ in such way that

$$\frac{\partial^\beta v(x,t)}{\partial |x|^\beta} = \frac{\partial m(x,t)}{\partial t}, \quad \forall (x,t) \in \Omega. \tag{2.6}$$

Definition 2.2. Let u, m be a pair of functions satisfying the initial-boundary-value problem (2.5). The mass density of the system is given by the expression $\mathcal{M}(x,t) = |u(x,t)|^2$, for each $(x,t) \in \Omega$. In turn, the total mass at the time $t \in [0,T]$ is calculated through $\mathcal{M}(t) = \|u\|_{x,2}^2$. Let us define the Hamiltonian of our fractional Zakharov equations as

$$\mathcal{H}(x,t) = \left| \frac{\partial u}{\partial t} \right|^2 + \mathcal{H}_F(x,t), \quad \forall (x,t) \in \Omega. \tag{2.7}$$

Here,

$$\mathcal{H}_F(x,t) = \left| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right|^2 + |u|^2 + m|u|^2 + \frac{1}{2} \left| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right|^2 + \frac{1}{2} m^2 + \frac{1}{2}|u|^4, \quad \forall (x,t) \in \Omega \tag{2.8}$$

denotes the Higgs' free local energy density component, and v satisfies Eq. (2.6). For the sake of simplification of the nomenclature, we obviated the dependence of all the functions on the right-hand side of this identity with respect to (x,t). In turn, the associated total energy of the system at the time $t \in [0,T]$ is provided then by

$$\mathcal{E}(t) = \int_{-\infty}^{\infty} \mathcal{H}(x,t) dx = \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \mathcal{E}_F(t), \tag{2.9}$$

where

$$\mathcal{E}_F = \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2} \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4 \tag{2.10}$$

represents the Higgs' free energy at the time t.

Theorem 2.3 (*Conservation of Mass*). *If u and m satisfy the problem (2.5), then the total mass is conserved.*

Proof. Take the imaginary part of the inner product between the first equation of (2.5) with u to obtain that

$$0 = \mathrm{Im} \left\langle i \frac{\partial u}{\partial t} + \frac{\partial^\alpha u}{\partial |x|^\alpha} - u - mu - |u|^2 u, u \right\rangle_x = \frac{1}{2} \frac{d}{dt} \|u\|_{x,2}^2, \quad \forall t \in (0,T). \tag{2.11}$$

The property of conservation of mass readily follows now from these identities. □

Theorem 2.4 (*Conservation of Free Energy*). *If u and m satisfy (2.5), then the free energy is non-negative and constant.*

Proof. Using the first equation of (2.5), it follows that

$$\begin{aligned} 0 &= \mathrm{Re} \left\langle i \frac{\partial u}{\partial t}, \frac{\partial u}{\partial t} \right\rangle_x = \mathrm{Re} \left\langle - \frac{\partial^\alpha u}{\partial |x|^\alpha} + u + mu + |u|^2 u, \frac{\partial u}{\partial t} \right\rangle_x \\ &= \frac{1}{2} \frac{d}{dt} \left(\left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \langle m, |u|^2 \rangle_x + \frac{1}{2} \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4 \right), \quad \forall t \in (0,T). \end{aligned} \tag{2.12}$$

We conclude from this that $\mathcal{E}_F'(t) = 0$, for each $t \in [0,T]$, as desired. The non-negativity of the function $\mathcal{E}_F$ readily follows from its definition and the fact that $\langle m, |u|^2 \rangle_x \leq \frac{1}{2}\|m\|_{x,2}^2 + \frac{1}{2}\|u\|_{x,4}^4$ by Young's inequality. □

Corollary 2.5 (*Boundedness*)**.** *Assume that u and m satisfy the initial-boundary-value problem (2.5). Suppose also that $u, \partial^2 u/\partial x^2 \in L_{x,2}(\overline{\Omega})$. Then there exists a constant C which depends only on the initial conditions, such that*

$$\left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 + \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \|m\|_{x,2}^2 \leq C, \quad \forall t \in [0, T]. \tag{2.13}$$

Moreover, the functions (2.9) and (2.10) are both non-negative.

Proof. Notice that Theorem 2.4 assures that there exists a constant $C_0 \in \mathbb{R}$ such that $\mathcal{E}_F(t) = C_0$, for each $t \in [0, T]$. It is worth pointing out that $C_0 = \mathcal{E}_F(0)$, which is entirely expressed in terms of the initial conditions. On the other hand, observe that $|\langle m, |u|^2 \rangle| \leq \frac{1}{2}(\|m\|_2^2 + \|u\|_4^4)$ holds for all $t \in [0, T]$. Therefore, it follows that

$$\begin{aligned} 3C_0 &\geq \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 - \left| \langle m, |u|^2 \rangle_x \right| + \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \frac{3}{2}\|m\|_{x,2}^2 + \frac{3}{2}\|u\|_{x,4}^4 \\ &\geq \left\| \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}} \right\|_{x,2}^2 + \|u\|_{x,2}^2 + \left\| \frac{\partial^{\beta/2} v}{\partial |x|^{\beta/2}} \right\|_{x,2}^2 + \|m\|_{x,2}^2 + \|u\|_{x,4}^4. \end{aligned} \tag{2.14}$$

Finally, we readily reach the conclusion of this result by letting $C = 3C_0$. □

Theorem 2.6 (*Dissipation of Energy*)**.** *The total energy of the system (2.5) is dissipated.*

Proof. We compute firstly the derivative of the first equation of (2.5). Next, we take the imaginary part of the inner product between that derivative and $\frac{\partial u}{\partial t}$ to obtain, for each $t \in [0, T]$, that

$$0 = \operatorname{Im} \left\langle i \frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t} \left(\frac{\partial^\alpha u_t}{\partial |x|^\alpha} - u - mu - |u|^2 u \right), \frac{\partial u}{\partial t} \right\rangle_x = \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u}{\partial t} \right\|_{x,2}^2 - \operatorname{Im} \left\langle u \left(\frac{\partial m}{\partial t} + u \frac{\partial \overline{u}}{\partial t} \right), \frac{\partial u}{\partial t} \right\rangle_x . \tag{2.15}$$

Using the property on the conservation of free energy and the last identity, we notice that

$$\mathcal{E}'(t) = 2 \operatorname{Im} \left\langle u \left(\frac{\partial m}{\partial t} + u \frac{\partial \overline{u}}{\partial t} \right), \frac{\partial u}{\partial t} \right\rangle_x , \quad \forall t \in (0, T). \tag{2.16}$$

We conclude that the total energy of the system (2.5) is dissipated, as desired. □

Before we close this section, we introduce the concept of fractional centered differences which will be the cornerstone to provide consistent a discretization for Riesz-type fractional partial derivatives. For the remainder, we will employ the discrete spatial step-size $h = (x_R - x_L)/J$.

Definition 2.7 (*Ortigueira [31]*)**.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function, and let α and h be real numbers such that $\alpha \in (0, 1) \cup (1, 2]$ and $h > 0$. Let $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ be the two-sided infinite sequence given by

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{Z}. \tag{2.17}$$

When it exists, the *fractional-order centered difference* of order α of f at the point x is defined as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R}, \tag{2.18}$$

It is well known [40] that the sequence $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ satisfies the following properties when $\alpha \in (0, 1) \cup (1, 2]$:

(i) $g_0^{(\alpha)} \geq 0$,
(ii) $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ for all $k \geq 1$, and
(iii) $\displaystyle\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$.

Moreover, if all the derivatives of the function $f : \mathbb{R} \to \mathbb{R}$ up to the order five are integrable over $\mathbb{R}$ and $h > 0$, then the following consistency property holds true [40]:

$$-\frac{1}{h^\alpha}\Delta_h^\alpha f(x) = \frac{d^\alpha f(x)}{d|x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}. \tag{2.19}$$

3. Numerical model

For the remainder, we will use the symbol $\mathbb{F}$ to denote any of $\mathbb{R}$ or $\mathbb{C}$. Let J and N be arbitrary natural numbers, and introduce the computational constant $\tau = T/N$. Fix regular partitions of $[x_L, x_R]$ and $[0, T]$, respectively, in the following way:

$$x_L = x_0 < x_1 < \cdots < x_j < \cdots < x_J = x_R, \quad \forall j \in \overline{I}_J, \tag{3.1}$$

and

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T, \quad \forall n \in \overline{I}_N. \tag{3.2}$$

Let us set $u_j^n = u(x_j, t_n)$ and $m_j^n = m(x_j, t_n)$, for each $(j, n) \in \overline{I}_J \times \overline{I}_N$, and agree that U_j^n and M_j^n denote computational estimates for the exact values of u_j^n and m_j^n, respectively. We employ the notation $\mathcal{V}_h$ to represent the vector space of all $\mathbb{F}$-valued functions defined on the set $\{x_j : j \in \overline{I}_J\}$ which vanish at x_0 and x_J. If $V \in \mathcal{V}_h$, we agree that $V_j = V(x_j)$, for each $j \in \overline{I}_J$. Finally, set $U^n = (U_j^n)_{j\in\overline{I}_J} \in \mathcal{V}_h$ and $M^n = (M_j^n)_{j\in\overline{I}_J} \in \mathcal{V}_h$, and let $U = (U^n)_{n\in\overline{I}_N}$ and $M = (M^n)_{n\in\overline{I}_N}$.

Definition 3.1. Let $1 \le q < \infty$. The functions $\langle\cdot,\cdot\rangle : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{C}$ and $\|\cdot\|_q, \|\cdot\|_\infty : \mathcal{V}_h \to \mathbb{R}$ are defined by

$$\langle U, V\rangle = h\sum_{j\in\overline{I}_J} U_j\overline{V_j}, \quad \forall U, V \in \mathcal{V}_h, \tag{3.3}$$

$$\|U\|_q^q = h\sum_{j\in\overline{I}_J} |U_j|^q, \quad \forall U \in \mathcal{V}_h, \tag{3.4}$$

$$\|U\|_\infty = \max\left\{|U_j| : j \in \overline{I}_J\right\}, \quad U \in \mathcal{V}_h. \tag{3.5}$$

Moreover, for any $V = (V^n)_{n\in\overline{I}_N} \subseteq \mathcal{V}_h$ we define $|\|V\||_\infty = \sup\{\|V^n\|_\infty : n \in \overline{I}_N\}$.

Definition 3.2. Let V be U or M, and assume $\alpha \in (0, 1) \cup (1, 2]$. Introduce the discrete operators

$$\delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h}, \qquad \forall (j, n) \in \overline{I}_{J-1} \times \overline{I}_N, \tag{3.6}$$

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \qquad \forall (j, n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{3.7}$$

$$\mu_t V_j^n = \frac{V_j^{n+1} + V_j^n}{2}, \quad \forall (j, n) \in \overline{I}_J \times \overline{I}_{N-1}, \tag{3.8}$$

$$\mu_t^{(1)} V_j^n = \frac{V_j^{n+1} + V_j^{n-1}}{2}, \quad \forall (j, n) \in \overline{I}_J \times I_{N-1}. \tag{3.9}$$

Using these definitions, we introduce the operators $\delta_x^{(2)} V_j^n = \delta_x \circ \delta_x V_{j-1}^n$, $\delta_t^{(1)} V_j^n = \mu_t \circ \delta_t V_j^{n-1}$, $\delta_t^{(2)} V_j^n = \delta_t \circ \delta_t V_j^{n-1}$ and $\mu_t^{(2)} V_j^n = \mu_t \circ \mu_t V_j^{n-1}$, for each $(j, n) \in I_{J-1} \times \overline{I}_N$. Moreover, let

$$\delta_x^{(\alpha)} V_j^n = -\frac{1}{h^\alpha}\sum_{k\in\overline{I}_J} g_{j-k}^{(\alpha)} V_k^n, \qquad \forall (j, n) \in I_{J-1} \times \overline{I}_N. \tag{3.10}$$

Lemma 3.3 (*Macías-Díaz [19]*)**.** *Assume that $\alpha \in (1, 2]$ and $U, V \in \mathcal{V}_h$. Then $\langle -\delta_x^{(\alpha)} U, V\rangle = \langle \delta_x^{(\alpha/2)} U, \delta_x^{(\alpha/2)} V\rangle$.* □

With this nomenclature, the discrete model proposed in the present manuscript to approximate the solutions of (2.5) is summarized as the following coupled system of algebraic equations:

$$\begin{gathered} i\delta_t^{(1)}U_j^n + \delta_x^{(\alpha)}\mu_t^{(1)}U_j^n - \mu_t^{(1)}U_j^n - M_j^n\mu_t^{(1)}U_j^n - \left(\mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right) = 0, \quad \forall (j,n) \in I, \\ \delta_t^{(2)}M_j^n - \delta_x^{(\beta)}M_j^n - \delta_x^{(\beta)}|U_j^n|^2 = 0, \quad \forall (j,n) \in I, \\ \text{such that} \begin{cases} U_j^0 = u_0(x_j), & M_j^0 = m_0(x_j), & \forall j \in \overline{I}_J, \\ \mu_t^{(1)}U_j^0 = u_0(x_j), & \delta_t^{(1)}M_j^0 = m_1(x_j), & \forall j \in I_{J-1}, \\ U_0^n = U_J^n = 0, & M_0^n = M_J^n = 0, & \forall n \in \overline{I}_N. \end{cases} \end{gathered} \tag{3.11}$$

The first equation of this system yields an expression with complex parameters in which the only unknown is U_j^{n+1}. Moreover, the second equation of (3.11) is a fully explicit difference equation which can be easily solved for M_j^{n+1}, for each $(j,n) \in I$. Using then the initial data, we readily obtain that for each $j \in I_{J-1}$, the following identities hold:

$$U_j^1 = u_0(x) + i\tau\left[\delta_x^{(\alpha)}u_0(x_j) - u_0(x_j)\left(1 + M_j^0 + \frac{1}{2}\left(|U_j|^2 + |2u_0(x_j) - U_j^1|^2\right)\right)\right], \tag{3.12}$$

$$M_j^1 = m_0(x_j) + \tau m_1(x_j) + \frac{\tau^2}{2}\delta_x^{(\beta)}\left(m_0(x_j) + |u_0(x_j)|^2\right). \tag{3.13}$$

For the remainder of this manuscript, we will employ the sequence $(V^n)_{n\in\overline{I}_N}$ in $\mathring{\mathcal{V}}_h$ which satisfies $\delta_x^{(\beta)}V_j^n = \delta_t M_j^n$, for each $(j,n) \in I_{J-1} \times \overline{I}_{N-1}$. Under these circumstances, (U, M) will denote a solution of (3.11).

Lemma 3.4 (*Macías-Díaz [19]*). *If $V \in \mathcal{V}_h$ and $\alpha \in (1,2]$ then*

(a) $\|\delta_x^{(\alpha/2)}V\|_2^2 \le 2\, g_0^{(\alpha)}h^{1-\alpha}\|V\|_2^2$,
(b) $\|\delta_x^{(\alpha)}V\|_2^2 = \|\delta_x^{(\alpha/2)}\delta_x^{(\alpha/2)}V\|_2^2$,
(c) $\|\delta_x^{(\alpha)}V\|_2^2 \le 2\, g_0^{(\alpha)}h^{1-\alpha}\|\delta_x^{(\alpha/2)}V\|_2^2 \le 4\left(g_0^{(\alpha)}h^{1-\alpha}\right)^2\|V\|_2^2$.

In a first stage, we will prove the existence of solutions for the numerical model (3.11). The cornerstone in our proof will be the following fixed-point result from the standard literature.

Lemma 3.5 (*Browder Fixed-point [5]*). *Let $(H, \langle\cdot,\cdot\rangle)$ be a finite-dimensional inner-product space, let $\|\cdot\| : H \to H$ be the norm induced by $\langle\cdot,\cdot\rangle$, and suppose that $F : H \to H$ is continuous. Assume that there exists $\lambda > 0$ such that $\mathrm{Re}\langle F(z), z\rangle > 0$, for all $z \in H$ with $\|z\| = \lambda$. Then, there is $z^* \in H$, such that $F(z^*) = 0$ and $\|z^*\| \le \lambda$.*

Theorem 3.6 (*Solubility*). *The model (3.11) is solvable for any set of initial conditions.*

Proof. Notice that the approximation (U^0, M^0) is defined by the initial conditions. Proceeding inductively, suppose that (U^{n-1}, M^{n-1}) and (U^n, M^n) have been already obtained for some $n \in I_{N-1}$. In a first stage, observe that the second equation of (3.11) can be written as $A\Psi = b$, where Ψ is the unknown vector of approximations at time t_{n+1}, and the matrix A and the vector b, are given by

$$A = \frac{1}{\tau^2}\begin{pmatrix} \tau^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \tau^2 \end{pmatrix} \tag{3.14}$$

and

$$b = \begin{pmatrix} 0 \\ \delta_x^{(\beta)}|U_1^n|^2 + \delta_x^{(\beta)}M_1^n + \frac{1}{\tau^2}\left(2M_1^n - M_1^{n-1}\right) \\ \delta_x^{(\beta)}|U_2^n|^2 + \delta_x^{(\beta)}M_2^n + \frac{1}{\tau^2}\left(2M_2^n - M_2^{n-1}\right) \\ \vdots \\ \delta_x^{(\beta)}|U_{J-1}^n|^2 + \delta_x^{(\beta)}M_{J-1}^n + \frac{1}{\tau^2}\left(2M_{J-1}^n - M_{J-1}^{n-1}\right) \\ 0 \end{pmatrix}. \tag{3.15}$$

Since A is nonsingular, there exists a vector M^{n+1} which satisfies the system consisting of all the second difference equations of (3.11) at time t_n. On the other hand, observe that we can rewrite the first equation of (3.11) as

$$\mu_t^{(1)}U_j^n - U_j^{n-1} + i\tau\left[-\delta_x^{(\alpha)}\mu_t^{(1)}U_j^n + \mu_t^{(1)}U_j^n + M_j^n\mu_t^{(1)}U_j^n + \left(\mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] = 0. \tag{3.16}$$

Now, let us consider the function $F : \mathcal{V}_h \to \mathcal{V}_h$, where each of the component functions of F is given by

$$F_j(\eta) = \eta_j^n - U_j^{n-1} + i\tau\left[-\delta_x^{(\alpha)}\eta_j^n + \eta_j^n + M_j^n\eta_j^n + \left(\mu_t^{(1)}|U_j^n|^2\right)\eta_j^n\right], \quad \forall \eta \in \mathcal{V}_h,\ j \in I_{J-1}. \tag{3.17}$$

Then, taking the real part of the inner product between above identity and η, we obtain

$$\mathrm{Re}\langle F(\eta), \eta\rangle = \|\eta\|_2^2 - \mathrm{Re}\langle U^{n-1}, \eta\rangle \geq \|\eta\|_2^2 - |\langle U^{n-1}, \eta\rangle| \geq \frac{1}{2}\left(\|\eta\|_2^2 - \|U^{n-1}\|_2^2\right). \tag{3.18}$$

Applying Browder's fixed-point theorem with $\lambda = \|U^{n-1}\|_2^2 + 1$, it follows that there exists a vector $U^{n+1} \in \mathcal{V}_h$ which satisfies the remaining equation of (3.11). □

Definition 3.7. Let (U, M) be a solution of (3.11). The discrete mass density of (3.11) at the point x_j and time t_n is given by $\mu_t|U_j^n|^2$, for each $(j, n) \in I_{J-1} \times I_{N-1}$. In turn, the total discrete mass of the system at time t_n is given by $\mu_t\|U^n\|_2^2$, for each $n \in I_{N-1}$. The discrete energy density at the point x_j and time t_n is given by

$$\begin{aligned} H_j^n &= |\delta_t U_j^n|^2 + \mu_t|\delta_x^{(\alpha/2)}U_j^n|^2 + \mu_t|U_j^n|^2 + \frac{1}{2}|\delta_x^{(\beta/2)}V_j^n|^2 \\ &\quad + \frac{1}{2}M_j^{n+1}M_j^n + \frac{1}{2}\mu_t|U_j^n|^4 + \frac{1}{2}\left[M_j^n|U_j^{n+1}|^2 + M_j^{n+1}|U_j^n|^2\right], \quad \forall (j, n) \in I_{J-1} \times I_{N-1}. \end{aligned} \tag{3.19}$$

In turn, the total discrete energy at the time t_n is defined, for each $n \in \overline{I}_{N-1}$, by

$$E^n = h\sum_{j\in\overline{J}} H_j^n = \|\delta_t U^n\|_2^2 + E_F^n, \tag{3.20}$$

where

$$\begin{aligned} E_F^n &= \mu_t\|\delta_x^{(\alpha/2)}U^n\|_2^2 + \mu_t\|U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)}V^n\|_2^2 + \frac{1}{2}\langle M^{n+1}, M^n\rangle + \frac{1}{2}\mu_t\|U^n\|_4^4 \\ &\quad + \frac{1}{2}\left[\langle M^n, |U^{n+1}|^2\rangle + \langle M^{n+1}, |U^n|^2\rangle\right], \end{aligned} \tag{3.21}$$

Here, the nomenclature $|U^n|^2 = (|U_j^n|^2)_{j\in\overline{I}_J}$ is observed, for each $n \in \overline{I}_N$.

Theorem 3.8 (*Conservation of Discrete Mass*). *If (U, M) is a solution of (3.11), then the total discrete mass is conserved with respect to the discrete time.*

Proof. Rewrite the first difference equation of the discrete model (3.11), compute the inner product on both sides with $\mu_t^{(1)}U_j^n$ and take imaginary parts. As a consequence, we readily check that

$$0 = \mathrm{Im}\left\langle -i\delta_t^{(1)}U^n, \mu_t^{(1)}U^n\right\rangle = \frac{1}{2}\delta_t^{(1)}\|U^n\|_2^2 = \frac{1}{2}\delta_t\mu_t\|U^n\|_2^2, \quad \forall n \in I_{N-1}, \tag{3.22}$$

which yields what we wanted to prove. □

Theorem 3.9 (*Conservation of Discrete Free Energy*)**.** *Suppose that (U, M) is a solution of (3.11). Then the discrete free energy E_F^n is constant. Moreover, if $\tau^2 g_0^{(\beta)} h^{1-\beta} < 1$, then $E_F^n \geq 0$ and $E^n \geq 0$, for each $n \in I_{N-1}$.*

Proof. Notice that, for each $n \in I_{N-1}$, the following identities are satisfied:

$$0 = \mathrm{Re}\left\langle i\delta_t^{(1)} U^n, \delta_t^{(1)} U^n \right\rangle = \mathrm{Re}\left\langle -\delta_x^{(\alpha)} \mu_t^{(1)} U^n + \mu_t^{(1)} U^n + M^n \mu_t^{(1)} U^n + \left(\mu_t^{(1)} |U^n|^2\right)\left(\mu_t^{(1)} U^n\right), \delta_t^{(1)} U^n \right\rangle, \tag{3.23}$$

$$0 = \langle -\delta_t^{(2)} M^n + \delta_x^{(\beta)} M^n + \delta_x^{(\beta)} |U^n|^2, \mu_t V^{n-1} \rangle = \frac{1}{2}\delta_t \left(\|\delta_x^{(\beta/2)} V^{n-1}\|_2^2 + \langle M^n, M^{n-1} \rangle \right) + \langle |U^n|^2 \delta_t^{(1)} M^n \rangle. \tag{3.24}$$

Using identities of [26] and calculating the right-hand side of the first of these identities, we obtain that

$$\delta_t \mu_t \left(\|\delta_x^{(\alpha/2)} U^{n-1}\|_2^2 + \|U^{n-1}\|_2^2 + \frac{1}{2}\|U^{n-1}\|_4^4 \right) + \langle M^n, \delta_t^{(1)} U^n \rangle = 0, \quad \forall n \in I_{N-1}. \tag{3.25}$$

Observe now that the identity $\langle M^n, \delta_t^{(1)} U^n \rangle + \langle |U^n|^2, \delta_t^{(1)} M^n \rangle = \frac{1}{2}\delta_t \left[\langle M^{n-1}, |U^n|^2 \rangle + \langle M^n, |U^{n-1}|^2 \rangle \right]$ is satisfied, for each $n \in I_{N-1}$. Finally, sum (3.24) and (3.25) to reach $\delta_t E_F^{n-1} = 0$. For the second part, notice that Lemma 3.4(c) implies that $h^{\beta-1}\|\delta_t M^n\|_2^2 \leq 2g_0^{(\beta)} \|\delta_x^{(\beta/2)} V^n\|_2^2$. Moreover, rearranging terms, using some algebraic simplifications and applying the Cauchy–Schwarz inequality, we may observe that

$$\langle M^{n+1}, M^n \rangle = \mu_t \|M^n\|_2^2 - \frac{\tau^2}{2}\|\delta_t M^n\|_2^2, \quad \forall n \in I_{N-1}, \tag{3.26}$$

$$\langle M^n, |U^{n+1}|^2 \rangle + \langle M^{n+1}, |U^n|^2 \rangle| \leq \mu_t \|M^n\|_2^2 + \mu_t \|U^n\|_4^4, \quad \forall n \in I_{N-1}. \tag{3.27}$$

As a consequence,

$$E_F^n \geq \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{4}\left(\frac{h^{\beta-1}}{g_0^{(\beta)}} - \tau^2 \right) \|\delta_t M^n\|_2^2 \geq 0, \quad \forall n \in I_{N-1}. \tag{3.28}$$

Moreover, $E^n = \|\delta_t U^n\|_2^2 + E_F^n \geq 0$, whence the conclusion of this result readily follows. □

Theorem 3.10 (*Boundedness*)**.** *Let $u_0, m_0 \in H^1$ and $u_1, m_1 \in L^2(\overline{B})$, and suppose that (U, M) is the solution of (3.11) corresponding to the initial data u_0, m_0, u_1 and m_1. If $g_0^{(\beta)}\tau^2 h^{1-\beta} < 1$ holds, then the sequences $(\|\delta_x^{(\alpha/2)} U^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|U^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|\delta_x^{(\beta/2)} V^n\|_2)_{n\in\overline{I}_{N-1}}$, $(\|M^n\|_2)_{n\in\overline{I}_{N-1}}$ and $(\|U^n\|_4)_{n\in\overline{I}_{N-1}}$ are bounded by a common constant.*

Proof. Proceeding as in Theorem 3.9, we have

$$\frac{1}{2}\mu_t \|M^n\|_2^2 = \frac{\tau^2}{4}\|\delta_x^{(\beta)} V^n\|_2^2 + \frac{1}{2}\langle M^{n+1}, M^n \rangle \leq \frac{1}{2}\tau^2 g_0^{(\beta)} h^{1-\beta} \left\|\delta_x^{(\beta/2)} V^n\right\|_2^2 + \frac{1}{2}\langle M^{n+1}, M^n \rangle, \quad \forall n \in \overline{I}_{N-1}. \tag{3.29}$$

From the previous theorem, we know that there is a constant C_0 such that $E_F^n = C_0$, for all $n \in \overline{I}_{N-1}$. Then

$$\begin{aligned} C_0 \geq{} & \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2 + \frac{1}{2}\langle M^{n+1}, M^n \rangle \\ & + \frac{1}{2}\mu_t \|U^n\|_4^4 - \frac{1}{2}\left|\langle M^n, |U^{n+1}|^2 \rangle\right| - \frac{1}{2}\left|\langle M^{n+1}, |U^n|^2 \rangle\right| \end{aligned} \tag{3.30}$$

Applying Young's inequality two times, we obtain

$$\frac{1}{2}\left|\langle M^n, |U^{n+1}|^2 \rangle\right| + \frac{1}{2}\left|\langle M^{n+1}, |U^n|^2 \rangle\right| \leq \frac{1}{2}\mu_t \|M^n\|_2^2 + \frac{1}{2}\mu_t \|U^n\|_4^4, \quad \forall n \in \overline{I}_{N-1}, \tag{3.31}$$

$$\frac{1}{2}\left|\langle M^n, |U^{n+1}|^2 \rangle\right| + \frac{1}{2}\left|\langle M^{n+1}, |U^n|^2 \rangle\right| \leq \frac{1}{4}\mu_t \|M^n\|_2^2 + \mu_t \|U^n\|_4^4, \quad \forall n \in \overline{I}_{N-1}. \tag{3.32}$$

Using (3.31) in (3.30) and simplifying algebraically, it is easy to see that

$$\mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\|\delta_x^{(\beta/2)} V^n\|_2^2 + \frac{1}{2}\langle M^{n+1}, M^n \rangle - \frac{1}{2}\mu_t \|M^n\|_2^2 \leq C_0, \quad \forall n \in \overline{I}_{N-1}. \tag{3.33}$$

Now, take the sum between both sides of (3.29) and (3.33) and simplify again. As a consequence, it follows that

$$C_0 \geq \mu_t \|\delta_x^{(\alpha/2)} U^n\|_2^2 + \mu_t \|U^n\|_2^2 + \frac{1}{2}\left(1 - \tau^2 g_0^{(\beta)} h^{1-\beta}\right) \|\delta_x^{(\beta/2)} V^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}. \tag{3.34}$$

Using Lemmas 4.3 and 4.4 of [25], it readily follows that $\|\delta_x^{(\beta/2)} V^n\|_2^2 + \langle M^{n+1}, M^n \rangle \leq \frac{3}{5}\left(\|\delta_x^{(\beta/2)} V^n\|_2^2 + \mu_t \|M^n\|_2^2\right)$. Removing now the first two terms on right-hand side of (3.30), and using (3.32) and the previous remark yields

$$C_0 \geq \frac{3}{10}\|\delta_x^{(\beta/2)} V^n\|_2^2 + \frac{3}{10}\mu_t \|M^n\|_2^2 - \frac{1}{4}\|M^n\|_2^2 - \frac{1}{2}\mu_t \|U^n\|_4^4, \quad \forall n \in \overline{I}_{N-1}. \tag{3.35}$$

Since $\mu_t \|U^n\|_2$ is bounded, then $\mu_t \|U^n\|_4$ is also bounded. Therefore, there is a constant C_1 such that

$$C_1 \geq \frac{1}{2}\mu_t \|U^n\|_4^4 + C_0 \geq \frac{1}{20}\mu_t \|M^n\|_2^2, \quad \forall n \in \overline{I}_{N-1}. \tag{3.36}$$

The conclusion of this theorem is obtained now by letting $C = C_0 + 20C_1$. □

4. Numerical properties

In this section, we establish the main properties of the finite-difference method (3.11). More precisely, we prove the consistency, the stability and the convergence of our numerical model. Some additional nomenclature will be required to that end. For example, we will employ the continuous differential operators

$$\mathcal{L}_u(x,t) = i\frac{\partial u(x,t)}{\partial t} + \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} - u(x,t) - m(x,t)u(x,t) - |u(x,t)|^2 u(x,t), \quad \forall (x,t) \in \Omega, \tag{4.1}$$

$$\mathcal{L}_m(x,t) = \frac{\partial^2\, m(x,t)}{\partial t^2} - \frac{\partial^\beta m(x,t)}{\partial |x|^\beta} - \frac{\partial^\beta \left(|u(x,t)|^2\right)}{\partial |x|^\beta}, \quad \forall (x,t) \in \Omega. \tag{4.2}$$

Set $\mathcal{L}(x,t) = (\mathcal{L}_u(x,t), \mathcal{L}_m(x,t))$, for each $(x,t) \in \Omega$. Moreover, define $\mathcal{L}_j^n = \mathcal{L}(x_j, t_n)$, for each $(j,n) \in \overline{I}_J \times \overline{I}_N$. For the sake of convenience, we let $\mathcal{L}^n = (\mathcal{L}_j^n)_{j \in \overline{I}_J}$, for each $n \in \overline{I}_N$, and convey $\mathcal{L} = (\mathcal{L}^n)_{n \in \overline{I}_N}$.

On the other hand, let us introduce the discrete difference operators

$$L_U(x_j,t_n) = i\delta_t^{(1)} U_j^n + \delta_x^{(\alpha)} \mu_t^{(1)} U_j^n - \mu_t^{(1)} U_j^n - M_j^n \mu_t^{(1)} U_j^n - \left(\mu_t^{(1)} |U_j^n|^2\right)\left(\mu_t^{(1)} U_j^n\right) = 0, \quad \forall (j,n) \in I, \tag{4.3}$$

$$L_M(x_j,t_n) = \delta_t^{(2)} M_j^n - \delta_x^{(\beta)} M_j^n - \delta_x^{(\beta)} |U_j^n|^2 = 0, \quad \forall (j,n) \in I. \tag{4.4}$$

As in the continuous case, we agree that $L(x_j,t_n) = (L_U(x_j,t_n), L_M(x_j,t_n))$, for each $(j,n) \in \overline{I}_J \times \overline{I}_N$, and define $L_j^n = L(x_j,t_n)$. Let us set $L^n = (L_j^n)_{j \in \overline{I}_J}$, for each $n \in \overline{I}_N$, and let $L = (L^n)_{n \in \overline{I}_N}$.

Theorem 4.1 (*Consistency*)**.** *Suppose that $u, m \in \mathcal{C}_{x,t}^{5,4}(\overline{\Omega})$. Then there exist constants C and C' which are independent of τ and h, such that $|||\mathcal{L} - L|||_\infty \leq C(\tau^2 + h^2)$ and $|||\mathcal{H} - H|||_\infty \leq C'(\tau^2 + h^2)$.*

Proof. Using Taylor's theorem, the mean value theorem and the regularity of the functions u and m, it is possible to show that there are constants $C_i \in \mathbb{R}^+$ independent of τ and h, for each $i \in I_5$, such that

$$\left|\frac{\partial u(x_j,t_n)}{\partial t} - \delta_t^{(1)} u_j^n\right| \leq C_1(\tau^2 + h^2), \quad \forall (j,n) \in I, \tag{4.5}$$

$$\left|\frac{\partial^\alpha u(x_j,t_n)}{\partial |x|^\alpha} - \delta_x^{(\alpha)} \mu_t^{(1)} u_j^n\right| \leq C_2(\tau^2 + h^2), \quad \forall (j,n) \in I, \tag{4.6}$$

$$\left|u(x_j,t_n) - \mu_t^{(1)} u_j^n\right| \leq C_3 \tau^2, \quad \forall (j,n) \in I, \tag{4.7}$$

$$\left|m(x_j,t_n)u(x_j,t_n) - m_j^n \mu_t^{(1)} u_j^n\right| \leq C_4 \tau^2, \quad \forall (j,n) \in I, \tag{4.8}$$

$$\left||u(x_j,t_n)|^2 u(x_j,t_n) - \left(\mu_t^{(1)} |u_j^n|^2\right)\left(\mu_t^{(1)} u_j^n\right)\right| \leq C_5 \tau^2, \quad \forall (j,n) \in I. \tag{4.9}$$

From the triangle inequality, there exists a constant $C^* \in \mathbb{R}^+$ which is independent of τ and h, with the property that $|||\mathcal{L}_U - L_U|||_\infty < C^*(\tau^2 + h^2)$. In similar fashion, there exist constants $C_6, C_7, C_8 \in \mathbb{R}^+$ which are independent of both τ and h, for which the inequalities

$$\left|\frac{\partial^2 m(x_j,t_n)}{\partial t^2} - \delta_t^{(2)} m_j^n\right| \leq C_6(\tau^2+h^2), \quad \forall (j,n) \in I, \tag{4.10}$$

$$\left|\frac{\partial^\beta m(x_j,t_n)}{\partial |x|^\beta} - \delta_x^{(\beta)} m_j^n\right| \leq C_7(\tau^2+h^2), \quad \forall (j,n) \in I, \tag{4.11}$$

$$\left|\frac{\partial^\beta \left(|u(x_j,t_n)|^2\right)}{\partial |x|^\beta} - \delta_x^{(\beta)} |u_j^n|^2\right| \leq C_8 h^2, \quad \forall (j,n) \in I, \tag{4.12}$$

are satisfied. Again, we use the triangle inequality to show that there is a constant $C^{**} \in \mathbb{R}^+$ which is independent of τ and h, with the property that $\||\mathcal{L}_M - L_M\||_\infty < C^{**}(\tau^2+h^2)$. The conclusion is following letting C as the maximum of C^* and C^{**}. The second inequality of this result can be obtained in similar fashion. $\Box$

In view to show the stability and convergence properties of (2.5), assume that (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$ are two sets of initial conditions of (2.5). Moreover, suppose that the initial data for (3.11) are provided exactly.

Definition 4.2. If $f : \mathbb{F} \to \mathbb{F}$ and $V \in \mathcal{V}_h$ then we define $\widetilde{\delta}(f(V_j)) = f(\widetilde{V}_j) - f(V_j)$, for each $j \in I_{J-1}$.

Lemma 4.3 (*Pen-Yu [33]*). *Let $(\omega^n)_{n=0}^N$ and $(\rho^n)_{n=0}^N$ be finite sequences of nonnegative real numbers, assume that $\tau > 0$ and suppose that there exists $C \geq 0$ such that*

$$\omega^k \leq \rho^k + C\tau \sum_{n=0}^{k} \omega^n, \quad \forall k \in \overline{I}_N. \tag{4.13}$$

If τ is sufficiently small then $\omega^n \leq \rho^n e^{Cn\tau}$ for each $n \in \overline{I}_N$.

Theorem 4.4 (*Stability*). *Let $u_0, m_0, \widetilde{u}_0, \widetilde{m}_0 \in H^1(\overline{B})$ and $u_1, m_1, \widetilde{u}_1, \widetilde{m}_1 \in L_2(\overline{B})$. Suppose that (U, M) and $(\widetilde{U}, \widetilde{M})$ are the solutions of (3.11) corresponding to (u^0, u^1, m^0, m^1) and $(\widetilde{u}^0, \widetilde{u}^1, \widetilde{m}^0, \widetilde{m}^1)$, respectively. Let $\varepsilon^n = \widetilde{U}^n - U^n$, $\zeta^n = \widetilde{M}^n - M^n$ and $\upsilon^n = \widetilde{V}^n - V^n$, for each $n \in \overline{I}_N$, and define*

$$\omega^n = \mu_t \left(\|\varepsilon^n\|_2^2 + \|\zeta^n\|_2^2\right) + \|\delta_x^{(\beta/2)} \upsilon^n\|_2^2, \quad \forall n \in I_{N-1}. \tag{4.14}$$

For τ sufficiently small, there exists $C \in \mathbb{R}^+$ independent of h and τ, such that $\omega^n \leq \omega^0 e^{Cn\tau}$, for each $n \in \overline{I}_{N-1}$.

Proof. Clearly the sequence (ε, ζ) satisfies the system

$$\begin{aligned} i\delta_t^{(1)}\varepsilon_j^n + \delta_x^{(\alpha)}\mu_t^{(1)}\varepsilon_j^n - \mu_t^{(1)}\varepsilon_j^n - \widetilde{\delta}\left[\left(M_j^n + \mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right] &= 0, \quad \forall (j,n) \in I, \\ \delta_t^{(2)}\zeta_j^n - \delta_x^{(\beta)}\zeta_j^n - \widetilde{\delta}\left(\delta_x^{(\beta)}|U_j^n|^2\right) &= 0, \quad \forall (j,n) \in I, \\ \text{subject to } \varepsilon_0^n = \varepsilon_J^n = 0 \text{ and } \zeta_0^n = \zeta_J^n = 0, \quad \forall n \in \overline{I}_N.& \end{aligned} \tag{4.15}$$

Solving the first equation of (4.15) for $i\delta_t^{(1)}\varepsilon_j^n$, computing the inner product on both sides of that identity with $2\mu_t^{(1)}\varepsilon^n$, taking imaginary parts, and using algebraic arguments, there exists $C_1 > 0$ such that, for each $n \in \overline{I}_{N-1}$,

$$\begin{aligned} \mu_t\delta_t\|\varepsilon^{n-1}\|_2^2 &= 2\,\mathrm{Im}\left\langle \widetilde{\delta}\left[\left(M_j^n + \mu_t^{(1)}|U_j^n|^2\right)\left(\mu_t^{(1)}U_j^n\right)\right], \mu_t^{(1)}\varepsilon^n \right\rangle \\ &\leq C_1\left(\|\varepsilon^{n-1}\|_2^2 + \|\varepsilon^n\|_2^2 + \|\varepsilon^{n+1}\|_2^2 + \|\zeta^{n-1}\|_2^2 + \|\zeta^n\|_2^2 + \|\zeta^{n+1}\|_2^2\right). \end{aligned} \tag{4.16}$$

Now, since $\delta_t\zeta^n = \delta_x^{(\beta)}\upsilon^n$ for each $n \in \overline{I}_{N-1}$, is easy to check that

$$\begin{aligned} 2\left[\langle -\delta_t^{(2)}\zeta^n, \mu_t\upsilon^{n-1}\rangle + \langle \delta_x^{(\beta)}\zeta^n, \mu_t\upsilon^{n-1}\rangle\right] &= \delta_t\left(\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 + \langle\zeta^{n-1}, \zeta^n\rangle\right) \\ &\geq \frac{1}{2}\delta_t\left(\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 + \mu_t\|\zeta^{n-1}\|_2^2\right). \end{aligned} \tag{4.17}$$

Take the inner product between the second equation of (4.15) and $\mu_t\upsilon^{n-1}$, use the above inequality and the fact that $\|\delta_t\zeta^n\|_2^2 \leq 2g_0^{(\beta)}h^{1-\beta}\|\delta_x^{(\beta/2)}\upsilon^n\|_2^2$. It is possible to show then that there is a constant $C_2 > 0$ such that

$$\mu_t\delta_t\|\zeta^{n-1}\|_2^2 + \delta_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2 \leq C_2\left(\|\varepsilon^n\|_2^2 + \mu_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2\right). \tag{4.18}$$

Adding (4.16) and (4.18), and taking the sum from $n = 1$ to m on both sides of the resulting inequality, we obtain that

$$\begin{aligned}\mu_t\left(\|\varepsilon^m\|_2^2+\|\zeta^m\|_2^2\right)+\|\delta_x^{(\beta/2)}\upsilon^m\|_2^2 &\leq \mu_t\left(\|\varepsilon^0\|_2^2+\|\zeta^0\|_2^2\right)+\|\delta_x^{(\beta/2)}\upsilon^0\|_2^2+C_2\tau\sum_{n=1}^m\left(\|\varepsilon^n\|_2^2+\mu_t\|\delta_x^{(\beta/2)}\upsilon^{n-1}\|_2^2\right)\\ &\quad+C_1\tau\sum_{n=1}^m\left(\|\varepsilon^{n-1}\|_2^2+\|\varepsilon^n\|_2^2+\|\varepsilon^{n+1}\|_2^2+\|\zeta^{n-1}\|_2^2+\|\zeta^n\|_2^2+\|\zeta^{n+1}\|_2^2\right)\\ &\leq (1+(6C_1+2C_2)\tau)\left[\mu_t\left(\|\varepsilon^0\|_2^2+\|\zeta^0\|_2^2\right)+\|\delta_x^{(\beta/2)}\upsilon^0\|_2^2\right]\\ &\quad+(6C_1+2C_2)\tau\sum_{n=1}^m\left[\mu_t\left(\|\varepsilon^n\|_2^2+\|\zeta^n\|_2^2\right)+\|\delta_x^{(\beta/2)}\upsilon^n\|_2^2\right].\end{aligned} \tag{4.19}$$

Let now $C = 6C_1 + 2C_2$ and $\rho = (1 + (6C_1 + 2C_2)\tau)\left[\mu_t\left(\|\varepsilon^0\|_2^2+\|\zeta^0\|_2^2\right)+\|\delta_x^{(\beta/2)}\upsilon^0\|_2^2\right]$, and apply Lemma 4.3 to reach the conclusion of this theorem. $\square$

The following is a straight-forward consequence from the stability property of (3.11).

Corollary 4.5 (*Uniqueness*)**.** *Let (u_0, u_1, m_0, m_1) be a set of initial conditions satisfying $u_0, \tilde{u}_0, m_0 \in H^1$ and $u_1, \tilde{u}_1 \in L_2$. For sufficiently small values of τ, the finite-difference scheme (3.11) is uniquely solvable.* $\square$

Definition 4.6. If $f : \mathbb{F} \to \mathbb{F}$ and $V \in \mathcal{V}_h$ then we define $\widehat{\delta}(f(v_j)) = f(v_j) - f(V_j)$, for each $j \in I_{J-1}$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

We establish next the convergence property of our numerical model.

Theorem 4.7 (*Convergence*)**.** *Suppose that $u, m \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$. Then the solution of the problem (3.11) converges to that of (2.5) with order $\mathcal{O}(\tau^2 + h^2)$.*

Proof. Consider the local truncation errors of the finite-difference system (3.11) at (x_j, t_n), given by

$$\begin{aligned}\rho_j^n &= i\delta_t^{(1)}u_j^n+\delta_x^{(\alpha)}\mu_t^{(1)}u_j^n-\mu_t^{(1)}u_j^n-m_j^n\mu_t^{(1)}u_j^n-\left(\mu_t^{(1)}|u_j^n|^2\right)\left(\mu_t^{(1)}u_j^n\right), \quad \forall(j,n)\in I,\\ \sigma_j^n &= \delta_t^{(2)}m_j^n-\delta_x^{(\beta)}m_j^n-\delta_x^{(\beta)}|u_j^n|^2=0, \quad \forall(j,n)\in I.\end{aligned} \tag{4.20}$$

By Theorem 4.1, we know that $|\rho_j^n| + |\sigma_j^n| = \mathcal{O}(\tau^2 + h^2)$. Then, let (u, m) be a solution of (2.5) and (U, M) a solution of (3.11), and define $\epsilon_j^n = u_j^n - U_j^n$, $\eta_j^n = m_j^n - M_j^n$ and $\theta_j^n = \upsilon_j^n - V_j^n$, $\forall(j, n) \in I$. Notice that, $\delta_x^{(\beta)}\theta_j^n = \delta_t\eta_j^n$, $\forall(j, n) \in I$. Moreover, the pair (ϵ, η) satisfies the system

$$\begin{aligned}i\delta_t^{(1)}\epsilon_j^n+\delta_x^{(\alpha)}\mu_t^{(1)}\epsilon_j^n-\mu_t^{(1)}\epsilon_j^n-\widehat{\delta}\left[\left(m_j^n+\mu_t^{(1)}|u_j^n|^2\right)\left(\mu_t^{(1)}u_j^n\right)\right] &= \rho_j^n, \quad \forall(j,n)\in I,\\ \delta_t^{(2)}\eta_j^n-\delta_x^{(\beta)}\eta_j^n-\widehat{\delta}\left(\delta_x^{(\beta)}|u_j^n|^2\right) &= \sigma_j^n, \quad \forall(j,n)\in I,\\ \text{subject to } \epsilon_0^n=\epsilon_J^n=0 \text{ and } \eta_0^n=\eta_J^n=0, &\quad \forall n\in\overline{I}_N.\end{aligned} \tag{4.21}$$

Proceeding as in Theorem 4.4, we can check that there exist constants C_3 and C_4 such that

$$\mu_t\delta_t\|\epsilon^{n-1}\|_2^2 \leq C_3\left(\|\rho^n\|_2^2+\|\epsilon^{n-1}\|_2^2+\|\epsilon^n\|_2^2+\|\epsilon^{n+1}\|_2^2+\|\eta^{n-1}\|_2^2+\|\eta^n\|_2^2+\|\eta^{n+1}\|_2^2\right), \tag{4.22}$$

$$\mu_t\delta_t\|\eta^{n-1}\|_2^2+\delta_t\|\delta_x^{(\beta/2)}\theta^{n-1}\|_2^2 \leq C_4\left(\|\sigma^n\|_2^2+\|\epsilon^n\|_2^2+\mu_t\|\delta_x^{(\beta/2)}\theta^{n-1}\|_2^2\right). \tag{4.23}$$

For each $k \in I_{N-1}$, take $\omega^k = \mu_t\left(\|\epsilon^k\|_2^2 + \|\eta^k\|_2^2\right) + \|\delta_x^{(\beta/2)}\theta^k\|_2^2$ and

$$\rho^k = (1+(6C_3+2C_4)\tau)\left[\mu_t\left(\|\epsilon^0\|_2^2+\|\eta^0\|_2^2\right)+\|\delta_x^{(\beta/2)}\theta^0\|_2^2\right]+(C_3+C_4)\tau\sum_{n=0}^m\left(\|\rho^n\|_2^2+\|\sigma^n\|_2^2\right). \tag{4.24}$$

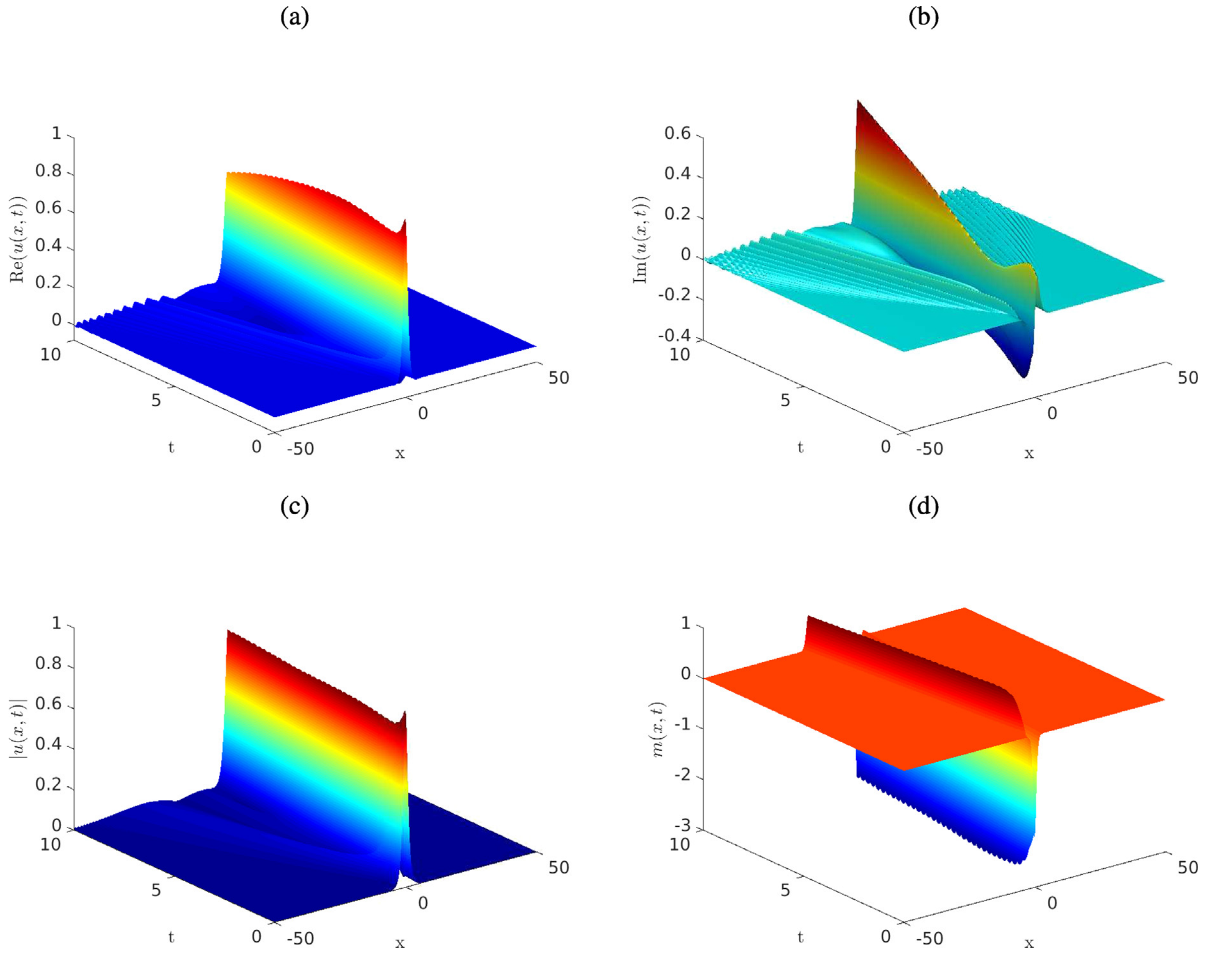

Fig. 1. Approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$ versus x and t. The approximations were obtained using the finite-difference method (3.11) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$ and $\alpha = \beta = 2$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

It follows that there exists a constant $C \geq 0$, with the property that $\omega^k \leq C\rho^k$, for each $k \in I_{N-1}$. As a consequence, $\|\epsilon^n\|_2, \|\eta^n\|_2 \leq \sqrt{C}(\tau^2 + h^2)$, which implies that the solutions of (3.11) converge quadratically to those of (2.5). □

5. Computer simulations

The purpose of this section is to provide computer simulations using a Matlab implementation of the numerical model (3.11) to solve the Zakharov system (2.5). The computer code was employed a fixed-point approach to approximate the solution of the first discrete equation of (3.11) at each iteration. Meanwhile, the second equation of our numerical model was solved explicitly and exactly. For the sake of convenience, we provide a computer program coded in Matlab at the end of this work, in Appendix.

To produce our simulations, we will impose homogeneous Neumann conditions on the boundary of B, along with the following set of initial conditions:

$$u_0(x) = \frac{\sqrt{10} - \sqrt{2}}{2} \operatorname{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right) \exp\left(i\sqrt{\frac{2}{1+\sqrt{5}}}x\right), \tag{5.1}$$

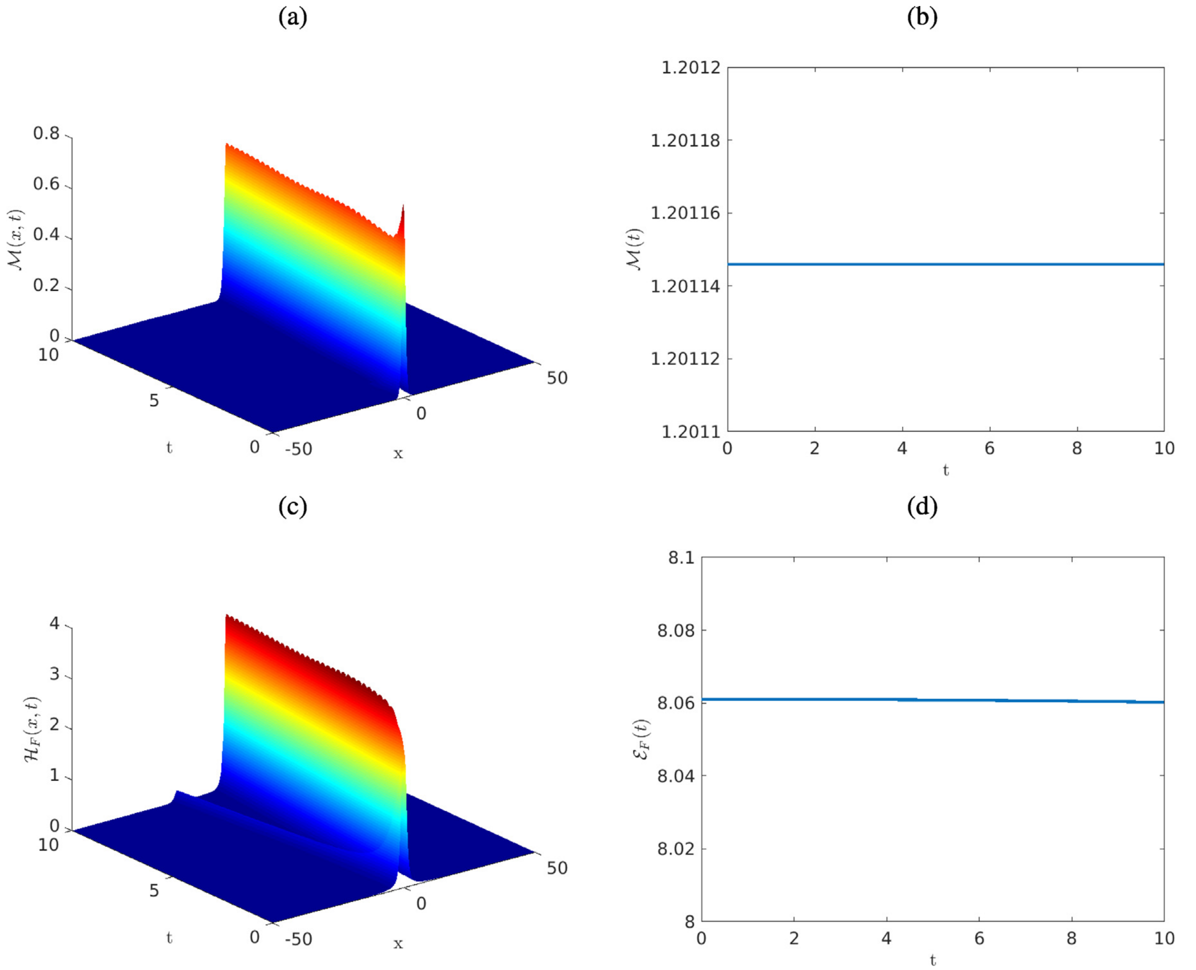

Fig. 2. Left column: approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t. Right column: approximate solutions for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$. The approximations were obtained using the finite-difference method (3.11) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$ and $\alpha = \beta = 2$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

$$m_0(x) = -2\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right), \tag{5.2}$$

$$m_1(x) = -4\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right). \tag{5.3}$$

As a matter of fact, it is worth pointing out that these functions are initial conditions for an exact solution of the well-known Klein–Gordon–Zakharov equations which describe the propagation of Langmuir waves in plasma physics. That exact solution is actually provided by the set of functions (see [17,18])

$$u(x,t) = \frac{\sqrt{10}-\sqrt{2}}{2}\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right)\exp\left[i\left(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\right)\right], \quad \forall (x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}, \tag{5.4}$$

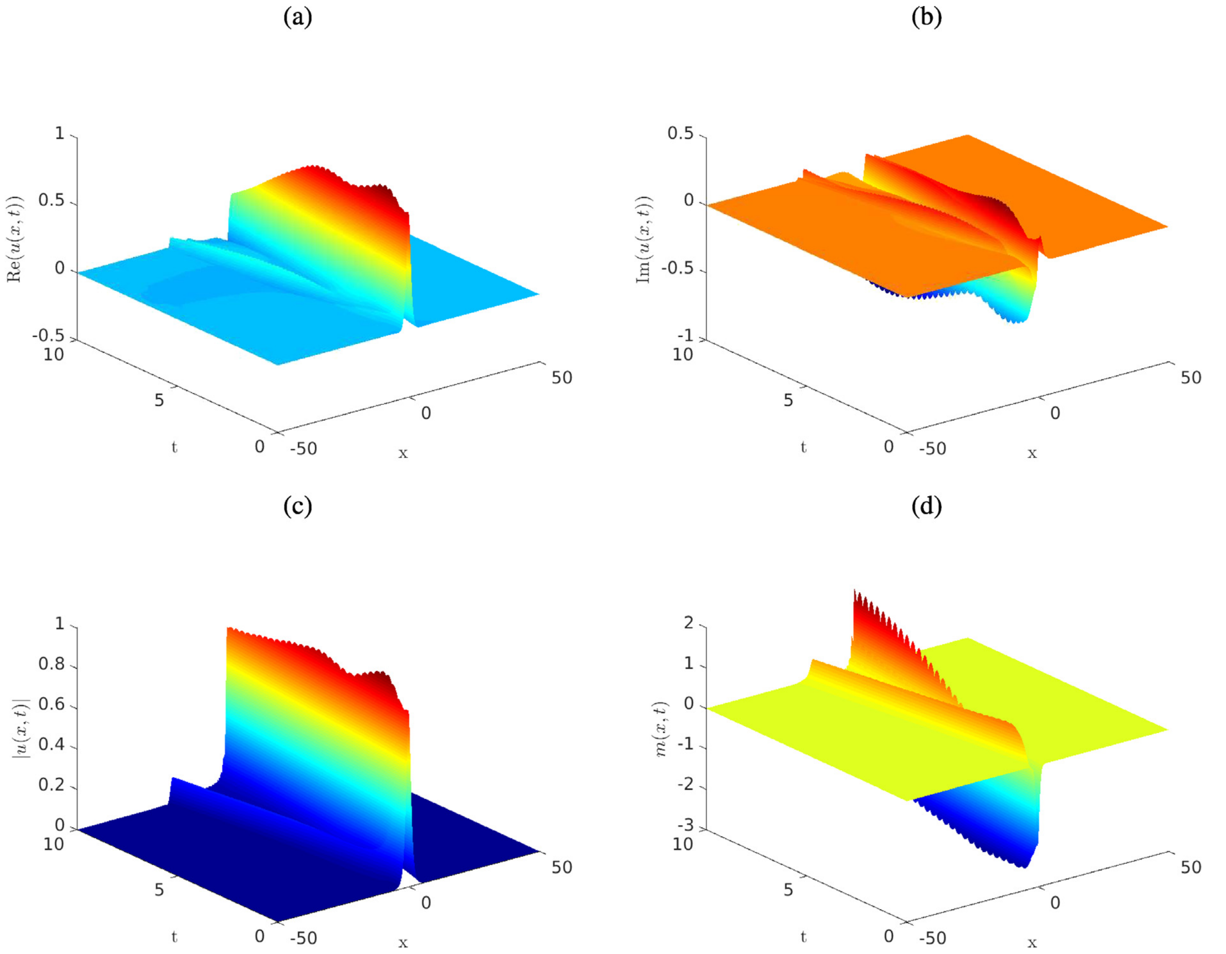

Fig. 3. Approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$ versus x and t. The approximations were obtained using the finite-difference method (3.11) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$, $\alpha = 1.2$ and $\beta = 1.8$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

$$m(x,t) = -2\operatorname{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\right), \quad \forall (x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}. \tag{5.5}$$

In a first approach, we consider the system (2.5) with $\alpha = \beta = 2$, and defined over the space–time domain $\Omega = (-50, 50) \times (0, 10)$. Computationally, we let $h = 0.5$ and $\tau = 0.01$. As we mentioned previously, the mathematical model will be solved using the finite-difference scheme (3.11), which will require a computational implementation of a fixed-point method to solve the first difference equation at each iteration. To that end, we will set a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30. In the absence of a known exact solution for the Zakharov system, we will obtain the first approximations of our methodology using the exact solutions (5.4)–(5.5). Under these circumstances, Fig. 1 provides the approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$ versus x and t. In turn, Fig. 2 shows graphs of the approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t, and for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$ versus t. From these results, we can readily observe that the total mass and the Higgs' free energy are approximately conserved in the discrete domain, in agreement with the theoretical results presented in this work.

Before closing this section, we will provide a new set of simulations using now $\alpha = 1.2$ and $\beta = 1.8$. All the initial and boundary conditions along with the model and computational parameters are as before. With these conventions, Fig. 3 shows the approximate solutions for (a) $\operatorname{Re} u(x,t)$, (b) $\operatorname{Im} u(x,t)$, (c) $|u(x,t)|$ and (d) $m(x,t)$

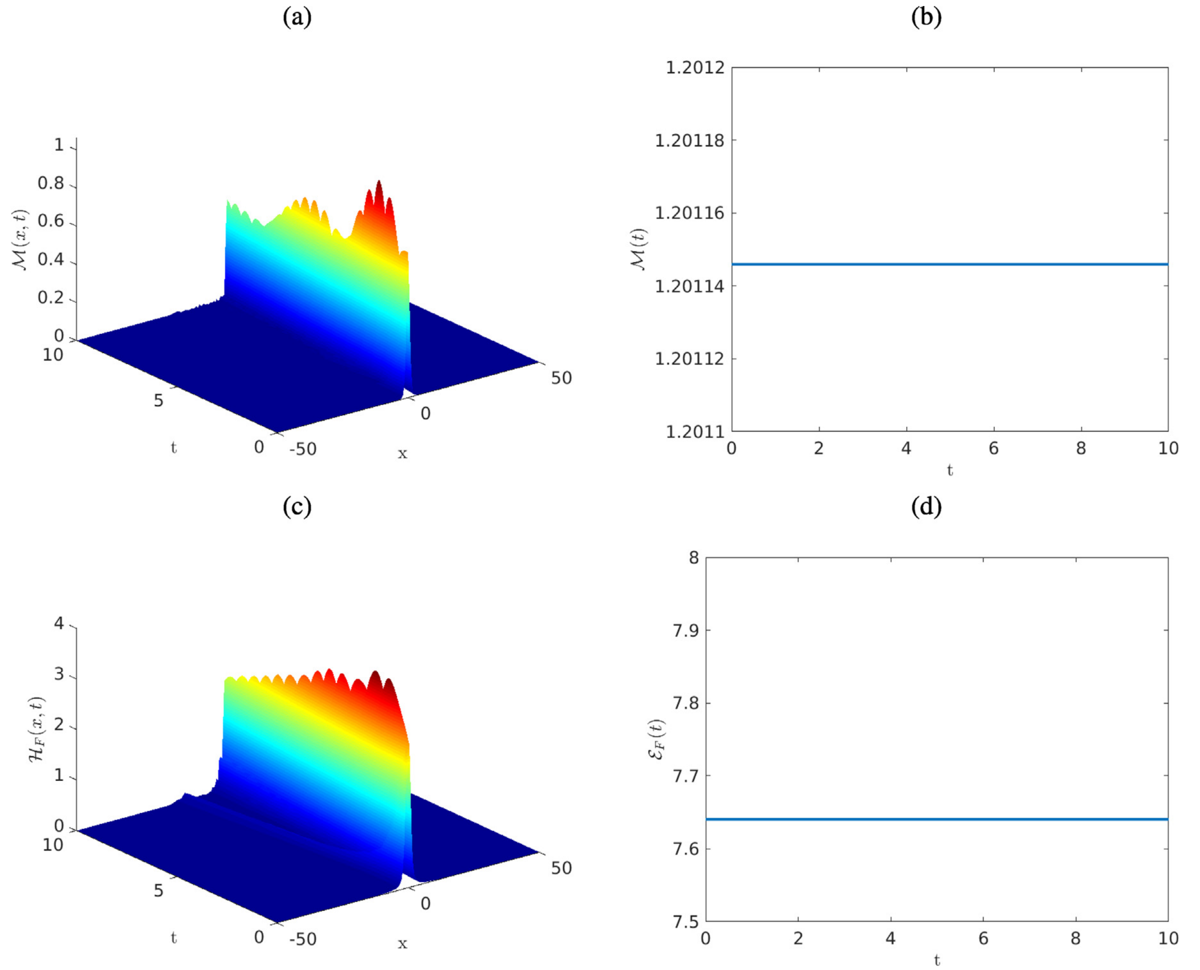

Fig. 4. Left column: approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t. Right column: approximate solutions for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$. The approximations were obtained using the finite-difference method (3.11) with parameters $h = 0.5$, $\tau = 0.01$, $\Omega = (-50, 50) \times (0, 10)$, $\alpha = 1.2$ and $\beta = 1.8$. Computationally, we used a tolerance in the infinity norm equal to 1×10^{-12}, and a maximum number of iterations equal to 30.

versus x and t. On the other hand, Fig. 4 shows graphs of the approximate solutions for (a) $\mathcal{M}u(x,t)$ and (c) $\mathcal{H}_F u(x,t)$ versus x and t, and for (b) $\mathcal{M}(t)$ and (d) $\mathcal{E}_F(t)$ versus t. The results show again the capability of the finite-difference scheme to preserve the total mass of the system and the Higgs' free energy in the discrete scenario. Again, this is in agreement with the theoretical results provided in this work.

6. Conclusions

A space-fractional extension of the Zakharov system was introduced and investigated in this study from analytical and numerical points of views. The system consists of two partial differential equations with nonlinear coupling, and initial and boundary conditions are imposed on a bounded interval of real numbers. It was proven that the fractional system is capable of preserving the mass and Higgs' free energy throughout time, and that the total energy is dissipated. Moreover, the total mass, the total free energy and the total energy are non-negative functions of time. Consequently, the boundedness of the solutions of the system we established. Motivated by these results, we proposed a finite-difference scheme to solve this system via fractional-order central difference approximations. The discrete model proposed is a three-level scheme whose implementation was implemented by using both vector equations and fixed-point techniques. The existence of solutions was proven rigorously through Browder's fixed-point theorem, and proposed discrete expressions for the total mass, Higgs' free energy and the total energy. It was shown theoretically that the numerical model is capable of preserving the discrete mass and the discrete Higgs' free energy. Moreover, the positivity of the mass, the free energy and the total energy was also verified. From the numerical analysis point of view, we proved systematically properties of consistency, stability and convergence of the algorithm. As a consequence of these investigations, the uniqueness of the numerical solutions was also validated. Computer simulations based on the discrete model were presented. The computational experiments illustrate important properties of our numerical solution, including its capability to preserve the mass and Higgs' free energy.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data that support the findings of this study are available from the corresponding author (J.E.M.-D.) upon reasonable request.

Acknowledgments

The authors wish to thank the anonymous reviewers for their invaluable comments, which substantially improved the quality of this work.

Funding

One of the authors (J.E.M.-D.) was funded by the National Council of Science and Technology of Mexico through grant A1-S-45928. The last author (Q.S.) was supported in part by a University Research Council Award (No. 2021-CAS03), Baylor University, USA.

Appendix. Matlab code

The following is a preliminary version of the Matlab code used to approximate the solutions of the mathematical model (2.5). The final version of the code is available from the corresponding authors upon reasonable request.

```
function [t,E]=zakharov

  function surfplot(X,Y,Z,title)
    figure;
    surf(X,Y,Z);
    shading interp;
    colormap jet;
    xlabel('x','interpr','latex','fontsize',14);
    ylabel('t','interpr','latex','fontsize',14);
    zlabel(title,'interpr','latex','fontsize',14);
    grid off;
    lighting gouraud
    set(gca,'fontsize',12);
    myfig=gcf;
    myfig.RendererMode = 'manual';
  end

  function lineplot(t,F,title)
    figure
    plot(t,F,'linewidth',2)
    xlabel('t','interpr','latex','fontsize',14);
```

```
  ylabel(title,'interpr','latex','fontsize',14);
  grid off;
  set(gca,'fontsize',12);
  myfig=gcf;
  myfig.RendererMode = 'manual';
end

function u=uexact(x,t)
  u=0.5.*(sqrt(10)-sqrt(2)).*sech(sqrt(0.5.*(1+sqrt(5))).*x-t).*...
    exp(1i.*((sqrt(2)./(1+sqrt(5))).*x-t));
end

function m=mexact(x,t)
  m=-2.*sech(sqrt(0.5.*(1+sqrt(5))).*x-t).^2;
end

xL=-50;
xR=50;
T0=10;

alpha=2;
beta=2;

h=0.5;
tau=0.01;
tol=1e-8;
NumIt=20;

x=xL:h:xR;
t=0:tau:T0;

M=length(x);
N=length(t);

[X,T]=meshgrid(x,t);
u=zeros(size(X));
m=zeros(size(X));
u(1,:)=uexact(x,t(1));
m(1,:)=mexact(x,t(1));
u(2,:)=uexact(x,t(2));
m(2,:)=mexact(x,t(3));

ga=zeros(1,M);
gb=zeros(1,M);
gha=zeros(1,M);
ghb=zeros(1,M);
ga(1)=gamma(alpha+1)/gamma(0.5*alpha+1)^2/h^alpha;
gb(1)=tau*tau*gamma(beta+1)/gamma(0.5*beta+1)^2/h^beta;
gha(1)=gamma(0.5*alpha+1)/gamma(0.25*alpha+1)^2/h^(0.5*alpha);
ghb(1)=gamma(0.5*beta+1)/gamma(0.25*beta+1)^2/h^(0.5*beta);
for k=1:M-1
  ga(k+1)=(1-(alpha+1)/(0.5*alpha+k))*ga(k);
```

```
    gb(k+1)=(1-(beta+1)/(0.5*beta+k))*gb(k);
    gha(k+1)=(1-(0.5*alpha+1)/(0.25*alpha+k))*gha(k);
    ghb(k+1)=(1-(0.5*beta+1)/(0.25*beta+k))*ghb(k);
  end

  I=eye(M);
  Ha=zeros(M,M);
  Hb=zeros(M,M);
  Hha=zeros(M,M);
  Hhb=zeros(M,M);
  for i=1:M
    for j=1:M
      Ha(i,j)=-ga(abs(i-j)+1);
      Hb(i,j)=-gb(abs(i-j)+1);
      Hha(i,j)=-gha(abs(i-j)+1);
      Hhb(i,j)=-ghb(abs(i-j)+1);
    end
  end

  for n=3:N
    m(n,:)=2.*m(n-1,:)-m(n-2,:)+abs(u(n-1,:)).^2*Hb+m(n-1,:)*Hb;

    k=0;
    diff=1;
    p0=u(n-1,:);
    while (k<NumIt)&&(diff>tol)
      p1=u(n-2,:)+tau.*1i.*((p0+u(n-2,:))*(Ha-I)-(p0+u(n-2,:)).*...
        (m(n-1,:)+0.5.*(abs(p0).^2+abs(u(n-2,:)).^2)));
      diff=norm(p0-p1,Inf);
      p0=p1;
      k=k+1;
    end
    u(n,:)=p0;

    m(n,1)=0;
    m(n,M)=0;
    u(n,1)=0;
    u(n,M)=0;
  end

  W=(m(2:N,:)-m(1:N-1,:))/Hhb./tau;

  MassDensity=0.5.*(abs(u(1:N-1,:)).^2+abs(u(2:N,:)).^2);
  Hf=0.5.*(abs(u(2:N,:)*Hha).^2+abs(u(1:N-1,:)*Hha).^2)...
    +0.5.*(abs(u(2:N,:)).^2+abs(u(1:N-1,:)).^2)...
    +0.5.*abs(W).^2+0.5.*m(2:N,:).*m(1:N-1,:)...
    +0.25.*(abs(u(2:N,:)).^4+abs(u(1:N-1,:)).^4)...
    +0.5.*(m(1:N-1,:).*abs(u(2:N,:)).^2+m(2:N,:).*abs(u(1:N-1,:)).^2);
  H=Hf+abs(m(2:N,:)-m(1:N-1,:)).^2./tau./tau;

  Mass=h.*sum(MassDensity,2);
  Ef=h.*sum(Hf,2);
```

```
  E=h.*sum(H,2);

  surfplot(X,T,real(u),'$\mathrm{Re}(u(x,t))$');
  surfplot(X,T,imag(u),'$\mathrm{Im}(u(x,t))$');
  surfplot(X,T,abs(u),'$\vert u(x,t)\vert$');
  surfplot(X,T,m,'$m(x,t)$');

  surfplot(X(1:N-1,:),T(1:N-1,:),MassDensity,'$\mathcal{M}(x,t)$');
  surfplot(X(1:N-1,:),T(1:N-1,:),Hf,'$\mathcal{H}_F(x,t)$');
  surfplot(X(1:N-1,:),T(1:N-1,:),H,'$\mathcal{H}(x,t)$');

  lineplot(t(1:N-1),Mass,'$\mathcal{M}(t)$');
  lineplot(t(1:N-1),Ef,'$\mathcal{E}_F(t)$');
  lineplot(t(1:N-1),E,'$\mathcal{E}(t)$');
end
```

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