Yamabe Invariants, Homogeneous Spaces, and Rational Complex Surfaces

Claude LEBRUN

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651, USA
E-mail: *claude@math.stonybrook.edu*
URL: http://www.math.stonybrook.edu/~claude/

Received February 23, 2023, in final form May 02, 2023; Published online May 07, 2023
https://doi.org/10.3842/SIGMA.2023.027

Abstract. The Yamabe invariant is a diffeomorphism invariant of smooth compact manifolds that arises from the normalized Einstein–Hilbert functional. This article highlights the manner in which one compelling open problem regarding the Yamabe invariant appears to be closely tied to static potentials and the first eigenvalue of the Laplacian.

Key words: scalar curvature; conformal structure; Yamabe problem; diffeomorphism invariant

2020 Mathematics Subject Classification: 53C21; 53C18; 14J26; 58J50

1 Introduction

The *Yamabe invariant* of a smooth compact n-manifold M, $n \geq 3$, is a fascinating real-valued diffeomorphism invariant $\mathscr{Y}(M)$ that is defined by applying a minimax procedure to the *normalized Einstein–Hilbert functional*

$$\mathscr{E}\big(M^n, g\big) = \frac{\int_M s_g \, \mathrm{d}\mu_g}{\left(\int_M \mathrm{d}\mu_g\right)^{1-\frac{2}{n}}},$$

on the space of Riemannian metrics g; here s_g and $\mathrm{d}\mu_g$ respectively denote the scalar curvature and n-dimensional volume measure of any Riemannian metric g. One key reason for studying this functional is that the critical points of $\mathscr{E}$ are exactly the *Einstein metrics* [4], or in other words the metrics of constant Ricci curvature. However, $\mathscr{E}$ is neither bounded above nor below, so one can never find a critical point by minimizing or maximizing the functional. Nonetheless, Yamabe discovered that $\mathscr{E}$ does become bounded below whenever one restricts it to a conformal class

$$\gamma = [g_0] = \big\{ g = f g_0 \,|\, f \colon M \xrightarrow{C^\infty} \mathbb{R}^+ \big\},$$

and his influential posthumous paper [27] moreover claimed to prove that the *Yamabe constant* of each conformal class γ, as defined by

$$Y(M, \gamma) := \inf \mathscr{E}|_\gamma,$$

is always achieved by some minimizing metric. Although Yamabe's article made a perilous analytic mistake, a series of fundamental papers by Trudinger [26], Aubin [3], and Schoen [22] eventually proved that such minimizers, now called *Yamabe metrics*, do exist in every γ.

This paper is a contribution to the Special Issue on Differential Geometry Inspired by Mathematical Physics in honor of Jean-Pierre Bourguignon for his 75th birthday. The full collection is available at https://www.emis.de/journals/SIGMA/Bourguignon.html

The *Yamabe invariant* (originally introduced by Kobayashi [12] and Schoen [23], but under different names) is defined by

$$\mathscr{Y}(M) = \sup_{\gamma} Y(M,\gamma) = \sup_{\gamma} \inf_{g\in\gamma} \mathscr{E}(M,g),$$

where one takes the infimum of the Einstein–Hilbert functional in each conformal class, and then takes the supremum of these infima over all conformal classes on M. The existence of minimizers in every conformal class then allows us to reword this, more geometrically, as

$$\mathscr{Y}(M) = \sup\{s_g \mid g \text{ is a unit-volume Yamabe metric on } M\}.$$

This supremum is moreover finite, because one of Aubin's fundamental contributions to the Yamabe problem is that any conformal class automatically satisfies

$$Y\big(M^n,\gamma\big) \leq \mathscr{E}\big(S^n, g_{\text{unit}}\big) = n(n-1)\pi \left[\frac{2\sqrt{\pi}}{\Gamma(\frac{n+1}{2})}\right]^{2/n},$$

where g_{unit} is the standard metric on the unit n-sphere; it therefore follows that any smooth compact n-manifold satisfies

$$\mathscr{Y}\big(M^n\big) \leq \mathscr{Y}\big(S^n\big) = \mathscr{E}\big(S^n, g_{\text{unit}}\big).$$

This "mountain-pass" definition thus attaches a real number $\mathscr{Y}(M)$ to every smooth compact manifold. Because $\mathscr{Y}(M) > 0$ iff M carries a metric of positive scalar curvature, the quest to compute Yamabe invariants represents a quantitative refinement of the problem of determining which manifolds admit positive-scalar-curvature metrics.

While any Yamabe metric has constant scalar curvature, it must be emphasized that the converse is not generally true. Nonetheless, there are two important basic results that run in the converse direction. First, metrics of constant scalar curvature $s \leq 0$ are always Yamabe, and indeed are, up to constant rescalings, the unique Yamabe metrics in their conformal classes. Second, a theorem of Obata [19] guarantees that, on a compact manifold M, any Einstein metric g is necessarily a Yamabe metric; indeed, up to constant rescalings, it is the *only* constant-scalar-curvature metric in its conformal class, except in the special case where $\big(M^n,[g]\big)$ is the standard sphere $\big(S^n,[g_{\text{unit}}]\big)$.

The fact that a metric of constant positive scalar curvature is not necessarily Yamabe makes it surprisingly difficult to calculate the Yamabe invariant $\mathscr{Y}(M)$ in the positive case. As a consequence, there are very few manifolds where the Yamabe invariant is positive, known, and different from that of the sphere. By contrast, there are large classes of manifolds with $\mathscr{Y} \leq 0$ where the precise value of the Yamabe invariant is currently known. For example, the Yamabe invariant is known [1, 16] for the underlying 4-manifold M of any compact complex surface $\big(M^4,J\big)$ of Kodaira dimension $\neq -\infty$. One key ingredient is that Seiberg–Witten theory allows one to show [14] that if $\big(M^4,J\big)$ admits a Kähler–Einstein metric g of non-positive Ricci curvature, then

$$\mathscr{Y}(M) = \mathscr{E}(M,g) = -4\pi\sqrt{2c_1^2(M,J)}.$$

On the other hand, $\mathscr{Y}$ turns out to be invariant under blowing up or down in this context, so that [1, 14, 16]

$$\mathscr{Y}\big(M\#k\overline{\mathbb{CP}}_2\big) = \mathscr{Y}(M)$$

for any complex surface $\big(M^4,J\big)$ of Kodaira dimension $\neq -\infty$.

A faint echo of the above story about Kähler–Einstein metrics can still be heard in the positive case. Indeed, two different arguments involving spin^c Dirac operators [11, 15] show that the Fubini–Study metric g_{FS} on $\mathbb{CP}_2$ similarly achieves the manifold's Yamabe invariant:

$$\mathscr{Y}(\mathbb{CP}_2) = \mathscr{E}(\mathbb{CP}_2, g_{\text{FS}}) = 12\pi\sqrt{2} < 8\pi\sqrt{6} = \mathscr{Y}\big(S^4\big).$$

By contrast, however, the other conformally Kähler, positive Einstein metrics on 4-manifolds [7, 17, 20, 21, 25] all have energies $\mathscr{E}$ less than corresponding the Yamabe invariants. Indeed, most of these manifolds are blow-ups of the complex projective plane, and in all these cases induction on an inequality due to Kobayashi [12, Theorem 2] implies that

$$\mathscr{Y}\big(\mathbb{CP}_2\#k\overline{\mathbb{CP}_2}\big) \geq \min\big\{\mathscr{Y}(\mathbb{CP}_2), \mathscr{Y}\big(\overline{\mathbb{CP}_2}\big), \ldots, \mathscr{Y}\big(\overline{\mathbb{CP}_2}\big)\big\} = \mathscr{Y}(\mathbb{CP}_2),$$

which is larger than the energy of any of the Einstein metrics in question. In fact, there is only one smooth compact 4-manifold that carries a conformally Kähler, positive Einstein metric that is not covered by this argument, namely the spin manifold $\mathbb{CP}_1 \times \mathbb{CP}_1$. This case is harder, and demands a very different set of techniques. However, Böhm, Wang, and Ziller [5] were able to settle this outstanding case by showing that the Kähler–Einstein metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$ can be perturbed to yield nearby Yamabe metrics of slightly higher energy.

When M is the underlying smooth 4-manifold of a compact complex surface $\big(M^4, J\big)$ of Kodaira dimension $\neq -\infty$, we have already pointed out that the Yamabe invariant $\mathscr{Y}(M)$ is unchanged if M is blown up or blown down. This makes it natural to ask whether this pattern also holds for the compact complex surfaces that can be obtained from $\mathbb{CP}_2$ by some sequence of blow-ups and blow-downs; these are the *rational complex surfaces* of the title. As a step in this direction, this article will focus on the following looser question:

Main Question. If a compact complex surface $\big(M^4, J\big)$ is obtained from $\mathbb{CP}_2$ by blowing up and down, is it necessarily true that $\mathscr{Y}(M) \geq \mathscr{Y}(\mathbb{CP}_2)$?

Once again, the only hard case is $S^2 \times S^2$, so a positive answer would immediately follow from an affirmative answer to the following:

Technical Question. Let h_2 be the homogeneous metric on $S^2 \times S^2$ that is the Riemannian product of two standard round metrics on S^2, one of radius 1, and one of radius $1/\sqrt{2}$. Is h_2 a Yamabe metric?

This metric, which satisfies $\mathscr{E}(S^2 \times S^2, h_2) = \mathscr{Y}(\mathbb{CP}_2)$, also carries a static potential [2, 8, 13], and thus enjoys an interestingly different special status for the behavior of the scalar curvature. Our approach to the above technical question combines elements of the Böhm–Wang–Ziller argument with some simple observations about the spectrum of the Laplacian. I hope that Jean-Pierre will enjoy the way this connects these questions to his own investigations of the scalar curvature.

2 The bass-note of a Yamabe metric

One of Jean-Pierre Bourguignon's early contributions [4, 6] to the study of the scalar curvature was the discovery that, on a compact manifold M^n, the linearized behavior of the scalar curvature s at a constant-scalar-curvature metric g depends on whether $s/(n-1)$ is an eigenvalue of the Laplace–Beltrami operator Δ_g. However, it is less often noted that this question also reveals important information about whether the given metric g is a Yamabe metric:

Lemma 2.1. *If g is a Yamabe metric on a smooth compact manifold M of dimension $n \geq 3$, then*

$$\lambda_1 \geq \frac{s}{n-1}, \tag{2.1}$$

where λ_1 is the smallest positive eigenvalue of the Laplace–Beltrami operator $\Delta_g = \mathrm{d}^*\mathrm{d}$, and where s is the scalar curvature of g.

Proof. If (M, g) is a compact Riemannian manifold of scalar curvature $s = \text{const}$, it suffices to prove the contrapositive statement that

$$\lambda_1 < \frac{s}{n-1} \implies g \text{ is not a Yamabe metric.}$$

To see this, first recall that, after setting $p := \frac{2n}{n-2}$, so that any metric $\widetilde{g}$ conformal to g and its corresponding metric volume measure can be simultaneously expressed as

$$\widetilde{g} = u^{p-2} g \qquad \text{and} \qquad \widetilde{\mathrm{d}\mu} = u^p \, \mathrm{d}\mu,$$

the scalar curvature $\widetilde{s} = s_{\widetilde{g}}$ of the conformally rescaled metric can then be read off from the *Yamabe equation*

$$\widetilde{s} u^{p-1} = [(p+2)\Delta + s]\, u. \tag{2.2}$$

Now assume that $s > (n-1)\lambda_1 > 0$, and suppose that $f \not\equiv 0$ belongs to the λ_1-eigenspace of the Laplacian. In particular, since

$$\Delta f = \lambda_1 f,$$

where $\lambda_1 > 0$, it then follows that $\int_M f \, \mathrm{d}\mu_g = \frac{1}{\lambda_1} \int_M (\Delta f) \, \mathrm{d}\mu_g = 0$. Now set $u_t = 1 + tf$, and, for every small real number t, let $g_t = u_t^{p-2} g$. Setting $V = \mathrm{Vol}(M, g)$ and letting $\fint = V^{-1} \int_M$, we then have

$$\begin{aligned}
\mathcal{E}(g_t) &= \frac{\int_M u_t \left[(p+2)\Delta u_t + s u_t\right] \mathrm{d}\mu_g}{\left(\int_M u_t^p \mathrm{d}\mu_g\right)^{2/p}} \\
&= \frac{V \fint \left[t^2(p+2)\lambda_1 f^2 + s + s t^2 f^2\right] \mathrm{d}\mu_g}{\left(V \fint [1 + t^2 \frac{p(p-1)}{2} f^2 + O(t^3)] \mathrm{d}\mu_g\right)^{2/p}} \\
&= s V^{1-\frac{2}{p}} \frac{\left[1 + t^2\big(1 + (p+2)\frac{\lambda_1}{s}\big) \fint f^2 \mathrm{d}\mu_g\right]}{\left[1 + \frac{p(p-1)}{2} t^2 \fint f^2 \mathrm{d}\mu_g + O\big(t^3\big)\right]^{2/p}} \\
&= s V^{2/n} \left[1 + t^2 \left(1 + (p+2)\frac{\lambda_1}{s}\right) \fint f^2 \mathrm{d}\mu_g\right] \left[1 - (p-1) t^2 \fint f^2 \mathrm{d}\mu_g + O\big(t^3\big)\right] \\
&= s V^{2/n} \left[1 + t^2 \left((2-p) + (p+2)\frac{\lambda_1}{s}\right) \fint f^2 \mathrm{d}\mu_g + O\big(t^3\big)\right] \\
&= s V^{2/n} \left[1 - \frac{4t^2}{n-2} \left(1 - \frac{\lambda_1}{s/(n-1)}\right) \fint f^2 \mathrm{d}\mu_g + O\big(t^3\big)\right] \\
&< s V^{2/n} = \mathcal{E}(g)
\end{aligned}$$

for all sufficiently small $t \neq 0$. Thus g is not a Yamabe metric, and the Lemma therefore follows by contraposition. ∎

Example 2.2. For any integer $k \geq 2$, let (X^ℓ, h) be a compact Einstein manifold of Ricci curvature $k-1$, and, for an arbitrary real parameter $\mathtt{t} \geq 1$, consider the Riemannian product metric

$$h_{\mathtt{t}} = g_{\text{unit}} \oplus \mathtt{t}^{-1} h \tag{2.3}$$

on $S^k \times X^\ell$, where g_{unit} is the standard metric on the unit k-sphere $S^k \subset \mathbb{R}^{k+1}$. Because $\big(S^k \times X^\ell, h_{\mathtt{t}}\big)$ has Ricci curvature $\geq k-1$, Lichnerowicz's eigenvalue estimate [18] tells us that it satisfies $\lambda_1 \geq k$; and this space therefore actually has $\lambda_1 = k$, since any linear functional $\mathbb{R}^{k+1} \to \mathbb{R}$, restricted to the unit k-sphere, and then pulled back to $S^k \times X^\ell$, is a Laplace eigenfunction with eigenvalue k. On the other hand, the scalar curvature of $\big(S^k \times X^\ell, h_{\mathtt{t}}\big)$ is $s = (k-1)(k+\mathtt{t}\ell)$. Thus, inequality (2.1) reads $(k+\ell-1)k \geq (k-1)(k+\mathtt{t}\ell)$, and so is satisfied iff $\mathtt{t} \in \big[1, \frac{k}{k-1}\big]$. Consequently, Lemma 2.1 implies that $h_{\mathtt{t}}$ is *not* a Yamabe metric if $\mathtt{t} > \frac{k}{k-1}$. By contrast, when $\mathtt{t} = 1$, the metric h_1 is Einstein, and so, by Obata's theorem [19], is certainly a Yamabe metric.

These last observations fit into an interesting broader discussion. On a compact n-manifold M, $n \geq 3$, Obata's theorem guarantees that any Einstein metric g is actually a Yamabe metric, so Lemma 2.1 implies that (2.1) holds for any compact Einstein manifold. However, Lichnerowicz gave a Bochner–Weitzenböck proof [18] in this Einstein case that is not only more elementary, but also sharpens the result, as it shows that any compact Einstein manifold $\big(M^n, g\big)$ for which equality holds in (2.1) must, up to isometry and rescaling, be the standard n-sphere $\big(S^n, g_{\text{unit}}\big)$. By contrast, however, we will now see that equality holds in (2.1) for many other Yamabe metrics.

Lemma 2.3. *On a given compact n-manifold M, let $g(t) = [u(t)]^{p-2} g$ be a smooth one-parameter family of metrics in some conformal class $[g]$. If all of these metrics have the same constant scalar curvature $s = const$, and if $u(t) = 1 + vt + O\big(t^2\big)$, then the function $v = \dot{u}(0)$ satisfies*

$$\Delta v = \frac{s}{n-1} v$$

with respect to the background metric $g = g(0)$.

Proof. Under these circumstances, the Yamabe equation (2.2) becomes

$$s[u(t)]^{p-1} = (p+2)\Delta u(t) + s u(t),$$

and differentiating with respect to the parameter t thus yields

$$s(p-1)[u(t)]^{p-2}\dot{u}(t) = (p+2)\Delta \dot{u}(t) + s\dot{u}(t).$$

Setting $t = 0$, where $u(0) = 1$ and $\dot{u}(0) = v$ by assumption, we obtain

$$\Delta v = s\left(\frac{p-2}{p+2}\right) v,$$

and the substitution $p := \frac{2n}{n-2}$ therefore, upon simplification, yields

$$\Delta v = \frac{s}{n-1} v.$$

If $v \not\equiv 0$, this function is therefore a Laplace–Beltrami eigenfunction, with eigenvalue $\frac{s}{n-1}$. ∎

Lemma 2.4. *Let g be a Yamabe metric on a smooth compact manifold M of dimension $n \geq 3$, and suppose that (M, g) carries a conformal Killing field ξ that is not actually Killing. Then (M, g) satisfies*

$$\lambda_1 = \frac{s}{n-1},$$

and so saturates the bound provided by inequality (2.1).

Proof. If Φ_t is the 1-parameter family of diffeomorphisms generated by ξ, the pull-backs $\Phi_t^* g = [u(t)]^{p-2} g$ form a family of constant-scalar-curvature metrics as in Lemma 2.3. Moreover $v := \dot{u}(0) \not\equiv 0$, because $(p-2)vg = \mathscr{L}_\xi g \not\equiv 0$. Since $\Delta v = \frac{s}{n-1} v$ by Lemma 2.3, it thus follows that $\frac{s}{n-1}$ belongs to the Laplace spectrum of (M, g). Next, observe that, since the $\Phi_t^* g$ provide us with non-homothetic, constant-scalar-curvature metrics in the conformal class $[g]$, the scalar curvature s of g must be positive. But since λ_1 is by definition the smallest positive Laplace eigenvalue of (M, g), this implies that $\lambda_1 \leq \frac{s}{n-1}$. However, since g is by assumption a Yamabe metric, Lemma 2.1 also tells us that $\lambda_1 \geq \frac{s}{n-1}$. It therefore follows that $\lambda_1 = \frac{s}{n-1}$, as claimed. ■

Lemma 2.5. *Let g be a unit-volume homogeneous metric on a smooth compact n-manifold M, $n \geq 3$, and let $\tilde{g} \in [g]$ be a Yamabe metric of unit volume in the conformal class of g. If $\tilde{g} \neq g$, then the scalar curvature s and the first Laplace eigenvalue λ_1 of $\tilde{g}$ must satisfy*

$$\lambda_1 = \frac{s}{n-1}.$$

Proof. By hypothesis, there is a compact connected Lie group G that acts transitively and isometrically on (M, g). The Lie algebra $\mathfrak{g}$ of G thus acts on M by Killing fields of g, and these are therefore all conformal Killing fields of the Yamabe metric $\tilde{g}$. However, since $\tilde{g} \neq g$ by assumption, and since the homogeneous metric g is the unique unit-volume, G-invariant metric in its conformal class, there must be some $\xi \in \mathfrak{g}$ which is not a Killing field for $\tilde{g}$. The claim thus follows by applying Lemma 2.4 to $\tilde{g}$. ■

Theorem 2.6. *Let $M = \mathsf{G}/\mathsf{H}$ be a compact homogeneous manifold of dimension $n \geq 3$, where G is a compact connected Lie group and $\mathsf{H} \subset \mathsf{G}$ is a compact Lie subgroup. Consider the finite-dimensional manifold*

$$\mathscr{M} = \left\{ \textit{unit-volume } \mathsf{G}\textit{-invariant metrics } g \textit{ on } M \mid \lambda_1 > \frac{s}{n-1} \right\},$$

and define subsets $\mathscr{U} \subset \mathscr{V} \subset \mathscr{M}$ by

$$\mathscr{V} = \{ g \in \mathscr{M} \mid g \textit{ is a Yamabe metric} \}, \quad \text{and}$$

$$\mathscr{U} = \{ g \in \mathscr{M} \mid g \textit{ is the } \mathsf{unique} \textit{ unit-volume Yamabe metric in } [g] \}.$$

Then $\mathscr{U} \subset \mathscr{M}$ is open, while $\mathscr{V} \subset \mathscr{M}$ is closed.

Proof. The complement $\mathscr{U}^{\complement} = \mathscr{M} - \mathscr{U}$ of $\mathscr{U}$ consists of those $g \in \mathscr{M}$ for which there is a unit-volume Yamabe metric $\tilde{g} \in [g]$ with $\tilde{g} \neq g$. By Lemma 2.5, this complement can therefore also be described as

$$\mathscr{U}^{\complement} = \left\{ g \in \mathscr{M} \mid \exists \text{ unit-volume Yamabe } \tilde{g} \in [g] \text{ for which } \lambda_1 = \frac{s}{n-1} \right\},$$

since, by definition, any $g \in \mathscr{M}$ satisfies $\lambda_1 \neq \frac{s}{n-1}$.

We will now use this observation to prove that $\mathscr{U}^{\complement}$ is closed. Indeed, suppose that $g_j \in \mathscr{U}^{\complement}$ is a sequence of unit-volume homogeneous metrics with $g_j \to g$ for some $g \in \mathscr{M}$. By definition,

we may then choose a sequence $\tilde{g}_j \in [g_j]$ of unit-volume, non-homogeneous Yamabe metrics in the corresponding conformal classes. However, since $g \in \mathscr{M}$ satisfies $\lambda_1 > \frac{s}{n-1}$ by hypothesis, it follows that (M, g) is certainly not a constant-sectional-curvature n-sphere, and a compactness argument [5, p. 716] therefore guarantees that a subsequence $\tilde{g}_{j_i}$ of these Yamabe metrics must converge in C^2 to some unit-volume Yamabe metric $\tilde{g} \in [g]$; cf. [23, Proposition 2.1]. It then follows that $\lambda_1(\tilde{g}_{j_i}) \to \lambda_1(\tilde{g})$ and $s(\tilde{g}_{j_i}) \to s(\tilde{g})$. However, since $\lambda_1(\tilde{g}_{j_i}) = \frac{s}{n-1}(\tilde{g}_{j_i})$, this then implies that $\lambda_1(\tilde{g}) = \frac{s}{n-1}(\tilde{g})$, too. But $\lambda_1(g) > \frac{s}{n-1}(g)$ by our definition of $\mathscr{M}$, so it follows that $\tilde{g} \neq g$. Hence $g \in \mathscr{U}^{\complement}$. We have thus shown that $\mathscr{U}^{\complement}$ is closed, and hence that $\mathscr{U}$ is open.

On the other hand [4, Proposition 4.31], the Yamabe constant $Y(M, [g])$ is a continuous function of g, as is, more obviously, the scalar curvature s_g. Since we have normalized each $g \in \mathscr{M}$ to have unit volume, the subset $\mathscr{V} \subset \mathscr{M}$ is exactly the zero locus of the continuous function $Y(M, [g]) - s_g$. It thus follows that $\mathscr{V}$ is closed, as claimed. ∎

Dropping the convenient but unnecessary unit-volume constraint, and recalling our discussion of (2.3) in Example 2.2, we now apply this theorem to the family $\big(S^2 \times S^2, h_{\mathsf{t}}\big)$, $\mathsf{t} \in [1, 2]$, where $h_{\mathsf{t}} = g_{\text{unit}} \oplus \mathsf{t}^{-1} g_{\text{unit}}$. It follows that there is a maximal $\mathsf{T} \in (1, 2]$ such that h_{t} is, up to homothety, the unique Yamabe metric in $[h_{\mathsf{t}}]$ for every $\mathsf{t} \in [1, \mathsf{T})$. However, it then also follows, by continuity, that h_{T} is itself a Yamabe metric. To get an affirmative answer to the the questions posed in the Introduction, it would thus suffice to show, if $\mathsf{T} < 2$, that h_{T} is necessarily, up to homothety, the unique Yamabe metric in $[h_{\mathsf{T}}]$. The fact that h_{T} is a Yamabe, Kähler metric on a 4-manifold does seem to offer some hope of proving just this, especially in light of the suggestive results of [10, 11]. Nonetheless, this problem remains, for the present, just a topic for further investigation.

3 Static potentials

We now conclude this article with a few remarks relating our discussion to the fascinating topic of static potentials, which originally arose from the work of Bourguignon [6] and Fischer–Marsden [9], and later led to the beautiful results of Lafontaine [13] and others [2, 8, 24]. The motivating question here is to ask for the precise circumstances under which the linearization

$$Ds\colon\ \Gamma\big(\odot^2 T^*M\big) \to C^\infty(M) \tag{3.1}$$

of the scalar curvature, at a given Riemannian metric g on a smooth compact n-manifold M, fails to be surjective. Since the t-derivative of the scalar curvature s, at $t = 0$, for a family $g + th$ is

$$\dot{s} = \Delta\big(h_{ab} g^{ab}\big) + \nabla^a \nabla^b h_{ab} - r^{ab} h_{ab},$$

integration by parts reveals that the co-kernel of (3.1) exactly consists of functions f that satisfy the overdetermined linear equation

$$0 = (\Delta f) g_{ab} + \nabla_a \nabla_b f - f r_{ab}. \tag{3.2}$$

A solution $f \not\equiv 0$ of (3.2) is called a *static potential* on (M, g), in order to highlight the remarkable fact [4, Corollary 9.107, with $p = 1$, $\hat{\lambda} = 0$] that (3.2) is precisely equivalent to demanding that

$$\widehat{g} = g + f^2\, \mathrm{d}t^2 \tag{3.3}$$

be an Einstein metric on $(M - Z) \times S^1$, where $Z := f^{-1}(0)$.

One simple strategy for guessing a solution f of (3.2) would be to take the function $f \not\equiv 0$ to be constant. However, inspection reveals that this ansatz works if and only if (M, g) is Ricci-flat. While this does show that the linearized scalar curvature (3.1) fails to be surjective on any

compact Ricci-flat manifold, there is really nothing else to say about this "trivial" case. We will therefore assume henceforth that f is non-constant.

Next, notice that taking the trace of (3.2) yields the identity

$$\Delta f = \frac{s}{n-1} f. \tag{3.4}$$

We may therefore rewrite (3.2) as

$$\nabla_a \nabla_b f = \varrho_{ab} f, \tag{3.5}$$

where ϱ is defined, in terms of the scalar curvature s and the Ricci tensor r or trace-free Ricci tensor $\mathring{r}$, to be

$$\varrho_{ab} := r_{ab} - \frac{s}{n-1} g_{ab} = \mathring{r}_{ab} - \frac{s}{n(n-1)} g_{ab}.$$

Equivalently, f satisfies the linear second-order ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f = \varrho(\mathsf{v}, \mathsf{v}) f \tag{3.6}$$

along every unit-speed geodesic, where $\mathsf{v} = \mathrm{d}/\mathrm{d}t$ denotes the unit tangent vector. Since we have assumed that our connected Riemannian manifold (M, g) is compact, and hence geodesically complete, and since $f \not\equiv 0$, it follows that df must be non-zero at every point of $Z = f^{-1}(0) \subset M$; otherwise, any $q \in M$ could be joined by some geodesic segment to a point $p \in Z$ at which the initial-value pair $(f(p), f'(p))$ for (3.6) vanished, thereby forcing f to vanish at q, too. In particular, Z is a smooth hypersurface. Moreover, Z is totally geodesic, because the same initial-value argument for (3.6) shows that any geodesic that is initially tangent to Z must remain confined to $Z = f^{-1}(0)$. Finally, notice that, since

$$\nabla_a |\nabla f|^2 = 2\big(\nabla^b f\big) \nabla_a \nabla_b f = 2 f \varrho_{ab} \big(\nabla^b f\big)$$

vanishes along Z, the length of the normal vector field ∇f is actually constant on each connected component of Z.

We next apply the divergence operator $-\nabla^a$ to both sides of (3.2). Remembering Bochner's formula for the Hodge Laplacian on 1-forms, and recalling that d commutes with the Hodge Laplacian, the Bianchi identity then yields

$$\begin{aligned} 0 &= -\nabla^a (g_{ab} \Delta f) - \nabla^a \nabla_a \nabla_b f + \nabla^a (f r_{ab}) - \nabla_b \Delta f + \nabla^* \nabla (\nabla_b f) + r_{ab} \nabla^a f + f \nabla^a r_{ab} \\ &= -\nabla_b (\mathrm{d} + \mathrm{d}^*)^2 f + (\mathrm{d} + \mathrm{d}^*)^2 \nabla_b f + \tfrac{1}{2} f \nabla_b s \\ &= \tfrac{1}{2} f \nabla_b s. \end{aligned}$$

This shows that $\nabla s = 0$ on the set $M - Z$ where $f \neq 0$. However, since Z is just a hypersurface, $M - Z \subset M$ is dense, and it therefore follows by continuity that $\nabla s = 0$ everywhere; and since M has been assumed to be connected, this then implies that the scalar curvature s is constant. Consequently, (3.4) now tells us that f is an eigenfunction of the Laplacian, with eigenvalue $\frac{s}{n-1}$. This in particular gives the scalar curvature a Rayleigh-quotient interpretation:

$$\frac{s}{n-1} = \frac{\int_M f \Delta f \, \mathrm{d}\mu}{\int_M f^2 \, \mathrm{d}\mu} = \frac{\int_M |\nabla f|^2 \, \mathrm{d}\mu}{\int_M f^2 \, \mathrm{d}\mu}.$$

Our assumption that f is *non-constant* therefore constrains the scalar curvature s to be positive, and so forces the average value of f to vanish:

$$⨍_M f \, \mathrm{d}\mu = \frac{n-1}{s} ⨍_M \Delta f \, \mathrm{d}\mu = \frac{n-1}{sV} \int_M \Delta f \, \mathrm{d}\mu = 0.$$

Since the smooth function f must achieve its average value, there must be some point $p \in M$ where f vanishes, and we therefore conclude that the totally geodesic hypersurface $Z = f^{-1}(0)$ is necessarily non-empty.

A particularly attractive special case of the above occurs when Z is connected. The Mayer–Vietoris sequence then guarantees that $M - Z$ must have exactly two connected components, namely the two regions respectively defined by $f > 0$ and $f < 0$. On the other hand, the connectedness of Z implies that $|\nabla f|$ is a non-zero constant on Z, so the linearity of (3.2) allows us, by replacing f with a constant multiple, to arrange that $|\nabla f| \equiv 1$ along Z. Setting $M_+ = f^{-1}([0,\infty))$, we can now construct a smooth compact connected $(n+1)$-manifold by setting $N = \big(M_+ \times S^1\big)/\!\sim$, where the equivalence relation collapses $\partial\big(M_+ \times S^1\big) = Z \times S^1$ down to Z, and (3.3) then defines a smooth Einstein metric $\widehat{g}$ on N^{n+1}. Of course, replacing f with $-f$ also gives us a second Einstein $(n+1)$-manifold, which in principle might not be isometric to the first. Note that the connectedness of Z plays an important role in this construction, since otherwise $|\nabla f|$ could take different values on different connected components, leading to unavoidable edge-cone singularities in $\big(N, \widehat{g}\big)$; for details, see [2].

Following in the tracks of Lafontaine [13, Section B], we next observe that the metrics of (2.3), for the critical parameter value $\mathsf{t} = \frac{k}{k-1}$, provide us with examples of manifolds that carry static potentials. Thus, for any positive integer k, let $\big(S^k, g_{\text{unit}}\big)$ be the standard unit k-sphere, and let $f\colon S^k \to \mathbb{R}$ be the restriction of the Euclidean x^1 coordinate to $S^k \subset \mathbb{R}^{k+1}$. The gradient $\eta = \nabla f$ is then a conformal Killing field on S^k with

$$\mathscr{L}_\eta g_{\text{unit}} = -2 f g_{\text{unit}},$$

and one therefore has

$$\nabla\nabla f = -f g_{\text{unit}}$$

on $\big(S^k, g_{\text{unit}}\big)$. Now let $\big(X^\ell, \widehat{h}\big)$ be a compact ℓ-dimensional Einstein manifold of Ricci curvature k, and set

$$\big(M^n, g\big) = \big(S^k, g_{\text{unit}}\big) \times \big(X^\ell, \widehat{h}\big),$$

so that $n = k + \ell$ and $g = g_{\text{unit}} \oplus \widehat{h}$. We now reinterpret f as a function $f\colon M \to \mathbb{R}$, by pulling it back from the first factor of the Riemannian product. In this context, the gradient $\eta = \nabla f$ of this function then satisfies

$$\mathscr{L}_\eta g = -2 f g_{\text{unit}},$$

so that we still have

$$\nabla\nabla f = -f g_{\text{unit}} \tag{3.7}$$

on (M, g). On the other hand, $\big(S^k \times X, g\big)$ has Ricci tensor

$$r = (k-1) g_{\text{unit}} \oplus k\widehat{h} = kg - g_{\text{unit}}$$

and hence has scalar curvature

$$s = k(k-1) + k\ell = k(n-1).$$

As a consequence, $M = S^k \times X$ has

$$\varrho := r - \frac{s}{n-1} g = (kg - g_{\text{unit}}) - kg = -g_{\text{unit}}.$$

Equation (3.7) can therefore be written as

$$\nabla\nabla f = \varrho f,$$

so that f satisfies (3.5), and is therefore a static potential on (M, g).

Finally, what are the Einstein $(n+1)$-manifolds that arise from (3.3) in the context of these specific examples? They are in fact just the the Riemannian products

$$\left(N, \widehat{g}\right) = \left(S^{k+1}, g_{\rm unit}\right) \times \left(X, \widehat{h}\right).$$

For example, to get the metric h_2 on $S^2 \times S^2$ featured in the technical question of the introduction, one first divides a product Einstein metric on $S^3 \times S^2$ by a rotation of the S^3 factor around an equatorial S^1, and then doubles the result. In any such example, the upstairs metric $\widehat{g}$ is Yamabe by Obata's theorem, and this then implies an interesting weighted Sobolev–Kondrakov inequality downstairs. Unfortunately, however, this inequality does not by itself appear sufficient to imply that the downstairs metric g is itself Yamabe.

Acknowledgements

It is a pleasure to dedicate this article to Jean-Pierre Bourguignon, on the occasion of his seventy-fifth birthday. This research was supported in part by NSF grant DMS–2203572.

References

[1] Albanese M., LeBrun C., Kodaira dimension and the Yamabe problem, II, arXiv:2106.14333.

[2] Ambrozio L., On static three-manifolds with positive scalar curvature, *J. Differential Geom.* **107** (2017), 1–45, arXiv:1503.03803.

[3] Aubin T., Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.* **55** (1976), 269–296.

[4] Besse A.L., Einstein manifolds, *Ergeb. Math. Grenzgeb. (3)*, Vol. 10, Springer, Berlin, 1987.

[5] Böhm C., Wang M., Ziller W., A variational approach for compact homogeneous Einstein manifolds, *Geom. Funct. Anal.* **14** (2004), 681–733.

[6] Bourguignon J.-P., Une stratification de l'espace des structures riemanniennes, *Compositio Math.* **30** (1975), 1–41.

[7] Chen X., LeBrun C., Weber B., On conformally Kähler, Einstein manifolds, *J. Amer. Math. Soc.* **21** (2008), 1137–1168, arXiv:0705.0710.

[8] Corvino J., Scalar curvature deformation and a gluing construction for the Einstein constraint equations, *Comm. Math. Phys.* **214** (2000), 137–189.

[9] Fischer A.E., Marsden J.E., Manifolds of Riemannian metrics with prescribed scalar curvature, *Bull. Amer. Math. Soc.* **80** (1974), 479–484.

[10] Gursky M.J., The Weyl functional, de Rham cohomology, and Kähler–Einstein metrics, *Ann. of Math.* **148** (1998), 315–337.

[11] Gursky M.J., LeBrun C., Yamabe invariants and Spin^c structures, *Geom. Funct. Anal.* **8** (1998), 965–977, arXiv:dg-ga/9708002.

[12] Kobayashi O., Scalar curvature of a metric with unit volume, *Math. Ann.* **279** (1987), 253–265.

[13] Lafontaine J., Sur la géométrie d'une généralisation de l'équation différentielle d'Obata, *J. Math. Pures Appl.* **62** (1983), 63–72.

[14] LeBrun C., Four-manifolds without Einstein metrics, *Math. Res. Lett.* **3** (1996), 133–147, arXiv:dg-ga/9511015.

[15] LeBrun C., Yamabe constants and the perturbed Seiberg–Witten equations, *Comm. Anal. Geom.* **5** (1997), 535–553, arXiv:dg-ga/9605009.

[16] LeBrun C., Kodaira dimension and the Yamabe problem, *Comm. Anal. Geom.* **7** (1999), 133–156, arXiv:dg-ga/9702012.

[17] LeBrun C., On Einstein, Hermitian 4-manifolds, *J. Differential Geom.* **90** (2012), 277–302, arXiv:1010.0238.

[18] Lichnerowicz A., Géométrie des groupes de transformations, *Travaux Rech. Math.*, Vol. 3, Dunod, Paris, 1958.

[19] Obata M., The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geometry* **6** (1971), 247–258.

[20] Odaka Y., Spotti C., Sun S., Compact moduli spaces of del Pezzo surfaces and Kähler–Einstein metrics, *J. Differential Geom.* **102** (2016), 127–172, arXiv:1210.0858.

[21] Page D., A compact rotating gravitational instanton, *Phys. Lett. B* **79** (1978), 235–238.

[22] Schoen R., Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Differential Geom.* **20** (1984), 479–495.

[23] Schoen R.M., Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, in Topics in Calculus of Variations (Montecatini Terme, 1987), *Lecture Notes in Math.*, Vol. 1365, Springer, Berlin, 1989, 120–154.

[24] Seshadri H., On Einstein four-manifolds with S^1-actions, *Math. Z.* **247** (2004), 487–503.

[25] Tian G., Yau S.-T., Kähler–Einstein metrics on complex surfaces with $C_1 > 0$, *Comm. Math. Phys.* **112** (1987), 175–203.

[26] Trudinger N.S., Remarks concerning the conformal deformation of riemannian structures on compact manifolds, *Ann. Scuola Norm. Sup. Pisa* **22** (1968), 265–274.

[27] Yamabe H., On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* **12** (1960), 21–37.