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On the Long Time Dynamics of the Landau-De Gennes Gradient Flow

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Abstract

We study the gradient flow, generated by the Landau-De Gennes energy functional, in the physically relevant spatial dimensions $d = 2, 3$. We establish global well-posedness and global exponential time decay bounds for large H^1 data in the 2D case, and uniform bounds for large data in 3D. This is indeed the best possible outcome for unrestricted coefficients in 3D, given that steady states do exist, at least for some coefficient configurations. We also establish leading order terms and in particular sharp asymptotics for the said dynamics in 2D. In 3D, we similarly isolate the leading order term, under the necessary assumption that a given, possibly large, solution converges to zero as $t \to \infty$. As a corollary, we prove an asymptotic formula for the correlation functional, $c(y, t) = \frac{\int_{\mathbf{R}^d} \mathrm{tr}(\mathbb{Q}(x+y,t)\mathbb{Q}(x,t))dx}{\int_{\mathbf{R}^d} \mathrm{tr}(\mathbb{Q}^2(x,t))dx} = e^{-\frac{|y|^2}{8t}} + O_{L^\infty_y}(t^{-\frac{1}{2}})$, $d = 2, 3$ for (potentially large) solutions $\mathbb{Q}(t)$ obeying a natural asymptotic condition. Such a formula was established in Kirr (J Stat Phys 155:625–657 (2014) for $d = 3$ and small initial data $\mathbb{Q}_0$, subject to the non-degeneracy condition $\int_{\mathbf{R}^3} \mathbb{Q}_0(x)dx \neq 0$.

Keywords Gradient flow · Landau-De Gennes nematic liquid crystals · Asymptotics

Mathematics Subject Classification 82D30 · 35Q74 · 76A15

1 Introduction

The dynamic behavior of nematic liquid crystals is an ubiquitous and well-studied problem. We invite the reader to consult the early and perhaps more accessible introduction to the

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modeling aspects in the book [4], while more recent accounts can be found in [9, 10, 12]. The main object of study is a second order moment of the probability density function of a nematic liquid crystal embedded in $\mathbf{R}^d$. This is a symmetric zero-trace $n \times n$ matrix $\mathbb{Q}$, which describes the orientation of liquid crystal molecules. In the work of Berris-Edwards, [2], a more sophisticated model is proposed, which includes the interaction of a of a non Newtonian fluid with the liquid crystal. Mathematically, this is a forced Navier-Stokes equation, coupled with a parabolic system, posed on $\mathbf{R}^d$—we direct the reader to the papers, [13, 14], as well as [5, 6, 8, 12], where the well-posedness issues and long term behavior of this and related models, has been studied in detail. As the standard Navier-Stokes equation is embedded in these models (when $\mathbb{Q} = 0$), it goes without saying that the question for the global well-posedness and the global dynamic of the system is a complicated one, especially in the case of spatial dimensions $\mathbf{R}^d$, $d \geq 3$. As our interest is in the global dynamics of nematic models of this type, we consider the pure Landau-De Gennes model, without the fluid interaction. Even in this simplified framework, there are numerous related models that capture its main properties, but we choose to concentrate on the Landau-De Gennes form of the bulk energy.

In order to properly introduce the objects of interest, consider the space of symmetric, traceless $d \times d$ real-valued matrices

$$S_0(d) = \left\{ \mathbb{Q} \in \mathcal{M}_{d\times d} : \mathbb{Q}_{ij} = \mathbb{Q}_{ji}, \mathrm{tr}(\mathbb{Q}) = \sum_{i=1}^{d} \mathbb{Q}_{ii} = 0 \right\}.$$

Additionally, we look at real-valued functions $\mathbb{Q} : \mathbf{R}^d \to S_0(d)$, and we introduce the relevant norms

$$\|\mathbb{Q}\|_{L^p} = \left(\sum_{i,j=1}^{d} \int_{\mathbf{R}^d} |\mathbb{Q}_{ij}(x)|^p dx \right)^{\frac{1}{p}} ; 1 \leq p < \infty \quad \|\nabla\mathbb{Q}\|_{L^2} = \left(\sum_{i,j,k=1}^{d} \int_{\mathbf{R}^d} |\partial_k \mathbb{Q}_{ij}(x)|^2 dx \right)^{\frac{1}{2}}.$$

More precisely, our main object of study will be the following energy functional

$$\begin{aligned} \mathcal{E}_{LdG}[\mathbb{Q}] &= \int_{\mathbf{R}^d} E(\mathbb{Q}(x)) dx; \\ E(\mathbb{Q}) &= \frac{1}{2}|\nabla\mathbb{Q}|^2 + \frac{a}{2}\mathrm{tr}(\mathbb{Q}^2) - \frac{b}{3}\mathrm{tr}(\mathbb{Q}^3) + \frac{c}{4}(\mathrm{tr}(\mathbb{Q}^2))^2 \end{aligned}$$

A standard, but technical calculation, [8, 13, 14] shows that the corresponding gradient flow, supplemented by initial condition, is in the form

$$\begin{cases} \mathbb{Q}_t = -\frac{\partial \mathcal{E}}{\partial \mathbb{Q}} = \Delta\mathbb{Q} - a\mathbb{Q} + b\left(\mathbb{Q}^2 - \frac{1}{d}\mathrm{tr}(\mathbb{Q}^2) Id\right) - c\,\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q} \\ \mathbb{Q}(0, x) = \mathbb{Q}_0(x) \end{cases} \tag{1}$$

In co-ordinate form

$$\partial_t \mathbb{Q}_{ij} = \Delta\mathbb{Q}_{ij} - a\mathbb{Q}_{ij} + b\left(\sum_{k=1}^{d} \mathbb{Q}_{ik}\mathbb{Q}_{kj} - \frac{1}{d}\mathrm{tr}(\mathbb{Q}^2) Id \right) - c\,\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q}_{ij}, i, j \in [1, \dots, d]. \tag{2}$$

From the physics of the problem and other modeling considerations, the parameters a, c are necessarily positive, but the problem makes sense even without this requirement, so we will keep them as general as possible for the time being.

Our main objective in this paper is to study the local and indeed global well-posedness properties of the dynamics of (1). The classical approach is to consider the corresponding Cauchy problem and to establish Hadamard well-posedness in some function space, say X.

That mostly works for local solutions, as it was shown in many recent works [3], but see also our Proposition 1 below for a sample statement of such result. A more ambitious goal is to establish that these local solutions do not blow up in finite time—such results are also available in the literature [8, 13, 14], although many of them do require either small data or else, some restrictions in the parameter space (a, b, c). Yet, there are further results that seek to establish global bounds on the solution $\mathbb{Q}(t)$ for all times $0 < t < \infty$—a sample and essentially sharp result of this kind appears in Proposition 3 below, for the case $d = 2$, and in Proposition 4 for the case $d = 3$. Note that these results apply for any H^1 large data,[1] whenever $c > 0$, while the case $c < 0$ likely leads to finite time blow up, at least for some sets of initial data [11, 12].

Another major contribution of our work is an asymptotic and explicit leading order term formula for $\mathbb{Q}(t)$ for large times t. Namely, we show that for any reasonable large data in $d = 2$, the free solution represents the main term contribution, while the non-linearity contributes to lower order and in a very controlled way, depending on the $L^p, 2 \le p \le \infty$ norms. For the case $d = 3$, under the assumption that $\lim_{t\to\infty} \|\mathbb{Q}(t)\|_{L^\infty} = 0$, we show that for all reasonable (and possibly large) data, the main contribution is the corresponding free solution.

This allows us to address the question for the conjectured self-similar behavior of the correlation functional, defined as follows

$$c(y,t) = \frac{\int_{\mathbf{R}^d} \mathrm{tr}(\mathbb{Q}(x+y,t)\mathbb{Q}(x,t))dx}{\int_{\mathbf{R}^d} \mathrm{tr}(\mathbb{Q}^2(x,t))dx}.$$

This question was partially addressed in the case $d = 3$, [8] for small data. More precisely, the authors have shown the following asymptotics

$$c(y,t) = e^{-\frac{|y|^2}{8t}} + r(t,y), \ \|r(t,\cdot)\|_{L^\infty(\mathbf{R}^3)} \le Ct^{-1/2}, \tag{3}$$

which applies to the solution emanating from a generic set[2] of small data $\mathbb{Q}_0 : \int_{\mathbf{R}^3} \mathbb{Q}_0(x)dx \ne 0$ in some weighted Sobolev space.

In this paper, by using our asymptotics we extend (3) in several ways: in the case $d = 2$ and for all (possibly large) data $\mathbb{Q}_0 \in H^1(\mathbf{R}^2)$, which has the asymptotic (5), we establish (3). For the case $d = 3$, and under the extra assumption that $\mathbb{Q}(t)$ has the asymptotic (9), we establish (3) for all localized (possibly large) data. It is worth pointing out that our result for $d = 3$ implies the statement of [8]. In fact, we show that *all* sufficiently small initial data $\mathbb{Q}_0$, with the property $\int_{\mathbf{R}^3} \mathbb{Q}_0(x)dx \ne 0$ generates solutions $\mathbb{Q}(t)$ which necessarily have the asymptotic (9) (Theorem 4), and hence by Theorem 5 has the property (3).

We now state the main contributions of this work. We split our results to the 2D case and the 3D case.

1.1 The 2D Case

Theorem 1 *(*Global bounds for the 2D problem)

Let $d = 2$,$c > 0$. Then, (1) is globally well-posed in $H^1(\mathbf{R}^2)$. More concretely, for each $\mathbb{Q}_0 \in H^1(\mathbf{R}^2, S_0(2))$, there is an unique global solution,

$$\mathbb{Q}(t) \in L^\infty_t([0,+\infty); H^1(\mathbf{R}^2, S_0(2)) \cap L^\infty_t[(\delta,\infty), H^\infty(\mathbf{R}^2, S_0(2))$$

[1] With some modest point-wise assumptions in the case $d = 3$.

[2] Defined in appropriate sense.

for every $\delta > 0$. Moreover, $\mathbb{Q}$ obeys the bounds,

$$|\mathbb{Q}(t,x)| \leq e^{-at}\sqrt{[e^{t\Delta}\mathrm{tr}(\mathbb{Q}_0^2)](x)}, \quad \|\nabla\mathbb{Q}(t)\|_{L^2} \leq C(\|\mathbb{Q}_0\|_{H^1})e^{-at}.$$

In particular,[3]

$$\|\mathbb{Q}(t)\|_{L^\infty} \leq C(\|\mathbb{Q}_0\|_{H^1})\frac{e^{-at}}{\sqrt{1+t}}, \ \|\mathbb{Q}(t)\|_{L^2} \leq C(\|\mathbb{Q}_0\|_{H^1})e^{-at}. \tag{4}$$

One can actually derive asymptotic expansion, provided $\mathbb{Q}_0$ is assumed to be slightly more localized. To this end, we introduce the weighted L^2 spaces $L^2_m(\mathbf{R}^d)$, $m \geq 2$, as follows

$$L^2_m(\mathbf{R}^d) = \left\{ f : \mathbf{R}^d \to \mathbf{C}^1 : \|f\|_m = \left(\int_{\mathbf{R}^d} (1+|\eta|^2)^m |f(\eta)|^2 d\eta\right)^{\frac{1}{2}} \right\}.$$

Theorem 2 *(*Sharp time asymptotics for the 2D case)
Let $d = 2$,$c > 0$. Let also $\mathbb{Q}_0 \in L^2_2(\mathbf{R}^2) \cap H^1(\mathbf{R}^2)$. Then, the global solution guaranteed by Theorem 1 is in the following form

$$\mathbb{Q}(t) = e^{-at}t^{-1}e^{-\frac{|x|^2}{4t}}\mu_0 + O_{L^2}(e^{-at}(1+t)^{-1}), \ \mu_0 \in S_0(2). \tag{5}$$

In particular, the solution $\mathbb{Q}(t)$ obeys the following dichotomy:

(1) If $\mu_0 \neq 0$, then

$$\|\mathbb{Q}(t)\|_{L^2} \sim \frac{e^{-at}}{\sqrt{t}}, t >> 1 \tag{6}$$

(2) If $\mu_0 = 0$, then

$$\|\mathbb{Q}(t)\|_{L^2} \leq C\frac{e^{-at}}{t}, t >> 1 \tag{7}$$

Finally, assuming either one of the following

- At least one of the components of $\mathbb{Q}_0$ is either non-negative or non-positive,
- $\mathbb{Q}_0 : \int_{\mathbf{R}^2} \mathbb{Q}_0(x)dx \neq 0$ is sufficiently small

then, one has that $\mu_0 \neq$ and hence the alternative $\|\mathbb{Q}(t)\|_{L^2} \sim \frac{e^{-at}}{\sqrt{t}}, t >> 1$ holds true.

Remarks • It is not clear to us whether or not the alternative (6) does not reduce to just the statement $\|\mathbb{Q}(t)\|_{L^2} \sim \frac{e^{-at}}{\sqrt{t}}$. In fact, we conjecture this to be the case either for all data $\mathbb{Q}_0$ or at least for a generic subset of the phase space.

The next results are about the 3D model.

1.2 The 3 D Case

We start with a result that applies to any (possibly large) reasonably smooth and localized data. Here, as in the 2D case, we expect exponential growth for $a < 0$. As our interest is in bounded global solutions (and even decaying to zero ones, see Theorems 4 and 5 below), we restrict our attention to the case $a > 0$.

[3] Here the assumption is only that $\mathbb{Q}_0 \in H^1(\mathbf{R}^2)$, and not necessarily that $\mathbb{Q}_0 \in L^1(\mathbf{R}^2)$, in which case there are better bounds available.

Theorem 3 *(*Global behavior for the solutions in the 3D problem)

Let $a > 0, c > 0, b \in \mathbf{R}$. Let $\mathbb{Q}_0 \in H^1(\mathbf{R}^3)$, and $|\mathbb{Q}_0(x)| \leq C(1+|x|)^{-\frac{1}{2}}$, i.e. $\mathbb{Q}_0 \in L^\infty_{1/2}$. Then, the problem (1) has an unique global solution $\mathbb{Q}(t)$, which satisfies

$$\|\mathbb{Q}(t)\|_{L^2(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)} + \|\nabla\mathbb{Q}(t)\|_{L^2(\mathbf{R}^3)} \leq C(\|\mathbb{Q}_0\|_{H^1\cap L^\infty_{1/2}}). \tag{8}$$

Remark The uniform estimates (8) may be the best optimal results for generic data $\mathbb{Q}_0$. That is, one does not expect all data to produce solutions which decay to zero (in which case the asymptotics (9) would apply). In fact, and unlike the 2D case, it is expected that for some values of $a > 0, b, c > 0$, there are static solutions of (1), which clearly do not tend to zero as $t \to \infty$. In fact, we conjecture that for any fixed $a > 0, c > 0$, there exists $b^* = b^*(a, c)$ so that for all $b > b^*$, one has a stationary solution of (1). Such a region in parameter space was suggested in [8], see the region $\mathcal{D} = \{(a, b, c) \in \mathbf{R}^3_+ : b^2 > 24ac\}$ on page 630 [8]. Note that for the case $a > 0, c < 0$, static solutions for (1) were recently constructed in [1], by relying on a mountain pass approach.

In the case, when the solution $\mathbb{Q}(t)$ actually vanishes at infinity, we have the following asymptotic results.

Theorem 4 *(*Asymptotics for vanishing solutions in the 3D case)

In addition to the assumptions of Theorem 3, assume that $\mathbb{Q}_0 \in L^2_2(\mathbf{R}^3)$ and the solution decays to zero, i.e. $\lim_{t\to\infty} \|\mathbb{Q}(t)\|_{L^\infty} = 0$, then it decays to zero exponentially fast. In fact, there is the asymptotic expansion formula

$$\mathbb{Q}(t) = e^{-at}t^{-\frac{3}{2}}e^{-\frac{|x|^2}{4t}}\mu_0 + O_{L^2}(e^{-at}t^{-\frac{5}{4}}), t >> 1 \tag{9}$$

where $\mu_0 \in S_0(3)$. In particular, there is the following alternative

(1) If $\mu_0 \neq 0$, then

$$\|\mathbb{Q}(t)\|_{L^2} \sim e^{-at}t^{-\frac{3}{4}}, t >> 1 \tag{10}$$

(2) If $\mu_0 = 0$, then

$$\|\mathbb{Q}(t)\|_{L^2} \leq Ce^{-at}t^{-\frac{5}{4}}t >> 1 \tag{11}$$

In the following two cases,

- At least one of the components of $\mathbb{Q}_0$ is either non-negative or non-positive,
- $\mathbb{Q}_0 : \int_{\mathbf{R}^3} \mathbb{Q}_0(x)dx \neq 0$ is sufficiently small

the asymptotic expression (9) holds with $\mu_0 \neq 0$, so in particular (10) is valid.

Finally, we provide asymptotics for the correlation functional

Theorem 5 *(*Asymptotics for the correlation functional)

Let either one of the following holds

- $d = 2$, and $\mathbb{Q}(t)$ is a solution, which has the asymptotic (5),
- $d = 3$, and similarly $\mathbb{Q}(t)$ obeys (9).

Then,

$$c(y, t) = e^{-\frac{|y|^2}{8t}} + O_{L^\infty}(t^{-\frac{1}{2}}). \tag{12}$$

That is, there is a constant C, so that for each $t > 1$,

$$\|c(\cdot, t) - e^{-\frac{|\cdot|^2}{8t}}\|_{L^\infty_y(\mathbf{R}^d)} \leq C(1 + t)^{-\frac{1}{2}}, d = 2, 3.$$

Note that this covers the result from [8] as such asymptotic formula for c is established for small 3D solutions with the property $\int_{\mathbf{R}^3} \mathbb{Q}_0(x)dx \neq 0$. However, according to Theorem 4, such solutions have the asymptotic (9) with $\mu_0 \neq 0$, and hence our result (12) applies to it.

We plan our presentation as follows. In Sect. 2 below, we state some standard facts about the Cauchy problem (1). Next, we develop some global a priori estimates for the solutions $\mathbb{Q}(t)$, in the cases $d = 2, 3$, which provide unique and global solutions. The main contribution here, proven under the assumption $c > 0$, is the sharp exponential rate of decay for the 2D model, as well as the uniform bound for the solutions in the 3D case. We also provide some weighted estimates, which are useful in the sequel. In Sect. 3, we start with a short introduction of the scaled variables approach and appropriate spectral information as well as estimates in function spaces. These are the main technical tools for establishing the sharp asymptotics. Sharp asymptotics in the 2D case are presented in Sect. 3.2, while the details of the proof for the asymptotics for the 3D case are given in Sect. 3.3. In Sect. 4, we give a sharp asymptotic formula for the correlation functional, based on the asymptotics developed in Sect. 3.

2 Local and Global Well-Posedness

We start with some preliminaries. We adopt the following definition of the Fourier transformation

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}dx, \quad f(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{f}(\xi)e^{i\langle x,\xi\rangle}d\xi.$$

As $\widehat{\partial_j f}(\xi) = i\xi_j \hat{f}(\xi)$, we deduce that $\widehat{\Delta f}(\xi) = -|\xi|^2 \hat{f}(\xi)$.

For future reference, note that the solution of the in-homogeneous heat equation

$$u_t - \Delta u = F, u(0, x) = u_0(x), \tag{13}$$

is given by the formula

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}F(s, \cdot)ds.$$

Introducing the heat kernel function $G : \widehat{G}(\xi) = e^{-|\xi|^2}$ or equivalently $G(x) = (4\pi)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4}}$, we may represent the solutions to the heat equation as a convolutions operator with an appropriate re-scale of the heat kernel. To this end, let $G_t(x) := t^{-\frac{d}{2}}G\left(\frac{x}{\sqrt{t}}\right) = (4\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$. For sufficiently decaying functions f, say $f \in L^2(\mathbf{R}^d)$, the solution to (13) is given by

$$e^{t\Delta}f(x) = G_t * f = (4\pi t)^{-\frac{d}{2}} \int_{\mathbf{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Regarding the calculus of matrices, note that

$$\|\mathbb{Q}\|_{L^2}^2 = \sum_{i,j=1}^d \int_{\mathbf{R}^d} \mathbb{Q}_{ij}^2(x)dx = \int_{\mathbf{R}^d} tr(\mathbb{Q}^2)dx$$

The non-negative scalar quantity $y(t, x) := tr(\mathbb{Q}^2(t, x))$ becomes very useful in controlling the evolution of $\mathbb{Q}$. In fact, multiplying the equation (1) by $\mathbb{Q}_{ij}$ and summing in $i, j \in$

$[1, \ldots, d]$ leads to the following PDE for y,

$$y_t = \Delta y - 2|\nabla \mathbb{Q}|^2 - 2ay + 2b\mathrm{tr}(\mathbb{Q}^3) - 2cy^2, \tag{14}$$

which will be helpful in the sequel.

Note that we can recast the evolution problem (1) as

$$\mathbb{Q}(t, \cdot) = e^{t(\Delta - a)}\mathbb{Q}_0 + \int_0^t e^{(t-s)(\Delta-a)} \left(b\left(\mathbb{Q}^2(s) - \frac{1}{d}\mathrm{tr}(\mathbb{Q}^2(s))Id\right) - c\mathrm{tr}(\mathbb{Q}^2(s))\mathbb{Q}(s)\right) ds \tag{15}$$

We now proceed to establish the local existence and uniqueness of the Cauchy problem (1). This result is more or less standard, but we include its precise statement for future reference.

Proposition 1 *Let $d = 2, 3$ and $\mathbb{Q}_0 \in H^1(\mathbf{R}^d, S_0(d))$. Then, there exists a time $T^* = T^*(\|\mathbb{Q}_0\|_{H^1})$, $0 < T^* \leq \infty$, so that there exists unique solution $\mathbb{Q}$ of the integral equation (1) in a ball of $L^\infty((0, T^*), H^1(\mathbf{R}^d, S_0(d)))$. Moreover, for every $t_0 > 0$, we have that the solution is infinitely smooth, i.e. $\mathbb{Q} \in L^\infty((t_0, T^*), H^\infty(\mathbf{R}^d, S_0(d)))$.*

Finally, there is the blowup alternative. More specifically,

$$\text{Either } T^* = \infty \text{ or } \limsup_{t \to T^*-} \|\mathbb{Q}(t, \cdot)\|_{H^1(\mathbf{R}^d)} = \infty. \tag{16}$$

Proof The proof is standard and proceeds via a fixed point argument for the map

$$\Lambda(\mathbb{Q}) := e^{t(\Delta-a)}\mathbb{Q}_0 + \int_0^t e^{(t-s)(\Delta-a)} \left(b\left(\mathbb{Q}^2(s) - \frac{1}{d}\mathrm{tr}(\mathbb{Q}^2(s))Id\right) - c\mathrm{tr}(\mathbb{Q}^2(s))\mathbb{Q}(s)\right) ds$$

in the ball $\mathbb{B} = \{\mathbb{Q} \in L^\infty((0, T^*), H^1(\mathbf{R}^d, S_0(d))) : \|\mathbb{Q}\|_{H^1} \leq 2\|\mathbb{Q}_0\|_{H^1}\}$ for some small time $T^* = T^*(\|\mathbb{Q}_0\|)$ to be determined. This argument follows easily from the convolution estimates

$$\begin{aligned}
&\|e^{t(\Delta-a)}\mathbb{Q}_0\|_{H^1} \leq e^{t|a|}\|G_t\|_{L^1}\|\mathbb{Q}_0\|_{H^1} = e^{t|a|}\|\mathbb{Q}_0\|_{H^1}, \\
&\|\int_0^t e^{(t-s)(\Delta-a)} \left(b\left(\mathbb{Q}^2(s) - \frac{1}{d}\mathrm{tr}(\mathbb{Q}^2(s))Id\right) - c\mathrm{tr}(\mathbb{Q}^2(s))\mathbb{Q}(s)\right) ds\|_{H^1} \\
&\leq C\int_0^t e^{-a(t-s)}\|K_{t-s}\|_{L^2}\left(\|\mathbb{Q}^2\|_{L^1} + \|\nabla(\mathbb{Q}^2)\|_{L^1} + \|\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q}\|_{L^1} + \|\nabla[\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q}]\|_{L^1}\right) ds \\
&\leq C\|\mathbb{Q}_0\|_{H^1}^2(1 + \|\mathbb{Q}_0\|_{H^1})\int_0^t \frac{e^{-a(t-s)}}{(t-s)^{\frac{d}{4}}} ds.
\end{aligned}$$

where we have used the Sobolev embedding $H^1(\mathbf{R}^d) \hookrightarrow L^4(\mathbf{R}^d)$, $d = 2, 3$, and $\|\mathbb{Q}\|_{H^1} \leq 2\|\mathbb{Q}_0\|_{H^1}$. Choosing small enough[4] T^* will ensure that $\Lambda : \mathbb{B} \to \mathbb{B}$ Similarly, the contractivity of Λ on $\mathbb{B}$ follows as well. One establishes the higher regularity on $[t_0, \infty)$ in a standard manner, by induction on the order of the derivative, as one is always able to place one derivative on the heat kernel K_{t-s}. Finally, the blow up alternative follows via the usual contradiction argument. Indeed, assume that the some high norm of the solution blows up at T^*, while $M := \limsup_{t \to t^*} \|\mathbb{Q}(t, \cdot)\|_{H^1} < \infty$. Then, applying the local well-posedness theory to an arbitrary $\bar{t} < T^*$, we have that for each $\bar{t} < T^*$, the solution may be extended for at least $C_a(1+M)^{-\frac{1}{4}}$ units of time. Moreover, in this time interval, there are the higher regularity estimates for $\|\mathbb{Q}\|$. But this and the uniqueness then contradicts the blow up assumption

[4] In fact, from the estimates above, one can take $T^* \geq C_a(1 + \|\mathbb{Q}_0\|^{-\frac{1}{4}})$.

at T^*, as we can easily overshoot it as the time interval is now at least $\bar{t} + C_a(1+M)^{-\frac{1}{4}}$ for arbitrary $\bar{t} < T^*$. □

By the local existence and uniqueness the semi-linear parabolic problem it follows that the solutions to (1) conserve $\mathbb{Q}(t) \in S_d(0)$, if $\mathbb{Q}_0 \in S_d(0)$.

Next, we state an useful preliminary result, which is classical in the literature, see for example Proposition 1 [13].

Lemma 1 *Let $d = 2, 3$, $\mathbb{Q}_0 \in H^1$ and T^* is the maximal time of existence for the solutions of (1), guaranteed by Proposition 1. Then, for all $0 < t < T^*$, there is*

$$\partial_t \mathcal{E}(t) = -\int_{\mathbf{R}^d} \left(\Delta\mathbb{Q} - a\mathbb{Q} + b[\mathbb{Q}^2 - \frac{1}{d} tr(\mathbb{Q}^2)] - ctr(\mathbb{Q}^2)\mathbb{Q} \right)^2 dx, \tag{17}$$

where $\mathcal{E}(t) := \mathcal{E}(\mathbb{Q}(t))$.

Proof The derivation of (17) is immediate from (1), as

$$\partial_t \mathcal{E}(t) = \frac{\partial \mathcal{E}}{\partial \mathbb{Q}}[\mathbb{Q}_t] = -\int_{\mathbf{R}^d} \mathbb{Q}_t^2 dx = -\int_{\mathbf{R}^d} \left(\Delta\mathbb{Q} - a\mathbb{Q} + b[\mathbb{Q}^2 - \frac{1}{d}\mathrm{tr}(\mathbb{Q}^2)] - c\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q} \right)^2 dx$$

The regime $c < 0$ is an interesting from a mathematical point of view, as Proposition 1 guarantees that at least short time solutions do exist. Based on modeling considerations however, see [11, 12], as well as the form of the energy functional[5] $\mathcal{E}$, it looks likely that a finite time blow up is a possibility, at least for some initial data. We henceforth assume the coercivity condition $c > 0$.

2.1 Positivity of Solutions for Quasi-Linear Heat Equations with Positive Initial Data

We state a result which might be considered classical and yet, we do not find its specific formulation below in the literature. In fact, we first state a general result about linear equations, and then, we give applications to the semi-linear case.

Proposition 2 *Suppose $d \geq 1$, $V : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}$ is a bounded from below and continuous function. Assume that*

$$u \in C^2(\mathbf{R}^d) \cap L^2(\mathbf{R}^d) : \|u(t, \cdot)\|_{L^2(\mathbf{R}^d)} \leq Ce^{\lambda_0 t}, \quad \text{for some } \lambda_0 > 0 \tag{18}$$

is a classical solution of

$$u_t - \Delta u + V(t, x)u = 0, u(x, 0) = f(x).$$

Then, $u \geq 0$, whenever $f \geq 0$.

Proof The result may be obtained via the Feynman-Kac formula, but we prefer a more elementary proof, kindly provided to us by Phan [15]. First, we can assume without loss of generality that $V(x, t) > 0$. If not, find some $\lambda > 0$, $V(t, x) + \lambda > 0$. Then, we replace $u = e^{\lambda t} w$, so that we obtain the following equation for w

$$w_t - \Delta w + (V(t, x) + \lambda)w = 0.$$

Assume now for contradiction that the function w is negative somewhere. As

$$\lim_{|x|+|t| \to \infty} |w(x, t)| = \lim_{|x|+|t| \to \infty} e^{-\lambda t}|u(x, t)| = 0,$$

[5] Which makes it unbounded from below.

the minimum is achieved at $(t_0, x_0) \in \mathbf{R}_+ \times \mathbf{R}^d$, as for $t_0 = 0$, $u(0, x_0) = f(x_0) \geq 0$ by assumption. It follows that $w_t(x_0, t_0) = 0$, while $\Delta w(t_0, x_0) \geq 0$, whereas $(V(t_0, x_0) + \lambda)w(t_0, x_0) < 0$. This is however a contradiction with the PDE at (t_0, x_0). □

We now have a straightforward corollary.

Corollary 1 *Assume that F is a smooth function, and $u : \mathbf{R}_+ \times \mathbf{R}^d \to \mathbf{R}$ is a classical solution of*

$$u_t - \Delta u + F(u, \nabla u)u = 0, u(x, 0) = f(x).$$

which satisfies the growth condition (18) and $u, \nabla u \in L^\infty_{tx}(\mathbf{R}_+ \times \mathbf{R}^d)$. Then $u \geq 0$, whenever $f \geq 0$

For the proof, apply Proposition 2 to the case $V(t, x) = F(u(t, x), \nabla u(t, x))$, which is bounded from below since $u, \nabla u \in L^\infty_{tx}$.

2.2 Global a Priori Estimates in the Case $d = 2$

The case $d = 2$ is simplifies matters quite a bit, if one uses the formulation (14).

Proposition 3 *Let $d = 2$, $c > 0$. Then, the solution $\mathbb{Q}$ is global, for any initial data $\mathbb{Q}_0 \in H^1(\mathbf{R}^2)$. In addition, there are the following estimates*

$$|\mathbb{Q}(t, x)| \leq e^{-at}\sqrt{[e^{t\Delta} tr(\mathbb{Q}_0^2)](x)}, \tag{19}$$

$$\|\nabla \mathbb{Q}(t, \cdot)\|_{L^2} \leq C(\|\mathbb{Q}_0\|_{H^1})e^{-at} \tag{20}$$

In particular, the gradient flow (1) is globally well-posed on $H^1(\mathbf{R}^2)$.

Remarks (1) For the case $a > 0$, the estimates (19) and (20) imply exponential decay for all $\|\mathbb{Q}(t, \cdot)\|_{L^p}$, $2 \leq p \leq \infty$ and H^1 norms, namely

$$\|\mathbb{Q}(t, \cdot)\|_{L^p} \leq Ce^{-at}\|\mathbb{Q}_0\|_{L^p}, \|\mathbb{Q}(t, \cdot)\|_{H^1} \leq Ce^{-at}\|\mathbb{Q}_0\|_{H^1} \tag{21}$$

We will establish much more precise results later on, see the proof of Theorem 2 in Sect. 3.2.2 below and more precisely, the proof of the asymptotic formula (55) therein.

(2) In [14] (see also the earlier work [13]), the authors work with a more general system, namely Navier-Stokes-Landau-De Gennes, which describes the interaction of the fluid with the nematics. They gave a proof of a similar result (see Proposition 2), but with a general exponential, parameter dependent rate of growth of $\|\mathbb{Q}(t, \cdot)\|_{H^1}$.

(3) In [3], the authors consider the Landau-De Gennes, posed on $\mathbf{R}^d$, $d \geq 2$. They provide existence and uniqueness of small global solutions of as well as Hölder bounds.

Proof We consider the formulation (14). Note that in $d = 2$, we have that $\mathrm{tr}(\mathbb{Q}^3) = 0$ for a symmetric traceless matrix $\mathbb{Q}$. Thus, estimating away various positive terms, (14) reduces to

$$y_t - \Delta y + 2ay \leq 0. \tag{22}$$

Note that (22) is a *point-wise* inequality. This is equivalent to

$$\partial_t(ye^{2at}) - \Delta(ye^{2at}) \leq 0.$$

By the positivity of the heat kernel, this implies the *point-wise* inequality

$$0 \leq |\mathbb{Q}(t, x)|^2 = y(t, x) \leq e^{-2at}[e^{t\Delta} y_0](x).$$

Taking square roots imply (19). For the bound (20), we split our considerations in two cases - $a > 0$ and $a < 0$. For the former case, we take derivatives in the integral equation (15) and then L^2 norms. We have, using the estimate (21).

$$\begin{aligned}
&\|\nabla e^{t(\Delta - a)}\mathbb{Q}_0\|_{L^2} \leq Ce^{-at}\|\nabla \mathbb{Q}_0\|_{L^2} \\
&\|\nabla \int_0^t e^{(t-s)(\Delta-a)}\left(b\left(\mathbb{Q}^2(s) - \frac{1}{d}\mathrm{tr}(\mathbb{Q}^2(s))Id\right) - c\mathrm{tr}(\mathbb{Q}^2(s))\mathbb{Q}(s)\right)ds\| \\
&\leq C\int_0^t e^{-a(t-s)}\|\nabla K_{t-s}\|_{L^1}(\|\mathbb{Q}\|^2_{L^\infty} + \|\mathbb{Q}\|^3_{L^\infty})ds \leq C\int_0^t e^{-a(t-s)}\frac{e^{-2as}}{\sqrt{t-s}}ds \leq Ce^{-at}.
\end{aligned}$$

For the case $a < 0$, the same method yields an estimate Ce^{-3at}, which is not optimal, as we shall see next. So, we approach it in a different manner.

We use the dissipation law (17). From it, it follows that $\mathcal{E}(t)$ is non-increasing, so $\mathcal{E}(t) \leq \mathcal{E}(0)$. However, note that in our case $d = 2$, whence this means, as $\mathrm{tr}(\mathbb{Q}^3) = 0$,

$$\int_{\mathbf{R}^2}\frac{1}{2}|\nabla\mathbb{Q}(t,x)|^2 + \frac{a}{2}\mathrm{tr}(\mathbb{Q}^2(t,x)) + \frac{c}{4}(\mathrm{tr}(\mathbb{Q}^2(t,x)))^2dx \leq \mathcal{E}(0)$$

Since $c > 0, a < 0$, this implies, according to (21)and $\mathcal{E}(0) \leq \|\mathbb{Q}_0\|^2_{H^1}(1 + \|\mathbb{Q}_0\|_{H^1})$,

$$\int_{\mathbf{R}^2}|\nabla\mathbb{Q}(t,x)|^2 \leq -a\int_{\mathbf{R}^2}\mathrm{tr}(\mathbb{Q}^2(t,x))dx + \mathcal{E}(0) \leq Ce^{-2at}$$

The bound (20) follows by taking square root of the previous estimate. □

2.3 Global a Priori Estimates in the Case $d = 3$

In the three dimensional case, we the term $\mathrm{tr}(\mathbb{Q}^3)$ in the Landau-De Gennes energy does not trivialize, so we need to take it into account. We have the following Proposition.

Proposition 4 *Let $\mathbb{Q}_0 \in H^1(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$, and $a > 0$. Then, the local solution of (1) guaranteed by Proposition 1 satisfies the bound*

$$\|\mathbb{Q}(t,\cdot)\|_{L^\infty(\mathbf{R}^3)} \leq C(\|\mathbb{Q}_0\|_{L^\infty(\mathbf{R}^3)}). \tag{23}$$

If in addition, we assume the point-wise bound $|\mathbb{Q}_0(x)| \leq \frac{C}{\sqrt{1+|x|}}$, that is $\mathbb{Q}_0 \in L^\infty_{\frac{1}{2}}$, then for every $2 \leq p < \infty$,

$$\|\mathbb{Q}(t,\cdot)\|_{L^p(\mathbf{R}^3)} \leq C(\|\mathbb{Q}_0\|_{L^\infty(\mathbf{R}^3)\cap H^1\cap L^\infty_{\frac{1}{2}}}). \tag{24}$$

Finally, one can also bound

$$\|\nabla\mathbb{Q}(t,\cdot)\|_{L^2(\mathbf{R}^3)} \leq C(\|\mathbb{Q}_0\|_{L^\infty(\mathbf{R}^3)\cap H^1\cap L^\infty_{\frac{1}{2}}}). \tag{25}$$

Remarks
- One can obtain the same results (24) under a much weaker assumption on $\mathbb{Q}_0$, namely $|\mathbb{Q}_0(x)| \leq \frac{C}{(1+|x|)^\epsilon}$, for any $\epsilon > 0$.
- One can obtain, under the same assumptions, bounds on $\|\mathbb{Q}(t,\cdot)\|_{L^p(\mathbf{R}^3)}$, $1 \leq p < \infty$ as well.

Proof Recall (14), so that it suffices to establish $\|y(t,\cdot)\|_{L^1(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)}$ bounds. Since

$$|\mathrm{tr}(\mathbb{Q}^3)| = |\sum_{j=1}^3\lambda_j^3| \leq (\sum_{j=1}^3\lambda_j^2)^{\frac{3}{2}} = (\mathrm{tr}(\mathbb{Q}^2))^{\frac{3}{2}},$$

we can derive the following inequality from (14),

$$y_t - \Delta y + 2ay \le 2|b|y^{\frac{3}{2}} - 2cy^2, \tag{26}$$

In fact, taking into account the structure of the left hand side, we see that it is negative, if $y > \frac{4|b|}{c^2}$, whence we can further bound it by

$$y_t - \Delta y + 2ay \le 16\frac{|b|^{\frac{5}{2}}}{c^3}. \tag{27}$$

Thus, we have the estimate

$$y(t,x) \le e^{t\Delta}y_0(x) + \int_0^t e^{(t-s)\Delta}e^{-2a(t-s)}16\frac{|b|^{\frac{5}{2}}}{c^3}ds.$$

whence we derive the L^∞ bound (recall that $y > 0$ by construction)

$$\|y(t,\cdot)\|_{L^\infty} \le \|y_0\|_{L^\infty} + \frac{|b|^{\frac{5}{2}}}{2ac^3} \tag{28}$$

One can also derive $L^1, 2 \le p < \infty$ bounds on y, but under some extra point-wise assumptions on the initial data $\mathbb{Q}_0$.

To this end, introduce a non-negative cutoff function $\varphi \in C_0^\infty : supp\varphi \subset (1/2, 4), \varphi(x) = 1, 1 \le |x| \le 2$ and set $y_k(t,x) := y(t,x)\varphi(2^{-k}x), k \ge 1$. We now need to establish inequalities for y_k, based on (26). Ultimately, this leads to weighted inequalities for y, which then can be bootstrapped to $L^p, 1 \le p$ estimates for y.

To this end, multiply (26) by $\varphi(2^{-k}x)$ results in

$$\partial_t y_k - \Delta y_k + 2ay_k \le 2|b|y^{\frac{3}{2}}\varphi(2^{-k}x) - 2^{-k+1}\nabla\varphi(2^{-k}x)\cdot\nabla y + 2^{-2k}|\Delta\varphi(2^{-k}x)|y \tag{29}$$

after taking into account $-\varphi(2^{-k}x)\Delta y = -\Delta(y_k) + 2^{-k+1}\nabla\varphi(2^{-k}x)\cdot\nabla y + 2^{-2k}\Delta\varphi(2^{-k}x)y$. Thus,

$$\begin{aligned} y_k(t) \le e^{t\Delta}y_k(0) + \int_0^t e^{(t-s)\Delta}e^{-2a(t-s)}[2|b|y^{\frac{3}{2}}\varphi(2^{-k}x) \\ -2^{-k+1}\nabla\varphi(2^{-k}x)\cdot\nabla y + 2^{-2k}\Delta\varphi(2^{-k}x)y]ds. \end{aligned}$$

Note that integration by parts yields

$$\begin{aligned} \int_0^t e^{(t-s)\Delta}\nabla\varphi(2^{-k}x)\cdot\nabla y(s,\cdot)ds = -2^{-k}\int_0^t e^{(t-s)\Delta}\Delta\varphi(2^{-k}x)y(s,\cdot)ds \\ -\int_0^t \nabla e^{(t-s)\Delta}\varphi(2^{-k}\cdot)y(s,\cdot)ds \end{aligned}$$

We estimate, using the a priori bound (28) and $\|e^{(t-s)\Delta}\|_{B(L^\infty)} = 1$, $\|\nabla e^{(t-s)\Delta}\|_{B(L^\infty)} \lesssim \frac{1}{\sqrt{t-s}}$,

$$\begin{aligned} \|\int_0^t e^{(t-s)\Delta}e^{-2a(t-s)}[\Delta\varphi(2^{-2k}x)y(s,\cdot)]ds\|_{L^\infty} \lesssim \int_0^t e^{-2a(t-s)}ds \lesssim 1 \\ \|\int_0^t \nabla e^{(t-s)\Delta}e^{-2a(t-s)}[\varphi(2^{-k}\cdot)y(s,\cdot)]ds\|_{L^\infty} \lesssim \int_0^t \frac{e^{-2a(t-s)}}{\sqrt{t-s}}ds \lesssim 1 \end{aligned}$$

So,

$$\|y_k(t,\cdot)\|_{L^\infty} \lesssim \|y_k(0)\|_{L^\infty} + 2^{-k} + \int_0^t e^{-2a(t-s)}[\|y_k(s,\cdot)\|_{L^\infty}$$
$$+\|y_{k-1}(s,\cdot)\|_{L^\infty} + \|y_{k+1}(s,\cdot)\|_{L^\infty}]^{\frac{3}{2}} ds.$$

Clearly, if we are working with initial data $\|y_k(0)\|_{L^\infty} \lesssim 2^{-k}$ (and we do assume that!), we can conclude that for large k, we have that $\|y_k(t,\cdot)\|_{L^\infty} \lesssim 2^{-k}$ or

$$|y(t,x)| \le C(1+|x|)^{-1}. \tag{30}$$

This already is a good enough to conclude $\sup_{0<t<\infty} \|y(t,\cdot)\|_{L^p} < C$ for all $p > 3$. In addition, this can be bootstrapped to a bound on the L^1 norm (and then by interpolation to all $1 \le p < \infty$. Indeed, take an integral over $\mathbf{R}^3$ in (26), we obtain

$$\partial_t \int_{\mathbf{R}^3} y(t,x)dx + 2a \int_{\mathbf{R}^3} y(t,x)dx \le 2|b| \int y^{\frac{3}{2}}(t,x)dx. \tag{31}$$

Let $m(t) := \int_{\mathbf{R}^3} y(t,x)dx$. We estimate by the log-convexity of the L^p norms $\|y\|_{L^{\frac{3}{2}}} \le \|y\|_{L^1}^{\frac{3}{5}} \|y\|_{L^6}^{\frac{2}{5}}$. Noting that (30) yields a bound on $\|y\|_{L^6}$, (31) becomes

$$m'(t) + 2am(t) \le C_0 m^{\frac{9}{10}}(t) \tag{32}$$

This yields a bound, by Gronwal's and Young's inequalities, on $m(t)$, namely $m(t) \le 2m(0) + \frac{2^{10}}{a}$, whence the $\|y(t,\cdot)\|_{L^1}$ bound is established. Finally, for the bound (25), we take ∇ in the (15), and we estimate the L^2 norms by using the a priori estimates already established for $\|Q\|_{L^p}, 2 \le p < \infty$. □

3 Detailed Asymptotics of $\mathbb{Q}$

We start our discussion with the introduction of the scaled variables, which is necessary in the derivation of the sharp asymptotics later on. We also report on the relevant estimates for the scaled Laplacians.

3.1 Scaled Variables and Scaled Laplacians

In order to obtain a detailed asymptotics expansion of the parabolic problems in consideration herein, we refer to the scaled variables. In fact, we illustrate this powerful method on the linear heat equation

$$v_t - \Delta v = 0, v(0,x) = v_0(x), x \in \mathbf{R}^d. \tag{33}$$

Clearly, one has the expression $v = e^{t\Delta} v_0$, but it turns out that there is a more precise way to write an asymptotic expansion, depending on the properties of v_0. To this end, we introduce the scaled variable formulation of (44) as in Gallay-Wayne [7]. More specifically,

$$\tau := \ln(1+t); \quad \eta := \frac{x}{\sqrt{1+t}} \tag{34}$$

$$v(t,x) = \frac{1}{\sqrt{1+t}} V\left(\frac{x}{\sqrt{1+t}}, \ln(1+t)\right). \tag{35}$$

Note that the pre-factor $\frac{1}{\sqrt{1+t}}$ is an arbitrary choice for the *linear* problem, and it will subsequently be chosen differently, depending on nonlinearity (if present) and dimensions

$$\mathscr{L} = \Delta + \frac{1}{2}\eta \cdot \nabla_\eta + \frac{1}{2}.$$

These operators, along with their spectra and the semigroups generated in weighted L^2 spaces, have been considered in the literature.

3.1.1 Properties of $\mathscr{L}$ in the 2D Case

We collect all the pertinent information for the two dimensional case in the following Proposition.

Proposition 5 *(Properties of $\mathscr{L}$ in the case $d = 2$)*
Let $m \geq 2$ is an integer and $\mathscr{L}$ be defined as a closed operator in its maximal domain, in the Hilbert space $L^2_m(\mathbf{R}^2)$. Then,

- *its spectrum is given by $\sigma(\mathscr{L}) = \sigma_{ess}(\mathscr{L}) \cup \sigma_{p.p.}(\mathscr{L})$, where*

$$\sigma_{p.p.}(\mathscr{L}) = \left\{-\frac{k}{2}, k = 1, \ldots, m-1\right\}, \sigma_{ess}(\mathscr{L}) = \left\{\lambda : \Re\lambda \leq -\frac{m}{2}\right\}.$$

  Moreover, the isolated eigenvalues $-\frac{k}{2}$ correspond to explicit eigenfunctions, namely $\partial^\alpha G, \alpha \in \mathbb{N}^2$, given as follows

$$\mathscr{L}[\partial^\alpha G] = -\frac{1+|\alpha|}{2}\partial^\alpha G,$$

  The highest e-value Riesz spectral projection is rank one, and it is given by

$$P_{-\frac{1}{2}} f = \left(\int_{\mathbf{R}^2} f(\eta) d\eta\right) G,$$

- *The operator $\mathscr{L}$ generates a C_0 semigroup $e^{\tau\mathscr{L}}$ on L^2_2, given in an explicit form by*

$$e^{\tau\mathscr{L}} f(\eta) = \frac{e^{\frac{\tau}{2}}}{4\pi a(\tau)} \int_{\mathbf{R}^2} e^{-\frac{|\eta-\eta'|^2}{4a(\tau)}} f(e^{\frac{\tau}{2}}\eta') d\eta'.$$

  where $a(\tau) = 1 - e^{-\tau}$. Moreover, there are the following estimates of $e^{\tau\mathscr{L}}$ a

$$\|e^{\tau\mathscr{L}} f\|_{L^p(\mathbf{R}^2)} \leq C e^{\tau\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^p(\mathbf{R}^2)}, 1 \leq p \leq \infty \tag{36}$$

$$\|e^{\tau\mathscr{L}} f\|_{L^2_2} \leq C e^{-\frac{\tau}{2}} \|f\|_{L^2_2}, \tag{37}$$

$$\|e^{\tau\mathscr{L}} f\|_{L^2_2} \leq C e^{-\tau} \|f\|_{L^2_2}, \quad \text{if } \hat{f}(0) = 0. \tag{38}$$

This proposition is a compilation of results given in Appendix A, [7] (Proposition A.2), see also [16] (Proposition 3). The reader should keep in mind that the specific operator L appearing in [7] is of the form $L = \mathscr{L} + \frac{1}{2}$, so the estimates stated herein always have an extra factor of $e^{-\frac{\tau}{2}}$.

As a corollary, we display the following result for the solutions to the linear heat equation (33) in the two dimensional case $d = 2$.

Corollary 2 *Let $v_0 \in L^2_2(\mathbf{R}^2)$. Then, the solution to (33) can be written as follows*

$$v(t,x) = \left(\int_{\mathbf{R}^2} v_0(y)dy\right) \frac{1}{1+t} G\left(\frac{x}{\sqrt{1+t}}\right) + O_{L^2}\left((1+t)^{-1}\right). \tag{39}$$

More specifically, $\|v - \left(\int_{\mathbf{R}^2} v_0(y)dy\right) \frac{1}{1+t} G\left(\frac{\cdot}{\sqrt{1+t}}\right)\|_{L^2} \leq C(1+t)^{-1}$. In particular, there is the estimate

In fact, assuming that initial data belongs to higher order weighted spaces, one can write the asymptotic expansions

$$v(t,x) = \sum_{0\leq|\alpha|\leq k} \left(\int_{\mathbf{R}^2} v_0(y)y^\alpha dy\right) \frac{1}{(1+t)^{\frac{|\alpha|+2}{2}}} \partial^\alpha G\left(\frac{x}{\sqrt{1+t}}\right) + O_{L^2}\left((1+t)^{-\frac{k+2}{2}}\right). \tag{40}$$

Proof We change variables according to (34), (35). We obtain a new function $V : V(\tau,\eta) = e^{\tau\mathscr{L}} V_0$. According to Proposition 5, we have that

$$V(\tau,\eta) = e^{-\frac{\tau}{2}} \left(\int V_0(\eta)d\eta\right) G(\eta) + O_{L^2_2}(e^{-\tau}).$$

Back in terms of the original v function, it follows that we have

$$v(t,x) = \left(\int v_0(x)dx\right) \frac{1}{1+t} G\left(\frac{x}{\sqrt{1+t}}\right) + O_{L^2}\left((1+t)^{-1}\right).$$

More generally for integer k, we have

$$V = \sum_{0\leq|\alpha|\leq k} \left(\int_{\mathbf{R}^2} V_0(\eta)\eta^\alpha dy\right) e^{-\frac{\tau(|\alpha|+1)}{2}} \partial^\alpha G(\eta) + O_{L^2_2}\left(e^{-\frac{\tau(k+2)}{2}}\right),$$

which implies (40). □

Next, we state the relevant properties of the operator $\mathscr{L}$ in the case of three spatial dimensions.

3.1.2 Properties of $\mathscr{L}$ in the 3D Case

We state a companion statement to Proposition 5 in the case $d = 3$. This again is a direct consequence of Theorem A.1, [7].

Proposition 6 *For $\mathscr{L} = \Delta + \frac{1}{2}\eta \cdot \nabla_\eta + \frac{1}{2}$, defined as a closed operator in its maximal domain, in the Hilbert space $L^2_2 = \{f : \|f\|_{L^2_2} := \left(\int_{\mathbf{R}^3} (1+|\eta|^2)^2 |f(\eta)|^2 d\eta\right)^{\frac{1}{2}}\}$, we have the following*

- *its spectrum is given by $\sigma(\mathscr{L}) = \sigma_{ess}(\mathscr{L}) \cup \sigma_{p.p.}(\mathscr{L})$, where*

$$\sigma_{p.p.}(\mathscr{L}) = \{-1\}, \sigma_{ess}(\mathscr{L}) = \left\{\lambda : \Re\lambda \leq -\frac{3}{2}\right\}.$$

  Moreover, the isolated eigenvalue -1 corresponds to the explicit eigenfunction G, namely $\mathscr{L}[G] = -G$. The highest e-value Riesz spectral projection is rank one, and it is given by $P_{-1}f = \langle f, 1\rangle G$.

- *The operator $\mathscr{L}$ generates a C_0 semigroup $e^{\tau\mathscr{L}}$ on $L^2_2(\mathbf{R}^3)$ as follows*

$$e^{\tau\mathscr{L}} f(\eta) = \frac{e^{\frac{\tau}{2}}}{(4\pi a(\tau))^{\frac{3}{2}}} \int_{\mathbf{R}^3} e^{-\frac{|\eta-\eta'|^2}{4a(\tau)}} f(e^{\frac{\tau}{2}}\eta')d\eta'.$$

There are the following estimates

$$\|e^{\tau\mathscr{L}} f\|_{L^p(\mathbf{R}^3)} \le Ce^{\tau\left(\frac{1}{2}-\frac{3}{2p}\right)}\|f\|_{L^p(\mathbf{R}^2)}, 1 \le p \le \infty \tag{41}$$

$$\|e^{\tau\mathscr{L}} f\|_{L^2_2} \le Ce^{-\tau}\|f\|_{L^2_2}, \tag{42}$$

$$\|e^{\tau\mathscr{L}} f\|_{L^2_2} \le Ce^{-\frac{3\tau}{2}}\|f\|_{L^2_2}, \text{ if } \hat{f}(0) = 0. \tag{43}$$

Next, we provide further detailed description of the asymptotic behavior of the solution $\mathbb{Q}$ of (1). We take on the two dimensional case first.

3.2 Sharp Asymptotics in the Case $d = 2$

Recall that in the case $a > 0, d = 2$, we have managed to show (see Proposition 3) that $\|\mathbb{Q}\|_{L^p}, 2 \le p \le \infty$ has exponential decay e^{-at}, provided $\mathbb{Q}_0 \in H^1$. So, in (1), we apply the integrating factor e^{at}, so that we obtain the following PDE for the variable $Y(t) = e^{at}\mathbb{Q}(t) \in S_)(2)$

$$Y_t - \Delta Y = -ce^{-2at}\mathrm{tr}(Y^2)Y, \tag{44}$$

where[6] the a priori bound (19) implies

$$\|Y(t)\|_{L^p} \le C\sqrt{\|e^{t\Delta}\mathbb{Q}_0^2\|_{L^{\frac{p}{2}}}} \le C\|\mathbb{Q}_0\|_{L^p}, 2 \le p \le \infty. \tag{45}$$

In fact, we can obtain better estimates for $\|Y(t)\|_{L^p}, 1 \le p \le \infty$, in the following way. By the Duhamel's formula,

$$Y(t) = e^{t\Delta}\mathbb{Q}_0 - c\int_0^t e^{(t-s)\Delta}e^{-2as}\mathrm{tr}(Y^2(s))Y(s)ds \tag{46}$$

Using the a priori bound (45) leads to the following estimate for the forcing term

$$\begin{aligned}\|\int_0^t e^{(t-s)\Delta}e^{-2as}\mathrm{tr}(Y^2(s))Y(s)ds\|_{L^p} \le C\int_0^{\frac{t}{2}} &\frac{1}{(t-s)^{1-\frac{1}{p}}}e^{-2as}\|Y(s)\|^3_{L^3}ds + \\ &+C\int_{\frac{t}{2}}^t e^{-2as}\|Y(s)\|^3_{L^\infty}ds\end{aligned}$$

for any $1 \le p \le \infty$. Applying the a priori bound (45) yields the estimate

$$\|Y(t)\|_{L^p} \le \frac{C}{(1+t)^{1-\frac{1}{p}}}. \tag{47}$$

In the next section, we establish weighted estimates for Y.

[6] Recall that the b term drops out in the case $d = 2$.

3.2.1 Weighted Estimates for $\mathbb{Q}$

Proposition 7 *Let* $\mathbb{Q}_0 \in L^2_2(\mathbf{R}^2)$. *Then,* $\mathbb{Q} \in L^\infty_t L^2_x(2)$. *That is, there exists an absolute constant* C*, so that for all* $t > 0$,

$$\|\mathbb{Q}(t,\cdot)\|_{L^2_2} \leq C\|\mathbb{Q}_0\|_{L^2_2} e^{-at}.$$

Proof We proceed similar to the weighted L^∞ estimate established earlier, see (29). Namely, introduce a non-negative cutoff function $\varphi \in C_0^\infty : supp \ \ \varphi \subset (1/2, 4), \varphi(x) = 1, 1 \leq |x| \leq 2$ and set $y = \mathrm{tr}(Y^2(t,x))$, $y_k(t,x) := y(t,x)\varphi(2^{-k}x)$, $k \geq 1$. It follows that

$$y_k(t) \leq e^{t\Delta} y_k(0) + \int_0^t e^{(t-s)\Delta} e^{-2as}[-2^{-k+1}\nabla\varphi(2^{-k}x) \cdot \nabla y + 2^{-2k}\Delta\varphi(2^{-k}x)y]ds. \tag{48}$$

Integration by parts and estimating in $L^1(\mathbf{R}^2)$, using the a priori bound $\|y(t)\|_{L^1} = \|\mathbb{Q}(t)\|^2 \leq C$, yields the bound, as in (29),

$$\|y_k(t,\cdot)\|_{L^1} \lesssim \|y_k(0)\|_{L^1} + 2^{-k}\left(\|y_{k-1}(t,\cdot)\|_{L^1} + \|y_k(t,\cdot)\|_{L^1} + \|y_{k+1}(t,\cdot)\|_{L^1}\right)$$

Due to the a priori bound $\|y_k(t)\|_{L^1} \leq \|Y\|^2_{L^2} \leq C$, we conclude

$$\|y_k(t,\cdot)\|_{L^1} \lesssim C\|y_k(0)\|_{L^1}$$

Multiplying by the appropriate weights, squaring and adding in k yields $\|Y\|_{L^2_2} \leq C\|Y_0\|_{L^2_2}$. Translating back in terms of $\mathbb{Q}$, we get the result. □

3.2.2 Asymptotics in the Scaling Variables Formulation

Consider the solution $Y(t)$ of (44), with data $Y_0 \in L^2_2 \cap L^\infty$. We now apply the scaling change of variables as described in (34), (35). Namely,

$$Y(t,x) = \frac{1}{\sqrt{1+t}} Z\left(\frac{x}{\sqrt{1+t}}, \ln(1+t)\right). \tag{49}$$

Note that as a consequence of Proposition 7, $Z \in L^2_2$ and also due to the bound (47), we have that

$$\|Z(\tau,\cdot)\|_{L^p} \leq Ce^{-\frac{\tau}{2}}, 1 \leq p \leq \infty. \tag{50}$$

Clearly, the unknown function $Z(\eta,\tau)$ satisfies the PDE

$$Z_\tau - \mathscr{L}Z = -ce^{2a}e^{-2ae^\tau}\mathrm{tr}(Z^2)Z.$$

Since the precise form of the constant ce^{2a} is inconsequential, we replace it with c_1. We get the initial value problem for Z,

$$\begin{cases} Z_\tau - \mathscr{L}Z = -c_1 e^{-2ae^\tau}\mathrm{tr}(Z^2)Z. \\ Z(0,\eta) = \mathbb{Q}_0(\eta) \end{cases} \tag{51}$$

Equivalently to (51),

$$Z(\tau) = e^{\tau\mathscr{L}}\mathbb{Q}_0 - c_1\int_0^\tau e^{(\tau-s)\mathscr{L}} e^{-2ae^s}\mathrm{tr}(Z^2(s))Z(s)ds. \tag{52}$$

As the operator $\mathscr{L}$ is dissipative, it is clearly beneficial to project (52) along the eigenvalues, as in Proposition 5. This would require interpreting (52) in the L^2_m context for large m. Instead, we would just judiciously use the estimates (36), (37), (38). To this end, apply (52) to the function 1. This is justified as $Z \in L^2_2(\mathbf{R}^2) \subset L^1(\mathbf{R}^2)$. As $\mathscr{L}^*[1] = -\frac{1}{2}$, we have

$$\langle Z(\tau), 1\rangle = \langle \mathbb{Q}_0, 1\rangle e^{-\frac{\tau}{2}} - c_1 \int_0^\tau e^{-\frac{(\tau-s)}{2}} e^{-2ae^s} \langle \mathrm{tr}(Z^2(s))Z(s), 1\rangle ds. \tag{53}$$

It follows that

$$e^{\frac{\tau}{2}} \langle Z(\tau), 1\rangle = \langle \mathbb{Q}_0, 1\rangle - c_1 \int_0^\tau e^{\frac{s}{2}} e^{-2ae^s} \langle \mathrm{tr}(Z^2(s))Z(s), 1\rangle ds.$$

whence, by (50),

$$\sigma_0 := \langle \mathbb{Q}_0, 1\rangle - c_1 \int_0^\infty e^{\frac{s}{2}} e^{-2ae^s} \langle \mathrm{tr}(Z^2(s))Z(s), 1\rangle ds.$$

exists and we have $\sigma_0 = \lim_{\tau\to\infty} e^{\frac{\tau}{2}} \langle Z(\tau), 1\rangle$. In fact, by a very rough estimate,

$$|e^{\frac{\tau}{2}} \langle Z(\tau), 1\rangle - \sigma_0| \le c_1 \int_\tau^\infty e^{\frac{s}{2}} e^{-2ae^s} \langle \mathrm{tr}(Z^2(s))Z(s), 1\rangle ds \le Ce^{-\tau},$$

so that $|\langle Z(\tau), 1\rangle - \sigma_0 e^{-\frac{\tau}{2}}| \le Ce^{-\tau}$. Writing $\tilde{Z}(\tau) := Z(\tau) - \langle Z(\tau), 1\rangle G$, we obtain

$$\tilde{Z}(\tau) = e^{\tau\mathscr{L}}[\mathbb{Q}_0 - \langle \mathbb{Q}_0, 1\rangle G] - c_1 \int_0^\tau e^{(\tau-s)\mathscr{L}} e^{-2ae^s} [\mathrm{tr}(Z^2(s))Z(s) - \langle \mathrm{tr}(Z^2(s))Z(s), 1\rangle G] ds.$$

Note that for each L^2_2 function $h \in L^2_2$, we have that $\int_{\mathbf{R}^2} [f - \langle f, 1\rangle G] d\xi = 0$, whence the estimate (38) is applicable to it. We obtain

$$\|\tilde{Z}(\tau)\|_{L^2_2} \lesssim e^{-\tau} + \int_0^\tau e^{-(\tau-s)} e^{-2ae^s} e^{-s} \|\tilde{Z}(\tau)\|_{L^2_2} ds$$

since $\|\mathrm{tr}(Z^2(s))Z(s)\|_{L^2_2} \le C\|Z(s)\|^2_{L^\infty} \|Z(s)\|_{L^2_2}$. It follows that $\|\tilde{Z}(\tau)\|_{L^2_2} \lesssim e^{-\tau}$.

Thus, if $\sigma_0 \neq 0$, we have the representation formula

$$Z(\tau) = \langle Z(\tau), 1\rangle G + \tilde{Z} = \sigma_0 e^{-\frac{\tau}{2}} G(\xi) + O_{L^2_2}(e^{-\tau}),$$

which implies that

$$\|Z(\tau) - \sigma_0 e^{-\frac{\tau}{2}} G(\xi)\|_{L^2_2} \le Ce^{-\tau}.$$

In particular,

$$Z(\tau, \xi) = \sigma_0 e^{-\frac{\tau}{2}} G(\xi) + O_{L^2}(e^{-\tau}). \tag{54}$$

In terms of the original variables

$$\begin{aligned} Y(t,x) &= \frac{1}{(1+t)} G\left(\frac{x}{\sqrt{1+t}}\right) \sigma_0 + O_{L^2}((1+t)^{-1}) = \frac{e^{-\frac{|x|^2}{t+1}}}{4\pi(1+t)} \sigma_0 + O_{L^2}((1+t)^{-1}) \\ &= t^{-1} e^{-\frac{|x|^2}{4t}} \sigma_0 + O_{L^2}((1+t)^{-1}). \end{aligned}$$

Finally, in terms of $\mathbb{Q}$, we get the formula

$$\mathbb{Q}(t) = e^{-at} t^{-1} e^{-\frac{|x|^2}{4t}} \mu_0 + O_{L^2}(e^{-at}(1+t)^{-1}). \tag{55}$$

as stated in (5). Clearly, this representation implies the required result. Unfortunately, it is hard to find general enough conditions on the initial data, which guarantee that $\mu_0 \neq 0$.

It is in fact, quite possible that this behavior is in some sense universal, and we conjecture that that it happens for every initial data $\mathbb{Q}_0$ or at least on a generic subset of it.

Note that, if it happens that $\mu_0 \neq 0$, we get the sharp asymptotic formula $\|\mathbb{Q}(t)\|_{L^2} \sim \frac{e^{-at}}{\sqrt{1+t}}$. while for $\mu_0 = 0$, we get the bound $\|\mathbb{Q}(t)\|_{L^2} \lesssim \frac{e^{-at}}{1+t}$. In the next section, we show that $\mu_0 \neq 0$ for a wide class of initial data.

3.2.3 Asymptotics for Positive Data

For this section, assume that at least some of the entries of $\mathbb{Q}_0$ are positive, say $\mathbb{Q}_0^{11}(x) \geq 0$. By the results of Proposition 2, $\mathbb{Q}^{11}(t, x) \geq 0$ for all $t > 0$. We now show that for large $\tau > \tau_0$, we have $\|Z^{11}(\tau)\|_{L^2} \geq \frac{1}{2}\|e^{\tau\mathscr{L}}\mathbb{Q}_0^{11}(\tau_0)\|_{L^2}$.

Take a small $0 < \epsilon < 1$, to be specified momentarily and large enough τ_0, so that $e^{-\frac{\tau_0}{2}} < \epsilon$. Consider the Duhamel's formulation (52) starting at time τ_0, with initial data $\mathbb{Q}^{11}(\tau_0) \geq 0$, that is

$$Z^{11}(\tau) = e^{\tau\mathscr{L}}\mathbb{Q}^{11}(\tau_0) - c_1 \int_{\tau_0}^{\tau} e^{(\tau-s)\mathscr{L}} e^{-2ae^s} \operatorname{tr}(Z^2(s)) Z^{11}(s) ds. \tag{56}$$

Clearly, for small increment times $\tau_0 < \tau < \tau_0 + \delta$, we certainly have the point-wise bounds

$$|Z^{11}(\tau)| \leq 2|e^{\tau\mathscr{L}}\mathbb{Q}^{11}(\tau_0)| = 2e^{\tau\mathscr{L}}\mathbb{Q}^{11}(\tau_0), \tag{57}$$

where we took into account the positivity of the semigroup $e^{\tau\mathscr{L}}$. We claim that $\delta = \infty$, that is we can extend the validity of (57) for all $\tau > \tau_0$. Indeed, up to any τ so that the estimate (57) holds, we have the point-wise bound

$$|Z^{11}(\tau)| \leq e^{\tau\mathscr{L}}\mathbb{Q}_0^{11}(\tau_0) + 2\epsilon \int_{\tau_0}^{\tau} e^{-2ae^s} e^{(\tau-s)\mathscr{L}}[e^{s\mathscr{L}}\mathbb{Q}_0^{11}(\tau_0)]ds \leq (1 + C\epsilon)e^{\tau\mathscr{L}}\mathbb{Q}_0^{11}(\tau_0)$$

where C is an absolute constant. Clearly, a choice of $\epsilon : C\epsilon < 2$ would suffice to conclude that (57) holds for all $\tau > \tau_0$. This of course implies the estimate

$$\frac{1}{2}\|e^{\tau\mathscr{L}}\mathbb{Q}_0^{11}(\tau_0)\|_{L^2} \leq \|Z^{11}(\tau)\|_{L^2} \leq 2\|e^{\tau\mathscr{L}}\mathbb{Q}_0^{11}(\tau_0)\|_{L^2},$$

for all sufficiently large τ. Thus, as $\mathbb{Q}_0^{11}(\tau_0) > 0$,

$$\|Z^{11}(\tau)\|_{L^2} \sim \|e^{\tau\mathscr{L}}\mathbb{Q}_0^{11}(\tau_0)\|_{L^2} \sim (\int \mathbb{Q}_0^{11}(\tau_0, \xi)d\xi)e^{-\frac{\tau}{2}}$$

In terms of the original variables

$$\|\mathbb{Q}(t)\|_{L^2} \sim \frac{e^{-at}}{\sqrt{1+t}}.$$

This clearly implies that the limiting matrix μ_0 identified above, has $\mu_0 \neq 0$.

3.2.4 Asymptotics for Small Data with Non-vanishing Mean

We now prove that if
$\int_{\mathbf{R}^2} \mathbb{Q}_0(x)dx \neq 0$ and $\mathbb{Q}_0$ is small enough, then $\mu_0 \neq 0$. This is in fact not hard to see from the Duhamel's representation (52).

Indeed, in such a case, we have the bound

$$\|e^{\tau\mathscr{L}}\mathbb{Q}_0\|_{L^2\cap L^\infty} - C\|Z\|^3_{L^2\cap L^\infty} \le \|Z\|_{L^2\cap L^\infty} \le \|e^{\tau\mathscr{L}}\mathbb{Q}_0\|_{L^2\cap L^\infty} + C\|Z\|^3_{L^2\cap L^\infty}.$$

As we assume small data, it follows that $\|Z\|_{L^2\cap L^\infty} << 1$, whence we have $\|Z\|_{L^2\cap L^\infty} \sim \|e^{\tau\mathscr{L}}\mathbb{Q}_0\|_{L^2\cap L^\infty}$. Furthermore, as is clear from our subsequent analysis,

$$e^{\tau\mathscr{L}}\mathbb{Q}_0 = \langle\mathbb{Q}_0, 1\rangle G e^{-\frac{\tau}{2}} + O_{L^2_2\cap L^\infty}(e^{-\tau})$$

whence we conclude $\|Z\|_{L^2\cap L^\infty} \sim |\langle\mathbb{Q}_0, 1\rangle| e^{-\frac{\tau}{2}} + O_{L^2_2}(e^{-\tau}) \sim |\langle\mathbb{Q}_0, 1\rangle| e^{-\frac{\tau}{2}}$, if $|\langle\mathbb{Q}_0, 1\rangle| \neq 0$. Translating back in terms of $\mathbb{Q}(t)$, we see that it implies the bound (6) for $\mathbb{Q}(t)$, which means that $\mu_0 \neq 0$.

3.3 Sharp Asymptotics for the Case $d = 3$

The analysis in this case is pretty similar to the previous one, with a few exceptions. First, we do not have an exponential decay rate, as stated in Proposition 3 for the case $d = 2$. Instead, we have an a priori uniform bounds as stated in (23), (24), (25), provided we require an additional point-wise bound on the initial data $\mathbb{Q}_0$. It has to be mentioned that this is the best one can do in general - as it is well-known, that at least for some values of $a > 0, b, c > 0$, there are stationary solutions $\mathbb{Q}$ of (1). Hence, no time decay is expected in general.

However, if it is a priori known that a solution $\mathbb{Q}(t)$ is attracted to zero (in a sense to be made precise below), then we can actually compute its leading order asymptotic, similar to the 2D case. To this end, assume that for a given $\mathbb{Q}_0$, its unique global solution guaranteed by Proposition 4 satisfies $\lim_{t\to+\infty} \|\mathbb{Q}(t,\cdot)\|_{L^\infty(\mathbf{R}^3)} = 0$. Then, for all $\epsilon > 0$, there is T_ϵ, so that $\|\mathbb{Q}(t,\cdot)\|_{L^\infty} < \epsilon$ for all $t > T_\epsilon$. This immediately leads to smallness and in fact time decay, for the $\|\mathbb{Q}(t,\cdot)\|_{L^p}, 2 \le p \le \infty$. Indeed, taking into account that $\lim_{t\to\infty} \|y(t)\|_{L^\infty} = 0$, we get that for all $\epsilon > 0$ and for all sufficiently large times $t > T_\epsilon, \|y(t)\|_{L^\infty} \le \epsilon$, whence from the inequality (26) for y, we derive the bound

$$y(t) - \Delta y + 2ay \le c_1 y^{\frac{3}{2}}.$$

Applying the Duhamel's formula in intervals $T_\epsilon, t)$ leads to the bound for $t > T_\epsilon$ and $1 \le q \le \infty$,

$$\|y(t)\|_{L^q} \le Ce^{-2a(t-T_\epsilon)}\|y(T_\epsilon)\|_{L^q} + C\sqrt{\epsilon}\int_{T_\epsilon}^t e^{-2a(t-s)}\|y(s)\|_{L^q}ds.$$

It follows from Gronwal's that we have, for all $\epsilon > 0$, the bound $\|y(t)\|_{L^q} \le C_\epsilon e^{-2(a-\epsilon)t}, 1 \le q \le \infty$. In terms of $\mathbb{Q}$, this means

$$\|\mathbb{Q}(t)\|_{L^p} \le Ce^{-(a-\epsilon)t}, 2 \le p \le \infty. \tag{58}$$

Then, we introduce, as in the two dimensional case, the variable $Y(t) = e^{at}\mathbb{Q}(t)$, which satisfies the equation

$$Y_t = \Delta Y + be^{-at}\left(Y^2 - \frac{1}{3}\mathrm{tr}(Y^2)Id\right) - ce^{-2at}\mathrm{tr}(Y^2)Y. \tag{59}$$

Note that as a consequence of (58), we have the *a priori* bound, valid for all $\epsilon > 0$, $\|Y(t)\|_{L^p} \le C_\epsilon e^{\epsilon t}, 2 \le p \le \infty$. This can be of course immediately bootstrapped to $\|Y(t)\|_{L^p} \le C, 2 \le p \le \infty$ (using the Duhamel's re-formulation of (59)), if say the initial data $\mathbb{Q}_0 = Y_0 \in L^1 \cap L^2$.

At this point, we basically repeat the argument from Sect. 3.2.2. We use the transformation (49), which leads to the equation

$$Z_\tau - \mathscr{L} Z = b e^{a} e^{\frac{\tau}{2}} e^{-a e^{\tau}} \left(Z^2 - \frac{1}{3} \operatorname{tr}(Z^2) Id \right) - c e^{2a} e^{-2 a e^{\tau}} \operatorname{tr}(Z^2) Z. \tag{60}$$

Again, the precise form of the constants be^{a}, ce^{2a} are unimportant and we denote them generically by b_1, c_1. Also, the formula (49) allows us to translate the bound $\|Y(t)\|_{L^p} \leq C, 2 \leq p \leq \infty$ to (the very inefficient!) a priori bound

$$\|Z(\tau)\|_{L^p} \leq C e^{\frac{\tau}{2}(1-\frac{3}{p})}, 2 \leq p \leq \infty \tag{61}$$

The Duhamel form of (60) reads as follows

$$Z = e^{\tau \mathscr{L}} \mathbb{Q}_0 + \int_0^\tau e^{(\tau - s)\mathscr{L}} \left[b_1 e^{\frac{s}{2}} e^{-a e^{s}} \left(Z^2 - \frac{1}{3} \operatorname{tr}(Z^2) Id \right) - c_1 e^{-2 a e^{s}} \operatorname{tr}(Z^2) Z \right] ds \tag{62}$$

We again project along the eigen-direction, corresponding to the largest in real part e-value of $\mathscr{L}$, which is -1. We have, similar to (52),

$$\begin{aligned} \langle Z(\tau), 1\rangle = \langle \mathbb{Q}_0, 1\rangle e^{-\tau} + \int_0^\tau & e^{-(\tau - s)} e^{\frac{s}{2}} e^{-a e^{s}} \\ & \langle \left(b_1 \left(Z^2 - \frac{1}{3} \operatorname{tr}(Z^2) Id \right) - c_1 e^{-\frac{s}{2}} e^{-a e^{s}} \operatorname{tr}(Z^2) Z \right), 1\rangle ds. \end{aligned}$$

The a priori bound (61) for Z implies that $\sigma_0 := \lim_{\tau \to \infty} e^{\tau} \langle Z(\tau), 1\rangle$ exists and in fact it can be calculated

$$\sigma_0 = \langle \mathbb{Q}_0, 1\rangle + \int_0^\infty e^{\frac{3s}{2}} e^{-a e^{s}} \langle \left(b_1 \left(Z^2 - \frac{1}{3} \operatorname{tr}(Z^2) Id \right) - c_1 e^{-\frac{s}{2}} e^{-a e^{s}} \operatorname{tr}(Z^2) Z \right), 1\rangle ds.$$

It follows that

$$|e^{\tau} \langle Z(\tau), 1\rangle - \sigma_0| \leq \int_\tau^\infty e^{\frac{3s}{2}} e^{-a e^{s}} \max(e^{\frac{3s}{2}(1-\frac{3}{p})}, 1) ds \leq C e^{-\tau}.$$

Thus, using the decomposition $Z(\tau) = \langle Z(\tau), 1\rangle G + \tilde{Z}(\tau)$, we have the equation

$$\tilde{Z}(\tau) = e^{\tau \mathscr{L}} [\mathbb{Q}_0 - \langle \mathbb{Q}_0, 1\rangle G] + \int_0^\tau e^{(\tau - s)\mathscr{L}} \left[F(Z[s]) - \langle F(Z(s)), 1\rangle G \right] ds$$

where $F(Z) := b_1 e^{\frac{s}{2}} e^{-a e^{s}} \left(Z^2 - \frac{1}{3} \operatorname{tr}(Z^2) Id \right) - c_1 e^{-2 a e^{s}} \operatorname{tr}(Z^2) Z$. As in the 2D case, we have the estimate (43), which allows us to gain better decay estimates for $\|e^{\tau \mathscr{L}} h\|_{L^2_2}$, when $h : \hat{h}(0) = 0$. Specifically, we have the bound

$$\|\tilde{Z}(\tau)\|_{L^2_2} \leq C e^{-\frac{3\tau}{2}} + \int_0^\tau e^{-\frac{3(\tau - s)}{2}} e^{\frac{3s}{2}} e^{-a e^{s}} \|\tilde{Z}(s)\|_{L^\infty} (1 + \|\tilde{Z}(s)\|_{L^\infty}) \|\tilde{Z}(s)\|_{L^2_2} ds$$

Due to the a priori estimates for $\|Z\|_{L^\infty}$, and hence for $\|\tilde{Z}\|_{L^\infty}$, we conclude that $\tilde{Z}(\tau) \in L^2_2$ for all times, and moreover

$$\|\tilde{Z}(\tau)\|_{L^2_2} \leq C e^{-\frac{3\tau}{2}}.$$

It follows that

$$Z(\tau, \xi) = \sigma_0 e^{-\tau} G(\xi) + O_{L^2}(e^{-\frac{3\tau}{2}}).$$

This is then easily translatable in terms of $\mathbb{Q}(t)$, recalling $\mathbb{Q}(t) = e^{-at}(1+t)^{-\frac{1}{2}} Z\left(\frac{x}{\sqrt{1+t}}, \ln(1+t)\right)$, as follows

$$\mathbb{Q}(t) = e^{-at} \frac{1}{(1+t)^{\frac{3}{2}}} G\left(\frac{x}{\sqrt{1+t}}\right) \sigma_0 + O_{L^2}(e^{-at}(1+t)^{-\frac{5}{4}})$$
$$= e^{-at} t^{-\frac{3}{3}} e^{-\frac{|x|^2}{4t}} \mu_0 + O_{L^2}(e^{-at}(1+t)^{-\frac{5}{4}}), \mu_0 \in S_0(3).$$

Again, a condition that guarantees $\mu_0 \neq 0$, especially in terms of initial data, is not easy to come by. However, the proofs of the claims that $\mu_0 \neq 0$, provided that either $\mathbb{Q}_0$ has an entry with a positive data or $\mathbb{Q}_0$ is for sufficiently small, proceed in an identical manner as those in the 2D case, see Sects. 3.2.3 and 3.2.4.

4 The Asymptotics of the Correlation Functional: Proof of Theorem 5

We consider the denominator first, and we show the following result.

Lemma 2 *Let $d = 2$ and $\mathbb{Q}(t)$ has the asymptotic (5). Then,*

$$\int_{\mathbf{R}^2} \mathbb{Q}^2(x,t)dx = 2\pi e^{-2at} t^{-1} \mu_0^2 + O(e^{-2at} t^{-\frac{3}{2}}) \tag{63}$$

If $d = 3$, and similarly $\mathbb{Q}(t)$ has the asymptotic (9), then

$$\int_{\mathbf{R}^3} \mathbb{Q}^2(x,t)dx = (2\pi)^{\frac{3}{2}} e^{-2at} t^{-\frac{3}{2}} \mu_0^2 + O(e^{-2at} t^{-2}). \tag{64}$$

Proof We have from either (5) that

$$\int_{\mathbf{R}^2} \mathbb{Q}^2(x,t)dx = e^{-2at} \int_{\mathbf{R}^2} \left(t^{-1} e^{-\frac{|x|^2}{4t}} \mu_0 + R(t,x)\right)^2 dx$$
$$= e^{-2at} t^{-2} \mu_0^2 \int_{\mathbf{R}^2} e^{-\frac{|x|^2}{2t}} dx + \tilde{R}(t),$$

where

$$\tilde{R}(t) = e^{-2at} \int \left(2t^{-1} e^{-\frac{|x|^2}{4t}} \mu_0 R(t,x) + R^2(t,x)\right) dx$$

As $\|R(t,\cdot)\|_{L^2} \le Ct^{-1}$, we can bound the residual by

$$|\tilde{R}(t)| \le Ce^{-2at} t^{-\frac{1}{2}} \|R(t,\cdot)\|_{L^2} + e^{-2at} \|R(t,\cdot)\|_{L^2}^2 \le Ce^{-2at} t^{-\frac{3}{2}}.$$

The main term is thus provided by the expression

$$e^{-2at} t^{-2} \mu_0^2 \int_{\mathbf{R}^2} e^{-\frac{|x|^2}{2t}} dx = 2\pi e^{-2at} t^{-1} \mu_0^2$$

For the case $d = 3$, we use the asymptotic formula (9) and a similar analysis of the one presented We obtain

$$\int_{\mathbf{R}^3} \mathbb{Q}^2(x,t)dx = e^{-2at} t^{-3} \mu_0^2 \int_{\mathbf{R}^3} e^{-\frac{|x|^2}{2t}} dx + \tilde{R}(t) = (2\pi)^{\frac{3}{2}} e^{-2at} t^{-\frac{3}{2}} \mu_0^2 + O(e^{-2at} t^{-2}).$$

□

The next step is to compute the precise asymptotics of the numerator in the correlation functional. To this end, we have the following lemma.

Lemma 3 *Let $d = 2$ and $\mathbb{Q}(t)$ has the asymptotic formula (5). Then,*

$$\int_{\mathbf{R}^2} \mathbb{Q}(x,t)\mathbb{Q}(x+y,t)dx = 2\pi t^{-1} e^{-2at} \mu_0^2 e^{-\frac{|y|^2}{8t}} + O_{L_y^\infty}(e^{-2at} t^{-\frac{3}{2}}). \tag{65}$$

If $d = 3$, and $\mathbb{Q}(t)$ has the asymptotic formula (9), then

$$\int_{\mathbf{R}^3} \mathbb{Q}(x,t)\mathbb{Q}(x+y,t)dx = (2\pi)^{\frac{3}{2}} t^{-\frac{3}{2}} e^{-2at} \mu_0^2 e^{-\frac{|y|^2}{8t}} + O_{L_y^\infty}(e^{-2at} t^{-2}). \tag{66}$$

Proof For the case $d = 2$, we start with the same asymptotic expression provided by (5). We obtain

$$\begin{aligned}
&\int_{\mathbf{R}^2} \mathbb{Q}(x,t)\mathbb{Q}(x+y,t)dx \\
&= e^{-2at} \int_{\mathbf{R}^2} \left(t^{-1} e^{-\frac{|x|^2}{4t}} \mu_0 + R(t,x)\right)\left(t^{-1} e^{-\frac{|x+y|^2}{4t}} \mu_0 + R(t,x+y)\right) dx \\
&= e^{-2at} t^{-2} \mu_0^2 \int_{\mathbf{R}^2} e^{-\frac{|x|^2}{4t}} e^{-\frac{|x+y|^2}{4t}} dx + \tilde{R}(t,y),
\end{aligned}$$

Similarly, we can estimate, *uniformly in y*,

$$\|\tilde{R}(t,\cdot)\|_{L_y^\infty} \le C(e^{-2at} t^{-\frac{1}{2}} \|R(t,\cdot)\|_{L^2} + e^{-2at} \|R(t,\cdot)\|_{L^2}^2) \le C e^{-2at} t^{-\frac{3}{2}}$$

Thus, it remains to compute the main term. We have

$$\begin{aligned}
e^{-2at} t^{-2} \mu_0^2 \int_{\mathbf{R}^2} e^{-\frac{|x|^2}{4t}} e^{-\frac{|x+y|^2}{4t}} dx &= e^{-2at} t^{-2} \mu_0^2 e^{-\frac{|y|^2}{8t}} \int_{\mathbf{R}^2} e^{-\frac{|x-\frac{y}{2}|^2}{2t}} dx \\
&= 2\pi t^{-1} e^{-2at} \mu_0^2 e^{-\frac{|y|^2}{8t}}.
\end{aligned}$$

Similarly, in the 3D case,

$$\begin{aligned}
&\int_{\mathbf{R}^3} \mathbb{Q}(x,t)\mathbb{Q}(x+y,t)dx \\
&= e^{-2at} t^{-3} \mu_0^2 \int_{\mathbf{R}^3} e^{-\frac{|x|^2}{4t}} e^{-\frac{|x+y|^2}{4t}} dx + \tilde{R}(t,y) = (2\pi)^{\frac{3}{2}} t^{-\frac{3}{2}} e^{-2at} \mu_0^2 e^{-\frac{|y|^2}{8t}} + O_{L_y^\infty}(e^{-2at} t^{-2}).
\end{aligned}$$

□

Putting together the asymptotics from Lemma 2 and Lemma 3 easily implies the asymptotic formula for $c(y,t)$, $t >> 1$. Indeed, for the 2D case, we have

$$c(y,t) = \frac{2\pi t^{-1} e^{-\frac{|y|^2}{8t}} \operatorname{tr}(\mu_0^2) + O_{L_y^\infty}(t^{-\frac{3}{2}})}{2\pi t^{-1} \operatorname{tr}(\mu_0^2) + O(t^{-\frac{3}{2}})} = e^{-\frac{|y|^2}{8t}} + O_{L_y^\infty}(t^{-\frac{1}{2}}).$$

while for the 3D case, we similarly have

$$c(y,t) = \frac{(2\pi)^{\frac{3}{2}} t^{-\frac{3}{2}} e^{-\frac{|y|^2}{8t}} \operatorname{tr}(\mu_0^2) + O_{L_y^\infty}(t^{-2})}{(2\pi)^{\frac{3}{2}} t^{-\frac{3}{2}} \operatorname{tr}(\mu_0^2) + O(t^{-2})} = e^{-\frac{|y|^2}{8t}} + O_{L_y^\infty}(t^{-\frac{1}{2}}).$$

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article. The authors declare that there is no associated data with this manuscript.

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