

Research Article

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Conformal Transformation of Uniform Domains Under Weights That Depend on Distance to The Boundary

<https://doi.org/10.1515/agms-2022-0141>

Received April 4, 2022; accepted August 10, 2022

Abstract: The sphericalization procedure converts a Euclidean space into a compact sphere. In this note we propose a variant of this procedure for locally compact, rectifiably path-connected, non-complete, unbounded metric spaces by using conformal deformations that depend only on the distance to the boundary of the metric space. This deformation is locally bi-Lipschitz to the original domain near its boundary, but transforms the space into a bounded domain. We will show that if the original metric space is a uniform domain with respect to its completion, then the transformed space is also a uniform domain.

Keywords: Uniform domains; conformal change in metric; distance to the boundary

MSC: Primary: 30L05; Secondary: 30L10

1 Introduction

The stereographic projection identifies the one-point compactified complex plane with the unit sphere, and this identity has been exploited in the study of analytic functions and conformal maps between planar regions and their behavior at infinity. Higher dimensional stereographic projections also identify the one-point compactification of $\mathbb{R}^n$ with the n -dimensional unit sphere in $\mathbb{R}^{n+1}$. The work of [5, 6] formulated a fruitful generalization of stereographic projection to more general metric spaces, and this formulation has been used in the literature to study Gromov hyperbolic spaces, uniform domains, and quasi-Möbius maps, see for example [2, 5, 6, 8, 9, 11–13, 17]. A metric space *inversion* about a point p in a metric space X turns a bounded metric space $X \setminus \{p\}$ into an unbounded metric space, and the *sphericalization* of X with respect to the base point $p \in X$ turns an unbounded metric space X into a bounded space whose completion is topologically the one-point compactification of X . It was also shown in [6] that the sphericalization and inversion operations of uniform domains yield uniform domains. Moreover, they show that if the metric space satisfies a geometric condition called annular quasiconvexity, then there is control over how the uniformity constant is transformed by these operations.

Apart from the study of quasiconformal geometry (see [4, 16] for a tiny sampling of offerings from this line of enquiry), uniform domains also play a key role in potential theory as uniform domains in a complete doubling metric measure space supporting a p -Poincaré inequality are also known to support a p -Poincaré inequality [3] and admit a description of traces of Sobolev-class functions on the domain as belonging to certain Besov classes [15]. Hence, much of the potential theory and Dirichlet problems on smooth domains are extendable to uniform domains. An additional sampling of the vast literature on potential theory related

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to uniform domains can be found in [1, 14]. For these reasons, a systematic study of metric transformations that preserve the uniform domain property is desirable, and the goal of the present note is to contribute to this study.

While sphericalization and inversion are analogues of stereographic projection and its inverse, these operators distort the metric on X everywhere, including near the boundary of X if X is not complete. In certain circumstances we would wish not to distort the metric, at least locally, near the boundary of X ; for example, if X is a uniform domain (and hence is locally compact but non-complete), we would wish to preserve the local nature of the metric on X near $\partial X := \bar{X} \setminus X$, where $\bar{X}$ denotes the completion of X , while transforming X into a bounded space. Such transformation is desirable if we are interested in studying boundary-value problems on X in terms of boundary value problems on bounded domains (see for example [7]) and so find information about growth-at-infinity behavior of solutions in the unbounded domain. The purpose of this note is to propose a range of modifications of the sphericalization procedure of [6] so that the modification does not perturb the inner length metric, locally, near ∂X . Here, by the inner length metric, we mean the metric d_{inn} given by $d_{\text{inn}}(x, y) = \inf_{\gamma} \ell_d(\gamma)$, where the infimum is over all curves in Ω with end points x and y .

To this end, we consider (Ω, d) to be a locally compact, non-complete metric space such that Ω is a uniform domain in $\bar{\Omega}$ (that is, Ω is a uniform space in the language of [4]). We also fix a monotone decreasing continuous function $\varphi : (0, \infty) \rightarrow (0, \infty)$ such that $\varphi(t) = 1$ when $0 < t \leq 1$,

$$\int_0^{\infty} \varphi(t) dt < \infty,$$

and there is a constant $C_{\varphi} \geq 1$ such that we have $\varphi(t) \leq C_{\varphi} \varphi(2t)$ for all $t > 0$. As Ω is a uniform domain, it is rectifiably connected, that is, pairs of points in Ω can be connected by curves in Ω of finite length. Hence, we use φ to construct a new metric d_{φ} on Ω by setting

$$d_{\varphi}(x, y) := \inf_{\gamma} \int_{\gamma} \varphi \circ d_{\Omega} ds,$$

with the infimum ranging over all rectifiable curves in Ω with end points $x, y \in \Omega$. Here, $\int_{\gamma} h ds := \int_{\gamma} h(\gamma(\cdot)) ds$ is the path integral with respect to the arc-length parametrization of the rectifiable curve γ , see for example [10, Chapter 5]. In the above, d_{Ω} is defined by $d_{\Omega}(x) = \text{dist}(x, \partial\Omega)$, see Definition 2.1 below.

We will show in Section 2 that d_{φ} and d are locally bi-Lipschitz near $\partial\Omega$ and that the completion of $\bar{\Omega}$ with respect to d_{φ} is topologically a one-point compactification of $\bar{\Omega}$. We denote $\Omega_{\varphi} := \bar{\Omega} \cup \partial\bar{\Omega}^{\varphi} \setminus \partial\Omega$, where $\bar{A}^{\varphi}$ is the completion of $A \subset \bar{\Omega}$ with respect to the metric d_{φ} . The following is the main theorem of this note.

Theorem 1.1. *The domain Ω_{φ} , equipped with the metric d_{φ} , is a uniform domain with $\partial\Omega_{\varphi} = \partial\Omega$ and uniformity constant depending only on the constant C_{φ} and the uniformity constant C_U associated with the metric d . Furthermore, the natural identity map $I : \Omega \rightarrow \Omega_{\varphi} \setminus \{\infty\}$ is a local bi-Lipschitz map, and it is also uniformly locally bi-Lipschitz near $\partial\Omega$. If Ω is a length space, then I is a local isometry near $\partial\Omega$.*

One additional advantage of the above is that we do not need annular quasiconvexity of Ω with respect to the metric d in order to gain quantitative control over the uniformity constant with respect to the metric d_{φ} .

We end the discussion in this section by describing a simple illustrative example. Recall that the upper half-plane $\Omega := \mathbb{R} \times (0, \infty)$ is a uniform domain and that the sphericalization procedure on Ω gives an isometric copy of a spherical cap in $\mathbb{S}^2$. Fixing $\beta > 1$, with the choice of $\varphi(t) = t^{-\beta}$ for $t > 1$ and $\varphi(t) = 1$ for $0 < t \leq 1$, we have that as a set, $\Omega_{\varphi} = (\mathbb{R} \times (0, \infty)) \cup \{\infty\}$. Note that the Euclidean boundary $\partial\Omega = \mathbb{R} \times \{0\}$. For each $z \in \mathbb{R} \times \{0\}$, a calculation shows that $d_{\varphi}(z, \infty) = \frac{\beta}{\beta-1}$, and so ∞ is not an accumulation point (with respect to the metric d_{φ}) of $\mathbb{R} \times \{0\}$; this is in contrast to the sphericalization of Ω , which has an accumulation point of $\mathbb{R} \times \{0\}$. Note also that for each $z \in \mathbb{R} \times \{0\}$ we have that $d_{\varphi}(z, z+1) = 1$, and hence, in this example, $\bar{\Omega}_{\varphi}^{\varphi}$ is not compact.

The above example illustrates the fact that given Ω and φ as in this note, there is a positive distance (with respect to the metric d_{φ}) between ∞ and $\partial\Omega_{\varphi} = \partial\Omega$; see Lemma 2.6 and its proof below.

2 Preliminaries

Consider an unbounded metric space (Ω, d) with completion $\overline{\Omega}$ and boundary $\partial\Omega := \overline{\Omega} \setminus \Omega$. We say that Ω is a *uniform domain* if it is locally compact and non-complete, and there exists a constant $C_U \geq 1$ for which each pair $x, y \in \Omega$ with $x \neq y$ can be connected by a C_U -uniform curve γ . That is, γ satisfies the following:

- its length (with respect to the metric d) satisfies $\ell_d(\gamma) \leq C_U d(x, y)$;
- for each z in the trajectory of γ , we have

$$\min\{\ell_d(\gamma_{x,z}), \ell_d(\gamma_{z,y})\} \leq C_U \operatorname{dist}(z, \partial\Omega).$$

By increasing the value of C_U if need be, we can assume that subcurves of uniform curves are also uniform, see [4]. Moreover, for each point $\zeta \in \partial\Omega$ and $x \in \Omega$ we can find a C_U -uniform curve $\gamma : [0, L) \rightarrow \Omega$ such that $\gamma(0) = x$ and $\lim_{t \rightarrow L^-} \gamma(t) = \zeta$.

More generally, given $x \in \Omega$ and $\zeta \in \partial\Omega$, we say that a curve $\beta : [0, L) \rightarrow \Omega$ has end points x and ζ with respect to the metric d if $\beta(0) = x$ and $\lim_{t \rightarrow L^-} \beta(t) = \zeta$, the limit being taken with respect to d .

Definition 2.1. For $x \in \Omega$, we set $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$. We let

$$\Omega_0 := \{x \in \Omega : d_\Omega(x) \leq 1\},$$

and, for positive integers n , we set

$$\Omega_n := \{x \in \Omega : 2^{n-1} < d_\Omega(x) \leq 2^n\}.$$

Note that $\Omega = \bigcup_{n=0}^\infty \Omega_n$.

We fix a monotone decreasing continuous function $\varphi : (0, \infty) \rightarrow (0, \infty)$ such that $\varphi(t) = 1$ when $0 < t \leq 1$, there is a constant $C_\varphi \geq 1$ such that we have $\varphi(t) \leq C_\varphi \varphi(2t)$ for all $t > 0$, and

$$\int_0^\infty \varphi(t) dt < \infty.$$

This condition ensures, by quasiconvexity of Ω , that the metric space $(\Omega_\varphi, d_\varphi)$ is bounded, see Lemma 2.10. The above condition is equivalent to the condition we will use frequently in this note:

$$\sum_{n=0}^\infty 2^n \varphi(2^n) < \infty. \quad (2.2)$$

The prototype function φ to keep in mind is $\varphi(t) = t^{-\beta}$ for some $\beta > 1$ when $t > 1$, or $\varphi(t) = t^{-\beta}(1 + \log(t))^{-\kappa}$ for some $\beta > 1$ and $\kappa > 0$ when $t > 1$. The first prototype function is used in [7, Section 7] to convert an unbounded uniform domain $Z \times [0, \infty)$, with Z a compact length space, into a bounded uniform domain.

Standing assumptions: In summary, we will assume that (Ω, d) is an unbounded uniform domain and that subcurves of all uniform curves are also uniform. Moreover, $\varphi : (0, \infty) \rightarrow (0, \infty)$ is a monotone decreasing function such that $\varphi(t) = 1$ for $t \leq 1$, $\int_0^\infty \varphi(t) dt$ is finite, and there is a constant $C_\varphi \geq 1$ such that for all $t > 0$ we have $\varphi(t) \leq C_\varphi \varphi(2t)$.

We use φ to, in the language of [4], dampen the metric d on Ω by modifying it to d_φ . We define this new metric by setting

$$d_\varphi(x, y) := \inf_{\gamma} \int_{\gamma} \varphi \circ d_\Omega ds =: \inf_{\gamma} \int_{\gamma} \varphi(d_\Omega(\gamma(t))) dt,$$

with the infimum ranging over all rectifiable curves γ in Ω with end points x and y ; we consider the arc-length (with respect to the original metric d) parametrization of γ . As $\varphi(t) \leq 1$ for all $t > 0$, we have that $d_\varphi(x, y) \leq C_U d(x, y)$ whenever $x, y \in \Omega$.

Now we have two identities for the set Ω ; namely, (Ω, d) and (Ω, d_φ) . Since $\varphi = 1$ on Ω_0 , both metrics d and d_φ extend as metrics to $\Omega \cup \partial\Omega$. We will show this below in Lemma 2.6.

We denote $\Omega_\varphi := \overline{\Omega \cup \partial\Omega}^\varphi \setminus \partial\Omega$, where $\overline{A}^\varphi$ is the completion of $A \subset \overline{\Omega}$ with respect to the metric d_φ . First, we show in the following lemma that there is only one point in the completion of $\overline{\Omega}$ with respect to d_φ that is not in the completion of Ω with respect to d . Denoting this point by ∞ , it follows that $\Omega_\varphi = \Omega \cup \{\infty\}$.

Lemma 2.3. *There is a sequence in Ω that is Cauchy with respect to the metric d_φ but not with respect to d . Any two d_φ -Cauchy sequences that are not d -Cauchy sequences must be equivalent with respect to the metric d_φ .*

Proof. We fix $x_0 \in \Omega_2$, and we choose $x_j \in \Omega_j$ for each integer $j \geq 3$. Let β_j be a C_U -uniform curve in Ω with end points x_0, x_j . Since Ω is locally compact, we can exhaust Ω by a sequence of proper subdomains D_k of Ω such that $x_0 \in D_k \subseteq D_{k+1}$. Here $D_k \subseteq D_{k+1}$ means that the closure of D_k is a compact subset of D_{k+1} . By the Arzelà-Ascoli theorem, we can then find a curve β_∞ and a subsequence of β_j , also denoted β_j , such that for each k the segments of the curves β_j lying in D_k converge uniformly to the segment of β_∞ in D_k . Since each β_j is a C_U -uniform curve, so is β_∞ and each of its subcurves.

We now use β_∞ to construct a sequence that is Cauchy with respect to d_φ but not Cauchy with respect to d . Since $d(x_0, x_j) \rightarrow \infty$, it follows that $\ell_d(\beta_j) \rightarrow \infty$ as $j \rightarrow \infty$, and so $\ell_d(\beta_\infty) = \infty$. Hence, by the uniformity of β_∞ , we know that for each $j \geq 3$ the curve β_∞ intersects Ω_j . For each $j \geq 3$, we set $y_j = \beta_\infty(t_j)$ to be the first time β_∞ intersects Ω_j . Then for each $m \in \mathbb{N}$, $d(y_j, y_{j+m}) \geq 2^{j+m-1} - 2^j$, showing that $(y_j)_j$ is not Cauchy with respect to d . Let γ_j be a subcurve of β_∞ with end points y_j and y_{j+1} . Then, as φ is monotone decreasing, we have

$$d_\varphi(y_j, y_{j+1}) \leq \ell_\varphi(\gamma_j) = \int_{\gamma_j} \varphi(d_\Omega(\gamma_j(t))) dt \leq \varphi(2^j/C_U) \ell_d(\gamma_j) \leq \varphi(2^j/C_U) C_U d_\Omega(y_{j+1}) \leq 2C_U 2^j \varphi(2^j/C_U).$$

By the reverse doubling property of φ , and from the above inequality it follows that there is a positive constant C depending solely on C_U and C_φ such that

$$d_\varphi(y_j, y_{j+1}) \leq C 2^j \varphi(2^j).$$

It follows that for each $j \geq 3$ and $m \in \mathbb{N}$,

$$d_\varphi(y_j, y_{j+m}) \leq C \sum_{n=j}^{j+m-1} 2^n \varphi(2^n).$$

The above inequality and (2.2) guarantee that the sequence $(y_j)_j$ is Cauchy with respect to d_φ .

Let $(z_j)_j$ be another sequence in Ω that is Cauchy with respect to d_φ but not with respect to d . Then we know that $\lim_j d_\Omega(z_j) = \infty$. Indeed, if there is some $k_0 \in \mathbb{N}$ such that each $z_j \in \bigcup_{n=0}^{k_0} \Omega_n$, then for $j, k \in \mathbb{N}$ and γ any curve connecting z_j and z_k with $\ell_\varphi(\gamma) \leq \frac{11}{10} d_\varphi(z_j, z_k)$ we have either that γ lies in $\bigcup_{n=0}^{k_0+1} \Omega_n$, in which case

$$\ell_\varphi(\gamma) = \int_{\gamma} \varphi(d_\Omega(\gamma(t))) dt \geq \varphi(2^{k_0+1}) \ell_d(\gamma) \geq \varphi(2^{k_0+1}) d(z_j, z_k),$$

or else that γ intersects Ω_{k_0+2} , in which case

$$\ell_\varphi(\gamma) \geq \int_{\gamma \cap \Omega_{k_0+1}} \varphi(d_\Omega(\gamma(t))) dt \geq \varphi(2^{k_0+1}) \ell_d(\gamma \cap \Omega_{k_0+1}) \geq 2^{k_0} \varphi(2^{k_0+1}).$$

In both of these cases, we get that $\limsup_{j,k \rightarrow \infty} d_\varphi(z_j, z_k) > 0$, violating the d_φ -Cauchy property.

To complete the proof of the lemma, we show that $(z_j)_j$ is d_φ -equivalent to $(y_j)_j$, that is, $\lim_j d_\varphi(z_j, y_j) = 0$. Since $\lim_j d_\Omega(z_j) = \infty$ and $\lim_j d_\Omega(y_j) = \infty$, for each integer $m \geq 2$ both z_j and y_j belong to $\bigcup_{n=m}^{\infty} \Omega_n$ for sufficiently large j . So, for sufficiently large j , with α_j a C_U -uniform curve in Ω with end points z_j and y_j , let $\alpha_j : [0, L] \rightarrow \Omega$ be the standard arclength parametrization of α_j (with respect to the metric d). Then with n_0

the positive integer for which $2^{n_0-1} < C_U \leq 2^{n_0}$,

$$\begin{aligned} \ell_\varphi(\alpha_j|_{[0, L/2]}) &= \sum_{n=m-n_0}^{\infty} \ell_\varphi(\alpha_j \cap \Omega_n) \leq \sum_{n=m-n_0}^{\infty} \varphi(2^n) \ell_d(\alpha_j \cap \Omega_n) \\ &\leq \sum_{n=m-n_0}^{\infty} \varphi(2^n) C_U 2^n. \end{aligned}$$

An analogous treatment of $\alpha_j|_{[L/2, L]}$ then tells us that

$$\ell_\varphi(\alpha_j) \leq 2 C_U \sum_{n=m-n_0}^{\infty} 2^n \varphi(2^n),$$

which goes to zero as $m \rightarrow \infty$ by (2.2). $\square$

Remark 2.4. Here, and in the rest of the paper, n_0 is the positive integer such that $2^{n_0-1} \leq C_U < 2^{n_0}$. The above proof also yields an additional property, namely, that if $y, z \in \bigcup_{n=m}^{\infty} \Omega_n$ for some $m \geq n_0$, then with γ a C_U -uniform curve with end points x and y , we have

$$d_\varphi(y, z) \leq \ell_\varphi(\gamma) \leq 2 C_U \sum_{n=m-n_0}^{\infty} 2^n \varphi(2^n).$$

Remark 2.5. From the fact that $\int_0^\infty \varphi(t) dt$ is finite, we see that whenever $c > 0$, then there is a positive integer N_c such that whenever $n \in \mathbb{N}$ with $\int_{2^n}^\infty \varphi(t) dt \geq c$, we have $n \leq N_c$.

Lemma 2.6. $\partial\Omega_\varphi = \partial\Omega$. Moreover, the extensions of the metrics d and d_φ to $\partial\Omega$ and $\partial\Omega_\varphi$ are locally bi-Lipschitz.

We point out here that this lemma does not require Ω to be a uniform domain, but we do need Ω to be C_q -quasiconvex with respect to the metric d ; in this case the reader should replace C_U in the following proof with C_q , the quasiconvexity constant.

Proof. If $(x_j)_j$ is a sequence in Ω that converges to a point $\zeta \in \partial\Omega$ with respect to the metric d , then we have that this sequence is also Cauchy in Ω_φ . Moreover, for sufficiently large j, k we have that $d(x_j, x_k) < \frac{1}{10C_U}$ and $d(x_j, \zeta) < \frac{1}{10C_U}$. It follows that for sufficiently large j , $x_j \in \Omega_0$, and if γ is a curve in Ω with end points x_j and x_k such that $\ell_\varphi(\gamma) \leq \frac{11}{10} d_\varphi(x_j, x_k)$, then γ cannot leave Ω_0 ; for if it does, then

$$\frac{11}{10} d_\varphi(x_j, x_k) \geq \ell_\varphi(\gamma) = \int_\gamma \varphi(d_\Omega(\gamma(t))) dt \geq \ell_d(\gamma_0) \geq \frac{9}{10},$$

where γ_0 is the largest subcurve of γ with one end point x_j and such that $\gamma_0 \subset \Omega_0$. Since $d_\varphi(x_j, x_k) \leq C_U d(x_j, x_k) < \frac{1}{10}$, this leads to a contradiction.

We now show that the sequence $(x_j)_j$, which is Cauchy with respect to both d and d_φ , cannot be d_φ -equivalent to any d_φ -Cauchy sequence $(y_j)_j$ that is not a d -Cauchy sequence (see Lemma 2.3 for the existence and uniqueness of such a sequence $(y_j)_j$, which is denoted in this paper by ∞). Indeed, as $(y_j)_j$ is not Cauchy with respect to d , it is not equivalent to $(x_j)_j$ with respect to the metric d . Hence there is some $0 < c < 1/(10C_U)$ such that for sufficiently large j (perhaps after passing to a subsequence if necessary), we have that $d(x_j, y_j) > c$. Now, if β is any curve in Ω connecting x_j to y_j , we must then have that β starts from x_j and leaves the ball $B_d(x_j, c) \subset \Omega_0$, and so

$$\ell_\varphi(\beta) \geq \int_{\beta \cap B_d(x_j, c/2)} ds \geq c/2.$$

Taking the infimum over all such β gives us $d_\varphi(x_j, y_j) \geq c/2$. It follows that $(x_j)_j$ cannot be equivalent to $(y_j)_j$ with respect to the metric d_φ , that is, $(x_j)_j$ cannot converge to ∞ in the metric d_φ .

Moreover, if $(x_j)_j$ and $(y_j)_j$ are two non-equivalent d -Cauchy sequences in Ω , converging to two distinct points $\zeta, \eta \in \partial\Omega$, then for sufficiently large j we have that $d(x_j, y_j) \geq \tau = d(\zeta, \eta)/2 > 0$. In this case, any curve γ connecting x_j to y_j in Ω must have length $\ell_d(\gamma) \geq \tau$. If such γ does not stay within Ω_0 , then an argument as above tells us that $\ell_\varphi(\gamma) \geq \frac{9}{10}$ when j is large. If γ stays entirely within Ω_0 , then $\ell_\varphi(\gamma) = \ell_d(\gamma) \geq \tau$. It follows that $d_\varphi(x_j, y_k) \geq \min\{\tau, \frac{9}{10}\} > 0$, and thus the two sequences are not Cauchy-equivalent with respect to the metric d_φ either. That is, $\partial\Omega \subset \partial\Omega_\varphi$.

Now suppose that $(x_j)_j$ is a Cauchy sequence in Ω_φ that does not converge. Then, in particular, there is some $r > 0$ such that for sufficiently large j we have that $x_j \notin B_\varphi(\infty, r)$. It follows then from Remark 2.5 that there is some $k_0 \in \mathbb{N}$ such that when j is sufficiently large, we have $x_j \in \bigcup_{n=0}^{k_0} \Omega_n$. For such sufficiently large $j, k \in \mathbb{N}$, let γ be a curve in Ω with end points x_j, x_k such that $d_\varphi(x_j, x_k) \leq \frac{11}{10} \ell_\varphi(\gamma)$, then we consider two cases. Either γ is entirely inside $\bigcup_{n=0}^{2k_0} \Omega_n$, in which case we have

$$\ell_\varphi(\gamma) \geq \varphi(2^{2k_0}) \ell_d(\gamma) \geq \varphi(2^{2k_0}) d(x_j, x_k),$$

or else γ intersects Ω_{2k_0+1} , in which case, we have that

$$\ell_\varphi(\gamma) \geq \varphi(2^{2k_0}) \ell_d(\gamma \cap \Omega_{2k_0}) \geq 2^{2k_0-1} \varphi(2^{2k_0}).$$

This latter case is not possible for sufficiently large j and k , since by choice, $\lim_{j,k \rightarrow \infty} d_\varphi(x_j, x_k) = 0$. The former case is therefore the only possibility for sufficiently large j, k , and hence $(x_j)_j$ is Cauchy with respect to the original metric d . As this sequence does not converge with respect to d_φ , it follows that it does not converge in Ω with respect to d either; hence, $\partial\Omega_\varphi \subset \partial\Omega$.

The above argument also shows that if $\zeta, \eta \in \partial\Omega$ with $d(\zeta, \eta) \leq \frac{1}{10}$, then $d(\zeta, \eta) \leq d_\varphi(\zeta, \eta) \leq C_U d(\zeta, \eta)$, where we used the quasiconvexity of Ω with respect to the metric d . Thus, the two metrics are locally bi-Lipschitz. $\square$

For $x \in \Omega_\varphi$, we set

$$d_{\Omega_\varphi}(x) := \text{dist}_\varphi(x, \partial\Omega_\varphi) := \inf\{d_\varphi(x, \zeta) : \zeta \in \partial\Omega_\varphi\}.$$

We now consider some preliminary lemmas that will be useful in showing that Ω_φ is uniform.

When m is a non-negative integer and γ is a curve in Ω that intersects both Ω_m and Ω_{m+2} , then

$$\ell_\varphi(\gamma) \geq \frac{\varphi(2^m)}{C_\varphi^2} \ell_d(\gamma \cap \Omega_{m+1}) \geq \frac{\varphi(2^m)}{C_\varphi^2} 2^m. \quad (2.7)$$

Lemma 2.8. *Let $x \in \Omega_m$ for some integer $m \geq 0$. If $y \in \Omega$ is such that*

$$d_\varphi(x, y) < \left[\min \left\{ \frac{10}{11C_\varphi^2} 2^{-2}, \frac{10}{22C_\varphi^2} \right\} \right] \varphi(2^m) 2^m,$$

then

$$\varphi(2^{m+1}) d(x, y) \leq \frac{11}{10} d_\varphi(x, y) \leq C_A \varphi(2^m) d(x, y).$$

In particular, $C_A^{-1} \varphi(2^m) d(x, y) \leq d_\varphi(x, y) \leq C_A \varphi(2^m) d(x, y)$.

Here the constant C_φ is the reverse doubling constant, preventing uncontrolled decay of the dampening function φ . It follows from the above lemma that the two metrics are locally bi-Lipschitz equivalent in Ω , and generate the same topology there. This lemma is applicable even when Ω is not a uniform domain, but we need Ω to be quasiconvex.

Proof. Let γ be a curve in Ω connecting x, y such that $\ell_\varphi(\gamma) \leq \frac{11}{10} d_\varphi(x, y)$. Then

$$\ell_\varphi(\gamma) < \varphi(2^m) 2^{m-2} / C_\varphi^2,$$

and so by (2.7) we have that γ does not intersect Ω_{m+2} . If $m \geq 2$ and γ intersects Ω_{m-2} , then by (2.7) again, we would have $\ell_\varphi(\gamma) \geq \varphi(2^{m-2}) 2^{m-2} / C_\varphi^2 \geq \varphi(2^m) 2^{m-2} / C_\varphi^2$, which again violates the above inequality. It follows that $\gamma \subset \Omega_{m-1} \cup \Omega_m \cup \Omega_{m+1}$, where, for convenience, we set $\Omega_n = \emptyset$ when n is a negative integer. Hence

$$\frac{11}{10} d_\varphi(x, y) \geq \ell_\varphi(\gamma) \geq \varphi(2^{m+1}) \ell_d(\gamma) \geq \varphi(2^{m+1}) d(x, y), \quad (2.9)$$

which proves the first inequality of the desired double inequality claimed in the lemma.

On the other hand, as Ω is a quasiconvex space, we can find a curve β with end points x, y such that $\ell_d(\beta) \leq C_q d(x, y)$ where C_q is the quasiconvexity constant of the metric d on Ω . We consider two cases, $C_q < 2C_\varphi$ and $C_q \geq 2C_\varphi$.

In the first case, $C_q < 2C_\varphi$. Here we use that $d_\varphi(x, y) < \frac{10}{11C_\varphi^2} \varphi(2^m) 2^{m-2}$ and so (2.9) implies that $d(x, y) < C_\varphi^{-1} 2^{m-2}$. Hence, for each z in the trajectory of β ,

$$d_\Omega(z) \leq d_\Omega(x) + \ell_d(\beta) < 2^m + \frac{C_q}{C_\varphi} 2^{m-2} = \left(1 + \frac{C_q}{4C_\varphi}\right) 2^m < 2^{m+1}$$

and

$$d_\Omega(z) \geq d_\Omega(x) - \ell_d(\beta) \geq 2^{m-1} - \frac{C_q}{C_\varphi} 2^{m-2} = A_1 2^{m-1}.$$

Here, $A_1 = 1 - \frac{C_q}{2C_\varphi} > 0$.

In the second case $C_q \geq 2C_\varphi$. Here we use that $d_\varphi(x, y) < \frac{10}{22C_q^2} \varphi(2^m) 2^m$, and so it follows from (2.9) that $d(x, y) < C_\varphi C_q^{-2} 2^{m-1}$. Hence, for each z in the trajectory of β ,

$$d_\Omega(z) \leq d_\Omega(x) + \ell_d(\beta) \leq 2^m + \frac{C_\varphi}{C_q} 2^m = \left(1 + \frac{C_\varphi}{C_q}\right) 2^m \leq 2^{m+1}$$

and

$$d_\Omega(z) \geq d_\Omega(x) - \ell_d(\beta) \geq 2^{m-1} - \frac{C_\varphi}{C_q} 2^{m-1} = A_2 2^{m-1}.$$

Here, $A_2 = 1 - \frac{C_\varphi}{C_q} > 0$.

Let k_0 be the positive integer such that $2^{-k_0} < A_1 \leq 2^{1-k_0}$ if $C_q < 2C_\varphi$ or the positive integer such that $2^{-k_0} < A_2 \leq 2^{1-k_0}$ if $C_q \geq 2C_\varphi$. In either case, it follows that $\beta \subset \bigcup_{n=m-k_0}^{m+1} \Omega_n$, and so

$$d_\varphi(x, y) \leq \ell_\varphi(\beta) \leq \varphi(2^{m-k_0-1}) \ell_d(\beta) \leq C_q C_\varphi^{k_0+1} \varphi(2^m) d(x, y).$$

We set $C_A = C_q C_\varphi^{k_0+1}$ to complete the proof. $\square$

Note that if $x \in \Omega_n$ and $d(x, y) < s 2^n \varphi(2^n)$ for sufficiently small $s > 0$, then $d_\varphi(x, y)$ satisfies the hypothesis of Lemma 2.8. Hence the lemma also tells us that we should think of balls $B_d(x, s 2^n \varphi(2^n))$ as Whitney-type balls in d_φ ; note that the doubling property of φ guarantees that φ satisfies a Harnack-type condition on these balls, as outlined in [4].

Lemma 2.10. *Let $x \in \Omega_m$ for some integer $m \geq n_0 + 2$. Then*

$$C_U C_\varphi \sum_{n=m-n_0}^{\infty} 2^n \varphi(2^n) \geq d_\varphi(x, \infty) \geq \frac{5}{11} \sum_{n=m+1}^{\infty} 2^n \varphi(2^n).$$

Here n_0 is the positive integer such that $2^{n_0-1} \leq C_U < 2^{n_0}$ with C_U the uniformity constant associated with the uniform domain (Ω, d) .

Proof. Let $(x_j)_{j>m}$ be a sequence of points in Ω such that $x_j \in \Omega_j$. Then this sequence is not convergent in Ω . Let β_j be a uniform curve in Ω with end points x, x_j . Note that $\ell_d(\beta_j) \leq C_U d(x, x_j)$ where C_U is the uniformity constant of Ω . Recall from Section 2 that we only consider C_U -uniform curves (with respect to the metric d) whose each subcurve is also C_U -uniform, see [4] for more on this. It follows from the local compactness of Ω and the Arzelà-Ascoli theorem that there is a locally uniformly convergent subsequence of the sequence of curves β_j , that converges to a curve β with one end point x and leaving each compact subdomain of $\overline{\Omega}$; note that each β_j lies in $\Omega \setminus \Omega_0$, and hence so does β . We also have $\lim_{t \rightarrow \infty} d(x, \beta(t)) = \infty$, and so β connects x to ∞ . For each positive integer n let $\widehat{\beta}_n = \beta \cap \Omega_n$. By the uniformity of each β_j we know that β_j does not intersect

Ω_{m-n_0-1} , and hence neither does β . Then

$$\begin{aligned} d_\varphi(x, \infty) &\leq \ell_\varphi(\beta) = \sum_{n=m-n_0}^{\infty} \ell_\varphi(\widehat{\beta}_n) \leq \sum_{n=m-n_0}^{\infty} \varphi(2^{n-1}) \ell_d(\widehat{\beta}_n) \\ &\leq \sum_{n=m-n_0}^{\infty} \varphi(2^{n-1}) C_U d_\Omega(z_n) \\ &\leq C_U C_\varphi \sum_{n=m-n_0}^{\infty} \varphi(2^n) 2^n, \end{aligned}$$

where z_n is a point in $\widehat{\beta}_n$. In particular, this also means that $d_\varphi(x, \infty)$ is finite by (2.2).

Now let $\gamma : [0, \infty) \rightarrow \Omega$ be any curve in Ω such that $\gamma(0) = x$ and $\lim_{n \rightarrow \infty} d(x, \gamma(t)) = \infty$ and $\ell_\varphi(\gamma) \leq \frac{11}{10} d_\varphi(x, \infty)$. Letting $\gamma_n = \gamma \cap \Omega_n$, we see that

$$\frac{11}{10} d_\varphi(x, \infty) \geq \sum_{n=m+1}^{\infty} \ell_\varphi(\gamma_n) \geq \sum_{n=m+1}^{\infty} \varphi(2^n) \ell_d(\gamma_n) \geq \sum_{n=m+1}^{\infty} \varphi(2^n) 2^{n-1}. \quad \square$$

Thanks to the above lemma, we know that Ω_φ is a bounded domain. Indeed, when $x, y \in \Omega$, we have that

$$d_\varphi(x, y) \leq d_\varphi(x, \infty) + d_\varphi(y, \infty) \leq 2C_U C_\varphi \sum_{n=0}^{\infty} 2^n \varphi(2^n).$$

Lemma 2.11. *Let $x \in \Omega_m$ for some non-negative integer m . If $m > 0$, then*

$$C_U C_\varphi \sum_{n=0}^{m+n_0} 2^n \varphi(2^n) \geq d_{\Omega_\varphi}(x) \geq \left(\frac{50}{121} \right) \sum_{n=0}^{m-1} 2^n \varphi(2^n).$$

If $m = 0$, then $d_{\Omega_\varphi}(x) = d_\Omega(x)$. Here, as usual, n_0 is the positive integer such that $2^{n_0-1} \leq C_U < 2^{n_0}$ with C_U the uniformity constant associated with the uniform domain (Ω, d) .

Proof. By Lemma 2.6 we know that $\partial\Omega_\varphi = \partial\Omega$. Let $\zeta \in \partial\Omega$ be such that $d_\Omega(x) = d(x, \zeta)$, and let β be a C_U -uniform curve (with respect to the metric d) with end points x and ζ . Then,

$$2^{m-1} \leq d_\Omega(x) \leq \ell_d(\beta) \leq C_U d(x, \zeta) \leq C_U 2^m \leq 2^{m+n_0}.$$

It follows that the trajectory of β lies in $\bigcup_{n=0}^{m+n_0} \Omega_n$. Therefore,

$$\begin{aligned} \ell_\varphi(\beta) &\leq \sum_{n=0}^{m+n_0} \varphi(2^{n-1}) \ell_d(\beta \cap \Omega_n) \leq C_\varphi \sum_{n=0}^{m+n_0} \varphi(2^n) C_U d_\Omega(z_n) \\ &\leq C_U C_\varphi \sum_{n=0}^{m+n_0} \varphi(2^n) 2^n \\ &= C_U C_\varphi \sum_{n=0}^{m+n_0} \varphi(2^n) 2^n, \end{aligned}$$

where z_n is a point in $\beta \cap \Omega_n$. Hence,

$$d_{\Omega_\varphi}(x) \leq \ell_\varphi(\beta) \leq C_U C_\varphi \sum_{n=0}^{m+n_0} \varphi(2^n) 2^n.$$

If $m \geq 1$, then let $\zeta \in \partial\Omega$ such that $d_{\Omega_\varphi}(x) \geq \frac{10}{11} d_\varphi(x, \zeta)$, and let γ be a curve in Ω connecting x to ζ such that $\ell_\varphi(\gamma) \leq \frac{11}{10} d_\varphi(x, \zeta)$. Then

$$\ell_\varphi(\gamma) \geq \sum_{n=0}^{m-1} \ell_\varphi(\gamma \cap \Omega_n) \geq \sum_{n=0}^{m-1} \varphi(2^n) \ell_d(\gamma \cap \Omega_n) \geq \sum_{n=0}^{m-1} \varphi(2^n) 2^{n-1}.$$

It follows that

$$d_{\Omega_\varphi}(x) \geq \frac{1}{2} \left(\frac{10}{11} \right)^2 \sum_{n=0}^{m-1} \varphi(2^n) 2^n.$$

If $m = 0$, then $\varphi(t) = 1$ for $0 < t \leq 1$ tells us that $d_{\Omega_\varphi}(x) \geq d_\Omega(x)$. $\square$

3 Uniform domain property of Ω_φ .

This section is devoted to the proof of Theorem 1.1, the main theorem of this note.

Lemma 3.1. Suppose that $x \in \Omega_m$ and $y \in \Omega_k$ with $0 \leq m \leq k$. If

$$d_\varphi(x, y) < \frac{5}{44 C_\varphi^{n_0+1} C_q^2 C_U C_A} 2^m \varphi(2^m),$$

then any C_U -uniform curve with respect to the original metric d with end points x, y is a C_1^φ -uniform curve with respect to the metric d_φ , where

$$C_1^\varphi = \max \left\{ C_A C_\varphi^{n_0+1} C_U, \frac{363 C_U}{50} \right\}.$$

Here C_A is the constant from Lemma 2.8, which depends only on C_φ and the quasiconvexity constant C_q .

Proof. We first consider the case $m \geq 1$. By Lemma 2.8, we have that $\varphi(2^m) d(x, y) \leq C_A d_\varphi(x, y)$. It follows that

$$d(x, y) \leq \frac{C_A}{\varphi(2^m)} d_\varphi(x, y) \leq \frac{5}{44 C_\varphi^{n_0+1} C_q^2 C_U} 2^m,$$

and so

$$2^{m-1} \leq d_\Omega(y) \leq d(x, y) + d_\Omega(x) \leq \frac{5}{44 C_\varphi^{n_0+1} C_q^2 C_U} 2^m + 2^{m+1} \leq C_* 2^m.$$

It follows that if β is a C_U -uniform curve with respect to d with end points x, y that $\beta \subset \bigcup_{n=m-n_0}^{m+k_0+n_0} \Omega_n$, where n_0 is the positive integer such that $2^{n_0-1} \leq C_U < 2^{n_0}$, see Remark 2.4, and k_0 is the positive integer such that $2^{k_0-1} \leq C_* < 2^{k_0}$. Hence,

$$\begin{aligned} \ell_\varphi(\beta) &= \sum_{n=m-n_0}^{m+k_0+n_0} \int_{\beta \cap \Omega_n} \varphi(d_\Omega(\beta(t))) dt \leq \sum_{n=m-n_0}^{m+k_0+n_0} \varphi(2^n) \ell_d(\beta \cap \Omega_n) \\ &\leq \varphi(2^{m-n_0-1}) \ell_d(\beta) \\ &\leq C_\varphi^{n_0+1} \varphi(2^m) C_U d(x, y). \end{aligned}$$

Using Lemma 2.8 again, we conclude that

$$d_\varphi(x, y) \geq C_A^{-1} \varphi(2^m) d(x, y) \geq \frac{1}{C_A C_\varphi^{n_0+1} C_U} \ell_\varphi(\beta);$$

that is, β is a quasiconvex curve with respect to the metric d_φ .

Next, if z is a point in the trajectory of β , then by Lemma 2.11,

$$\begin{aligned} d_{\Omega_\varphi}(z) &\geq d_{\Omega_\varphi}(x) - d_\varphi(x, z) \geq \frac{50}{121} \sum_{n=0}^{m-1} 2^n \varphi(2^n) - \ell_\varphi(\beta) \\ &\geq \frac{50}{121} 2^{m-1} \varphi(2^{m-1}) - C_A C_\varphi^{n_0+1} C_U d_\varphi(x, y) \\ &\geq \frac{50}{121} 2^{m-1} \varphi(2^{m-1}) - \frac{5}{44 C_q^2} 2^m \varphi(2^m) \\ &\geq \frac{50}{121} 2^{m-1} \varphi(2^{m-1}) - \frac{5}{22 C_q^2} 2^{m-1} \varphi(2^{m-1}) \\ &= \frac{45}{242} 2^{m-1} \varphi(2^{m-1}). \end{aligned}$$

As

$$\ell_\varphi(\beta) \leq C_A C_\varphi^{n_0+1} C_U d_\varphi(x, y) \leq \frac{5}{44C_q^2} 2^m \varphi(2^m) \leq \frac{5}{22C_q^2} 2^{m-1} \varphi(2^{m-1}),$$

it follows that β is a C_1^φ -uniform curve with respect to the metric d_φ .

Now we consider the case $m = 0$; that is, $x \in \Omega_0$. Then, by the assumption on y , we must have that $y \in \Omega_0 \cup \Omega_1$. If not, then any curve in Ω that connects x to y must have a segment in Ω_1 with length at least 2, and therefore the d_φ -length of all such curves are at least $2\varphi(2) \geq 2/C_\varphi$ which is larger than the assumed bound on $d_\varphi(x, y)$. Moreover, by Lemma 2.8 we have that $d(x, y) \leq 1/(4C_\varphi)$. Hence any C_U -uniform curve (in the metric d) with end points x, y must lie in $\bigcup_{n=0}^{n_0+1} \Omega_n$. Let β be such a curve. We have that $\ell_\varphi(\beta) \leq \ell_d(\beta)$. This implies, by Lemma 2.8, that

$$d_\varphi(x, y) \geq C_A^{-1} d(x, y) \geq C_A^{-1} C_U^{-1} \ell_d(\beta) \geq C_A^{-1} C_U^{-1} \ell_\varphi(\beta),$$

meaning that β is quasiconvex with respect to the metric d_φ .

For z in the trajectory of β , consider the segment $\beta[x, z]$ of β with end points x, z . As we require that subcurves of chosen uniform curves (with respect to the metric d) also be uniform, β has no loops, and so there is only one such segment. If $z \in \Omega_0$, then

$$d_{\Omega_\varphi}(z) = d_\Omega(z) \geq C_U^{-1} \ell_d(\beta[x, z]) \geq C_U^{-1} \ell_\varphi(\beta[x, z]).$$

If $z \in \Omega_j$ for some $0 < j \leq n_0 + 1$, then

$$\begin{aligned} \ell_\varphi(\beta[x, z]) &= \sum_{n=0}^j \int_{\beta[x, z] \cap \Omega_n} \varphi(d_\Omega(\beta(t))) dt \leq \sum_{n=0}^j \varphi(2^n) \ell_d(\beta[x, z] \cap \Omega_n) \\ &\leq \sum_{n=0}^j \varphi(2^n) \ell_d\left(\bigcup_{i=1}^n \beta[x, z] \cap \Omega_i\right) \\ &\leq C_U \sum_{n=0}^j \varphi(2^n) 2^n. \end{aligned}$$

Thus, noting that $2^{j-1} \varphi(2^{j-1}) + 2^j \varphi(2^j) \leq 3(2^{j-1} \varphi(2^{j-1}))$, from Lemma 2.11 it follows that

$$\begin{aligned} d_{\Omega_\varphi}(z) &\geq \frac{50}{121} \sum_{n=0}^{j-1} 2^n \varphi(2^n) = \frac{50}{121} \left[\sum_{n=0}^{j-2} 2^n \varphi(2^n) + 2^{j-1} \varphi(2^{j-1}) \right] \\ &\geq \frac{50}{363} \sum_{n=0}^j 2^n \varphi(2^n) \\ &\geq \frac{50}{363 C_U} \ell_\varphi(\beta_{x,z}). \end{aligned}$$

This shows that β is a C_1^φ -uniform curve with respect to d_φ . □

From equation (2.2) it follows that we can fix a positive integer $m_0 > n_0 + 2$ such that

$$\sum_{n=m_0-n_0}^{\infty} 2^n \varphi(2^n) < \frac{1}{8C_U C_\varphi}. \quad (3.2)$$

Lemma 3.3. Suppose that $x \in \Omega_m$ and $y \in \Omega_k$ with $m_0 \leq m \leq k$. If γ is a curve in Ω with end points x and y such that $\ell_\varphi(\gamma) \leq \frac{11}{10} d_\varphi(x, y)$, then γ is a $\frac{1331}{669}$ -uniform curve with respect to the metric d_φ .

Proof. Suppose that $x \in \Omega_m$ and $y \in \Omega_k$ with $k \geq m \geq m_0$. Then by Lemma 2.10, $d_\varphi(x, y) \leq 2C_U C_\varphi \sum_{n=m-n_0}^{\infty} 2^n \varphi(2^n) < \frac{1}{4}$, and moreover, by Lemma 2.11 we also have

$$d_{\Omega_\varphi}(x) \geq \frac{50}{121} \sum_{n=0}^{m-1} 2^n \varphi(2^n) \geq \frac{50}{121}.$$

Similar statement holds also for $d_{\Omega_\varphi}(y)$. Let γ be a curve in Ω with end points x, y such that $\ell_\varphi(\gamma) \leq \frac{11}{10}d_\varphi(x, y)$. Then $\ell_\varphi(\gamma) < \frac{11}{40}$. Let z be a point in the trajectory of γ ; then,

$$d_{\Omega_\varphi}(z) \geq d_{\Omega_\varphi}(x) - d_\varphi(x, z) \geq \frac{50}{121} - \ell_\varphi(\gamma) \geq \frac{50}{121} - \frac{11}{40} = \frac{669}{4840}.$$

It follows that

$$d_{\Omega_\varphi}(z) \geq \frac{669}{1331} \ell_\varphi(\gamma),$$

that is, γ is a $\frac{1331}{669}$ -uniform curve with respect to the metric d_φ . $\square$

In what follows, we denote by λ and Λ the numbers

$$\lambda = \min_{0 \leq n \leq m_0 + n_0} 2^n \varphi(2^n) \quad \text{and} \quad \Lambda = \max_{0 \leq n \leq m_0 + n_0} 2^n \varphi(2^n). \quad (3.4)$$

Lemma 3.5. Suppose that $x \in \Omega_m$ and $y \in \Omega_k$ with $0 \leq m \leq k \leq m_0$, and

$$\frac{5}{22C_\varphi^2} 2^m \varphi(2^m) \leq d_\varphi(x, y) < C 2^m \varphi(2^m).$$

Any C_U -uniform curve with respect to the original metric d with end points x, y lying entirely in $\bigcup_{j=0}^{m_0+n_0} \Omega_j$ is a C_2^φ -uniform curve with respect to the metric d_φ .

If the uniform curve is not entirely contained in $\bigcup_{j=0}^{m_0+n_0} \Omega_j$, then with z_1, z_2 two points in the trajectory of the curve with the segment between x and z_1 , and the segment between z_2 and y lying in $\bigcup_{j=0}^{m_0+n_0} \Omega_j$, we can replace the segment between z_1 and z_2 by a $11/10$ -quasiconvex curve with respect to d_φ with end points z_1, z_2 to obtain a C_2^φ -uniform curve with respect to the metric d_φ .

Here

$$C_2^\varphi = \frac{2000}{669} \frac{2CC_1^\varphi}{T} \frac{\Lambda}{\lambda} \left(2T_0 + \frac{121C_\varphi^2}{20\lambda} \right).$$

Moreover, in both cases, for each point z in the trajectory of β (resp. γ), we have that $C_2^\varphi d_{\Omega_\varphi}(z)$ is minorized by the length of the entire curve with respect to the metric d_φ .

Proof. Let β be a C_U -uniform curve (with respect to d) with end points x, y with arclength (with respect to d) parametrization $\beta : [0, L] \rightarrow \Omega$. We can find $t_1, t_2, \dots, t_{J-1} \in (0, L)$ such that

$$0 = t_0 < t_1 < t_2 < \dots < t_{J-1} < t_J = L$$

and for $j = 1, \dots, J$,

$$d_\varphi(\beta(t_j), \beta(t_{j-1})) < \frac{5}{22C_\varphi^{n_0+1}C_U C_A} 2^{m_j} \varphi(2^{m_j}) = T 2^{m_j} \varphi(2^{m_j})$$

with

$$d_\varphi(\beta(t_j), \beta(t_{j-1})) \geq T 2^{m_j-1} \varphi(2^{m_j}).$$

Here m_j is chosen such that either $\beta(t_j) \in \Omega_{m_j}$ or $\beta(t_{j-1}) \in \Omega_{m_j}$. With $z_j = \beta(t_j)$, note by the hypotheses of the lemma that when $m_j \leq m_0 + n_0$,

$$\begin{aligned} d_\varphi(z_j, z_{j-1}) &< T 2^{m_j} \varphi(2^{m_j}) \leq T \Lambda \frac{22C_\varphi^2}{5 \cdot 2^m \varphi(2^m)} d_\varphi(x, y) \\ &\leq \frac{22C_\varphi^2}{5} T \frac{\Lambda}{\lambda} d_\varphi(x, y). \end{aligned}$$

The remaining proof is split into two cases.

Case 1: $\beta \subset \bigcup_{j=0}^{m_0+n_0} \Omega_j$. It follows that $0 \leq m_j \leq m_0 + n_0$. Then by the C_U -uniformity of β with respect to the metric d , we have that for the midpoint $z \in \beta$,

$$2^{m_0+n_0} \geq d_\Omega(z) \geq \frac{1}{C_U} \frac{\ell_d(\beta)}{2} \geq \frac{1}{2C_U} d(x, y),$$

that is, $d(x, y) \leq 2^{m_0+n_0+1} C_U$. Hence

$$J \leq \frac{\ell_d(\beta)}{T\lambda} \leq \frac{C_U d(x, y)}{T\lambda} \leq \frac{2^{m_0+n_0+1} C_U^2}{T\lambda}.$$

Applying Lemma 3.1 to each subcurve β_j connecting $z_j = \beta(t_j)$ and $z_{j-1} = \beta(t_{j-1})$, we have that

$$\begin{aligned} \ell_\varphi(\beta) &= \sum_{j=1}^J \ell_\varphi(\beta_j) \leq C_1^\varphi \sum_{j=1}^J d_\varphi(z_j, z_{j-1}) \leq C_1^\varphi J \frac{22C_\varphi^2}{5} T \frac{\Lambda}{\lambda} d_\varphi(x, y) \\ &\leq C_1^\varphi \frac{22C_\varphi^2}{5} T \frac{\Lambda}{\lambda} \frac{2^{m_0+n_0+1} C_U^2}{T\lambda} d_\varphi(x, y). \end{aligned}$$

We set

$$T_0 := C_1^\varphi \frac{22C_\varphi^2}{5} \frac{\Lambda}{\lambda} \frac{2^{m_0+n_0+1} C_U^2}{\lambda}.$$

Moreover, any z in the trajectory of β is in the trajectory of β_j for some j and so, applying Lemma 3.1 to this curve,

$$\begin{aligned} d_{\Omega_\varphi}(z) &\geq \frac{1}{C_1^\varphi} \ell_\varphi(\beta_j) \geq \frac{1}{C_1^\varphi} d_\varphi(z_j, z_{j-1}) \geq \frac{1}{C_1^\varphi} \frac{T}{2} \lambda \frac{1}{C 2^m \varphi(2^m)} d_\varphi(x, y) \\ &\geq \frac{T}{2C C_1^\varphi} \frac{\lambda}{\Lambda} d_\varphi(x, y) \\ &\geq \frac{T}{2C C_1^\varphi} \frac{\lambda}{\Lambda} \frac{1}{T_0} \ell_\varphi(\beta). \end{aligned}$$

Case 2: There is some z in the trajectory of β such that $d_\Omega(z) > 2^{m_0+n_0}$. Let z_1, z_2 be two points in the trajectory of β such that $d_\Omega(z_1) = d_\Omega(z_2) = 2^{m_0+n_0}$ and $\beta[x, z_1], \beta[z_2, y]$ lie entirely in $\bigcup_{j=0}^{m_0+n_0} \bar{\Omega}_j$. In this case, we replace $\beta[z_1, z_2]$ with a curve $\hat{\beta}$ with end points z_1, z_2 such that $\ell_\varphi(\hat{\beta}) \leq \frac{11}{10} d_\varphi(z_1, z_2)$. By Lemma 2.10 and by (3.2), we have that

$$\ell_\varphi(\hat{\beta}) \leq \frac{11}{10} [d_\varphi(z_1, \infty) + d_\varphi(z_2, \infty)] \leq \frac{11}{40} \leq \frac{11}{40} \frac{22C_\varphi^2}{\lambda} d_\varphi(x, y).$$

Considering the subdivisions of $\beta[x, z_1]$ and $\beta[z_2, y]$ as before, we get

$$\ell_\varphi(\beta[x, z_1]) \leq \sum_{j=1}^{J_1} d_\varphi(z_j, z_{j-1}) \leq T_0 d_\varphi(x, y)$$

and

$$\ell_\varphi(\beta[z_2, y]) \leq \sum_{j=J_2}^J d_\varphi(z_j, z_{j-1}) \leq T_0 d_\varphi(x, y).$$

Here we used the fact that both J_1 and $J - J_2$ satisfy the estimates given in Case 1 for J . Thus, with γ the concatenation of the three curves $\beta[x, z_1], \hat{\beta}$, and $\beta[z_2, y]$, we obtain

$$\ell_\varphi(\gamma) \leq \left(2T_0 + \frac{121C_\varphi^2}{20\lambda} \right) d_\varphi(x, y).$$

Let $z \in \gamma$. If $z \in \beta[x, z_1]$ or if $z \in \beta[z_2, y]$, then as in Case 1 above, we obtain

$$d_{\Omega_\varphi}(z) \geq \frac{T}{2C C_1^\varphi} \frac{\lambda}{\Lambda} d_\varphi(x, y) \geq \frac{T}{2C C_1^\varphi} \frac{\lambda}{\Lambda} \left(2T_0 + \frac{121C_\varphi^2}{20\lambda} \right)^{-1} \ell_\varphi(\gamma).$$

If $z \in \hat{\beta}$, then by Lemma 3.3,

$$d_{\Omega_\varphi}(z) \geq \frac{669}{1331} \ell_\varphi(\hat{\beta}).$$

Hence

$$d_{\Omega_\varphi}(z) \geq d_{\Omega_\varphi}(z_1) - d_\varphi(z, z_1) \geq d_{\Omega_\varphi}(z_1) - \ell_\varphi(\hat{\beta})$$

and so, by the inequality above, we have

$$d_{\Omega_\varphi}(z) \geq \frac{669}{2000} d_{\Omega_\varphi}(z_1) \geq \frac{669}{2000} \frac{T}{2C} \frac{\lambda}{C_1^\varphi} \frac{1}{\Lambda} \left(2T_0 + \frac{121C_\varphi^2}{20\lambda} \right)^{-1} \ell_\varphi(\gamma). \quad \square$$

Recall the definition of λ and Λ from (3.4) above.

Lemma 3.6. Suppose that $x \in \Omega_m$ and $y \in \Omega_k$ with $0 \leq m \leq k \leq m_0$. Then,

$$d_\varphi(x, y) \leq \frac{2^{m_0+n_0+1} C_U^2}{\lambda} 2^m \varphi(2^m).$$

Proof. Suppose that x, y are as in the hypothesis of the lemma, and that

$$d_\varphi(x, y) > \frac{2^{m_0+n_0+1} C_U^2}{\lambda} 2^m \varphi(2^m).$$

Let β be a C_U -uniform curve (with respect to d) with end points x, y . By the above supposition, there is some point z in the trajectory of β such that $d_\Omega(z) > 2^{m_0+n_0}$. Let z_1, z_2 be two points in the trajectory of β such that $d_\Omega(z_1) = d_\Omega(z_2) = 2^{m_0}$ and $\beta[x, z_1], \beta[z_2, y]$ lie entirely in $\bigcup_{j=0}^{m_0} \overline{\Omega_j}$. We replace $\beta[z_1, z_2]$ with a curve $\hat{\beta}$ with end points z_1, z_2 such that $\ell_\varphi(\hat{\beta}) \leq \frac{11}{10} d_\varphi(z_1, z_2)$. By the supposition assumed at the beginning of the proof again, with $C = \frac{2^{m_0+n_0+1} C_U^2}{\lambda}$, we have

$$\ell_\varphi(\beta[x, z_1]) \leq \ell_d(\beta[x, z_1]) \leq C_U d_\Omega(z_1) = C_U 2^{m_0} \leq \frac{C_U 2^{m_0}}{\lambda C} d_\varphi(x, y).$$

Similarly, we get

$$\ell_\varphi(\beta[z_2, y]) \leq \frac{C_U 2^{m_0}}{\lambda C} d_\varphi(x, y).$$

Moreover, by Lemma 2.10 and (3.2),

$$\ell_\varphi(\hat{\beta}) \leq \frac{11}{10} d_\varphi(z_1, z_2) \leq \frac{11}{40} \leq \frac{11}{40} \frac{1}{C\lambda} d_\varphi(x, y).$$

It follows that

$$\ell_\varphi(\gamma) \leq \left(\frac{2^{m_0+1} C_U}{\lambda C} + \frac{11}{40C\lambda} \right) d_\varphi(x, y) < d_\varphi(x, y),$$

which is not possible. □

Lemma 3.7. Suppose that $x \in \Omega_m$ and $y \in \Omega_k$ with $0 \leq m < m_0 < k$. If

$$d_\varphi(x, y) \geq \frac{5\lambda}{44C_\varphi^{n_0+1} C_q^2 C_U C_A}$$

then with β a C_U -uniform curve (with respect to the metric d) with end points x, y and with z_1 a point in the trajectory of β such that $d_\Omega(z_1) = 2^{m_0}$ and $\beta[x, z_1]$ contained in $\bigcup_{j=0}^{m_0} \overline{\Omega_j}$, and $\hat{\beta}$ a curve with end points z_1 and y such that $\ell_\varphi(\hat{\beta}) \leq \frac{11}{10} d_\varphi(z_1, y)$, the concatenation γ of $\beta[x, z_1]$ and $\hat{\beta}$ is a C_3^φ -uniform curve with respect to the metric d_φ with end points x, y .

Here C_3^φ is the larger of the two following numbers:

$$C_2^\varphi + \frac{44C_\varphi^{n_0+1} C_q^2 C_U C_A}{5\lambda} \left(\frac{C_2^\varphi}{2C_\varphi} + \frac{11}{40} \right),$$

$$C_2^\varphi \left[1 + \left(\frac{11}{40C_2^\varphi} + \frac{1}{2C_\varphi} \right) \frac{2000 C_2^\varphi}{669\lambda} \right].$$

Proof. Let β , z_1 , $\widehat{\beta}$, and γ be as in the statement of the lemma. Then by Lemma 2.10, equation (3.2), and Lemma 3.5,

$$\begin{aligned}\ell_\varphi(\gamma) &= \ell_\varphi(\beta[x, z_1]) + \ell_\varphi(\widehat{\beta}) \leq C_2^\varphi d_\varphi(x, z_1) + \frac{11}{40} \\ &\leq C_2^\varphi [d_\varphi(x, y) + d_\varphi(z_1, y)] + \frac{11}{40} \\ &\leq C_2^\varphi d_\varphi(x, y) + \left(\frac{C_2^\varphi}{2C_\varphi} + \frac{11}{40} \right) \\ &\leq \left[C_2^\varphi + \left(\frac{C_2^\varphi}{2C_\varphi} + \frac{11}{40} \right) \frac{44C_\varphi^{n_0+1}C_q^2C_UC_A}{5\lambda} \right] d_\varphi(x, y),\end{aligned}$$

showing that γ is quasiconvex with respect to d_φ .

Now, if z is a point in the trajectory of $\beta[x, z_1]$, then by Lemma 3.5 we have that

$$d_{\Omega_\varphi}(z) \geq \frac{1}{C_2^\varphi} \ell_\varphi(\beta[x, z_1]).$$

If z is a point in $\widehat{\beta}$, then

$$\begin{aligned}d_{\Omega_\varphi}(z) &\geq d_{\Omega_\varphi}(z_1) - d_\varphi(z, z_1) \geq \frac{1}{C_2^\varphi} \ell_\varphi(\beta[x, z_1]) - \frac{1}{2C_\varphi} \\ &\geq \frac{1}{C_2^\varphi} \ell_\varphi(\gamma) - \left(\frac{\ell_\varphi(\widehat{\beta})}{C_2^\varphi} + \frac{1}{2C_\varphi} \right) \\ &\geq \frac{1}{C_2^\varphi} \ell_\varphi(\gamma) - \left(\frac{11}{40C_2^\varphi} + \frac{1}{2C_\varphi} \right).\end{aligned}$$

Also, by Lemma 3.3,

$$d_{\Omega_\varphi}(z) \geq \frac{669}{1331} \ell_\varphi(\widehat{\beta}),$$

and so

$$d_{\Omega_\varphi}(z) \geq \frac{1}{C_2^\varphi} \ell_\varphi(\beta[x, z_1]) - \ell_\varphi(\widehat{\beta}) \geq \frac{2^{m_0}\varphi(2^{m_0})}{C_2^\varphi} - \frac{1331}{661} d_{\Omega_\varphi}(z),$$

from whence we obtain

$$\frac{2000}{669} d_{\Omega_\varphi}(z) \geq \frac{\lambda}{C_2^\varphi}.$$

Thus, we finally get

$$\frac{1}{C_2^\varphi} \ell_\varphi(\gamma) \leq \left[1 + \left(\frac{11}{40C_2^\varphi} + \frac{1}{2C_\varphi} \right) \frac{2000C_2^\varphi}{669\lambda} \right] d_{\Omega_\varphi}(z),$$

implying that γ is a C_3^φ -uniform curve with respect to d_φ . $\square$

Lemma 3.8. Suppose that $x \in \Omega$ and $y = \infty$. Then, there exists a C_4^φ -uniform curve with respect to d_φ with end points x, y . Here

$$C_4^\varphi = \max \left\{ \frac{1331}{669}, C_3^\varphi \right\}.$$

Proof. Let $x \in \Omega_m$ for some non-negative integer m . If $m \geq m_0$, then as in the proof of Lemma 2.10 we can find a curve β beginning from x and with $\lim_{t \rightarrow \infty} \beta(t) = \infty$, such that $\ell_\varphi(\beta) \leq \frac{11}{10} d_\varphi(x, \infty) < \frac{11}{80}$. Here $\beta : [0, \infty) \rightarrow \Omega$. By considering $x, \beta(t)$, and $\beta|_{[0, t]}$ in Lemma 3.3, we see that $\beta[0, t]$ is a $\frac{1331}{669}$ -uniform curve with respect to the metric d_φ for each $t > 0$. It follows that β is a $\frac{1331}{669}$ -uniform curve with respect to d_φ as well.

Now we consider the case that $m < m_0$. Let β be a C_U -uniform curve (with respect to the metric d) as constructed in Lemma 2.3 such that $\beta : [0, \infty) \rightarrow \Omega$ with $\lim_{t \rightarrow \infty} \beta(t) = \infty$. We fix $k \geq m_0 + n_0$ such that for each $z \in \Omega_k$ we have $d_\varphi(z, x) \geq \frac{5\lambda}{44C_\varphi^{n_0+1}C_q^2C_UC_A}$ as in Lemma 3.7. If no such k exists, then we can directly

apply Lemma 3.7 to β to see that β is a C_3^φ -uniform curve. With the choice of such k , let $\tau = \inf\{t > 0 : \beta(t) \in \bigcup_{j=k}^\infty \Omega_j\}$, and we set γ to be the concatenation of $\beta|_{[0,\tau]}$ with a curve $\hat{\beta}$ with end points $\beta(\tau)$ and ∞ such that $\ell_\varphi(\hat{\beta}) \leq \frac{11}{10}d_\varphi(\beta(\tau), \infty)$. An application of Lemma 3.7 now tells us that γ is a C_3^φ -uniform curve with respect to the metric d_φ .

By combining the above two cases, we obtain a C_4^φ -uniform curve with respect to the metric d_φ and connecting x to ∞ ; here

$$C_4^\varphi = \max\left\{\frac{1331}{669}, C_3^\varphi\right\}. \quad \square$$

Now we are ready to prove the main theorem of this note.

Proof of Theorem 1.1. The second claim of the theorem was established in Section 2, and so we now focus on proving that Ω_φ is a uniform domain. To this end, let $x, y \in \Omega_\varphi$ with $x \neq y$. If $x = \infty$ or $y = \infty$, then by Lemma 3.8 we have a C_4^φ -uniform curve with respect to d_φ connecting x to y . So it only remains to consider when $x, y \in \Omega_\varphi \setminus \{\infty\} = \Omega$.

Let m, k be two non-negative integers such that $x \in \Omega_m$ and $y \in \Omega_k$. Without loss of generality, we assume that $m \leq k$.

With n_0 and m_0 positive integers such that $2^{n_0-1} \leq C_U < 2^{n_0}$ and $m_0 \geq n_0 + 2$ with $\sum_{n=m_0-n_0}^\infty 2^n \varphi(2^n) < (8C_U C_\varphi)^{-1}$ as in (3.2), we consider three cases.

1. $m_0 \leq m \leq k$. In this case, by Lemma 3.3 we have a $\frac{1331}{669}$ -uniform curve with respect to d_φ connecting x to y .
2. $0 \leq m \leq k \leq m_0$. In this case, Lemma 3.6 we know that $d_\varphi(x, y) \leq 2^{m_0+n_0+1} C_U^2 \lambda^{-1} 2^m \varphi(2^m)$. Hence, by Lemma 3.1 and by Lemma 3.5 (with $C = 2^{m_0+n_0+1} C_U^2 \lambda^{-1}$), there is a $\max\{C_1^\varphi, C_2^\varphi\}$ -uniform curve, with respect to the metric d_φ , connecting x to y .
3. $0 \leq m < m_0 < k$. Then by Lemma 3.1 and Lemma 3.7 there is a $\max\{C_1^\varphi, C_3^\varphi\}$ -uniform curve with respect to the metric d_φ with end points x and y .

Since the above cases exhaust all the possibilities of $x, y \in \Omega$, it follows that Ω_φ is A_φ -uniform with respect to the metric d_φ , with

$$A_\varphi = \max\{C_1^\varphi, C_2^\varphi, C_3^\varphi, C_4^\varphi, \frac{1331}{669}\}. \quad \square$$

Acknowledgement: N.S. is partially supported by the NSF (U.S.A.) grant DMS #2054960. Part of the work on this paper was done while N.S. was visiting MSRI in Spring 2022 to participate in a program supported by the NSF (U.S.A.) grant DMS #1928930. She wishes to thank MSRI for its kind hospitality. We also thank the kind referee for valuable comments pointing out inaccuracies in early manuscripts of the paper. Authors state no conflict of interest.

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