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Herbert Spohn

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Herbert Spohn[a]

AFFILIATIONS

Departments of Mathematics and Physics, Technical University Munich, Boltzmannstr. 3, 85747 Garching, Germany

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[a]**Author to whom correspondence should be addressed:** spohn@ma.tum.de

ABSTRACT

Ablowitz and Ladik discovered a discretization that preserves the integrability of the nonlinear Schrödinger equation in one dimension. We compute the generalized free energy of this model and determine the generalized Gibbs ensemble averaged fields and their currents. They are linked to the mean-field circular unitary matrix ensemble. The resulting hydrodynamic equations follow the pattern already known from other integrable many-body systems. The discretized modified Korteweg–de-Vries equation is also studied, which turns out to be related to the beta Jacobi log gas.

I. INTRODUCTION

A famous integrable classical field theory in 1 + 1 dimensions is the nonlinear Schrödinger equation (NLS). In the defocusing case, the wave field, $\psi(x,t) \in \mathbb{C}$, is governed by

$$\mathrm{i}\partial_t \psi = -\partial_x^2 \psi + 2|\psi|^2 \psi. \tag{1.1}$$

While many properties of this equation have been studied,[1] given the more recent interest in generalized hydrodynamics,[2–5] our goal is to investigate the Euler type spacetime scale for this nonlinear wave equation.

In hydrodynamics, one considers random initial data with an energy far above the ground state energy. The resulting physical picture is based on the notion of local equilibrium. In a small cell, still containing many particles, the system is in one of its equilibrium states. In approximation, the equilibrium parameters are changing slowly on the scale set by the inter-particle distance and evolve according to a system of coupled hyperbolic conservation laws. In a conventional strongly interacting fluid in one dimension, equilibrium is labeled by three parameters. In strong contrast, for integrable systems, the time-stationary states require an extensive number of parameters. Thus, the first step in any derivation of hydrodynamic equations consists of a detailed study of stationary states.

For the NLS, the densities of the locally conserved fields are known (see Ref. 6 and references therein). The beginning of the list reads

$$Q^{[0]}(x) = |\psi(x)|^2, \qquad Q^{[1]}(x) = -\mathrm{i}\bar{\psi}(x)\partial_x\psi(x), \qquad Q^{[2]}(x) = |\partial_x\psi(x)|^2 + |\psi(x)|^4. \tag{1.2}$$

For a given bounded interval $\Lambda \subset \mathbb{R}$, the total conserved quantities then become

$$Q_\Lambda^{[n]} = \int_\Lambda \mathrm{d}x Q^{[n]}(x), \quad n = 0, 1, \ldots. \tag{1.3}$$

Formally, the time-stationary generalized Gibbs ensembles (GGE) are of the form

$$\exp\left[-\sum_{n=0}^{\infty} \mu_n Q_\Lambda^{[n]}\right] \prod_{x\in\Lambda} \mathrm{d}^2\psi(x), \tag{1.4}$$

where μ_n's are suitable chemical potentials. For the thermal case, $n = 0, 1, 2$, much work has been invested to construct a proper probability measure (see Refs. 7–10 for a very partial account). The basic idea of the construction is easily explained. Obviously, the Lebesgue measure makes sense upon lattice discretizing Λ. However, the limit of zero lattice spacing is ill-defined within standard measure theory. Multiplying lattice Lebesgue with $\exp\left(-\beta \int |\partial_x \psi(x)|^2\right)$, the normalized limit converges to a $\mathbb{R}^2$-valued Brownian motion over Λ. Now, $x \mapsto |\psi(x)|^4$ is a somewhat singular function on path space. The technical challenge is to prove that its exponential is integrable with respect to Brownian motion. As a separate issue, one has to show that the so-constructed measure is invariant under the NLS dynamics. Recently, such a method has been extended to a much larger class of generalized Gibbs measures.[11] While the precise statement in Ref. 11 is more complicated, the sum in (1.4) is restricted to some highest even n, and as *a priori* measure, Brownian motion is replaced by the well-defined Gaussian measure with energy,

$$Q_\Lambda^{[0]} + \int_\Lambda \mathrm{d}x |\partial_x^{n/2} \psi(x)|^2. \tag{1.5}$$

As for $n = 2$, the technical part is to establish that for an appropriate choice of chemical potentials, the exponential of all lower order terms can be integrated with respect to this Gaussian measure. Very roughly, the such constructed measure is concentrated on $(n-2)$ times differentiable paths. Time-stationarity is established separately. To go beyond such an existence result seems to be a difficult problem.

In numerical simulations of NLS, one discretizes the equation. While generically this would break integrability, surprisingly enough, in many cases, there is one very specific discretization for which integrability is maintained. For NLS, such a discretization was discovered by Ablowitz and Ladik.[12–14] For convenience, we will use here AL as acronym [rather than, integrable nonlinear Schroedinger equation, as in Ref. 1], and our contribution is focused on the Ablowitz–Ladik system.

A further example in the same spirit is the classical sinh-Gordon equation,

$$\partial_t^2 \phi - \partial_x^2 \phi + \sinh\, \phi = 0, \tag{1.6}$$

with $\phi(x, t)$ being a real-valued wave field. In Ref. 15, the hydrodynamic equations of this nonlinear wave equation are derived and studied. The reported numerical simulations are based on an analytic continuation of the discretized sine-Gordon equation discovered in Ref. 16. A further case is the integrable Landau–Lifshitz model of a one-dimensional magnet. Here, the spin field is a three-vector, $\vec{S}(x, t)$, with $|\vec{S}| = 1$, and the equations of motion are

$$\partial_t \vec{S} = \vec{S} \times \partial_x^2 \vec{S} + \vec{S} \times \mathrm{J}\vec{S}, \quad \mathrm{J} = \mathrm{diag}(0, 0, \delta). \tag{1.7}$$

The naive discretization is non-integrable. For recent investigations, we refer to Refs. 17 and 18 and references therein. The integrable discretization was discovered in Refs. 19 and 20 (see also Ref. 21). A numerical simulation of the integrable chain is reported in Ref. 22.

As we will see, the AL lattice equations are structurally rather similar to the classical Toda lattice, for which fairly detailed notes are available.[23] When pointing out such similarity, we merely refer to these notes. However, our text is essentially self-contained and the reader may as well just ignore the cross links.

As other integrable wave equations, the NLS admits soliton solutions. Rather than assuming an initial state that is locally GGE, random initial conditions can be imposed in the form of a solitons gas, meaning that the location and velocity of a soliton are random, similar to a classical gas of point particles. In approximation, on large scales, the time evolution of such a gas can be described by a coupled set of hyperbolic conservation laws. For more details, see a very recent review article by El,[24] who used the Korteweg–de-Vries equation as his most prominent example.

II. THE PERIODIC AL SYSTEM

Upon discretization, the wave field is over the one-dimensional lattice $\mathbb{Z}$, $\psi_j(t) \in \mathbb{C}$, and is governed by

$$\mathrm{i}\frac{d}{dt}\psi_j = -\psi_{j-1} + 2\psi_j - \psi_{j+1} + |\psi_j|^2(\psi_{j-1} + \psi_{j+1}). \tag{2.1}$$

Hence,

$$\mathrm{i}\frac{d}{dt}\psi_j = -(1-|\psi_j|^2)(\psi_{j-1}+\psi_{j+1}) + 2\psi_j. \tag{2.2}$$

Setting $\alpha_j(t) = \mathrm{e}^{2\mathrm{i}t}\psi_j(t)$, one arrives at the standard version

$$\frac{d}{dt}\alpha_j = \mathrm{i}\rho_j^2(\alpha_{j-1}+\alpha_{j+1}), \quad \rho_j^2 = 1-|\alpha_j|^2. \tag{2.3}$$

Clearly, the natural phase space is $\alpha_j \in \mathbb{D}$ with the unit disk $\mathbb{D} = \{z \| z| \le 1\}$. In principle, whenever $\alpha_j(t)$ hits the boundary of $\mathbb{D}$, it freezes and thereby decouples the system. As we will discuss, a conservation law ensures that, if initially away from the boundary, the solution will stay so forever.

Our main focus is the generalized free energy, for which the standard setup is a finite ring of N sites, labeled $j = 0, \dots, N-1$ with periodic boundary conditions, $\alpha_{j+N} = \alpha_j$. While the periodic system is integrable, there seems to be no method for obtaining its generalized free energy in the limit $N \to \infty$. However, employing a judicious choice of boundary conditions, such a task becomes feasible. Therefore, we have to separately discuss the ring and a segment with boundary conditions.

Conserved fields. We consider a ring of N sites. The evolution equations are of Hamiltonian form by regarding α and its complex conjugate $\bar{\alpha}$ as canonically conjugate variables and introducing the weighted Poisson bracket,

$$\{f,g\}_{\mathrm{AL}} = \mathrm{i}\sum_{j=0}^{N-1}\rho_j^2\left(\frac{\partial f}{\partial\bar{\alpha}_j}\frac{\partial g}{\partial\alpha_j} - \frac{\partial f}{\partial\alpha_j}\frac{\partial g}{\partial\bar{\alpha}_j}\right). \tag{2.4}$$

The Hamiltonian of the AL system reads

$$H_N = -\sum_{j=0}^{N-1}(\alpha_{j-1}\bar{\alpha}_j + \bar{\alpha}_{j-1}\alpha_j). \tag{2.5}$$

One readily checks that indeed

$$\frac{d}{dt}\alpha_j = \{\alpha_j, H_N\}_{\mathrm{AL}} = \mathrm{i}\rho_j^2(\alpha_{j-1}+\alpha_{j+1}). \tag{2.6}$$

The next step is to find out the locally conserved fields. Guided by other integrable models, the convenient tool is a Lax matrix, if available. Nenciu[25,26] discovered that this role is played by a Cantero–Moral–Velázquez (CMV) matrix.[27–30] The basic building blocks are the 2×2 matrices, which requires N to be *even* because of periodic boundary conditions. One defines

$$\Xi_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} \tag{2.7}$$

and forms the $N\times N$ matrices

$$L_N = \mathrm{diag}(\Xi_0, \Xi_2, \dots, \Xi_{N-2}), \quad (M_N)_{i,j=1,\dots,N-2} = \mathrm{diag}(\Xi_1, \Xi_3, \dots, \Xi_{N-3}), \tag{2.8}$$

together with $(M_N)_{0,0} = -\alpha_{N-1}$, $(M_N)_{0,N-1} = \rho_{N-1}$, $(M_N)_{N-1,0} = \rho_{N-1}$, and $(M_N)_{N-1,N-1} = \bar{\alpha}_{N-1}$. More pictorially, L_N corresponds to the blocking $(0,1),\dots,(N-2,N-1)$, while M_N uses the by 1 shifted blocking $(1,2),\dots,(N-1,0)$. The CMV matrix associated with the coefficients $\alpha_0,\dots,\alpha_{N-1}$ is then given by

$$C_N = L_N M_N. \tag{2.9}$$

Obviously, L_N, M_N are unitary and so is C_N. The eigenvalues of C_N are denoted by $\mathrm{e}^{\mathrm{i}\vartheta_j}$, $\vartheta_j \in [0,2\pi]$, $j = 1,\dots,N$. Of course, the eigenvalues depend on N, which is suppressed in our notation.

Next, for a general matrix, A, we define the + operation as

$$(A_+)_{i,j} = \begin{cases} A_{i,j} & \text{if } i < j, \\ \frac{1}{2}A_{i,j} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases} \tag{2.10}$$

Then, one version of the Lax pair reads

$$\{C_N, \mathrm{tr}(C_N)\}_{\mathrm{AL}} = \mathrm{i}[C_N, (C_N)_+], \qquad \{C_N, \mathrm{tr}(C_N^*)\}_{\mathrm{AL}} = \mathrm{i}[C_N, (C_{N+})^*]. \tag{2.11}$$

Since the Poisson bracket acts as a derivative, one deduces

$$\{(C_N)^n, \mathrm{tr}(C_N)\}_{\mathrm{AL}} = \sum_{m=0}^{n-1} (C_N)^m \mathrm{i}[C_N, C_{N+}](C_N)^{n-m-1} = \mathrm{i}[(C_N)^n, C_{N+}], \tag{2.12}$$

and similarly,

$$\{(C_N)^n, \mathrm{tr}(C_N^*)\}_{\mathrm{AL}} = \mathrm{i}[(C_N)^n, (C_{N+})^*]. \tag{2.13}$$

Hence, the locally conserved fields are given by

$$Q^{[n],N} = \mathrm{tr}\big[(C_N)^n\big]. \tag{2.14}$$

By a similar argument, it can be shown that the mutual Poisson brackets vanish,

$$\{Q^{[n],N}, Q^{[n'],N}\}_{\mathrm{AL}} = 0. \tag{2.15}$$

The fields $Q^{[n],N}$ are complex-valued. Physically real-valued phase functions are preferred, which are achieved through taking real and imaginary parts,

$$\begin{aligned} Q^{[n,+],N} &= \frac{1}{2}\mathrm{tr}\big[(C_N)^n + (C_N^*)^n\big] = \mathrm{tr}\big[\cos((C_N)^n)\big], \\ Q^{[n,-],N} &= -\frac{1}{2}\mathrm{i}\,\mathrm{tr}\big[(C_N)^n - (C_N^*)^n\big] = \mathrm{tr}\big[\sin((C_N)^n)\big], \end{aligned} \tag{2.16}$$

with $n = 1, \ldots, N/2$. These fields have a density, respectively, given by

$$Q_j^{[n],N} = Q_j^{[n,+],N} + \mathrm{i}Q_j^{[n,-],N} = ((C_N)^n)_{j,j}. \tag{2.17}$$

Although the matrices L_N, M_N have a basic 2×2 structure, the densities of the conserved fields are shift covariant by 1. Let us introduce the left shift, τ, of the sequence $\alpha = (\alpha_0, \ldots, \alpha_{N-1})$ by $(\tau\alpha)_j = \alpha_{j+1}$. Then, as established in the Appendix,

$$Q_{j+1}^{[n,\sigma],N}(\alpha) = Q_j^{[n,\sigma],N}(\tau\alpha) \tag{2.18}$$

with the convention $\sigma = \pm$.

Later on, we will consider the infinite volume limit, $N \to \infty$. This will always be understood as a two-sided limit. For example, the infinite volume limit of L_N, denoted by L, is $L = \mathrm{diag}(\ldots, \Xi_{-2}, \Xi_0, \Xi_2, \ldots)$ and correspondingly $M = \mathrm{diag}(\ldots, \Xi_{-1}, \Xi_1, \Xi_3, \ldots)$. L, M are unitary operators on the Hilbert space $\ell_2(\mathbb{Z})$, and so is $C = LM$. The traces in (2.10) have no limit, but densities do. The matrix elements of C^n can be expanded as the sum

$$(C^n)_{i,j} = \sum_{j_1 \in \mathbb{Z}} \cdots \sum_{j_{2n-1} \in \mathbb{Z}} L_{i,j_1} M_{j_1,j_2} \ldots L_{j_{2n-2},j_{2n-1}} M_{j_{2n-1},j}, \tag{2.19}$$

which consists of a finite number of terms only. For the infinite system, the index n runs over all positive integers. The infinite volume densities, $Q_j^{[n,\sigma]}$, are strictly local functions of α with support of at most $2n$ sites. The sum in (2.19) can be viewed as resulting from a nearest neighbor $2n$ step random walk from left to right (see Fig. 1). For this purpose, one considers a checkerboard on $[0, 2n] \times \mathbb{R}$. The unit square with corners $(0,0), (1,0), (1,1), (0,1)$ is white. Single steps of the walk are either horizontal, $j \rightsquigarrow j$, or up-down, $j \rightsquigarrow j \pm 1$. Such diagonal steps are permitted only on white squares. The matrix element $(C^n)_{i,j}$ is then the sum over all $2n$ step walks starting at i and ending at j. Each walk represents a particular polynomial obtained by taking the product of local weights along the walk. The weights are

ρ_j for the diagonal steps $j \rightsquigarrow j+1$ and $j+1 \rightsquigarrow j$,
$\bar{\alpha}_j$ for the horizontal step $j \rightsquigarrow j$ in case its lower square is black, and
$-\alpha_{j-1}$ for the horizontal step $j \rightsquigarrow j$ in case its upper square is black.

As examples, $C_{j,j} = -\alpha_{j-1}\bar{\alpha}_j$, $H_j = C_{j,j} + \bar{C}_{j,j}$, and $(C^2)_{j,j} = \alpha_{j-1}^2\bar{\alpha}_j^2 - \rho_{j-1}^2\alpha_{j-2}\bar{\alpha}_j - \rho_j^2\alpha_{j-1}\bar{\alpha}_{j+1}$. Note that densities are not unique, in general, while the total conserved fields, $Q^{[n],N}$, are unique. To illustrate, in the previous formula, an equivalent density would be $\alpha_{j-1}^2\bar{\alpha}_j^2 - 2\rho_j^2\alpha_{j-1}\bar{\alpha}_{j+1}$.

The CMV matrix misses one physically very important field, namely,

$$Q^{[0],N} = -\sum_{j=0}^{N-1} \log(\rho_j^2). \tag{2.20}$$

To simplify notation, we set $[0] = [0, \sigma]$ and $0 = 0\sigma$. The time-derivative of $Q^{[0],N}$ yields a telescoping sum, which vanishes on a ring. In lack of a common name, we call $Q^{[0],N}$ the log intensity. The log intensity vanishes for small amplitudes $|\alpha_j|^2$ and diverges at the maximal value, $|\alpha_j|^2 = 1$. Note also that

$$\exp\left(-Q^{[0],N}\right) = \prod_{j=0}^{N-1} \rho_j^2 \tag{2.21}$$

is conserved. Thus, if initially $\exp\left(-Q^{[0],N}\right) > 0$, it stays so for all times, guaranteeing that the phase space boundary is never reached.

Generalized Gibbs ensemble. Hydrodynamics is based on the propagation of local equilibrium. For the micro-canonical equilibrium measure, the statistical mechanics rule is to adopt the uniform measure on the hypersurface defined through fixing the values of all conserved fields. For nonintegrable chains of N sites, its codimension would be $1, 2, 3$, depending on the model. As claimed by the integrable systems community, the statistical mechanics rule applies even in case of an extensive number conservation laws. In fact, in favorable situations, one can control the Hamiltonian written in terms of action variables. If this function has no flat pieces, then the uniform measure on invariant tori is approached in the long-time limit, almost surely. The AL system has a phase space of dimension $2N$. $Q^{[n,\sigma],N}$'s constitute N conservation laws. Together with $Q^{[0],N}$, such a rule means the uniform measure on an invariant torus of dimension $N-1$. As for other problems in statistical mechanics, more accessible is the grand-canonical version, a route also adopted here. One would expect that for large N, this hardly makes any difference, provided only averages of local observables are considered. In our context, to prove such an equivalence of ensembles stays as one open problem.

Because of the Hamiltonian structure, the volume measure $\prod_{j=0}^{N-1} \mathrm{d}^2\alpha_j$ is stationary under the AL dynamics. Structurally, it turns out to be more convenient to also include $\exp\left(-PQ^{[0],N}\right)$ in the *a priori* measure, which then becomes the product measure

$$\prod_{j=0}^{N-1} \mathrm{d}^2\alpha_j (\rho_j^2)^{P-1} = \prod_{j=0}^{N-1} \mathrm{d}^2\alpha_j (\rho_j^2)^{-1} \exp\left(-PQ^{[0],N}\right) \tag{2.22}$$

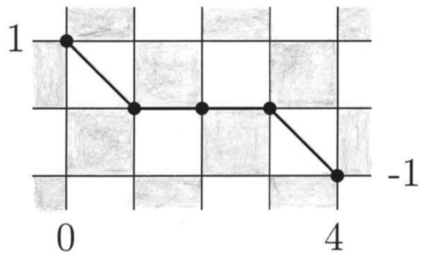

FIG. 1. An admissible walk from $(0,1)$ to $(4,-1)$. According to the rules, its weight is given by $\rho_0(-\alpha_{-1})\bar{\alpha}_0\rho_{-1}$.

on $\mathbb{D}^N$. To normalize the measure, $P > 0$ is required. The log intensity is controlled by the parameter P, which, in analogy to the Toda lattice, is called pressure. Small P corresponds to the maximal log intensity, i.e., $|\alpha_j|^2 \to 1$, and large P corresponds to low log intensity, i.e., $|\alpha_j|^2 \to 0$. In the grand-canonical ensemble, the Boltzmann weight is constructed from a linear combination of the conserved fields, which is written as

$$\sum_{n\in\mathbb{Z}} \mu_n \mathrm{tr}\big[(C_N)^n\big] = \mathrm{tr}\big[\hat{V}(C_N)\big], \qquad \hat{V}(z) = \sum_{n\in\mathbb{Z}} \mu_n z^n. \tag{2.23}$$

The chemical potentials, μ_n, are assumed to be independent of N. To have the trace real-valued, one imposes $\mu_n = \bar{\mu}_{-n}$. The normalization $\mu_0 = 0$ is also adopted. Combining (2.22) and (2.23) then yields the generalized Gibbs ensemble (GGE) as

$$Z_N(P,V)^{-1} \prod_{j=0}^{N-1} \mathrm{d}^2\alpha_j (\rho_j^2)^{P-1} \exp\big(-\mathrm{tr}[\hat{V}(C_N)]\big), \quad P > 0. \tag{2.24}$$

$Z_N(P,V)$ is the normalizing partition function. As label, the more natural object turns out to be the Fourier transform of the sequence $\{\mu_n, n \in \mathbb{R}\}$,

$$V(w) = \sum_{n\in\mathbb{Z}} \mu_n \mathrm{e}^{\mathrm{i}nw} = \hat{V}(\mathrm{e}^{\mathrm{i}w}). \tag{2.25}$$

V is a real-valued function on $[0, 2\pi]$. For the Toda lattice, the corresponding object lives on $\mathbb{R}$ and is called confining potential because it confines the eigenvalues of the Lax matrix. For the CMV matrix, the eigenvalues are on the unit circle and there is nothing to confine. To distinguish from other potentials, for convenience, we still stick to "confining." If $\hat{V}$ is given by a finite sum, then the interaction of the Gibbs measure in (2.24) is of finite range. In this case, the infinite volume limit can be controlled through transfer matrix methods. In particular, the limit measure is expected to have a finite correlation length. Presumably, a larger set of confining potentials could be allowed, as studied in Ref. 31 for the Toda lattice. Finite volume expectations with respect to the measure in (2.24) are denoted by $\langle\cdot\rangle_{P,V,N}$ and their infinite volume limit is denoted by $\langle\cdot\rangle_{P,V}$. The GGE is labeled by the pressure P and some smooth function on the unit circle.

The generalized free energy, F_{AL}, is defined through

$$\lim_{N\to\infty} -\frac{1}{N} \log Z_N(P,V) = F_{\mathrm{AL}}(P,V). \tag{2.26}$$

In the hydrodynamic context of particular interest is the empirical density of states (DOS),

$$\rho_{Q,N}(w)\mathrm{d}w = \frac{1}{N}\sum_{j=1}^{N} \delta(w - \vartheta_j)\mathrm{d}w \tag{2.27}$$

with $\mathrm{e}^{\mathrm{i}\vartheta_j}$'s eigenvalues of C_N. $\rho_{Q,N}$ is a probability measure on $[0, 2\pi]$ and has an almost sure limit as

$$\lim_{N\to\infty} \rho_{Q,N}(w) = \rho_Q(w). \tag{2.28}$$

To see the significance of the DOS, we first introduce the trigonometric functions $\varsigma_0(w) = 1$, $\varsigma_{n-}(w) = \sin(nw)$, and $\varsigma_{n+}(w) = \cos(nw)$, $n = 1, 2, \ldots$. They span the Hilbert space $L^2([0, 2\pi], \mathrm{d}w)$. Then, for the trigonometric moments of $\rho_{Q,N}(w)$,

$$\langle \rho_{Q,N}\varsigma_{n\sigma}\rangle = N^{-1}\langle Q^{[n,\sigma],N}\rangle_{P,V,N}, \quad \lim_{N\to\infty} N^{-1}\langle Q^{[n,\sigma],N}\rangle_{P,V,N} = \langle Q_0^{[n,\sigma]}\rangle_{P,V} = \langle \rho_Q\varsigma_{n\sigma}\rangle. \tag{2.29}$$

Here, $\langle\cdot\rangle$ is simply a short hand for the integration over $[0, 2\pi]$. The limit value can also be expressed as a variational derivative of the generalized free energy per site,

$$\frac{\mathrm{d}}{\mathrm{d}\kappa} F_{\mathrm{AL}}(P, V + \kappa\varsigma_{n\sigma})|_{\kappa=0} = \langle Q_0^{[n,\sigma]}\rangle_{P,V}. \tag{2.30}$$

In addition, one introduces for the average log intensity, denoted by ν, for which

$$\nu = \langle Q_0^{[0]}\rangle_{P,V} = \partial_P F_{\mathrm{AL}}(P,V). \tag{2.31}$$

For $V = 0$, one readily obtains $F_{\mathrm{AL}}(P,0) = \log(P/\pi)$ with log intensity $\nu(P) = P^{-1} > 0$. Hence, there is no high pressure phase as known for the Toda lattice [see Ref. 23 (Sec. 8) and Ref. 32]. Thermal equilibrium corresponds to $V(w) = \beta \cos w$, with β being the inverse temperature. Then, the Gibbs measure in (2.24) has nearest neighbor interactions and explicit expressions seem no longer to be available; see, however, the note at the end of Sec. V.

While the existence of the infinite volume limit is reassuring, more computable expressions are needed so to write down the hydrodynamic equations. In demand would be the joint probability density for the eigenvalues and the resulting DOS. This looks difficult. Fortunately, Killip and Nenciu[33] discovered that through a suitable modification of the boundary conditions, the corresponding volume element can be transformed to only depend on the eigenvalues.

III. CIRCULAR MATRICES WITH SLOWLY VARYING PRESSURE RAMP

Following Ref. 33, we modify the CMV matrix at the two boundaries. As before, the number N of sites is even. L_N remains unchanged, and M_N is modified to $M_N^\diamond$, where $(M_N^\diamond)_{0,0} = 1$, $(M_N^\diamond)_{0,N-1} = 0$, $(M_N^\diamond)_{N-1,0} = 0$, and $(M_N^\diamond)_{N-1,N-1} = \mathrm{e}^{\mathrm{i}\phi}$, $\phi \in [0, 2\pi]$. This leads to the particular CMV matrix,

$$C_N^\diamond = L_N M_N^\diamond. \tag{3.1}$$

For the *a priori* measure (2.22), the pressure P is constant, which is now modified to a linearly changing pressure with arbitrary slope $-\frac{1}{2}\beta$, $\beta > 0$, as

$$\prod_{j=0}^{N-2} \mathrm{d}^2\alpha_j \mathrm{d}\phi \prod_{j=0}^{N-2} (\rho_j^2)^{-1} (\rho_j^2)^{\beta(N-1-j)/2}. \tag{3.2}$$

Surprisingly, relative to this measure, the joint distribution of eigenvalues of $C_N^\diamond$ can be computed in a concise way.[33] We define the Vandermonde determinant as

$$\Delta(z_1, \dots, z_N) = \prod_{1 \le i < j \le N} (z_j - z_i). \tag{3.3}$$

Denoting the eigenvalues of $C_N^\diamond$ by $\mathrm{e}^{\mathrm{i}\vartheta_1}, \dots, \mathrm{e}^{\mathrm{i}\vartheta_N}$, their joint (un-normalized) distribution under the measure in (3.2) is given by

$$\zeta_N^\diamond(\beta) |\Delta(\mathrm{e}^{\mathrm{i}\vartheta_1}, \dots, \mathrm{e}^{\mathrm{i}\vartheta_N})|^\beta \prod_{j=1}^{N} \mathrm{d}\vartheta_j, \qquad \zeta_N^\diamond(\beta) = 2^{(1-N)} \frac{1}{N!} \frac{\Gamma(\beta/2)^N}{\Gamma(N\beta/2)}. \tag{3.4}$$

Since β is a free parameter, one can choose specifically

$$\beta = \frac{2P}{N}. \tag{3.5}$$

Now, the ramp has slope $-P/N$, and in the limit $N \to \infty$, close to the lattice point $(1-u)N$, $0 < u < 1$, the measure of (3.2) will converge to the product measure of (2.22) with pressure uP. Since

$$\mathrm{tr}\big[\hat{V}(C_N^\diamond)\big] = \sum_{j=1}^{N} V(\vartheta_j), \tag{3.6}$$

the Boltzmann weight can be naturally included in (3.2). Hence, the partition function of the system with boundary conditions is defined by

$$\begin{aligned} Z_N^\diamond(P,V) &= \int_{[0,2\pi]^{N-1}} \prod_{j=0}^{N-2} \mathrm{d}^2\alpha_j \int_0^{2\pi} \mathrm{d}\phi \prod_{j=0}^{N-2} (\rho_j^2)^{-1} (\rho_j^2)^{P(N-1-j)/N} \exp\big(-\mathrm{tr}[V(C_N^\diamond)]\big) \\ &= \zeta_N^\diamond(P) \int_0^{2\pi} \mathrm{d}\vartheta_1 \dots \int_0^{2\pi} \mathrm{d}\vartheta_N \exp\left(-\sum_{j=1}^{N} V(\vartheta_j) + P \frac{1}{N} \sum_{j,\ell=1, j\neq\ell}^{N} \log|\mathrm{e}^{\mathrm{i}\vartheta_\ell} - \mathrm{e}^{\mathrm{i}\vartheta_j}| \right). \end{aligned} \tag{3.7}$$

In statistical mechanics, the probability distribution

$$(Z_{\log,N})^{-1} \exp\left(-\sum_{j=1}^{N} V(\vartheta_j) + P\frac{1}{N}\sum_{j,\ell=1,j\neq\ell}^{N} \log|e^{i\vartheta_\ell} - e^{i\vartheta_j}|\right) \tag{3.8}$$

is known as a circular unitary ensemble (CUE) or circular log-gas.[34] Since the coupling strength is proportional to $1/N$, it is the much studied mean-field version of the log-gas (see Refs. 35 and 36 and references therein). In the statistical mechanics interpretation, β is the inverse temperature and the regime defined through (3.5) can be viewed as high temperature.

The AL model with boundary conditions has the free energy per site defined through

$$F^{\diamond}(P,V) = \lim_{N\to\infty} -\frac{1}{N}\log Z_N^{\diamond}(P,V). \tag{3.9}$$

Because the pressure ramp has slope $1/N$, in the limit, the free energies merely add up as

$$F^{\diamond}(P,V) = \int_0^1 du F_{\mathrm{AL}}(uP,V). \tag{3.10}$$

Before studying the infinite volume free energy, we remark that the CMV matrix $C_N^{\diamond}$ is still linked to a suitably modified AL dynamics governed by the Hamiltonian

$$H_N^{\diamond} = \mathrm{tr}\left[C_N^{\diamond} + C_N^{\diamond *}\right]. \tag{3.11}$$

Working out the Poisson brackets leads to the evolution equation

$$\frac{d}{dt}\alpha_j = i\rho_j^2(\alpha_{j-1} + \alpha_{j+1}), \quad \rho_j^2 = (1 - |\alpha_j|^2), \tag{3.12}$$

$j = 0, \ldots, N-2$, with the boundary conditions $\alpha_{-1} = -1$ and $\alpha_{N-1} = e^{i\phi}$. As before, $\mathrm{tr}\left[\left(C_N^{\diamond}\right)^n\right]$ is preserved under the dynamics. However, the *a priori* measure (3.2) is no longer stationary. The long time dynamics with tied down boundary conditions differs qualitatively from the one on the ring.

The prefactor in (3.7) can be easily handled with the result

$$\lim_{N\to\infty} -\frac{1}{N}\log \zeta_N^{\diamond}(2P/N) = \log(2P). \tag{3.13}$$

For the main term of (3.7), one notes that the exponential can be written in terms of the empirical density as

$$\varrho_N(w) = \frac{1}{N}\sum_{j=1}^{N} \delta(w - \vartheta_j). \tag{3.14}$$

The term reflecting the confining potential V is linear in ϱ_N, while the interaction term is quadratic up to the diagonal contribution. Thus, the limiting free energy is determined by a variational principle. We first define the mean-field free energy functional

$$\mathcal{F}^{\mathrm{MF}}(\varrho) = \int_0^{2\pi} dw \varrho(w) V(w) - P\int_0^{2\pi} dw \int_0^{2\pi} dw' \log|e^{iw} - e^{iw'}|\varrho(w)\varrho(w') + \int_0^{2\pi} dw \varrho(w)\log\varrho(w). \tag{3.15}$$

This functional has to be varied over all densities ϱ, with the constraints $\varrho(w) \geq 0$ and $\langle\varrho\rangle = 1$. The minimizer is known to be unique[37] and will be denoted by $\varrho^{\star}$. One then arrives at

$$F^{\diamond}(P,V) = \log(2P) + \mathcal{F}^{\mathrm{MF}}(\varrho^{\star}), \tag{3.16}$$

and hence, using (3.10),

$$F_{\mathrm{AL}}(P,V) = \partial_P(P\mathcal{F}^{\mathrm{MF}}(\varrho^\star)) + \log(2P) + 1. \tag{3.17}$$

It turns out to be more convenient to absorb P into ϱ by setting $\rho = P\varrho$. Then, $P\mathcal{F}^{\mathrm{MF}}(P^{-1}\rho) = \mathcal{F}(\rho) - P\log P$ with the transformed free energy functional

$$\mathcal{F}(\rho) = \int_0^{2\pi} \mathrm{d}w \rho(w) V(w) - \int_0^{2\pi} \mathrm{d}w \int_0^{2\pi} \mathrm{d}w' \log|\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{\mathrm{i}w'}|\rho(w)\rho(w') + \int_0^{2\pi} \mathrm{d}w \rho(w) \log \rho(w). \tag{3.18}$$

$\mathcal{F}$ has to be minimized under the constraint

$$\rho(w) \geq 0, \quad \int_0^{2\pi} \mathrm{d}w \rho(w) = P \tag{3.19}$$

with the minimizer denoted by $\rho^\star$. Then,

$$F_{\mathrm{AL}}(P,V) = \partial_P \mathcal{F}(\rho^\star) + \log 2. \tag{3.20}$$

The constraint (3.19) is removed by introducing the Lagrange multiplier μ as

$$\mathcal{F}_\mu(\rho) = \mathcal{F}(\rho) - \mu \int_0^{2\pi} \mathrm{d}w \rho(w). \tag{3.21}$$

A minimizer of $\mathcal{F}_\mu(\rho)$ is denoted by ρ_μ and determined as a solution of the Euler–Lagrange equation,

$$V(w) - \mu - 2\int_0^{2\pi} \mathrm{d}w' \log|\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{\mathrm{i}w'}|\rho_\mu(w') + \log \rho_\mu(w) = 0. \tag{3.22}$$

The Lagrange parameter μ has to be adjusted such that

$$P = \int_0^{2\pi} \mathrm{d}w \rho_\mu(w). \tag{3.23}$$

To obtain the Ablowitz–Ladik free energy, we differentiate as

$$\begin{aligned}\partial_P \mathcal{F}(\rho^\star) = &\int_0^{2\pi} \mathrm{d}w \partial_P\rho^\star(w) V(w) - 2\int_0^{2\pi} \mathrm{d}w \int_0^{2\pi} \mathrm{d}w' \log|\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{\mathrm{i}w'}|\rho^\star(w)\partial_P\rho^\star(w') \\ &+ 1 + \int_0^{2\pi} \mathrm{d}w (\partial_P\rho^\star(w)) \log \rho^\star(w).\end{aligned} \tag{3.24}$$

Integrating (3.21) against $\partial_P\rho^\star$, one arrives at

$$\partial_P \mathcal{F}(\rho_\mu) = \mu + \log 2, \tag{3.25}$$

and thus,

$$F_{\mathrm{AL}}(P,V) = \mu(P,V) + \log 2. \tag{3.26}$$

Sharing with other many-body integrable models, the Ablowitz–Ladik lattice has the property that its free energy is determined through an explicit variational problem.

Added note. Independently, Mazzuca and Grava[38] discovered the construction just presented. In the case of a confining potential specified by a finite polynomial, for the Ablowitz–Ladik model, they prove the existence of the infinite volume limit DOS and its connection to the minimizer of the generalized free energy as in (4.6). For thermal equilibrium, they show that the minimizer is a solution of the double confluent Heun equation.

In a follow-up, Mazzuca and Memin[39] proved the almost sure limit of the Lax density of states, its relation to the variational problem, and the uniqueness of the corresponding minimizer. These results hold for any continuous confining potential V, which from the perspective of generalized hydrodynamics is a most welcome advance: the confining potential and the associated DOS is the natural characterization of a GGE.

IV. DENSITY OF STATES

For the derivation of the hydrodynamic equations, the GGE average of the conserved fields, $\langle Q_0^{[n,\sigma]}\rangle_{P,V}$, is required for which purpose there are two equivalent methods. One can start from the microscopic definition and use that $Q^{[n,\sigma],N}$ depends only on the eigenvalues of the CMV matrix. The other method, employed here, is to simply differentiate the free energy per site. We start with $n = 0$ and note that the average log intensity

$$\nu = \langle Q_0^{[0]}\rangle_{P,V} = \partial_P F_{\mathrm{AL}}(P,V) = \partial_P \mu(P,V) = \left(\int_0^{2\pi} \mathrm{d}w \partial_\mu \rho_\mu(w)\right)^{-1}, \tag{4.1}$$

where the last equality results from differentiating Eq. (3.19) as $1 = (\int \partial_\mu \rho_\mu)\mu'(P)$. For $n \geq 1$, we perturb V as $V_\kappa(w) = V(w) + \kappa\varsigma_{n\sigma}(w)$ and differentiate the free energy at $\kappa = 0$. Then,

$$\langle Q_0^{[n,\sigma]}\rangle_{P,V} = \partial_\kappa F_{\mathrm{AL}}(P,V_\kappa)\big|_{\kappa=0} = \partial_P \partial_\kappa \mathcal{F}(\rho^\star(P,V_\kappa))\big|_{\kappa=0}, \tag{4.2}$$

and first introducing the linearization of $\rho^\star$ as

$$\partial_\kappa \rho^\star(P,V_\kappa)\big|_{\kappa=0} = \rho^{\star\prime}, \tag{4.3}$$

one obtains

$$\begin{aligned}\partial_\kappa \mathcal{F}(\rho^\star(P,V_\kappa))\big|_{\kappa=0} &= \int_0^{2\pi} \mathrm{d}w \rho^\star(w,P,V)\varsigma_{n\sigma} + \int_0^{2\pi} \mathrm{d}w V(w)\rho^{\star\prime}(w) \\ &-2\int_0^{2\pi}\mathrm{d}w \int_0^{2\pi}\mathrm{d}w' \log|\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{\mathrm{i}w'}|\rho^{\star\prime}(w)\rho^\star(w',P,V) + \int_0^{2\pi}\mathrm{d}w \rho*'(w)\log\rho^\star(w,P,V)\end{aligned} \tag{4.4}$$

using that $\int_0^{2\pi}\mathrm{d}w\rho^{\star\prime}(w) = 0$. Integrating the Euler–Lagrange equation (3.22) at $\mu = \mu(P)$ against $\rho^{\star\prime}$, the terms on the right-hand side of (4.4) vanish and

$$\langle Q_0^{[n,\sigma]}\rangle_{P,V} = \int_0^{2\pi} \mathrm{d}w \big(\partial_P \rho^\star(w,P,V)\big)\varsigma_{n\sigma}(w). \tag{4.5}$$

Thus, in the limit $N \to \infty$, the density of states is given by

$$\rho_{\mathrm{Q}}(w) = \partial_P \rho^\star(w). \tag{4.6}$$

Naively one might have guessed that the DOS equals $\varrho^\star$. However, the linear pressure ramp in the Killip–Nenciu identity amounts to a slightly deviating result.

In the literature, an Euler–Lagrange equation of the type (3.22) is written differently by formally introducing a Boltzmann weight through

$$\rho_\mu(w) = \mathrm{e}^{-\varepsilon(w)} \tag{4.7}$$

with quasi-energy $\varepsilon(w)$. Then,

$$\varepsilon(w) = V(w) - \mu - 2\int_0^{2\pi}\mathrm{d}w'\mathrm{d}w' \log|\sin(\tfrac{1}{2}(w-w'))|\mathrm{e}^{-\varepsilon(w')}. \tag{4.8}$$

In fact, comparing with (2.25), one could absorb μ into V. In general, one has to obtain the solution numerically. As discussed in Ref. 40, the most efficient method appears to use the nonlinear Fokker–Planck equation related to Dyson's Brownian motion on the circle. If $V = 0$, the solution is uniform on the interval $[0, 2\pi]$.

We pause for a while to collect a number standard identities, together with their notations. The Hilbert space of square integrable functions on $[0,2\pi]$ is denoted by $L^2([0,2\pi],\mathrm{d}w)$ with the scalar product

$$\langle f,g\rangle = \int_0^{2\pi} \mathrm{d}w \bar{f}(w) g(w). \tag{4.9}$$

There will be many integrals over $[0,2\pi]$ and a convenient shorthand is

$$\langle f\rangle = \langle 1,f\rangle = \int_0^{2\pi} \mathrm{d}w f(w). \tag{4.10}$$

Let us define the integral operator

$$T\psi(w) = 2\int_0^{2\pi} \mathrm{d}w' \log|\sin(\frac{1}{2}(w-w'))|\psi(w'), \quad w\in[0,2\pi]. \tag{4.11}$$

Then, the Euler–Lagrange equation (4.8) can be rewritten as

$$\varepsilon(w) = V(w) - \mu - (T\mathrm{e}^{-\varepsilon})(w). \tag{4.12}$$

One introduces the dressing of a function ψ through

$$\psi^{\mathrm{dr}} = \psi + T\rho_\mu \psi^{\mathrm{dr}}, \quad \psi^{\mathrm{dr}} = (1-T\rho_\mu)^{-1}\psi, \tag{4.13}$$

where ρ_μ is regarded as multiplication operator, i.e., $(\rho_\mu\psi)(w) = \rho_\mu(w)\psi(w)$. With our improved notation, the DOS in (4.6) can be written as

$$\rho_{\mathrm{Q}} = \partial_P \rho_\mu = (\partial_P \mu)\partial_\mu \rho_\mu = v\rho_{\mathrm{P}}, \quad \partial_\mu \rho_\mu = \rho_{\mathrm{P}}, \quad v\langle\rho_{\mathrm{P}}\rangle = 1. \tag{4.14}$$

Differentiating (4.13) with respect to μ, we conclude

$$\rho_{\mathrm{P}} = (1-\rho_\mu T)^{-1}\rho_\mu = \rho_\mu(1-T\rho_\mu)^{-1}\varsigma_0 = \rho_\mu \varsigma_0^{\mathrm{dr}}. \tag{4.15}$$

For later purposes, we also state

$$q_{n\sigma} = \langle Q_0^{[n,\sigma]}\rangle_{P,V} = v\langle\rho_{\mathrm{P}}\varsigma_{n\sigma}\rangle. \tag{4.16}$$

Of physical relevance are v and $v\rho_{\mathrm{P}}$, since they encode the GGE average of the conserved fields.

V. AVERAGE CURRENTS AND HYDRODYNAMIC EQUATIONS

GGE averaged currents. Returning to a ring of N sites with periodic boundary conditions, for the conserved field with index $[n,\sigma]$, the current from $j-1$ to j is denoted by $J_j^{[n,\sigma],N}$ and the current from j to $j+1$ by $J_{j+1}^{[n,\sigma],N}$. Then, the conserved fields satisfy a continuity equation of the form

$$\frac{d}{dt}Q_j^{[n,\sigma],N} = J_j^{[n,\sigma],N} - J_{j+1}^{[n,\sigma],N} = \{Q_j^{[n,\sigma],N}, H_N\}, \qquad J_j^{[n],N} = J_j^{[n,+],N} + \mathrm{i}J_j^{[n,-],N}. \tag{5.1}$$

As explained in more detail in Appendix, the current densities are local and shift invariant in the sense that

$$J_{j+1}^{[n],N}(\alpha) = J_j^{[n],N}(\tau\alpha). \tag{5.2}$$

The lowest index current densities are

$$J_j^{[0],N} = 2Q_j^{[1,-],N}, \quad J_j^{[1],N} = \mathrm{i}\big(-\rho_{j-1}^2\alpha_{j-2}\tilde{\alpha}_j + |\alpha_{j-1}|^2\big). \tag{5.3}$$

For higher currents, one has to rely on an abstract argument (see the Appendix). Actually, such expressions are not so helpful when trying to compute the GGE averaged currents. Fortunately, there is a generic argument[23,41] that applies also to the AL system.

We start with the fields and define the infinite volume correlator

$$C_{m\sigma,n\sigma'}(j-i) = \langle Q_j^{[m,\sigma]} Q_i^{[n,\sigma']}\rangle_{P,V}^{\mathrm{c}}, \tag{5.4}$$

where the superscript $^{\mathrm{c}}$ denotes truncation or connected correlation, $\langle gf\rangle^{\mathrm{c}} = \langle gf\rangle - \langle g\rangle\langle f\rangle$. Truncated correlations decay rapidly to zero, and the field–field susceptibility matrix is given by

$$C_{m\sigma,n\sigma'} = \sum_{j\in\mathbb{Z}} C_{m\sigma,n\sigma'}(j) = \langle Q^{[m,\sigma]}; Q^{[n,\sigma']}\rangle_{P,V}, \tag{5.5}$$

where the right-hand side is merely a convenient notation for the sum. $C_{m\sigma,n\sigma'}$ is the matrix of second derivatives of the generalized free energy. Correspondingly, we introduce the field–current correlator

$$B_{m\sigma,n\sigma'}(j-i) = \langle J_j^{[m,\sigma]} Q_i^{[n,\sigma']}\rangle_{P,V}^{\mathrm{c}}, \quad B_{m\sigma,n\sigma'} = \sum_{j\in\mathbb{Z}} B_{m\sigma,n\sigma'}(j). \tag{5.6}$$

Despite its apparently asymmetric definition, B satisfies

$$B_{m\sigma,n\sigma'}(j) = B_{n\sigma',m\sigma}(-j). \tag{5.7}$$

To prove, we employ the conservation law and spacetime stationarity to arrive at

$$\begin{aligned}\partial_j\langle J_j^{[m,\sigma]}(t) Q_0^{[n,\sigma']}(0)\rangle_{P,V}^{\mathrm{c}} &= -\partial_t\langle Q_j^{[m,\sigma]}(t) Q_0^{[n,\sigma']}(0)\rangle_{P,V}^{\mathrm{c}} \\ &= -\partial_t\langle Q_0^{[m,\sigma]}(0) Q_{-j}^{[n,\sigma']}(-t)\rangle_{P,V}^{\mathrm{c}} = \partial_j\langle Q_0^{[m,\sigma]}(0) J_{-j}^{[n,\sigma']}(-t)\rangle_{P,V}^{\mathrm{c}},\end{aligned} \tag{5.8}$$

denoting the difference operator by $\partial_j f(j) = f(j+1) - f(j)$. Setting $t = 0$, the difference $\langle J_j^{[m,\sigma]} Q_0^{[n,\sigma']}\rangle_{P,V}^{\mathrm{c}} - \langle J_{-j}^{[n,\sigma']} Q_0^{[m,\sigma]}\rangle_{P,V}^{\mathrm{c}}$ is constant in j. Since truncated correlations decay to zero, this constant has to vanish, which yields (5.7). In particular, the field–current susceptibility matrix is symmetric,

$$B_{m\sigma,n\sigma'} = B_{n\sigma',m\sigma}. \tag{5.9}$$

Using this symmetry, we consider the P-derivative of the average current,

$$\partial_P\langle J_0^{[n,\sigma]}\rangle_{P,V} = -B_{n\sigma,0} = -B_{0,n\sigma} = -2\langle Q^{[1,-]}; Q^{[n,\sigma]}\rangle_{P,V}, \tag{5.10}$$

since $J^{[0]} = 2Q^{[1,-]}$ by (5.3). We easily arrived at a very surprising identity. The P-derivative of the average current equals a particular covariance of the eigenvalue fluctuations. In Ref. 23 (Sec. IV), the fluctuations of eigenvalues for the GUE mean-field log-gas are handled. To switch from GUE to CUE in essence amounts to a notational change. The joint distribution of eigenvalues is stated in (3.8). Their asymptotic density is $\varrho^\star$, as a minimizer of the variational problem (3.15). The corresponding fluctuation field is defined through

$$\phi_N(f) = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\big(f(\vartheta_j) - \langle\rho^\star f\rangle\big) = \int_0^{2\pi}\mathrm{d}w f(w)\phi_N(w), \tag{5.11}$$

with f being some smooth test function on the circle $[0, 2\pi]$. As $N \to \infty$, ϕ_N converges to a Gaussian field with covariance,

$$\langle \tilde{f}, C^\sharp f\rangle = \langle \tilde{f}, (1 - P\varrho^\star T)^{-1}\varrho^\star f\rangle - \nu P\langle \tilde{f}, (1 - P\varrho^\star T)^{-1}\varrho^\star\rangle\langle (1 - P\varrho^\star T)^{-1}\varrho^\star, f\rangle, \tag{5.12}$$

where $\varrho^\star$ is regarded as a multiplication operator. The subtraction arises because the number of eigenvalues does not fluctuate.

As in the case of the free energy, since the pressure is varying as $1/N$, the fluctuation covariance is adding up, resulting in

$$\int_0^1 \mathrm{d}u\langle Q^{[1,-]}; Q^{[n,\sigma]}\rangle_{uP,V} = \langle \varsigma_{1-}, C^\sharp \varsigma_{n\sigma}\rangle. \tag{5.13}$$

Therefore, using identity (5.10), one arrives at

$$\partial_P\Big(\langle J_0^{[n,\sigma]}\rangle_{P,V} + 2P\langle \varsigma_{1-}, C^\sharp \varsigma_{n\sigma}\rangle\Big) = 0, \tag{5.14}$$

implying that the round bracket has to be independent of P, in particular, equal to its value at $P = 0$. Since $\langle \varsigma_{1-}, C^\sharp \varsigma_{n\sigma}\rangle$ is bounded in P, the second summand vanishes at $P = 0$. For the first summand, one notes that in the limit $P \to 0$, for each j, the *a priori* measure (2.22) becomes uniform on the unit circle and the CMV matrix turns diagonal, since $\rho_j^2 \to 0$. Denoting $\alpha_j = \mathrm{e}^{\mathrm{i}\phi_j}$, $\phi_j \in [0, 2\pi]$, in the limit $P \to 0$, the GGE (2.19) converges to

$$(Z_N)^{-1}\prod_{j=0}^{N-1}\mathrm{d}\phi_j\,\exp\Big(-\sum_{j=0}^{N-1} V(\phi_{j+1} - \phi_j)\Big), \quad \phi_N = \phi_0. \tag{5.15}$$

Using (5.3), one observes that $\langle J_0^{[1]}\rangle_{0,V} = \mathrm{i}$. By a direct computation, $\langle J_0^{[2]}\rangle_{0,V} = 0$. To extend the average to general n seems to be difficult, since a sufficiently explicit formula for $J_0^{[n]}$ is missing. We assume that $\langle J_0^{[n]}\rangle_{0,V} = d_n$ with some constant d_n independent of V. Next, we substitute $P\varrho^* = \rho_\mu$ with the result

$$P\langle \varsigma_{1-}, C^\sharp \varsigma_{n\sigma}\rangle = \langle \varsigma_{1-}, (1 - \rho_\mu T)^{-1}\rho_\mu \varsigma_{n\sigma}\rangle - \nu\langle \varsigma_{1-}, (1 - \rho_\mu T)^{-1}\rho_\mu\rangle\langle \varsigma_{n\sigma}, (1 - \rho_\mu T)^{-1}\rho_\mu\rangle. \tag{5.16}$$

Noting that $(1 - \rho_\mu T)^{-1}\rho_\mu$ is a symmetric operator, one finally arrives at

$$\langle J_0^{[n,\sigma]}\rangle_{P,V} - d_n = -2\Big(\langle \rho_\mu \varsigma_{1-}^{\mathrm{dr}}\varsigma_{n\sigma}\rangle - q_{1-}\langle \rho_{\mathrm{p}}\varsigma_{n\sigma}\rangle\Big), \qquad q_{1-} = \nu\langle \rho_{\mathrm{p}}\varsigma_{1-}\rangle. \tag{5.17}$$

In the conventional scheme of generalized hydrodynamics, the average currents are written somewhat differently. First, by linearity, there is some function $\rho_J(w)$ on $[0, 2\pi]$ such that

$$\langle J_0^{[n,\sigma]}\rangle_{P,V} - d_n = \langle \varsigma_{n\sigma}\rho_J\rangle. \tag{5.18}$$

For the currents, ρ_J plays the same role as ρ_Q for the conserved fields. However, ρ_J cannot have a definite sign. Second, one writes

$$\rho_J = -2\rho_{\mathrm{p}}\Big(v^{\mathrm{eff}} - q_{1-}\Big), \quad q_{1-} = \langle Q_0^{1,-}\rangle_{P,V}, \tag{5.19}$$

with v^{eff} being the effective velocity. The effective velocity can be written more concisely as

$$v^{\mathrm{eff}} = \frac{\varsigma_{1-}^{\mathrm{dr}}}{\varsigma_0^{\mathrm{dr}}}, \tag{5.20}$$

(see Ref. 23, Sec. VI).

Hydrodynamic equations. On the hydrodynamic scale, the local GGE is characterized by the log intensity v and the CMV density of states $v\rho_{\mathrm{p}}$, both of which now become spacetime dependent. Merely inserting the average currents and since d_n has been assumed to be a constant, one arrives at the Euler type hydrodynamic evolution equations,

$$\begin{aligned}&\partial_t v(x,t) + 2\partial_x q_{1-}(x,t) = 0,\\ &\partial_t\big(v(x,t)\rho_{\mathrm{p}}(x,t;v)\big) - 2\partial_x\Big(\big(v^{\mathrm{eff}}(x,t;v) - q_{1-}(x,t)\big)\rho_{\mathrm{p}}(x,t;v)\Big) = 0.\end{aligned} \tag{5.21}$$

This equation is based on the assumption of local GGE. To actually establish such an equation from the underlying AL model seems to be a difficult task.

As a most remarkable feature of generalized hydrodynamics, the equations can be transformed explicitly to a quasilinear form (see Ref. 23, Sec. VI). For this purpose we rewrite the identity in (4.15) as

$$\rho_\mu = \rho_{\mathrm{p}}\big(1 + (T\rho_{\mathrm{p}})\big)^{-1}, \tag{5.22}$$

now regarded as the nonlinear mapping $\rho_{\mathrm{p}} \mapsto \rho_\mu$. Then, Eq. (5.21) assumes the normal form

$$\partial_t\rho_\mu - 2v^{-1}(v^{\mathrm{eff}} - q_{1-})\partial_x\rho_\mu = 0. \tag{5.23}$$

Thus, the linearization operator is, in fact, merely multiplication by $-2v^{-1}(v^{\mathrm{eff}} - q_{1-})$; in other words, the operator is diagonal.

In (2.1), we followed a standard convention, which amounts to the free dispersion relation $E(p) = 2(1 - \cos p)$. Upon adopting $E(p) = 1 - \cos p$, the extra factors of 2 in (5.21) and (5.23) would be removed.

VI. MODIFIED KORTEWEG–dE-VRIES EQUATION

Instead of the Hamiltonian H_N of (2.5), one can choose

$$\check{H}_N = -\mathrm{i}\sum_{j=0}^{N-1}\big(\alpha_{j-1}\bar{\alpha}_j - \bar{\alpha}_{j-1}\alpha_j\big) = -\mathrm{i}\,\mathrm{tr}\big[C_N - C_N^*\big]. \tag{6.1}$$

Then,

$$\frac{d}{dt}\alpha_j = \{\alpha_j, \check{H}_N\}_{\mathrm{AL}} = \rho_j^2(\alpha_{j+1} - \alpha_{j-1}), \tag{6.2}$$

which is known as Schur flow.[42] Through a formal Taylor expansion, in Ref. 13, it is argued that the continuum limit of Eq. (6.2) yields the modified Korteweg–de-Vries equation,

$$\partial_t u = \partial_x^3 u - 6u^2\partial_x u, \tag{6.3}$$

which is a good reason to briefly touch upon (6.2).

As before, $\alpha_j \in \mathbb{D}$ and the conservation laws remain unchanged. However, the currents have to be modified from $J^{[n]}$ to $\check{J}^{[n]}$. For the log intensity current, one finds

$$\check{J}_j^{[0],N} = 2Q_j^{[1,+],N} = H_{N,j}. \tag{6.4}$$

The arguments of Sec. V can be repeated, *ad verbatim*. In the hydrodynamic equation (5.21), q_{1-} is replaced by q_{1+} and the effective velocity turns to

$$v^{\mathrm{eff}} = \frac{\varsigma_{1+}^{\mathrm{dr}}}{\varsigma_0^{\mathrm{dr}}}. \tag{6.5}$$

Such interchange of the roles of momentum and energy is already familiar from the relativistic sinh-Gordon quantum field theory.[2]

Such a discussion misses, however, an interesting point. The wave field of the modified Korteweg–de-Vries equation is real-valued, a feature that is maintained in the discrete approximation. If in (6.2) one chooses initially a real field α, then it stays real throughout time. From the perspective of GGE, such initial conditions amount to a set of measure zero and one has to reconsider the analysis. Fortunately, the relevant transformation formula is already proved in Ref. 33. To avoid duplication of symbols, in the remainder of this section, $\alpha_j \in \mathbb{R}$, and hence, $\bar{\alpha}_j = \alpha_j$ everywhere. The equations of motion read

$$\frac{d}{dt}\alpha_j = \rho_j^2(\alpha_{j+1} - \alpha_{j-1}), \quad \alpha_j \in [-1,1], \quad \rho_j^2 = 1 - \alpha_j^2. \tag{6.6}$$

While an obvious Hamiltonian structure is lost, one readily checks that the *a priori* measure

$$\prod_{j=0}^{N-1} \mathrm{d}\alpha_j (\rho_j^2)^{P-1} = \prod_{j=0}^{N-1} \mathrm{d}\alpha_j (\rho_j^2)^{-1} \exp\left(-PQ^{[0],N}\right) \tag{6.7}$$

is still stationary under the dynamics. As before, the densities of the conserved fields are

$$Q_j^{[m]} = (C^m)_{j,j}, \tag{6.8}$$

$m = 1, 2, \ldots$. In particular,

$$Q_j^{[0]} = -\log \rho_j^2, \quad Q_j^{[1]} = -\alpha_{j-1}\alpha_j, \quad Q_j^{[2]} = \alpha_{j-1}^2\alpha_j^2 - \rho_{j-1}^2\alpha_{j-2}\alpha_j - \rho_j^2\alpha_{j-1}\alpha_{j+1}. \tag{6.9}$$

There is no longer a distinction of $\pm$. For the log intensity current, $J_j^{[0]} = 2Q_j^{[1]}$. Thus, $J^{[0]}$ is conserved and the indirect method for computing the average currents is still in place (see Sec. V).

Since now C_N is a real matrix, its eigenvalues come in pairs. If $\mathrm{e}^{\mathrm{i}\vartheta_j}$ is an eigenvalue, so is $\mathrm{e}^{-\mathrm{i}\vartheta_j}$. For a system of size N, there are only $n = N/2$ independent eigenvalues. Rather than using a DOS reflecting such symmetry, it is more effective to restrict the eigenvalues as $0 \le \vartheta_j \le \pi$, subsequently setting $y_j = \cos\vartheta_j$. The empirical DOS is given by

$$\rho_{Q,n}(w)\mathrm{d}w = \frac{1}{n}\sum_{j=1}^{n} \delta(w - y_j)\mathrm{d}w, \quad w \in [-1,1], \tag{6.10}$$

where the more convenient n as the size parameter is used. In the limit $n \to \infty$, $\rho_{Q,n}(w)$ converges to the deterministic limit $\rho_Q(w)$. The GGE expectations are then

$$\langle Q_0^{[m]} \rangle_{P,V} = 2\int_{-1}^{1} \mathrm{d}w \rho_Q(w)\varsigma_m(w), \qquad \varsigma_m(w) = \cos(m\vartheta), \quad w = \cos\vartheta, \tag{6.11}$$

i.e., being the mth Chebyshev polynomial. The confining potential transforms to

$$V_{\mathrm{kdv}}(w) = \sum_{m=1}^{\infty} \mu_m \varsigma_m(w) \tag{6.12}$$

with real chemical potentials μ_m.

As proved in Ref. 33, under the measure

$$\prod_{j=0}^{2n-2} (1-\alpha_j^2)^{-1}(1-\alpha_j^2)^{\beta(2n-j-1)/4}(1-\alpha_j)^{a+1-(\beta/4)}(1+(-1)^j\alpha_j)^{b+1-(\beta/4)}\mathrm{d}\alpha_j \tag{6.13}$$

on $[-1,1]^{2n-1}$, $\beta > 0$, and $a, b > -1 + (\beta/4)$, the joint (un-normalized) distribution of the eigenvalues of C_N, imposing $\alpha_{2n-1} = 1$, is given by

$$\zeta_n^{\diamond}(\beta)2^{\kappa}|\Delta(2y_1,\ldots,2y_n)|^{\beta}\prod_{j=1}^{n}(1-y_j)^a(1+y_j)^b\mathrm{d}y_j \tag{6.14}$$

on $[-1,1]^n$. One notes β times the energy of the repulsive log gas with the single site *a priori* weight given by the Jacobi polynomial on $[-1,1]$. The normalization $\zeta_n^\diamond$ is defined in (3.4) and $\kappa = (n-1)(-\frac{1}{2}\beta + a + b + 2)$. To achieve a pressure ramp of slope $-P/2n$, one has to set

$$\beta = \frac{2P}{n}, \quad a = b = -1 + \frac{1}{4}\beta. \tag{6.15}$$

Thus, as before, one has to study the high temperature regime, this time for the β Jacobi ensemble. Since $\kappa = 0$, (6.14) becomes

$$\zeta_n^\diamond(\beta)|\Delta(2y_1,\dots,2y_n)|^\beta \prod_{j=1}^n (1-y_j)^{-1+(\beta/2)}(1+y_j)^{-1+(\beta/2)}\mathrm{d}y_j, \tag{6.16}$$

while (6.13) turns to

$$\prod_{j=0}^{2n-2}(1-\alpha_j^2)^{-1}(1-\alpha_j^2)^{P(2n-j-1)/2n}\mathrm{d}\alpha_j. \tag{6.17}$$

Now, the strategy of Sec. III is in force. We add a confining potential. Then, the asymptotic DOS is obtained by minimizing the mean-field free energy,

$$\begin{aligned}\mathcal{F}^{\mathrm{KdV}}(\varrho) = &\int_{-1}^1 \mathrm{d}w\varrho(w)V_{\mathrm{kdv}}(w) + \int_{-1}^1 \mathrm{d}w\varrho(w)\log(1-w^2)\\ &-P\int_{-1}^1 \mathrm{d}w\int_{-1}^1 \mathrm{d}w'\,\log(2|w-w'|)\varrho(w)\varrho(w') + \int_{-1}^1 \mathrm{d}w\varrho(w)\log\varrho(w).\end{aligned} \tag{6.18}$$

$\mathcal{F}^{\mathrm{KdV}}$ has to be minimized over all $\rho \geq 0$ with $\int \mathrm{d}w\varrho(w) = 1$ and the boundary condition $\rho(-1) = \rho(1)$.

Actually, our mean-field limit is somewhat singular, since at $a = -1 = b$ the *a priori* measure is not integrable. The quadratic energy term is repulsive at short distances, but the linear term with $\log(1-w^2)$ pushes the eigenvalues toward the two end points. It is not so obvious whether and how the two terms balance. Fortunately, the particular case $V_{\mathrm{kdv}} = 0$ has been studied in the recent contributions on the β-Jacobi ensemble.[43,35] The asymptotic DOS (6.10) with parameters $\beta = 2\alpha/n$ is studied, and hence, $\alpha = P$, while the parameters $a > -1$ and $b > -1$. The exact density of states in terms of the hypergeometric function ${}_2F_1$ is obtained. As communicated by Trinh, their proof works also for the limiting cases of interest here. Simply inserting the values $a = -1, b = -1$ in the general formula, a well-defined DOS is obtained. A confining potential will modify the DOS, but the balance between terms should persist.

Surprisingly, the confining potential V_{kdv} is corrected by the $\log(1-w^2)$ potential, which is attractive and favors the accumulation of eigenvalues near the two boundary points ± 1. The prior computation of the average currents is carried out for fixed V, which has now to be corrected to by $V_{\mathrm{cor}}(w) = V_{\mathrm{kdv}}(w) + \log(1-w^2)$. ρ_μ and ρ_{p} depending on V_{cor}. In addition, the dressing operator T is changed to

$$T\psi(w) = \int_{-1}^1 \mathrm{d}w'\,\log(2|w-w'|)\psi(w'), \quad w \in [-1,1]. \tag{6.19}$$

With these modifications, the hydrodynamic equations are derived along the standard route. In particular, the effective velocity is still given by

$$v^{\mathrm{eff}} = \frac{\varsigma_1^{\mathrm{dr}}}{\varsigma_0^{\mathrm{dr}}}, \tag{6.20}$$

where now the dressing operator T from (6.19) has to be used. In addition, q_1 is modified to

$$q_1 = 2\nu\int_{-1}^1 \mathrm{d}w\rho_{\mathrm{p}}(w)w. \tag{6.21}$$

VII. DISCUSSION

For the defocusing discrete NLS in one dimension, we established the form of the hydrodynamic equations. As a novel feature, their structure is determined by the mean-field version of the log gas corresponding to CUE random matrices. Our analysis is pretty much on the same level as the one for the Toda lattice. Only the handling of average currents in the limit $P \to 0$ is incomplete. We hope to return to this point in the future.

The Toda lattice is linked to the β-GUE ensemble at small β, i.e., high temperatures, such that the energy and entropy balance. Our results explain how the Ablowitz–Ladik system is connected to the β-CUE ensemble at small repulsion. Thus, one might wonder whether other classical matrix ensembles are linked to yet to be identified integrable dynamics. For the discrete modified Korteweg–deVries (KdV) equation, as an unexpected feature, the TBA equations pick up a correction to the confining potential V_{kdv}.

The reported results leave me with a puzzle. In generalized hydrodynamics, the accepted expression for the effective velocity is

$$v^{\mathrm{eff}} = \frac{[E']^{\mathrm{dr}}}{[p']^{\mathrm{dr}}} \tag{7.1}$$

with the parametrically given dispersion relation (p, E).[2] For the Toda lattice, $E(p) = \frac{1}{2}p^2$ and $E'(p) = p, p' = 1$. The kernel defining the dressing transformation is the two-particle scattering shift, which for Toda is $2\log|w - w'|$. Hence, what is the scattering shift for the discretized nonlinear Schrödinger equation and how is the rule (7.1) connected to (5.20), (6.5), or (6.20)?

DEDICATION

Freeman Dyson pioneered the investigation of random matrices. For a glimpse on the early founding period of the subject, one can consult his introductory chapter in the 2011 Oxford Handbook on Random Matrices. Our contribution is related, since we elucidate a close connection between the high temperature limit of beta CUE and the Ablowitz–Ladik discretized version of the nonlinear Schrödinger equation in one spatial dimension. In deep appraisal of an outstanding scientist, this article is dedicated to Freeman Dyson.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: THE GENERAL ABLOWITZ–LADIK SYSTEM

The defocusing AL model is studied in Refs. 25 and 26. In particular, it is proved that the CMV matrix C_N determines the local conservation laws. Here, we remark that the algebra in Ref. 25 actually holds at greater generality.

Already in the original contribution[14] (see also Ref. 1), it was noted that an often more convenient formulation is the coupled system,

$$\begin{aligned} &\frac{d}{dt}q_j = \mathrm{i}\rho_j^2(q_{j-1} + q_{j+1}),\\ &\frac{d}{dt}r_j = -\mathrm{i}\rho_j^2(r_{j-1} + r_{j+1}), \qquad \rho_j^2 = 1 - q_j r_j, \end{aligned} \tag{A1}$$

with $q_j \in \mathbb{C}$, $r_j \in \mathbb{C}$. (Following the quantum convention and Ref. 26, for us the Schrödinger equation reads $\mathrm{i}\partial_t\psi = -\partial_x^2\psi$, which amounts to reversing time when compared to Ref. 1.) As before, we consider a ring with N sites, N even. For better readability, we will drop the index N. The defocusing case corresponds to setting $q_j = \alpha_j$, $r_j = \tilde{\alpha}_j$, while the physically equally interesting focusing case corresponds to $q_j = \alpha_j$, $r_j = -\tilde{\alpha}_j$.

We introduce two classes of CMV matrices. Their building blocks are

$$\Xi_j = \begin{pmatrix} r_j & \rho_j \\ \rho_j & -q_j \end{pmatrix}, \qquad \tilde{\Xi}_j = \begin{pmatrix} q_j & \rho_j \\ \rho_j & -r_j \end{pmatrix}. \tag{A2}$$

The matrices $L, \tilde{L}$ and $M, \tilde{M}$ are constructed by the same scheme as before and the CMV matrix is defined through

$$C = LM, \qquad \tilde{C} = \tilde{L}\tilde{M}, \tag{A3}$$

i.e., the tilde operation amounts to interchanging q and r. The pair (q_j, r_j) is viewed as canonical coordinates. Then, the weighted Poisson bracket generalizes to

$$\{f, g\} = \mathrm{i}\sum_{j=0}^{N-1} \rho_j^2 \left(\frac{\partial f}{\partial r_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial r_j} \right). \tag{A4}$$

The time evolution in (A1) is generated by the Hamiltonian

$$H = -\sum_{j=0}^{N-1} (q_j r_{j+1} + r_j q_{j+1}) = \mathrm{tr}\big[C + \tilde{C}\big]. \tag{A5}$$

As before, the log intensity field,

$$Q^{[0]} = -\sum_{j=0}^{N-1} \log(\rho_j) = \tilde{Q}^{[0]}, \tag{A6}$$

is conserved. The complete tower of conserved fields is of the form

$$Q^{[n]} = \mathrm{tr}\big[C^n\big], \qquad \tilde{Q}^{[n]} = \mathrm{tr}\big[\tilde{C}^n\big] \tag{A7}$$

with densities

$$Q_j^{[n]} = (C^n)_{j,j}, \qquad \tilde{Q}_j^{[n]} = (\tilde{C}^n)_{j,j}. \tag{A8}$$

These densities are shift covariant in the sense that

$$Q_{j+1}^{[n]}(q, r) = Q_j^{[n]}(\tau q, \tau r), \qquad \tilde{Q}_{j+1}^{[n]}(q, r) = \tilde{Q}_j^{[n]}(\tau q, \tau r), \tag{A9}$$

where $(\tau q)_j = q_{j+1} \bmod(N)$. To confirm, one introduces the unitary shift matrix S through $(S^* A S)_{i,j} = A_{i+1,j+1} \bmod(N)$. Then, denoting by $^{\mathrm{T}}$ the transpose,

$$S^{\mathrm{T}} C(q, r) S = S^{\mathrm{T}} L(q, r) S S^{\mathrm{T}} M(q, r) S = M(\tau q, \tau r) L(\tau q, \tau r) = C(\tau q, \tau r)^{\mathrm{T}}, \tag{A10}$$

which implies (A9).

To extend the Lax pair relations (2.11), one notes that $(C_N)_{i,j}(\alpha, \bar{\alpha})$ is a polynomial in $\alpha = (\alpha_0, \ldots, \alpha_{N-1})$ and $\bar{\alpha} = (\bar{\alpha}_0, \ldots, \bar{\alpha}_{N-1})$, which in Ref. 25 are regarded as independent variables. The Poisson bracket generates some other polynomial and the operation $(\tilde{C}_N)_{i,j}$ interchanges the roles of $\alpha, \bar{\alpha}$. Therefore, the algebra in Ref. 25 persists upon replacing α_j by q_j and $\bar{\alpha}_j$ by r_j and properly translating the operations * and $^-$. Thereby, one arrives at

$$\{C, \mathrm{tr}(C)\} = \mathrm{i}[C, C_+], \qquad \{C, \mathrm{tr}(\tilde{C})\} = \mathrm{i}[C, (\tilde{C}_+)^{\mathrm{T}}]. \tag{A11}$$

Since the Poisson bracket acts as a derivative, one deduces

$$\{C^n, \mathrm{tr}(C)\} = \sum_{m=0}^{n-1} C^m \mathrm{i}[C, C_+] C^{n-m-1} = \mathrm{i}[C^n, C_+], \tag{A12}$$

and similarly,

$$\{C^n, \mathrm{tr}(\tilde{C})\} = \mathrm{i}[C^n, (\tilde{C}_+)^{\mathrm{T}}]. \tag{A13}$$

As claimed, $Q^{[n]}$ and $\tilde{Q}^{[n]}$ are conserved. In addition, mutual Poisson brackets vanish,

$$\{Q^{[n]}, Q^{[n']}\} = 0, \qquad \{Q^{[n]}, \tilde{Q}^{[n']}\} = 0, \qquad \{\tilde{Q}^{[n]}, \tilde{Q}^{[n']}\} = 0. \tag{A14}$$

The densities (A8) can still be expanded as a sum over weighted random walks on a checkerboard. The matrix element $(C^n)_{i,j}$ is then the sum over all $2n$ step walks starting at i and ending at j. Each walk represents a particular polynomial obtained by taking the product of local weights along the walk. The weights are

ρ_j for the diagonal steps $j \rightsquigarrow j+1$ and $j+1 \rightsquigarrow j$,
r_j for the horizontal step $j \rightsquigarrow j$ in case its lower square is black, and
$-q_{j-1}$ for the horizontal step $j \rightsquigarrow j$ in case its upper square is black.

For $\tilde{Q}^{[n]}$, the roles of q_j and r_j are exchanged.

In Ref. 1 (Chaps. 3.2 and 3.4), also conserved fields are discussed and a recursion relation iteratively determining conserved fields is determined. However, these fields are nonlocal and one still would have to follow the "logarithmic subtraction procedure" to arrive at their local version (see Ref. 23, Sec. 11).

Of particular interest are the current densities. In (A14), the matrix C_+ has nonvanishing matrix elements only for $(j, j+\ell)$ with $\ell = 0, 1, 2$. The terms with $\ell = 0$ cancel and

$$\begin{aligned}&\{(C^n)_{j,j}, H\}\\ &\quad = \sum_{\ell=1,2} \mathrm{i}\big(C_{j-\ell,j}(C^n)_{j,j-\ell} - C_{j,j+\ell}(C^n)_{j+\ell,j} + \tilde{C}_{j,j+\ell}(C^n)_{j,j+\ell} - \tilde{C}_{j-\ell,j}(C^n)_{j-\ell,j}\big).\end{aligned} \tag{A15}$$

This looks like a shift difference, but it is not, since the off-diagonal matrix elements are only two-periodic. For $n = 1$, one obtains

$$\{C_{j,j}, \mathrm{tr}(C)\} = \mathrm{i}\big(-\rho_{j-1}^2 q_{j-2} r_j + \rho_j^2 q_{j-1} r_{j+1}\big), \tag{A16}$$

while

$$\{C_{j,j}, \mathrm{tr}(\tilde{C})\} = \mathrm{i}\big(\rho_j^2 q_{j+1} r_{j+1} - \rho_{j-1}^2 q_{j-2} r_{j-2} + \rho_j^2 \rho_{j+1}^2 - \rho_{j-2}^2 \rho_{j-1}^2\big) = \mathrm{i}(q_{j-1} r_{j-1} - q_j r_j) \tag{A17}$$

for even j and

$$\{C_{j,j}, \mathrm{tr}(\tilde{C})\} = \mathrm{i}\big(\rho_j^2 q_{j-1} r_{j-1} - \rho_{j-1}^2 q_j r_j\big) = \mathrm{i}(q_{j-1} r_{j-1} - q_j r_j) \tag{A18}$$

for odd j. Therefore, the first current reads

$$J_j^{[1]} = \mathrm{i}\big(-\rho_{j-1}^2 q_{j-2} r_j + q_{j-1} r_{j-1}\big). \tag{A19}$$

This computation illustrates the difficulties when taking the $P \to 0$ limit. To reach Eq. (A19) still requires explicit cancellations to yield the actual current. For general n, one has to rely on an abstract argument.

Existence of local currents. We want to establish that there is a local current local, $J_j^{[n]}$, such that

$$\{(C^n)_{j,j}, H\} = J_j^{[n]} - J_{j+1}^{[n]}. \tag{A20}$$

We fix n and choose $N > 4n$. $\{Q_0^{[n]}, H\}$ is a polynomial of degree at most $2n+2$. This polynomial is decomposed into *patterns* consisting of monomials and their spatial shifts, denoted by ω_j. An example would be $\omega_j = r_{j-1}(q_{j+3})^2$. Then, $\{Q_0^{[n]}, H\}$ is a sum of terms of the form

$$\sum_{|\ell| \le 2n} a^{(\ell)} \omega_\ell \tag{A21}$$

with some complex coefficients $a^{(\ell)}$, which may vanish. Since $\{Q_j^{[n]}, H\}$ is shift covariant, from the conservation law,

$$\sum_{j=0}^{N-1} \sum_{|\ell| \le 2n+2} a^{(\ell)} \omega_{\ell+j} = 0. \tag{A22}$$

Relabeling the sum over j, one arrives at

$$\sum_{|\ell|\leq 2n+2} a^{(\ell)} = 0. \tag{A23}$$

To have a one-shift covariant current density means

$$\sum_{|\ell|\leq 2n+2} b^{(\ell)}(\omega_\ell - \omega_{\ell+1}) = J_0^\omega - J_1^\omega. \tag{A24}$$

Using (A23), the coefficients $b^{(\ell)}$ are uniquely determined through $a^{(\ell)}$'s. The total fields $Q^{[n],N}$ are unique, but the one-shift covariant local densities $Q_j^{[n],N}$ are not. Once the densities are fixed, the corresponding one-shift covariant current density is determined.

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