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Limiting Current Distribution for a Two Species Asymmetric Exclusion Process

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Abstract: We study current fluctuations of a two-species totally asymmetric exclusion process, known as the Arndt–Heinzel–Rittenberg model. For a step-Bernoulli initial condition with finite number of particles, we provide an explicit multiple integral expression for a certain joint current probability distribution. By performing an asymptotic analysis we prove that the joint current distribution is given by a product of a Gaussian and a GUE Tracy–Widom distribution in the long time limit, as predicted by non-linear fluctuating hydrodynamics.

1. Introduction and Main Results

Asymmetric exclusion processes on $\mathbb{Z}$, in which many particles perform asymmetric random walks with only one particle allowed on each lattice site, are considered to be among the most fundamental processes in the theory of stochastic interacting particle systems [66,67,88]. Originally introduced as biophysical models for protein synthesis on RNA [68,69] they also have many other applications in biology and in other disciplines such as physics and engineering [42]. These processes have attracted much attention over the years and various large scale behaviours have been studied such as the hydrodynamic limit, large deviations and other properties [52].

More recently many studies have focused on the non-Gaussian fluctuation properties of asymmetric exclusion processes, which are related to the Kardar-Parisi-Zhang (KPZ) universality class [7,48]. In 2000, Johansson showed that the current distribution of the totally asymmetric simple exclusion process (TASEP) with the step initial condition is given by the GUE Tracy–Widom distribution in the long time limit [47]. As a corollary he proved the fluctuation exponent 1/3, which is characteristic of the KPZ class in one dimension. [1] The KPZ dynamical exponent was also observed in Bethe ansatz studies of the spectral gap of the asymmetric simple exclusion process (ASEP) with periodic

[1] In two dimensions the ASEP has a $\log(t)^{2/3}$ law [100].

and open boundary conditions [31–33,43,44,50]. Since Johansson's result there have been a number of generalisations of his results for various types of asymmetric exclusion processes, including the ASEP and q-TASEP, under several different initial conditions, see e.g. [1,6,12,24,79,83–86,94].

Most of these results on limiting distributions have been established based on explicit exact formulas for appropriate quantities before taking the long time limit. For example, for the case of TASEP with step initial condition, the current distribution is written as a multiple integral related to random matrix theory [47,76]. For ASEP and q-TASEP, certain q-deformed moments admit a multiple integral representation, which leads to a Fredholm determinant expression for the q-deformed Laplace transform [12,13]. The reason for the existence of such explicit formulas for certain classes of models is related to the underlying integrability of such models. In fact, almost all models for which limiting distributions have been studied have turned out to be a special or a limiting case of a stochastic higher spin six vertex model (HS6VM) [18,29,71,78], whose similarity transformed non-stochastic version has long been known to be (Yang-Baxter) integrable [53,59]. Very recently a method to prove universality without resorting to integrability has been proposed [81] (cf. also [98]).

Despite these remarkable developments, most of the asymptotic results have so far only been obtained for models in which there is only a single species of particles. It is a quite natural and important problem to try to generalise the analysis to multi-species models. Given the above situation for single species models, it would be natural to start our studies on integrable multi-species models. In fact several multi-species models have been already known. Multi-species asymmetric exclusion processes were introduced a long time ago, see e.g. [34,49,70], and several other multi-species stochastic processes have been proposed recently in [61]. Among these, those related to $U_q(sl_n)$ (n-ASEP) have been most studied. Their stationary measures on both the infinite line and the ring were studied in [36,80] and in [26] these were put in the context of Macdonald polynomial theory, the Knizhnik–Zamolodchikov equation and bosonic solutions of the Yang-Baxter equation. The n-TASEP was shown to be related to the combinatorial R-matrix and solutions of the tetrahedron equation in [60,61]. In [8,9,28,54–56] methods have been developed to construct multi-species duality functionals, and many other algebraic properties and connections to non-symmetric Macdonald polynomials and partition functions are discussed in [19]. The transition probability has also been discussed [57,64,96].

Much less is known rigorously about dynamic properties for multi-species models. The fluctuation exponent of n-ASEP was addressed in finite size scaling of the gap of its generator [2], and the Bethe ansatz for the 2-ASEP with open boundaries was considered in [101]. Limit distributions for a single second class particle have been studied in [35,77]. To our knowledge, full limiting distributions for multi-species models have not been derived other than where there is a relation to a single species model such as shift invariance and colour-position symmetry [11,17,21–23,40,58].

In this paper we consider the two-species Arndt–Heinzel–Rittenberg model on $\mathbb{Z}$. This is a solvable lattice model and we first derive its transition probability (or Green's function) in the form of multiple contour integrals over two families of integration variables by diagonalising the time evolution generator using the nested Bethe ansatz method. Summing over initial and final positions leads to a joint current distribution in the same form. Then we use a proposition which allows us to rewrite one set of multiple integral into a Fredholm determinant. Finally we rigorously derive a limiting joint current distribution for late times from an exact formula for a total crossing probability.

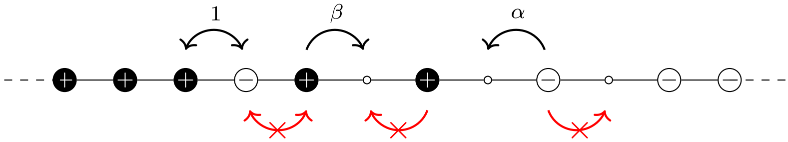

Fig. 1. Configuration of + and − particles and hopping rates in the AHR model on $\mathbb{Z}$

This limiting distribution factorises in a Tracy–Widom GUE distribution function and a Gaussian.

1.1. The Arndt–Heinzel–Rittenberg model. In this paper we study a two-species asymmetric exclusion process which is different from n-ASEP, and establish a result on the long time limit for a certain joint distribution of currents. The model we consider was first introduced in 1997 by Arndt–Heinzel–Rittenberg in [3] and we refer to it as the AHR model below. We study (a special case of) the AHR exclusion process [2] on the one dimensional lattice $\mathbb{Z}$, with two species of particles. Namely, each site can be either empty or occupied by one of the two kinds of particles, which are called "+" (plus) and "−" (minus) particles. The plus particle can hop forward while the minus particle can hop backward. In addition, an adjacent $+-$ pair of particles can swap to $-+$, but the swap can only occur in one direction (see the explicit jumping rates below). Obviously, since the AHR model is defined to be a Markov process, the hopping and swapping occurs according to exponential clocks. Additionally, the hopping and swapping is suppressed if the neighbouring site is occupied, i.e. a site cannot be occupied with more than one particle so that the model is a proper exclusion process. The explicit jumping rates are listed below.

$$\begin{aligned}(+,0) &\to (0,+) \ \text{ with rate } \beta,\\ (0,-) &\to (-,0) \ \text{ with rate } \alpha,\\ (+,-) &\to (-,+) \ \text{ with rate } 1.\end{aligned} \tag{1.1}$$

See Fig. 1.

The Yang-Baxter integrability of (this case of) the AHR model was proved in [25]. In [3], the authors studied the stationary state of the model on a ring and observed an interesting condensation phenomena [4,5], which was further studied in [82] by using a connection to the Al-Salam-Chihara polynomials. The hydrodynamics of the AHR was studied numerically in [51], where it was observed that the model can be described by two coupled Burgers equations that decouple at large length scales. Here, we specifically consider the case where the rates α and β sum up to unity, $\alpha + \beta = 1$. It is known that in this case the stationary measure is factorised. In addition, this condition drastically simplifies technicalities in the construction of the eigenfunctions of the time evolution generator in the form of a product of plane waves, and hence also in the derivation of the transition probability of the AHR model. It would be a quite interesting and challenging problem to generalise our results to the case with more general values of the parameters.

[2] In the original AHR model, the reverse exchange $(-,+) \to (+,-)$ is also allowed with rate $q (\geq 0)$. In addition we impose $\alpha + \beta = 1$ as mentioned below.

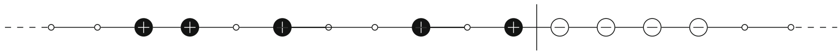

Fig. 2. Step-Bernoulli initial configuration of the AHR model on $\mathbb{Z}$

1.2. Transition probability. In Sect. 2 we give a formula for the transition probability (or the Green's function) for the AHR model on $\mathbb{Z}$ in the form of a multiple integral. This formula is a generalisation of the formula for the single species TASEP, first given by Schütz in [87], see also [93]. Our proof will also be similar to the ones in [87,93], i.e. we show explicitly that the multiple integral satisfies the correct time evolution equation and initial condition.

The origin of the integrand of the formula may be understood by considering a connection to the Bethe ansatz, as explained in Appendix A. A big difference from the single species case is that the form of the transition probability depends strongly on the ordering of particles at initial and final times. In this paper we will focus on a special case in which initially all + particles are to the left of all − particles, they swap their positions completely and at final time all − particles are to the left of all + particles. For this particular case, the interaction effects between the + and − particles are contained in a nice product form in the integrand of the transition probability. For the moment it seems difficult to do asymptotics for general orderings.

As for the initial condition, we consider a mix of random and step initial conditions, namely, we consider the situation in which initially there are n of + particles with density ρ to the left of the origin while m of − particles occupy the first m sites to the right of the origin (see (3.4) for a precise definition). See Fig. 2. Let $\mathbb{P}_{n,m}$ denote the probability measure for this initial condition and $N_{\pm}(t)$ the number of $\pm$ particles which passed the origin up to time t. For this setup, our first main result is the following theorem.

Theorem 1.1. *For the AHR model with $\alpha + \beta = 1$ and the above step-Bernoulli initial condition, the probability that all particles passed the origin at time t is given as the following multiple integral,*

$$\mathbb{P}_{n,m}[N_+(t) = n, N_-(t) = m] = (-1)^{n+m} \oint_{C_{\vec{z},\vec{w}}} \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}}\ \mathrm{e}^{\Lambda_{n,m}t}$$
$$\times \frac{\rho^n \prod_{1\le i<j\le n}(z_i - z_j) \prod_{1\le k<\ell\le m}(w_\ell - w_k) \prod_{j=1}^{n} z_j^{n-j} \prod_{k=1}^{m} w_k^{k-1}}{\prod_{j=1}^{n}(z_j-1)^{n+1-j}\big(1-(1-\rho)z_j\big)\prod_{k=1}^{m}(w_k-1)^k \prod_{j=1}^{n}\prod_{k=1}^{m}\Big(\alpha z_j + \beta w_k\Big)}, \quad (1.2)$$

where $\Lambda_{n,m} = \beta \sum_{j=1}^{n}(z_j^{-1}-1) + \alpha \sum_{k=1}^{m}(w_k^{-1}-1)$, and the contours $C_{\vec{z},\vec{w}}$ are chosen such that $\vec{z}$-contours enclose only the poles at 0 and $\vec{w}$-contours enclose only the poles at 0 and $\{-\alpha z_j/\beta\}_{j=1}^{n}$.

A proof of this theorem will be given in Sect. 3.

Remark 1.2. To ease description we sometimes employ the notation $\oint_{x_1,\dots,x_l}$ for the contour integral along the contour enclosing only the poles at $x_1, \dots, x_l$. For example, integrals in the statement of Theorem 1.1 are written as

$$\oint_{C_{\vec{z},\vec{w}}} \prod_{j=1}^{n} \mathrm{d}z_j \prod_{k=1}^{m} \mathrm{d}w_k = \oint_0 \prod_{j=1}^{n} \mathrm{d}z_j \oint_{0,\{-\alpha z_j/\beta\}_{j=1}^{n}} \prod_{k=1}^{m} \mathrm{d}w_k = \oint_0 \prod_{k=1}^{m} \mathrm{d}w_k \oint_0 \prod_{j=1}^{n} \mathrm{d}z_j.$$

Note that in the right-most expression the poles round the origin only need to be taken.

We can also choose the opposite contours $C_{\vec{w},\vec{z}}$ such that $\vec{w}$-contours enclose only the poles at 0 and $\vec{z}$-contours enclose only the poles at 0 and $\{-\beta w_k/\alpha\}_{k=1}^m$. See a few sentences after (3.3) in the proof.

1.3. Transformation to Fredholm determinant. Let us recall the definition of the continuous and discrete versions of the Fredholm determinant, which we will use in this paper. Let K be an integral operator acting on functions $f \in L^2(s, \infty)$ with kernel $K(\zeta, \xi)$ given by

$$(Kf)(\zeta) := \int_s^\infty K(\zeta, \xi) f(\xi) \mathrm{d}\xi.$$

A Fredholm determinant of the operator $1 + \lambda K$ is formally defined as the series

$$\det(1+\lambda K)_{L^2(s,\infty)} = 1 + \sum_{k=1}^\infty \frac{\lambda^k}{k!} \int_s^\infty \int_s^\infty \cdots \int_s^\infty \det_{1\le i,j\le k} \left[K(\xi_i, \xi_j)\right] \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_k. \tag{1.3}$$

A discrete analogue for a kernel acting on $\ell^2(\mathbb{N}) = \ell^2(1, 2, \ldots)$ is defined by

$$\det(1+\lambda K)_{\ell^2(\mathbb{N})} = 1 + \sum_{k=1}^\infty \frac{\lambda^k}{k!} \sum_{x_1=1}^\infty \sum_{x_2=1}^\infty \cdots \sum_{x_k=1}^\infty \det_{1\le i,j\le k} \left[K(x_i, x_j)\right]. \tag{1.4}$$

The series converge when K satisfies certain conditions, e.g. when it is trace-class. For further details on Fredholm determinants in the context of Macdonald processes we refer to [12], and to [41,63] for more general theory.

In the case of single species models, a useful approach to establish late time results is to rewrite multiple integral expressions into a Fredholm determinant for which asymptotics is easier to perform. For example, for the most standard case of the GUE random matrix eigenvalues, or in fact for general determinantal point processes, such a rewriting is well known and is explained in [39,72,87,92].

In our multi-species case it has not been known whether the AHR model is associated with a determinantal process or not, but the following rewriting in terms of auxiliary variables (or Fourier modes) has turned out to be useful in our analysis. A very similar and related manipulation was stated in [46] for the case related to the q-Whittaker function (see also [45] for its application to a more general setting), but here we give a statement and its proof (in Section 4) for the case with several parameters but without q. In the following, we denote by $[N, M]$ the set $\{N, \ldots, M\} \subset \mathbb{N}$ for $N, M \in \mathbb{N}$ satisfying $N < M$.

Proposition 1.3. *Set*

$$g^c(\zeta, s; \kappa) = \prod_{j=1}^{\nu} \frac{1}{1 - u_j/\zeta} \prod_{k=1}^{\mu} \frac{1}{1 + v_k \zeta} \times \zeta^{\mu-\nu-s} (\zeta + c)^\kappa \, \mathrm{e}^{\gamma\zeta}, \tag{1.5}$$

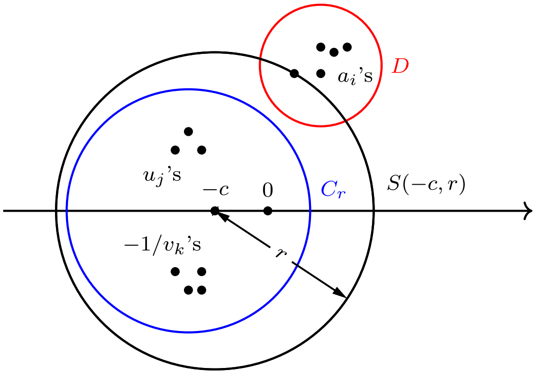

Fig. 3. An illustration of the contours and poles in Proposition 1.3. The black circle is the boundary of the disk $S(-c, r)$, the blue circle is C_r and the red circle is D

where $\nu, \mu, \kappa \in \mathbb{N}$, $s \in \mathbb{Z}$ *and* $\zeta, c, u_j, v_k, \gamma \in \mathbb{C}$ *for* $j \in [1, \nu], k \in [1, \mu]$ *and* $g(\zeta, s) = g^c(\zeta, s; 0)$. *Let* I_ν *be a multiple integral of a form*

$$I_\nu = \frac{1}{\nu!} \oint_C \prod_{i=1}^{\nu} \frac{\mathrm{d}\zeta_i}{2\pi\mathrm{i}\, a_i^s \zeta_i} \frac{\prod_{1\le i\neq j\le\nu}(1-\zeta_i/\zeta_j)}{\prod_{1\le i,j\le\nu}(1-a_i/\zeta_j)} \prod_{i=1}^{\nu} \frac{g(\zeta_i, s)}{g(a_i, s)}, \tag{1.6}$$

where $a_i \in \mathbb{C}$, $i \in [1, \nu]$ *and the contours* C *include* $0, a_i, u_j, -1/v_k$ *for* $i \in [1, \nu]$, $j \in [1, \nu]$, $k \in [1, \mu]$.

We assume that there exists $c \in \mathbb{C}$ *such that an open disc of radius* $r = \min_{1\le j\le\nu}(|a_j + c|)$ *centered at* $-c$, *denoted by* $S(-c, r)$, *includes the poles at* $0, u_j$ *and* $-1/v_k$ *for* $j \in [1, \nu]$ *and* $k \in [1, \mu]$.

Then this multiple integral can be written as a Fredholm determinant

$$I_\nu = \left(\prod_{l=1}^{\mu} a_l^{-s}\right) \det\left(1 - K^c\right)_{\ell^2(\mathbb{N})}, \tag{1.7}$$

where the kernel is written in the form

$$K^c(x, y) = \sum_{k=0}^{\nu-1} \phi_k^c(x)\psi_k^c(y)$$

with

$$\phi_k^c(x) = \oint_D \frac{\mathrm{d}\xi}{2\pi\mathrm{i}} \frac{\xi^{k-\nu}}{g^c(\xi, s; x)(\xi - a_1)\cdots(\xi - a_{k+1})}, \tag{1.8}$$

$$\psi_k^c(x) = a_{k+1} \oint_{C_r} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,(\zeta + c)\zeta^{k-\nu+1}} g^c(\zeta, s; x)(\zeta - a_1)\cdots(\zeta - a_k). \tag{1.9}$$

The contour D *includes the poles at* a_i *for* $i \in [1, \nu]$ *and the contour* C_r *includes the poles at* $0, u_j, -1/v_k$ *for* $j \in [1, \nu]$, $k \in [1, \mu]$ *(see Fig. 3).*

Existing applications of Proposition 1.3 so far have used only the case $c = 0$ where the conditions on the contours D, C_r translate to those used in [46]. In this paper, c is introduced to prevent the right hand side of (1.7) from diverging when a_j goes to unity for all $j \in [1, \nu]$ and v_k approaches 1, which later makes it possible to perform the asymptotic analysis of I_ν in Sect. 6. Concretely, introducing c guarantees that the Fredholm determinant formula in (6.1) converges and that the inequalities (6.20) hold, which plays a significant role in the proof of Proposition 6.3.

1.4. Limiting distribution and nonlinear fluctuating hydrodynamics. We can perform an asymptotic analysis of the multiple integral formula from Theorem 1.1 using the rewriting in terms of a Fredholm determinant as in Proposition 1.3. A novel feature compared to the single species case is that one encounters dynamic poles, i.e. poles located at additional integration variables in the kernel of a Fredholm determinant. In our case we will see that, by taking certain poles at the beginning, one can evaluate the effects of the interaction and observe that in fact the two sets of variables decouple asymptotically as the parameter t tends to infinity. As a consequence we can study the long time limit of the joint distribution, which is our second main result.

To state our result let us recall the definitions (1.4) and (1.3) of the Fredholm determinant, and set $\mathrm{Ai}(x)$ to denote the Airy function [20,38,74,91] defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi \mathrm{i}} \int_C \exp\Big(z^3/3 - zx\Big) \mathrm{d}z, \tag{1.10}$$

where the contour C starts at $\infty\,\mathrm{e}^{-\pi \mathrm{i}/3}$ and goes to $\infty\,\mathrm{e}^{\pi \mathrm{i}/3}$.

Definition 1.4. [91] *The Fredholm determinant $F_2(s) := \det(1 - A)_{L^2(s,\infty)}$ is a distribution function called GUE Tracy–Widom distribution. Its kernel is the Airy kernel:*

$$A(x, y) = \int_0^\infty \mathrm{Ai}(x + \lambda)\,\mathrm{Ai}(y + \lambda)\mathrm{d}\lambda. \tag{1.11}$$

With these preparations, our result on the limiting distribution is stated as follows. Some comments about the scalings of variables will be given right after the theorem.

Theorem 1.5. *For the AHR model with $\alpha = \beta = \frac{1}{2}$ and the step-Bernoulli initial condition above, we have, in the long time limit,*

$$\lim_{t\to\infty} \mathbb{P}_{n,m}[N_+(t) = n, N_-(t) = m] = F_2(s_2) \cdot F_G(s_{\mathrm{g}}), \tag{1.12}$$

where on the left hand side we use the scaling

$$n = j_+(\rho)t - \frac{1}{12(1-\rho)}(2\rho c_2 s_2 t^{1/3} + (3-\rho)c_{\mathrm{g}} s_{\mathrm{g}} t^{1/2}),$$
$$m = j_-(\rho)t - \frac{1}{12(1-\rho)}(2(2-\rho)c_2 s_2 t^{1/3} + (1+\rho)c_{\mathrm{g}} s_{\mathrm{g}} t^{1/2}),$$

with c_2, c_{g} and $j_\pm(\rho)$ defined in (5.11) and (5.13).

F_2 and F_G on the right hand side are the distribution functions of the GUE Tracy–Widom and the standard Gaussian distributions respectively.

The specialisation to $\alpha = \beta = \frac{1}{2}$ is just for a simplicity. One can generalise our whole calculations and arguments to the case with $\alpha + \beta = 1$.

This theorem can be proved as written, but one may wonder what the meaning are of the scalings $t^{1/3}$, $t^{1/2}$ and variables s_2, $s_{\rm g}$ and also the reason for the appearance of the limiting distribution in (1.12). In fact, these can be understood from nonlinear fluctuating hydrodynamics (NLFHD) which is a heuristic physics theory for studying the long time behaviour of one dimensional multi-component systems.

NLFHD was first proposed in [89,97] to provide concrete predictions for the long time behaviour of one dimensional Hamiltonian dynamics with nonlinear interaction. One first writes down the hydrodynamic equations for three conserved quantities of the original system, stretch, momentum and energy, and takes fluctuation effects into account by adding white noise. By taking appropriate linear combinations of the three conserved quantities one switches to normal modes which have intrinsic propagating speeds.

It is natural to study fluctuations of the normal modes. A key idea of NLFHD is that if the speeds of the normal modes are different, then the interaction among them should be irrelevant in the long time limit and thus the fluctuations for each mode would be described by the single species noisy-Burgers equation (a.k.a. KPZ equation). Fluctuations in the long time limit would then generically be of order $\mathcal{O}(t^{1/3})$ and given by the Tracy–Widom type distributions.

NLFHD has also been formulated for stochastic models and gives concrete predictions for the distribution of currents for multispecies models [37]. A prototypical model is in fact the AHR model, which has two obvious conserved quantities, the number of + and that of − particles. The hydrodynamic equation is given by

$$\frac{\partial \vec{\rho}(x,t)}{\partial t} + \frac{\partial \vec{j}(\vec{\rho}(x,t))}{\partial x} = 0, \tag{1.13}$$

where $\vec{\rho}(x,t) = (\rho_+(x,t), \rho_-(x,t))$ is the density vector and $\vec{j}(\vec{\rho}) = (\mathsf{j}_+(\vec{\rho}), \mathsf{j}_-(\vec{\rho}))$ denotes the macroscopic current of $\pm$ particles given by

$$\mathsf{j}_+(\vec{\rho}) = \rho_+(1-\rho_+-\rho_-) + 2\rho_+\rho_-, \tag{1.14}$$
$$\mathsf{j}_-(\vec{\rho}) = -(1-\rho_+-\rho_-)\rho_- - 2\rho_+\rho_- \,. \tag{1.15}$$

The normal modes for the AHR model follow from diagonalisation of the Jacobian matrix $\partial\vec{j}/\partial\vec{\rho}$.

In [73], a step initial condition was studied. As a particular case related to our study, let us consider a mix of Bernoulli and step initial conditions, namely, we consider the situation in which initially the left half of $\mathbb{Z}$ is filled randomly with + particles with density ρ[3] while the right half of $\mathbb{Z}$ is filled with − particles, as in Fig. 2. This seems the simplest nontrivial initial condition for the AHR model. Note that the special case with $\rho = 1$ is simpler but this case can be easily seen to be equivalent to the single species of TASEP with step initial conditions since there are no holes and "−" particles behave as holes.

For the step-Bernoulli initial condition, one can consider fluctuations of the normal modes, and NLFHD predicts the following. Firstly, the variables $s_{\rm g}(n,m,t)$ and $s_2(n,m,t)$ defined below (5.13) are just the scaling variables for the fluctuations in the

[3] This ρ should be distinguished from $\vec{\rho}$, which appears only in this subsection.

normal directions, and the resulting prediction of NLFHD relevant to our problem is the following:

$$\begin{aligned}&\mathbb{P}_{\infty,\infty}[N_+(t)=n, N_-(t)=m]\\&\simeq\left|\det\begin{pmatrix}\frac{\partial s_2(n,m;t)}{\partial n} & \frac{\partial s_2(n,m;t)}{\partial m}\\ \frac{\partial s_{\mathrm{g}}(n,m;t)}{\partial n} & \frac{\partial s_{\mathrm{g}}(n,m;t)}{\partial m}\end{pmatrix}\right|\partial F_2(s_2(n,m;t))\cdot\partial F_{\mathrm{G}}(s_{\mathrm{g}}(n,m;t)).\end{aligned}\tag{1.16}$$

For the s_{g} mode, one expects the Gaussian rather than Tracy–Widom due to the strong effects from the initial randomness.

In the case of single species TASEP, the study of the current distribution for the step initial condition is reduced to a problem of finite number of particles because each particle in TASEP can not affect the dynamics of particles in front. For the AHR model this kind of property does not hold any more and therefore the problem for the step type initial condition in the previous paragraph cannot be reduced to one with only finite number of particles. We can therefore not directly establish the above prediction of NLFHD using the formula (1.2) for the transition probability because this holds only for a finite number of particles.

Instead we can generalise the prediction of the NLFHD to the case with large but finite number of particles. This generalisation is nontrivial and a full discussion will be given elsewhere. The main idea is that we may assume that after the two species of particles become separate the fluctuations are transported by the hydrodynamics. Then the probability of interest $\lim_{t\to\infty}\mathbb{P}_{n,m}[N_+(t)=n, N_-(t)=m]$ would be written as a sum of contributions of $\lim_{t\to\infty}\mathbb{P}_{\infty,\infty}[N_+(t)=i, N_-(t)=j]$ from a certain region only. Integrating (1.16) over the independent modes s_2, s_{g} leads to a product of F_2 and F_G, which is exactly the limiting distribution appearing in Theorem 1.5. In this way, the probability $\mathbb{P}_{n,m}[N_+(t)=n, N_-(t)=m]$ studied in this paper has a direct relation to predictions of NLFHD, providing a justification of our investigation of this quantity.

The main results in this paper were announced in [27]. The purpose of this paper is to provide full details of the intricate and elaborate calculations as well as the mathematical proofs that establish our results. As already mentioned above, details of the physics oriented discussion of NLFHD for the case with finite number of particles falls outside the scope of this paper and will be given in a separate publication.

The rest of the paper is organised as follows. In Sect. 2 we give a multiple integral formula for the transition probability, or Green's function. In Sect. 3, we give a multiple integral formula for a certain joint distribution of the currents. In Sect. 4 we explain a rewriting of a multiple integral to a Fredholm determinant. In Sects. 5, 6 and 7 we perform asymptotics and establish Theorem 1.5. In the appendices we provide further technical details about asymptotic estimates as well as details related to the Bethe ansatz for the AHR model, and symmetrisation identities for multi-variable rational functions. In addition we provide a detailed derivation of a crucial decoupling identity. Details of some calculations and proofs are given in Appendices from A through F.

2. Transition Probability

We consider the AHR model with n particles of one type, denoted by plus, and m particles of a second type, denoted by minus. Define the set of coordinates by

$$\mathbb{W}^k := \{\vec{x}=(x_1,\ldots,x_k)\in\mathbb{Z}^k : x_1<x_2<\cdots<x_k\}.\tag{2.1}$$

Suppose $P(\vec{x}, \vec{y}; t)$ is the probability that at time t, plus particles are sitting at position $\vec{x} = (x_1, \dots, x_n) \in \mathbb{W}^n$, while the minus particles are at $\vec{y} = (y_1, \dots, y_m) \in \mathbb{W}^m$, and let Ω^{n+m} be the physical domain of $(\vec{x}, \vec{y})$, i.e. we write $(\vec{x}, \vec{y}) \in \Omega^{n+m}$ if $(\vec{x}, \vec{y}) \in \mathbb{W}^n \times \mathbb{W}^m$ with the additional condition that $\vec{x} \cap \vec{y} = \emptyset$. From the dynamics of the system described above, we can write down the master equation for $P(\vec{x}, \vec{y}; t)$ in the AHR model for the cases where all particles are far apart from each other,

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\vec{x}, \vec{y}; t) = \beta \sum_{i=1}^{n} P(\vec{x}_i^-, \vec{y}; t) + \alpha \sum_{j=1}^{m} P(\vec{x}, \vec{y}_j^+; t) - (\beta n + \alpha m) P(\vec{x}, \vec{y}; t), \quad (2.2)$$

where $\vec{x}_i^{\pm} := (x_1, \dots, x_i \pm 1, \dots, x_n)$. We emphasise again that (2.2) only describes the time evolution of hops but without swaps and exclusions.

On the right hand side of (2.2), the positive terms describe the arrival part of the change in $P(\vec{x}, \vec{y}; t)$. Namely, the state $(\vec{x}_i^-, \vec{y})$ turns into the state $(\vec{x}, \vec{y})$ at rate β, while $(\vec{x}, \vec{y}_j^+)$ turns into $(\vec{x}, \vec{y})$ at rate α. The negative term on the right hand side comes from the loss part of the change in $P(\vec{x}, \vec{y}; t)$. Specifically, the state $(\vec{x}, \vec{y})$ evolves into the state $(\vec{x}_i^+, \vec{y})$ at rate β and the state $(\vec{x}, \vec{y}_j^-)$ at rate α.

In order to describe the interactions between particles, i.e. the exclusions and swaps, one could include appropriate Kronecker delta functions on the right hand side of (2.2). Alternatively, one can impose the following boundary conditions. For all appropriate i, j,

- Interactions between plus particles:

$$P(x_1, \dots, x_i, x_{i+1} = x_i, \dots, x_n; \vec{y}) = P(x_1, \dots, x_i, x_{i+1} = x_i + 1, \dots, x_n; \vec{y}). \quad (2.3)$$

- Interactions between minus particles:

$$P(\vec{x}; y_1, \dots, y_{i-1} = y_i, y_i, \dots, y_m) = P(\vec{x}; y_1, \dots, y_{i-1} = y_i - 1, y_i, \dots, y_m). \quad (2.4)$$

- Interactions between plus and minus particles:

$$\begin{aligned} P(\vec{x}; y_1, \dots, y_j = x_i + 1, \dots, y_m) &= \beta P(\vec{x}; y_1, \dots, y_j = x_i, \dots, y_m) \\ &+ \alpha P(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_n; y_1, \dots, y_j = x_i + 1, \dots, y_m). \end{aligned} \quad (2.5)$$

We observe that (2.2)–(2.5) are well defined for $P(\vec{x}; \vec{y})$ on $\mathbb{Z}^n \times \mathbb{Z}^m$. However, we know that the position states are defined only on the physical coordinates $(\vec{x}, \vec{y}) \in \Omega^{n+m}$. The solution of (2.2)–(2.5) indeed gives the transition probability of the AHR model on Ω^{n+m}, but $P(\vec{x}; \vec{y})$ is no longer a probability on $(\mathbb{Z}^n \times \mathbb{Z}^m) \setminus (\Omega^{n+m})$. We refer to Appendix A for further details on the boundary conditions.

Imposing the initial condition that the positions of plus and minus particles at $t = 0$ are given by $\vec{x}^{(0)}$ and $\vec{y}^{(0)}$, respectively, the probability $P(\vec{x}, \vec{y}; t)$ is called the *transition probability* or *Green's function* of the model. We emphasise that this probability is independent of the absolute values of positions, but depends only on the relative position of the initial and final states, and we denote it by $G(\vec{x}, \vec{y}, t; \vec{x}^{(0)}, \vec{y}^{(0)})$. The initial condition is given by

$$G(\vec{x}, \vec{y}, 0; \vec{x}^{(0)}, \vec{y}^{(0)}) = \prod_{j=1}^{n} \delta(x_j - x_j^{(0)}) \prod_{k=1}^{m} \delta(y_k - y_k^{(0)}). \quad (2.6)$$

A transition probability $G(\vec{x}, \vec{y}, t; \vec{x}^{(0)}, \vec{y}^{(0)})$ is the function that satisfies the master equation (2.2), the boundary conditions (2.3)–(2.5), as well as the initial condition (2.6). Generally, the transition probability is constructed as an integral form

$$G(\vec{x}, \vec{y}, t; \vec{x}^{(0)}, \vec{y}^{(0)}) = \oint_C \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}} \, \mathrm{e}^{\Lambda_{n,m} t} \, A(\vec{z}, \vec{w}) \, \psi(\vec{x}; \vec{y}), \tag{2.7}$$

where $\psi(\vec{x}; \vec{y})$ and $\Lambda_{n,m}$ are the Bethe wave function and eigenvalue of the Markov generator found by the Bethe ansatz, respectively, see (A.12) and (A.13). $A(\vec{z}, \vec{w})$ and the contour C are chosen so that the initial condition is satisfied where $\vec{z}$ and $\vec{w}$ denote the collection of variables z_j's with $j \in [1, n]$ and w_k's with $k \in [1, m]$, respectively. This construction of the transition probability can be seen as a generalisation of those for the single species TASEP and single species ASEP in [87] and [93], respectively. In Appendix A we provide details of the construction of the transition probability using the Bethe ansatz. Here we give the final explicit integral form for $G(\vec{x}, \vec{y}, t; \vec{x}^{(0)}, \vec{y}^{(0)})$ and prove that it satisfies all desired properties (2.2)–(2.6).

Theorem 2.1. *Consider the AHR model on $\mathbb{Z}$ with $\alpha + \beta = 1$ which has n plus and m minus particles. Define r_j as the number of plus particles to the right of the j^{th} minus particles at time t, and $r_j^{(0)}$ the same quantity at $t = 0$, i.e.,*

$$r_j := \#\{x_i \in \vec{x} \mid x_i \geq y_j\}, \qquad r_j^{(0)} := \#\{x_i^{(0)} \in \vec{x}^{(0)} \mid x_i^{(0)} \geq y_j^{(0)}\}.$$

Then the transition probability is given by (2.7) where

$$A(\vec{z}, \vec{w}) = \prod_{k=1}^{m} (\beta w_k)^{r_k^{(0)}} \prod_{j=1}^{n} z_j^{-x_j^{(0)}-1} \prod_{k=1}^{m} w_k^{y_k^{(0)}-1} \prod_{j=1}^{n} (1 - z_j)^j \prod_{k=1}^{m} (1 - w_k)^{-k}, \tag{2.8}$$

$$\Lambda_{n,m} = \beta \sum_{j=1}^{n} (z_j^{-1} - 1) + \alpha \sum_{k=1}^{m} (w_k^{-1} - 1), \tag{2.9}$$

and $\psi(\vec{x}; \vec{y})$ is given by (A.12), and the contours C in (2.7) is chosen as $C_{\vec{z},\vec{w}}$ defined below (1.2).

Proof. The proof consists of four parts. We show below that (2.7) satisfies the master equation (2.2), the boundary conditions (2.3)–(2.5), the initial condition (2.6) and finally that the solution is unique.

(i) **Proof of master equation** (2.2)
Clearly the integrand is C^1 continuous in time. By interchanging $\mathrm{d}/\mathrm{d}t$ with the integral and the explicit form of $\Lambda_{n,m}$, the evolution equation (2.2) follows.

(ii) **Proof of boundary conditions** (2.3)–(2.5) The proof of the boundary conditions (2.3)–(2.5) is elementary but detailed, we give it in Appendix B.

(iii) **Proof of initial condition** (2.6) Firstly, we show by mathematical induction that (2.7) at $t = 0$ has vanishing residues unless $\pi_i = i$ and $x_i = x_i^{(0)}$ for all i by carrying out the $\vec{z}$-integrations before $\vec{w}$-integrations. To see this, we first prove the case for $i = 1$.
If $x_1 \geq x_1^{(0)}$, then since the components of $\vec{x}$ are well ordered, $x_j \geq x_1^{(0)}$ for all j. Consider now an arbitrary permutation π such that $\pi_k = 1$. The exponent of z_1 in

the integrand of (2.7) is given by $z_1^{x_k - x_1^{(0)} - 1}$, and the exponent in this expression is always non-negative unless $k = 1$ and $x_1 = x_1^{(0)}$. If these conditions are not met, then at time $t = 0$ the integrand is analytic at $z_1 = 0$ and therefore has a vanishing residue.

For the induction step we assume first that (2.7) vanishes at $t = 0$ unless $x_i = x_i^{(0)}$ and $\pi_i = i$ for all $i \le \ell - 1$. If $\pi_k = \ell$, the exponent of z_ℓ is $x_k - x_\ell^{(0)} - 1$, and by the induction hypothesis we have that $k \ge \ell$, and hence the exponent is non-negative. This implies that (2.7) is zero unless $\pi_\ell = \ell$ and $x_\ell = x_\ell^{(0)}$. Therefore, we can conclude that $G(\vec{x}, \vec{y}, 0; \vec{x}^{(0)}, \vec{y}^{(0)})$ is nonzero only when $x_i = x_i^{(0)}$ for all i and $\pi = \mathsf{id}$.

When $x_i = x_i^{(0)}$ for all $i \in [1, n]$ and $\pi = \mathsf{id}$, the after integration over the $\vec{z}$-variables the transition probability (2.7) at $t = 0$ becomes

$$G(\vec{x}, \vec{y}, 0; \vec{x}^{(0)}, \vec{y}^{(0)}) = \oint_0 \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}} \sum_{\sigma \in S_m} \operatorname{sign}(\sigma) \prod_{k=1}^{m} \frac{(\beta w_k)^{r_k^{(0)}}}{(\beta w_{\sigma_k})^{r_k}} \prod_{k=1}^{m} \left(\frac{w_k - 1}{w_{\sigma_k} - 1} \right)^{-k} w_{\sigma_k}^{-y_k} w_k^{y_k^{(0)} - 1},$$

where the integral symbol is used as explained in Remark 1.2, namely the $\vec{w}$-contours include only the poles at the origin. We now prove that the above function is nonzero only when $y_j = y_j^{(0)}$ for all $j \in [1, m]$ and $\sigma = \mathsf{id}$. While plus particles hop to the right, minus particles always hop to the left and so we have that $y_j^{(0)} \ge y_j$ for all $j \in [1, m]$. Consider an arbitrary permutation σ such that $\sigma_k = j$ with $j > k$. At time $t = 0$, the exponent of w_j in the integrand in (2.7) is given by $w_j^{y_j^{(0)} - y_k + r_j - r_k - 1}$ because $r_k = r_k^{(0)}$ for all $k \in [1, m]$ when $x_i = x_i^{(0)}$ and $y_j = y_j^{(0)}$ for all $i \in [1, n]$ and $j \in [1, m]$. By the definition of r_j, we must have $y_j - y_k + r_j - r_k > 0$ for any $j > k$. Hence $y_j^{(0)} - y_k + r_j - r_k - 1 \ge y_j - y_k + r_j - r_k - 1 > 0$. Therefore, the exponent of w_j is always positive unless $\sigma_j = j$. When $\sigma = \mathsf{id}$, the exponent of w_j becomes $y_j^{(0)} - y_j - 1$, which results in a vanishing residue unless $y_j^{(0)} = y_j$.

It remains to show that the transition probability is normalised as $G(\vec{x}^{(0)}, \vec{y}^{(0)}, 0; \vec{x}^{(0)}, \vec{y}^{(0)}) = 1$. This can be easily seen by the residue theorem.

(iv) **Proof of uniqueness** The transition probability is a solution of a master equation with bounded initial condition, and the number of particle jumps for a given time t in the AHR model is bounded by a Poisson random variable with parameter given by a constant times t. The global existence and uniqueness is therefore guaranteed by general considerations as provided in Proposition 4.9 and Appendix C of [13].

This concludes the proof for the Green's function (2.7). □

3. Joint Current Distribution

In the following, we are interested in the probability that all plus and minus particles have crossed the origin at time t. Exact expressions for such joint current distributions under different initial conditions can be derived using the transition probability. We will focus here on the case in which initially n plus particles are distributed by the Bernoulli measure with density ρ at negative integers, and the first m sites at the non-negative integers are occupied by minus particles. We called such initial condition the step-Bernoulli initial condition in the Sect. 1. Note that setting $\rho = 1$ corresponds to an initial condition which we may call a step-step initial condition because initial conditions are step type for both plus and minus particles. In the following we will denote the probability that all particles passed the origin at time t by $P_{n,m,\rho}(t)$, i.e.

$$P_{n,m,\rho}(t) := \mathbb{P}_{n,m}[N_+(t) = n, N_-(t) = m], \tag{3.1}$$

and call it the joint current distribution. In terms of $P_{n,m,\rho}(t)$, we recall Theorem 1.1:

Theorem 1.1. *For the AHR model with $\alpha + \beta = 1$ and the step-Bernoulli initial condition with density ρ, the joint current distribution $P_{n,m,\rho}(t)$ is given by*

$$P_{n,m,\rho}(t) = (-1)^{n+m} \oint_{C_{\vec{z},\vec{w}}} \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\, \mathrm{e}^{\Lambda_{n,m}t}$$
$$\times \frac{\rho^n \prod_{1\le i<j\le n}(z_i - z_j) \prod_{1\le k<\ell\le m}(w_\ell - w_k) \prod_{j=1}^{n} z_j^{n-j} \prod_{k=1}^{m} w_k^{k-1}}{\prod_{j=1}^{n}(z_j-1)^{n+1-j}\big(1-(1-\rho)z_j\big) \prod_{k=1}^{m}(w_k - 1)^k \prod_{j=1}^{n}\prod_{k=1}^{m}\Big(\alpha z_j + \beta w_k\Big)}, \tag{3.2}$$

where $\Lambda_{n,m}$ is given by (2.9), and the contours $C_{\vec{z},\vec{w}}$ are chosen such that $\vec{z}$-contours enclose only the pole at 0 and $\vec{w}$-contours enclose only the poles at 0 and $\{-\alpha z_j/\beta\}_{j=1}^{n}$.

This theorem is proved by summing over the initial and final coordinates of the particles in the transition probability, and by making use of the following lemma. We provide the proof of this lemma in Appendix C. Remark that it can also be obtained by specialisation of the equation (9) in [95].

Lemma 3.1.

$$\sum_{\pi\in S_n} \mathrm{sign}(\pi) \prod_{i=1}^{n} \left(\frac{z_{\pi_i}-1}{z_{\pi_i}}\right)^i \frac{1}{1-(1-\rho)\prod_{j=1}^{i} z_{\pi_j}} = \frac{\prod_{1\le i<j\le n}(z_j - z_i)\prod_{i=1}^{n}(z_i-1)}{\prod_{i=1}^{n} z_i^n (1-(1-\rho)z_i)}.$$

Proof of Theorem 1.1. For the given initial and final coordinates, all the plus particles initially are to the left of all the minus particles, and at time t end up to the right of all minus particles. We therefore have that $r_k = n$, $r_k^{(0)} = 0$ for all $k \in [1, m]$ and the final transition probability is given by

$$
\begin{aligned}
G(\vec{x}, \vec{y}, t; \vec{x}^{(0)}, \vec{y}^{(0)}) = \oint \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}}\ \mathrm{e}^{\Lambda_{n,m} t} \prod_{k=1}^{m} \prod_{j=1}^{n} \frac{1}{\alpha z_j + \beta w_k} \\
\times \sum_{\pi \in S_n} \operatorname{sign}(\pi) \prod_{j=1}^{n} \left(\frac{z_j - 1}{z_{\pi_j} - 1} \right)^{j} z_{\pi_j}^{x_j} z_j^{-x_j^{(0)} - 1} \\
\times \sum_{\sigma \in S_m} \operatorname{sign}(\sigma) \prod_{k=1}^{m} \left(\frac{w_k - 1}{w_{\rho_k} - 1} \right)^{-k} w_{\rho_k}^{-y_k} w_k^{y_k^{(0)} - 1}.
\end{aligned}
\tag{3.3}
$$

When $r_k = n$ and $r_k^{(0)} = 0$ hold for all $k \in [1, m]$, the calculations carried out in the third step of the proof of Theorem 2.1 are valid also for the case where the contours are chosen as $C_{\vec{w},\vec{z}}$ defined in Remark 1.2 by carrying out the $\vec{w}$-integrations before the $\vec{z}$-integrations. Hence it is allowed to exchange the order of $\vec{z}$-integrals and $\vec{w}$-integrals in (3.3).

In the Bernoulli measure, the distances among the initial positions of the plus particles are independently distributed and each is distributed as a geometric random variable with parameter $1 - \rho$, i.e., the probability that plus particles locating at $\vec{x}^{(0)}$ is given by

$$
\Pi(\vec{x}^{(0)}; 0) = \rho^n \prod_{j=1}^{n-1} (1-\rho)^{x_{j+1}^{(0)} - x_j^{(0)} - 1} \cdot (1-\rho)^{-x_n^{(0)} - 1} = \left(\frac{\rho}{1 - \rho} \right)^{n} (1-\rho)^{-x_1^{(0)}}. \tag{3.4}
$$

The joint current distribution $P_{n,m,\rho}(t)$ thus is the sum of

$$
G(\vec{x}, \vec{y}, t; \vec{x}^{(0)}, \vec{y}^{(0)}) \Pi(\vec{x}^{(0)}; 0),
$$

over all final coordinates $0 \le x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 \cdots < y_m \le -1$ and all initial coordinates $x_1^{(0)} < \cdots < x_n^{(0)} \le -1$, with $G(\vec{x}, \vec{y}, t; \vec{x}^{(0)}, \vec{y}^{(0)})$ given by (3.3) and $y_j^{(0)} = j - 1$. The sums of initial and final coordinates are calculated by taking geometric series, and the sums of permutations π, σ in (3.3) are computed using Lemma 3.1 with $\rho = 0$, which coincides with the totally asymmetric case of (1.6) in [93], resulting in,

$$
\begin{aligned}
P_{n,m,\rho}(t) = (-1)^{n+m} \oint_0 \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \oint_0 \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}}\ \mathrm{e}^{\Lambda_{n,m} t} \\
\times \frac{\rho^n \prod_{1 \le i < j \le n} (z_j - z_i) \prod_{1 \le k < \ell \le m} (w_\ell - w_k) \prod_{j=1}^{n} z_j^{n-j} \prod_{k=1}^{m} w_k^{k-1}}{\prod_{j=1}^{n} (z_j - 1)^{n+1-j} (1-(1-\rho) \prod_{i=1}^{j} z_i) \prod_{k=1}^{m} (w_k - 1)^k \prod_{j=1}^{n} \prod_{k=1}^{m} \left(\alpha z_j + \beta w_k \right)}.
\end{aligned}
$$

Since the change of variables $z_j \to z_{\sigma_j}$ for all $j \in [1, n]$ does not affect the value of $\vec{z}$-integrals for all permutations $\sigma \in S_n$, summing up $\vec{z}$-integrals in which $\vec{z}$-variables are changed in such a way over $\sigma \in S_n$ is equivalent to multiplying $\vec{z}$-integrals by $n!$. Because of $\prod_{1 \le i < j \le n} (z_{\sigma_j} - z_{\sigma_i}) = \operatorname{sign}(\sigma) \prod_{1 \le i < j \le n} (z_j - z_i)$, the sum over permutations $\sigma \in S_n$ can be computed using Lemma 3.1 and we obtain

$$
\begin{aligned}
P_{n,m,\rho}(t) = (-1)^{n+m} \frac{\rho^n}{n!} \oint_0 \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \oint_0 \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi \mathrm{i}}\ \mathrm{e}^{\Lambda_{n,m} t} \frac{\prod_{1 \le k < \ell \le m} (w_\ell - w_k) \prod_{k=1}^{m} w_k^{k-1}}{\prod_{k=1}^{m} (w_k - 1)^k} \\
\times \frac{\prod_{j=1}^{n} z_j^n \left[\prod_{1 \le i < j \le n} (z_j - z_i) \right]^2}{\prod_{j=1}^{n} (z_j - 1)^n (1 - (1 - \rho) z_j) \prod_{j=1}^{n} \prod_{k=1}^{m} \left(\alpha z_j + \beta w_k \right)}.
\end{aligned}
\tag{3.5}
$$

Consider the symmetrisation identity

$$\sum_{\pi\in S_N}\mathrm{sign}(\pi)\prod_{j=1}^{N}\left(\frac{1}{z_{\pi_j}-1}\right)^{j}=\frac{\prod_{1\le i<j\le N}(z_i-z_j)}{\prod_{j=1}^{N}(z_j-1)^N}, \tag{3.6}$$

which can be proved using a Vandermonde determinant. Since the right hand side of (3.6) for $N=n$ appears in (3.5), we can substitute the left hand side of (3.6) into it. By the change of variables $z_j\to z_{\pi_j}$ for all $j\in[1,n]$ for each term of the sum with respect to $\pi\in S_n$, it turns out that all terms are equal and we can get the conclusion (3.2). □

In order to prove our main result, Theorem 1.5, we rewrite the joint current distribution $P_{n,m,\rho}(t)$ in the form (3.2) as (5.2) in Sect. 5. It is suitable for asymptotic analyses when $n<m$, which is compatible with our scaling (5.12). In the remaining part of this section let us consider the opposite case $n\ge m$, because this case is simpler to analyse and is still useful to understand what kind of asymptotics will be considered for the $n<m$ case. When $n\ge m$, at late times we would only observe plus particles crossing the origin while all minus particles have already crossed. Hence $P_{n,m,\rho}(t)$ in this region is expected to be asymptotically close to the current distribution of the single species TASEP [15,47,79].

Suppose now that $n\ge m$ holds (in fact the following arguments are valid when $n\ge m-2$ holds). In this case we can evaluate the contour integrals over the $\vec{w}$-variables in (3.2) for the following reasons. The eigenvalue $\Lambda_{n,m}$ given by (2.9) introduces an essential singularity at the origin. It is therefore convenient to replace the contours as enclosing all other possible poles except the origin, and including ∞. First let us consider the residue at ∞. The degree of w_k near ∞ in the integrand of (3.2) is $(m-1+k-1)-(k+n)=m-2-n\le 0$ so that the residue at ∞ is zero. Next consider the simple poles at $w_k=-\alpha z_j/\beta$. Such residues will cancel the variable z_j in $\Lambda_{n,m}$, making the integrand analytic in z_j and hence give rise to a zero residue at $z_j=0$.

The only poles with non-zero contribution are therefore at $w_k=1$. The residue of the simple pole at $w_1=1$ can be easily evaluated. The factor $\prod_{\ell=2}^{m}(w_\ell-w_1)$ in such residues will decrease the order of each pole at $w_k=1$ by one. Hence the second order pole at $w_2=1$ becomes a simple pole and its residue can therefore be simply evaluated subsequently. The integration of w_2 again reduces the order of the pole at $w_3=1$ by one, and hence it also becomes a simple pole. Evaluating all poles at $w_k=1$ sequentially, we arrive at the following result,

$$P_{n,m,\rho}(t)=\frac{\rho^n}{n!}\oint_0\prod_{j=1}^{n}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}}\,\mathrm{e}^{\Lambda_{n,0}t}\frac{\prod_{1\le i<j\le n}(z_i-z_j)^2}{\prod_{j=1}^{n}(1-(1-\rho)z_j)(z_j-1)^n\prod_{j=1}^{n}\left(\alpha z_j+\beta\right)^m},\ \text{for } n\ge m-2, \tag{3.7}$$

and we have made use of the symmetrisation identity

$$\sum_{\pi\in S_n}\mathrm{sign}(\pi)\prod_{j=1}^{n}z_{\pi_j}^{n-j}(z_{\pi_j}-1)^{j-1}=\prod_{1\le j<k\le n}(z_j-z_k),$$

which can be proved using a Vandermonde determinant.

Standard asymptotic analysis [15,47] then shows that the long time limit of $P_{n,m,\rho}(t)$ is governed by the GUE Tracy–Widom distribution [39,72,91], which is the same as the asymptotic current distribution for the single species TASEP, as expected.

For a special case, we also have an exact correspondence to the single species TASEP.

Corollary 3.2. *When $n = m$, $\alpha = \beta = 1/2$ and $\rho = 1$, we have, by applying the change of variable $z_j = x_j/(2 - x_j)$ in (3.7),*

$$P_{n,n,\rho}(t) = \frac{1}{n!} \oint_0 \prod_{j=1}^{n} \frac{\mathrm{d}x_j}{2\pi \mathrm{i}} \, \mathrm{e}^{\mathcal{E}t} \frac{\prod_{1 \le i < j \le n} (x_i - x_j)^2}{\prod_{j=1}^{n} (x_j - 1)^n}, \tag{3.8}$$

where

$$\mathcal{E} = \sum_{j=1}^{n} (x_j^{-1} - 1),$$

and this integral can also be evaluated by reversing the orientations of the contours and considering the contributions from the poles at $1, \infty$.

This recovers the distribution for the single species TASEP for the step initial condition, namely, the second equation in the remark following the proof of the corollary of Theorem 5.2 in [93]. This is understood easily because the AHR model for this particular setting is equivalent to the single species TASEP by regarding + particle as a particle and and − particle as a hole.

4. Fredholm Determinant

In this section, we describe a general method of converting a multiple contour integral into a Fredholm determinant. In other words, we will give a proof of Proposition 1.3, which was stated in Sect. 1. The main idea is to transform the integrand into determinants using the Cauchy determinant identity. The integral can then be converted into a determinant according to the Cauchy–Binet formula, and subsequently into a Fredholm determinant.

For $\nu \in \mathbb{N}$, let us now first write down again the ν-fold integral (1.6) we want to consider, which reads

$$I_\nu = \frac{1}{\nu!} \oint_C \prod_{i=1}^{\nu} \frac{\mathrm{d}\zeta_i}{2\pi \mathrm{i}\, a_i^s \zeta_i} \frac{\prod_{1 \le i \ne j \le \nu} (1 - \zeta_i/\zeta_j)}{\prod_{1 \le i, j \le \nu} (1 - a_i/\zeta_j)} \prod_{i=1}^{\nu} \frac{g(\zeta_i, s)}{g(a_i, s)}, \tag{4.1}$$

where $g(\zeta, s) = g^c(\zeta, s; 0)$ with

$$g^c(\zeta, s; \kappa) = \prod_{j=1}^{\nu} \frac{1}{1 - u_j/\zeta} \prod_{k=1}^{\mu} \frac{1}{1 + v_k \zeta} \times \zeta^{\mu - \nu - s} (\zeta + c)^\kappa \, \mathrm{e}^{\gamma \zeta}, \tag{4.2}$$

and $\mu, \kappa \in \mathbb{N}$, $s \in \mathbb{Z}$ and $\zeta, c, a_j, u_j, v_k, \gamma \in \mathbb{C}$ for $j \in [1, \nu]$ and $k \in [1, \mu]$. The contours C include $0, a_j, u_j, -1/v_k$ for $j \in [1, \nu]$ and $k \in [1, \mu]$.

Let us assume temporarily that all a_j's are distinct. The factor $\prod_{i \ne j} (1 - \zeta_i/\zeta_j) / \prod_{i,j} (1 - a_i/\zeta_j)$ in the integrand can be written as a product of two determinants via the Cauchy determinant identity,

$$\frac{\prod_{i \ne j} (1 - \zeta_i/\zeta_j)}{\prod_i \zeta_i \prod_{i,j} (1 - a_i/\zeta_j)} = \frac{\prod_{i \ne j} (\zeta_i - \zeta_j)}{\prod_{i,j} (\zeta_i - a_j)} = \det \left(\frac{1}{\zeta_i - a_j} \right)_{1 \le i,j \le \nu} \det \left(\frac{1}{\zeta_k - a_\ell} \right)_{1 \le k,\ell \le \nu} \frac{\prod_{i,j} (\zeta_i - a_j)}{\prod_{i \ne j} (a_i - a_j)}.$$

Recall the Cauchy–Binet identity (or Andreief identity):

$$\frac{1}{\nu!}\int \det\left(f_i(x_j)\right)_{1\le i,j\le\nu} \det\left(h_i(x_j)\right)_{1\le i,j\le\nu} \prod_{i=1}^{\nu} \mathrm{d}u(x_i) = \det\left(\int f_i(x)h_j(x)\mathrm{d}u(x)\right)_{1\le i,j\le\nu},$$

from which, the ν-fold integral is written as a single determinant,

$$\begin{aligned} I_\nu &= \frac{1}{\nu!}\oint_C \prod_{j=1}^{\nu}\left[\frac{\mathrm{d}\zeta_j}{2\pi \mathrm{i}\, a_j^s}\frac{g(\zeta_j,s)}{g(a_j,s)}\frac{\prod_\ell(\zeta_j-a_\ell)}{\prod_{\ell\neq j}(a_j-a_\ell)}\right]\det\left(\frac{1}{\zeta_i-a_j}\right)_{1\le i,j\le\nu}\det\left(\frac{1}{\zeta_k-a_\ell}\right)_{1\le k,\ell\le\nu} \\ &= \det\left(\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\frac{a_j^{-s}}{(\zeta-a_j)(\zeta-a_k)}\frac{g(\zeta,s)}{g(a_j,s)}\frac{\prod_\ell(\zeta-a_\ell)}{\prod_{\ell\neq j}(a_j-a_\ell)}\right)_{1\le j,k\le\nu}. \end{aligned}$$

From this explicit form it is clear that the poles at $\zeta = a_j$ and $\zeta = a_k$ are removable unless $j = k$, hence after evaluating residues at $\zeta = a_j$ for all j, we obtain

$$I_\nu = \prod_{l=1}^{\nu} a_l^{-s}\det\left(\delta_{jk} + \oint_{C_r}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\frac{1}{(\zeta-a_j)(\zeta-a_k)}\frac{g(\zeta,s)}{g(a_j,s)}\frac{\prod_\ell(\zeta-a_\ell)}{\prod_{\ell\neq j}(a_j-a_\ell)}\right)_{1\le j,k\le\nu}, \tag{4.3}$$

where the contour C_r includes only the poles at $0, u_j$ and $-1/v_k$, but not those at a_j. To transform the integral around C_r into a product of two operators (or matrices) we rewrite the factor $-1/(\zeta - a_j)$ in the form of a geometric series,

$$-1/(\zeta-a_j) = \sum_{x=1}^{\infty}(\zeta+c)^{x-1}/(a_j+c)^x. \tag{4.4}$$

Therefore the integral I_ν becomes

$$I_\nu = \prod_{l=1}^{\nu} a_l^{-s}\det\left(\delta_{jk} - \sum_{x=1}^{\infty}\oint_{C_r}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}(\zeta+c)}\frac{g^c(\zeta,s;x)}{g^c(a_j,s;x)}\frac{\prod_{\ell\neq k}(\zeta-a_\ell)}{\prod_{\ell\neq j}(a_j-a_\ell)}\right)_{1\le j,k\le\nu}. \tag{4.5}$$

Note that, by the assumptions of Proposition 1.3, one can deform the contour C_r and find c such that $|(\zeta + c)/(a_j + c)| \le \epsilon < 1$ holds for some positive constant ϵ and all $j \in [1, \nu]$, so that the geometric sum converges uniformly.

The sum over $x \ge 1$ in (4.5) can be interpreted as representing a product of two matrices, A^c and B^c, of dimensions $\nu \times \infty$ and $\infty \times \nu$. Namely one can write

$$I_\nu = \left(\prod_{l=1}^{\nu} a_l^{-s}\right)\det\left(1 - A^c B^c\right)_{1\le j,k\le\nu},$$

where the matrices A^c and B^c are given explicitly by

$$\begin{aligned} A^c(k,x) &= \oint_{C_r}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}(\zeta+c)}g^c(\zeta,s;x)\prod_{\ell\neq k}(\zeta-a_\ell), \\ B^c(x,j) &= \left(a_j^s g^c(a_j,s;x)\prod_{\ell\neq j}(a_j-a_\ell)\right)^{-1}. \end{aligned}$$

Then by swapping the product order of these two matrices, the integral I_ν is converted into a Fredholm determinant,

$$I_\nu = \left(\prod_{l=1}^{\nu} a_l^{-s}\right) \det\bigl(1 - K^c\bigr)_{\ell^2(\mathbb{N})},$$

where the kernel of the operator $K^c := B^c A^c$ is given by

$$K^c(x, y) = \sum_{j=1}^{\nu} B^c(x, j) A^c(j, y). \tag{4.6}$$

Note that, although A^c, B^c and the kernel $B^c A^c$ depend on c, I_ν is independent of c because c stemmed from the non-unique way that $1/(\zeta - a_j)$ can be represented as a geometric series in (4.4).

For the purpose of asymptotic analyses using steepest descent method in Sects. 6 and 7, we will show that the kernel of the Fredholm determinant can be written into a product of two contour integrals. The kernel K^c is given by

$$\begin{aligned}
K^c(x, y) &= \sum_{j=1}^{\nu} B^c(x, j) A^c(j, y) \\
&= \sum_{j=1}^{\nu} \frac{1}{g^c(a_j, s; x) \prod_{\ell \neq j}(a_j - a_\ell)} \oint_{C_r} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}(\zeta + c)} g^c(\zeta, s; y) \prod_{\ell \neq j} (\zeta - a_\ell) \\
&= \oint_{C_r} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}(\zeta + c)} \oint_D \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{1}{\zeta - \xi} \prod_{\ell=1}^{\nu} \frac{\zeta - a_\ell}{\xi - a_\ell} \frac{g^c(\zeta, s; y)}{g^c(\xi, s; x)} \\
&= \oint_{C_r} \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}(\zeta + c)} \oint_D \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{1}{\zeta - \xi} \left(\frac{\xi^\nu}{\zeta^\nu} \prod_{\ell=1}^{\nu} \frac{\zeta - a_\ell}{\xi - a_\ell} - 1 \right) \frac{\zeta^\nu g^c(\zeta, s; y)}{\xi^\nu g^c(\xi, s; x)}.
\end{aligned} \tag{4.7}$$

In the sequel the condition $|(\zeta + c)/(a_j + c)| \le \epsilon < 1$ is not necessary for the contour C_r. The contour D for the ξ integration includes the poles at a_j and should therefore be separated from C_r, namely, be chosen such that D does not intersect with C_r and also that neither D nor C_r encloses each other. The term -1 inside the parentheses is inserted for convenience of the next step and does not change the value of the integral.

Using a simple identity in [46],

$$\frac{1}{\zeta - \xi} \left(\frac{\xi^\nu}{\zeta^\nu} \prod_{\ell=1}^{\nu} \frac{\zeta - a_\ell}{\xi - a_\ell} - 1 \right) = \sum_{k=0}^{\nu-1} a_{k+1} \frac{(\zeta - a_1) \cdots (\zeta - a_k) \xi^k}{(\xi - a_1) \cdots (\xi - a_{k+1}) \zeta^{k+1}}, \tag{4.8}$$

the kernel can be written in the form,

$$K^c(x, y) = \sum_{k=0}^{\nu-1} \phi_k^c(x) \psi_k^c(y), \tag{4.9}$$

with

$$\phi_k^c(x) = \oint_D \frac{\mathrm{d}\xi}{2\pi \mathrm{i}} \frac{\xi^{k-\nu}}{g^c(\xi, s; x)(\xi - a_1) \cdots (\xi - a_{k+1})}, \tag{4.10}$$

$$\psi_k^c(x) = a_{k+1} \oint_{C_r} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\,(\zeta + c)\zeta^{k-\nu+1}} g^c(\zeta, s; x)(\zeta - a_1)\cdots(\zeta - a_k). \tag{4.11}$$

Note that at this stage one can set some of a_j's to be the same in these expressions. In the proof we have assumed that the a_j are distinct but Proposition 1.3 remains valid also when some or all of the a_j are equal. Also after the use of (4.8) the contours C_r and D do not need to be separate but D excludes zero for the final result.

5. Asymptotics: Preparations

In this section, we will rewrite the joint current distribution in Theorem 1.1 to a form which is suitable for an asymptotic analysis. For technical simplicity, from now on, we set $\beta = \frac{1}{2}$. Using shorthand $\rho' = 1 - \rho$, recall the integral formula of the joint current distribution given in Theorem 1.1 (with $\alpha = \beta = \frac{1}{2}$):

$$\begin{aligned} P_{n,m,\rho}(t) =&(-1)^{n+m} \oint_0 \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \oint_0 \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi\mathrm{i}}\ \mathrm{e}^{\Lambda_{n,m}t} \\ &\times \frac{\rho^n \prod_{1\le i<j\le n} (z_i - z_j) \prod_{1\le k<\ell\le m} (w_\ell - w_k) \prod_{j=1}^{n} z_j^{n-j} \prod_{k=1}^{m} w_k^{k-1}}{\prod_{j=1}^{n} (z_j - 1)^{n+1-j}(1-\rho' z_j) \prod_{k=1}^{m} (w_k - 1)^k \prod_{j=1}^{n}\prod_{k=1}^{m} \left(\frac{1}{2}(z_j + w_k)\right)} \\ =&(-1)^{n+m} \frac{\rho^n}{n!m!} \oint_0 \mathrm{d}^n z \oint_0 \mathrm{d}^m w\, \frac{\mathrm{e}^{\Lambda_{n,m}t} \Delta_n(z)\Delta_n(-z)\Delta_m(w)\Delta_m(-w)}{\prod_{j=1}^{n}(1-\rho' z_j) \prod_{j=1}^{n} (z_j - 1)^n \prod_{k=1}^{m} (w_k - 1)^m S_{n,m}(z, w)}, \end{aligned} \tag{5.1}$$

with $\Lambda_{n,m}$ defined as (2.9) where the second line follows from symmetrisation using the Vandermonde determinant in the $\vec{z}$-variables as well as in the $\vec{w}$-variables, and we defined the following abbreviations to make the formulas more compact,

$$\mathrm{d}^n z\, \mathrm{d}^m w = \prod_{j=1}^{n} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \prod_{k=1}^{m} \frac{\mathrm{d}w_k}{2\pi\mathrm{i}}, \quad \Delta_n(z) = \prod_{1\le i<j\le n} (z_i - z_j),$$

$$S_{n,m}(z, w) = \prod_{j=1}^{n}\prod_{k=1}^{m} \left(\tfrac{1}{2}(z_j + w_k)\right).$$

Before performing an asymptotic analysis, we first perform some rearrangements in (5.1) to disentangle the integral over the $\vec{z}$-variables from those over the $\vec{w}$-variables.

We choose the contours in (5.1) to be $C_{\vec{w},\vec{z}}$, explained at the end of Remark 1.2. The $\vec{z}$-contours enclose the poles at 1, $1/\rho'$ and ∞ with the anti-clockwise orientation hence we can evaluate the integrals in (5.1) by reversing the orientations of the $\vec{z}$-contours and considering the contributions from these poles. Since we will be interested in the asymptotics with the scaling (5.12), we can focus on the case when $n < m+2$, for which one can check that there is no pole at $z_j = \infty$ and the only other poles are located at $z_j = 1$ and $z_j = 1/\rho'$. Therefore in this case the target distribution becomes

$$P_{n,m,\rho}(t) = \frac{(-1)^m \rho^n}{n!m!} \oint_0 \mathrm{d}^m w \oint_{1,1/\rho'} \mathrm{d}^n z\, \frac{\mathrm{e}^{\Lambda_{n,m}t} \Delta_n(z)\Delta_n(-z)\Delta_m(w)\Delta_m(-w)}{\prod_{j=1}^{n}(1-\rho' z_j) \prod_{j=1}^{n}(z_j - 1)^n \prod_{k=1}^{m}(w_k - 1)^m S_{n,m}(z, w)},$$

where the integral symbols are used as explained in Remark 1.2, namely the $\vec{z}$-contours include only the poles at $z_j = 1$ and $z_j = 1/\rho'$ and the $\vec{w}$-contours include only the poles at the origin. The poles at $z_j = 1/\rho'$ are simple poles and hence can be easily evaluated. Since the $\vec{z}$-integrand is a symmetric function with respect to an exchange of $\vec{z}$-variables, the contributions of the poles at $z_j = 1/\rho$ for all $j \in [1, n]$ are the same, and thus the sum of them coincides with n times that of the pole at $z_n = 1/\rho$. It can be easily seen that after evaluating the pole at $z_n = 1/\rho'$, the Vandermonde product produces a factor $\prod_{j\neq n}(z_j - 1/\rho')^2$, which cancels all other poles at $z_j = 1/\rho'$. Therefore we obtain, when $n < m + 2$,

$$\begin{aligned} P_{n,m,\rho}(t) = \;& \frac{(-1)^m \rho^n}{n!m!} \oint_0 \mathrm{d}^m w \oint_1 \mathrm{d}^n z \frac{\mathrm{e}^{\Lambda_{n,m}t}\Delta_n(z)\Delta_n(-z)\Delta_m(w)\Delta_m(-w)}{\prod_{j=1}^n(1-\rho' z_j)\prod_{j=1}^n(z_j-1)^n\prod_{k=1}^m(w_k-1)^m S_{n,m}(z,w)} \\ & -\frac{(-1)^{n+m-1}\mathrm{e}^{-\rho t/2}}{(\rho')^{n-1}(n-1)!m!} \oint_0 \mathrm{d}^m w \oint_1 \mathrm{d}^{n-1} z \frac{\mathrm{e}^{\Lambda_{n-1,m}t}\Delta_{n-1}(z)\Delta_{n-1}(-z)\Delta_m(w)\Delta_m(-w)}{\prod_{j=1}^{n-1}(z_j-1)^n\prod_{k=1}^m(w_k-1)^m} \\ & \times \frac{\prod_{j=1}^{n-1}(1-\rho' z_j)}{S_{n-1,m}(z,w)\prod_{k=1}^m\left(\frac{1}{2}(1/\rho'+w_k)\right)}. \end{aligned} \tag{5.2}$$

In the next two sections, we will determine the asymptotic behaviours of each term of (5.2). To that end we first rewrite them into a form which is more suitable for asymptotics. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ denote the first term and the second term of (5.2), respectively. Then the results of rewritings are summarised as follows.

Lemma 5.1. *The probability $P_{n,m,\rho}(t)$ can be written as*

$$P_{n,m,\rho}(t) = \mathcal{I}_1 - \mathcal{I}_2, \tag{5.3}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are given by

$$\mathcal{I}_1 = \frac{(-1)^m}{m!}\oint_0 \mathrm{d}^m w \frac{\mathrm{e}^{\Lambda_{0,m}t}\,\Delta_m(w)\Delta_m(-w)}{\prod_{k=1}^m(w_k-1)^m S_{n,m}(1,w)}, \tag{5.4}$$

$$\mathcal{I}_2 = \frac{\mathrm{e}^{-\rho t/2}}{(\rho')^{n-1}}\left(\frac{2(1-\rho)}{2-\rho}\right)^m \frac{(-1)^{n-1}}{(n-1)!}\oint_1 \mathrm{d}^{n-1} z \frac{\mathrm{e}^{\Lambda_{n-1,0}t}\,\Delta_{n-1}(z)\Delta_{n-1}(-z)\prod_{j=1}^{n-1}(1-\rho' z_j)}{\prod_{j=1}^{n-1}(z_j-1)^n S_{n-1,m}(z,1)} I_w(\vec{z}) \tag{5.5}$$

$$= \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) I_w(\vec{z}), \tag{5.6}$$

with

$$I_w(\vec{z}) := \frac{(-1)^m}{m!}\oint_0 \mathrm{d}^m w \frac{\mathrm{e}^{\Lambda_{0,m}t}\,\Delta_m(w)\Delta_m(-w)S_{n-1,m}(z,1)}{\prod_{k=1}^m(w_k-1)^m S_{n-1,m}(z,w)}\prod_{k=1}^m \frac{1+1/\rho'}{w_k+1/\rho'}, \tag{5.7}$$

$$L(\vec{z}) := (-1)^{n-1}\frac{\mathrm{e}^{-\rho t/2}}{(\rho')^{n-1}}\left(\frac{2(1-\rho)}{2-\rho}\right)^m \frac{\mathrm{e}^{\Lambda_{n-1,0}t}\,\Delta_{n-1}(-z)\prod_{j=1}^{n-1}(1-\rho' z_j)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}S_{n-1,m}(z,1)}. \tag{5.8}$$

As stated below (2.7), here and in the following $\vec{z}$ denotes the collection of variables z_j's with $j \in [1, n-1]$.

Proof. First, we show that the first term $\mathcal{I}_1$ is written as (5.4). Consider the symmetrisation identity (3.6). Since the right hand side of (3.6) for $N = n$ appears in the first term of (5.2), substituting the left hand side of (3.6) into it and considering that a Vandermonde product $\Delta_n(-z)$ is anti-symmetric under the exchange $z_i \leftrightarrow z_j$ for any pairs $(i, j) \in [1, n]^2$, we obtain

$$\begin{aligned} &\frac{(-1)^m \rho^n}{n!m!} \oint_0 \mathrm{d}^m w \oint_1 \mathrm{d}^n z \frac{\mathrm{e}^{\Lambda_{n,m} t}\, \Delta_n(z)\Delta_n(-z)\Delta_m(w)\Delta_m(-w)}{\prod_{j=1}^n (z_j-1)^n \prod_{k=1}^m (w_k-1)^m \prod_{j=1}^n (1-\rho' z_j) S_{n,m}(z,w)} \\ &= \frac{(-1)^m \rho^n}{m!} \oint_0 \mathrm{d}^m w \oint_1 \mathrm{d}^n z \frac{\mathrm{e}^{\Lambda_{n,m} t}\, \Delta_n(-z)\Delta_m(w)\Delta_m(-w)}{\prod_{j=1}^n (z_j-1)^j \prod_{k=1}^m (w_k-1)^m \prod_{j=1}^n (1-\rho' z_j) S_{n,m}(z,w)}. \end{aligned} \tag{5.9}$$

On the right hand side, the pole at $z_1 = 1$ is first order and its residue can be evaluated in an easy way. In doing so, the Vandermonde product produces a factor $z_2 - 1$, making the pole at $z_2 = 1$ first order. This pole can be subsequently evaluated in an easy way. Proceeding successively all poles at $z_j = 1$ can be easily evaluated from $j = 1$ to $j = n$, giving (5.4).

Second, we show that the second term $\mathcal{I}_2$ can be written as either (5.5) or (5.6). (5.5) follows immediately from the second term of (5.2). Since the right hand side of (3.6) for $N = n - 1$ appears in the right hand side of (5.5), in the same fashion as the derivation of (5.9), we can obtain the anti-symmetric formula (5.6). Unlike the right hand side of (5.9), the pole whose order is the lowest is at $z_1 = 1$ and its order is 2 so that we can not evaluate the poles at $z_j = 1$ explicitly in a compact manner. □

5.1. Scaling limit. Nonlinear fluctuating hydrodynamics [30,62,75,90] for this model, predicts that integrated currents for the scaled normal modes defined by

$$\begin{aligned} \eta_2(t) &= \frac{1}{c_2 t^{1/3}}\Big((1+\rho)N_+(t) - (3-\rho)N_-(t) + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t\Big), \\ \eta_{\mathrm{g}}(t) &= \frac{1}{c_{\mathrm{g}} t^{1/2}}\Big(-2(2-\rho)N_+(t) + 2\rho N_-(t) + (2-\rho)(1-\rho)\rho t\Big), \end{aligned} \tag{5.10}$$

where the constants c_2 and c_{g} are given by

$$\begin{aligned} c_2 &= (3/32)^{1/3}(1-\rho)(3-\rho)^{2/3}(1+\rho)^{2/3}, \\ c_{\mathrm{g}} &= 2^{-1/2} 3(1-\rho)^{3/2}\sqrt{\rho(2-\rho)}, \end{aligned} \tag{5.11}$$

would tend to finite limiting random variables in the large time limit. Put differently the joint distribution for the scaled normal modes, $\mathbb{P}[\eta_2 \le s_2, \eta_{\mathrm{g}} \le s_{\mathrm{g}}]$ is equivalent to that for the original variable $\mathbb{P}[N_+(t) \ge n, N_-(t) \ge m]$ if we take the scaling for n and m as

$$\begin{aligned} n &= j_+(\rho)t - \frac{1}{12(1-\rho)}(2\rho c_2 s_2 t^{1/3} + (3-\rho)c_{\mathrm{g}} s_{\mathrm{g}} t^{1/2}), \\ m &= j_-(\rho)t - \frac{1}{12(1-\rho)}(2(2-\rho)c_2 s_2 t^{1/3} + (1+\rho)c_{\mathrm{g}} s_{\mathrm{g}} t^{1/2}), \end{aligned} \tag{5.12}$$

where the macroscopic currents $j_\pm$ are defined by

$$j_+(\rho) = \frac{\rho(3-\rho)^2}{16}, \qquad j_-(\rho) = \frac{(1+\rho)^2(2-\rho)}{16}. \tag{5.13}$$

Denote the result of solving (5.12) for s_{g} and s_2 as functions of n, m, t by $s_{\mathrm{g}}(n, m, t)$ and $s_2(n, m, t)$ respectively. Then they are nothing but the right hand sides of (5.10) with $N_+(t), N_-(t)$ replaced with n, m.

The scalings with $t^{1/3}$ and $t^{1/2}$ in (5.10) and (5.12) stem from the KPZ and Gaussian fluctuations respectively, which are predicted in nonlinear fluctuating hydrodynamics. Moreover, we will show that, in the long time limit with the scaling described above, the probability (5.3) tends to a product of the Gaussian distribution and the GUE Tracy–Widom distribution. In other words, we will prove the following theorem.

Theorem 5.2. *With the scaling* (5.12)*, the terms $\mathcal{I}_1$ and $\mathcal{I}_2$ given by* (5.4) *and* (5.5)*, respectively, tend to the following limits,*

$$\lim_{t\to\infty} \mathcal{I}_1 = F_2(s_2), \tag{5.14a}$$

$$\lim_{t\to\infty} \mathcal{I}_2 = \left[1 - F_G(s_{\mathrm{g}})\right] F_2(s_2). \tag{5.14b}$$

Here we recall that F_2 and F_G denote the cumulative distribution functions of the GUE Tracy–Widom distribution in Definition 1.4 and the Gaussian distribution, respectively.

As a corollary of this theorem, we obtain the final result.

Corollary 5.3. *With the scaling* (5.12)*, the probability $P_{n,m,\rho}(t)$ converges to the product of the Gaussian distribution and the GUE Tracy–Widom distribution, i.e.,*

$$\lim_{t\to\infty} P_{n,m,\rho}(t) = \lim_{t\to\infty} (\mathcal{I}_1 - \mathcal{I}_2) = F_G(s_{\mathrm{g}}) F_2(s_2). \tag{5.15}$$

The proofs of (5.14a) and (5.14b) will constitute the contents of Sects. 6 and 7, respectively.

6. Limit of $\mathcal{I}_1$: Proof of (5.14a) in Theorem 5.2

Let us start from the simplified formula of $\mathcal{I}_1$, (5.4). The asymptotic behaviour of the right hand side of (5.4) can be obtained in a standard way with an extra parameter c where we first transform it to a Fredholm determinant according to the procedure outlined in Sect. 4, and then perform a steepest descent analysis of the Fredholm kernel. The change of variables $w_k \to 1/w_k$ maps the origin to ∞ and the contours after this mapping now enclose the poles at $-1, 0, 1$ but with the clockwise orientation. By changing the orientations of the contours, we get a -1 sign for each variable. Thus we have

$$\begin{aligned}
\mathcal{I}_1 &= \frac{(-1)^m}{m!} \oint_0 \mathrm{d}^m w \, \frac{\mathrm{e}^{\Lambda_{0,m} t}\, \Delta_m(w) \Delta_m(-w)}{\prod_{k=1}^m (w_k - 1)^m S_{n,m}(1, w)} \\
&= \frac{1}{m!} \oint_{0,1,-1} \mathrm{d}^m w \, \frac{\mathrm{e}^{\tilde{\Lambda}_{0,m} t} \prod_{1\le k\ne\ell\le m} (w_\ell - w_k)}{\prod_{k=1}^m w_k^m (w_k - 1)^m \prod_{k=1}^m (\frac{1}{2}(1 + 1/w_k))^n},
\end{aligned}$$

where $\tilde{\Lambda}_{n,m} = \frac{1}{2}\sum_{j=1}^n (z_j - 1) + \frac{1}{2}\sum_{k=1}^m (w_k - 1)$. This expression can be written as the standard m-fold integral (1.6) in Proposition 1.3,

$$\mathcal{I}_1 = \frac{1}{m!} \oint_{0,\{a_j\}_{j=1}^m,-1} \frac{\mathrm{d}^m w}{\prod_{k=1}^m w_k} \frac{\prod_{1\le k\ne\ell\le m} (1 - w_k/w_\ell)}{\prod_{k,\ell=1}^m (1 - a_\ell/w_k)} \prod_{k=1}^m \frac{g(w_k, 0)}{g(a_k, 0)},$$

with

$$\begin{aligned}\nu &= m,\ \mu = n, \quad \gamma = t/2, \quad s = 0,\\ u_i &= 0, \ \ 1 \le i \le m,\\ a_i &= 1, \ \ 1 \le i \le m,\\ v_k &= 1, \ \ 1 \le k \le n,\end{aligned}$$

where $g(w_k, x) = g^c(w_k, x; 0)$ and $g^c(w_k, x; y)$ is given in (1.5),

$$g^c(w, x; y) = \left(\frac{w}{w+1}\right)^n w^{-m-x}(w+c)^y\, \mathrm{e}^{wt/2}\,.$$

We take an arbitrary $c > 0$ and let $S(-c, |1+c|)$ be the open disc of radius $|1+c|$ centred at $-c$. Then one can check that the conditions for c in Proposition 1.3 are satisfied, namely, $S(-c, |1+c|)$ includes poles at $0, -1$ since $|-c| < |1+c|$ and $|-1+c| < |1| + |c| = |1+c|$. Thus applying Proposition 1.3, the integral $\mathcal{I}_1$ defined in (5.4) can thus be written as a Fredholm determinant:

$$\mathcal{I}_1 = \det\bigl(1 - K^c\bigr)_{\ell^2(\mathbb{N})}, \qquad K^c(x, y) = \sum_{k=0}^{m-1} \phi_k^c(x)\psi_k^c(y), \tag{6.1}$$

with

$$\phi_k^c(x) = \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{(z+c)^x(z-1)} \left(\frac{1+z}{z}\right)^n \left(\frac{z}{1-z}\right)^k \mathrm{e}^{-zt/2}, \tag{6.2a}$$

$$\psi_k^c(x) = \oint_{0,-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{(w+c)^{x-1}}{w} \left(\frac{w}{1+w}\right)^n \left(\frac{1-w}{w}\right)^k \mathrm{e}^{wt/2}, \tag{6.2b}$$

in which the pole at $w = 1$ is separated from the poles at $w = -1, 0$. As will be mentioned, we choose the contours of the integrals with respect to z and w as Γ, more precisely Γ', and Σ that will be introduced in Lemma 6.1. By performing the sum $\sum_{k=0}^{m-1}$ in (6.1), we obtain a double contour integral expression of $K^c(x, y)$ as

$$\begin{aligned}K^c(x, y) = &\oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{(z+c)^x} \left(\frac{1+z}{z}\right)^n \left(\frac{z}{1-z}\right)^m \mathrm{e}^{-zt/2}\\ &\times \oint_{0,-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} (w+c)^{y-1} \left(\frac{w}{1+w}\right)^n \left(\frac{1-w}{w}\right)^m \mathrm{e}^{wt/2} \frac{1}{w-z}.\end{aligned} \tag{6.3}$$

In the following, we will show a rigorous asymptotic analysis on the first term $\mathcal{I}_1$, (6.1), by computing the scaling limit of the kernel $K^c(x, y)$, (6.3). Recall that the scaling of n and m we consider has been set in (5.12). Accordingly, we rescale (x, y) as

$$x = \lambda_c t^{1/3}\xi, \qquad y = \lambda_c t^{1/3}\zeta, \tag{6.4}$$

where

$$\lambda_c = (1 - \rho + 2c)\left(\frac{3}{4(1+\rho)(3-\rho)}\right)^{1/3}. \tag{6.5}$$

The scaling $t^{1/3}$ in (6.4) is chosen according to (5.12), which may be found by nonlinear fluctuating hydrodynamics (NLFHD), and shows the KPZ nature of fluctuations. The constant λ_c in (6.5) is chosen for convenience to ease notation in Proposition 6.2. The basic strategies can be taken from the previous works [15,16] and are given as follows:

– First in Sect. 6.1, we define a steepest descent contour of K^c with the scaling (5.12) and (6.4).
– Using the steepest descent, we prove the uniform convergence of $K^c(x, y)$ to the Airy kernel (1.11) for bounded ξ, ζ in Sect. 6.2.
– Then we evaluate the bounds of $K^c(x, y)$ for large ξ, ζ in Sect. 6.3.
– Finally by the convergence and bounds of the kernel $K^c(x, y)$, we are able to obtain the long time behaviour of $\mathcal{I}_1$ with (5.12) given in (6.23).

However, we can not simply follow the arguments in the previous works. A novelty in our treatment of Fredholm determinants is that we introduce an additional parameter c (greater than 0) to avoid divergence of the Fredholm determinant stemming from the singularity at -1. After that, as will be stated in Proposition 6.3, we evaluate the whole kernel K^c instead of ϕ_k^c and ψ_k^c to show that it is bounded by an exponential decreasing function in a similar manner to the previous works [14–16].

Before continuing to a rigorous analysis, let us first introduce the following notations for convenience. We rescale kernel by

$$\mathcal{K}_t^c(\xi, \zeta) := (w_* + c)^{\lambda_c t^{1/3}(\xi-\zeta)} \lambda_c t^{1/3} K^c(\lambda_c t^{1/3}\xi, \lambda_c t^{1/3}\zeta), \tag{6.6}$$

where $K^c(x, y)$ is given in (6.3), and $w_* := (1 - \rho)/2$ is the saddle point (which we will see in Lemma 6.1). Then with the scaling (5.12), the rescaled kernel (6.6) can be rewritten by collecting the terms according to the order of t:

$$\mathcal{K}_t^c(\xi, \zeta) = \lambda_c t^{1/3} \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)+g_\phi(z)} \oint_{0,-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \mathrm{e}^{-f(w,t,\zeta)+f(w_*,t,\zeta)+g_\psi(w)} \frac{1}{w-z}, \tag{6.7}$$

where $f(z, t, \xi) = g_1(z)t + g_2(z)t^{1/2} + g_3(z, \xi)t^{1/3}$ with the functions $g_1(z)$, $g_2(z)$, $g_3(z, \xi)$, $g_\phi(z)$ and $g_\psi(z)$ defined by

$$g_1(z) = \frac{\rho(3-\rho)^2}{16} \ln\left(\frac{1+z}{z}\right) + \frac{(1+\rho)^2(2-\rho)}{16} \ln\left(\frac{z}{1-z}\right) - \frac{z}{2}, \tag{6.8a}$$

$$g_2(z) = -\frac{(3-\rho)c_\mathrm{g}s_\mathrm{g}}{12(1-\rho)} \ln\left(\frac{1+z}{z}\right) - \frac{(1+\rho)c_\mathrm{g}s_\mathrm{g}}{12(1-\rho)} \ln\left(\frac{z}{1-z}\right), \tag{6.8b}$$

$$g_3(z, \xi) = \frac{-\rho c_2 s_2}{6(1-\rho)} \ln\left(\frac{1+z}{z}\right) - \frac{(2-\rho)c_2 s_2}{6(1-\rho)} \ln\left(\frac{z}{1-z}\right) - \xi\lambda_c \ln(z+c), \tag{6.8c}$$

$$g_\psi(z) = -\ln(z+c), \quad g_\phi(z) = 0. \tag{6.8d}$$

The appearance of $\mathcal{O}(t^{1/2})$ term in $f(z, t, \xi)$ above is related to Gaussian nature of fluctuations, as mentioned after (5.12).

In the vicinity of a saddle point w_*, the functions $g_1(z)$, $g_2(z)$, $g_3(z, \xi)$ and $g(z)$ can be represented as the following Taylor expansions which are useful in the later analysis:

$$g_1(z) = g_1(w_*) + 2a_1(z - w_*)^3 + (z - w_*)^4 h_1(z), \tag{6.9a}$$

$$g_2(z) = g_2(w_*) + b_2(z - w_*)^2 + (z - w_*)^3 h_2(z), \tag{6.9b}$$

$$g_3(z, \xi) = g_3(w_*, \xi) + b_{3,\xi}(z - w_*) + (z - w_*)^2 h_3(z, \xi), \tag{6.9c}$$

$$g_\psi(z) = g_\psi(w_*) + b_\psi(z - w_*) + (z - w_*)^2 h_\psi(z), \tag{6.9d}$$

$$g_\phi(z) = g_\phi(w_*) + b_\phi(z - w_*) + (z - w_*)^2 h_\phi(z), \tag{6.9e}$$

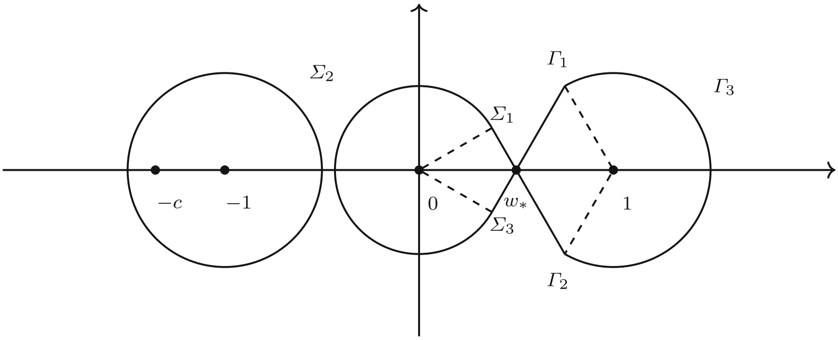

Fig. 4. Steepest descent contour of $\mathcal{K}_t^c$

where h_1, h_2, h_3, h_ψ and h_ϕ are some functions, and constants $a_1, b_2, b_{3,\xi}, b_\psi$ and b_ϕ are given by

$$a_1 = \frac{1}{(3-\rho)(1+\rho)}, \tag{6.10a}$$

$$b_2 = -\frac{4c_g s_g}{3(1-\rho)^2(1+\rho)(3-\rho)}, \tag{6.10b}$$

$$b_{3,\xi} = -\frac{4c_2 s_2}{(1-\rho)(1+\rho)(3-\rho)} - \frac{\xi\lambda_c}{w_* + c}, \tag{6.10c}$$

$$b_\psi = -\frac{1}{w_* + c}, \quad b_\phi = 0. \tag{6.10d}$$

Note that a_1 is positive.

6.1. Steepest descent. To obtain the scaling limit of $\mathcal{K}_t^c$, (6.7), we first need to find the critical point as well as the corresponding steepest descent contour. Since $t \to \infty$, we only consider the term in the integrand of (6.7) that is of the largest order of t:

$$\mathcal{K}_t^c(\xi, \zeta) = \lambda_c t^{1/3} \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \oint_{0,-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \exp\Big(g_1(z)t - g_1(w)t + \mathcal{O}\Big(t^{1/2}\Big)\Big),$$

where $g_1(z)$ is given in (6.8a). Regarding to $g_1(w)$, we define the following descent contour, along which $\mathrm{Re}(g_1(w))$ has a global maximum.

Lemma 6.1 (Steepest descent contour of $\mathcal{K}_t^c$). *For* $0 < \rho < 1$ *set*

$$g_1(w) = \frac{\rho(3-\rho)^2}{16} \ln\left(\frac{1+w}{w}\right) + \frac{(1+\rho)^2(2-\rho)}{16} \ln\left(\frac{w}{1-w}\right) - \frac{w}{2}.$$

Then $g_1'(w) = 0$ *has a double root at* $w_* = (1-\rho)/2 = \rho'/2$ *and a single root* $w_2 = \rho - 1 = -\rho'$. *The path* $\Gamma = \bigcup_{i=1}^3 \Gamma_i$ *(see Fig. 4) given by (6.11) below is a steepest descent path of* $g_1(w)$ *passing through* w_*. *Namely,* $w = w_*$ *is the strict global maximum point of* $\mathrm{Re}(g_1)$ *along* Γ, *i.e.,* $\mathrm{Re}(g_1(w)) < \mathrm{Re}(g_1(w_*))$ *except when* $w = w_*$. *Moreover,* $\mathrm{Re}(g_1)$ *is monotone along* Γ *except two points where it reaches its maximum and minimum. Meanwhile, the path* $\Sigma = \bigcup_{i=1}^3 \Sigma_i$ *(see Fig. 4) given by (6.12) below is a steepest descent path of* $-g_1(w)$ *passing through* w_*.

$$\Gamma_1 = \left\{ w \in \mathbb{C} \middle| w = \frac{1-\rho}{2} - s e^{i\pi/3}, s \in \left[-\frac{1+\rho}{2}, 0 \right] \right\}, \tag{6.11a}$$

$$\Gamma_2 = \left\{ w \in \mathbb{C} \middle| w = \frac{1-\rho}{2} + s e^{-i\pi/3}, s \in \left[0, \frac{1+\rho}{2} \right] \right\}, \tag{6.11b}$$

$$\Gamma_3 = \left\{ w \in \mathbb{C} \middle| w = 1 + \frac{1+\rho}{2} e^{i\theta}, \theta \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3} \right] \right\}, \tag{6.11c}$$

$$\Sigma_1 = \left\{ w \in \mathbb{C} \middle| w = \frac{1-\rho}{2} + s e^{2\pi i/3}, s \in \left[0, \frac{(1-\rho)}{4} \right] \right\}, \tag{6.12a}$$

$$\Sigma_2 = \left\{ w \in \mathbb{C} \middle| w = \frac{\sqrt{3}}{4}(1-\rho) e^{i\theta}, \theta \in \left[\frac{\pi}{6}, \frac{11\pi}{6} \right] \right\} \cup \left\{ w \in \mathbb{C} \middle| w = -1 + \frac{1+\rho}{2} e^{i\theta}, \theta \in [0, 2\pi] \right\}, \tag{6.12b}$$

$$\Sigma_3 = \left\{ w \in \mathbb{C} \middle| w = \frac{1-\rho}{2} - s e^{-2\pi i/3}, s \in \left[-\frac{(1-\rho)}{4}, 0 \right] \right\}. \tag{6.12c}$$

The proof is given in the Appendix D.1.

6.2. $\mathcal{K}_t^c$ on a bounded set. With the steepest descent contour given above, we arrive at the uniform convergence of $\mathcal{K}_t^c$ for bounded ξ, ζ. We first show the contribution from the contour away from the saddle point $w_* = \rho'/2$ vanishes as $t \to \infty$, then the convergence is obtained by a Taylor expansion near $w_* = \rho'/2$ and a change of a variable.

Proposition 6.2 (Uniform convergence of $\mathcal{K}_t^c$ on a bounded set). *Let m, n be scaled as* (5.12). *Then for any $c > 0$ and fixed $L > 0$, the rescaled kernel $\mathcal{K}_t^c(\xi, \zeta)$ defined in* (6.7) *converges uniformly on $\xi, \zeta \in [-L, L]$ to*

$$\lim_{t\to\infty} \mathcal{K}_t^c(\xi, \zeta) = A(\xi + s_2, \zeta + s_2), \tag{6.13}$$

where $A(x, y)$ is the Airy kernel defined in (1.11) *and s_2 is given in* (5.10).

Proof. We give a sketch of the proof idea and provide a rigorous proof in Appendix D.2. Owing to (6.9) and (6.10), the function $f(z, t, \xi)$ can be expanded with respect to z in the vicinity of a saddle point w_* as

$$f(z,t,\xi) = f(w_*,t,\xi) - b_{3,\xi}(z-w_*)t^{1/3} + (z-w_*)^2\mathcal{O}(t^{1/2}) + 2a_1(z-w_*)^3(t+\mathcal{O}(t^{1/2})) + \ldots,$$

where a_1 and $b_{3,\xi}$ is given in (6.10). The same is true for $f(w, t, \zeta)$.

To show that the uniform convergence of kernel rigorously, concretely to evaluate an upper bound of $1/(w - z)$ in the integrand of the kernel $\mathcal{K}_t^c$ given by (6.7), we need to choose z-contour such that it does not touch w-contour. Hence we choose z-contour as Γ' which is defined by (D.1) and obtained by slightly deforming Γ such that it does not pass a saddle point w_*. Let us define the scaled variables v and u along the contours $\Gamma' \cap \{z \in \mathbb{C} \mid |z - w_*| \le (1+\rho)/2\}$ and $\Sigma_{1,3}$, respectively, by

$$z - w_* = \frac{v}{\lambda t^{1/3}}, \qquad w - w_* = \frac{u}{\lambda t^{1/3}}.$$

We choose λ such that $6a_1 = \lambda^3$, namely $\lambda = (6a_1)^{1/3}$. Recall the coeffcients given in (6.10), we obtain

$$\lambda = \left(\frac{6}{(1+\rho)(3-\rho)}\right)^{1/3}. \tag{6.14}$$

Therefore by simple calculation, we have $b_{3,\xi}\lambda^{-1} = -s_2 - \xi\lambda_c(w_*+c)^{-1}\lambda^{-1}$. We now choose λ_c such that $\lambda_c(w_*+c)^{-1}\lambda^{-1} = 1$, i.e.,

$$\lambda_c = (w_*+c)\lambda = (1-\rho+2c)\left(\frac{3}{4(1+\rho)(3-\rho)}\right)^{1/3},$$

which agrees with (6.5). From the integral form of the Airy function (1.10) and the identity $1/a = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-ax}$ for $\mathrm{Re}(a) > 0$, we arrive at the final result,

$$\begin{aligned}
&\lim_{t\to\infty} \lambda_c t^{1/3} \oint_{\Gamma'} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)} \times \lambda t^{1/3} \oint_{\Sigma} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \mathrm{e}^{-f(w,t,\zeta)+f(w_*,t,\zeta)+g_\psi(w)} \frac{1}{\lambda t^{1/3}(w-z)} \\
&= \lim_{t\to\infty} \int_{\gamma^{\Delta t^{1/3}}} \frac{\mathrm{d}v}{2\pi \mathrm{i}}\, \mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v+\mathcal{O}(t^{-1/6})} \int_{\sigma^{\Delta t^{1/3}}} \frac{\mathrm{d}u}{2\pi \mathrm{i}}\, \mathrm{e}^{-\frac{1}{3}u^3+(s_2+\zeta)u+\mathcal{O}(t^{-1/6})} \int_0^\infty \mathrm{d}\kappa\, \mathrm{e}^{-\kappa(v-u)} \\
&= \int_0^\infty \mathrm{Ai}\,(s_2+\xi+\kappa)\, \mathrm{Ai}\,(s_2+\zeta+\kappa)\mathrm{d}\kappa.
\end{aligned}$$

In the second line, the paths of contour integrals are defined as $\gamma^{\Delta t^{1/3}} := \{v \in \mathbb{C} \mid w_* + v/(\lambda t^{1/3}) \in \Gamma'^{\Delta}\}$ and $\sigma^{\Delta t^{1/3}} := \{u \in \mathbb{C} \mid w_* + u/(\lambda t^{1/3}) \in \Sigma^{\Delta}\}$ where Γ'^{Δ} and Σ^{Δ} are given by (D.2) and (D.3), respectively, and we can choose $\Delta = t^{-1/9}$ (see Appendix D.2). Above, we showed only pointwise convergence on a bounded set $[-L, L]^2$. In Appendix D.2, we prove uniform convergence rigorously. □

It is enough to show that (6.13) holds for $\xi, \zeta \in [0, L]$ in order to prove the first claim of Theorem 5.2 given by (5.14a). In Proposition 6.2 we have shown (6.13) holds for ξ, ζ on any bounded set, to keep the statement as general as possible.

6.3. Estimate of kernel. In the following, we will give an estimate of the rescaled function $\mathcal{K}_t^c(\xi, \zeta)$ for unbounded ξ, ζ, namely, we will prove Proposition 6.3. Unlike Proposition 6.2, an additional condition $c > 1$ is imposed in Proposition 6.3 in order to guarantee the inequality associated with w of (6.20) to hold. Actually, we can make the condition stricter but we choose sufficient one for simplicity (see Appendix D.4).

Proposition 6.3 (Estimate of $\mathcal{K}_t^c(\xi, \zeta)$ for unbounded ξ, ζ). *Let m, n be scaled as (5.12). Then for any $c > 1$ and large enough L and t, the rescaled kernel $\mathcal{K}_t^c(\xi, \zeta)$ defined in (6.7) is bounded by*

$$\left|\mathcal{K}_t^c(\xi, \zeta)\right| \le \mathrm{e}^{-(\zeta+\xi)},$$

when $(\xi, \zeta) \in [0, \infty)^2 \backslash [0, L]^2$.

Proof. We first need to deform the contours. In the proof of Proposition 6.2, which is provided in Appendix D.2, we deformed only Γ to Γ', since we only needed to bound $|w - z|$ there. Here we need to deform both of Γ, Σ to Γ', Σ' to be away from w_* (see Fig. 5), because now we want to estimate terms which involve both ζ and ξ, i.e. $\left|\frac{w_*+c}{z+c}\right|$ and $\left|\frac{w+c}{w_*+c}\right|$. The deformed contour Γ' is obtained by replacing, the segments which are within $2w_*\delta$ from the saddle point w_*, with a vertical line through $w_*(1+\delta)$. Similarly we have the deformed contour Σ'. Explicitly they are given by

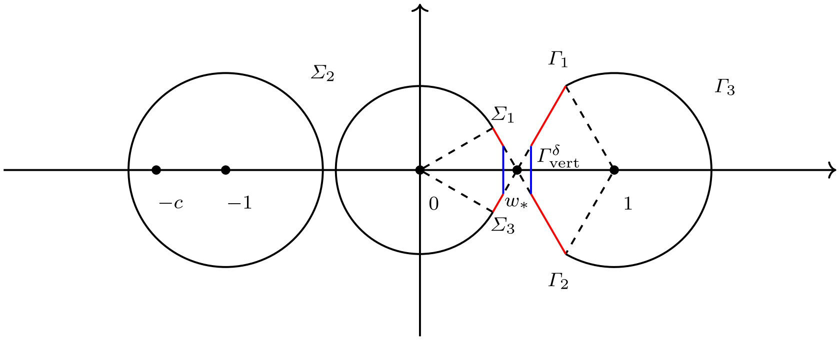

Fig. 5. Deformed contours Γ' and Σ'. Γ' is obtained by replacing the edge of $\Gamma(=\bigcup_{i=1}^3 \Gamma_i)$ near the point w_* by a vertical line $\Gamma_{\rm vert}$ (the blue line), and Σ' is replacing the edge of $\Sigma(=\bigcup_{i=1}^3 \Sigma_i)$ near the point w_* by $\Sigma_{\rm vert}$ (the blue line)

$$\Gamma' = \left\{z \in \Gamma \middle| |z-w_*| > 2w_*\delta\right\} \cup \left\{z \in \mathbb{C} \middle| z = w_* + w_*\delta(1-s\mathrm{i}), s \in \left[-\sqrt{3}, \sqrt{3}\right]\right\}, \tag{6.15a}$$

$$\Sigma' = \left\{z \in \Sigma \middle| |z-w_*| > 2w_*\delta\right\} \cup \left\{z \in \mathbb{C} \middle| z = w_* - w_*\delta(1-s\mathrm{i}), s \in \left[-\sqrt{3}, \sqrt{3}\right]\right\}. \tag{6.15b}$$

Then we will separate the integrand into two parts: those containing ζ or ξ, and those independent of ζ and ξ. We first estimate the second one (independent of ζ and ξ) along the deformed contour in three parts: $\Gamma' = (\Gamma' \backslash \Gamma'^{\Delta}) \cup (\Gamma'^{\Delta} \backslash \Gamma_{\rm vert}) \cup \Gamma_{\rm vert}$, where

$$\Gamma_{\rm vert} = \left\{z \in \mathbb{C} \middle| z = w_* + w_*\delta(1-s\mathrm{i}), s \in \left[-\sqrt{3}, \sqrt{3}\right]\right\}, \quad \Gamma^{\Delta} = \left\{z \in \Gamma' \middle| |z-w_*| \le \Delta\right\}.$$

Respectively, we have $\Sigma' = (\Sigma' \backslash \Sigma'^{\Delta}) \cup (\Sigma'^{\Delta} \backslash \Sigma_{\rm vert}) \cup \Sigma_{\rm vert}$. We require that $2w_*\delta < \Delta$, so that $\Gamma_{\rm vert}$ is inside Γ'^{Δ} (and $\Sigma_{\rm vert}$ is inside Σ'^{Δ}). The reason we separate the contour Γ' into three parts is that $g_1(z)$ decays exponentially far away from the saddle point $z = w_*$ (the black arcs in Fig. 5), while for those near the saddle point, the contour along the vertical line (the blue line in Fig. 5) and the one along the direction $\mathrm{e}^{\pm \mathrm{i}\pi/3}$ (the red part in Fig. 5) need to be considered individually. Here we choose $\Delta = t^{-1/9}$ for t is large enough. Then we bound the factors which contain ζ or ξ by its maximal value along Γ' (and Σ'). Combining these two results, one can arrive with the bound $\mathrm{e}^{-\zeta-\xi}$ by choosing appropriate value of δ according to ζ and ξ.

We let $\bar{g}_3(z) := g_3(z,\xi) + \xi\lambda_c \ln(z+c)$, i.e., $\bar{g}_3$ is the first two terms in g_3, which is defined as (6.8c), that do not involve ζ or ξ. Correspondingly, $\bar{f}(z,t) := g_1(z)t + g_2(z)t^{1/2} + \bar{g}_3(z)t^{1/3}$. Then the Taylor expansion in (6.9) now becomes $\bar{g}_3(z) = \bar{g}_3(w_*) + \bar{b}_3(z-w_*) + (z-w_*)^3\bar{h}_3(z)$, where $\bar{b}_3 = b_{3,\xi} + \xi\lambda_c/(w_*+c) = -4c_2s_2/[(1-\rho)(1+\rho)(3-\rho)]$. Now the rescaled kernel is written as

$$\mathcal{K}_t^c(\xi,\zeta) = \lambda_c t^{1/3} \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)+g_\phi(z)} \oint_{0,-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \mathrm{e}^{-\bar{f}(w,t)+\bar{f}(w_*,t)+g_\psi(w)} \frac{1}{w-z} \times \left(\frac{w_*+c}{z+c}\right)^{\xi\lambda_c t^{1/3}} \left(\frac{w+c}{w_*+c}\right)^{\zeta\lambda_c t^{1/3}}. \tag{6.16}$$

Clearly from Fig. 5,

$$\left|\frac{1}{w-z}\right| \leq \frac{1}{2w_*\delta} \tag{6.17}$$

holds along $\Gamma' \times \Sigma'$. In the following, we will give estimate of $\bar{f}(z,t)$ along Γ' and Σ' first, and then the estimate of $\left|\frac{w_*+c}{z+c}\right|$ and $\left|\frac{w+c}{w_*+c}\right|$.

(i) Estimate of terms independent of ξ, ζ. Recall the Taylor expansion of the g's functions given in (6.9). Assuming that z is not close to the critical point w_*, i.e., given a restriction on δ, one can bound the term independent of ξ, ζ by $c' \, \mathrm{e}^{c|z-w_*|^3 t}$, where c and c' are some constants to be fixed. We will give the results directly and a detailed calculation can be found in Appendix D.3. When $\delta > \sqrt{\frac{12|\bar{b_3}|}{a_1 w_*^2}} t^{-1/3} := c_1' t^{-1/3}$ where a_1 is given in (6.10),

$$\left|\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)+g_\phi(z)}\right| \leq \mathrm{e}^{28a_1(w_*\delta)^3 t}\,\mathrm{e}^{g_\phi(w_*)}, \qquad \text{for } z \in \Gamma_{\mathrm{vert}}, \tag{6.18a}$$

$$\left|\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)+g_\phi(z)}\right| \leq \mathrm{e}^{-a_1 v^3 t/2}\,\mathrm{e}^{g_\phi(w_*)}, \qquad \text{for } z \in \Gamma'^{\Delta}\backslash\Gamma_{\mathrm{vert}}, \tag{6.18b}$$

$$\left|\mathrm{e}^{f(z,t)-f(w_*,t)+g_\phi(z)}\right| \leq \mathrm{e}^{-a_1 \Delta^3 t/2}, \qquad \text{for } z \in \Gamma'\backslash\Gamma'^{\Delta}, \tag{6.18c}$$

where in the second inequality, $z = w_* + v\,\mathrm{e}^{\pm \mathrm{i}\pi/3}$. It follows that

$$t^{1/3}\int_{\Gamma'}\left|\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right|\left|\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)+g_\phi(z)}\right| \leq t^{1/3}|\Gamma'\backslash\Gamma'^{\Delta}|\,\mathrm{e}^{-a_1\Delta^3 t/2} + 2t^{1/3}e^{g_\phi(w_*)}\int_0^{\Delta}\mathrm{e}^{-a_1 v^3 t/2}\mathrm{d}v + t^{1/3}e^{g_\phi(w_*)}\,\mathrm{e}^{28a_1(w_*\delta)^3}\,|\Gamma_{\mathrm{vert}}|\,.$$

We consider the first term on the right hand side of the equation. Choosing t large enough and substituting $\Delta = t^{-1/9}$, we can see $t^{1/3}\,\mathrm{e}^{-a_1\Delta^3 t/2}$ is bounded by some constant. Then for the second term, we use the change of variable $vt^{1/3} = u$. Thus we have $t^{1/3}\int_0^{t^{-1/9}}\mathrm{e}^{-a_1 v^3 t/2}\,\mathrm{d}v = \int_0^{t^{2/9}}\mathrm{e}^{-a_1 u^3/2}\,\mathrm{d}u < \infty$. The second term is also bounded by some constant. We assume the sum of the first two terms is bounded by a positive constant r_3. Let us consider the last term $t^{1/3}e^{g_\phi(w_*)}\,\mathrm{e}^{28a_1(w_*\delta)^3}\,|\Gamma_{\mathrm{vert}}| = t^{1/3}e^{g_\phi(w_*)}\,\mathrm{e}^{28a_1(w_*\delta)^3 t}\,2\sqrt{3}w_*\delta := r_1 t^{1/3}\delta\,\mathrm{e}^{r_2\delta^3 t}$, where r_1, r_2 are some positive constants. Specifically, $r_1 = 2\sqrt{3}w_* e^{g_\phi(w_*)}$, $r_2 = 28a_1 w_*^3$. Collecting all the results, we have

$$\begin{aligned} t^{1/3}\oint_{\Gamma'}\left|\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right|\left|\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)+g_\phi(z)}\right| &\leq r_3 + r_1\delta t^{1/3}\,\mathrm{e}^{r_2\delta^3 t} \\ &\leq (r_3 + r_1\delta t^{1/3})\,\mathrm{e}^{r_2\delta^3 t} \\ &\leq R(1+\delta t^{1/3})\,\mathrm{e}^{r_2\delta^3 t}, \end{aligned}$$

where $R = \max\{r_1, r_3\}$. In order to put the δ and t into the exponential, we use the inequality $1 + x < \mathrm{e}^x$ and $\mathrm{e}^x < \mathrm{e}^{x^3}$ when $x > 1$. Namely, if $\delta > t^{-1/3}$,

$$t^{1/3}\oint_{\Gamma'}\left|\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right|\left|\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)+g_\phi(z)}\right| \leq R\,\mathrm{e}^{\delta t^{1/3}}\,\mathrm{e}^{r_2\delta^3 t} \leq R\,\mathrm{e}^{r\delta^3 t},$$

where $r = r_2 + 1$ is a positive constant. Please bear in mind that the above inequality holds only when δ satisfies the restriction: $\delta > c_1 t^{-1/3}$ where $c_1 = \max\{c_1', 1\}$.

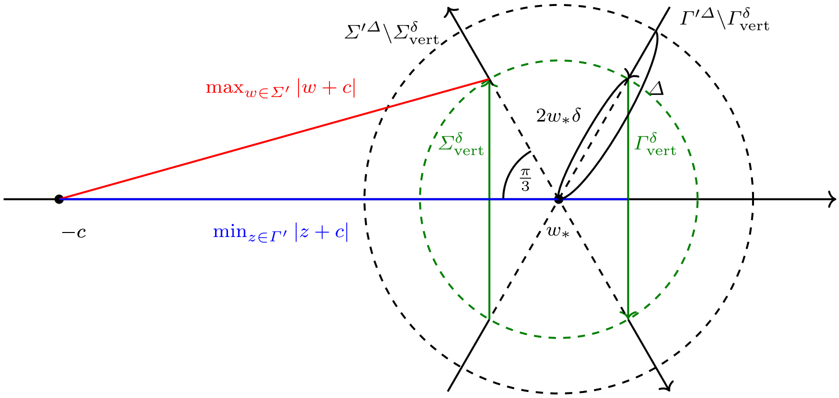

Fig. 6. The dotted green and black curves are circles of radius $2w_*\delta$ and Δ, respectively, centred at w_*. The green lines represent the paths $\Gamma^{\delta}_{\mathrm{vert}}$ and $\Sigma^{\delta}_{\mathrm{vert}}$, while the black lines in the black dotted circle are the paths $\Gamma'^{\Delta}\backslash\Gamma^{\delta}_{\mathrm{vert}}$ and $\Sigma^{\Delta}\backslash\Sigma^{\delta}$. The length of blue line is the minimum values of $|z+c|$ on Γ' and that of red line is the maximum value of $|w+c|$ on Σ

Such estimate can be repeated for the deformed contour Σ'. Combining these two results together, one obtains

$$t^{2/3}\oint_{\Sigma'}\left|\frac{\mathrm{d}w}{2\pi\mathrm{i}}\right|\left|\mathrm{e}^{-\bar{f}(w,t)+\bar{f}(w_*,t)+g_\psi(w)}\right|\oint_{\Gamma'}\left|\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right|\left|\mathrm{e}^{-\bar{f}(z,t)+\bar{f}(w_*,t)+g_\phi(z)}\right|\leq R\,\mathrm{e}^{r\delta^3 t},\tag{6.19}$$

where R and r absorb the constants from the deformed contour Σ'.
(ii) Estimate of terms dependent on ξ,ζ. The estimate of these two terms is straightforward. We simply take the minimum value of $|z+c|$ along Γ', and the maximum value of $|w+c|$ along Σ'. The reason why we introduce the parameter c is that $\max_{w\in\Sigma'}|w|>\max_{w\in\Sigma'}|w_*|$, i.e., $|\frac{w}{w_*}|$ can not be bounded by some number that is smaller than 1. But introducing an appropriate extra parameter c, we have $|\frac{w+c}{w_*+c}|<1$ for $w\in\Sigma'$ (see Fig. 6).

From Fig. 6, one can see that (we refer to Appendix D.4)

$$\left|\frac{w_*+c}{z+c}\right|\leq\mathrm{e}^{-\frac{1}{2}\frac{w_*}{w_*+c}\delta},\quad\left|\frac{w+c}{w_*+c}\right|\leq\mathrm{e}^{-\frac{1}{2}\frac{w_*}{w_*+c}\delta},\quad\text{for }(z,w)\in\Gamma'\times\Sigma'\text{ and }\delta\in(0,1/4).\tag{6.20}$$

In conclusion, the integrand depending on ξ,ζ has the following bound along $\Gamma'\times\Sigma'$. Given $0<\delta<1/4$, we have

$$\left|\left(\frac{w_*+c}{z+c}\right)^{\xi\lambda_c t^{1/3}}\left(\frac{w+c}{w_*+c}\right)^{\zeta\lambda_c t^{1/3}}\right|\leq\mathrm{e}^{-\frac{1}{2}\lambda_c\frac{w_*}{w_*+c}(\xi+\zeta)\delta t^{1/3}}.\tag{6.21}$$

(iii) Total estimate. In part (i), we have the bound of $\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)+g_\phi(z)}$ (and $\mathrm{e}^{-\bar{f}(w,t)+\bar{f}(w_*,t)+g_\psi(w)}$) along Γ' (and Σ'), given in (6.19). Then the bound of $|(w_*+c)/(z+c)|$ and $|(w+c)/(w_*+c)|$ is analysed in part (ii) and given in (6.21). The integrand

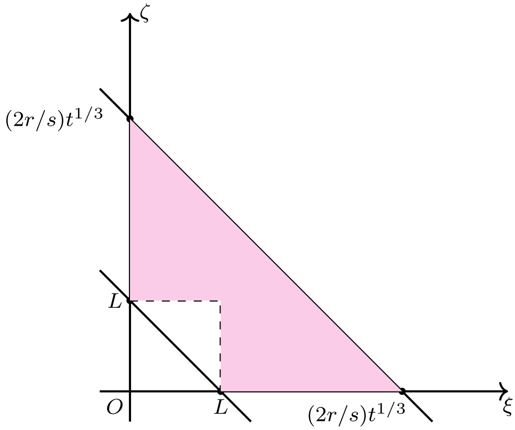

Fig. 7. $[0,\infty)^2\backslash[0,L]^2$, $L<\xi+\zeta\le(2r/s)t^{1/3}$

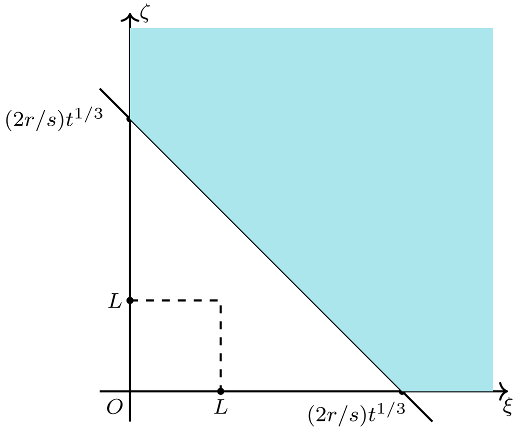

Fig. 8. $[0,\infty)^2\backslash[0,L]^2$, $\xi+\zeta>(2r/s)t^{1/3}$

of the rescaled kernel is left with $1/(w-z)$, which is bounded by $(2w_*\delta)^{-1}$ given in (6.17). It follows that when $c_1t^{-1/3}<\delta<1/4$,

$$\left|\mathcal{K}_t^c(\xi,\zeta)\right|\le\lambda_c t^{-1/3}R\,\mathrm{e}^{r\delta^3t}\,\frac{1}{2w_*\delta}\,\mathrm{e}^{-\frac{1}{2}\lambda_c\frac{w_*}{w_*+c}(\xi+\zeta)\delta t^{1/3}}\le\mathrm{e}^{r\delta^3t}\,\mathrm{e}^{-s(\xi+\zeta)\delta t^{1/3}},$$

where $s:=\frac{\lambda_c w_*}{2(w_*+c)}$ and we impose another restriction that $\delta>\frac{\lambda_c R}{2w_*}t^{-1/3}$. The condition on δ now becomes $c't^{-1/3}<\delta<1/4$ where c' is given by $c'=\max\{\frac{\lambda_c R}{2w_*},c_1\}$. We are only left with showing that the left hand side of (6.3) is smaller than $\mathrm{e}^{-\xi-\zeta}$. To achieve this result, we choose a different value of δ depending on the value of $\xi+\zeta$ (see Figs. 7, 8).

• When $\xi+\zeta\le\frac{2r}{s}t^{1/3}$, we choose $\delta=\sqrt{\frac{(\xi+\zeta)s}{2r}}t^{-1/3}$. Then $\sqrt{Ls/(2r)}t^{-1/3}<\delta<t^{-1/6}$. Then restriction $c't^{-1/3}<\delta<1/4$ is satisfied if we choose L large enough such that $L>(2c'r)^2$ and choose t large enough such that $t^{-1/6}<1/4$. Therefore from

(6.3),

$$\begin{aligned}\left|\mathcal{K}_t^c(\xi,\zeta)\right| &\le \mathrm{e}^{(\xi+\zeta)^{3/2}(s/2)^{3/2}r^{-1/2}}\,\mathrm{e}^{-(\xi+\zeta)^{3/2}(s)^{3/2}(2r)^{-1/2}} = \mathrm{e}^{-(s(\xi+\zeta)/2)^{3/2}r^{-1/2}}\\ &\le \mathrm{e}^{-(\xi+\zeta)(s/2)^{3/2}(L/r)^{1/2}} \le \mathrm{e}^{-\xi-\zeta},\end{aligned}$$

where in the second line, $L < \xi+\zeta$ since $(\xi,\zeta)\in[0,\infty)^2\backslash[0,L]^2$. The final inequality follows by choosing L large enough such that $L > r(2/s)^3$.

• When $\xi+\zeta > \frac{2r}{s}t^{1/3}$, i.e., $r < s(\xi+\zeta)/(2t^{1/3})$. We choose $\delta = t^{-1/6}$. If t is large enough, then the restriction $c't^{-1/3} < t^{-1/6} < 1/4$ is automatically satisfied. From (6.3), we have

$$\left|\mathcal{K}_t^c(\xi,\zeta)\right| \le \mathrm{e}^{rt^{1/2}}\,\mathrm{e}^{-s(\xi+\zeta)t^{1/6}} \le \mathrm{e}^{s(\xi+\zeta)t^{1/2}/(2t^{1/3})}\,\mathrm{e}^{-s(\xi+\zeta)t^{1/6}} = \mathrm{e}^{-(\xi+\zeta)t^{1/6}s/2} \le \mathrm{e}^{-\xi-\zeta},$$

where the last inequality holds when t is large enough such that $t > (2s)^6$. □

6.4. Long time limit of the first term in (5.3). We now are ready to conclude the limiting kernel is the *Airy kernel* [91] given in (1.11). From Sects. 6.2 and 6.3, we obtain the following theorem.

Theorem 6.4. *Consider the rescaled kernel defined in* (6.6)

$$\mathcal{K}_t^c(\xi,\zeta) = (w_*+c)^{\lambda_c(\xi-\zeta)t^{1/3}}\lambda_c t^{1/3}K^c(\lambda_c t^{1/3}\xi, \lambda_c t^{1/3}\zeta),$$

with λ_c given in (6.5)*. Then we have*

(i) *We know*
$$\lim_{t\to\infty}\det\big(1-\mathcal{K}_t^c\big)_{L^2(0,\infty)} = \lim_{t\to\infty}\det\big(1-\mathcal{K}_t^c\big)_{\ell^2(\mathbb{N}/(\lambda_c t^{1/3}))}, \tag{6.22}$$
where $\mathbb{N}/(\lambda_c t^{1/3}) := \{x\in\mathbb{R} \mid x = n/(\lambda_c t^{1/3}), n\in\mathbb{N}\}$. Here note that the operator $\mathcal{K}_t^c$ on the right hand side is regarded as acting on $\ell^2(\mathbb{N}/(\lambda_c t^{1/3}))$ while the one on the left hand side on $L^2(0,\infty)$.

(ii) *For any fixed $L > 0$*
$$\lim_{t\to\infty}\mathcal{K}_t^c(\xi,\zeta) = A(\xi+s_2,\zeta+s_2),$$
uniformly on $(\xi,\zeta)\in[0,L]^2$.

(iii) *For t large enough,*
$$|\mathcal{K}_t^c(\xi,\zeta)| \le C\,\mathrm{e}^{-(\xi+\zeta)},$$
for some constant $C > 0$ and $(\xi,\zeta)\in[0,\infty)^2$.

Proof. One can show (i) by rescaling $(x,y) = (\lambda_c t^{1/3}\xi, \lambda_c t^{1/3}\zeta)$ in the Fredholm kernel in (1.4) and replacing the Riemann sums $\lim_{t\to\infty}(\lambda_c t^{1/3})^{-1}\sum_{x\in\mathbb{N}}$ with the integral $\int_0^\infty d\xi$. Statement (ii) is obtained by replacing $[-L,L]^2$ with $[0,L]^2$ in Proposition 6.2. Statement (iii) is obtained by combining Propositions 6.2 and 6.3 because the Airy kernel is bounded above by an exponential decreasing function $\mathrm{e}^{-(\xi+\zeta)}$ when ξ and ζ are positive. □

A general statement on the convergence of a Fredholm determinant is given below.

Lemma 6.5. *Suppose a kernel $\mathcal{K}_t$ satisfies*

(i) *For any fixed $L > 0$*

$$\lim_{t\to\infty} \mathcal{K}_t(\xi, \zeta) = H(\xi, \zeta),$$

uniformly on $(\xi, \zeta) \in [-L, L]^2$.

(ii) *For any fixed $L > 0$ and t large enough,*

$$|\mathcal{K}_t(\xi, \zeta)| \le C\, \mathrm{e}^{-\max\{0,\xi\}-\max\{0,\zeta\}},$$

for some constant $C > 0$ and $(\xi, \zeta) \in [-L, \infty)^2$.

Then we have

$$\lim_{t\to\infty} \det(1 - \mathcal{K}_t) = \det(1 - H).$$

See Lemma C.2 in [46] for a detailed proof. Lemma 6.5 holds even if we replace $[-L, L]$ and $[-L, \infty)$ with $[0, L]$ and $[0, \infty)$, respectively. Consequently, we can conclude that, with the scaling (5.12) in Sect. 5.1, the Fredholm determinant $\mathcal{I}_1$ with kernel K^c defined in (6.1) satisfies

$$\begin{aligned}\lim_{t\to\infty} \mathcal{I}_1 &= \lim_{t\to\infty} \det\bigl(1 - \mathcal{K}_t^c\bigr)_{\ell^2(\mathbb{N}/(\lambda_c t^{1/3}))}\\ &= \lim_{t\to\infty} \det\bigl(1 - \mathcal{K}_t^c\bigr)_{L^2(0,\infty)} = \det(1 - A)_{L^2(s_2,\infty)} = F_2(s_2),\end{aligned} \tag{6.23}$$

where A is the Airy kernel defined in (1.11) and s_2 is given in (5.10). Obviously, the equality (6.23) itself coincides with the first claim of Theorem 5.2.

7. Limit of $\mathcal{I}_2$: Proof of (5.14b) in Theorem 5.2

In this section, which consists of three Sects. 7.1, 7.2 and 7.3, we consider the second term in (5.3), i.e., $\mathcal{I}_2$, given by (5.5). In Sect. 7.1, we will rewrite $\mathcal{I}_2$ of (5.5) to a form suitable for studying asymptotics with the scaling (5.12). We will state and prove Proposition 7.2 which allows to decouple the $\vec{z}$ dependence of K^c into a fairly simple rank one perturbation. This is the key to establish the asymptotic decoupling of the two modes in our paper. Using this we can divide $\mathcal{I}_2$ into a main contribution $\mathcal{I}_2^{(1)}$ and a remaining part $\mathcal{I}_2^{(2)}$ as in (7.27). In Sects. 7.2 and 7.3, we will show that the main term converges to the GUE Tracy–Widom distribution whilst the remainder tends to zero.

7.1. Rewriting of $\mathcal{I}_2$. As we just saw in the analysis of $\mathcal{I}_1$ in Sect. 6, a Fredholm determinant formula which appears as a result of procedures in Sect. 4 is often suitable for studying asymptotics with the scaling like (5.12). For $\mathcal{I}_2$, this approach cannot be applied at least directly, due to the presence of a factor involving both w_j and z_k variables. But as we will see below, $I_w(\vec{z})$ can be transformed into a Fredholm determinant following the procedures in Sect. 4, and we can show that the long time limit of $I_w(\vec{z})$ is dominated by the same multiple integral with $\vec{z} = \vec{1}$. It follows then that $\mathcal{I}_2$ can be asymptotically the same as the product of I_z defined in (7.9) below and $I_w(\vec{1})$, and asymptotics of each multiple integral can be valuated independently by following similar arguments as in the previous section.

Let us start from the multi-fold integral formula of $I_w(\vec{z})$, namely (5.7). In order to rewrite $I_w(\vec{z})$ to a Fredholm determinant, we first show that the right hand side of (5.7) fits the standard form (1.6). Changing the variables $w_k \to 1/w_k$ in $I_w(\vec{z})$, the $\vec{w}$-contours around the origin are deformed to contours around the infinity. Hence, we can replace these contours to surround the poles other than the ones at the infinity, namely $w_k = 0, 1, -\rho', -z_j^{-1}$ for all $j \in [1, n-1]$, resulting in

$$I_w(\vec{z}) = \frac{(-1)^m}{m!} \oint_0 \mathrm{d}^m w \frac{\mathrm{e}^{\Lambda_{0,m} t} \Delta_m(w) \Delta_m(-w) S_{n-1,m}(z,1)}{\prod_{k=1}^m (w_k - 1)^m S_{n-1,m}(z,w)} \prod_{k=1}^m \frac{1+1/\rho'}{w_k + 1/\rho'}$$
$$= \frac{1}{m!} \oint_{0,1,-\rho',\{-z_j^{-1}\}_{j=1}^{n-1}} \mathrm{d}^m w \frac{\mathrm{e}^{\tilde{\Lambda}_{0,m} t} \prod_{1\le k\ne \ell \le m} (w_\ell - w_k) \prod_{j=1}^{n-1} (1+z_j)^m}{\prod_{k=1}^m w_k^{m-n-1} (w_k - 1)^m \prod_{j=1}^{n-1} \prod_{k=1}^m (1 + z_j w_k)} \prod_{k=1}^m \frac{1+1/\rho'}{1 + w_k/\rho'},$$

where $\tilde{\Lambda}_{n,m} = \frac{1}{2}\sum_{j=1}^n (z_j - 1) + \frac{1}{2}\sum_{k=1}^m (w_k - 1)$. This expression can be written as the standard m-fold integral (1.6) in Proposition 1.3,

$$I_w(\vec{z}) = \frac{1}{m!} \oint_{0,\{a_j\}_{j=1}^m,-\rho',\{-z_j^{-1}\}_{j=1}^{n-1}} \frac{\mathrm{d}^m w}{\prod_{k=1}^m w_k} \frac{\prod_{1\le k\ne \ell\le m}(1 - w_k/w_\ell)}{\prod_{k,\ell=1}^m (1 - a_\ell/w_k)} \prod_{k=1}^m \frac{g(w_k,0)}{g(a_k,0)},$$

with

$$\begin{aligned} \nu &= m, \ \mu = n, \quad \gamma = t/2, \quad s = 0, \\ u_i &= 0, \ \ 1 \le i \le m, \\ a_i &= 1, \ \ 1 \le i \le m, \\ v_k &= z_j, \ 1 \le k \le n-1, \quad v_n = 1/\rho', \end{aligned}$$

where $g(w_k, x) = g^c(w_k, x; 0)$ and $g^c(w, x; y)$ is given in (1.5), namely

$$g^c(w, x; y) = \prod_{j=1}^{n-1} \frac{w}{1 + z_j w} \frac{w}{1 + w/\rho'} w^{-m-x} (w+c)^y \,\mathrm{e}^{wt/2}.$$

As is clear from (5.5) and (5.6), the contours of $\vec{z}$-integrals include only the poles at unity, and hence it is allowed to choose them as ones which lie in the vicinity of unity. When we choose such contours, $-1/z_j$'s are close to -1 and then, as with the case of $\mathcal{I}_1$ treated in Sect. 6, there exists $c \in \mathbb{C}$ for which $S(-c, |1+c|)$ includes $0, -\rho', -z_j^{-1}$ for all $j \in [1, n-1]$ and excludes unity. Therefore, the conditions stated in the claim of Proposition 1.3 are satisfied. Thus applying Proposition 1.3, the integral $I_w(\vec{z})$ defined in (5.7) can be written as a Fredholm determinant:

$$I_w(\vec{z}) = \det\big(1 - K^c(\vec{z})\big)_{\ell^2(\mathbb{N})} \tag{7.1}$$

with kernel

$$K^c(x, y, \vec{z}) = \sum_{k=0}^{m-1} \phi_k^c(x, \vec{z}) \psi_k^c(y, \vec{z}), \tag{7.2}$$

and

$$\phi_k^c(x, \vec{z}) = \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{1 + z/\rho'}{(z+c)^x z(z-1)} \prod_{j=1}^{n-1} \frac{1 + z_j z}{z} \left(\frac{z}{1-z}\right)^k \mathrm{e}^{-zt/2}, \tag{7.3}$$

$$\psi_k^c(x,\vec{z}) = \oint_{0,-\rho',\{-z_j^{-1}\}_{j=1}^{n-1}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{(w+c)^x}{w(1+w/\rho')} \prod_{j=1}^{n-1} \frac{w}{1+z_j w} \left(\frac{1-w}{w}\right)^k \mathrm{e}^{wt/2}. \quad (7.4)$$

In the right hand side of (7.2), the summation over $k \in [0, m-1]$ can be performed easily as

$$\sum_{k=0}^{m-1} \left[\frac{z(1-w)}{w(1-z)}\right]^k = \frac{w(1-z)}{w-z}\left[1-\left(\frac{z(1-w)}{w(1-z)}\right)^m\right].$$

The first term does not contribute to the kernel because the pole at $z=1$ of the integral with respect to z is removed. Hence, by performing the summation $\sum_{k=0}^{m-1}$ in (7.2), we obtain

$$K^c(x,y,\vec{z}) = \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \prod_{j=1}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',\{-z_j^{-1}\}_{j=1}^{n-1}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \prod_{j=1}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z},$$

where the functions $F^c(z,x)$ and $G^c(w,y)$ are defined as

$$F^c(z,x) = \frac{z+\rho'}{z+1}\mathrm{e}^{f^c(z,x)}, \qquad G^c(w,y) = \frac{w+1}{(w+\rho')(w+c)}\mathrm{e}^{-f^c(w,y)} \quad (7.5)$$

with $f^c(z,x)$ given by

$$f^c(z,x) = n\ln\left(\frac{1+z}{z}\right) + m\ln\left(\frac{z}{1-z}\right) - x\ln(z+c) - \frac{zt}{2}. \quad (7.6)$$

Let us substitute the Fredholm determinant formula of $I_w(\vec{z})$ (7.1) into the non-symmetric expression of $\mathcal{I}_2$ given by (5.6). Recalling the definition of the Fredholm determinant (1.4), $\mathcal{I}_2$ now reads

$$\mathcal{I}_2 = \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \left[1 + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty}\cdots\sum_{x_k=1}^{\infty} \det\left[K^c(x_i,x_j,\vec{z})\right]_{1\le i,j\le k}\right]. \quad (7.7)$$

Here $L(\vec{z})$ is defined in (5.8). Next we show that the summations over $x_i \in \mathbb{N}$ for any $i \in [1,m]$ and z-integrations commute, i.e., we prove the following Lemma.

Lemma 7.1. *For any $\rho \in (0,1)$, $t > 0$ and $n, m \in \mathbb{N}$, the following equality holds:*

$$\oint_1 \mathrm{d}^{n-1}z L(\vec{z}) \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty}\cdots\sum_{x_k=1}^{\infty} \det\left[K^c(x_i,x_j,\vec{z})\right]_{1\le i,j\le k}$$
$$= \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty}\cdots\sum_{x_k=1}^{\infty} \oint_1 \mathrm{d}^{n-1}z L(\vec{z}) \det\left[K^c(x_i,x_j,\vec{z})\right]_{1\le i,j\le k}.$$

Proof. See Appendix E.1. □

Owing to this Lemma, we can rewrite the right hand side of (7.7) as

$$\mathcal{I}_2 = I_z + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \det\left[K^c(x_i, x_j, \vec{z})\right]_{1\le i,j\le k}, \quad (7.8)$$

where

$$I_z := \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}). \quad (7.9)$$

In the following, we will prove that the asymptotic behaviour of $\mathcal{I}_2$ given by the right hand side of (7.8) is same as that with $K^c(x_i, x_j, \vec{z})$ replaced by $K^c(x_i, x_j, \vec{1})$. To establish this, the following Proposition for each term in (7.8) plays a central role.

Proposition 7.2. *For any $(x_1, x_2, \dots, x_k) \in \mathbb{N}^k$, $\rho \in (0, 1)$, $t > 0$ and $n, m \in \mathbb{N}$, the following equality holds:*

$$\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \det\left[K^c(x_i, x_j, \vec{z})\right]_{1\le i,j\le k}$$
$$= \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \det\left\{\bar{K}^c(x_i, x_j) - \left[\sum_{l=1}^{n-1} \prod_{k=1}^{l} (z_k - 1) A_l^c(x_i)\right] B^c(x_j)\right\}_{1\le i,j\le k}.$$

Here the functions, $\bar{K}^c(x, y)$, $A_j^c(x)$ and $B^c(x)$ are defined as

$$\bar{K}^c(x, y) := K^c(x, y, \vec{1}) = \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z, x) \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w, y) \frac{1}{w - z},$$

$$A_j^c(x) := \oint_1 \frac{\mathrm{d}w}{2\pi\mathrm{i}} F^c(w, x) \left(\frac{w}{1+w}\right)^{j-1} \frac{1}{1+w}, \quad (7.10)$$

$$B^c(y) := \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w, y) \frac{1}{w+1}, \quad (7.11)$$

where the functions $F^c(z, x)$ and $G^c(w, y)$ are defined in (7.5).

Here let us recall $L(\vec{z})$, defined in (5.8), and rewrite it as

$$L(\vec{z}) = \frac{\mathrm{e}^{-\rho t/2}}{(\rho')^{n-1}} \left(\frac{2(1-\rho)}{2-\rho}\right)^m \prod_{j=1}^{n-1} h(z_j) \frac{(-1)^{n-1} \Delta_{n-1}(-z)}{\prod_{j=1}^{n-1} (z_j - 1)^{j+1}}, \quad (7.12)$$

where $\Delta_{n-1}(z) = \prod_{1\le i<j\le n-1} (z_i - z_j)$ and

$$h(z) := (1 - \rho' z) \left(\frac{2}{1+z}\right)^m \mathrm{e}^{(z^{-1}-1)\frac{t}{2}}.$$

In order to prove Proposition 7.2, we need to take some steps which consist of proving Lemmas 7.3, 7.4 and 7.5. In the proofs of these Lemmas, the asymmetric part of $L(\vec{z})$, $\Delta_{n-1}(-z)/\prod_{j=1}^{n-1}(z_j - 1)^{j+1}$, plays a crucial role in algebraic manipulations. We utilise the properties that $\Delta_{n-1}(-z)$ is anti-symmetric under exchanges $z_i \leftrightarrow z_j$ for any pairs $(i, j) \in [1, n-1]^2$ and z-integrand, namely $L(\vec{z})$, has the poles of the order $j + 1$ at $z_j = 1$ for any $j \in [1, n-1]$. On the other hand, the detailed form of the other part of $L(\vec{z})$, which is symmetric function of z_js, is not important on algebraic manipulations because it does not produce any additional singularities at unity, whereas it is important on asymptotic analysis.

As a first step, we separate each entry of the determinant into the main contribution coming from $z_j = 1$ and the other.

Lemma 7.3. *For any* $(x, y) \in \mathbb{N}^2$, $\rho \in (0, 1)$, $t > 0$ *and* $n, m \in \mathbb{N}$, *the following equality holds:*

$$\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z})K^c(x, y, \vec{z}) = \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z})\left[\bar{K}^c(x, y) - (z_1 - 1)A^c(x, \vec{z})B^c(y)\right], \tag{7.13}$$

where $A^c(x, \vec{z})$ *is defined as*

$$A^c(x, \vec{z}) := \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z, x)\frac{1}{1+z}\prod_{j=2}^{n-1}\frac{1+z_j z}{1+z}.$$

Proof. We carefully consider the contributions of the poles at $z_j = 1$. Considering $L(\vec{z})$ is expressed as (7.12), the left hand side of (7.13) except for the constant factor, $\frac{\mathrm{e}^{-\rho t/2}}{(-\rho')^{n-1}}\left\{\frac{2(1-\rho)}{2-\rho}\right\}^m$, is explicitly given by

$$\oint_1 \prod_{j=1}^{n-1}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}}\frac{\Delta_{n-1}(-z)}{\prod_{j=1}^{n-1}(z_j - 1)^{j+1}}\prod_{j=1}^{n-1}h(z_j)$$
$$\times \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}}F^c(z, x)\prod_{j=1}^{n-1}\frac{1+z_j z}{1+z}\oint_{C_1}\frac{\mathrm{d}w}{2\pi\mathrm{i}}G^c(w, y)\prod_{j=1}^{n-1}\frac{1+w}{1+z_j w}\frac{1}{w-z}, \tag{7.14}$$

where C_1 is the contour obtained by substituting $k = 1$ into C_k which is defined as the contour enclosing the poles in w at $0, -\rho'$, -1 and $\{-z_j^{-1}\}_{j=k}^{n-1}$. When $k = n$, C_n is the contour enclosing the poles at $0, -\rho'$ and -1.

In the following, we choose the contours of $\vec{z}$-integrals such that they are included in the open polydisc $S(1, 1)^{n-1}$ where $S(a, r) := \{z \in \mathbb{C} \mid |z - a| < r\}$. Since the integral with respect to w has an essential singularity at the infinity and its contour is chosen such that it excludes the infinity, the contour of the integral in terms of the variable $1/w$ includes $\infty, -1/\rho', -1, -z_j$'s and excludes the origin. For $\vec{z} \in S(1, r)^{n-1}$ with $r \in (0, 1)$, to compute the residues at $z_j = 1$, we choose the contour such that $r < |1 + 1/w|$ holds. On such a contour, the following expansion converges uniformly,

$$\frac{1+w}{1+z_j w} = \sum_{i=0}^{\infty}\left(\frac{w(1-z_j)}{1+w}\right)^i, \qquad \max_{1\le j\le n-1}\left|\frac{w(1-z_j)}{w+1}\right| \le \varepsilon < 1, \tag{7.15}$$

and sum in (7.15) and integration with respect to z_j's can be interchanged. First, using (7.15) for $j = 1$, we perform the expansion of integrand in (7.14) with respect to z_1 around unity. Because the pole at $z_1 = 1$ is of order 2, only the terms corresponding to $i = 0$ and $i = 1$ in the expansion (7.15) give a nonzero contribution to (7.14). Therefore (7.14) can be written as $I_1^{(2)} + I_2^{(2)}$ where we define $I_1^{(k)}$ and $I_2^{(k)}$ for $k \in [2, n]$ as

$$
\begin{aligned}
I_1^{(k)} &= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \\
&\quad \times \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \prod_{j=1}^{n-1} \frac{1+z_j z}{1+z} \oint_{C_k} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \prod_{j=k}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z}, \\
I_2^{(k)} &= -\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}(z_1-1)} \prod_{j=1}^{n-1} h(z_j) \\
&\quad \times \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(x,z) \prod_{j=1}^{n-1} \frac{1+z_j z}{1+z} \oint_{C_k} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \frac{w}{1+w} \prod_{j=k}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z}.
\end{aligned}
$$

We should explain more in detail how we get this decomposition. As a convention we set the factor $\prod_{j=k}^{n-1} \frac{1+w}{1+z_j w}$ to unity for $k=n$.

Secondly, we will show that $I_1^{(2)}+I_2^{(2)}$ is the same as $I_1^{(n)}+I_2^{(n)}$, in which w-integrands do not depend on z_j's. To show this, we prove that the equality

$$
I_1^{(k)} + I_2^{(k)} = I_1^{(k+1)} + I_2^{(k+1)} \tag{7.16}
$$

holds for any $k \in [2, n-1]$. We expand $\frac{1+w}{1+z_k w}$ in $I_1^{(k)}$ around $z_k = 1$ as in (7.15). The terms of order $k+1$ or higher in this expansion make the integrand analytic in z_k and hence give zero residue. The remaining terms become

$$
\begin{aligned}
I_1^{(k)} &= \sum_{i=0}^{k} \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=1,j\neq k}^{n-1}(z_j-1)^{j+1}(z_k-1)^{k+1-i}} \prod_{j=1}^{n-1} h(z_j) \\
&\quad \times \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \prod_{j=1}^{n-1} \frac{1+z_j z}{1+z} \oint_{C_{k+1}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \left(\frac{-w}{1+w}\right)^i \prod_{j=k+1}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z}.
\end{aligned}
$$

The terms with $i = 1, \ldots, k-1$ vanish because for each of those terms the integrand is anti-symmetric under exchange $z_{k-i} \leftrightarrow z_k$. Hence only the terms with $i=0$ and $i=k$ survive,

$$
\begin{aligned}
I_1^{(k)} &= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=1,j\neq k}^{n-1}(z_j-1)^{j+1}(z_k-1)} \prod_{j=1}^{n-1} h(z_j) \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \prod_{j=1}^{n-1} \frac{1+z_j z}{1+z} \\
&\quad \times \oint_{C_{k+1}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \left(\frac{1}{(z_k-1)^k} + \left(\frac{-w}{1+w}\right)^k \right) \prod_{j=k+1}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z}.
\end{aligned} \tag{7.17}
$$

Likewise, expanding $\frac{1+w}{1+z_k w}$ around $z_k = 1$ in $I_2^{(k)}$ and making use of analytic and symmetry properties of the integrand, we obtain

$$
\begin{aligned}
I_2^{(k)} &= -\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=2,j\neq k}^{n-1}(z_j-1)^{j+1}(z_k-1)^2(z_1-1)} \prod_{j=1}^{n-1} h(z_j) \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \prod_{j=1}^{n-1} \frac{1+z_j z}{1+z} \\
&\quad \times \oint_{C_{k+1}} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \frac{w}{1+w} \left(\frac{1}{(z_k-1)^{k-1}} + \left(\frac{-w}{1+w}\right)^{k-1} \right) \prod_{j=k+1}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z}.
\end{aligned} \tag{7.18}
$$

The second terms of (7.17) and (7.18) differ only in terms of the orders of the poles at $z_1 = 1$ and $z_k = 1$. It is easy to see that the integrand of the sum of these terms is anti-symmetric under the exchange $z_1 \leftrightarrow z_k$ and thus it is equal to zero. On the other hand, the first terms of (7.17) and (7.18) coincide with $I_1^{(k+1)}$ and $I_2^{(k+1)}$, respectively. Thus, it turns out that the equality (7.16) holds for any $k \in [2, n-1]$. Using the equality (7.16) from $k = 2$ to $k = n-1$ consecutively, we obtain

$$I_1^{(2)} + I_2^{(2)} = I_1^{(n)} + I_2^{(n)}.$$

Thirdly, we further manipulate $I_1^{(n)}$ and $I_2^{(n)}$. Considering the equality $\frac{1+z_1 z}{1+z} = 1 + \frac{z(z_1-1)}{1+z}$, $I_1^{(n)}$ can be divided into two terms as

$$\begin{aligned} I_1^{(n)} &= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \\ &\quad \times \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \prod_{j=2}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \frac{1}{w-z} \\ &\quad + \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}(z_1-1)} \prod_{j=1}^{n-1} h(z_j) \\ &\quad \times \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \left(\frac{z}{1+z}\right) \prod_{j=2}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \frac{1}{w-z}. \end{aligned} \tag{7.19}$$

Applying the expansion

$$\prod_{j=2}^{n-1} \frac{1+z_j z}{1+z} = 1 + \sum_{j=1}^{n-2} \left(\frac{z}{1+z}\right)^j \sum_{2 \le \ell_1 < \ell_2 < \cdots < \ell_j \le n-1} \prod_{i=1}^{j} (z_{\ell_i} - 1) \tag{7.20}$$

to the first term of (7.19), unity remains and all terms in the sums vanish because for each term there exists a pair $(i, j) \in [1, n-1]^2$ such that the $\vec{z}$-integrand is anti-symmetric under the exchange $z_i \leftrightarrow z_j$. Indeed, since $(z_{\ell_1} - 1)$ in the numerator of each term of (7.20) reduces the order of the pole at $z_{\ell_1} = 1$ to ℓ_1 and the order of the pole at $z_{\ell_1 - 1} = 1$ is definitely ℓ_1 because of $\ell_1 \ge 2$, the factor depending on z_{ℓ_1-1} and z_{ℓ_1} in the denominator becomes $(z_{\ell_1-1} - 1)^{\ell_1}(z_{\ell_1} - 1)^{\ell_1}$ and then the $\vec{z}$-integrand becomes anti-symmetric under the exchange $z_{\ell_1-1} \leftrightarrow z_{\ell_1}$. Hence, $(i, j) = (\ell_1 - 1, \ell_1)$ is one of such pairs. In addition, if $\ell_i - \ell_{i-1} \ge 2$ were satisfied for some $i \in [2, n]$, the factor depending on z_{ℓ_i-1} and z_{ℓ_i} in the denominator becomes $(z_{\ell_i-1} - 1)^{\ell_i}(z_{\ell_i} - 1)^{\ell_i}$ for the same reason as above and then $(i, j) = (\ell_i - 1, \ell_i)$ is also one of pairs for which the $\vec{z}$-integrand is anti-symmetric under the exchange $z_{\ell_i-1} \leftrightarrow z_{\ell_i}$. Thus, the first term of (7.19) reduces to

$$\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} F^c(z,x) \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} G^c(w,y) \frac{1}{w-z}, \tag{7.21}$$

and it coincides with the first term of the right hand side of (7.13).

Likewise, considering the equality $\frac{1+z_1 z}{1+z} = 1 + \frac{z(z_1-1)}{1+z}$, $I_2^{(n)}$ can be divided into two terms and the second term vanishes because the z_1-integrand is regular in the vicinity of $z_1 = 1$, i.e.

$$I_2^{(n)} = -\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}(z_1-1)} \prod_{j=1}^{n-1} h(z_j)$$
$$\times \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} F^c(z,x) \prod_{j=2}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi \mathrm{i}} G^c(w,y) \frac{w}{1+w} \frac{1}{w-z}.$$

From the above, the sum of $I_2^{(n)}$ and the second term of (7.19) can be described as the product of a function of x and that of y, i.e.

$$\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}(z_1-1)} \prod_{j=1}^{n-1} h(z_j)$$
$$\times \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} F^c(z,x) \prod_{j=2}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi \mathrm{i}} G^c(w,y) \left[\frac{z}{1+z} - \frac{w}{1+w}\right] \frac{1}{w-z}$$
$$= -\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}(z_1-1)} \prod_{j=1}^{n-1} h(z_j)$$
$$\times \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} F^c(z,x) \frac{1}{1+z} \prod_{j=2}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi \mathrm{i}} G^c(w,y) \frac{1}{1+w},$$

and it coincides with the second term of the right hand side of (7.13). Putting everything together, it follows that

$$\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) K^c(x,y,\vec{z})$$
$$= I_1^{(2)} + I_2^{(2)} = I_1^{(n)} + I_2^{(n)}$$
$$= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\Delta_{n-1}(-z)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left[\bar{K}^c(x,y) - (z_1-1) A^c(x,\vec{z}) B^c(y)\right].$$

□

From the claim of Lemma 7.3, we can easily obtain the following Lemma 7.4. Although this looks similar to Lemma 7.3, they will be applied in different manners in the proof of Proposition 7.2.

Lemma 7.4. *For any* $(x,y) \in \mathbb{N}^2$*,* $\rho \in (0,1)$*,* $t > 0$ *and* $n, m \in \mathbb{N}$*, the following equality holds:*

$$\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \left[\bar{K}^c(x,y) - (z_1-1)A^c(x,\vec{z})B^c(y)\right]$$
$$= \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \left\{ \bar{K}^c(x,y) - \left[\sum_{j=1}^{n-1}\prod_{i=1}^{j}(z_i-1)A_j^c(x)\right] B^c(y)\right\}, \quad (7.22)$$

where $A_j^c(x)$ *is defined as* (7.10).

Proof. Applying the expansion (7.20) to $A^c(x,\vec{z})$ in the left hand side of (7.22), for any $j \in [1, n-2]$, the only terms of which $(\ell_1, \ell_2, \ldots, \ell_j)$ is the consecutive numbers $(2, 3, \ldots, j+1)$ remain and the others vanish for the following reason. All the terms satisfying $\ell_1 \geq 3$ vanish because $\vec{z}$-integrand turns out to be anti-symmetric at least under exchange $z_{\ell_1-1} \leftrightarrow z_{\ell_1}$ by the same discussion as that used in the derivation of (7.21). On the other hand, all the terms of which $(2, \ell_2, \ldots, \ell_j)$ are not anti-symmetric under exchange $z_i \leftrightarrow z_2$ because (z_2-1) in the numerator reduces the order of the pole at $z_2 = 1$ from 3 to 2 and the factor depending on z_1 and z_2 in the denominator becomes a non-symmetric function, $(z_1-1)(z_2-1)^2$. By the same discussion as above, we can sequentially show in the order from $i=1$ to $i=j$ the fact that the terms satisfying $\ell_i \geq i+2$ vanish for all $i \in [1, j]$. Hence, the only terms satisfying $(\ell_1, \ell_2, \ldots, \ell_j) = (2, 3, \ldots, j+1)$ remain. That is, the second term of the left hand side of (7.22) except for the constant factor can be described as

$$-\oint_1 \prod_{j=1}^{n-1}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)\prod_{j=1}^{n-1}h(z_j)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}(z_1-1)} \oint_1 \frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{F^c(w,x)}{1+w}\left\{1+\sum_{j=1}^{n-2}\left(\frac{w}{1+w}\right)^j \prod_{i=2}^{j+1}(z_i-1)\right\}B^c(y)$$
$$= -\oint_1 \prod_{j=1}^{n-1}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\Delta_{n-1}(-z)\prod_{j=1}^{n-1}h(z_j)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \left[\sum_{j=1}^{n-1}\prod_{i=1}^{j}(z_i-1)\oint_1\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{F^c(w,x)}{1+w}\left(\frac{w}{1+w}\right)^{j-1}\right]B^c(y).$$
$$(7.23)$$

From (7.10), the definition of $A_j^c(x)$, it turns out that the right hand side of (7.23) coincides with the second term of the right hand side of (7.22). Therefore, the equality (7.22) holds. □

As a second step, we see that $K^c(x_i, x_j, \vec{z})$ is a holomorphic function in the vicinity of $\vec{z} = \vec{1}$. It guarantees that Lemma 7.4 is extended to the determinant as will be mentioned in the proof of Proposition 7.2.

Lemma 7.5. *For any* $(x,y) \in \mathbb{N}^2$, $\rho \in (0,1)$, $t > 0$ *and* $n, m \in \mathbb{N}$, *the kernel* $K^c(x,y,\vec{z})$ *defined as* (7.1) *is a holomorphic function of* $\vec{z} = (z_1, \ldots, z_{n-1})$ *in* $S(1,1)^{n-1}$, *where* $S(a,r) := \{z \in \mathbb{C} \mid |z-a| < r\}$ *for* $a \in \mathbb{C}$.

Proof. See Appendix E.2. □

Proof of Proposition 7.2. It now follows from Lemmas 7.3, 7.4 and 7.5. Details will be given in Appendix E.3. □

In the following, we will divide $\mathcal{I}_2$ given by (5.5) into the dominant contribution and the subdominant term, which converge to the GUE Tracy–Widom distribution $F_2(s_2)$ and zero in the long time limit, respectively, with the scaling (5.12). Applying Proposition 7.2 to each term of the series in the right hand side of (7.8), we obtain

$$
\begin{aligned}
\mathcal{I}_2 &= I_z + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \det \left[K^c(x_i, x_j, \vec{z}) \right]_{1\le i,j\le k} \\
&= I_z + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \\
&\quad \times \det \left\{ \bar{K}^c(x_i, x_j) - \left[\sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1) A_p^c(x_i) \right] B^c(x_j) \right\}_{1\le i,j\le k} .
\end{aligned} \tag{7.24}
$$

Here, we note the Matrix Determinant Lemma:

$$
\det \left(a_{i,j} - u_i v_j \right)_{1\le i,j\le n} = \det \left(a_{i,j} \right)_{1\le i,j\le n} - \sum_{\ell=1}^{n} \sum_{k=1}^{n} (-1)^{\ell+k} u_\ell v_k \times \det \left(a_{i,j} \right)_{\substack{1\le i,j\le n \\ i\neq \ell, j\neq k}} \tag{7.25}
$$

holds for any matrix whose entries are given as $a_{i,j} \in \mathbb{C}, i, j \in [1, n]$ and any vectors whose components are given as $u_i, v_j \in \mathbb{C}, i, j \in [1, n]$. (7.25) can be easily proved by the cofactor expansion (or Laplace expansion) for the determinant. Applying the identity (7.25) to the determinant in the right hand side of the final equality of (7.24), we obtain

$$
\begin{aligned}
&\det \left\{ \bar{K}^c(x_i, x_j) - \left[\sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1) A_p^c(x_i) \right] B^c(x_j) \right\}_{1\le i,j\le k} \\
&= \det \left[\bar{K}^c(x_i, x_j) \right]_{1\le i,j\le k} - \sum_{r,\ell=1}^{k} (-1)^{r+\ell} \det \left[\bar{K}^c(x_i, x_j) \right]_{\substack{1\le i,j\le k \\ i\neq \ell, j\neq r}} \left[\sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1) A_p^c(x_\ell) \right] B^c(x_r).
\end{aligned} \tag{7.26}
$$

It follows from (7.24) and (7.26) that $\mathcal{I}_2$ can be written as

$$
\begin{aligned}
\mathcal{I}_2 = I_z &+ \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} I_z \times \det \left[\bar{K}^c(x_i, x_j) \right]_{1\le i,j\le k} \\
&- \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \sum_{r,\ell=1}^{k} (-1)^{r+\ell} \\
&\times \det \left[\bar{K}^c(x_i, x_j) \right]_{\substack{1\le i,j\le k \\ i\neq \ell, j\neq r}} \left[\sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1) A_p^c(x_\ell) \right] B^c(x_r).
\end{aligned}
$$

As for the second term, I_z can be moved to outside of the sums from $x_i = 1$ to ∞ for any $i \in [1, n-1]$ because $\det \left[\bar{\mathcal{K}}_t^c(x_i, x_j) \right]_{1\le i,j\le k}$ does not depend on z_j's. For the same reason, about the final term, $\oint_1 \mathrm{d}^{n-1} z\, L(\vec{z})$ can be moved to the inside of the sum from $p = 1$ to $n-1$. Then, we arrive at the following proposition.

Proposition 7.6. *The integral $\mathcal{I}_2$ given in* (5.5) *and* (5.6) *can be written into*

$$
\mathcal{I}_2 = \mathcal{I}_2^{(1)} - \mathcal{I}_2^{(2)}, \tag{7.27}
$$

where

$$
\mathcal{I}_2^{(1)} = I_z \times \det \left(1 - \bar{K}^c \right)_{\ell^2(\mathbb{N})}, \tag{7.28}
$$

$$\mathcal{I}_2^{(2)} = \sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} \sum_{r,\ell=1}^{k} (-1)^{r+\ell}$$
$$\times \det\left[\bar{K}^c(x_i,x_j)\right]_{\substack{1\le i,\, j\le k\\ i\neq \ell,\, j\neq r}} \left[\sum_{p=1}^{n-1} \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \prod_{q=1}^{p} (z_q-1) A_p^c(x_\ell)\right] B^c(x_r). \quad (7.29)$$

7.2. Evaluation of $\mathcal{I}_2^{(1)}$. We can separate the estimate of $\mathcal{I}_2^{(1)}$ into I_z and $\det\left(1-\bar{K}^c\right)_{\ell^2(\mathbb{N})}$ since $\bar{K}^c(x,y)$ does not contain $\vec{z}$. The analysis of $\det\left(1-\bar{K}^c\right)_{\ell^2(\mathbb{N})}$ follows exactly the same arguments as in Sect. 6, while the limit of I_z is obtained by the same idea as in the proofs of an exponentially decreasing bound and uniform convergence of a kernel employed in [14–16].

7.2.1. Evaluation of $\det\left(1-\bar{K}_t^c\right)_{\ell^2(\mathbb{N})}$. In order to perform asymptotic analysis, we define the rescaled functions which depend on the positive real numbers $\xi = x/\lambda_c t^{1/3}$ and $\zeta = y/\lambda_c t^{1/3}$ with $x, y \in \mathbb{N}$ as

$$\bar{\mathcal{K}}_t^c(\xi,\zeta) = (w_* + c)^{\lambda_c t^{1/3}(\xi-\zeta)} \lambda_c t^{1/3} \bar{K}_t^c(\lambda_c t^{1/3}\xi, \lambda_c t^{1/3}\zeta), \quad (7.30)$$

where λ_c is defined in (6.5). The rescaled kernel is explicitly described as

$$\bar{\mathcal{K}}_t^c(\xi,\zeta) = \lambda_c t^{1/3} \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left(\frac{z+\rho'}{z+1}\right) \mathrm{e}^{\,f(z,t,\xi)-f(w_*,t,\xi)} \oint_{0,-\rho',-1} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \left(\frac{w+1}{w+\rho'}\right)$$
$$\times\ \mathrm{e}^{\,-f(w,t,\zeta)+f(w_*,t,\zeta)+g(w)} \frac{1}{w-z},$$

where $g(w) = -\ln(w+c)$ and $f(z,t,\xi) = g_1(z)t + g_2(z)t^{1/2} + g_3(z,\xi)t^{1/3}$ with the functions $g_1(z)$, $g_2(z)$ and $g_3(z,\xi)$ defined as (6.8). Note that $\bar{\mathcal{K}}_t^c$ differs from $\mathcal{K}_t^c$ in the sense that it has $(z+\rho')(w+1)/[(z+1)(w+\rho')]$ in the integrands and its contour includes also $w = -\rho'$, and this fact leads to the following two Propositions that correspond to Propositions 6.2 and 6.3.

Proposition 7.7 (Uniform convergence of $\bar{\mathcal{K}}_t^c$ on a bounded set). *Let m, n be scaled as* (5.12). *Then for any $c > 0$ and fixed $L > 0$, the rescaled kernel $\bar{\mathcal{K}}_t^c(\xi,\zeta)$ defined in* (7.30) *converges uniformly on $\xi, \zeta \in [-L, L]$ to*

$$\lim_{t\to\infty} \bar{\mathcal{K}}_t^c(\xi,\zeta) = A(\xi+s_2, \zeta+s_2),$$

where $A(x,y)$ is defined as (1.11) *and s_2 is given in* (5.10).

Proposition 7.8 (Estimate of kernel $\bar{\mathcal{K}}_t^c(\xi,\zeta)$ for unbounded ξ,ζ). *Let m, n be scaled as* (5.12). *Then for any $c > 1$ and large enough L and t, the rescaled kernel $\bar{\mathcal{K}}_t^c(\xi,\zeta)$ defined in* (7.30) *is bounded by*

$$\left|\bar{\mathcal{K}}_t^c(\xi,\zeta)\right| \le \mathrm{e}^{-(\xi+\zeta)}$$

holds for $(\xi,\zeta) \in [0,\infty)^2 \backslash [0,L]^2$.

Proof of Propositions 7.7 and 7.8. The contours of the integrals with respect to z and w used in the proofs of Propositions 6.2 and 6.3 include also the point $w = -\rho'$, and $(z+\rho')(w+1)/[(z+1)(w+\rho')]$ is a regular function on the contours and equals unity at the point $(z, w) = (w_*, w_*)$. Hence, using the same arguments as for Propositions 6.2 and 6.3, we can prove Propositions 7.7 and 7.8, respectively. □

From Lemma 6.5, it turns out that $\det\big(1 - \bar{K}^c\big)_{\ell^2(\mathbb{N})}$ converges to the GUE Tracy–Widom distribution in the long time limit with the scaling (5.12), i.e.,

$$\lim_{t\to\infty} \det\big(1 - \bar{\mathcal{K}}_t^c\big)_{\ell^2(\mathbb{N}/(\lambda_c t^{1/3}))} = \lim_{t\to\infty} \det\big(1 - \bar{\mathcal{K}}_t^c\big)_{L^2(0,\infty)} = \det(1 - A)_{L^2(s_2,\infty)} = F_2(s_2). \quad (7.31)$$

7.2.2. Evaluation of I_z. We are now left with the $\vec{z}$-integrals, namely I_z, defined in (7.9). From the definition of $L(\vec{z})$ given in (5.8), I_z turns out to be expressed as

$$I_z = \frac{\mathrm{e}^{-\rho t/2}}{(\rho')^{n-1}} \left(\frac{2(1-\rho)}{2-\rho}\right)^m \frac{(-1)^{n-1}}{(n-1)!} \oint_1 \mathrm{d}^{n-1}z \frac{\mathrm{e}^{\Lambda_{n-1,0}t}\, \Delta_{n-1}(z)\Delta_{n-1}(-z) \prod_{i=1}^{n-1}(1-\rho' z_i)}{\prod_{i=1}^{n-1}(z_i-1)^n[\frac{1}{2}(z_i+1)]^m}.$$

It is hard to apply the method introduced in Sect. 4, since the order of the pole at $z_j = 1$ is one more than the number of $\vec{z}$-variables. But the difficulty can be overcome by considering the following n-fold integral instead of I_z,

$$J^{(n)} = \frac{\rho^n}{n!} \oint_{1,1/\rho'} \mathrm{d}^n z \frac{\mathrm{e}^{\Lambda_{n,0}t}\, \Delta_n(z)\Delta_n(-z)}{\prod_{i=1}^{n}(1-\rho' z_i)(z_i-1)^n[\frac{1}{2}(z_i+1)]^m} =: \oint_{1,1/\rho'} \mathrm{d}^n z J^{(n)}(\vec{z}). \quad (7.32)$$

Lemma 7.9. *The integral I_z defined in (7.9) can be written into $I_z = 1 - J^{(n)}$ with $J^{(n)}$ given by (7.32).*

Proof. Calculating the residue of the simple pole at $z_n = 1/\rho'$ of the z_n-integral, we have

$$\begin{aligned}
&\oint_{1,1/\rho'} \mathrm{d}^{n-1}z \,\mathrm{Res}_{\,z_n=1/\rho'} J^{(n)}(\vec{z}) \\
&= -\frac{\mathrm{e}^{-\rho t/2}}{(\rho')^{n-1}} \left(\frac{2(1-\rho)}{2-\rho}\right)^m \frac{(-1)^{n-1}}{n!} \oint_1 \mathrm{d}^{n-1}z \frac{\mathrm{e}^{\Lambda_{n-1,0}t}\, \Delta_{n-1}(z)\Delta_{n-1}(-z) \prod_{i=1}^{n-1}(1-\rho' z_i)}{\prod_{i=1}^{n-1}(z_i-1)^n[\frac{1}{2}(z_i+1)]^m} \\
&= -I_z/n.
\end{aligned}$$

Since $\mathrm{Res}_{z_n=1/\rho'} J^{(n)}(\vec{z})$ has no poles at $z_j = 1/\rho'$, and all the other contributions of poles at $z_j = 1/\rho'$ is just n times $\mathrm{Res}_{z_n=1/\rho'} J^{(n)}(\vec{z})$, we therefore obtain

$$\begin{aligned}
-I_z &= J^{(n)} - \frac{\rho^n}{n!} \oint_1 \mathrm{d}^n z \frac{\mathrm{e}^{\Lambda_{n,0}t}\, \Delta_n(z)\Delta_n(-z)}{\prod_{i=1}^{n}(1-\rho' z_i)(z_i-1)^n[\frac{1}{2}(z_i+1)]^m} \\
&= J^{(n)} - \rho^n \oint_1 \mathrm{d}^n z \frac{\mathrm{e}^{\Lambda_{n,0}t}\, \Delta_n(z)}{\prod_{i=1}^{n}(1-\rho' z_i)(z_i-1)^i[\frac{1}{2}(z_i+1)]^m} \\
&= J^{(n)} - 1.
\end{aligned}$$

The second equality follows from anti-symmetrisation with the identity (3.6). The third line is obtained by evaluating poles at $z_j = 1$ sequentially. Starting with the simple pole at $z_1 = 1$, its residue will decrease the order of other poles at $z_j = 1$ by one, and hence all the poles can be evaluated sequentially. □

The asymptotic behaviour of I_z is now given by the asymptotics of $J^{(n)}$, which is given by

$$\lim_{t\to\infty} J^{(n)} = F_G(s_{\rm g}), \tag{7.33}$$

where n is scaled as (5.12) and can be analysed via the previous method. Specifically, after rearrangements, we have

$$J^{(n)} = \frac{1}{n!}\oint_{1,\rho'} \frac{{\rm d}^n z}{\prod_{i=1}^n z_i} \frac{\prod_{1\le i\ne j\le n}(1-z_i/z_j)}{\prod_{i=1}^n(1-1/z_i)^n} \prod_{i=1}^n \frac{g(z_i,0)}{g(1,0)}, \tag{7.34}$$

where

$$g(z,x) = \frac{z^{-x-n}}{1-\rho'/z}\left(\frac{z}{1+z}\right)^m {\rm e}^{zt/2}.$$

The right hand side of (7.34) fits the standard form (1.6) with

$$\begin{aligned} \nu &= n,\ \mu = m, \quad \gamma = t/2, \quad s = 0,\\ v_i &= 1,\ 1\le i\le m,\\ a_k &= 1,\ 1\le k\le n,\\ u_k &= 0,\ 1\le k\le n-1, \quad u_n = \rho'. \end{aligned}$$

In this case, Proposition 1.3 holds for $c = 0$. According to Sect. 4, $J^{(n)}$ is thus written as a Fredholm determinant with the kernel,

$$K(x,y) = \sum_{k=0}^{n-1}\phi_k(x)\psi_k(y), \tag{7.35}$$

where

$$\begin{aligned} \phi_k(x) &= \oint_1 \frac{{\rm d}w}{2\pi{\rm i}}\frac{w^k(1-\rho'/w)}{w^x(w-1)^{k+1}}\left(\frac{w+1}{w}\right)^m {\rm e}^{-wt/2},\\ \psi_k(x) &= \oint_{\rho'} \frac{{\rm d}w}{2\pi{\rm i}}\frac{w^x(w-1)^k}{(1-\rho'/w)w^{k+2}}\left(\frac{w}{w+1}\right)^m {\rm e}^{wt/2}. \end{aligned}$$

The pole $w = \rho'$ in $\psi_k(x)$ is of order 1 and hence can be evaluated easily.

By performing the contour integration in $\psi_k(x)$ and the summation $\sum_{k=0}^{n-1}$ in (7.35), we obtain

$$K(x,y) = -(\rho')^y\left(\frac{\rho'}{1+\rho'}\right)^m\left(\frac{1-\rho'}{\rho'}\right)^n {\rm e}^{\rho' t/2}\oint_1\frac{{\rm d}w}{2\pi{\rm i}}\frac{1}{w^{x+1}}\left(\frac{w+1}{w}\right)^m\left(\frac{w}{1-w}\right)^n{\rm e}^{-wt/2}. \tag{7.36}$$

Like before, with the setting $(x,y) = (\lambda_2 t^{1/2}\xi, \lambda_2 t^{1/2}\zeta)$ where the scaling power $t^{1/2}$ stems from the Gaussian fluctuation and λ_2 is given by

$$\lambda_2 = \frac{c_{\rm g}}{2\rho(2-\rho)}, \tag{7.37}$$

we rescale the kernel by

$$\mathcal{K}_t(\xi,\zeta) := \rho'^{\lambda_2(\xi-\zeta)t^{1/2}}\lambda_2 t^{1/2}K(\lambda_2 t^{1/2}\xi, \lambda_2 t^{1/2}\zeta), \tag{7.38}$$

where $K(x, y)$ is given in (7.36). Then with the scaling (5.12), the rescaled kernel (7.38) can be rewritten by collecting the terms according to the order of t:

$$\mathcal{K}_t(\xi, \zeta) = -\lambda_2 t^{1/2} \oint_1 \frac{\mathrm{d}w}{2\pi \mathrm{i}}\, \mathrm{e}^{f(w,t,\xi)-f(\rho',t,\xi)+g_4(w)}, \tag{7.39}$$

where $f(w, t, \xi) = g_1(w)t + g_2(w, \xi)t^{1/2} + g_3(w)t^{1/3} + g_4(w)$ with $g_1(w)$, $g_2(w, \xi)$, $g_3(w)$ and $g_4(w)$ defined by

$$g_1(w) = \frac{(1+\rho)^2(2-\rho)}{16} \ln\left(\frac{w+1}{w}\right) + \frac{\rho(3-\rho)^2}{16} \ln\left(\frac{w}{1-w}\right) - \frac{w}{2}, \tag{7.40a}$$

$$g_2(w, \xi) = -\frac{(1+\rho)c_{\mathrm{g}}s_{\mathrm{g}}}{12(1-\rho)} \ln\left(\frac{w+1}{w}\right) - \frac{(3-\rho)c_{\mathrm{g}}s_{\mathrm{g}}}{12(1-\rho)} \ln\left(\frac{w}{1-w}\right) - \xi\lambda_2 \ln(w), \tag{7.40b}$$

$$g_3(w) = -\frac{(2-\rho)c_2 s_2}{6(1-\rho)} \ln\left(\frac{w+1}{w}\right) - \frac{\rho c_2 s_2}{6(1-\rho)} \ln\left(\frac{w}{1-w}\right), \tag{7.40c}$$

$$g_4(w) = -\ln(w). \tag{7.40d}$$

For the rescaled kernel $\mathcal{K}_t$ given by (7.39), we can prove the uniform convergence which corresponds to Proposition 6.2.

Proposition 7.10 (Uniform convergence of $\mathcal{K}_t$ on a bounded set). *Let m, n be scaled as* (5.12). *Then for any fixed $L > 0$, the rescaled kernel $\mathcal{K}_t(\xi, \zeta)$ defined in* (7.38) *converges uniformly on $\xi \in [-L, L]$ to*

$$\lim_{t\to\infty} \mathcal{K}_t(\xi, \zeta) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\frac{1}{2}(s_{\mathrm{g}}+\xi)^2},$$

where s_{g} is given in (5.10).

Proof. See Appendix F. □

Following the same arguments as in [14–16], which are also similar to the method used in the proof of Proposition 6.3, we have a corresponding estimate of the rescaled kernel $\mathcal{K}_t$ defined in (7.39).

Proposition 7.11 (Estimate of kernel $\mathcal{K}_t(\xi, \zeta)$ for unbounded ξ, ζ). *Let m, n be scaled as* (5.12). *Then for any large enough L and t, the rescaled kernel $\mathcal{K}_t(\xi, \zeta)$ defined in* (7.38) *is bounded by*

$$|\mathcal{K}_t(\xi, \zeta)| \leq \mathrm{e}^{-\xi}$$

holds for $\xi > L$.

With a deformed contour given in Fig. 9, the proof for this proposition follows exactly the same method as in the proof of Proposition 6.3. Therefore we will not repeat it here.

Consequently, these two Propositions 7.10 and 7.11 give the following theorem.

Theorem 7.12. *Consider the rescaled kernel defined in* (7.38)

$$\mathcal{K}_t(\xi, \zeta) = \rho'^{\lambda_2(\xi-\zeta)t^{1/2}} \lambda_2 t^{1/2} K(\lambda_2 t^{1/2}\xi, \lambda_2 t^{1/2}\zeta),$$

with λ_2 given in (7.37). *Then we have*

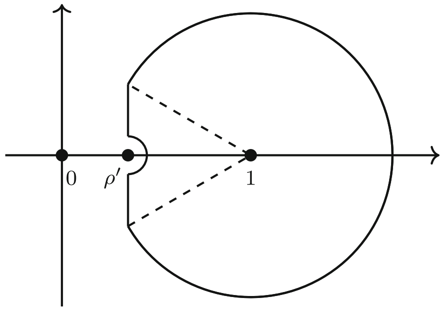

Fig. 9. Deformed steepest descent contour of integration in $\mathcal{K}_t$

(i) *We know*

$$\lim_{t\to\infty} \det(1-\mathcal{K}_t)_{L^2(0,\infty)} = \lim_{t\to\infty} \det(1-\mathcal{K}_t)_{\ell^2(\mathbb{N}/(\lambda_2 t^{1/2}))}, \tag{7.41}$$

where $\mathbb{N}/(\lambda_2 t^{1/2}) := \{x \in \mathbb{R} \mid x = n/(\lambda_2 t^{1/2}), n \in \mathbb{N}\}$. *Here note that the operator* $\mathcal{K}_t$ *on the right hand side is regarded as acting on* $\ell^2(\mathbb{N}/(\lambda_2 t^{1/2}))$ *while the one on the left hand side on* $L^2(0,\infty)$.

(ii) *For any fixed* $L > 0$

$$\lim_{t\to\infty} \mathcal{K}_t(\xi,\zeta) = \frac{e^{-(s_g+\xi)^2/2}}{\sqrt{2\pi}},$$

uniformly on $(\xi,\zeta) \in [0,L]^2$.

(iii) *For t large enough,*

$$|\mathcal{K}_t(\xi,\zeta)| \le C\, e^{-\xi},$$

for some constant $C > 0$ *and* $(\xi,\zeta) \in [0,\infty)^2$.

Proof. A proof falls into the same pattern as in Theorem 6.4. One can show (i) by rescaling $(x,y) = (\lambda_2 t^{1/2}\xi, \lambda_2 t^{1/2}\zeta)$ in the Fredholm kernel in (1.4) and replacing the Riemann sums $\lim_{t\to\infty} (\lambda_2 t^{1/2})^{-1} \sum_{x\in\mathbb{N}}$ with the integral $\int_0^\infty d\xi$. Statement (ii) is obtained by replacing $[-L,L]^2$ with $[0,L]^2$ in Proposition 7.10. Statement (iii) is obtained by combining Propositions 7.10 and 7.11 because the Gaussian function is bounded above by an exponential decreasing function $e^{-\xi}$ when ξ is positive. □

The above results with Lemma 6.5 allow us to take the limit of the Fredholm determinant into the kernel, i.e.,

$$\lim_{t\to\infty} \det(1-\mathcal{K}_t)_{\ell^2(\mathbb{N}/(\lambda_2 t^{1/2}))} = \det\Big(1 - \lim_{t\to\infty}\mathcal{K}_t\Big)_{L^2(0,\infty)} = 1 - \int_0^\infty \frac{e^{-(s_g+\xi)^2/2}}{\sqrt{2\pi}} d\xi$$
$$= \int_{-\infty}^0 \frac{e^{-(s_g+\xi)^2/2}}{\sqrt{2\pi}} d\xi = \int_{-\infty}^{s_g} \frac{e^{-\xi^2/2}}{\sqrt{2\pi}} d\xi = F_G(s_g),$$

which implies the asymptotics (7.33).

Putting all things in Sect. 7.2 together, we have the following theorem.

Theorem 7.13. *With the scaling* (5.12)*, the long time limit of the term* $\mathcal{I}_2^{(1)}$ *defined in* (7.28) *is given by*

$$\lim_{t\to\infty} \mathcal{I}_2^{(1)} = \left[1 - F_G(s_{\rm g})\right] F_2(s_2), \tag{7.42}$$

where $s_{\rm g}$ *and* s_2 *are given in* (5.10)*.*

Proof. Applying Lemma 7.9 to the right hand side of (7.28), we obtain

$$\mathcal{I}_2^{(1)} = (1 - J^{(n)}) \times \det\left(1 - \bar{\mathcal{K}}_t^c\right)_{\ell^2(\mathbb{N}/\lambda_c t^{1/3})}.$$

Since the long time limit of $\det\left(1 - \bar{\mathcal{K}}_t^c\right)_{\ell^2(\mathbb{N}/\lambda_c t^{1/3})}$ and $J^{(n)}$ with the scaling (5.12) are given by (7.31) and (7.33), respectively, we arrive at the result (7.42). □

7.3. Evaluation of $\mathcal{I}_2^{(2)}$. We will show that $\mathcal{I}_2^{(2)}$ tends to zero at long time limits with the scaling (5.12). As a preparation for the asymptotic analyses, we define the rescaled functions as

$$\mathcal{A}_{t,p}^c(\xi) = \mathrm{e}^{-f^c(w_*,\lambda_c t^{1/3}\xi)}\, A_p^c(\lambda_c t^{1/3}\xi), \tag{7.43}$$

$$\mathcal{B}_t^c(\xi) = \lambda_c t^{1/3}\, \mathrm{e}^{f^c(w_*,\lambda_c t^{1/3}\xi)}\, B^c(\lambda_c t^{1/3}\xi), \tag{7.44}$$

where $A_p^c(x)$ and $B^c(y)$ are defined in (7.10) and (7.11), respectively, with $f^c(z,x)$ given in (7.6). Using these functions, we rewrite $\mathcal{I}_2^{(2)}$ defined in (7.29) as

$$\begin{aligned}\mathcal{I}_2^{(2)} = &\sum_{k=1}^{m} \frac{(-1)^k}{k!} \sum_{x_1=1}^{\infty} \cdots \sum_{x_k=1}^{\infty} \sum_{r,\ell=1}^{k} (-1)^{r+\ell} \left(\frac{1}{\lambda_c t^{1/3}}\right)^k \\ &\times \det\left[\bar{\mathcal{K}}_t^c(x_i', x_j')\right]_{\substack{1\le i,j\le k\\ i\ne \ell,\, j\ne r}} \left[\sum_{p=1}^{n-1} \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \prod_{q=1}^{p} (z_q - 1) \mathcal{A}_{t,p}^c(x_\ell')\right] \mathcal{B}_t^c(x_r').\end{aligned} \tag{7.45}$$

Proposition 7.14. *For any fixed* $\rho \in (0,1)$ *and large enough* t*, the absolute value of the sum with respect to* p *in* (7.45) *satisfies the following bound*

$$\left| \sum_{p=1}^{n-1} \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \prod_{q=1}^{p} (z_q - 1) \mathcal{A}_{t,p}^c(\xi) \mathcal{B}_t^c(\zeta) \right| \le D_1 t^{-1/3}\, \mathrm{e}^{-(\xi+\zeta)}, \tag{7.46}$$

for $(\xi,\zeta) \in [0,\infty)^2$*.* D_1 *is some positive constant where* $L(\vec{z})$*,* $\mathcal{A}_{t,p}^c(\xi)$ *and* $\mathcal{B}_t^c(\zeta)$ *are defined in* (5.8)*,* (7.43) *and* (7.44)*, respectively.*

Proof. The proof consists of the four steps. As the first step, we perform the summation over $p \in [1, n-1]$ and derive the formula which is useful for the asymptotic analysis. In fact, it turns out that the integrals obtained as the result of the first step share similar integrands to the ones treated in Propositions 6.2, 6.3, 7.10 and 7.11. In the second step, we show the asymptotic analysis of the Gaussian part (7.54) for which we can apply a similar method to Proposition 7.10 and 7.11. In the third step, we investigate the asymptotic behaviours of the Airy parts (7.55) in the same fashion as Propositions 6.2 and 6.3. In the final step, combining the results of the previous steps, we arrive at the conclusion (7.46).

1. Rewrite of $\sum_{p=1}^{n-1}\oint_1 \mathrm{d}^{n-1}z L(\vec{z})\prod_{q=1}^{p}(z_q-1)\mathcal{A}^c_{t,p}(x)$. We will rewrite the sum over $p\in[1,n-1]$ into a suitable formula for the asymptotic analysis. Recall the function $L(\vec{z})$ defined in (5.8), the $\vec{z}$-integrals are written as

$$\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z})\prod_{q=1}^{p}(z_q-1)$$
$$=(-1)^{n-1}\frac{\mathrm{e}^{-\rho t/2}}{\rho'^{n-1}}\left(\frac{2(1-\rho)}{2-\rho}\right)^m\oint_1 \mathrm{d}^{n-1}z\frac{\mathrm{e}^{\Lambda_{n-1,0}t}\,\Delta_{n-1}(-z)\prod_{i=1}^{n-1}(1-\rho' z_i)}{\prod_{i=1}^{p}(z_i-1)^i\prod_{i=p+1}^{n-1}(z_i-1)^{i+1}[\frac{1}{2}(z_i+1)]^m}.$$

On the right hand side, all poles at $z_i=1$ can be easily evaluated from $i=1$ to $i=p$ in the same way as the proof of Lemma 5.1, i.e.,

$$\oint_1\prod_{j=1}^{n-1}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\frac{t}{2}\sum_{j=1}^{n-1}(z_j^{-1}-1)}\prod_{1\le i<j\le n-1}(z_j-z_i)\prod_{i=1}^{n-1}(1-\rho' z_i)}{\prod_{i=1}^{p}(z_i-1)^i\prod_{i=p+1}^{n-1}(z_i-1)^{i+1}[\frac{1}{2}(z_i+1)]^m}$$
$$=\rho^p\oint_1\prod_{j=p+1}^{n-1}\frac{\mathrm{d}z_j}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\frac{t}{2}\sum_{j=p+1}^{n-1}(z_j^{-1}-1)}\prod_{p+1\le i<j\le n-1}(z_j-z_i)\prod_{i=p+1}^{n-1}(1-\rho' z_i)}{\prod_{i=p+1}^{n-1}(z_i-1)^{i+1-p}[\frac{1}{2}(z_i+1)]^m}.$$

Changing variables $z_{j+p}\to z_j$ for any $j\in[1,n-p-1]$, we obtain

$$\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z})\prod_{q=1}^{p}(z_q-1)$$
$$=(-1)^{n-1}\frac{\mathrm{e}^{-\rho t/2}}{\rho'^{n-1}}\left(\frac{2(1-\rho)}{2-\rho}\right)^m\rho^p\oint_1 \mathrm{d}^{n-p-1}z\frac{\mathrm{e}^{\Lambda_{n-p-1,0}t}\,\Delta_{n-p-1}(-z)\prod_{i=1}^{n-p-1}(1-\rho' z_i)}{\prod_{i=1}^{n-p-1}(z_i-1)^{i+1}[\frac{1}{2}(z_i+1)]^m}.$$

Since the change of variables $z_j\to z_{\pi_j}$ for all $j\in[n-p-1]$ does not affect the value of the $\vec{z}$-integrals for all permutations $\pi\in S_{n-p-1}$, summing up $\vec{z}$-integrals in which $\vec{z}$-variables are changed in such a way over $\pi\in S_{n-p-1}$ is equivalent to multiplying $\vec{z}$-integrals by $(n-p-1)!$, i.e., we have

$$\oint_1 \mathrm{d}^{n-p-1}z\frac{\mathrm{e}^{\Lambda_{n-p-1,0}t}\prod_{1\le i<j\le n-p-1}(z_j-z_i)\prod_{i=1}^{n-p-1}(1-\rho' z_i)}{\prod_{i=1}^{n-p-1}(z_i-1)^{i+1}\left[\frac{1}{2}(z_i+1)\right]^m}$$
$$=\frac{1}{(n-p-1)!}\sum_{\pi\in S_{n-p-1}}\oint_1 \mathrm{d}^{n-p-1}z\frac{\mathrm{e}^{\Lambda_{n-p-1,0}t}\prod_{1\le i<j\le n-p-1}(z_{\pi_j}-z_{\pi_i})\prod_{i=1}^{n-p-1}(1-\rho' z_i)}{\prod_{i=1}^{n-p-1}(z_{\pi_i}-1)^{i+1}\left[\frac{1}{2}(z_i+1)\right]^m}.$$

Consider $\prod_{1\le i<j\le n-p-1}(z_{\pi_j}-z_{\pi_i})=\mathrm{sign}(\pi)\prod_{1\le i<j\le n-p-1}(z_j-z_i)$ and (3.6), the right hand side is rewritten as

$$\frac{1}{(n-p-1)!}\oint_1 \mathrm{d}^{n-p-1}z\frac{\mathrm{e}^{\Lambda_{n-p-1,0}t}\prod_{1\le i<j\le n-p-1}(z_j-z_i)\prod_{i=1}^{n-p-1}(1-\rho' z_i)}{\prod_{i=1}^{n-p-1}(z_i-1)^2\left[\frac{1}{2}(z_i+1)\right]^m}\sum_{\pi\in S_{n-p-1}}\frac{\mathrm{sign}(\pi)}{\prod_{i=1}^{n-p-1}(z_{\pi_i}-1)^{i-1}}$$
$$=\frac{1}{(n-p-1)!}\oint_1 \mathrm{d}^{n-p-1}z\frac{\mathrm{e}^{\Lambda_{n-p-1,0}t}\Delta_{n-p-1}(-z)\Delta_{n-p-1}(z)\prod_{i=1}^{n-p-1}(1-\rho' z_i)}{\prod_{i=1}^{n-p-1}(z_i-1)^{n-p}\left[\frac{1}{2}(z_i+1)\right]^m}.$$

From the above, it turns out that the $\vec{z}$-integrals are rewritten as the symmetrised formula:

$$\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \prod_{q=1}^{p} (z_q - 1)$$
$$= \left(\frac{-\rho}{\rho'}\right)^p (-1)^{n-p-1} \frac{\mathrm{e}^{-\rho t/2}}{\rho'^{n-p-1}} \left(\frac{2(1-\rho)}{2-\rho}\right)^m \frac{1}{(n-p-1)!}$$
$$\times \oint_1 \mathrm{d}^{n-p-1}z\, \frac{\mathrm{e}^{\Lambda_{n-p-1,0}t} \Delta_{n-p-1}(z)\Delta_{n-p-1}(-z) \prod_{i=1}^{n-p-1}(1-\rho' z_i)}{\prod_{i=1}^{n-p-1}(z_i-1)^{n-p}[\frac{1}{2}(z_i+1)]^m}.$$

The formula in the right hand side except for the factor $(-\rho/\rho')^p$ coincides with I_z, defined in (7.9), in which n is replaced with $n-p$. Therefore, the $\vec{z}$-integrals can be written as

$$\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \prod_{q=1}^{p}(z_q - 1) = -\left(\frac{-\rho}{\rho'}\right)^p \left(J^{(n-p)} - 1\right), \tag{7.47}$$

where $J^{(n)}$ is defined in (7.32). Following the arguments in Sect. 4, $J^{(n-p)}$ is thus written as a Fredholm determinant with the kernel

$$K(x,y) = \oint_1 \frac{\mathrm{d}w}{2\pi \mathrm{i}} \left(\frac{w-1}{w}\right)^p (1-\rho'/w)\, \mathrm{e}^{h(w,x)} \oint_{\rho'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{-h(z,y)} \frac{z^p}{z(w-z)(1-\rho'/z)(z-1)^p}$$
$$= \mathrm{e}^{-h(\rho',y)} \left(\frac{-\rho'}{\rho}\right)^p \oint_1 \frac{\mathrm{d}w}{2\pi \mathrm{i} w} \left(\frac{w-1}{w}\right)^p \mathrm{e}^{h(w,x)},$$

where $\mathrm{e}^{h(w,x)} := w^{-x} \left(\frac{w}{w-1}\right)^n \left(\frac{w+1}{w}\right)^m \mathrm{e}^{-wt/2}$. Since the operator with the above kernel is a rank one operator, the $k \geq 2$ terms of the series (1.4), which defines its Fredholm determinant, vanish. Therefore, we obtain

$$J^{(n-p)} = 1 - \sum_{x=1}^{\infty} K(x,x) = 1 - \left(\frac{\rho'}{-\rho}\right)^p \sum_{x=1}^{\infty} \int_1 \frac{\mathrm{d}w}{2\pi \mathrm{i} w} \left(\frac{w-1}{w}\right)^p \mathrm{e}^{h(w,x)-h(\rho',x)}. \tag{7.48}$$

Since the contour includes only the pole at $w = 1$, it is allowed to choose the contour on which $|w| > \rho'$ always holds. Therefore, we can perform the summation over $x \in \mathbb{N}$ in the right hand side of (7.48), and we obtain

$$\left(\frac{-\rho}{\rho'}\right)^p (J^{(n-p)} - 1) = \oint_1 \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{\rho'}{w(w-\rho')} \left(\frac{w-1}{w}\right)^p \mathrm{e}^{h(w,0)-h(\rho',0)}. \tag{7.49}$$

Note that the contour does not include the pole at $w = \rho'$. Recalling the definition of $A_p^c(x)$ given in (7.10) and combine (7.47) and (7.49), we have

$$\sum_{p=1}^{n-1} \oint_1 \mathrm{d}^{n-1}z\, L(\vec{z}) \prod_{q=1}^{p} (z_q-1) A_p^c(x)$$
$$= \sum_{p=1}^{n-1} \oint_1 \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\rho'}{w-\rho'} \left(\frac{w-1}{w}\right)^p \mathrm{e}^{h(w,0)-h(\rho',0)+g_J(w)} \oint_1 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{f^c(z,x)-f^c(w_*,x)+g_\phi(z)} \left(\frac{z}{1+z}\right)^p$$

with $g_J(w) := -\ln(w)$, $g_\phi(z) := \ln(z+\rho')/(z(z+1))$ and $f^c(z,x)$ defined in (7.6). In the right hand side, we can perform the summation over $p \in [1, n-1]$ as

$$\sum_{p=1}^{n-1} \left[\frac{z(w-1)}{w(z+1)}\right]^p = \frac{z(w-1)}{z+w} \left[1 - \left(\frac{z(w-1)}{w(z+1)}\right)^{n-1}\right], \tag{7.50}$$

and the factor $(w-1)^n$ in the second term removes the pole of the nth order at $w=1$ of the integrand $\mathrm{e}^{h(w,0)}$, i.e., the second term of (7.50) does not contribute. If we choose the contours such that $\mathrm{Re}\,(z+w)>0$ holds for any w and z on the contours, the factor $1/(z+w)$ can be written as $1/(z+w)=\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-k(z+w)}$. Therefore, we obtain

$$\sum_{p=1}^{n-1}\oint_1 \mathrm{d}^{n-1}z\,L(\vec{z})\prod_{q=1}^{p}(z_q-1)A_p^c(x)$$
$$=\oint_1\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{\rho'}{w-\rho'}\mathrm{e}^{h(w,0)-h(\rho',0)+g_J'(w)}\oint_1\frac{\mathrm{d}z}{2\pi\mathrm{i}}\mathrm{e}^{f^c(z,x)-f^c(w_*,x)+g_\phi'(z)}\int_0^\infty \mathrm{d}k\,\mathrm{e}^{-k(z+w)} \tag{7.51}$$

with $g_J'(w):=g_J(w)+\ln(w-1)$ and $g_\phi'(z):=g_\phi(z)+\ln(z)$.

In the next step, we will investigate the asymptotic behaviour in the long time limit with the scaling (5.12). To perform the asymptotic analyses, we scale $(x,y)=(\lambda_c t^{1/3}\xi,\lambda_c t^{1/3}\zeta)$ and $k=t^{1/3}\kappa$, and choose the contours of the integrals with respect to z and w in (7.51) as Γ' defined in (6.15a) and Θ' given in Fig. 9, respectively. Θ' can be expressed as

$$\Theta'=\left\{w\in\Theta\mid|w-\rho'|>\epsilon\right\}\cup\left\{w\in\mathbb{C}\mid w=\rho'+\epsilon\,\mathrm{e}^{-\mathrm{i}s},\,s\in[-\pi/2,\pi/2]\right\}$$

with $\epsilon\ll t^{-1/6}$ where Θ is defined in (F.1). It is easy to see that $\mathrm{Re}\,(z+w)>0$ holds for any $(z,w)\in\Gamma'\times\Theta'$. Rewriting (7.51) in terms of $\mathcal{A}_{t,p}^c(\xi)$ and putting the integral with respect to κ outside the ones with respect to z and w, we get

$$\sum_{p=1}^{n-1}\oint_1 \mathrm{d}^{n-1}z\,L(\vec{z})\prod_{q=1}^{p}(z_q-1)\mathcal{A}_{t,p}^c(\xi)$$
$$=t^{1/3}\int_0^\infty \mathrm{d}\kappa\,\mathrm{e}^{-3\rho' t^{1/3}\kappa/2}\oint_{\Theta'}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{\rho'}{w-\rho'}\mathrm{e}^{\bar{h}(w,t)-\bar{h}(\rho',t)-\kappa(w-\rho')t^{1/3}+g_J'(w)}$$
$$\times\oint_{\Gamma'}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)-\kappa(z-w_*)t^{1/3}+g_\phi'(z)}\left(\frac{w_*+c}{z+c}\right)^{\xi\lambda_c t^{1/3}}, \tag{7.52}$$

where $\bar{f}(z,t)$ is defined above (6.16) and $\bar{h}(w,t)$ is defined as

$$\bar{h}(w,t):=g_1(w)t+g_2(w,0)t^{1/2}+g_3(w)t^{1/3}$$

with $g_i(w)$ and $g_2(w,\xi)$ given in (7.40). Similarly, the rescaled function $\mathcal{B}_t^c(\zeta)$ can be written as

$$\mathcal{B}_t^c(\zeta)=\lambda_c t^{1/3}\oint_{\Sigma'}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\mathrm{e}^{-\bar{f}(w,t)+\bar{f}(w_*,t)+g_\psi(w)}\left(\frac{w+c}{w_*+c}\right)^{\zeta\lambda_c t^{1/3}} \tag{7.53}$$

with $g_\psi(w)=-\ln(w+\rho')(w+c)$ and Σ' defined in (6.15b). It follows from (7.52) and (7.53) that the function which is to be evaluated is written as

$$\sum_{p=1}^{n-1}\oint_1 \mathrm{d}^{n-1}z\,L(\vec{z})\prod_{q=1}^{p}(z_q-1)\mathcal{A}_{t,p}^c(\xi)\mathcal{B}_t^c(\zeta)=\int_0^\infty \mathrm{d}\kappa\,\mathrm{e}^{-3\rho' t^{1/3}\kappa/2}G_{\mathrm{g}}(\kappa,s_{\mathrm{g}})G_2(\kappa,s_2,\xi,\zeta),$$

where the functions G_{g} and G_2 are defined as

$$G_{\mathrm{g}}(\kappa, s_{\mathrm{g}}) := \oint_{\Theta'} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\rho'}{w-\rho'} \mathrm{e}^{\bar{h}(w,t)-\bar{h}(\rho',t)-\kappa(w-\rho')t^{1/3}+g'_J(w)}, \tag{7.54}$$

$$G_2(\kappa, s_2, \xi, \zeta) := t^{1/3} \oint_{\Gamma'} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)-\kappa(z-w_*)t^{1/3}+g'_\phi(z)} \left(\frac{w_*+c}{z+c}\right)^{\xi\lambda_c t^{1/3}} \\ \times \lambda_c t^{1/3} \oint_{\Sigma'} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \mathrm{e}^{-\bar{f}(w,t)+\bar{f}(w_*,t)+g_\psi(w)} \left(\frac{w+c}{w_*+c}\right)^{\zeta\lambda_c t^{1/3}}. \tag{7.55}$$

Note that the two integrals in $G_2(\kappa, s_2, \xi, \zeta)$ are independent but we denote it as a single function so as to compare it with $\mathcal{K}^c_t(\xi, \zeta)$ given by (6.7). We observe that the integral in $G_{\mathrm{g}}(\kappa, s_{\mathrm{g}})$ has a similar form to the kernel $\mathcal{K}_t(\xi, \zeta)$ given by (7.39), which converges to the Gaussian function. It is easy to see that the integrals in $G_2(\kappa, s_2, \xi, \zeta)$ share similar integrands to those with respect to z and w in (6.3) except for $1/(w-z)$.
2. Estimate of $G_{\mathrm{g}}(\kappa, s_{\mathrm{g}})$. By using a steepest descent analysis, in a similar way to that used in Propositions 7.10 and 7.11, we can show that

$$\left|G_{\mathrm{g}}(\kappa, s_{\mathrm{g}})\right| \le C_1 \tag{7.56}$$

holds for t large enough with some positive constant C_1 independent of κ. Since $\bar{h}(w,t)$ has a saddle point at $w=\rho'$ as shown in Appendix F, we divide the contour Θ' into $\Theta^\delta := \left\{w \in \Theta' \mid |w-\rho'| \le \delta\right\}$ and $\Theta' \backslash \Theta^\delta$ with $\delta = t^{-1/6}$.

First, we focus on the main contribution from Θ^δ. Since the Taylor expansion of $\bar{h}(w,t)$ around $w=\rho'$ is given by (F.2), we obtain

$$\int_{\Theta^\delta} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\rho'}{w-\rho'} \mathrm{e}^{\bar{h}(w,t)-\bar{h}(\rho',t)-\kappa(w-\rho')t^{1/3}+g'_J(w)} \\ = \mathrm{e}^{-\frac{1}{2}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)^2} \int_{\theta^{\delta t^{1/2}}} \frac{\mathrm{d}v}{2\pi\mathrm{i}} \left(\frac{-\rho}{v}\right) \mathrm{e}^{-\frac{1}{4}\left[v+\sqrt{2}\mathrm{i}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)\right]^2} + R_v, \tag{7.57}$$

where the remainder R_v is given by

$$R_v = \mathrm{e}^{-\frac{1}{2}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)^2} \int_{\theta^{\delta t^{1/2}}} \frac{\mathrm{d}v}{2\pi\mathrm{i}} \left(\frac{-\rho}{v}\right) \mathrm{e}^{-\frac{1}{4}\left[v+\sqrt{2}\mathrm{i}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)\right]^2} \\ \times \left(\mathrm{e}^{\mathcal{O}(v^3t^{-1/2}, v^2t^{-1/6}, vt^{-1/2})} - 1\right)$$

with $\theta^{\delta t^{1/2}} := \{v \in \mathbb{C} \mid \rho' + \mathrm{i}v\rho'/(\sqrt{2t}\lambda_2) \in \Theta^\delta\}$ and λ_2 defined in (7.37). In the same fashion as Appendix D.2, we can show

$$|R_v| \le c_1 t^{-1/6},$$

where c_1 is some positive constant independent of κ. Moreover, the first term of the right hand side of (7.57) is rewritten as

$$\begin{aligned}
&\mathrm{e}^{-\frac{1}{2}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)^2}\int_{\theta\delta t^{1/2}}\frac{\mathrm{d}v}{2\pi\mathrm{i}}\left(\frac{-\rho}{v}\right)\mathrm{e}^{-\frac{1}{4}\left[v+\sqrt{2}\mathrm{i}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)\right]^2}\\
&\quad=\mathrm{e}^{-\frac{1}{2}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)^2}\int_{-\sqrt{2}\lambda_2\delta t^{1/2}/\rho'}^{\sqrt{2}\lambda_2\delta t^{1/2}/\rho'}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{\rho}{u-\sqrt{2}\mathrm{i}\beta}\,\mathrm{e}^{-\frac{1}{4}\left[u+\sqrt{2}\mathrm{i}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2-\beta)\right]^2}+R_z
\end{aligned} \tag{7.58}$$

for some $\beta>0$, and R_z satisfies

$$|R_z|\le c_2\,\mathrm{e}^{-a_1\delta^2 t}$$

with some positive constant c_2 independent of κ and $a_1:=\lambda_2{}^2/(2\rho'^2)$. Incidentally, the equality (7.58) can be derived by considering the contour integration with the contour shown in Fig. 10, and R_z is contribution from paths C_+ and C_-.

Second, we focus on the contribution from $\Theta'\backslash\Theta^\delta$. Obviously, we have

$$\begin{aligned}
&\left|\int_{\Theta'\backslash\Theta^\delta}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{\rho'}{w-\rho'}\,\mathrm{e}^{\bar{h}(w,t)-\bar{h}(\rho',t)-\kappa(w-\rho')t^{1/3}+g'_J(w)}\right|\\
&\qquad\le\int_{\Theta'\backslash\Theta^\delta}\left|\frac{\mathrm{d}w}{2\pi\mathrm{i}}\right|\left|\frac{\rho'}{w-\rho'}\,\mathrm{e}^{\bar{h}(w,t)-\bar{h}(\rho',t)-\kappa(w-\rho')t^{1/3}+g'_J(w)}\right|.
\end{aligned} \tag{7.59}$$

In the integrand, $|\,\mathrm{e}^{-\kappa(w-\rho')t^{1/3}}\,|\le 1$ holds because $\mathrm{Re}\,(w-\rho')\ge 0$ holds for any $w\in\Theta'$, hence we can evaluate the exponent of the right hand side of (7.59) in the same way as Proposition 7.10. That is to say, $g_1(w)$ given in (7.40) takes the maximum value at $w=\rho'\pm\mathrm{i}\delta$ along the contour $\Theta'\backslash\Theta^\delta$, thus

$$\mathrm{Re}\,(g_1(w)-g_1(\rho'))=\mathrm{Re}(-a_1\delta^2+\mathcal{O}(\delta^3))\le -a_1\delta^2/2$$

holds for any $w\in\Theta'\backslash\Theta^\delta$. The other terms in the exponent behaves as $\mathcal{O}(t^{1/2},t^{1/3})$ because the functions $g_2(w,0,0)$ and $g_3(w)$ given in (7.40) are regular for any $w\in\Theta'\backslash\Theta^\delta$. Owing to $\delta=t^{-1/6}$, $a_1\delta^2 t$ is dominant in the exponent. Therefore, we obtain

$$\left|\int_{\Theta'\backslash\Theta^\delta}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{\rho'}{w-\rho'}\,\mathrm{e}^{\bar{h}(w,t)-\bar{h}(\rho',t)-\kappa(w-\rho')t^{1/3}+g'_J(w)}\right|\le c_3\,\mathrm{e}^{-a_1\delta^2 t/2},$$

where c_3 is some positive constant independent of κ. From the above, it turns out that

$$\begin{aligned}
&\oint_{\Theta'}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{\rho'}{w-\rho'}\,\mathrm{e}^{\bar{h}(w,t)-\bar{h}(\rho',t)-\kappa(w-\rho')t^{1/3}+g'_J(w)}\\
&\quad=\mathrm{e}^{-\frac{1}{2}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)^2}\int_{-\sqrt{2}\lambda_2\delta t^{1/2}/\rho'}^{\sqrt{2}\lambda_2\delta t^{1/2}/\rho'}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{\rho}{u-\sqrt{2}\mathrm{i}\beta}\,\mathrm{e}^{-\frac{1}{4}\left[u+\sqrt{2}\mathrm{i}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2-\beta)\right]^2}+\mathcal{O}(t^{-1/6},\,\mathrm{e}^{-a_1\delta^2 t/2})
\end{aligned}$$

holds and, in the long time limit with the scaling (5.12), it behaves as

$$\mathrm{e}^{-\frac{1}{2}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2)^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{\rho}{u-\sqrt{2}\mathrm{i}\beta}\,\mathrm{e}^{-\frac{1}{4}\left[u+\sqrt{2}\mathrm{i}(s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2-\beta)\right]^2}. \tag{7.60}$$

To show that (7.60) is bounded above by some constant independent of κ, we introduce some small constant $\epsilon>0$ and a variable $x:=s_{\mathrm{g}}+\rho'\kappa t^{-1/6}/\lambda_2$. When $-\epsilon\le x$, we can straightforwardly evaluate the upper bound of (7.60). Since $|u-\sqrt{2}\mathrm{i}\beta|\ge\sqrt{2}\beta$, the

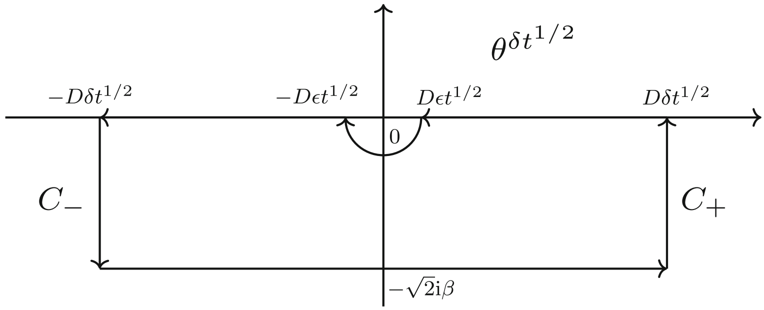

Fig. 10. The contour of the integral chosen to derive (7.58) where $D := \sqrt{2}\lambda_2/\rho'$. The path $\theta^{\delta t^{1/2}}$ consists of the lines along the real axis and semi-circle centered at the origin

absolute value of u-integral is bounded above by the product of $\mathrm{e}^{\frac{1}{2}(x-\beta)^2}$ and the Gaussian integral. Hence, due to $-\epsilon \le x$, (7.60) turns out to be bounded above by $C_\beta\, \mathrm{e}^{\epsilon\beta}$ where C_β depends only on β. On the other hand, when $-\epsilon > x$, we consider the complex integral of $\frac{1}{z-\sqrt{2}\mathrm{i}\beta}\, \mathrm{e}^{-\frac{1}{4}[z+\sqrt{2}\mathrm{i}(x-\beta)]^2}$ along a contour which encloses the pole at $\sqrt{2}\mathrm{i}\beta$. Choosing the contour as the rectangle whose vertices are $(-R, 0)$, $(R, 0)$, $(R, \sqrt{2}\mathrm{i}\{\beta - x\})$ and $(-R, \sqrt{2}\mathrm{i}\{\beta - x\})$ with $R := \sqrt{2}\lambda_2\delta t^{1/2}/\rho'$, we see that (7.60) is bounded above by some constant which depends only on ϵ by evaluating contributions from each edges and using the residue theorem. From the above, we see that (7.60) can be bounded by some constant independent of κ, and this leads to the result (7.56).

3. Estimate of $G_2(\kappa, s_2, \xi, \zeta)$. In a similar way to Propositions 6.2 and 6.3, we can show that

$$|G_2(\kappa, s_2, \xi, \zeta)| \le C_2\, \mathrm{e}^{-(\xi+\zeta)} \tag{7.61}$$

holds for t large enough and $(\xi, \zeta) \in [0, \infty)^2$ with some positive constant C_2 independent of κ. As with the proofs of Propositions 6.2 and 6.3, we separate the proof into two cases of $(\xi, \zeta) \in [0, L]^2$ and $(\xi, \zeta) \in [0, \infty)^2 \backslash [0, L]^2$ for L large enough but independent of t.

(i) Estimate for $(\xi, \zeta) \in [0, L]^2$. Performing a steepest descent analysis, we can prove

$$\lim_{t\to\infty} G_2(\kappa, s_2, \xi, \zeta) = c_4 \mathrm{Ai}(\xi + s_2 + \kappa/\lambda)\mathrm{Ai}(\zeta + s_2), \tag{7.62}$$

where $c_4 := 1/[\lambda(w_*+1)]$ for any $\kappa \in (0, \infty)$ and $(\xi, \zeta) \in [0, L]^2$, and this yields (7.61) because $|\mathrm{Ai}(x)| \le C_a\, \mathrm{e}^{-ax}$ holds for $x \in \mathbb{R}$ with some $a > 0$ and $C_a > 0$ depending on a.

Since the integral with respect to w in (7.55) does not depend on κ, by the same arguments as in Proposition 6.2, we can prove that it converges to the Airy function uniformly for $\zeta \in [0, L]$ and $\kappa \in (0, \infty)$, i.e.,

$$\lim_{t\to\infty} \lambda_c t^{1/3} \oint_{\Sigma'} \frac{\mathrm{d}w}{2\pi\mathrm{i}}\, \mathrm{e}^{-\bar{f}(w,t)+\bar{f}(w_*,t)+g_\psi(w)} \left(\frac{w+c}{w_*+c}\right)^{\zeta\lambda_c t^{1/3}} = c_5 \mathrm{Ai}(\zeta + s_2), \tag{7.63}$$

where $c_5 := 1/(w_* + \rho')$.

On the other hand, the integral with respect to z in (7.55) depends on κ, and hence we will evaluate it by slightly different calculations from Proposition 6.2. First, we focus on the main contribution from Γ'^{Δ} defined in (D.2) where $\Delta = t^{-1/9}$. Since the Taylor expansion of $\bar{f}(z,t)$ around $z = w_* = \rho'/2$ is given by (6.9), we obtain

$$\begin{aligned} t^{1/3} \int_{\Gamma'^{\Delta}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \,\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)-\kappa(z-w_*)t^{1/3}+g'_{\phi}(z)} \left(\frac{w_*+c}{z+c}\right)^{\xi\lambda_c t^{1/3}} \\ = c_6 \int_{\gamma^{\Delta t^{1/3}}} \frac{\mathrm{d}u}{2\pi \mathrm{i}} \,\mathrm{e}^{\frac{u^3}{3}-(s_2+\xi+\kappa/\lambda)u} + R_u, \end{aligned}$$

where $c_6 := (w_* + \rho')/[\lambda(w_* + 1)]$ and the remainder R_u is given by

$$R_u = c_6 \int_{\gamma^{\Delta t^{1/3}}} \frac{\mathrm{d}u}{2\pi \mathrm{i}} \,\mathrm{e}^{\frac{u^3}{3}-(s_2+\xi+\kappa/\lambda)u} \left(\mathrm{e}^{\mathcal{O}(u^4 t^{-1/3}, u^2 t^{-1/6}, u t^{-1/3})} - 1\right)$$

with $\gamma^{\Delta t^{1/3}} := \{u \in \mathbb{C} \mid w_* + u/(\lambda t^{1/3}) \in \Gamma'^{\Delta}\}$ and λ defined in (6.14). In the integrand, $\left|\mathrm{e}^{-(\xi+\kappa/\lambda)u}\right| \leq 1$ holds because ξ, κ, λ are positive and $\mathrm{Re}(u) \geq 0$ holds for any $u \in \gamma^{\Delta t^{1/3}}$. Therefore, in the same fashion as Appendix D.2, we can show

$$|R_u| \leq c_7 t^{-1/6},$$

where c_7 is some positive constant independent of κ. Second, we focus on the contribution from $\Gamma' \backslash \Gamma'^{\Delta}$. Since $|\,\mathrm{e}^{-\kappa(z-w_*)t^{1/3}}\,| \leq 1$ holds for any $\kappa \in (0, \infty)$ and $z \in \Gamma'$, we can evaluate an upper bound in the same way as Proposition 6.2, i.e.,

$$\left| t^{1/3} \int_{\Gamma' \backslash \Gamma'^{\Delta}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \,\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)-\kappa(z-w_*)t^{1/3}+g'_{\phi}(z)} \left(\frac{w_*+c}{z+c}\right)^{\xi\lambda_c t^{1/3}} \right| \leq c_8 \,\mathrm{e}^{-a_1 \Delta^3 t/2}$$

with some positive constant c_8 independent of κ and a_1 given in (6.10).

Consequently, the integral with respect to z in (7.55) converges to the Airy function uniformly for $\xi \in [0, L]$ and $\kappa \in (0, \infty)$, i.e.,

$$\lim_{t\to\infty} t^{1/3} \oint_{\Gamma'} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \,\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)-\kappa(z-w_*)t^{1/3}+g'_{\phi}(z)} \left(\frac{w_*+c}{z+c}\right)^{\xi\lambda_c t^{1/3}} = c_6 \mathrm{Ai}(\xi+s_2+\kappa/\lambda). \tag{7.64}$$

Combining (7.63) and (7.64), we arrive at (7.62).

(ii) Estimate for $(\xi, \zeta) \in [0, \infty)^2 \backslash [0, L]^2$. In the same way as Proposition 6.3, we can prove that (7.61) holds for large enough t and large enough L but independent of t. Obviously, we have

$$\begin{aligned} |G_2(\kappa, s_2, \xi, \zeta)| \leq t^{1/3} \oint_{\Gamma'} \left|\tfrac{\mathrm{d}z}{2\pi \mathrm{i}}\right| \left|\,\mathrm{e}^{\bar{f}(z,t)-\bar{f}(w_*,t)-\kappa(z-w_*)t^{1/3}+g'_{\phi}(z)}\right| \left|\tfrac{w_*+c}{z+c}\right|^{\xi\lambda_c t^{1/3}} \\ \times \lambda_c t^{1/3} \oint_{\Sigma'} \left|\tfrac{\mathrm{d}w}{2\pi \mathrm{i}}\right| \left|\,\mathrm{e}^{-\bar{f}(w,t)+\bar{f}(w_*,t)+g_{\psi}(w)}\right| \left|\tfrac{w+c}{w_*+c}\right|^{\zeta\lambda_c t^{1/3}}. \end{aligned} \tag{7.65}$$

In the integrand, $|\,\mathrm{e}^{-\kappa(z-w_*)t^{1/3}}\,| \leq 1$ holds for any $\kappa \in (0, \infty)$ and $z \in \Gamma'$. Hence, we can apply the same arguments as in Proposition 6.3 to the right hand side of (7.65), which results in (7.61).

4. Conclusion. Combining the above two results (7.56) and (7.61), we arrive at

$$\begin{aligned}\left|\sum_{p=1}^{n-1}\oint_1 \mathrm{d}^{n-1}z\, L(\vec{z})\prod_{q=1}^{p}(z_q-1)\mathcal{A}^c_{t,p}(\xi)\mathcal{B}^c_t(\zeta)\right| &\le \int_0^\infty \mathrm{d}\kappa\, \mathrm{e}^{-\frac{3}{2}\rho' t^{1/3}\kappa}\left|G_g(\kappa,s_g)\right|\left|G_2(\kappa,s_2,\xi,\zeta)\right| \\ &\le C_1C_2\,\mathrm{e}^{-(\xi+\zeta)}\int_0^\infty \mathrm{d}\kappa\,\mathrm{e}^{-\frac{3}{2}\rho'\kappa t^{1/3}} \\ &= D_1 t^{-1/3}\,\mathrm{e}^{-(\xi+\zeta)}\end{aligned}$$

with $D_1 := 2C_1C_2/(3\rho')$. □

Utilising Propositions 7.8 and 7.14, we get the following theorem.

Theorem 7.15. *With the scaling* (5.12)*, the long time limit of the term* $\mathcal{I}_2^{(2)}$ *defined in* (7.29) *is given by*

$$\lim_{t\to\infty}\mathcal{I}_2^{(2)} = 0. \tag{7.66}$$

Proof. It follows from Proposition 7.8 and the Hadamard's inequality that the determinant part of $\mathcal{I}_2^{(2)}$, given in (7.45), is bounded above as

$$\left|\det\left[\bar{\mathcal{K}}^c_t(x'_i,x'_j)\right]_{\substack{1\le i,j\le k\\ i\ne l,j\ne r}}\right| \le (k-1)^{(k-1)/2}\prod_{\substack{1\le i\le k\\ i\ne l,r}}\mathrm{e}^{-x'_i}. \tag{7.67}$$

Combining Proposition 7.14 and (7.67), the absolute value of $\mathcal{I}_2^{(2)}$ turns out to be bounded above as

$$\left|\mathcal{I}_2^{(2)}\right| \le D_1 t^{-1/3}\sum_{k=1}^{m} k\frac{(k-1)^{(k-1)/2}}{(k-1)!}\prod_{i=1}^{k}\left\{\frac{1}{\lambda_c t^{1/3}}\sum_{x_i=1}^{\infty}\mathrm{e}^{-x_i/(\lambda_c t^{1/3})}\right\}. \tag{7.68}$$

From $1+x < \mathrm{e}^x$ for $x>0$ and $\sum_{k=1}^{\infty} k(k-1)^{(k-1)/2}/(k-1)! < \infty$, we obtain

$$\frac{1}{\lambda_c t^{1/3}}\sum_{x_i=1}^{\infty}\mathrm{e}^{-x_i/(\lambda_c t^{1/3})} = \frac{1}{\lambda_c t^{1/3}}\frac{1}{\mathrm{e}^{1/(\lambda_c t^{1/3})}-1} < 1,$$

and this inequality guarantees that the right hand side of (7.68) converges to zero in the long time limit with the scaling (5.12). Therefore, we have (7.66). □

Eventually, it follows from Theorems 7.13 and 7.15 that, with the scaling (5.12), $\mathcal{I}_2$ satisfies

$$\lim_{t\to\infty}\mathcal{I}_2 = \lim_{t\to\infty}\left(\mathcal{I}_2^{(1)} - \mathcal{I}_2^{(2)}\right) = \left[1-F_G(s_g)\right]F_2(s_2). \tag{7.69}$$

Obviously, the second equality (7.69) itself coincides with the second claim of Theorem 5.2. As mentioned below Theorem 5.2 in Sect. 5, we obtain the final result as Corollary 5.3.

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A. Bethe Wave Function

We consider the AHR model with n plus and m minus particles. Define a set of coordinates by $\mathbb{W}^k := \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{Z}^k : x_1 < x_2 < \cdots < x_k\}$, and let $\vec{x} = (x_1, \dots, x_n) \in \mathbb{W}^n$ and $\vec{y} = (y_1, \dots, y_m) \in \mathbb{W}^m$ be the positions of plus and minus particles, respectively. To each set of coordinates we associate a unit basis vector $|\vec{x}; \vec{y}\rangle$, and construct the state space as $S = \mathrm{Span}\{|\vec{x}; \vec{y}\rangle\}$. Let

$$|P(t)\rangle := \sum_{\vec{x},\vec{y}} P(\vec{x}, \vec{y}; t)|\vec{x}; \vec{y}\rangle$$

be the vector of probabilities to observe a configuration $|\vec{x}, \vec{y}\rangle$ of the AHR model at time t with plus particles at positions $\vec{x}$ and minus particles at positions $\vec{y}$. The vector $|P(t)\rangle$ satisfies the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = M|P(t)\rangle, \tag{A.1}$$

where M is the transition matrix encoding the jumping rates of the AHR model. The $\big((\vec{x}; \vec{y}), (\vec{x}'; \vec{y}')\big)$ entry of M is the transition rate from state $(\vec{x}'; \vec{y}')$ to $(\vec{x}; \vec{y})$, given by

$$M\big((\vec{x}; \vec{y}), (\vec{x}'; \vec{y}')\big) = \begin{cases} \beta, & \text{if } (\vec{x}'; \vec{y}') = (\vec{x}_i^-; \vec{y}), \forall i \in [1, n] \\ \alpha, & \text{if } (\vec{x}'; \vec{y}') = (\vec{x}; \vec{y}_j^+), \forall j \in [1, m] \\ -\beta n - \alpha m & \text{if } (\vec{x}'; \vec{y}') = (\vec{x}; \vec{y}) \\ 0, & \text{otherwise,} \end{cases} \tag{A.2}$$

where $\vec{x}_i^{\pm} := (x_1, \dots, x_i \pm 1, \dots, x_n)$. (A.1) is equivalent to (2.2).

To derive an expression for $P(\vec{x}, \vec{y}; t)$, or more precisely the transition probability (2.7) considered in Sect. 2, we first look at the eigenvalue problem of the master equation with eigenvector

$$|\psi\rangle = \sum_{\vec{x},\vec{y}} \psi(\vec{x}; \vec{y})|\vec{x}; \vec{y}\rangle, \qquad M|\psi\rangle = \Lambda_{n,m}|\psi\rangle,$$

where $\Lambda_{n,m}$ is the eigenvalue.

We will follow Bethe's method in [10] to obtain the Bethe wave function $\psi(\vec{x}; \vec{y})$ for the AHR model. First we consider the case of single species particles only, before considering the two species case. Then for each case we start with two particles and then generalise to the many particles case.

Such method of Bethe ansatz is first introduced in [10], and applied to many particles systems in [65,99,100]. The Bethe ansatz was especially used to solve the master equation of ASEP and TASEP in [43,87]. A more complicated formula would follow from the standard nested Bethe ansatz [25].

A.1. Single species. First we consider the case without minus particles and two plus particles, i.e., $n = 2$, $m = 0$. The corresponding eigenvalue equation for $x_1 + 1 < x_2$ is

$$\Lambda_{2,0}\psi(x_1, x_2) = \beta\psi(x_1 - 1, x_2) + \beta\psi(x_1, x_2 - 1) - 2\beta\psi(x_1, x_2), \tag{A.3}$$

while for $x_1 = x = x_2 - 1$ it is

$$\Lambda_{2,0}\psi(x, x + 1) = \beta\psi(x - 1, x + 1) - \beta\psi(x, x + 1). \tag{A.4}$$

These two equations can be solved by the trial wave function

$$\psi(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_1^{x_2} z_2^{x_1},$$

for which (A.3) results in the eigenvalue expression $\Lambda_{2,0} = \beta(z_1^{-1} + z_2^{-1} - 2)$, while (A.4) gives the scattering relation $A_{12}/A_{21} = -(1 - z_1)/(1 - z_2)$.

The number of equations increases as the number n of plus particles increases. To solve the resulting system of equations it is convenient to replace (A.4) by

$$\psi(x, x) = \psi(x, x + 1), \tag{A.5}$$

because by imposing (A.5) on (A.3) for $x_1 = x = x_2 - 1$, equation (A.4) is automatically satisfied. In other words, (A.3) along with the boundary condition (A.5) gives the same solution as that of the eigenvalue problem. This boundary condition (A.5) is the same as in [87], since the AHR model reduced to TASEP when there are only one species particles.

It can thus be seen that the eigenvalue problem corresponding to $m = 0$ and general n is equivalent to

$$\Lambda_{n,0}\psi(\vec{x}) = \beta\sum_{i=1}^{n}\psi(\vec{x}_i^-) - n\beta\psi(\vec{x}),$$
$$\psi(x_1, \ldots, x_i, x_{i+1} = x_i, \ldots, x_n) = \psi(x_1, \ldots, x_i, x_{i+1} = x_i + 1, \ldots, x_n),\ i \in [1, n - 1].$$

These equations are solved by the wave function

$$\psi(\vec{x}) = \sum_{\pi\in S_n} A_\pi \prod_{i=1}^{n} z_{\pi_i}^{x_i},$$

with eigenvalue $\Lambda_{n,0} = \beta\sum_{i=1}^{n}(z_i^{-1} - 1)$, and with scattering relation

$$\frac{A_\pi}{A_{s_i\pi}} = -\frac{1 - z_{\pi_i}}{1 - z_{\pi_{i+1}}},$$

where s_i $(i = 1, \dots, n-1)$ are simple transposition, i.e., the generators of the symmetric group S_n. This is the same ratio of amplitudes obtained in [87]. This ratio is satisfied if A_π is of the form

$$A_\pi = \mathrm{sign}(\pi) \prod_{i=1}^{n} \left(\frac{1}{1 - z_{\pi_i}} \right)^i .$$

The derivation of $\psi(\vec{y})$, when $n = 0$ and m general, is very similar to the case when $m = 0$ and n general. The corresponding eigenvalue problem is

$$\begin{aligned} \Lambda_{0,m} \psi(\vec{y}) =& \alpha \sum_{j=1}^{m} \psi(\vec{y}_j^{+}) - m\alpha\psi(\vec{y}), \\ \psi(y_1, \dots, y_{i-1} = y_i, y_i, \dots, y_m) =& \psi(y_1, \dots, y_{i-1} = y_i - 1, y_i, \dots, y_m), \quad i \in [2, m], \end{aligned}$$

and we will not repeat analogous details of derivation here. Naively this observation would suggest that a wave function for two species particles is of the form

$$\sum_{\pi \in S_n} \mathrm{sign}(\pi) \prod_{i=1}^{n} \left(\frac{1}{1 - z_{\pi_i}} \right)^i z_{\pi_i}^{x_i} \sum_{\sigma \in S_m} \mathrm{sign}(\sigma) \prod_{j=1}^{m} \left(\frac{1}{1 - w_{\sigma_j}} \right)^{-j} w_{\sigma_j}^{-y_j}, \tag{A.6}$$

with eigenvalue $\Lambda_{n,m} = \beta \sum_{j=1}^{n} (z_j^{-1} - 1) + \alpha \sum_{k=1}^{m} (w_k^{-1} - 1)$. Such a trial wave function would only work when there is no interaction between the plus and the minus particles. In the following, we shall consider a modification of this wave function so that the interaction between different types of particles can be taken into account.

As we have seen, the Bethe wave function of the AHR model for the single species cases is identical to the one of the TASEP [87]. The important detail of the AHR model's wave function lies in the interaction between different types of particles.

A.2. Two species case. Now consider the case that two types of particles are present. We start with the simplest case with one plus and one minus particles. When $x \neq y \pm 1$, the corresponding eigenvalue equation reads as

$$\Lambda_{1,1} \psi(x; y) = \alpha\psi(x; y + 1) + \beta\psi(x - 1; y) - (\beta + \alpha)\psi(x; y), \tag{A.7}$$

while for $x = y + 1$ and $y = x + 1$,

$$\Lambda_{1,1} \psi(y + 1; y) = \psi(y; y + 1) - (\beta + \alpha)\psi(y + 1; y), \tag{A.8}$$

$$\Lambda_{1,1} \psi(x; x + 1) = \beta\psi(x - 1; x + 1) + \alpha\psi(x; x + 2) - \psi(x; x + 1). \tag{A.9}$$

Expression (A.7) is satisfied by the wave function (A.6) and the eigenvalue $\Lambda_{1,1} = \beta(z^{-1} - 1) + \alpha(w^{-1} - 1)$. In the same way, (A.8) and (A.9) are satisfied by imposing appropriate boundary conditions. Comparing (A.7) and (A.8) gives the first condition,

$$\psi(y; y + 1) = \alpha\psi(y + 1; y + 1) + \beta\psi(y; y).$$

This indicates a piece-wise wave function

$$\psi(x; y) = \begin{cases} C_{+-} z^x w^{-y} & x < y \\ C_{-+} z^x w^{-y} & x \geq y \end{cases},$$

so that (A.8) is satisfied by setting the amplitude ratio to $C_{+-}/C_{-+} = \alpha z + \beta w$. Fortunately, when $\beta + \alpha = 1$ equation (A.9) agrees with (A.7) without any extra condition. Incidentally, the condition $\beta + \alpha = 1$ leads to a factorised stationary state [82]. The reasoning above suggests to modify (A.6) to

$$\psi(\vec{x};\vec{y}) = \sum_{\pi\in S_n} \mathrm{sign}\,(\pi) \prod_{i=1}^{n} \left(\frac{1}{1-z_{\pi_i}}\right)^{i} z_{\pi_i}^{x_i} \sum_{\sigma\in S_m} \mathrm{sign}\,(\sigma) \prod_{j=1}^{m} \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} w_{\sigma_j}^{-y_j} C_{\vec{x},\vec{y},\pi,\sigma}, \tag{A.10}$$

where $C_{\vec{x},\vec{y},\pi,\sigma}$ depends on the relative position of $\vec{x}$, $\vec{y}$. For general n, m, we only need to impose the following boundary conditions on (A.10),

$$\begin{aligned}\psi(\vec{x}; y_1,\dots,y_j = x_i+1,\dots,y_m) &= \beta\psi(\vec{x}; y_1,\dots,y_j = x_i,\dots,y_m)\\ &+\alpha\psi(x_1,\dots,x_{i-1},x_i+1,x_{i+1},\dots,x_n; y_1,\dots,y_j = x_i+1,\dots,y_m).\end{aligned} \tag{A.11}$$

As in the single species case, there are no extra boundary condition for sectors with more than two particles. Substituting (A.10) into (A.11) gives the ratio

$$\frac{C_{y_j>x_i}}{C_{y_j\le x_i}} = \alpha z_{\pi_i} + \beta w_{\sigma_j}.$$

This suggests a form of the coefficient $C_{\vec{x},\vec{y},\pi,\sigma}$ as

$$C_{\vec{x},\vec{y},\pi,\sigma} = \prod_{j=1}^{m}\prod_{k=1}^{r_j} \frac{1}{\alpha z_{\pi_{n-k+1}} + \beta w_{\sigma_j}},$$

where r_j is the number of plus particles to the right of the j^{th} minus particle, i.e., $r_j = \#\{x_i \in \vec{x} \mid x_i \ge y_j\}$.

In conclusion, the general Bethe wave function is given by

$$\psi(\vec{x};\vec{y}) = \sum_{\pi\in S_n} \mathrm{sign}\,(\pi) \prod_{i=1}^{n} \left(\frac{1}{1-z_{\pi_i}}\right)^{i} z_{\pi_i}^{x_i} \sum_{\sigma\in S_m} \mathrm{sign}\,(\sigma) \prod_{j=1}^{m} \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} w_{\sigma_j}^{-y_j} \prod_{k=1}^{r_j} \frac{1}{\alpha z_{\pi_{n-k+1}} + \beta w_{\sigma_j}}, \tag{A.12}$$

with eigenvalue

$$\Lambda_{n,m} = \beta\sum_{j=1}^{n}(z_j^{-1}-1) + \alpha\sum_{k=1}^{m}(w_k^{-1}-1). \tag{A.13}$$

This result leads to an integral formula for the transition probability, as stated in Sect. 2.

B. Boundary Conditions for the Transition Probability

In this appendix we provide details of the proof that (2.7) satisfies the boundary conditions (2.3)–(2.5). Throughout the entire proof, we shall call the factor $\prod_{k=1}^{m}\prod_{j=1}^{r_k}(\alpha z_{\pi_{n-j+1}} + \beta w_{\sigma_k})^{-1}$ the scattering factor because it corresponds to scatterings of plus and minus particles.

B.1. Proof of boundary condition (2.3). On both sides of (2.3), i.e., when $x_{i+1} = x_i$ on the left hand side and $x_{i+1} = x_i + 1$ on the right hand side, the scattering factor $\prod_{k=1}^{m} \prod_{j=1}^{r_k} (\alpha z_{\pi_{n-j+1}} + \beta w_{\sigma_k})^{-1}$ remains unchanged within the physical regions Ω^{n+m}. Moreover, the scattering factor is symmetric in $z_{\pi_i}, z_{\pi_{i+1}}$ since there is no minus particle between the i^{th} and the $i+1^{\text{st}}$ plus particles.

From (2.7) we observe that x_i and x_{i+1} only appear as exponents of z_{π_i} and $z_{\pi_{i+1}}$, and as there is no change in the scattering factor we only need to compare the z_{π_i} and $z_{\pi_{i+1}}$ factors

$$\left(\frac{1}{1-z_{\pi_i}}\right)^{i} \left(\frac{1}{1-z_{\pi_{i+1}}}\right)^{i+1} z_{\pi_i}^{x_i} z_{\pi_{i+1}}^{x_{i+1}} \tag{B.1}$$

in the integrand for both sides of (2.3). For the left hand side of (2.3), i.e., when $x_{i+1} = x_i$, the factor (B.1) reads as

$$LHS = \left(\frac{1}{1-z_{\pi_i}}\right)^{i} \left(\frac{1}{1-z_{\pi_{i+1}}}\right)^{i+1} (z_{\pi_i} z_{\pi_{i+1}})^{x_i}.$$

When $x_{i+1} = x_i + 1$, (B.1) becomes

$$RHS = \left(\frac{1}{1-z_{\pi_i}}\right)^{i} \left(\frac{1}{1-z_{\pi_{i+1}}}\right)^{i+1} (z_{\pi_i} z_{\pi_{i+1}})^{x_i} z_{\pi_{i+1}}.$$

It follows that

$$\begin{aligned} LHS - RHS &= \left[\frac{1}{(1-z_{\pi_i})(1-z_{\pi_{i+1}})}\right]^{i} (z_{\pi_i} z_{\pi_{i+1}})^{x_i} \left(\frac{1}{1-z_{\pi_{i+1}}} - \frac{z_{\pi_{i+1}}}{1-z_{\pi_{i+1}}}\right) \\ &= \left[\frac{1}{(1-z_{\pi_i})(1-z_{\pi_{i+1}})}\right]^{i} (z_{\pi_i} z_{\pi_{i+1}})^{x_i}. \end{aligned}$$

This factor is symmetric in $z_{\pi_i}, z_{\pi_{i+1}}$. Moreover, the scattering factor is also symmetric in $z_{\pi_i}, z_{\pi_{i+1}}$, and hence summing over $\pi \in S_n$ in (2.7) gives a zero integrand due to the factor $\operatorname{sign}(\pi)$.

B.2. Proof of boundary condition (2.4). The proof of this boundary condition is very similar to the one above. First we notice that when $y_{i-1} = y_i$ and $y_{i-1} = y_i - 1$, the scattering factor is unchanged and symmetric in $w_{\sigma_{i-1}}, w_{\sigma_i}$. Hence as before, we only need to consider

$$\left(\frac{1}{1-w_{\sigma_{i-1}}}\right)^{-i+1} \left(\frac{1}{1-w_{\sigma_i}}\right)^{-i} w_{\sigma_{i-1}}^{-y_{i-1}} w_{\sigma_i}^{-y_i}. \tag{B.2}$$

Then difference of this factor between the left hand side and right hand side of (2.4) is given by

$$\begin{aligned} LHS - RHS &= \left[\frac{1}{(1-w_{\sigma_{i-1}})(1-w_{\sigma_i})}\right]^{-i} (w_{\sigma_{i-1}} w_{\sigma_i})^{-y_i} \left(\frac{1}{1-w_{\sigma_{i-1}}} - \frac{w_{\sigma_{i-1}}}{1-w_{\sigma_{i-1}}}\right) \\ &= \left[\frac{1}{(1-w_{\sigma_{i-1}})(1-w_{\sigma_i})}\right]^{-i} (w_{\sigma_{i-1}} w_{\sigma_i})^{-y_i}, \end{aligned}$$

which is symmetric in $w_{\sigma_{i-1}}, w_{\sigma_i}$. Thus summing over $\sigma \in S_m$ in (2.7) gives a zero integrand, as required.

B.3. Proof of boundary condition (2.5). We consider two cases when $y_j = x_i$ and $y_j = x_i + 1$. From the integrand of (2.7), in these two cases, one only needs to check the factor

$$\left(\frac{1}{1-z_{\pi_i}}\right)^i \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} z_{\pi_i}^{x_i} w_{\sigma_j}^{-y_j} \prod_{k=1}^{r_j} \frac{1}{\alpha z_{\pi_{n-k+1}} + \beta w_{\sigma_j}},$$

for both sides of (2.5). For the left hand side of (2.5), $y_j = x_i + 1$, which indicates that the $i+1^{\text{st}}$ plus particle is sitting at the right hand side of y_j. Therefore $r_j = n - i$ and then

$$LHS = \left(\frac{1}{1-z_{\pi_i}}\right)^i \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} \left(\frac{z_{\pi_i}}{w_{\sigma_j}}\right)^{x_i} w_{\sigma_j}^{-1} \prod_{k=1}^{n-i} \frac{1}{\alpha z_{\pi_{i+k}} + \beta w_{\sigma_j}}.$$

Now let us consider the right hand side of (2.5) where $y_j = x_i$. In this case $r_j = n-i+1$, since the i^{th} plus particle is sitting at y_j. It follows that

$$\begin{aligned}
RHS =& \beta \left(\frac{1}{1-z_{\pi_i}}\right)^i \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} \left(\frac{z_{\pi_i}}{w_{\sigma_j}}\right)^{x_i} \prod_{k=0}^{n-i} \frac{1}{\alpha z_{\pi_{i+k}} + \beta w_{\sigma_j}} \\
&+ \alpha \left(\frac{1}{1-z_{\pi_i}}\right)^i \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} \left(\frac{z_{\pi_i}}{w_{\sigma_j}}\right)^{x_i+1} \prod_{k=0}^{n-i} \frac{1}{\alpha z_{\pi_{i+k}} + \beta w_{\sigma_j}} \\
=& \left(\frac{1}{1-z_{\pi_i}}\right)^i \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} \left(\frac{z_{\pi_i}}{w_{\sigma_j}}\right)^{x_i} \prod_{k=0}^{n-i} \frac{1}{\alpha z_{\pi_{i+k}} + \beta w_{\sigma_j}} \left(\beta + \alpha \frac{z_{\pi_i}}{w_{\sigma_j}}\right) \\
=& \left(\frac{1}{1-z_{\pi_i}}\right)^i \left(\frac{1}{1-w_{\sigma_j}}\right)^{-j} \left(\frac{z_{\pi_i}}{w_{\sigma_j}}\right)^{x_i} w_{\sigma_j}^{-1} \prod_{k=1}^{n-i} \frac{1}{\alpha z_{\pi_{i+k}} + \beta w_{\sigma_j}},
\end{aligned}$$

which is exactly the same as LHS.

C. Proof of Symmetrisation Identity Lemma 3.1

In this appendix we provide detail of the proof of Lemma 3.1. In Appendix C.1, we show the direct proof by induction. On the other hand, the identity Lemma 3.1 can also be derived by specialising equation (9) of [95], and we show detail of the proof in Appendix C.2.

C.1. Direct proof by induction. We first recall the statement of Lemma 3.1,

$$\sum_{\pi \in S_n} \operatorname{sign}(\pi) \prod_{i=1}^{n} \left(\frac{z_{\pi_i} - 1}{z_{\pi_i}}\right)^i \frac{1}{(1-(1-\rho)\prod_{j=1}^{i} z_{\pi_j})} = \frac{\prod_{1 \le i < j \le n}(z_j - z_i) \prod_{i=1}^{n}(z_i - 1)}{\prod_{i=1}^{n} z_i^n (1-(1-\rho)z_i)}. \tag{C.1}$$

We prove this identity by mathematical induction in n. One can easily see that (C.1) holds for $n = 1$. We assume that it holds for $n - 1$, and prove it for n. Let us denote the left hand side of the identity by $f_n(z_1, \dots, z_n)$. To make use of the induction assumption, we need to find the relation between f_n and f_{n-1}. The sum over S_n can be split into a sum over $k \in [1, n]$ such that $\pi_n = k$ and then sum over S_{n-1}. We observe that $\operatorname{sign}(\pi) = (-1)^{n-k} \operatorname{sign}(\sigma)$ where σ is the permutation π restricted on $\{1, 2, \dots, k-1, k+1, \dots, n\}$, and $(-1)^{n-k}$ is the signature of the permutation $(1, \dots, k-1, k+1, \dots, n, k)$. Therefore,

$$f_n(z_1, \dots, z_n) = \sum_{k=1}^{n} (-1)^{n-k} \left(\frac{z_k - 1}{z_k} \right)^n \frac{1}{1 - (1-\rho)\prod_{i=1}^n z_i} f_{n-1}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n).$$

Substituting our induction assumption for $f_{n-1}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n)$ and after some rearrangements, we obtain

$$f_n(z_1, \dots, z_n) = \frac{\prod_{1 \le i < j \le n}(z_j - z_i)}{1 - (1-\rho)\prod_{i=1}^n z_i} \prod_{i=1}^{n} \frac{z_i - 1}{z_i^{n-1}} \sum_{k=1}^{n} \frac{(z_k - 1)^{n-1}}{z_k} \prod_{i=1, i \ne k}^{n} \frac{1}{(z_k - z_i)(1 - (1-\rho)z_i)}.$$

In order to show the sum over k gives expected result, we consider the function defined by

$$F(z) = \frac{(z-1)^{n-1}(1 - (1-\rho)z)}{z \prod_{i=1}^n (z - z_i)(1 - (1-\rho)z_i)}.$$

By the residue theorem,

$$\begin{aligned}
\sum_{k=1}^{n} \operatorname{Res}_{z=z_k} F(z) &= \sum_{k=1}^{n} \frac{(z_k - 1)^{n-1}}{z_k} \prod_{i=1, i \ne k}^{n} \frac{1}{(z_k - z_i)(1 - (1-\rho)z_i)} \\
&= -\operatorname{Res}_{z=0} F(z) - \operatorname{Res}_{z=\infty} F(z) \\
&= \frac{1}{\prod_{i=1}^n z_i (1 - (1-\rho)z_i)} - \frac{(1-\rho)}{\prod_{i=1}^n (1 - (1-\rho)z_i)},
\end{aligned}$$

which gives the required result to complete the proof.

C.2. Alternative proof. We start from the identity, which is equation (9) of [95]:

$$\begin{aligned}
&\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{1 \le i < j \le n} \frac{1}{p + q\xi_{\sigma_i}\xi_{\sigma_j} - \xi_{\sigma_i}} \prod_{i=1}^{n} \frac{1}{(\prod_{j=i}^n \xi_{\sigma_j}) - (1-\rho) - \tau^{n-i+1}\rho} \\
&\quad = q^{n(n-1)/2} \frac{\prod_{1 \le i < j \le n}(\xi_i - \xi_j)}{\prod_{i=1}^n (\xi_i - (1-\rho) - \tau\rho) \prod_{1 \le i \ne j \le n}(p + q\xi_i\xi_j - \xi_i)},
\end{aligned}$$

where $0 < \rho \le 1$, $p + q = 1$ and $\tau = p/q$. Setting $p = 0$, $q = 1$, $\tau = 0$ and simplifying the product yield

$$\begin{aligned}
&\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} \left(\frac{1}{\xi_{\sigma_i}} \right)^{n-i} \left(\frac{1}{\xi_{\sigma_i} - 1} \right)^{i-1} \frac{1}{(\prod_{j=i}^n \xi_{\sigma_j}) - (1-\rho)} \\
&\quad = q^{n(n-1)/2} \frac{\prod_{1 \le i < j \le n}(\xi_i - \xi_j)}{\prod_{i=1}^n (\xi_i - (1-\rho))\xi_i^{n-1}(\xi_i - 1)^{n-1}}.
\end{aligned}$$

Changing variables $\xi_i = z_i^{-1}$ for all $i \in [1, n]$, this equation is written as

$$\sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} z_{\sigma_i}^{n-i} \left(\frac{z_{\sigma_i}}{1 - z_{\sigma_i}} \right)^{i-1} \frac{\prod_{j=i}^{n} z_{\sigma_j}}{1 - (1-\rho) \prod_{j=i}^{n} z_{\sigma_j}} = \frac{\prod_{i=1}^{n} z_i^n \prod_{1 \le i < j \le n} (z_j - z_i)}{\prod_{i=1}^{n} (1 - z_i)^{n-1} (1 - (1-\rho) z_i)}.$$

Replacing $\sigma = (\sigma_1, \dots, \sigma_n)$ with its reversal $\pi = (\sigma_n, \dots, \sigma_1)$, the left hand side is written as

$$\sum_{\pi \in S_n} (-1)^{n(n-1)/2} \mathrm{sign}(\pi) \prod_{i=1}^{n} \frac{z_{\pi_i}^{n-1} (1 - z_{\pi_i})^{i-n} \prod_{j=1}^{i} z_{\pi_j}}{1 - (1-\rho) \prod_{j=1}^{i} z_{\pi_j}},$$

and canceling like terms on both sides yields

$$\sum_{\pi \in S_n} (-1)^{n(n-1)/2} \mathrm{sign}(\pi) \prod_{i=1}^{n} \frac{(1 - z_{\pi_i})^i \prod_{j=1}^{i} z_{\pi_j}}{1 - (1-\rho) \prod_{j=1}^{i} z_{\pi_j}} = \frac{\prod_{i=1}^{n} z_i (1 - z_i) \prod_{1 \le i < j \le n} (z_j - z_i)}{\prod_{i=1}^{n} (1 - (1-\rho) z_i)}.$$

Considering the equality

$$\prod_{i=1}^{n} \prod_{j=1}^{i} z_{\pi_j} = \prod_{i=1}^{n} z_{\pi_i}^{n+1-i} = \prod_{i=1}^{n} z_i^{n+1} \prod_{i=1}^{n} z_{\pi_i}^{-i},$$

and dividing $\prod_{i=1}^{n} z_i^{n+1}$ into the both sides, we obtain

$$\sum_{\pi \in S_n} (-1)^{n(n-1)/2} \mathrm{sign}(\pi) \prod_{i=1}^{n} \left(\frac{1 - z_{\pi_i}}{z_{\pi_i}} \right)^i \frac{1}{1 - (1-\rho) \prod_{j=1}^{i} z_{\pi_j}}$$
$$= \frac{\prod_{i=1}^{n} (1 - z_i) \prod_{1 \le i < j \le n} (z_j - z_i)}{\prod_{i=1}^{n} z_i^n (1 - (1-\rho) z_i)},$$

and then, accounting for $(-1)^{n(n-1)/2}(-1)^{n(n+1)/2} = (-1)^{n^2} = (-1)^n$, we arrive at equation (C.1).

D. Asymptotic of the Rescaled Kernel

D.1. Proof of Lemma 6.1. The facts that w_* is a double root and w_2 is a single root can be checked easily. To prove Γ gives the steepest descent contour, we first show that $\mathrm{Re}(g_1)$ is decreasing along $\Gamma_2 \cup \Gamma_3$. On Γ_2, we find

$$\frac{\mathrm{d}\,\mathrm{Re}(g_1)(s)}{\mathrm{d}s}$$
$$= \frac{2s^2 \Big[-3(3-\rho)(1-\rho)^2(1+\rho) - 2s(1-\rho)(3+10\rho-5\rho^2) - 2s^2(11-6\rho+3\rho^2) - 12s^3(1-\rho) - 8s^4 \Big]}{\big(3s^2 + (1+\rho-s)^2\big)\big(3s^2 + (1+s-\rho)^2\big)\big(3s^2 + (3+s-\rho)^2\big)}.$$

One can verify that $\mathrm{d}\,\mathrm{Re}(g_1)(s)/\mathrm{d}s < 0$ for any $s \in [0, 2]$ and any $\rho \in (0, 1)$, implying that $\mathrm{Re}(g_1)$ is decreasing along Γ_2. It follows by symmetry that, $\mathrm{Re}(g_1)$ is increasing along Γ_1. In fact, $\mathrm{Re}(\ln(w)) = \ln(|w|)$, so $\mathrm{Re}(g_1)$ is symmetric with respect to the horizontal axis.

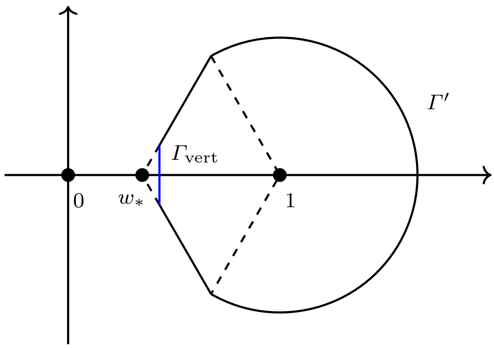

Fig. 11. The deformed contour Γ' with the blue part Γ_{vert}

It remains to check the monotonicity of $\mathrm{Re}(g_1)$ along Γ_3. Again taking the derivative along Γ_3, we have

$$\frac{\mathrm{d}\,\mathrm{Re}(g_1)(\theta)}{\mathrm{d}\theta} = \frac{2(1+\rho)^2 \sin(\theta)\cos(\theta/2)^2[17-5\rho+3\rho^2+\rho^3+8(1+\rho)\cos(\theta)]}{[5+2\rho+\rho^2+4(1+\rho)\cos(\theta)][17+2\rho+\rho^2+8(1+\rho)\cos(\theta)]},$$

which equals zero only at $\theta = 0, \pi$, i.e. $\mathrm{Re}(g_1)$ decreases when $s \in [-2\pi/3, 0]$, and increases when $s \in [0, 2\pi/3]$. In fact $17 - 5\rho + 3\rho^2 + \rho^3 + 8(1+\rho)\cos(\theta) \geq 9-13\rho+3\rho^2+\rho^3$. The derivative of $9-13\rho+3\rho^2+\rho^3$ with respect to ρ is $-13+6\rho+3\rho^2 \leq -13+6+3 < 0$, indicating that $9-13\rho+3\rho^2+\rho^3 \geq 9-13+3+1 = 0$. Similarly, one can see that $5+2\rho+\rho^2+4(1+\rho)\cos(\theta)$ and $17+2\rho+\rho^2+8(1+\rho)\cos(\theta)$ are non-negative for any θ and $\rho \in (0, 1)$.

Therefore, we conclude that $\mathrm{Re}(g_1)$ is strictly monotone along Γ except at its minimum point $w = 2 - \rho'/2$ and maximum point $w_* = \rho'/2$.

Similarly, the fact that the contour Σ is a steepest descent path for $-g_1(w)$ can be proved by calculating $\frac{\mathrm{d}\,\mathrm{Re}(g_1)(\theta)}{\mathrm{d}\theta}$ along Σ.

D.2. Proof of uniform convergence of the rescaled kernel, Proposition 6.2. Consider the steepest descent contour $\Gamma \times \Sigma = (\bigcup_{i=1}^{3} \Gamma_i) \times (\bigcup_{i=1}^{4} \Sigma_i)$ defined in (6.11) and (6.12). We observe that the integrand of the kernel contains the factor $(w-z)^{-1}$, which requires that the z, w-contours do not intersect with each other. As a consequence, we need to deform the contour Γ away from the saddle point w_*, and replaced by a vertical line through $w_*(1+\delta)$ (see Fig. 11 and (D.1)). We choose $\delta = t^{-1/3}$. Under such a deformed contour, we now can bound $|w-z|^{-1}$ by $(w_*\delta)^{-1}$ along $\Gamma' \times \Sigma$, where

$$\Gamma' = \left\{z \in \Gamma \,\middle|\, |z - w_*| > 2w_*\delta\right\} + \left\{z \in \mathbb{C} \,\middle|\, z = w_* + w_*\delta(1 - s\mathrm{i}),\, s \in \left[-\sqrt{3}, \sqrt{3}\right]\right\}. \tag{D.1}$$

To estimate the integrand of $\mathcal{K}_t^c(\xi, \zeta)$ in $\Gamma' \times \Sigma$, we separate the contour into two parts: far away from the point w_* and the neighbourhood of w_*, denoted by

$$\Gamma^{\Delta} = \left\{z \in \Gamma \,\middle|\, |z - w_*| \leq \Delta\right\}, \quad \Gamma'^{\Delta} = \left\{z \in \Gamma' \,\middle|\, |z - w_*| \leq \Delta\right\}, \tag{D.2}$$

$$\Sigma^{\Delta} = \left\{w \in \Sigma \,\middle|\, |w - w_*| \leq \Delta\right\}, \tag{D.3}$$

where we choose[4] $\Delta = t^{-1/9}$ so that the integrand along the distant parts can be bounded by $\mathrm{e}^{-a_1\Delta^3 t} \to 0$ as t tends to infinity. We will see this later in the proof. The deformed contour is now separated into four parts: $\Gamma'^{\Delta} \times \Sigma^{\Delta}$, $(\Gamma' \backslash \Gamma'^{\Delta}) \times \Sigma^{\Delta}$, $\Gamma'^{\Delta} \times (\Sigma \backslash \Sigma^{\Delta})$, $(\Gamma' \backslash \Gamma'^{\Delta}) \times (\Sigma \backslash \Sigma^{\Delta})$. As will be shown later, the main contribution in the long time limit is from the first part $\Gamma'^{\Delta} \times \Sigma^{\Delta}$, i.e., paths of integration passing near a saddle point w_*. It will be also shown that other parts and the error terms of the first parts converge uniformly to zero on $[-L, L]^2$.

The integrand in (6.7) is estimated in three parts: z-integrand $\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)+g_\phi(z)}$; w-integrand $\mathrm{e}^{-f(w,t,\zeta)+f(w_*,t,\zeta)+g_\psi(w)}$; and $(w-z)^{-1}$. We have the following bounds of the factor $|w-z|$ along the deformed contour,

$$\min_{(z,w)\in\Gamma'^{\Delta}\times\Sigma^{\Delta}} |w-z| = w_*\delta, \tag{D.4a}$$

$$\min_{(z,w)\in(\Gamma'\times\Sigma)\backslash(\Gamma'^{\Delta}\times\Sigma^{\Delta})} |w-z| \geq \frac{\sqrt{3}}{2}\Delta. \tag{D.4b}$$

(i) Main contribution on $\boldsymbol{\Gamma'^{\Delta}} \times \boldsymbol{\Sigma^{\Delta}}$. Let us now give the main contribution on $\Gamma'^{\Delta} \times \Sigma^{\Delta}$. Consider the z-integrand $\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)}$ and introduce a new variable: $z = w_* + \lambda^{-1}Z$, where λ is some constant which is defined in (6.14). Using the Taylor expansions given in (6.9), the z-integrand $\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)+g_\phi(z)}$ is rewritten as

$$\begin{aligned}
&\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)+g_\phi(z)} \\
&= \mathrm{e}^{\{g_1(z)-g_1(w_*)\}t+\{g_2(z)-g_2(w_*)\}t^{1/2}+\{g_3(z,\xi)-g_3(w_*,\xi)\}t^{1/3}+g_\phi(z)} \\
&= \mathrm{e}^{2a_1(\lambda^{-1}Z)^3 t+b_{3,\xi}\lambda^{-1}Zt^{1/3}+g_\phi(w_*)+\mathcal{O}(Z^4t,Z^2t^{1/2},Z^2t^{1/3},LZ^2t^{1/3},Z)} \\
&= \mathrm{e}^{\frac{1}{3}Z^3t-(s_2+\xi)Zt^{1/3}+\mathcal{O}(Z^4t,Z^2t^{1/2},Z^2t^{1/3},Z)},
\end{aligned} \tag{D.5}$$

where in the last line, $g_\phi(w_*) = 0$, $2a_1\lambda^{-3} = 1/3$ and $b_{3,\xi}\lambda^{-1} = -(s_2+\xi)$ with λ and λ_c defined in (6.14) and (6.5), respectively. Note that L is fixed and hence $\mathcal{O}(LZ^2t^{1/3})$ can be absorbed into $\mathcal{O}(Z^2t^{1/3})$.

With respect to the new variable Z, let γ^{Δ} be the corresponding path of the line integral: $\gamma^{\Delta} = \left\{Z \in \mathbb{C} \middle| w_* + \lambda^{-1}Z \in \Gamma'^{\Delta}\right\}$. It is easy to see that any $Z \in \gamma^{\Delta}$ satisfy $|Z| \leq \lambda\Delta$.

We then factorise the integrand into two terms: main contribution and the error term R_z:

$$\mathrm{e}^{\frac{1}{3}Z^3t-(s_2+\xi)Zt^{1/3}+\mathcal{O}(Z^4t,Z^2t^{1/2},Z^2t^{1/3},LZ^2t^{1/3},Z)} = \mathrm{e}^{\frac{1}{3}Z^3t-(s_2+\xi)Zt^{1/3}} + R_z,$$

where

$$R_z = \mathrm{e}^{\frac{1}{3}Z^3t-(s_2+\xi)Zt^{1/3}}\left(\mathrm{e}^{\mathcal{O}(Z^4t,Z^2t^{1/2},Z^2t^{1/3},Z)} - 1\right).$$

We claim that the error term R_z gives a zero integral in the long time limit, which will be shown in the second part (ii) of the proof. To get rid of the variable t in the

[4] We couldn't simply choose $\Delta = \delta = t^{-1/3}$, since we need to choose δ and Δ such that, as t goes to infinity, $\delta t^{1/3} < \infty$ and $\Delta t^{1/3} \to \infty$ are satisfied, i.e. the paths Γ'^{Δ} and Σ^{Δ} become the Airy contours defined in (1.10).

integrand, we change the variable Z to $v = t^{1/3}Z$. The deformed contour now becomes $\gamma^{\Delta t^{1/3}} = \left\{ v \in \mathbb{C} \mid w_* + v/(\lambda t^{1/3}) \in \Gamma'^{\Delta} \right\}$. The z-integrand is then rewritten into

$$\mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v} + R_v, \tag{D.6}$$

where

$$R_v = \mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v}\left(\mathrm{e}^{\mathcal{O}(v^4t^{-1/3},v^2t^{-1/6},v^2t^{-1/3},vt^{-1/3})} - 1\right). \tag{D.7}$$

Let us now consider the w-integrand $\mathrm{e}^{-f(w,t,\zeta)+f(w_*,t,\zeta)+g_\psi(w)}$. Similarly, we change the variable to $w = w_* + \lambda^{-1}W$ with $W = t^{-1/3}u$, then the contour now becomes $\sigma^{\Delta t^{1/3}} = \left\{ u \in \mathbb{C} \middle| w_* + u/(\lambda t^{1/3}) \in \Sigma^{\Delta} \right\}$ and the integrand is written as

$$\frac{1}{w_* + c}\mathrm{e}^{-\frac{1}{3}u^3+(s_2+\zeta)u} + R_u, \tag{D.8}$$

where

$$R_u = \frac{1}{w_* + c}\mathrm{e}^{-\frac{1}{3}u^3+(s_2+\zeta)u}\left(\mathrm{e}^{\mathcal{O}(u^4t^{-1/3},u^2t^{-1/6},u^2t^{-1/3},ut^{-1/3})} - 1\right). \tag{D.9}$$

Note that the coefficient $(w_* + c)^{-1}$ comes from the factor $\mathrm{e}^{g_\psi(w_*)} = (w_* + c)^{-1}$, while $\mathrm{e}^{g_\phi(w_*)} = 1$.

Here we focus on a part of the integral (6.7) whose integral path is limited to $\Gamma'^{\Delta} \times \Sigma^{\Delta}$. By the change of variables $v = t^{1/3}\lambda(z - w_*)$, $u = t^{1/3}\lambda(w - w_*)$, and combining (D.6), (D.8) and $(z - w)^{-1}$, a part of such an integral that does not involve error terms R_v and R_u now becomes

$$\int_{\overline{\gamma}^{\Delta t^{1/3}} \times \sigma^{\Delta t^{1/3}}} \frac{\mathrm{d}v}{2\pi\mathrm{i}}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{1}{v-u}\mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v-\frac{1}{3}u^3+(s_2+\zeta)u}, \tag{D.10}$$

where $\overline{\gamma}^{\Delta t^{1/3}}$ denotes a path obtained by reversing the orientation of $\gamma^{\Delta t^{1/3}}$. In the following, let $\overline{C}$ denote a path obtained by reversing the orientation of C.

Recall the Airy kernel $A(x, y)$ in (1.11). Considering the definition of the Airy function (1.10) and carrying out λ-integration by using the identity $\int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-\lambda a} = 1/a$ which holds for $a \in \mathbb{C}$ satisfying $\mathrm{Re}(a) > 0$, we can obtain another expression of $A(x, y)$ as

$$A(x, y) = \int_{\overline{\gamma}^{\infty} \times \sigma^{\infty}} \frac{\mathrm{d}v}{2\pi\mathrm{i}}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{1}{v-u}\mathrm{e}^{\frac{1}{3}v^3-xv-\frac{1}{3}u^3+yu}, \tag{D.11}$$

where

$$\begin{aligned}
\gamma^{\infty} :=& \left\{ v \in \mathbb{C} \mid v = -s\,\mathrm{e}^{\mathrm{i}\pi/3}, s \in \left(-\infty, -2w_*\delta\lambda t^{1/3}\right) \right\} \cup \left\{ v \in \mathbb{C} \mid v = w_*\delta\lambda t^{1/3}(1 - s\mathrm{i}), s \in \left[-\sqrt{3}, \sqrt{3}\right] \right\} \\
&\cup \left\{ v \in \mathbb{C} \mid v = s\,\mathrm{e}^{-\mathrm{i}\pi/3}, s \in \left(2w_*\delta\lambda t^{1/3}, \infty\right) \right\} \\
\sigma^{\infty} :=& \left\{ u \in \mathbb{C} \mid u = -s\,\mathrm{e}^{-\mathrm{i}2\pi/3}, s \in (-\infty, 0] \right\} \cup \left\{ u \in \mathbb{C} \mid u = s\,\mathrm{e}^{\mathrm{i}2\pi/3}, s \in (0, \infty) \right\}.
\end{aligned}$$

We choose such contours because they are convenient for the subsequent calculations.

Using similar arguments to those explained in [16], one can show that, in the long time limit, (D.10) behaves as the kernel of GUE Tracy–Widom distribution, i.e.,

$$\left| \int_{\overline{\gamma}^{\Delta t^{1/3}} \times \sigma^{\Delta t^{1/3}}} \frac{\mathrm{d}v}{2\pi\mathrm{i}}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{1}{v-u}\mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v-\frac{1}{3}u^3+(s_2+\zeta)u} - A(s_2 + \xi, s_2 + \zeta) \right| \le \mathcal{O}(\mathrm{e}^{-c_1\Delta^3 t}) \tag{D.12}$$

holds for t large enough where c_1 is some positive constant. To see this, we first notice, from the form of the Airy kernel (D.11), that the left hand side of (D.12) is bounded above by

$$\begin{aligned}&\left|\int_{\overline{\gamma}^{\infty}}\frac{\mathrm{d}v}{2\pi\mathrm{i}}\mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v}\int_{\sigma^{\infty}-\sigma^{\Delta t^{1/3}}}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\mathrm{e}^{-\frac{1}{3}u^3+(s_2+\zeta)u}\frac{1}{v-u}\right|\\&\quad+\left|\int_{\overline{\gamma}^{\infty}-\overline{\gamma}^{\Delta t^{1/3}}}\frac{\mathrm{d}v}{2\pi\mathrm{i}}\mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v}\int_{\sigma^{\Delta t^{1/3}}}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\mathrm{e}^{-\frac{1}{3}u^3+(s_2+\zeta)u}\frac{1}{v-u}\right|,\end{aligned}\tag{D.13}$$

where $\gamma_1-\gamma_2$ denotes the concatenation of the contours γ_1 and $\overline{\gamma_2}$. The Airy contour σ^{∞} differs from the contour $\sigma^{\Delta t^{1/3}}$ by

$$\sigma^{\infty}-\sigma^{\Delta t^{1/3}}=\left\{u\in\mathbb{C}\mid u=-s\,\mathrm{e}^{-\mathrm{i}2\pi/3},s\in\left(-\infty,-\lambda\Delta t^{1/3}\right)\right\}\cup\left\{u\in\mathbb{C}\mid u=s\,\mathrm{e}^{\mathrm{i}2\pi/3},s\in\left(\lambda\Delta t^{1/3},\infty\right)\right\}.\tag{D.14}$$

We can also find that, in the integrands of (D.13), $|v-u|\geq D\Delta t^{1/3}$ holds with some positive constant D. Then, the first term of (D.13) is bounded above by

$$\begin{aligned}&\int_{\overline{\gamma}^{\infty}}\frac{|\mathrm{d}v|}{2\pi}\left|\mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v}\right|\int_{\sigma^{\infty}-\sigma^{\Delta t^{1/3}}}\frac{|\mathrm{d}u|}{2\pi}\left|\mathrm{e}^{-\frac{1}{3}u^3+(s_2+\zeta)u}\right|\left|\frac{1}{v-u}\right|\\&\leq\frac{c'}{\Delta t^{1/3}}\int_0^{\infty}\mathrm{d}r\,\mathrm{e}^{-\frac{r^3}{3}-\frac{1}{2}(s_2+\xi)r}\int_{\lambda\Delta t^{1/3}}^{\infty}\mathrm{d}s\,\mathrm{e}^{-\frac{s^3}{3}-\frac{1}{2}(s_2+\zeta)s}.\end{aligned}\tag{D.15}$$

where c' is some positive constant. $\mathrm{e}^{-r^3/3}$ and $\mathrm{e}^{-s^3/3}$ are dominant in the integrands and it follows from $\int_{\lambda\Delta t^{1/3}}^{\infty}\mathrm{d}x\,\mathrm{e}^{-x^3/3}\leq\mathrm{e}^{-c_1\Delta^3t}$ with some positive constant c_1 that the first term of (D.13) is bounded above by $\mathcal{O}(\mathrm{e}^{-c_1\Delta^3t})$. For the second term of (D.13), $\overline{\gamma}^{\infty}$ also differs from $\overline{\gamma}^{\Delta t^{1/3}}$ by (D.14) ($\overline{\gamma}^{\infty}-\overline{\gamma}^{\Delta t^{1/3}}=$ (D.14)). Then, the second term of (D.13) can be evaluated in the same fashion as the first term. Hence it turns out that (D.13) is bounded above by $\mathcal{O}(\mathrm{e}^{-c_1\Delta^3t})$, and then we arrive at (D.12). Because $\Delta=t^{-1/9}$, $\mathcal{O}(\mathrm{e}^{-c_1\Delta^3t})$ decays exponentially with respect to t. Hence we have shown that without error terms, the integral in $\Gamma'^{\Delta}\times\Sigma^{\Delta}$ tends to an Airy kernel as t goes to ∞. Next we will prove the integral involving error terms vanishes.

(ii) Estimate of the error term $\boldsymbol{R}$ in $\boldsymbol{\Gamma'^{\Delta}}\times\boldsymbol{\Sigma^{\Delta}}$. Rewrite the target integral into the main contribution and the error terms:

$$\begin{aligned}&\lambda_c t^{1/3}\int_{\Gamma'^{\Delta}\times\Sigma^{\Delta}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\frac{1}{w-z}\frac{\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)+g_{\phi}(z)}}{\mathrm{e}^{f(w,t,\zeta)-f(w_*,t,\zeta)-g_{\psi}(w)}}-(D.10)\\&=\int_{\gamma^{\Delta t^{1/3}}}\frac{\mathrm{d}v}{2\pi\mathrm{i}}\int_{-\sigma^{\Delta t^{1/3}}}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{1}{u-v}\left(R_vR_u+R_u\,\mathrm{e}^{\frac{1}{3}v^3-(s_2+\xi)v}+R_v\,\mathrm{e}^{\frac{1}{3}u^3-(s_2+\xi)u}\right)\\&=\int_{\gamma^{\Delta t^{1/3}}}\frac{\mathrm{d}v}{2\pi\mathrm{i}}\int_{-\sigma^{\Delta t^{1/3}}}\frac{\mathrm{d}u}{2\pi\mathrm{i}}\frac{1}{u-v}\,\mathrm{e}^{h(v,\xi)+h(u,\zeta)}\left[\mathrm{e}^{O(v)+O(u)}-1\right],\end{aligned}\tag{D.16}$$

where $h(v,\xi)=\frac{1}{3}v^3-(s_2+\xi)v$, and $O(v)=\mathcal{O}(v^4t^{-1/3},v^2t^{-1/6},v^2t^{-1/3},vt^{-1/3})$. Let us now show that the above vanishes in the long time limit. First recall (D.4a), we have the estimate $|u-z|^{-1}\leq w_*^{-1}\lambda\delta^{-1}t^{-1/3}=\lambda/w_*$, as $\delta=t^{-1/3}$. Then by the inequality $|\,\mathrm{e}^x-1|\leq|x|\,\mathrm{e}^{|x|}$, we can bound (D.16) by the product of two line integrals as

$$(\mathrm{D}.16) \leq \frac{\lambda}{w_*}\int_{\gamma^{\Delta t^{1/3}}}\int_{-\sigma^{\Delta t^{1/3}}}\left|\frac{\mathrm{d}v}{2\pi\mathrm{i}}\right|\left|\frac{\mathrm{d}u}{2\pi\mathrm{i}}\right| \mathrm{e}^{\,h(v,\xi)+h(u,\zeta)}\,\mathrm{e}^{\,O(v)+O(u)}\,|O(u)+O(v)| := R. \tag{D.17}$$

We now only need to show that R goes to zero as t tends to infinity. Let us now consider $O(v)$. Clearly, $\mathcal{O}(v^4t^{-1/3})$ is dominated by $\mathcal{O}(v^4t^{-1/6})$ for large enough t. Likewise $\mathcal{O}(v^2t^{-1/6})$ dominates $\mathcal{O}(v^2t^{-1/3})$, and $\mathcal{O}(vt^{-1/6})$ dominates $\mathcal{O}(vt^{-1/3})$ when t is large enough. As a result, $\mathcal{O}(v^4t^{-1/3}, v^2t^{-1/6}, v^2t^{-1/3}, vt^{-1/3}) \ll t^{-1/6}\mathcal{O}(v^4, v^2, v)$. Then let us consider $\mathrm{e}^{O(v)}$. We observe that along $\gamma^{\Delta t^{1/3}}$, $\max|v| \leq \Delta t^{1/3} \leq t^{1/4}$ as we choose $\Delta = t^{-1/9}$. When t is large enough, $|v^4t^{-1/3}| \leq |v|^4t^{-1/3} \leq |v|^3t^{-1/12} < |v|^3/6$, i.e. $\mathcal{O}(v^4t^{-1/3}) < |v|^3/6$. Similarly, $\mathcal{O}(v^2t^{-1/6}, v^2t^{-1/3}) < |v|^2/6$ and $\mathcal{O}(vt^{-1/3}) < 1$ when t is large enough. Therefore, $\mathcal{O}(v^4t^{-1/3}, v^2t^{-1/6}, v^2t^{-1/3}, vt^{-1/3})$ is bounded by $(|v|^3 + |v|^2)/6 + 1$. Consequently, R is now bounded as

$$R < t^{-1/6}\frac{\lambda}{w_*}\int_{\gamma^{\Delta t^{1/3}}}\int_{-\sigma^{\Delta t^{1/3}}}\left|\frac{\mathrm{d}v}{2\pi\mathrm{i}}\right|\left|\frac{\mathrm{d}u}{2\pi\mathrm{i}}\right| \mathrm{e}^{\tilde{h}(v,\xi)+\tilde{h}(u,\zeta)}\left|\tilde{O}(v)+\tilde{O}(u)\right|, \tag{D.18}$$

where $\tilde{O}(v) = \mathcal{O}(v^4, v^2, v)$, and $\tilde{h}(v,\xi) = \frac{1}{3}v^3 - (s_2+\xi)v + \frac{1}{6}|v|^3 + \frac{1}{6}|v|^2 + 1$. One sees that the dependence of t only appears in the integration boundary $u = v = \lambda\Delta t^{1/3}e^{\pm \mathrm{i}\pi/3}$. Since $\Delta = t^{-1/9}$, i.e., $\Delta t^{1/3} \gg 1$, the integrand is dominated by the term $\mathrm{e}^{\frac{1}{3}v^3+\frac{1}{6}|v|^3}\,\mathrm{e}^{\frac{1}{3}u^3+\frac{1}{6}|u|^3} = \mathrm{e}^{-\frac{1}{3}\lambda^3\Delta^3 t}$ at the boundary. This implies that the integral (D.18), without the prefactor $t^{-1/6}$, is bounded in the limit $t \to \infty$. Namely, for large enough t, there exists a constant c_3 such that

$$R < c_3t^{-1/6}.$$

(iii) Estimate of $(\boldsymbol{\Gamma'} \times \boldsymbol{\Sigma})\backslash(\boldsymbol{\Gamma'^{\Delta}} \times \boldsymbol{\Sigma^{\Delta}})$. Next, we evaluate the rest of three terms in $(\Gamma' \times \Sigma)\backslash(\Gamma'^{\Delta} \times \Sigma^{\Delta})$: $(\Gamma'\backslash\Gamma'^{\Delta}) \times \Sigma^{\Delta}$, $\Gamma'^{\Delta} \times (\Sigma\backslash\Sigma^{\Delta})$, $(\Gamma'\backslash\Gamma'^{\Delta}) \times (\Sigma\backslash\Sigma^{\Delta})$. Here we only show the proof sketch of $(\Gamma'\backslash\Gamma'^{\Delta}) \times \Sigma^{\Delta}$, while the proofs of the other two follow exactly in the same way.

Let us consider $(\Gamma'\backslash\Gamma'^{\Delta}) \times \Sigma^{\Delta}$. By Lemma 6.1, we know Γ is the steepest descent contour along which the real part of $g_1(z)$ takes the maximum value at $z = w_*$ on Γ. For t large enough, the real part of $g_1(z)$ takes the maximum value at the points $z = w_* + \Delta\mathrm{e}^{\pm \mathrm{i}\pi/3}$ on $\Gamma'\backslash\Gamma'^{\Delta}$. By the Taylor expansion (6.8), one can see that along $\Gamma'\backslash\Gamma'^{\Delta}$, we have

$$\mathrm{Re}\,(g_1(z) - g_1(w_*)) \leq -2a_1\Delta^3 + \Delta^4\,\mathrm{Re}\,(\mathrm{e}^{\pm 4\mathrm{i}\pi/3}\,h_1(w_* + \Delta\mathrm{e}^{\pm i\pi/3})) \leq -a_1\Delta^3,$$

where $a_1 > 0$ is given in (6.10), and the second inequality holds when t is large enough. Since $\Delta = t^{-1/9}$ and the function $h_1(z)$ is bounded along Γ', then $\Delta|h_1(z)| \leq a_1$ for t large enough. Therefore, the z-integrand except $(w-z)^{-1}$ is bounded by

$$|\,\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)}\,| \leq \mathrm{e}^{-a_1\Delta^3 t+\mathcal{O}(t^{1/2})}.$$

Note that the bound is uniform for any fixed L. Then by (D.4b), we have the bound of $|w-z|^{-1} \leq 2/\sqrt{3}\Delta^{-1} \ll 2t^{1/3}$. Since $\Delta = t^{-1/9}$, then $\mathrm{e}^{-\Delta^3 t} = \mathrm{e}^{-t^{2/3}}$. Obviously $\mathrm{e}^{-t^{2/3}}\,t^{1/3}\,\mathrm{e}^{\mathcal{O}(t^{1/2})} \to 0$ as $t \to \infty$. In conclusion, for large enough t,

$$|\,\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)}(w-z)^{-1}| \leq \mathrm{e}^{-a_1\Delta^3 t/2}, \tag{D.19}$$

where $a_1 > 0$.

Now we are left with the w-integrand (except for $|w-z|^{-1}$). Using the change of variable again $u = t^{1/3}\lambda(w-w_*)$, we have

$$t^{1/3}\mathrm{e}^{-f(w,t,\zeta)+f(w_*,t,\zeta)+g_\psi(w)} = \frac{\lambda^{-1}}{w_*+c}\mathrm{e}^{h(u,\zeta)}\,\mathrm{e}^{O(u)},$$

where $h(u,\zeta) = \frac{1}{3}u^3 - (s_2+\zeta)u$ and $O(u) = \mathcal{O}(u^4 t^{-1/3}, u^2 t^{-1/6}, u^2 t^{-1/3}, u t^{-1/3})$. By the same analysis as in part (ii), one can see that

$$\int_{\sigma^{\Delta t^{1/3}}} \left|\frac{\mathrm{d}u}{2\pi \mathrm{i}}\right| \left|\frac{\lambda^{-1}}{w_*+c}\mathrm{e}^{h(u,\zeta)}\,\mathrm{e}^{O(u)}\right| \le c_4, \tag{D.20}$$

where c_4 is some positive constant independent of t.

Combining (D.19) and (D.20), we obtain

$$\left|\lambda_c t^{1/3}\int_{(\Gamma'\setminus\Gamma'^{\Delta})\times\Sigma^{\Delta}} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\mathrm{e}^{f(z,t,\xi)-f(w_*,t,\xi)+g_\phi(z)}\mathrm{e}^{-f(w,t,\zeta)+f(w_*,t,\zeta)+g_\psi(w)}\frac{1}{w-z}\right| \le c_5\mathrm{e}^{-\frac{1}{2}a_1\Delta^3 t}, \tag{D.21}$$

where c_5 is some positive constant, and hence (D.21) goes to zero as t goes to infinity.

D.3. Proof of (6.18). Since $g_1(z)$ is analytic along Γ' and Σ', there exists some positive constant H_1 such that $|h_1(z)| < H_1$ for z along Γ' and Σ'. Similarly, we also have $|h_2(z)| < H_2$, $|\bar{h}_3(z)| < H_3$, $|h_\phi(z)| < H_\phi$ and $|h_\psi(z)| < H_\psi$. Consider $g_1(z)$ near $z = w_*$, we have

$$\begin{aligned}|g_1(z)-g_1(w_*)| &= \left|2a_1(z-w_*)^3 + (z-w_*)^4 h_1(z)\right| \\ &\le 2a_1|z-w_*|^3 + H_1|z-w_*|^4 \le 3a_1|z-w_*|^3,\end{aligned} \tag{D.22}$$

where the last inequality holds under the restriction that $|z-w_*| \le H_1/a_1$. Consequently, if $|z-w_*| < A$ where $A := \max\{H_1/a_1, H_2/|b_2|, H_3/|\bar{b}_3|, H_\phi/|b_\phi|, H_\psi/|b_\psi|\}$, the following inequalities hold:

$$|g_1(z)-g_1(w_*)| \le 3a_1|z-w_*|^3, \tag{D.23a}$$

$$|g_2(z)-g_2(w_*)| \le 2|b_2||z-w_*|^2, \tag{D.23b}$$

$$|\bar{g}_3(z)-\bar{g}_3(w_*)| \le 2|\bar{b}_3||z-w_*|, \tag{D.23c}$$

$$\left|g_\phi(z)-g_\phi(w_*)\right| \le 2|b_\phi||z-w_*|, \tag{D.23d}$$

$$\left|g_\psi(z)-g_\psi(w_*)\right| \le 2|b_\psi||z-w_*|. \tag{D.23e}$$

Furthermore, if $|z-w_*| > 12|b_2|t^{-1/2}/a_1$, then $2|b_2||z-w_*|^2 t^{1/2} \le \frac{1}{6}a_1|z-w_*|^3 t$. Therefore, if $|z-w_*| > \max\{\frac{12|b_2|}{a_1}t^{-1/2}, \sqrt{\frac{12|\bar{b}_3|}{a_1}}t^{-1/3}, \sqrt{\frac{12|b_\phi|}{a_1}}t^{-1/2}, \sqrt{\frac{12|b_\psi|}{a_1}}t^{-1/2}\}$,

$$|g_2(z)-g_2(w_*)|\,t^{1/2} \le 2|b_2||z-w_*|^2 t^{1/2} \le \frac{1}{6}a_1|z-w_*|^3 t, \tag{D.24a}$$

$$|\bar{g}_3(z)-\bar{g}_3(w_*)|\,t^{1/3} \le 2|\bar{b}_3||z-w_*|t^{1/3} \le \frac{1}{6}a_1|z-w_*|^3 t, \tag{D.24b}$$

$$\left|g_\phi(z)-g_\phi(w_*)\right| \le 2|b_\phi||z-w_*| \le \frac{1}{6}a_1|z-w_*|^3 t, \tag{D.24c}$$

$$\left|g_\psi(z) - g_\psi(w_*)\right| \le 2|b_\psi||z - w_*| \le \frac{1}{6}a_1|z - w_*|^3 t. \tag{D.24d}$$

When t is large enough, $t^{-1/3} \gg t^{-1/2}$, i.e., we only require $|z - w_*| > \sqrt{\frac{12|\bar{b}_3|}{a_1}} t^{-1/3} := c_1 t^{-1/3}$, then the above inequalities hold. Using the above inequalities, we have the following estimations.

• Contribution from Γ_{vert}: (6.18a)

For $z \in \Gamma_{\text{vert}}$, $|z - w_*| \le 2w_*\delta \le \Delta = t^{-1/9}$. Since A is a fixed constant, one can choose t large enough such that $|z - w_*| \le \Delta < A$. Moreover, we restrict $\delta > c_1 t^{-1/3}/w_*$ so that $|z - w_*| \ge w_*\delta > c_1 t^{-1/3}$. Thus, from (D.23) and (D.24) we have,

$$\left|\mathrm{e}^{\bar{f}(z,t) - \bar{f}(w_*,t) + g_\phi(z)}\right| \le \mathrm{e}^{(3a_1 + a_1/2)|z - w_*|^3 t}\, \mathrm{e}^{g_\phi(w_*)} \le \mathrm{e}^{28a_1(w_*\delta)^3 t}\, \mathrm{e}^{g_\phi(w_*)}.$$

• Contribution from $\Gamma'^{\Delta} \backslash \Gamma_{\text{vert}}$: (D.25)

Note that we can also bound this part by $\mathrm{e}^{28a_1(w_*\delta)^3}$ using the same method as above. However, since the contour now is along the direction $\mathrm{e}^{\pm \mathrm{i}\pi/3}$, we could refine the bound of $g_1(z)$. We parameterise z along $\Gamma'^{\Delta} \backslash \Gamma_{\text{vert}}$ by $z = w_* + v\,\mathrm{e}^{\pm\mathrm{i}\pi/3}$ where $v \in (2w_*\delta, \Delta]$. Therefore,

$$\left|\mathrm{e}^{g_1(z) - g_1(w_*)}\right| = \left|\mathrm{e}^{-2a_1 v^3 + h_1(z) v^4\, \mathrm{e}^{\pm 4\mathrm{i}\pi/3}}\right| \le \mathrm{e}^{-a_1 v^3},$$

when $|z - w_*| < A$. In fact, we have $|z - w_*| \le \Delta = t^{-1/9} < A$ when t is large enough. Then we repeat the above estimate (D.24) for $g_2(z)t^{1/2} + \bar{g}_3(z)t^{1/3} + g_\phi(z)$ and we obtain for $\Gamma'^{\Delta} \backslash \Gamma_{\text{vert}}$,

$$\left|\mathrm{e}^{\bar{f}(z,t) - \bar{f}(w_*,t) + g_\phi(z)}\right| \le \mathrm{e}^{(-a_1 + a_1/2)|z - w_*|^3 t}\, \mathrm{e}^{g_\phi(w_*)} \le \mathrm{e}^{-a_1 v^3 t/2}\, \mathrm{e}^{g_\phi(w_*)}. \tag{D.25}$$

In fact, when $z \in \Gamma'^{\Delta} \backslash \Gamma_{\text{vert}}$, we have $|z - w_*| \ge 2w_*\delta > w_*\delta > c_1 t^{-1/3}$, which satisfies the restriction of (D.24).

• Contribution from $\Gamma' \backslash \Gamma'^{\Delta}$: (D.26)

This estimate is exactly the same as part (iii) in the Appendix D.2. Recall that we have proved in Appendix D.2, along $\Gamma' \backslash \Gamma'^{\Delta}$,

$$\left|\mathrm{e}^{\bar{f}(z,t) - \bar{f}(w_*,t) + g_\phi(z)}\right| \le \mathrm{e}^{-a_1 \Delta^3 t + \mathcal{O}(t^{1/2})} \le \mathrm{e}^{-a_1 \Delta^3 t/2}, \tag{D.26}$$

where $\mathrm{e}^{\mathcal{O}(t^{1/2})} < \mathrm{e}^{a_1 \Delta^3 t/2}$ for t large enough.

D.4. Proof of (6.20). Along Γ', the minimum value of $z + c$ is taken at the point $z = w_*(1 + \delta)$ (See Fig. 6). Therefore,

$$\left|\frac{w_* + c}{z + c}\right| \le \left|\frac{w_* + c}{w_*(1 + \delta) + c}\right| = \left|\frac{1}{1 + \frac{w_*\delta}{w_* + c}}\right| \le \mathrm{e}^{-\frac{1}{2}\frac{w_*}{w_* + c}\delta},$$

where the last inequality follows by $1/(1 + x) < \mathrm{e}^{-x/2}$ when $0 < x < 2$. This restriction is satisfied if we suppose $\delta < 1$. Similarly, we consider the term involving ζ. Along Σ', when $c > 1$, $|w+c|$ takes the maximum value at $w = w_* - w_*\delta(1 \pm \sqrt{3}\mathrm{i})$ and, as easily seen from Fig. 6, it is given by

$$|w+c| \le \left|\sqrt{(w_*+c)^2 + 4\delta^2 w_*^2 - 2\delta w_*(w_*+c)}\right| = \sqrt{w_*^2(4\delta^2 - 2\delta + 1) + 2w_* c(1-\delta) + c^2}.$$

This can be seen for the following reason. Actually, for $\delta \in (0, 1/(1-\rho))$, if c satisfied $0 < c < \{\delta(2\delta-1)(1-\rho)^2 - 4(1+\rho)\}/\{2\delta(1-\rho)-8\} =: C(\rho,\delta)$, $|w+c|$ takes the maximum value $(3+\rho)/2-c$ at $w = -(3+\rho)/2$. For fixed $\rho \in (0,1)$, $C(\rho,\delta)$ is an increasing function of δ in a region $\delta \in (0, 1/4)$ and hence $\max_{\delta\in(0,1/4)} C(\rho,\delta) = C(\rho, 1/4) = \{(15+\rho)^2 - 192\}/\{4(15+\rho)\}$. Since $C(\rho, 1/4)$ is also an increasing function of ρ in the region $\rho \in (0,1)$, it turns out that $\max_{\rho\in(0,1)} C(\rho, 1/4) = C(1, 1/4) = 1$. Therefore, when $c > 1$, $|w+c|$ always takes the maximum value at $w = w_* - w_*\delta(1 \pm \sqrt{3}\mathrm{i})$ on Σ' for any $\rho \in (0,1)$ and $\delta \in (0, 1/4)$. Then we bound $4\delta^2 - 2\delta + 1$ by $1-\delta$, which is valid when $0 < \delta < 1/4$. It follows that

$$\begin{aligned}
|w+c| \le& \sqrt{(w_*^2 + 2w_* c + c^2)(1-\delta) + \delta c^2} \\
=& \sqrt{(w_*+c)^2 - \delta[(w_*+c)^2 - c^2]} \\
\le& \sqrt{(w_*+c)^2 - \delta w_*(w_*+c)}.
\end{aligned}$$

Therefore,

$$\left|\frac{w+c}{w_*+c}\right| \le \sqrt{1 - \frac{\delta w_*}{w_*+c}} \le \mathrm{e}^{-\frac{1}{2}\frac{w_*}{w_*+c}\delta},$$

where the last line follows by $\sqrt{1-x} \le \mathrm{e}^{-x/2}$ when $x > 0$.

E. The proofs of Lemmas and Proposition in Sect. 7

In this appendix we prove Lemmas 7.1, 7.5 and Proposition 7.2 in Sect. 7.

E.1. Proof of Lemma 7.1, on commutativity between the integrals and the sums. In the following, we will prove that the determinant of the matrix $K^c(x_i, x_j, \vec{z})$ has the dominant series as

$$\left|\det\left[K^c(x_i, x_j, \vec{z})\right]_{1\le i,j\le k}\right| \le M \prod_{i=1}^{k} \mathrm{e}^{-\frac{w_*}{4(w_*+c)}\delta x_i}, \tag{E.1}$$

where M is some constant depending only on $n, m \in \mathbb{N}$, $t \in (0,\infty)$ and $k \in [1, m]$. Since the series in the right hand side of (E.1) is a geometric series, we can perform the infinite sums from $x_i = 1$ to ∞ for all $i \in [1, k]$ and it turns out to be finite:

$$\sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty}\cdots\sum_{x_k=1}^{\infty} M \prod_{i=1}^{k} \mathrm{e}^{-\frac{w_*}{4(w_*+c)}\delta x_i} = M\left[\frac{1}{\mathrm{e}^{(w_*\delta)/[4(w_*+c)]} - 1}\right]^k < \infty.$$

From the Weierstrass M-test, which is specialisation of the Lebesgue's dominated convergence theorem, it turns out that we can commute the sums over $x_i \in \mathbb{N}$ for all $i \in [1, k]$ and the integrals with respect to z_j for all $j \in [1, n-1]$.

In order to show (E.1), we will prove the inequality

$$\left|(w_* + c)^{x-y} K^c(x, y, \vec{z})\right| \le C\, \mathrm{e}^{-\frac{w_*}{4(w_*+c)}\delta(x+y)}, \tag{E.2}$$

where C is some constant depending on $n, m \in \mathbb{N}$ and $t \in (0, \infty)$. From the formula of $K^c(x, y, \vec{z})$, which is given by (7.1), we can find the following bound:

$$\left|(w_* + c)^{x-y} K^c(x, y, \vec{z})\right| \le \oint_{\Gamma_1} \frac{|\mathrm{d}z|}{2\pi} \left|\frac{z+\rho'}{z+1}\right| \mathrm{e}^{-\frac{t}{2}\mathrm{Re}\,(z)} \left|\frac{z}{z-1}\right|^m \left|\frac{1+z}{z}\right|^n \left|\frac{w_*+c}{z+c}\right|^x \prod_{j=1}^{n-1} \left|\frac{1+z_j z}{1+z}\right|$$
$$\times \oint_{\Gamma_{0,-\rho',\{-z_j^{-1}\}_{j=1}^{n-1}}} \frac{|\mathrm{d}w|}{2\pi} \left|\frac{w+1}{(w+\rho')(w+c)}\right| \mathrm{e}^{\frac{t}{2}\mathrm{Re}\,(w)} \left|\frac{w-1}{w}\right|^m \left|\frac{w}{1+w}\right|^n \left|\frac{w_*+c}{w+c}\right|^{-y} \prod_{j=1}^{n-1} \left|\frac{1+w}{1+z_j w}\right| \frac{1}{|w-z|}.$$

Let us choose the contours of z-integral and w-integral as Γ' and Σ' defined in the proof of Proposition 6.3, respectively. This implies that, for any z and w on the contours, the parts depending on $x \in \mathbb{N}$ and $y \in \mathbb{N}$ are bounded above as

$$\left|\frac{w_*+c}{z+c}\right|^x \le \mathrm{e}^{-\frac{w_*}{4(w_*+c)}\delta x},$$
$$\left|\frac{w_*+c}{w+c}\right|^{-y} \le \mathrm{e}^{-\frac{w_*}{4(w_*+c)}\delta y}.$$

Since the contour of z-integral does not pass the points at $z = -c$ and $|z|, |w| < \infty$ and $|w - z| > 0$ holds for any $(z, w) \in \Gamma' \times \Sigma'$, the other parts of integrands are bounded by some constant depending on $n, m \in \mathbb{N}$ and $t \in (0, \infty)$. In addition, considering that the lengths of the contours are finite, it turns out that (E.2) holds.

Since a determinant of a matrix whose (i, j) entry is given by $a_{i,j}$ and a matrix whose (i, j) entry is given by $b^{x_i - x_j} a_{i,j}$ are equivalent, we obtain the equality

$$\left|\det\left[K^c(x_i, x_j, \vec{z})\right]_{1\le i,j\le k}\right| = \left|\det\left[(w_* + c)^{x_i - x_j} K^c(x_i, x_j, \vec{z})\right]_{1\le i,j\le k}\right|. \tag{E.3}$$

Application of the Hadamard's inequality to the right hand side of (E.3) yields

$$\left|\det\left[(w_* + c)^{x_i - x_j} K^c(x_i, x_j, \vec{z})\right]_{1\le i,j\le k}\right| \le \prod_{i=1}^{k} \sqrt{\sum_{j=1}^{k} \left|(w_* + c)^{x_i - x_j} K^c(x_i, x_j, \vec{z})\right|^2}. \tag{E.4}$$

Finally, using the inequality (E.2), we get

$$\sqrt{\sum_{j=1}^{k} \left|(w_* + c)^{x_i - x_j} K^c(x_i, x_j, \vec{z})\right|^2} \le C \sqrt{\sum_{j=1}^{k} \mathrm{e}^{-\frac{w_*}{2(w_*+c)}\delta(x_i+x_j)}} = C\mathrm{e}^{-\frac{w_*}{4(w_*+c)}\delta x_i} \sqrt{\sum_{j=1}^{k} \mathrm{e}^{-\frac{w_*}{2(w_*+c)}\delta x_j}}. \tag{E.5}$$

Since $\frac{w_*}{2(w_*+c)}\delta > 0$ and $x_j \ge 1$ hold for any $j \in [1, k]$, it turns out that $\sum_{j=1}^{k} \mathrm{e}^{-\frac{w_*}{2(w_*+c)}\delta x_j} < k$ holds. Therefore, it follows from (E.5) that we obtain

$$\prod_{i=1}^{k} \sqrt{\sum_{j=1}^{k} \left|(w_* + c)^{x_i - x_j} K^c(x_i, x_j, \vec{z})\right|^2} \le k^{\frac{k}{2}} C^k \prod_{i=1}^{k} \mathrm{e}^{-\frac{w_*}{4(w_*+c)}\delta x_i}. \tag{E.6}$$

It follows from (E.3), (E.4) and (E.6) that the inequality (E.1) holds with a constant M set as $M = k^{\frac{k}{2}} C^k$.

E.2. Proof of the regularity of the kernel, Lemma 7.5. A complex function $f : \mathbb{C}^{n-1} \to \mathbb{C}$ is complex analytic if and only if it is holomorphic. In the following, we will show the kernel is holomorphic by proving it is complex analytic in $S(1,r)^{n-1}$ for $r \in (0,1)$. In other words, we will show that, for any $\vec{b} = (b_1, \ldots, b_{n-1}) \in S(1,r)^{n-1}$, there exists an open polydisc $S(b_1, r_1) \times \cdots \times S(b_{n-1}, r_{n-1}) \subset S(1,r)^{n-1}$ such that the kernel $K^c(x, y, \vec{z})$ has a power series expansion

$$K^c(x, y, \vec{z}) = \sum_{k_1,\ldots,k_{n-1}=0}^{\infty} c_{k_1,\ldots,k_{n-1}} \prod_{j=1}^{n-1} (z_j - b_j)^{k_j},$$

which converges for any $\vec{z} \in S(b_1, r_1) \times \cdots \times S(b_{n-1}, r_{n-1})$, where coefficients $c_{k_1,\ldots,k_{n-1}}$ are independent of $\vec{z}$. In this proof, we start from the form

$$K^c(x, y, \vec{z}) = \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} F^c(z, x) \prod_{j=1}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',\{-z_j^{-1}\}_{j=1}^{n-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} G^c(w, y) \prod_{j=1}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z}, \tag{E.7}$$

where the functions $F^c(z, x)$ and $G^c(w, y)$ are defined in (7.5).

As stated in the proof of Lemma 7.3, we can choose the contour such that

$$\left| \frac{1+w}{w} \right| > r > \left| b_j - 1 \right| \tag{E.8}$$

hold for any $\vec{b} = (b_1, \ldots, b_{n-1}) \in S(1,r)^{n-1}$ and w on the contour. Using the reverse triangle inequality and (E.8), we obtain

$$\left| \frac{1+w}{w} + (b_j - 1) \right| \geq \left| \frac{1+w}{w} \right| - |b_j - 1| > r - |b_j - 1| > 0. \tag{E.9}$$

Suppose that the open polydisc $S(b_1, r_1) \times \cdots \times S(b_{n-1}, r_{n-1})$ is included in $S(1,r)^{n-1}$, r_j and b_j satisfy

$$r_j < r - |b_j - 1| \tag{E.10}$$

for any $j \in [1, n-1]$. It follows from (E.9) and (E.10) that the equality and inequalities

$$\left| \frac{w(b_j - z_j)}{1 + b_j w} \right| = \left| \frac{b_j - z_j}{(1+w)/w + (b_j - 1)} \right| \leq \frac{|z_j - b_j|}{|(1+w)/w| - |b_j - 1|} < \frac{r_j}{r - |b_j - 1|} < 1 \tag{E.11}$$

hold for any $\vec{z} \in S(b_1, r_1) \times \cdots \times S(b_{n-1}, r_{n-1}) \subset S(1,r)^{n-1}$ and any w on the contour.

The right hand side of (E.7) depends on z_j via the factor $(1 + z_j z)/(1 + z_j w)$, and (E.11) guarantees that a Taylor series expansion of it around $z_j = b_j$:

$$\frac{1 + z_j z}{1 + z_j w} = \sum_{k=0}^{\infty} d_k(z, w, b_j)(z_j - b_j)^k \tag{E.12}$$

converges, where

$$d_k(z, w, b_j) = \begin{cases} \frac{1+b_j z}{1+b_j w} & \text{for } k = 0 \\ \frac{w-z}{w(1+b_j w)} \left(\frac{-w}{1+b_j w} \right)^k & \text{for } k \geq 1. \end{cases}$$

Since a Taylor series (E.12) converges uniformly for any z and w on the contours, we can move the sum over $k \in \mathbb{N} \cup \{0\}$ to the outside of z, w-integrals. Therefore, the kernel can be expanded as

$$K^c(x, y, \vec{z}) = \sum_{k_1,\ldots,k_{n-1}=0}^{\infty} c_{k_1,\ldots,k_{n-1}} \prod_{j=1}^{n-1} (z_j - b_j)^{k_j}, \tag{E.13}$$

where

$$c_{k_1,\ldots,k_{n-1}} = \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} \oint_{0,-\rho',\{-b_j^{-1}\}_{j=1}^{n-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{F^c(z,x)G^c(w,y)}{w-z} \left(\frac{1+w}{1+z}\right)^{n-1} \prod_{j=1}^{n-1} d_{k_j}(z, w, b_j).$$

To confirm convergence of a power series in the right hand side of (E.13), we will evaluate an upper bound of $|c_{k_1,\ldots,k_{n-1}}|$. Using the Cauchy's integral theorem, we can choose the contours of z, w-integrals such that the absolute value of integrand except for $\prod_{j=1}^{n-1} d_{k_j}(z, w, b_j)$ is finite for any $n, m \in \mathbb{N}$ and $t \in (0, \infty)$, i.e., the contour of z-integral does not pass the points at $z = -c$ and the lengths of both contours are finite. From (E.9) and (E.10), we also have

$$\left|\frac{w}{1+b_j w}\right| < \frac{1}{r - |b_j - 1|} \le \frac{1}{r_j},$$

hence we can find an upper bound of $|d_k(z, w, b_j)|$ as

$$\left|d_0(z, w, b_j)\right| = \left|\frac{1+b_j z}{1+b_j w}\right| < C_1, \tag{E.14a}$$

$$\left|d_k(z, w, b_j)\right| = \left|\frac{w-z}{w(1+b_j w)}\left(\frac{-w}{1+b_j w}\right)^k\right| < \frac{C_2}{r_j^k}, \tag{E.14b}$$

where C_1 and C_2 are some positive constants. Utilising (E.14), it turns out that $|c_{k_1,\ldots,k_{n-1}}|$ is bounded as

$$\begin{aligned}\left|c_{k_1,\ldots,k_{n-1}}\right| &\le \oint_1 \frac{|\mathrm{d}z|}{2\pi} \oint_{0,-\rho',\{-b_j^{-1}\}_{j=1}^{n-1}} \frac{|\mathrm{d}w|}{2\pi} \left|\frac{F^c(z,x)G^c(w,y)}{w-z}\right| \left|\frac{1+w}{1+z}\right|^{n-1} \prod_{j=1}^{n-1} \left|d_{k_j}(z, w, b_j)\right| \\ &< \frac{C}{r_1^{k_1} \cdots r_{n-1}^{k_{n-1}}},\end{aligned}$$

where C is some constant depending on $n, m \in \mathbb{N}$ and $t \in (0, \infty)$. This implies that a power series in the right hand side of (E.13) converges.

E.3. Proof of extension of decoupling to determinant, Proposition 7.2. In order to simplify the notations, we introduce the abbreviations

$$\begin{aligned} K_{ij}(\vec{z}) &= K^c(x_i, x_j, \vec{z}), & \bar{K}_{ij} &= \bar{K}^c(x_i, x_j), & A_i(\vec{z}) &= A^c(x_i, \vec{z}), \\ (A_p)_i &= A_p^c(x_i), & B_i &= B^c(x_i), \end{aligned}$$

and the vectors

$$\vec{K}_i^{(k)}(\vec{z}) = \left(K_{i1}(\vec{z}),\ K_{i2}(\vec{z}),\ \cdots,\ K_{ik}(\vec{z})\right),$$

$$\vec{\bar{K}}_i^{(k)} = \left(\bar{K}_{i1}, \bar{K}_{i2}, \cdots, \bar{K}_{ik}\right),$$
$$\vec{B}^{(k)} = \left(B_1, B_2, \cdots, B_k\right).$$

In the following, using k-dimensional row vectors $\vec{v}_i = (v_{i1}, \cdots, v_{ik})$, we represent the determinant of a $k \times k$ matrix $\det\left[v_{ij}\right]_{1\le i,j\le k}$ as

$$\left|\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_k \end{bmatrix}\right| = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix}.$$

Using these notations, the claim of Proposition 7.2 can be represented as

$$\begin{aligned} &\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left|\begin{bmatrix} \vec{K}_1^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix}\right| \\ &= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left|\begin{bmatrix} \vec{\bar{K}}_1^{(k)} - \sum_{p=1}^{n-1}\prod_{q=1}^{p}(z_q-1)(A_p)_1 \vec{B}^{(k)} \\ \vdots \\ \vec{\bar{K}}_k^{(k)} - \sum_{p=1}^{n-1}\prod_{q=1}^{p}(z_q-1)(A_p)_k \vec{B}^{(k)} \end{bmatrix}\right|. \end{aligned} \tag{E.15}$$

Since $K_{ij}(\vec{z})$ is invariant under exchanges of any two $\vec{z}$-variables and is a holomorphic function in the vicinity of $z_j = 1$ for any $j \in [1, n-1]$ as shown in Lemma 7.5, we can apply Lemma 7.4 to one row of $\det\left[K_{ij}(\vec{z})\right]_{1\le i,j\le k}$. Applying Lemma 7.3 to the first row and using the multilinearity of determinant, the left hand side of (E.15) is divided into two terms as

$$\begin{aligned} &\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left|\begin{bmatrix} \vec{K}_1^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix}\right| \\ &= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left|\begin{bmatrix} \vec{\bar{K}}_1^{(k)} - (z_1-1)A_1(\vec{z})\vec{B}^{(k)} \\ \vec{K}_2^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix}\right| \\ &= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left\{ \frac{1}{(z_1-1)^2} \left|\begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vec{K}_2^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix}\right| - \frac{1}{(z_1-1)} \left|\begin{bmatrix} A_1(\vec{z})\vec{B}^{(k)} \\ \vec{K}_2^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix}\right| \right\}. \end{aligned} \tag{E.16}$$

For the same reason, we can apply Lemma 7.3 to the second row of the first term of the right hand side of (E.16) and divide the term into further two terms. Carrying out the same calculation from the second row to the k^{th} row in order, the determinant is divided into $k+1$ terms as

$$\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left\| \begin{bmatrix} \vec{K}_1^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix} \right\|$$

$$= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=2}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left\{ \frac{1}{(z_1-1)^2} \left\| \begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vec{\bar{K}}_k^{(k)} \end{bmatrix} \right\| - \frac{1}{(z_1-1)} \sum_{\ell=1}^{k} \left\| \begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vdots \\ \vec{\bar{K}}_{\ell-1}^{(k)} \\ A_\ell(\vec{z})\vec{B}^{(k)} \\ \vec{K}_{\ell+1}^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix} \right\| \right\}. \tag{E.17}$$

For convenience, we define $K'^c(x,y,\vec{z})$ and $A'^c(x,\vec{z})$ as the function which are obtained by substituting 1 into z_1 of $K^c(x,y,\vec{z})$ and z_2 of $A(x,\vec{z})$, respectively. In other words, $K'^c(x,y,\vec{z})$ and $A'^c(x,\vec{z})$ are given by

$$\begin{aligned} K'^c(x,y,\vec{z}) &= K^c(x,y,\vec{z})\big|_{z_1=1} \\ &= \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} F^c(z,x) \prod_{j=2}^{n-1} \frac{1+z_j z}{1+z} \oint_{0,-\rho',-1,\{-z_j^{-1}\}_{j=2}^{n-1}} \frac{\mathrm{d}w}{2\pi \mathrm{i}} G^c(w,y) \prod_{j=2}^{n-1} \frac{1+w}{1+z_j w} \frac{1}{w-z}, \\ A'^c(x,\vec{z}) &= A^c(x,\vec{z})\big|_{z_2=1} = \oint_1 \frac{\mathrm{d}z}{2\pi \mathrm{i}} F^c(z,x) \frac{1}{1+z} \prod_{j=3}^{n-1} \frac{1+z_j z}{1+z}. \end{aligned}$$

In addition, we introduce the abbreviations

$$K'_{ij}(\vec{z}) = K'^c(x_i,x_j,\vec{z}), \quad A'_i(\vec{z}) = A'^c(x_i,\vec{z}),$$

and the vector

$$\vec{K}_i'^{(k)}(\vec{z}) = \left(K'_{i1}(\vec{z}),\ K'_{i2}(\vec{z}),\ \cdots,\ K'_{ik}(\vec{z}) \right).$$

We focus on each summand in the sum over $\ell \in [1,k]$ on the right hand side of (E.17). Since $K_{ij}(\vec{z})$ and $A_i(\vec{z})$ are holomorphic functions in the vicinity of $z_1 = 1$, the z_1-integrand is a function with a single pole of order 1 at $z_1 = 1$ and the z_1-integration is carried out by substituting 1 into z_1. Hence, the r^{th} summand ($r \in [1,k]$) in the sum over $\ell \in [1,k]$ on the right hand side of (E.17) is written as

$$h(1) \oint_1 \prod_{j=2}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{2\le i<j\le n-1}(z_j - z_i)}{\prod_{j=2}^{n-1}(z_j-1)^{j}} \prod_{j=2}^{n-1} h(z_j) \left\| \begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vdots \\ \vec{\bar{K}}_{r-1}^{(k)} \\ A_r(\vec{z})\vec{B}^{(k)} \\ \vec{K}_{r+1}'^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k'^{(k)}(\vec{z}) \end{bmatrix} \right\|. \tag{E.18}$$

Removing the variable z_{n-1} and shifting the indices of $\vec{z}$-variables by 1 as $\{z_1, \dots, z_{n-2}\} \to \{z_2, \dots, z_{n-1}\}$ in the claim of Lemma 7.3, we obtain

$$\begin{aligned}
&\oint_1 \prod_{j=2}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{2 \le i < j \le n-1} (z_j - z_i)}{\prod_{j=2}^{n-1} (z_j - 1)^j} \prod_{j=2}^{n-1} h(z_j) K'^{c}(x, y, \vec{z}) \\
&= \oint_1 \prod_{j=2}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{2 \le i < j \le n-1} (z_j - z_i)}{\prod_{j=2}^{n-1} (z_j - 1)^j} \prod_{j=2}^{n-1} h(z_j) \left[\bar{K}^c(x, y) - (z_2 - 1) A'^{c}(x, \vec{z}) B^c(y)\right].
\end{aligned} \tag{E.19}$$

Since $K'_{ij}(\vec{z})$ and $A_i(\vec{z})$ are symmetric under the exchange of any two variables in $\{z_2, \dots, z_{n-1}\}$, we can apply the equality (E.19) to one of the rows from $r + 1^{\text{st}}$ to k^{th} row of the determinant (E.18). Applying the equality (E.19) to $r + 1^{\text{st}}$ row of the determinant in (E.18), it is replaced with $\vec{\bar{K}}^{(k)}_{r+1} - (z_2 - 1) A'_{r+1}(\vec{z}) \vec{B}^{(k)}$ and (E.18) is divided into two terms as

$$\begin{aligned}
&h(1) \oint_1 \prod_{j=2}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{2 \le i < j \le n-1} (z_j - z_i)}{\prod_{j=2}^{n-1} (z_j - 1)^j} \prod_{j=2}^{n-1} h(z_j) \left| \begin{bmatrix} \vec{\bar{K}}^{(k)}_1 \\ \vdots \\ \vec{\bar{K}}^{(k)}_{r-1} \\ A_r(\vec{z}) \vec{B}^{(k)} \\ \vec{\bar{K}}^{(k)}_{r+1} - (z_2 - 1) A'_{r+1}(\vec{z}) \vec{B}^{(k)} \\ \vec{K}'^{(k)}_{r+2}(\vec{z}) \\ \vdots \\ \vec{K}'^{(k)}_{k}(\vec{z}) \end{bmatrix} \right| \\
&= h(1) \oint_1 \prod_{j=2}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{2 \le i < j \le n-1} (z_j - z_i)}{\prod_{j=3}^{n-1} (z_j - 1)^j} \prod_{j=2}^{n-1} h(z_j) \left\{ \frac{1}{(z_2 - 1)^2} \left| \begin{bmatrix} \vec{\bar{K}}^{(k)}_1 \\ \vdots \\ \vec{\bar{K}}^{(k)}_{r-1} \\ A_r(\vec{z}) \vec{B}^{(k)} \\ \vec{\bar{K}}^{(k)}_{r+1} \\ \vec{K}'^{(k)}_{r+2}(\vec{z}) \\ \vdots \\ \vec{K}'^{(k)}_{k}(\vec{z}) \end{bmatrix} \right| - \frac{A_r(\vec{z}) A'_{r+1}(\vec{z})}{(z_2 - 1)} \left| \begin{bmatrix} \vec{\bar{K}}^{(k)}_1 \\ \vdots \\ \vec{\bar{K}}^{(k)}_{r-1} \\ \vec{B}^{(k)} \\ \vec{B}^{(k)} \\ \vec{K}'^{(k)}_{r+2}(\vec{z}) \\ \vdots \\ \vec{K}'^{(k)}_{k}(\vec{z}) \end{bmatrix} \right| \right\}.
\end{aligned}$$

Obviously, the second term vanishes because the r^{th} row and the $r + 1^{\text{st}}$ row are equivalent. Carrying out the same calculations to the $r + 2^{\text{nd}}$ and subsequent rows of the first term, we obtain

$$h(1) \oint_1 \prod_{j=2}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{2 \le i < j \le n-1} (z_j - z_i)}{\prod_{j=2}^{n-1} (z_j - 1)^j} \prod_{j=2}^{n-1} h(z_j) \left| \begin{bmatrix} \vec{\bar{K}}^{(k)}_1 \\ \vdots \\ \vec{\bar{K}}^{(k)}_{r-1} \\ A_r(\vec{z}) \vec{B}^{(k)} \\ \vec{\bar{K}}^{(k)}_{r+1} \\ \vdots \\ \vec{\bar{K}}^{(k)}_{k} \end{bmatrix} \right|.$$

Since $\bar{K}_{ij}$ is independent of $\vec{z}$-variables and $A_i(\vec{z})$ is independent of z_1, it is allowed to revive z_1-integral as

$$\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left\| \begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vdots \\ \vec{\bar{K}}_{r-1}^{(k)} \\ (z_1-1)A_r(\vec{z})\vec{B}^{(k)} \\ \vec{\bar{K}}_{r+1}^{(k)} \\ \vdots \\ \vec{\bar{K}}_k^{(k)} \end{bmatrix} \right\| .$$

In addition, since $\bar{K}_{ij}$ is independent of $\vec{z}$-variables, we can apply Lemma 7.4 to r^{th} row and obtain

$$\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left\| \begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vdots \\ \vec{\bar{K}}_{r-1}^{(k)} \\ \sum_{p=1}^{n-1}\prod_{q=1}^{p}(z_q-1)(A_p)_r\vec{B}^{(k)} \\ \vec{\bar{K}}_{r+1}^{(k)} \\ \vdots \\ \vec{\bar{K}}_k^{(k)} \end{bmatrix} \right\| . \tag{E.20}$$

This implies that we can rewrite r^{th} summand in the sum over $\ell \in [1, k]$ on the right hand side of (E.17) as (E.20) for all $r \in [1, k]$. Therefore, we have

$$\begin{aligned} &\oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left\| \begin{bmatrix} \vec{K}_1^{(k)}(\vec{z}) \\ \vdots \\ \vec{K}_k^{(k)}(\vec{z}) \end{bmatrix} \right\| \\ &= \oint_1 \prod_{j=1}^{n-1} \frac{\mathrm{d}z_j}{2\pi \mathrm{i}} \frac{\prod_{1\le i<j\le n-1}(z_j - z_i)}{\prod_{j=1}^{n-1}(z_j-1)^{j+1}} \prod_{j=1}^{n-1} h(z_j) \left\{ \left\| \begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vec{\bar{K}}_k^{(k)} \end{bmatrix} \right\| - \sum_{\ell=1}^{k} \left\| \begin{bmatrix} \vec{\bar{K}}_1^{(k)} \\ \vdots \\ \vec{\bar{K}}_{\ell-1}^{(k)} \\ \sum_{p=1}^{n-1}\prod_{q=1}^{p}(z_q-1)(A_p)_\ell\vec{B}^{(k)} \\ \vec{\bar{K}}_{\ell+1}^{(k)} \\ \vdots \\ \vec{\bar{K}}_k^{(k)} \end{bmatrix} \right\| \right\} . \end{aligned} \tag{E.21}$$

By the Laplace expansion, we can see that the terms in the curly brackets the right hand side of (E.21) is equivalent to that of (7.25) with $a_{i,j} = \bar{K}_{ij}$, $u_i = \sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1)(A_p)_i$ and $v_j = B_j$. Hence we obtain

$$\left|\begin{bmatrix} \vec{K}_1^{(k)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vec{K}_k^{(k)} \end{bmatrix}\right| - \sum_{\ell=1}^{k} \left|\begin{bmatrix} \vec{K}_1^{(k)} \\ \vdots \\ \vec{K}_{\ell-1}^{(k)} \\ \sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1)(A_p)_\ell \vec{B}^{(k)} \\ \vec{K}_{\ell+1}^{(k)} \\ \vdots \\ \vec{K}_k^{(k)} \end{bmatrix}\right| = \left|\begin{bmatrix} \vec{K}_1^{(k)} - \sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1)(A_p)_1 \vec{B}^{(k)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vec{K}_k^{(k)} - \sum_{p=1}^{n-1} \prod_{q=1}^{p} (z_q - 1)(A_p)_k \vec{B}^{(k)} \end{bmatrix}\right| \tag{E.22}$$

and, finally, it follows from (E.21) and (E.22) that the equality (E.15) holds.

F. Proof of the Uniform Convergence of $\mathcal{K}_t$ on a Bounded Set, Proposition 7.10

A rigorous proof falls into the same pattern as the proof of Proposition 6.2 shown in Appendix D.2. To avoid reiterating ourselves, here we only give a basic idea of the proof. Recall that $\mathcal{K}_t(\xi, \zeta)$ is given by

$$\mathcal{K}_t(\xi, \zeta) = -\lambda_2 t^{1/2} \oint_1 \frac{\mathrm{d}w}{2\pi \mathrm{i}} \mathrm{e}^{f(w,t,\xi) - f(\rho',t,\xi) + g_4(w)},$$

where $f(w, t, \xi) = g_1(w)t + g_2(w, \xi)t^{1/2} + g_3(w)t^{1/3}$ with $g_i(w)$ and $g_2(w, \xi)$ given in (7.40). Solving $g_1'(w) = 0$ gives us $w_1 = \rho'$ and $w_2 = -\rho'/2$. One can obtain a steepest descent through $w_1 = \rho'$, since the one passing w_2 would include extra poles at origin and hence vary the estimate of the integral.

One can see that the contour $\Theta = \Theta_1 \cup \Theta_2$ (see Fig. 12), where

$$\Theta_1 = \left\{ w = \rho' - \frac{s}{\sqrt{3}}\mathrm{i}, \quad s \in [-\rho, \rho] \right\}, \tag{F.1a}$$

$$\Theta_2 = \left\{ w = 1 + \frac{2\rho}{\sqrt{3}} \mathrm{e}^{\mathrm{i}s}, \quad s \in [-5\pi/6, 5\pi/6] \right\}, \tag{F.1b}$$

is a steepest descent path of $g_1(w)$ passing through ρ'. Namely, $w = \rho'$ is the strict global maximum point of $\mathrm{Re}(g_1)$ along Θ. This can be proved by calculating $\frac{\mathrm{dRe}(g_1)(s)}{\mathrm{d}s}$ along Θ.

As shown in the proof of Proposition 6.2, we can prove that for large enough t, only the part $\Theta^\delta := \{w \in \Theta \mid |w - w_1| \le \delta\}$, where $\delta = t^{-1/6}$, contributes to the integral. Near $w_1 = \rho'$, the Taylor expansion of $g_i(w)$ are given by

$$g_1(w) - g_1(w_1) = \frac{9(1-\rho)(w-\rho')^2}{16(2-\rho)\rho} + \mathcal{O}[(w-\rho')^3], \tag{F.2a}$$

$$g_2(w, \xi) - g_2(w_1, \xi) = -\frac{c_{\mathrm{g}} s_{\mathrm{g}} + 2\rho(2-\rho)\lambda_2 \xi}{2\rho(1-\rho)(2-\rho)} (w - \rho') + \mathcal{O}[(w-\rho')^2], \tag{F.2b}$$

$$g_3(w) - g_3(w_1) = \mathcal{O}[(w-\rho')^2], \tag{F.2c}$$

$$g_4(w) - g_4(w_1) = \mathcal{O}[(w-\rho')]. \tag{F.2d}$$

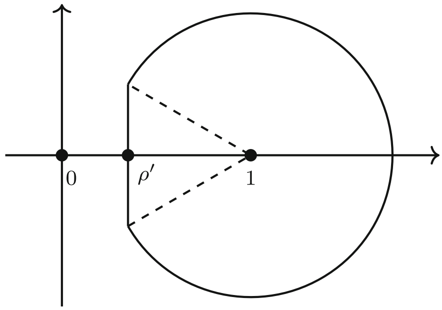

Fig. 12. steepest descent contour of integration in $\mathcal{K}_t$ passing the saddle point at $w_1 = \rho'$

Denote the function $g_i(w)$ without the error terms by $\bar{g}_i(w)$. As in Proposition 6.2, we can show that for large t, only the term $\mathrm{e}^{\bar{g}_1(w)t+\bar{g}_2(w,\xi_1,\kappa_1)t^{1/2}+\bar{g}_3(w)t^{1/3}+\bar{g}_4(w)}$ contributes to the integral. We re-parameterise Γ_δ by

$$w-\rho' = \mathrm{i}v\frac{2}{3}\sqrt{\frac{\rho(2-\rho)}{t(1-\rho)}},$$

where $-c\delta t^{1/2} \le v \le c\delta t^{1/2}$, and $c = \frac{3}{2}\sqrt{\frac{(1-\rho)}{\rho(2-\rho)}}$. Thus we are left with

$$\begin{aligned}
&\lim_{t\to\infty}(-\lambda_2 t^{1/2})\int_{\Gamma_\delta}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\,\mathrm{e}^{\bar{g}_1(w)t+\bar{g}_2(w,\xi)t^{1/2}+\bar{g}_3(w)t^{1/3}+\bar{g}_4(w)-f(\rho',t,\xi_1)}\\
&=\lim_{t\to\infty}(-\lambda_2 t^{1/2})\int_{\Gamma_\delta}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\exp\left(\frac{9(1-\rho)(w-\rho')^2}{16(2-\rho)\rho}t-\frac{c_g s_g+2\rho(2-\rho)\lambda_2\xi}{2\rho(1-\rho)(2-\rho)}(w-\rho')t^{1/2}-\ln(\rho')\right)\\
&=\lim_{t\to\infty}\frac{-1}{2\sqrt{2}\pi}\int_{c\delta t^{1/2}}^{-c\delta t^{1/2}}\mathrm{d}v\,\mathrm{e}^{-v^2/4-v\mathrm{i}(s_g+\xi)/\sqrt{2}}=\frac{1}{2\sqrt{2}\pi}\int_{-\infty}^{\infty}\mathrm{d}v\,\mathrm{e}^{-v^2/4-v\mathrm{i}(s_g+\xi)/\sqrt{2}}=\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-(s_g+\xi)^2/2},
\end{aligned}$$

where λ_2 is defined in (7.37).

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