HOMOGENEOUS FUNCTIONS WITH NOWHERE VANISHING HESSIAN DETERMINANT

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ABSTRACT. We prove that functions that are homogeneous of degree $\alpha \in (0, 1)$ on $\mathbb{R}^n$ and have nowhere vanishing Hessian determinant cannot change sign.

1. INTRODUCTION

Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be the cone over a domain $\Sigma \subset \mathbb{S}^{n-1}$ that has nonempty boundary. Let

$$p_n := \max\{1, (n-1)(n-2)\}.$$

In this paper we show:

Theorem 1.1. *If there exists a function $u \in C\left(\overline{\Omega}\right)$ that satisfies:*

(i) u is homogeneous of degree $\alpha \in (0, 1)$,
(ii) $u \in W^{2,p}_{loc}(\Omega) \cap W^{1,\infty}\left(\Omega \cap \left(B_2 \backslash \overline{B_1}\right)\right)$ for some $p > p_n$ (with $p = p_n$ allowed if $n \leq 3$),
(iii) $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, and
(iv) either $\det D^2 u$ or $-\det D^2 u$ is locally strictly positive in Ω,

then $\mathbb{R}^n \backslash \Omega$ is a convex cone, and Σ contains a closed hemisphere.

By locally strictly positive we mean bounded below by positive constants almost everywhere on compact sets, where the constants may depend on the sets. We in fact show in the course of proving Theorem 1.1 that $-\det D^2 u$ is locally strictly positive, and that $D^2 u$ has exactly one negative eigenvalue (see Remark 3.2).

An immediate consequence of Theorem 1.1 is:

Theorem 1.2. *If $u : \mathbb{R}^n \to \mathbb{R}$ satisfies:*

(i) u is homogeneous of degree $\alpha \in (0, 1)$,
(ii) $u \in W^{2,p}_{loc}\left(\mathbb{R}^n \backslash \{0\}\right)$ for some $p > p_n$, and
(iii) either $\det D^2 u$ or $-\det D^2 u$ is locally strictly positive in $\mathbb{R}^n \backslash \{0\}$,

then u does not change sign.

Indeed, if u changes sign then we may apply Theorem 1.1 to the sets $\Omega = \{u > 0\}$ and $\Omega = \{-u > 0\}$ to get a contradiction.

Remark 1.3. Theorem 1.2 is special to the cases $\alpha \in (0, 1)$. Indeed, if $\alpha \notin [0, 1]$ and $\alpha < k^2$ for some nonzero integer k, then the α-homogeneous function

$$u = r^\alpha \cos(k\theta)$$

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is sign-changing and has nowhere vanishing Hessian determinant on $\mathbb{R}^2\backslash\{0\}$. We also remark that 0-homogeneous functions have vanishing Hessian determinant on the rays where they achieve their maxima, and 1-homogeneous functions have identically vanishing Hessian determinant.

Apart from its own interest, Theorem 1.2 is motivated by the question of when interior gradient estimates hold for solutions to the special Lagrangian equation

$$F(D^2u) := \sum_{k=1}^{n} \tan^{-1}(\lambda_k(D^2u)) = \Theta(x) \in \left(-n\frac{\pi}{2}, n\frac{\pi}{2}\right). \tag{1}$$

Here u is a function on a domain in $\mathbb{R}^n$ and $\lambda_k(D^2u)$ denote the eigenvalues of D^2u. Equation (1), introduced in the seminal work of Harvey and Lawson [HL1], prescribes the mean curvature of the gradient graph of u in $\mathbb{R}^n \times \mathbb{R}^n$. In particular, this graph is volume-minimizing when Θ is constant. The existence of continuous viscosity solutions to the Dirichlet problem for (1) is known in certain situations (see e.g. [HL2], [HL3], [CP] and [DDT] for results concerning the Dirichlet problem for viscosity solutions, and [CNS], [CPW], and [BMS] for cases in which classical solutions can be obtained), and there are many fascinating open questions concerning the regularity of these solutions. For example, is not known whether viscosity solutions to (1) are locally Lipschitz if either Θ is a constant with $|\Theta| < (n-2)\frac{\pi}{2}$ (they are if Θ is a constant with $|\Theta| \geq (n-2)\frac{\pi}{2}$, see [WY], and more generally if $\Theta \in C^2$ and $|\Theta| \geq (n-2)\frac{\pi}{2}$, see [BMS]) or if Θ is Lipschitz. Classical proofs of interior gradient estimates for elliptic PDEs involve differentiating the equation once, so is reasonable to ask if interior gradient estimates for (1) hold under such conditions on Θ.

A first attempt to *disprove* the validity of such estimates could be to build a function u that is homogeneous of degree $\alpha \in (0, 1)$, smooth away from the origin, and has nowhere vanishing Hessian determinant. Then $F(D^2u)$ would behave near the origin like a multiple of $\frac{\pi}{2}$ plus a smooth function on the sphere times $|x|^{2-\alpha}$, which is C^1, while u has unbounded gradient. More precisely, if D^2u has k negative eigenvalues, then using the homogeneity of u and the expansion of $\tan^{-1}$ at $\pm\infty$ we calculate

$$F(D^2u(x)) = (n-2k)\frac{\pi}{2} - \sum_{j=0}^{\infty}\left[\frac{(-1)^j}{2j+1}\frac{\sigma_{n-1}[(D^2u(x/|x|))^{2j+1}]}{\det[(D^2u(x/|x|))^{2j+1}]}\right]|x|^{(2-\alpha)(2j+1)}$$

near the origin. Here $\sigma_k(D^2u)$ denotes the k^{th} symmetric polynomial of the eigenvalues of D^2u. Taking $u = |x|^\alpha$ appears to do the trick, but this function is not a viscosity solution at the origin; one needs u to change sign to prevent this issue. Theorem 1.2 precludes the existence of such functions, and hence can be viewed as evidence in favor of a positive result.

Remark 1.4. The most general results concerning the solvability of the Dirichlet problem for (1) in the class of continuous viscosity solutions have been established for continuous phases Θ that avoid the special values $\{(n-2k)\pi/2\}_{k=1}^{n-1}$ (see [HL3], [CP]). Moreover, it was recently shown that the comparison principle fails for certain continuous phases Θ that take a special value [Br]. In view of this one might expect weaker regularity results for solutions to (1) when the phase takes a special value. The possible examples we discussed above would have Lagrangian

phase $(n-2k)\pi/2$ at the origin for some k, in agreement with this expectation. However, in this work we rule out their existence.

Remark 1.5. One might also try to build one-homogeneous functions u on $\mathbb{R}^n$ such that $\sigma_{n-1}(D^2u)$ is nowhere vanishing, since in that case $F(D^2u)$ is Lipschitz. It is not hard to show that such functions are necessarily convex or concave (see Section 2), and thus do not solve the equation at the origin. Interestingly, there exist nonlinear one-homogeneous functions u on $\mathbb{R}^3$ whose Hessians are either indefinite or 0 at every point (see [M1]), so that $F(D^2u)$ tends to zero at the origin along rays, but $F(D^2u)$ is not continuous at the origin for these examples.

Remark 1.6. Theorem 1.2 can also be viewed as a rigidity result for homogeneous solutions to Monge-Ampère equations of mixed type. Such equations arise in challenging geometric problems such as the Minkowski problem for hedgehogs, for which basic existence and uniqueness questions remain open [M2].

Remark 1.7. Our results are also similar in spirit to those of H. Lewy about spherical harmonics in $\mathbb{R}^3$ [L1], [L2]. In those works Lewy first shows that a homogeneous polynomial P in $\mathbb{R}^3$ of degree $d > 2$ that has non-vanishing Hessian determinant away from the origin either doesn't change sign, or (up to multiplying by -1) the set $\{P > 0\}$ consists of two diametrically opposed convex cones. He then uses this to prove that the Hessian determinant of a spherical harmonic in $\mathbb{R}^3$ of degree > 2 must either vanish identically, or change sign. (Another proof of the latter result, using algebraic geometry, was given by Segre in [S]). Theorem 1.2 says that a similar conclusion holds in any dimension for functions that are homogeneous of a fractional degree and change sign. Moreover, the first step in Lewy's argument involves analyzing the geometry of level sets of P by considering the one-homogeneous function $P^{1/d}$. Likewise, in this work we consider the one-homogeneous function $u^{1/\alpha}$. However, we study the geometry of its gradient image. The author is grateful to a referee for bringing these results to his attention.

The paper is organized as follows. In Section 2 we recall some preliminary results about one-homogeneous functions and about maps with integrable dilatation, which are natural analogues of quasi-conformal maps in higher dimensions. In Section 3 we prove Theorem 1.1. The idea of the proof is to study the geometry of the gradient image of the one-homogeneous function $u^{\frac{1}{\alpha}}$. Our analysis is partly inspired by the beautiful arguments used in [HNY] to show the linearity of one-homogeneous functions on $\mathbb{R}^3$ that solve linear uniformly elliptic equations.

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2. Preliminaries

In this section we recall a few results about one-homogeneous functions and about maps with integrable dilatation.

Let v be a one-homogeneous function on $\mathbb{R}^n$ that, away from the origin, is locally $W^{2,p}$ for some $p \geq 1$. Euler's formula for homogeneous functions says that

$$v(x) = \nabla v(x) \cdot x. \tag{2}$$

Here and below we let $r := |x|$ and we denote points in $\mathbb{S}^{n-1}$ by ω. Writing

$$v = r\, g(\omega)$$

and choosing a coordinate system where ω is the last direction, we have

$$D^2 v(\omega) = \begin{bmatrix} \nabla^2_{\mathbb{S}^{n-1}} g + g\, I_{n-1\times n-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{3}$$

Here and below, $\nabla_{\mathbb{S}^{n-1}}$ and $\nabla^2_{\mathbb{S}^{n-1}}$ denote the usual gradient and Hessian operators on the sphere.

It is sometimes useful to represent v in $\{x_n > 0\}$ by a function $\bar{v}$ on $\mathbb{R}^{n-1}$ defined by

$$\bar{v}(y) := v(y,\, 1),$$

so that

$$v(x',\, x_n) = x_n \bar{v}\left(\frac{x'}{x_n}\right).$$

Taking the Hessian yields

$$D^2 v(y,\, 1) = \begin{bmatrix} D^2\bar{v} & -D^2\bar{v}\cdot y \\ -D^2\bar{v}\cdot y & y^T \cdot D^2\bar{v}\cdot y \end{bmatrix}. \tag{4}$$

Using this we calculate

$$\sigma_{n-1}(D^2 v)(y,\, 1) = \operatorname{tr}(\operatorname{cof}(D^2 v))(y,\, 1) = (1+|y|^2)\det D^2\bar{v}(y). \tag{5}$$

Here the operator σ_k denotes the k^{th} symmetric polynomial of the eigenvalues. It is easiest to verify this formula after rotating in the y variables so that $D^2\bar{v}$ is diagonal.

If v is C^2 in a neighborhood of e_n and $D^2\bar{v}(0)$ is nonsingular, then we can represent the gradient image of v near e_n as the graph of a function w using the relation

$$w(\nabla\bar{v}(y)) = \partial_n v(y,\, 1) = \bar{v} - y\cdot\nabla\bar{v},$$

i.e. w is the (negative) Legendre transform of $\bar{v}$. One differentiation gives

$$\nabla w(\nabla\bar{v}(y)) = -y,$$

and another gives

$$D^2 w(\nabla\bar{v}(y)) = -(D^2\bar{v})^{-1}(y). \tag{6}$$

In particular, up to a sign, the second fundamental form at $\nabla v(e_n)$ of the image under ∇v of a small ball around e_n is $(D^2\bar{v})^{-1}(0)$. From this it is easy to see that if v is locally C^2 away from the origin and $\sigma_{n-1}(D^2 v)$ is nowhere vanishing, then v is either convex or concave. Indeed, it suffices to show that either $D^2 v \geq 0$ or $-D^2 v \geq 0$ at some point. By (6) and (4) this is true at the inverse image under ∇v of any point on $\nabla v(\mathbb{S}^{n-1})$ that is touched from one side by a hyperplane.

We now recall a few facts about maps of integrable dilatation. Let

$$\varphi : U \subset \mathbb{R}^n \to \mathbb{R}^n$$

be a map in $W^{1,\,n}_{\text{loc}}(U)$ such that $\det D\varphi > 0$ almost everywhere. The dilatation $\mathbf{D}$ of φ is defined by the ratio

$$\mathbf{D}(x) := \frac{|D\varphi|^n}{\det D\varphi}.$$

If $\mathbf{D}$ is bounded and $n = 2$ then φ is quasi-conformal, hence continuous and open by classical results. Reshetnyak extended this result to mappings with bounded

dilatation in all dimensions [R]. The boundedness required in Reshetnyak's theorem has since been relaxed to integrability in certain L^p spaces:

Theorem 2.1 (**Iwaniec-Šverák**, [IS]). *If $n = 2$ and $\mathbf{D} \in L^1_{loc}(U)$, then φ is continuous and open.*

Theorem 2.2 (**Manfredi-Villamor**, [MV]). *If $n \geq 3$ and $\mathbf{D} \in L^p_{loc}(U)$ for some $p > n-1$, then φ is continuous and open.*

It is conjectured that the latter result holds in the equality case $p = n-1$ (see [IS]), and there are counterexamples when $p < n-1$ due to Ball (see [B]).

3. Proof of Theorem 1.1

In this final section we prove the main theorem.

Proof of Theorem 1.1. We write

$$u = r^\alpha f(\omega).$$

In a coordinate system where the last direction is $\omega \in \Sigma$, the Hessian of u at ω can be written

$$D^2 u = \begin{bmatrix} \nabla^2_{\mathbb{S}^{n-1}} f + \alpha\, f\, I_{n-1\times n-1} & (\alpha-1)\nabla_{\mathbb{S}^{n-1}} f \\ (\alpha-1)\nabla_{\mathbb{S}^{n-1}} f & \alpha(\alpha-1) f \end{bmatrix}. \tag{7}$$

Subtracting the multiple $\frac{(\nabla_{\mathbb{S}^{n-1}} f)_k}{\alpha f}$ of the last row from the k^{th} row in (7) for $k \leq n-1$ and taking the determinant we arrive at

$$\det D^2 u = \alpha(\alpha-1) f^{2-n} \cdot \left[\det\left(f\nabla^2_{\mathbb{S}^{n-1}} f + \left(\frac{1}{\alpha} - 1\right) \nabla_{\mathbb{S}^{n-1}} f \otimes \nabla_{\mathbb{S}^{n-1}} f + \alpha f^2 I_{n-1\times n-1} \right) \right]. \tag{8}$$

Now let v be the one-homogeneous function on $\mathbb{R}^n$ defined by

$$v(x) = \begin{cases} (u(x))^{1/\alpha}, & x \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

At $\omega \in \Sigma$ we compute the Hessian of v in the same coordinates as above, using the formula (3):

$$D^2 v = \frac{1}{\alpha} f^{\frac{1}{\alpha}-2} \begin{bmatrix} f\nabla^2_{\mathbb{S}^{n-1}} f + \left(\frac{1}{\alpha} - 1\right) \nabla_{\mathbb{S}^{n-1}} f \otimes \nabla_{\mathbb{S}^{n-1}} f + \alpha f^2 I_{n-1\times n-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{9}$$

We conclude from (8) and (9) that

$$\sigma_{n-1}(D^2 v) = \frac{f^{\frac{n-1}{\alpha}-n}}{\alpha^n(\alpha-1)} \det D^2 u \tag{10}$$

on Σ. In particular, either $\sigma_{n-1}(D^2 v)$ or $-\sigma_{n-1}(D^2 v)$ is locally strictly positive in Ω. We also have by standard embeddings that $v \in C^1_{loc}(\mathbb{R}^n \backslash \{0\})$. On $\mathbb{R}^n \backslash \Omega$ this follows from the fact that

$$\nabla v = \frac{1}{\alpha} u^{\frac{1}{\alpha}-1} \nabla u$$

in Ω, the facts that $1/\alpha > 1$ and $u = 0$ on $\partial\Omega$, and the local Lipschitz regularity of u up to $\partial\Omega$ away from the origin. In Ω, we use the observation that by homogeneity v can be viewed as a function of $n-1$ variables. When $n \geq 4$ the Sobolev exponent p_n is thus supercritical. In the case $n = 3$ it is critical and, denoting by $\bar{v}$ the

restriction of v to a hyperplane tangent to $\mathbb{S}^{n-1}$, we may apply the continuity assertion in Theorem 2.1 to either $\nabla\bar{v}$ or its reflection over a line. Here we used the fact that $\sigma_{n-1}(D^2v)$ or $-\sigma_{n-1}(D^2v)$ is locally strictly positive in $\{v > 0\}$ and the relation (5). In the case $n = 2$ we use that $W^{1,1}$ embeds to continuous on the line.

Now let $K := \nabla v\left(\mathbb{S}^{n-1}\right)$. For $\nu \in \mathbb{S}^{n-1}$, slide the hyperplane $\{x\cdot\nu = t\}$ (starting with t large, and decreasing t) until it touches K at some point p_ν. Since $0 \in K$ we have $p_\nu \cdot \nu \geq 0$. We claim that the following implication holds:

$$p_\nu \cdot \nu > 0 \Rightarrow (\nabla v)^{-1}(p_\nu) \cap \mathbb{S}^{n-1} = \{\nu\}. \tag{11}$$

To show the implication (11) it suffices to show that if $p_\nu \cdot \nu > 0$ then $(\nabla v)^{-1}(p_\nu) \cap \mathbb{S}^{n-1} \subset \{\nu, -\nu\}$, since by (2) we have

$$\nabla v(\omega) \cdot \omega = v(\omega) \geq 0$$

for all $\omega \in \mathbb{S}^{n-1}$. Assume by way of contradiction that $p_\nu \cdot \nu > 0$ but $(\nabla v)^{-1}(p_\nu) \cap \mathbb{S}^{n-1}$ is not contained in $\{\nu, -\nu\}$. After a rotation we may assume that $\nu = e_1$, and after another isometry in the x_2, ..., x_n variables we may assume that $\nabla v(\tilde{\nu}) = p_\nu$ for some $\tilde{\nu} \in \mathbb{S}^{n-1}$ such that $\tilde{\nu}_n > 0$. For $y \in \mathbb{R}^{n-1}$ we let

$$\bar{v}(y) = v(y, 1).$$

By construction, $\partial_1\bar{v}$ has a local maximum at $\tilde{\nu}/\tilde{\nu}_n$ (here we identify points on the hyperplane $\{x_n = 1\}$ with $\mathbb{R}^{n-1}$), and $\bar{v} > 0$ at this point, since $\nabla v(\tilde{\nu}) = p_\nu$ is nonzero. However, using (5) we see that either $\det D^2\bar{v}$ or $-\det D^2\bar{v}$ is locally strictly positive in $\{\bar{v} > 0\}$. Theorem 2.1 or 2.2, applied to either $\nabla\bar{v}$ or its reflection over a hyperplane, implies that $\nabla\bar{v}$ is an open mapping in $\{\bar{v} > 0\}$, which contradicts that $\partial_1\bar{v}$ has a local maximum in this set.

Finally, let $\mathrm{co}(K)$ denote the convex hull of K, and let w be the support function of $\mathrm{co}(K)$, that is,

$$w(x) := \sup_{y\in\mathrm{co}(K)} (y \cdot x).$$

The implication (11) implies that $v = w$. Indeed, it is clear that $0 \leq v \leq w$, and for $\nu \in \mathbb{S}^{n-1} \cap \{w > 0\}$ we have

$$w(\nu) = p_\nu \cdot \nu = \nabla v(\nu) \cdot \nu = v(\nu).$$

Because either $\sigma_{n-1}(D^2v)$ or $-\sigma_{n-1}(D^2v)$ is locally strictly positive in Ω, the set $\mathrm{co}(K)$ has non-empty interior. Indeed, if not, then $v = w$ is translation-invariant in some direction orthogonal to $\mathrm{co}(K)$, which along with the one-homogeneity of v implies that $\sigma_{n-1}(D^2v) \equiv 0$. We conclude that $\{v > 0\}\cap\mathbb{S}^{n-1}$ contains some closed hemisphere. Indeed, for some point z and some $\mu > 0$ we have $B_\mu(z) \subset \mathrm{co}(K)$, and by the definition of w we thus have

$$\{x \cdot z \geq 0\} \cap \mathbb{S}^{n-1} \subset \{w \geq \mu\} = \{v \geq \mu\}.$$

This completes the proof. □

Remark 3.1. The proof in fact shows that u is the α^{th} power of the support function of a convex set that has nonempty interior and 0 in its boundary (note that if 0 were not in the boundary of $\mathrm{co}(K)$ then $w = v > 0$ on all of $\mathbb{S}^{n-1}$), and that $\mathbb{R}^n\backslash\Omega$ is the reflection through the origin of the convex dual to the tangent cone of this set at the origin (see Figure 1).

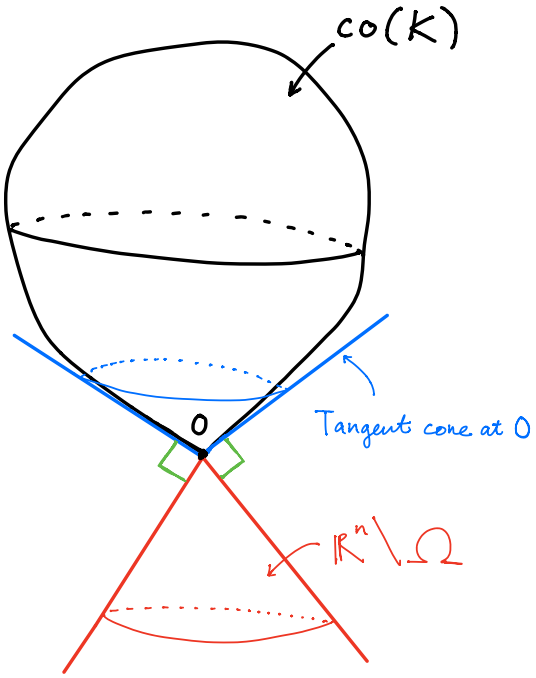

FIGURE 1

Remark 3.2. The convexity of v and relation (10) imply that $-\det D^2u$ is locally strictly positive in Ω. Moreover, D^2u has exactly one negative eigenvalue. Indeed, one sees using expressions (7) and (9) that D^2u can be written in appropriate coordinates at a point in Ω in the form

$$D^2u = M - p \otimes p,$$

where M has $n-1$ positive eigenvalues, $M_{nk} = 0$ for all k, and $p_n > 0$. The conclusion is clear when p is a multiple of e_n. Since

$$\det(M - q \otimes q) = -\sigma_{n-1}(M) q_n^2 < 0$$

for any $q \in \mathbb{R}^n$ with $q_n \neq 0$, the conclusion for general p with $p_n > 0$ follows from the continuity of the eigenvalues of $M - p \otimes p$ as a function of p.

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