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A nonconventional stability approach for a nonlinear Crank–Nicolson method solving degenerate Kawarada problems

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A B S T R A C T

Conventionally, the numerical stability of finite difference approximations of nonlinear Kawarada problems is shown only via frozen source terms, that is, ignoring potential jeopardization from quenching nonlinearities. The approach leaves an inadequacy behind even in the sense of localized stability analysis. This paper provides a much improved analysis of the numerical stability without freezing nonlinear source terms of the underlying equations. The strategy implemented can be extended for similar endeavors to higher dimensional cases. Simulation experiments are included.

1. Introduction

Let $\Omega_n \subset \mathbb{R}^n,\ n \in \mathbb{N}^+$, be a simply connected finite convex open domain, $\partial\Omega_n$ be its boundary and $\bar{\Omega}_n = \Omega_n \cup \partial\Omega_n$. A degenerate Kawarada problem, or quenching problem, can be formulated as

$$\sigma(x)u_t = \nabla^2 u + f(u), \quad x \in \Omega_n,\ t > t_0, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega_n,\ t > t_0, \tag{1.2}$$

$$u(x,t_0) = u_0(x), \quad x \in \bar{\Omega}_n, \tag{1.3}$$

where $\sigma(x) > 0,\ x \in \Omega_n;\ \sigma(x) \geq 0,\ x \in \partial\Omega_n$, is the degeneracy function, u is a thermal distribution, ∇^2 is the Laplacian, $f(0) = f_0 > 0,\ f_u(u) > 0,\ 0 \leq u_0 \ll b,\ b \in \mathbb{R}^+,\ t_0 \geq 0$ and $\lim_{u\to b^-} f(u) = \infty$ [1–3]. It has been shown that when the shape of Ω_n is fixed, then there exists a unique threshold $a^* > 0$ such that if a, the n-volume of Ω_n, is greater than a^* then there exists a finite time $T(a)$ such that u quenches at $T(a)$, that is,

$$\lim_{t\to T^-(a)} \max_{x\in\bar{\Omega}_n} u(x,t) = b.$$

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The above implies that u stops existing for $t \geq T(a);\ a \geq a^*$ [4–6]. Such a phenomenon is referred to as *quenching,* and the corresponding u is a *quenching solution.*

Numerous computational methods have been developed to solve (1.1)-(1.3) and its fractional order extensions in the past decades (for instance, see [7–11] and references therein). The study of numerical stability has been conducted in the von Neumann sense, since localized linear analysis has been proven to be sufficient for the type of nonlinear solvers presented [8,12–15]. Nonlinear source functions such as $f(u)$ in (1.1) are traditionally frozen in the analysis. This treatment, though helps avoid undesirable nonlinear complications that may disturb investigations, raises concerns if the overall trustfulness of the algorithms is impaired. Source function frozen has become a significant burden as more accurate simulations are in demand for applications. This motivates our study to break the nonlinearity barrier, that is, without freezing the nonlinear source term in (1.1) in the stability analysis.

For the simplicity of discussions, we set $b = 1$ and focus on a one-dimensional episode of (1.1)-(1.3) in this letter. Map $\Omega_1 = (0, a)$ to $(0, 1)$ to obtain the following reformulated Kawarada problem

$$u_t = a^{-2}\phi(x)u_{xx} + \psi(x, u), \quad x \in (0, 1),\ t > t_0, \tag{1.4}$$
$$u(0, t) = u(1, t) = 0, \quad t > t_0, \tag{1.5}$$
$$u(x, t_0) = u_0(x), \quad x \in [0, 1], \tag{1.6}$$

where $\phi(x) = 1/\sigma(x),\ \psi(x, u) = f(u)/\sigma(x),\ x \in (0, 1)$. We consider typical degenerate and source functions [1,3,8]

$$\sigma(x) = x^\alpha(1 - x)^{1-\alpha},\ f(u) = (1 - u)^{-\beta}, \quad 0 \leq \alpha \leq 1,\ \beta > 0,\ x \in (0, 1). \tag{1.7}$$

Our focused study is organized as follows. In the next section, a second order Crank–Nicolson type method for (1.4)-(1.7) is derived. Uniform spatial grids and adaptive temporal steps are used. Analysis of key approximation features are presented. The study of stability is carried out without freezing the nonlinear source term in the sense of Hairer and Iserles [9,16]. It improves all existing results by introducing a much relaxed constraint [7–10,12,13]. Section 3 focuses on multiple numerical experiments that highlight the stability of solution and computations under relaxed stability conditions. Illustrations are given for the adaptive steps generated and used. It is found that the analysis remarkably achieves its goal to ensure the overall stability of the numerical method. The new ideas in the proof can be further extended for solving multidimensional Kawarada problems and degenerate nonlinear systems, especially those modeling polydisperse sedimentations and multiclass traffic flow dynamics [17]. The spectral norm is used throughout our investigation unless otherwise specified.

2. Stability analysis without freezing the source term

Let $a \geq a^*,\ N \in \mathbb{N}^+$ and $N \gg 1$, and denote $x_k = kh,\ k = 0, 1, \ldots, N + 1,\ h = 1/(N + 1)$. We have

$$u_{xx}(x_k, t) = \frac{u(x_{k-1}, t) - 2u(x_k, t) + u(x_{k+1}, t)}{h^2} + \mathcal{O}(h^2), \quad k = 1, 2, \ldots, N,\ t > t_0. \tag{2.1}$$

Further, let $v_k = v_k(t)$ be an approximation of $v(x_k, t),\ k = 0, 1, \ldots, N + 1$. Drop the truncation error in (2.1). A semidiscretized system follows immediately from (1.4)-(1.6),

$$u' = Au + \psi, \quad t > t_0, \tag{2.2}$$
$$u(t_0) = u_0, \tag{2.3}$$

where $u = (u_1, u_2, \ldots, u_N)^\top,\ \psi = (\psi_1, \psi_2, \ldots, \psi_N)^\top,\ A = BT \in \mathbb{R}^{N\times N}$ with $B = \mathrm{diag}\,[\phi_1, \phi_2, \ldots, \phi_N],\ T = (ah)^{-2}\mathrm{tridiag}\,[1, -2, 1]$.

Now, a trapezoidal integrator for the solution of (2.2), (2.3) generates

$$u(t_{j+1}) = E(\tau_j A)u(t_j) + \frac{\tau_j}{2}\left[\psi(u(t_{j+1})) + E(\tau_j A)\psi(u(t_j))\right] + \mathcal{O}(\tau_j^2), \tag{2.4}$$

where $E(\cdot) = \exp(\cdot)$ is the matrix exponential and $\tau_j = t_{j+1} - t_j \ll 1,\ j = 0, 1, \ldots, J$, are variable temporal steps determined via an adaptation procedure, such as the exponentially evolving grids (EEG) formula [18,19]. We adopt the Courant constraints $c_1 \le \tau_j/h^2 \le d_1$, where $c_1, d_1 > 0$ are suitable constants, due to the variable steps used. Let $u^{(\ell)}$ denote an approximation of $u(t_\ell),\ \ell = 0, 1, 2, \ldots, J+1$. Approximating E by the A-acceptable [1/1] Padé approximant and dropping all truncation errors we acquire following fully discretized implicit method from (2.4),

$$u^{(j+1)} = \left(I - \frac{\tau_j}{2}A\right)^{-1}\left(I + \frac{\tau_j}{2}A\right)\left[u^{(j)} + \frac{\tau_j}{2}\psi\left(u^{(j)}\right)\right] + \frac{\tau_j}{2}\psi\left(u^{(j+1)}\right), \quad j = 0, 1, \ldots, J, \tag{2.5}$$

$$u^{(0)} = u_0. \tag{2.6}$$

Definition 2.1 (*[16,20]*)**.** Let the perturbed system corresponding to a numerical method such as (2.5), (2.6) be

$$\epsilon^{(j+1)} = M\epsilon^{(j)}, \quad j = 0, 1, \ldots,$$

where $\epsilon^{(j)} = u^{(j)} - \tilde{u}^{(j)}$, $\tilde{u}^{(j)}$ is a perturbed solution, and M is the perturbed coefficient matrix. We say that the numerical method is stable if there exists a uniform constant $c_0 > 0$ such that

$$\|M\|_2 \le 1 + c_0\tau,$$

where $\tau = \max_{0\le\ell\le J}\tau_\ell \to 0^+$.

Theorem 2.2. *Let $\varepsilon > 0$ be arbitrarily small. If*

$$\beta\tau_\ell < \beta\tau_\ell + \varepsilon \le 2\min_k\left\{\sigma(x_k)\left(1 - \hat{u}_k^{(\ell)}\right)^{\beta+1}\right\}, \quad \ell = 0, 1, \ldots, J,$$

where $\beta > 0$ is given in (1.7) and $\hat{u}_k^{(\ell)} \in \left(\min\left\{u_k^{(\ell)}, \tilde{u}_k^{(\ell)}\right\}, \max\left\{u_k^{(\ell)}, \tilde{u}_k^{(\ell)}\right\}\right) \subset (0,1)$, then the semi-adaptive nonlinear Crank–Nicolson method (2.5), (2.6) for solving (1.4)-(1.7) is stable.

Proof. The perturbed system corresponding to (2.5), (2.6) is

$$\epsilon^{(j+1)} - \frac{\tau_j}{2}\left(\psi\left(u^{(j+1)}\right) - \psi\left(\tilde{u}^{(j+1)}\right)\right) = \left(I - \frac{\tau_j}{2}A\right)^{-1}\left(I + \frac{\tau_j}{2}A\right)\left[\epsilon^{(j)} + \frac{\tau_j}{2}\left(\psi\left(u^{(j)}\right) - \psi\left(\tilde{u}^{(j)}\right)\right)\right], \quad j = 0, 1, \ldots, J.$$

We expand $\psi\left(u^{(\ell)}\right) - \psi\left(\tilde{u}^{(\ell)}\right),\ \ell = j, j+1$, in a remainder form. Hence, the above system is equivalent to

$$\left(I - \frac{\tau_j}{2}F^{(j+1)}\right)\epsilon^{(j+1)} = \left(I - \frac{\tau_j}{2}A\right)^{-1}\left(I + \frac{\tau_j}{2}A\right)\left(I + \frac{\tau_j}{2}F^{(j)}\right)\epsilon^{(j)}, \quad j = 0, 1, \ldots, J,$$

where $F^{(\ell)} = \psi_u\left(\hat{u}^{(\ell)}\right) \in \mathbb{R}^{N\times N},\ \ell = j, j+1$, are diagonal Jacobi matrices involved. Note that matrices $A,\ F^{(\ell)}$ do not, in general, commute. Thus,

$$\epsilon^{(j+1)} = M\epsilon^{(j)} = \left(I - \frac{\tau_j}{2}F^{(j+1)}\right)^{-1}\left(I - \frac{\tau_j}{2}A\right)^{-1}\left(I + \frac{\tau_j}{2}A\right)\left(I + \frac{\tau_j}{2}F^{(j)}\right)\epsilon^{(j)}, \quad j = 0, 1, \ldots, J. \tag{2.7}$$

It follows therefore

$$\begin{aligned}\left\|\epsilon^{(j+1)}\right\|_2 &= \left\|\left(I - \frac{\tau_j}{2}F^{(j+1)}\right)^{-1}\left(I - \frac{\tau_j}{2}A\right)^{-1}\left(I + \frac{\tau_j}{2}A\right)\left(I + \frac{\tau_j}{2}F^{(j)}\right)\epsilon^{(j)}\right\|_2 \\ &\le \left\|\left(I - \frac{\tau_j}{2}F^{(j+1)}\right)^{-1}\right\|_2\left\|\left(I - \frac{\tau_j}{2}A\right)^{-1}\left(I + \frac{\tau_j}{2}A\right)\right\|_2\left\|I + \frac{\tau_j}{2}F^{(j)}\right\|_2\left\|\epsilon^{(j)}\right\|_2. \end{aligned}\tag{2.8}$$

First, we recall that the Jacobians $F^{(j)}$, $F^{(j+1)}$ are diagonal and their nontrivial elements are positive due to the fact that $f_u > 0$. Denote $\gamma = 2\min_{1\le k\le N}\left\{\sigma(x_k)\left(1-\hat{u}_k^{(j)}\right)^{\beta+1}\right\}$. Thus, following the hypothesis of the theorem, we may claim that

$$\left\|\left(I-\frac{\tau_\ell}{2}F^{(j+1)}\right)^{-1}\right\|_2 \le \frac{\gamma}{\gamma-\beta\tau_j} = 1+\frac{\beta\tau_j}{\gamma-\beta\tau_j} \le 1+\frac{\beta\tau_j}{\varepsilon} \le 1+c_1\tau_j,$$
$$\left\|I+\frac{\tau_j}{2}F^{(j)}\right\|_2 \le 1+\frac{\beta\tau_j}{\gamma} \le 1+c_2\tau_j.$$

Secondly, for the Toeplitz symmetric tridiagonal (TST) matrix T, we have $\|T\|_2 \le 4(ah)^{-2}$. Further, since

$$B^{-1}-\frac{\tau_j}{2}T = \text{tridiag}\left(-\frac{\tau_j}{2a^2h^2}, \frac{\tau_j}{a^2h^2}+\frac{1}{\phi_k}, -\frac{\tau_j}{2a^2h^2}\right)$$

is tridiagonal, an application of the Gershgorin circle theorem yields the estimate

$$\left\|\left(B^{-1}-\frac{\tau_j}{2}T\right)^{-1}\right\|_2 \le \frac{2\tau_j}{a^2h^2}+\frac{1}{\phi_k} \le \frac{2d_1}{a^2}+\frac{1}{\min_{1\le k\le N}\phi_k} = d_0$$

due to the Courant constraints $\tau_j/h^2 \le d_1,\ j=0,1,\ldots,J$. Consequently, we find that

$$\left\|\left(I-\frac{\tau_j}{2}A\right)^{-1}\left(I+\frac{\tau_j}{2}A\right)\right\|_2 = \left\|\left(B^{-1}-\frac{\tau_j}{2}T\right)^{-1}\left(B^{-1}+\frac{\tau_j}{2}T\right)\right\|_2 = \left\|I+\tau_j\left(B^{-1}-\frac{\tau_j}{2}T\right)^{-1}T\right\|_2$$
$$\le 1+\tau_j\left\|\left(B^{-1}-\frac{\tau_j}{2}T\right)^{-1}\right\|_2\|T\|_2 \le 1+d_0\|T\|_2\tau_j \le 1+\frac{c_3\tau_j}{h^2}.$$

Thirdly, based on the above discussions, from (2.8) we arrive at

$$\left\|\epsilon^{(j+1)}\right\|_2 \le (1+c_1\tau_j)(1+c_2\tau_j)\left(1+\frac{c_3\tau_j}{h^2}\right)\left\|\epsilon^{(j)}\right\|_2 \le \left(1+\frac{c\tau_j}{h^2}\right)(1+c\tau_j)^2\left\|\epsilon^{(j)}\right\|_2, \quad j=0,1,\ldots,J,$$

where $c=\max\{c_1,c_2,c_3\}$. This completes our proof. ■

Corollary 2.3. *Denote $\lambda_j=\tau_jh^{-2},\ j=0,1,\ldots,J$, be the variable Courant numbers used. Then we have an accumulative error bound for the perturbed system:*

$$\left\|\epsilon^{(J+1)}\right\|_2 \le de^{2c\tilde{T}(a)}\left\|\epsilon^{(0)}\right\|_2,$$

where $\tau=\max\{\tau_0,\tau_1,\ldots,\tau_J\},\ d=\prod_{j=0}^{J}(1+c\lambda_j),\ c>0$, and $\tilde{T}(a)\approx T(a)$ is approximately the finite quenching time.

Proof. Recall (2.7). Recursively, we arrive at an accumulative perturbation system,

$$\epsilon^{(j+m+1)} = \prod_{\ell=j}^{j+m}\left(I-\frac{\tau_\ell}{2}F^{(\ell+1)}\right)^{-1}\left(I-\frac{\tau_\ell}{2}A\right)^{-1}\left(I+\frac{\tau_\ell}{2}A\right)\left(I+\frac{\tau_\ell}{2}F^{(\ell)}\right)\epsilon^{(j)}$$
$$= M_{j,m}\epsilon^{(j)}, \quad j=0,1,\ldots,J-m,\ m\ge 0. \tag{2.9}$$

Set $j=0,\ m=J$ and take a spectrum norm on both sides of (2.9) to yield

$$\left\|\epsilon^{(J+1)}\right\|_2 = \left\|\prod_{\ell=0}^{J}\left(I-\frac{\tau_\ell}{2}F^{(\ell+1)}\right)^{-1}\left(I-\frac{\tau_\ell}{2}A\right)^{-1}\left(I+\frac{\tau_\ell}{2}A\right)\left(I+\frac{\tau_\ell}{2}F^{(\ell)}\right)\epsilon^{(0)}\right\|_2$$
$$\le \prod_{\ell=0}^{J}\left\|\left(I-\frac{\tau_\ell}{2}F^{(\ell+1)}\right)^{-1}\right\|_2\left\|\left(I-\frac{\tau_\ell}{2}A\right)^{-1}\left(I+\frac{\tau_\ell}{2}A\right)\right\|_2\left\|I+\frac{\tau_\ell}{2}F^{(\ell)}\right\|_2\left\|\epsilon^{(0)}\right\|_2.$$

Utilizing the estimates derived in Theorem 2.2, we observe from the above that

$$\begin{aligned}\left\|\epsilon^{(J+1)}\right\|_2 &\le \prod_{\ell=0}^{J}(1+c_1\tau_\ell)(1+c_2\tau_\ell)\left(1+\frac{c_3\tau_\ell}{h^2}\right)\left\|\epsilon^{(0)}\right\|_2 \le \left[\prod_{\ell=0}^{J}\left(1+\frac{c\tau_\ell}{h^2}\right)\right]\left[\prod_{\ell=0}^{J}(1+c\tau)^2\right]\left\|\epsilon^{(0)}\right\|_2\\ &\le d\,(1+c\tau)^{2(J+1)}\left\|\epsilon^{(0)}\right\|_2,\end{aligned}$$

where $c=\max\{c_1,c_2,c_3\}$, $\tau=\max\{\tau_0,\tau_1,\ldots,\tau_J\}$. Note that $T(a)=\sum_{\ell=0}^{J}\tau_\ell$ is the finite quenching time when $a\ge a^*$. Hence, $T(a)\lessapprox\tilde{T}(a)=(J+1)\tau<\infty$. Subsequently,

$$(1+c\tau)^{2(J+1)}=\left[(1+c\tau)^{1/c\tau}\right]^{2c(J+1)\tau}=\left[(1+c\tau)^{1/c\tau}\right]^{2c\tilde{T}(a)}\le e^{2c\tilde{T}(a)}<\infty.$$

The above completes our proof. ■

Remark 2.4. Denote $\tau=\max\{\tau_0,\tau_1,\ldots,\tau_J\}$, $\lambda=\max\{\lambda_0,\lambda_1,\ldots,\lambda_J\}$. We have

$$d=\prod_{j=0}^{J}(1+c\lambda_j)\le e^{c(J+1)\lambda}<\infty,\quad c>0,$$

due to the fact that a quenching stop is always finite, that is, $J<\infty$.

Remark 2.5. In the event if

$$\left\|\left(I-\frac{\tau_\ell}{2}A\right)^{-1}\left(I+\frac{\tau_\ell}{2}A\right)\right\|_2\le 1,\quad \ell=0,1,\ldots,J,$$

then we have the following improved uniform upper bound,

$$\left\|\epsilon^{(J+1)}\right\|_2\le e^{2c\tilde{T}(a)}\left\|\epsilon^{(0)}\right\|_2.$$

3. Simulation experiments

Since the semi-adaptive method (2.5), (2.6) is nonlinear, iterative procedures are in general needed for advancing the solution. However, since our method is stable, an explicit intermediate solver may be employed to simplify the computational procedure satisfactorily [7,9,15]. Under this consideration, we adopt the following for simulation experiments,

$$u^{(j+1)}=\left(I-\frac{\tau_j}{2}A\right)^{-1}\left(I+\frac{\tau_j}{2}A\right)\left[u^{(j)}+\frac{\tau_j}{2}\psi\left(u^{(j)}\right)\right]+\frac{\tau_j}{2}\psi\left(v^{(j+1)}\right),\ j=0,1,\ldots,J, \tag{3.1}$$

$$u^{(0)}=u_0. \tag{3.2}$$

where $v^{(j+1)}$ is calculated via an Euler method and then a two-step explicit Nyström scheme [20].

A typical testing nonlinear source function $f(u)$ with $\beta=1$ is considered [3,8]. Temporal steps $\tau^{(\ell)}$ are kept uniformly for simplicity and efficiency until a pre-quenching phenomenon is detected. The spatial domain $\Omega_1=(0,a)$ with $a=5$ is selected. For a further clarity, numerical results will be displayed directly on either $\bar{\Omega}_1$ or Ω_1.

Fig. 1 shows the numerical solution and its first, and second temporal derivative functions. A uniform time step τ is used in the experiments until the solution almost quenches, for example, when $\max_{0\le x\le 5}u(x,t)$ reaches 0.98. Then the sequence of adaptive temporal steps, $\{\tau_j\}$, begins through

$$\tau_j=\max\left\{\min_j\left\{\tau_{j-1},c_0\min_k\left\{\left(1-u_k^{(j)}\right)^2\right\}\right\},m_0\right\},$$

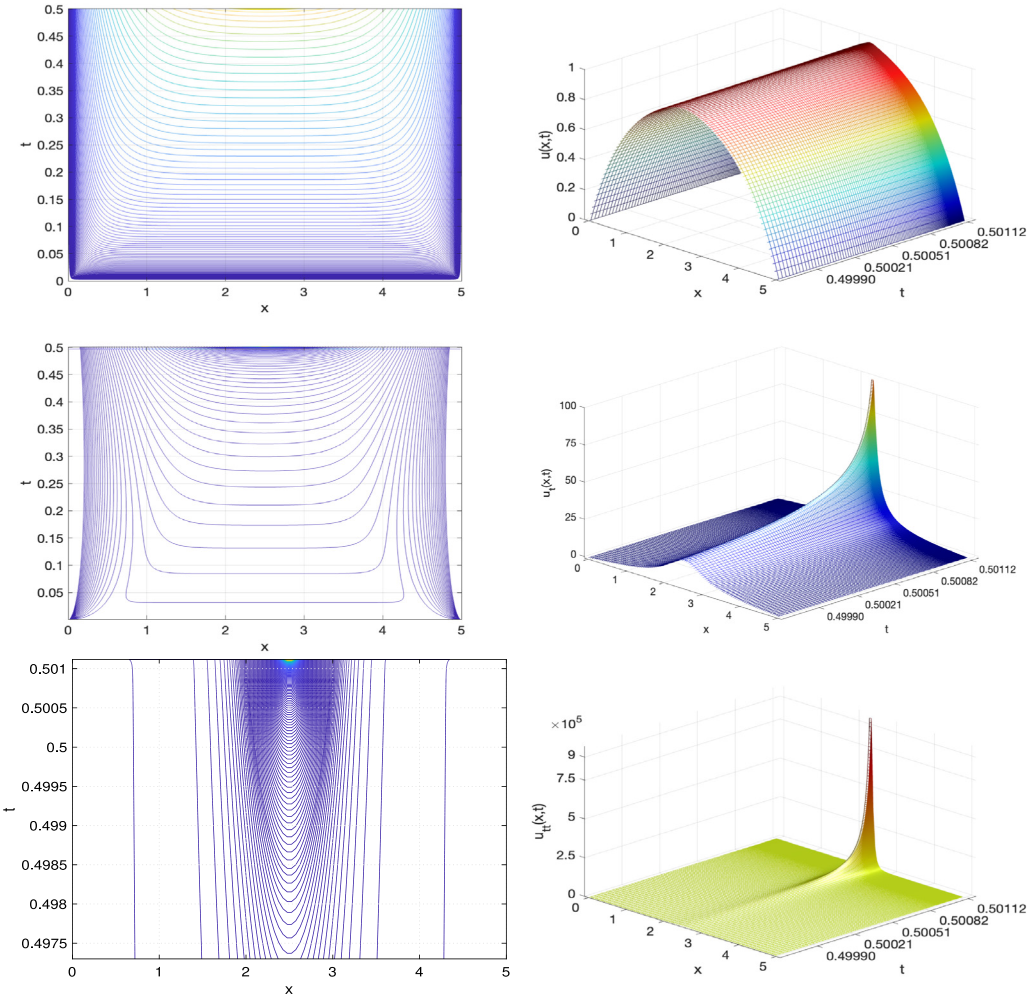

Fig. 1. Numerical solution u of (3.1)-(3.2) (TOP), its temporal derivative u_t (MIDDLE), and second derivative u_{tt} (BOTTOM) respectively. While the first two contour plots are full scaled, the rest of the figures are on the final 223 temporal steps immediately before quenching. Temporal grid adaptation is clearly visible. It is found that $\max_{0\le x\le 5} u(x,T(5)) \approx 0.99008661$, $\sup_{0<x<5} u_t(x,T(5)) \approx 99.77142399$, $\sup_{0<x<5} u_{tt}(x,T(5)) \approx 9.69478983 \times 10^5$, and $T(5) \approx 0.50111987$. The data agree satisfactorily with existing results [1,4,7,10,11].

where $c_0 > 0$ is a suitable speed controller, and m_0 is a minimum step size that may keep the ratio of τ_j/τ_{j-1} being bounded and smooth [10,11,19]. A quadratic function is being used to reflect the nonlinearity and determine the next step size which allows the actual quenching singularity to drive the process. The above monitoring function developed is different from classical arc-length formulas and is highly satisfactory. The weaker stability constraint stated in Theorem 2.2 is observed. A total of $J = 723$ temporal steps are executed. A single point quench is observed at $x = 2.5$ as predicted [1,12,21]. The numerical solution is clearly nonnegative, monotonically increasing, and stable as t increases at any $x \in [0, a]$. It can also be observed that while the solution u remains bounded throughout the computation, its rate of change function, that is, u_t, seems to shoot to infinity at $x = 2.5$ as quenching time is approached. The phenomenon is further illustrated by a second derivative u_{tt}, which blows up in Fig. 1. The numerical simulation reflects ideally a solid fuel combustion process facilitated through a single point ignition in the thermal physics [3,18].

Data availability

No data was used for the research described in the article.

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